# Estimating the $\phi(n)$ of Upper/Lower Bound in its RSA Cryptosystem

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**Abstract.** The RSA-768 (270 decimal digits) was factored by Kleinjung et al. on December 12 2009, and the RSA-704 (212 decimal digits) was factored by Bai et al. on July 2, 2012. And the RSA-200 (663 bits) was factored by Bahr et al. on May 9, 2005. Until right now, there is no body successful to break the RSA-210 (696 bits) currently. In this paper, we would discuss an estimation method to approach lower/upper bound of $\phi(n)$ in the RSA parameters. Our contribution may help researchers lock the $\phi(n)$ and the challenge RSA shortly.

**Keywords:** RSA cryptosystem; Euler's totient function; Factoring;

## 1 Introduction

Challenge RSA [12] is an interesting and difficult work. Recently, most scientists and researchers [1, 4, 8] using general number field sieve (GNFS) algorithm to factor RSA modulus $n$. In practical environment, it looks like if you to want to break the RSA, you may have best choice to choose GNFS when you already factor the modulus $n$. In theoretical, Wiener [17] first proposed a cryptanalysis of short secret exponents where the $d < N^{0.5}$ in 1990. Boneh [3] presented 'Twenty years of attacks on RSA cryptosystem' in 1999. He classified and described varieties attack. Followed by Boneh and Durfee [2], they suggested the provate key $d$ should be greater than $N^{0.292}$ for the security problem. Even though, some bodies focus on secret key $d$ or factor composite number $n$. Their purpose are clearly. We can not help but think, does it exist a general estimation way without factor to challenge RSA? In this article, we would introduce a new methodology where approach the lower bound and the upper bound of $\phi(n)$. For this generalize conception, it may match any bit length composite number $n$.

## 2 Review of RSA Conception

The signer prepares the prerequisite of a RSA signature: Two distinct large prime $p$ and $q$, $n = pq$, Let $e$ be a public key so that $\gcd(e, \phi(n)) = 1$, where $\phi(n) = (p-1)(q-1)$, then calculate the private key $d$ such that $ed \equiv 1 \pmod{\phi(n)}$. The signer publishes $(e, n)$ and keeps $(p, q, d)$ secretly. The notation as same in [12].

### 2.1 RSA Encryption and Decryption

In RSA public-key encryption, Alice encrypts a plaintext $M$ for Bob using Bob's public key $(n, e)$ by computing the ciphertext

$$C \equiv M^e \pmod{n} \tag{1}$$

where $n$, the modulus, is the product of two or more large primes, and $e$, the public exponent, is an (odd) integer $e \geq 3$ that is relatively prime to $\phi(n)$, the order of the multiplicative group $\mathbb{Z}_n^*$.

### 2.2 RSA Digital Signature

$$s \equiv M^d \pmod{n} \tag{2}$$

where $(n, d)$ is the signer's RSA private key. The signature is verified by recovering the message $M$ with the signer's RSA public key $(n, e)$:

$$M \equiv s^e \pmod{n} \tag{3}$$

## 3 Our Methodology

In this section, we would calculate the upper bound and the lower bound of $\phi(n)$ in RSA scheme. The detail described as below.

**Notation:**
$\ell$: means lower bound.
$u$: means upper bound.
$\varepsilon$: a decimal expansion number (e.g $99/100 = 0.99\cdots$).

### 3.1 Approaching $\phi(n)$

If $n$ is composite, hence

$$\phi(n) \leq n - \sqrt{n}, \tag{4}$$

Sierpinski [15] mentioned it in 1964. It is know that if equation (4) is a good upper bound for $\phi(n)$. Is there a good lower bound for $\phi(n)$? This question also be discussed by a newsgroup dialog between Ray Steiner and Bob Silverman in 1999 [16]. For $n > 30$, the $\phi(n) > n^{2/3}$, Kemdall and Osborn proved it [7]; for $n \geq 3$, the $\phi(n) > \frac{\log 2}{2} \frac{n}{\log n}$ given by Hatalova and Salat [6].

