A note on the practical complexity of the NFS in the medium prime case: Smoothness of Norms

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Abstract. During an ongoing examination of the practical complexity of the Number Field Sieve (NFS) in the medium prime case we have noticed numerous interesting patterns. In this paper we present findings on the complexity in practice of an aspect of the sieving stage. The contributions of these observations to the computational mathematics community are twofold: firstly, they bring us a step closer to understanding the true practical complexity of the algorithm and secondly, they enabled the development of a test for the effectiveness of the polynomials used in the NFS. The results of this work are of particular interest to cryptographers: the practical complexity of the NFS determines directly the security level of some discrete logarithm problem based protocols, such as those arising in pairing-based cryptography.

1 Introduction and Motivation

In cryptography the *computational* (or *practical*) complexity of particular algorithms is used to determine the security level of a protocol. More specifically, as the security of cryptographic protocols is based on the hardness of underlying mathematical problems, it follows that the security is reliant on the computational complexity of solving those problems in the given implementation context. It is therefore important that we have a sound knowledge of the behaviour of the algorithms used to solve these problems in practice in order to implement cryptographic protocols with reliable security settings. One particular hard problem, the Discrete Logarithm Problem, is the basis of security in numerous cryptographic protocols instantiated in two groups: in the additive group of points on an elliptic curve (ECDLP) and in a multiplicative group of a prime field or finite extension field (DLP). The most efficient known algorithms to solve the ECDLP have been thoroughly examined and the behaviour of these algorithms in practice is well understood, for example see [18, 20, 3, 4, 2]. For the DLP, however, the situation is quite different; little is understood about the behaviour of the relevant algorithms. The result of this is that we are not able to give precise implementation guidelines for protocols with security dependent on the DLP. In particular, in Pairing-Based Cryptography (PBC) often the underlying problems that form the basis for security are related to variations of both the ECDLP and

DLP – the hardness of the first problem is well understood but hardness of the second still remains elusive; this complicates parameter selection.

There are various known algorithms for solving the DLP, the most appropriate depends on the context of the problem. According to the complexity analysis in [12], the Number Field Sieve (NFS) in the medium prime case is the most efficient algorithm to solve the instances of finite field DLP occurring in PBC; these instances are in the "small prime" category [12, Section 3.1]. We are examining aspects of the practical complexity of this NFS; there are two motivating factors for examining this particular context:

1. It has been indicated that the behaviour of the NFS used for factorising varies widely depending on the particular implementation and that the theoretical complexity rarely gives a good indication of how the algorithm will perform in practice [21, 8]; this justifies a closer examination of other NFS versions.
2. Many Identity-based protocols use cryptographic pairings and given that it is desirable for such protocols to be implemented on small devices, such as smart cards and other embedded devices, it is important to have a more accurate measure of the required parameter sizes for a particular security level. In this particular setting, simply taking a larger field to compensate for "unknown capabilities" of the NFS is not a viable option, due to limited memory and processing power, so we need to better understand the capabilities of the NFS to make more precise security estimates.

To summarise our motivation: little is known about the practical complexity of the NFS in the medium prime case which limits our understanding of the security of protocols which are based on the associated DLP instances. The ultimate goal of this continuing project is to increase the general understanding of the practical complexity of the NFS. The analysis of the NFS algorithm in [12] is given for finite fields $\mathbb{F}_{p^k}$ as both p and k tend to infinity which necessitates loss of detail and generalisation to limiting cases. We wish to fine-tune this analysis to the cases relevant to cryptography, retaining as much detail as possible and with the aid of experimental results give more accurate estimates of the practical complexity in this context. One direct and important use of this information is to determine more precise estimates for appropriate PBC parameter sizes at a given security level. To contribute to this goal we outline here some of the practical observations made during our analysis. Using these observations we have developed a "pre-NFS" polynomial test on the effectiveness of a particular NFS instance. We also introduce a variation in the NFS polynomial selection. The results of our work will be of interest to implementers of the NFS and pairing-based protocols alike. The remainder of this paper is organised as follows: In section 2 we give a brief introduction to the NFS. In section 3 we examine one aspect of the theoretical complexity analysis, which solidifies our motivation and introduce the particular aspect of the NFS that this work focuses on. In section 4 we explain our experiment setting and the process of our investigation, presenting the results of this undertaking in section 5. Conclusions and open problems are presented in section 6.