**3.1.1 Estimate upper bound** Does the equation (4) a good upper bound? In follows, we would estimate a new value where its smaller than previous and optimize.

**Theorem 1.** *Assume $p, q$ are large prime numbers, where $n = pq$, then $\phi(n) = 4k$, $k \in \mathbb{Z}$ where $1 \leq k \leq \lfloor \frac{n - 2\lceil\sqrt{n}\rceil + 1}{4} \rfloor$.*

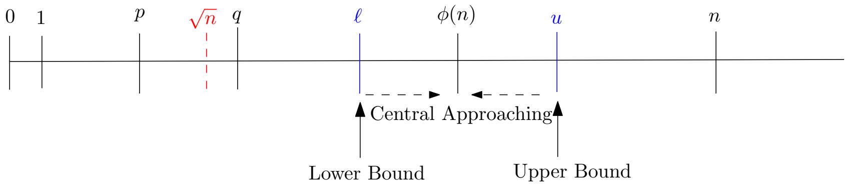


**Fig. 1.** The lower/upper bound of $\phi(n)$ in RSA.

*Proof.* As known two variant $p$ and $q$ are large prime numbers, and $p, q > 2$, since
$2 \nmid p,\ 2 \nmid q$,
therefore $2 \mid p-1,\ 2 \mid q-1$.
$4 \mid (p-1)(q-1),\ 4 \mid \phi(n)$.
$\phi(n) = 4k,\ k \in \mathbb{Z}^+$.
We will discuss how calculate the range of value $k$.

$$\begin{aligned}\phi(n) &= (p-1)(q-1)\\ &= pq-(p+q)+1\\ &= n-(p+q)=1.\end{aligned} \tag{5}$$

And

$$\begin{aligned}p+q &\geqslant 2\sqrt{n},\\ p+q &\in \mathbb{Z}^+,\\ 2 &\mid p+q.\\ p+q &\geqslant 2\lceil\sqrt{n}\rceil.\\ \phi(n) &\leq n+1-2\lceil\sqrt{n}\rceil.\\ \phi(n) &= 4k,\ k\in\mathbb{Z}^+.\\ \phi(n) &\leq 4\cdot\lfloor\frac{n+1-2\lceil\sqrt{n}\rceil}{4}\rfloor.\end{aligned} \tag{6}$$

Here, we know the maximum value (limit superior) for $k \leq \lfloor\frac{n-2\lceil\sqrt{n}\rceil+1}{4}\rfloor$, we call boundary value.
Consequently, according to above inference, we obtain a best upper bound $u$ of $\phi(n)$ where $\phi(n) \leq 4\lfloor\frac{n-2\lceil\sqrt{n}\rceil+1}{4}\rfloor$.

**Theorem 2.** *Assume $p, q$ are large prime numbers, and $p, q > 2$, $n = pq$, where $\phi(n) \leq t \leq n, t \in \mathbb{Z}$. Then $t = \phi(n) \iff x^2-(n+1-t)x+n = 0$ have two positive integer solutions.*

*Proof.* We describe two properties, necessary and sufficient conditions as follow:
Necessary condition:
If $t = \phi(n) = (p-1)(q-1) = pq-(p+q)+1 = n-(p+q) = 1$, then $n+1-t = p+q$,

in the same time, the formula be $x^2 - (p+q)x + pq = 0$. It is clearly, the equation of the two roots $p$ and $q$ are needed to set up.
Sufficient condition:
If equation $x^2 - (n+1-t)x + n = 0$ have two integer solutions.
Assume $x_1, x_2$ be the equation of the two roots, where $x_1, x_2 \in \mathbb{Z}^+$.
The equation could be transformed to $(x - x_1)(x - x_2) = 0$.
Promptly, $x^2 - (x_1 + x_2)x + x_1x_2 = 0$.
Then $n = x_1x_2$, according to $n$'s structure, there are two choice:
1) $x_1 = 1$ and $x_2 = n$ (or $x_1 = n$ and $x_2 = 1$).
2) $x_1 = p$ and $x_2 = q$ (or $x_2 = q$ and $x_2 = p$).
If $x_1, x_2$ one for 1 and $n$.
Now, $x_1 + x_2 = n + 1$, because $x_1 + x_2 = n + 1 - t$, hence $t = 0$.
However, in our assumption, the condition is $t > 0$, so this inference contradiction.
The equation have two integers $p$ and $q$ where