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2 NFS in a nutshell

In this section we give a brief introduction to the NFS. The NFS should not really be considered an algorithm, but a sequence of algorithms; it makes sense to assess the complexity of each of the "sub-algorithms" or "stages". The NFS can be divided into two main parts: the sieving stage and the linear algebra stage. Suppose we wish to solve a DLP instance in the finite field $\mathbb{F}_{p^k}$.

Sieving

The goal of the sieving stage is to find a set of linear relations in the logarithms of a fixed set of elements. This is done by selecting two isomorphic representations of a number field of dimension k over $\mathbb{Q}$, say $\mathcal{K}_1$ and $\mathcal{K}_2$, and fixing a subset of elements in each of the representations, called the *Factor Base*, denoted by $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. The factor bases contain the ideals of small norm, bounded by a *smoothness bound* $\mathcal{B}_i$, $i = [1, 2]$. Field elements are then 'sieved' to find pairs (α, β), $\alpha \in \mathcal{K}_1$ and $\beta \in \mathcal{K}_2$, with $\alpha \cong \beta$, and α and β *smooth* over their respective factor bases; that is, α and β can be completely decomposed into a product of elements from their respective factor bases. Such pairs are called *doubly smooth*. The sieving continues until enough relations

$$\alpha = \prod_{\gamma_i \in \mathcal{F}_1} \gamma_i^{a_i} \cong \prod_{\delta_j \in \mathcal{F}_2} \delta_j^{b_j} = \beta$$

are found, that is, at least the sum of the number of elements in the factor bases. These relations are converted to linear equations in the logarithms of the factor base elements by taking the discrete logarithms of each side with respect to fixed (isomorphic) elements in each representation. In the particular variation of the NFS we are concerned with here, the two number fields are constructed by adjoining to $\mathbb{Q}$ roots θ_1 and θ_2 of two different polynomials (f_1 and f_2 respectively, considered in $\mathbb{Z}[x]$) which have common root modulo p. The elements of these number fields are thus considered as polynomials of degree $t < n$ in θ_i. To compute the norm of an element $a = a(\theta_i)$ in $\mathcal{K}_i$ we consider a as a polynomial in x and take the resultant of $a(x)$ with $f_i(x)$.

Linear Algebra

The system of linear equations obtained in the sieving stage is solved using linear algebra to find the logarithms of the elements in the factor bases. Solving the linear equations is no trivial task. Though the matrix of equations is originally

sparse, after only a few operations it becomes congested. There are a few methods for reducing this matrix and solving for the unknown logarithms: Lanczos algorithm, Wiederman algorithm (complexity $L(1/3)$) and structured Gaussian elimination, some algorithms use a combination of these methods [15].