$$x_1 + x_2 = p + q, \tag{7}$$

then

$$p + q = n + 1 - t. \tag{8}$$

Promptly,

$$t = n + 1 - (p + q), \tag{9}$$

and

$$t = \phi(n). \tag{10}$$

Thus, for the sufficient condition is setting up.

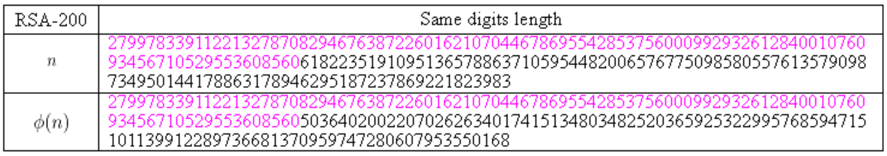
**Theorem 3.** *Assume $p, q$ are large prime numbers, and $p, q > 2$, $n = pq$, $t = 4k$ where $\phi(n) \leq t \leq n, t \in \mathbb{Z}$. If $\sqrt{(n+1-t)^2 - 4n}$ is an integer number, the equation $x^2 - (n+1-t)x + n = 0$ has two integer solutions.*

*Proof.* Since $p, q$ are both prime numbers where $n = pq$, $2 \nmid n$, but $2 \mid n + 1$.
And $t = 4k$, and $2 \mid t$. Thus $2 \mid n + 1 - t$.
If $\sqrt{(n+1-t)^2 - 4n} \in Z$, so $2 \mid \sqrt{(n+1-t)^2 - 4n}$.
The equation $x^2 - (n+1-t)x = 0$ of the two solutions are:
$x = \frac{n+1-t \pm \sqrt{(n+1-t)^2 - 4n}}{2}$.
Because $2 \mid n + 1 - t$, and while $\sqrt{(n+1-t)^2 - 4n} \in \mathbb{Z}$,
it exists $2 \mid \sqrt{(n+1-t)^2 - 4n}$.
When $\sqrt{(n+1-t)^2 - 4n} \in \mathbb{Z}$, $2 \mid n + 1 - t \pm \sqrt{(n+1-t)^2 - 4n}$.
Due to $\sqrt{(n+1-t)^2 - 4n} \in \mathbb{Z}$, the $x$ is also $\in \mathbb{Z}$.
Here, the two solutions of equation $x^2 - (n+1-t)x + n = 0$ are both integers.

**3.1.2 Estimate lower bound** Loomis et al. [10] thought the Shapiro's [13] lower bound $\phi(n) > n^{(\log 2)}/(\log 3)$ as a (naive) lower bound for $E_n$, they can determine when all members of a given $E_n$ have been found. Powell [11] pointed out that Konyagin and Shparlinksi's lower bound $N_1(n,p) > (p-1)/2 - p^{3/2}/n$ where $n > 1$ is a positive integer and that $p$ is an odd prime number with $p \equiv 1 \pmod{n}$; that is a good bound if $p$ is a small compared to $n$, and establishes that $N_1(n,p) \geq (\sqrt{\phi(n)}(\prod_{\substack{q\ prime \\ q|n}} q^{1/(q-1)})/n)p^{1-1/\phi(n)}$. Powell also discussed an improvement the upper and lower bounds in [11]. What is the optimal lower bound? The other discuss in following:

**Theorem 4.** *For all $n \geq 3$ we have $\phi(n) \geq \frac{n}{e^{\gamma}\log\log n} + \O(\frac{n}{(\log\log n)^2})$, where $\gamma$ is the Euler-Mascherone Constant, and the above holds with equality infinitely often.* ***Remark:****note in particular that since $loglogn \to \infty$ as $n$ grows large, we see that the result $\frac{n}{m} < \phi(n)$ can not hold for any fixed integer $m$.*

*Proof.* Consider *R*, set of all $n$ such that $m < n$ implies $\frac{\phi(n)}{n} < \frac{\phi(m)}{m}$. This set is then all of the 'record breaking' $n$. If $n \in$ *R* has $k$ prime factors, let $n^*$ be the product of the first $k$ prime factors. If $n \neq n^*$ and $\frac{\phi(n)^*}{n^*} \leq \frac{\phi(n)}{n}$, which is impossible. Hence, *R* consist entirely of $n$ of the form $n = \prod_{p\leq y} p$ for some $y$. Now for $n \in$ *R*, we can choose $y$ so that $\log n = \sum_{p\leq y} \log p = \theta(y)$. Then using one of Mertens estimates we see that $\frac{\phi(n)}{n} = \prod_{p\leq y}(1-\frac{1}{p}) = \frac{e^{-\gamma}}{\log y} + O(\frac{1}{(\log y)^2})$. Since $\log\log n = \log(\theta(y)) = \log y + \O(1)$ by Mertens estimates again, we have for $n \in$ *R*, $\phi(n) = \frac{ne^{-r}}{\log\log n} + O(\frac{1}{(\log\log n)^2}$.



| RSA-200 | Same digits length |
|---|---|
| $n$ | 27997833911221327870829467638722601621070446786955428537560009929326128400107609345671052955360856061822351910951365788637105954482006576775098580557613579098734950144178863178946295187237869221823983 |
| $\phi(n)$ | 27997833911221327870829467638722601621070446786955428537560009929326128400107609345671052955360856050364020022070262634017415134803482520365925322995768594715101139912289736681370959747280607953550168 |

**Fig. 2.** The same digits of $\phi(n)$ and modulus $n$ parameters in RSA.

From above, it seems so complexity. Does it exist a simple computation method? We observed the modulus $n$ with $\phi(n)$, there have some characteristics. An example for RSA-200, the modulus $n$ and the $\phi(n)$ are 200 decimal digits. We compared $n$ and $\phi(n)$ each other, there are the same digits from left to right **110** digits. The example showed in Figure 2. Discuss on RSA modulus number with haft of the bit prescribed, be introduced some literatures in [5, 9, 14]. To RSA-704, the $n$ and $\phi(n)$ had same

**Table 1.** Comparison of some types in RSA parameters. Unit: decimal digits

| type | Modulus $n$ length | $\phi(n)$ length | $p$ length | $q$ length | same digits $n$ & $\phi(n)$ | same digits $\phi(n)$ &$u$ |
|---|---|---|---|---|---|---|
| RSA-200 | 200 | 200 | 100 | 100 | 110 | 101 |
| RSA-210 | 200 | 200 | 105 | 105 | ? | ? |
| RSA-704 | 212 | 212 | 106 | 106 | 106 | 108 |
| RSA-768 | 232 | 232 | 116 | 116 | 115 | 120 |

digits $106$, it amounts same length with $p$ or $q$. We computed the upper bound value according to Theorem $1$, this upper bound had same $108$ digits with its $\phi(n)$. And analyzed the RSA-768, the $n$ had $115$ digits same with $\phi(n)$; the $\phi(n)$ had $120$ digits same with its upper bound $u$. Please see Table $1$ We observed the relationship of $\phi(n)$ and its boundary value $k$, when $\phi(n)$ divided by $k$, we obtained this quotient are very approaching to number $4$, these lower bounders are very close to multiples of number $4$. We say $3.\overline{999}$, and have $106$'s $9$ after decimal point for case of RSA-200 type. The lower bound approximation figure diagram be shown in figure $4$ and in Table $2$. As known as the modulus number $n$ of RSA-210, we re-estimated its lower/upper bounds. We assume:

$$(3+\varepsilon)\lfloor\frac{n-2\lceil\sqrt{n}\,\rceil+1}{4}\rfloor \leq \phi(n) \leq 4\lfloor\frac{n-2\lceil\sqrt{n}\,\rceil+1}{4}\rfloor, \tag{11}$$