3 Smoothness probability of norms

The run time of the sieving stage of the NFS is determined by the probability of finding doubly smooth relations. There exist varying methods in the literature to predict smoothness probabilities in different contexts. Let $\Psi(x, y)$ denote the number of integers $\leq x$ with no prime factor exceeding y; assuming that we are dealing with random integers, the probability of a given number of size approximately the size of x being y-smooth is therefore $\Psi(x, y)/x$. In [11] a comprehensive survey of estimates for $\Psi(x, y)$ is given; in the more recent [7] we are presented with a method for computing Ψ within an arbitrarily tight bound. The results of [14] are extensions of some of the theorems in [11] to the context of algebraic numbr fields – the relevant case for the NFS. The formulæ for calculating the probability is directly obtained from the $\Psi_K(x, y)$ estimate (number of integral ideals with norm $< x$, all of whose prime divisors have norm $< y$ in a number field K) [14, Satz 3] but computation of this probability is not feasible in practice and therefore can not be used in the complexity analysis of the NFS algorithm. The computation relies on knowledge of the class number of K and a result of the Dedekind Zeta function, both of which are known to be hard to compute in practice. This presents a major set back when trying to estimate the computational complexity of the NFS: are we unable to check the accuracy of the estimate (that is, for a given polynomial selection we can not know if the rate of smooth relation occurrence is as expected). Indeed, being able to compute the probability of smoothness of norms in a given number field (that is, for a given choice of f_1 and f_2) could be used to explain some of the behaviour noticed by Zajac [21] and therefore aid polynomial selection and enable optimisation of the factoring NFS. The smoothness probability estimate used for the complexity analysis in [12] is from a corollary [10, page 15]; this formula is the most appropriate choice, given that the algorithm is presented for extension degrees k and primes p both tending to infinity. The probability estimate formula is 'compacted' for use in the complexity analysis – as is common practice – which poses further issues when trying to estimate the computational complexity for a particular context. Much information is absorbed into an $O(1)$ or $O(k)$ term, thus obscuring factors which have a substantial effect on the run time of the algorithm. A concrete example of this: in [12] the authors specify that a constant factor of $6/\pi^2$ from the original estimate is absorbed into the L-notation for the theoretical complexity. This can not be neglected when assessing the practical behaviour of the sieving stage, this factor of $\sim 2/3$ means that we would overestimate the number of doubly-smooth relations found by %50, assuming we will find m relations when in fact we would only have $\frac{2}{3}m$ after a

given number of trials, this stage would have to run %50 longer then expected to reach completion.

As a result of these practices, the theoretical complexity is not an accurate estimate of the computational complexity; this issue is not isolated to this NFS algorithm, similar techniques are commonplace and the problem of the gap between theoretical and computational complexity was noted in [8] for the NFS used for factorisation. The main result presented in section 5 of this work is the examination of the computational complexity in this aspect of the NFS.

4 Experimental Context

In this section we outline our experimental context including a variation on the polynomial selection process.

Parameter sizes

As the contextual focus is PBC, we used the current field size suggestions in literature (from resources such as [17, 16, 1]) for the security levels 80, 128 and 192 which fixes the types of elliptic curves which can be used and therefore also fixes k. We used the complexity analysis of [12] to compute the corresponding values for t for each of the 3 instances. That gives us:

	80 bit security	128 bit security	192 bit security
curve	MNT6	BN	KSS18
size of p	160	256	512
k	6	12	18
t	2	3-4	4

We present our results using a working example of the MNT6 case and reserve and BN and KSS18 details for the appendix.

4.1 Selecting polynomials with coefficients of size around $\sqrt{p}$

In order to have the NFS polynomials of simple structure, we find two numbers $a_1,\ a_2$ of size $\sim \sqrt{p}$ of such that $a_1 \cdot a_2 = p + i$, where i is some small integer. We define the polynomials f_1 and f_2 to be:

$$f_1 = x^n + a_2$$
$$f_2 = a_1 x^n + i$$

when both are irreducible. It is straightforward to show that these polynomials have a common zero modulo p (in fact they have all zeros equal) and are therefore appropriate for use in the NFS. Using this polynomial selection means that the sieving space did not need to be skewed to balance the sizes of the norms of the elements in each number field (see [12] for details). One important detail supporting the use of binomials in this context is that for implementation ease

the prime p is usually chosen such that $p \equiv 1 \mod k$ (or the prime factors of k)[13] and thus it is always possible to find irreducible binomials for the NFS polynomial selection in this context. (For the BN case we were able to use irreducible polynomials as used for the tower constructions in [5].) Initial results showed that this new method is more likely to find smooth elements than the original polynomial selection; the reasons for this are still yet to be thoroughly investigated.

5 Our Contribution

In this section we present our experimental methodology and results of our work; the process is divided into three sections:

5.1 We used empirical data to compute the cumulative distribution function of the norms; we make observations about which parameters influence the distribution; we illustrate how the cumulative distribution affects the smoothness probability.
5.2 We show how to compute a closer estimate of the smoothness probability using the cumulative distribution function and the results of [7].
5.3 We present a test for the effectiveness of the f_1 and f_2 polynomials against the expected outcome.

5.1 Cumulative Distribution of Norms

Once the polynomials f_1 and f_2 have been selected, the computation of the norms can be reduced to the evaluation of a polynomial in the coefficients of the number field elements. The coefficients are taken from a fixed interval $[0, S]$ for a sieving bound S, computed following the instructions in [12]. We can therefore consider the coefficients as independent random variables $X_0, \ldots, X_t$ randomly selected from $[0, S]$ following a uniform distribution. The norm is also a random variable, N, which takes values in the range of the function on the random variables $X_0, \ldots, X_t$ defined by the sylvester matrix; we denote this function $n(X_0, \ldots, X_t)$. We are interested in the cumulative distribution of $N = n$. The reason for focussing on the cumulative distribution is that the estimates for the probability of smoothness are in fact cumulative probabilities: $\Psi(x, y)/x$ where $\Psi(x, y)$ is the number of integers $\leq x$ with no prime factor exceeding y. We will use the cumulative distribution of N to weight the smoothness probability estimate of the integers to find a more appropriate smoothness probability estimate for the norms. The resultant of polynomials of the form $Ax^6 + B$ with $X_0 + X_1 x + X_2 x^2$ (the MNT6 setting) is given by the evaluation of the function

$$n(X_0, X_1, X_2, X_3) = A^2 X_0^6 - 2ABX_0^3 X_2^3 + 9ABX_0^2 X_1^2 X_2^2 - 6ABX_0 X_1^4 X_2 + ABX_1^6 + B^2 X_2^6,$$

where $A = 1$ for f_1. Such a number clearly has much structure and numbers generated using this equation will have a distribution very different from the uniform, which is used in the complexity analysis. Trialling various theoretical methods to determine the probability distribution of this N from this function of

uniform random variables was unfruitful due to the complexity and large number of variables. We therefore use the empirical data directly to determine the cumulative distribution. In the following graph we see the cumulative distribution function as generated for the norms for the MNT6 case. Intuitively, we expect a "clumping" effect in the centre of the probability distribution of the norms due to the fact that we are repeatedly summing 6th powers of uniform variables from a bounded interval, which will increase the proportion (and therefore probability) of the low to middle range norm values. The result of this clumping effect in the probability distribution translates to a very steep cumulative distribution function over the low and middle range (plentiful) values, which tapers down to very small slope over the larger (rarer) values. This is exactly the result that the data presented. In figure 5.1 we see the cumulative distribution function as generated for the norms for the MNT6 case compared with the uniformly distributed variable. It is clear that this central clumping as discussed above results in the "mid-range" values being more probable in the norm distribution than for the integers: The number of integers in a particular interval increases constantly with the upper bound of the interval whereas the norms have a higher concentration in the mid-range of the same interval, a direct result of the range of the variables X_0, X_1, X_2.