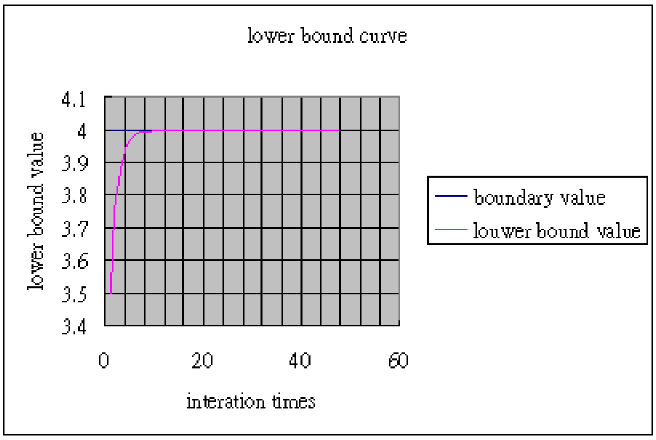
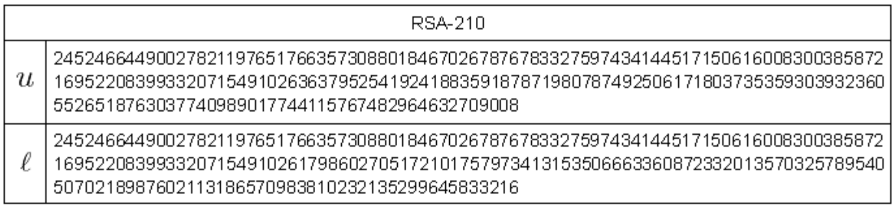


**Fig. 3.** The lower bound approximation curve status.

**Table 2.** The relationship of $\phi(n)$ and its boundary value $k$.

| type | $\phi(n)/k$ | statement |
|---|---|---|
| RSA-200 | $3.\,\overbrace{99999}^{99's\ 9}8$ | there have 99's 9 after the decimal point |
| RSA-210 | $3.\,\overbrace{99999}^{106's\ 9}2$ | Estimating have 106's 9 after decimal point |
| RSA-704 | $3.\,\overbrace{99999}^{107's\ 9}8$ | there have 107's 9 after decimal point |
| RSA-768 | $3.\,\overbrace{99999}^{117's\ 9}7$ | there have 117's 9 after decimal point |

where $\varepsilon = 0.\overbrace{99999}^{106's\ 9}$. We therefore compute the upper bound $u$ and lower bound $\ell$; those shown result in Figure 4.



| | RSA-210 |
|---|---|
| $u$ | 245246644900278211976517663573088018467026787678332759743414451715061600830038587216952208399332071549102636379525419241883591878719807874925061718037353593039323605526518763037740989017744115767482964632709008 |
| $\ell$ | 2452466449002782119765176635730880184670267876783327597434144517150616008300385872169522083993320715491026179860270517210175797341315350666336087233201357032578954050702189876021131865709838102321352996458333216 |

**Fig. 4.** The same digits of $\phi(n)$ and modulus $n$ parameters in RSA.

## 4 Conclusion

In this paper, we re-estimated a new lower/upper bound values of $\phi(n)$ in RSA-210, our methodology are easily, simply, clearly, no intricately and intuitive. It may be useful to researchers who would quickly reduce the searching ranges. Looking back, more researchers focus on secret $d$ or modulus $n$, such as well known short exponent attack, side channel attack, or common modulus and cyclic attacks. Our method is differ to previous literatures. Finally, We presented what we claimed actually.

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