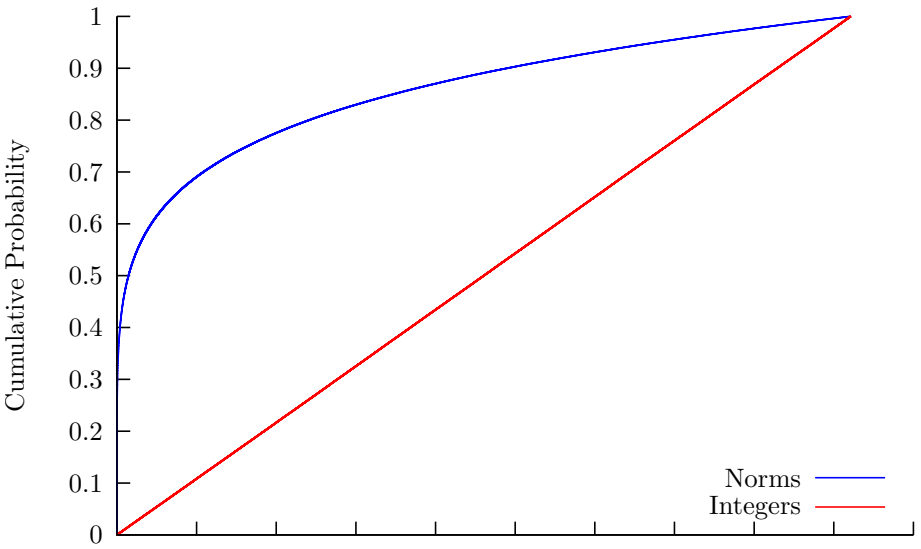

Fig. 5.1. Comparison of the cumulative distribution of norms and uniform integers in the MNT6 case. This particular graph generated using $p = 1125909838976401$ ($s = 349879$).

Another interesting observation we made, by varying the size of p for more tests, is that the distribution of the norms depends only on k and t. The size of p only dictates the *placement* of the distribution, as it directly determines the interval from which the variables X_0, X_1, X_2 are taken, not the *shape*. This is not surprising given that the shape of the elements of interest are the output of the above function $n(X_0, \ldots, X_t)$, a degree k function in t variables, with constants A and B being altered with each function. For this reason we use a smaller example case than would be used in practice: the MNT6 setting and a 50-bit prime (with $k = 6$ and $t = 2$). This allows us to perform more thorough and varied tests.

Fitting a line to the cumulative distribution The line which depicts the cumulative distribution as presented in figure 5.1 was generated using 2 million norm values for X_0, X_1, X_2 chosen uniformly at random within the given bounds as outlined in [12] for our working example with parameters as used for the generation of figure 5.1 (we used a range of primes and polynomials obtaining concurring results). To obtain the equation of this line we performed curve fitting using R, a language package for statistical computing [19]. Initial experiments indicated that this is not an exponential curve prompted us to use the *Box-Cox method* [9]. This method aids us in identifying a suitable power transformation on the response variable (in this case y, the proportion of norm values $\leq x$) to obtain a linear relationship between the explanatory variable (in this case x) and the modified response variable; that is, we wish to find a value λ such that $y^\lambda = ax+b$. Often the initial test range for λ is $[-5, 5]$ and the default setting in R is $[-2, 2]$ incrementing in steps of 0.1. The graph in fig 5.1 represents the density of norm values and as the norms are sums of monomials of degree 6 we expect λ to be close to 6 and therefore set our test range to be $[0, 10]$. Instead of using all 2 million points (which would have made the tests very slow) we repeated the test numerous times on random samples of 10000 points. The results for each of the tests gave clear indication that $\lambda = 6$, as indicated by the figure in appendix A.1. We then proceed to fit a linear model to the full data set of points of the transformed data (x, y^6) to find that $Y^6 = aX + b$ where $a = 2.171E - 106$ and $b = -1.868E - 5$. From a statistical perspective the fit is incredibly accurate as there is no noise in the data, given the smoothness of the curve generated from our empirical values we expect to have little error. Exact output of the linear model fitted in R is given in appendix A.1

How the cumulative distribution affects the smoothness probability To illustrate why the probability of smoothness is affected by the differing probability distributions we use the law of total probability. Let us examine the problem in terms of random variables: Let Z be a random variable which takes integer values (without loss of generality we assume that the interval is $[0, U]$ for some bound U). Let N be a randomly selected norm (also from the interval

$[0, U]$) and let S_Z (resp. S_N) be a binomial random variable such that

$$S_Z = \begin{cases} 1 \text{ if } Z = z \text{ is smooth;} \\ 0 \text{ if } Z = z \text{ is not smooth.} \end{cases} \tag{5.1}$$

We define S_N similarly where smooth in both cases is with respect to some fixed bound $\mathcal{S}$. Now the probability that a random integer (z) selected from the defined interval uniformly at random is smooth is expressed as

$$P(S_Z = 1 | Z = z).$$

Following the law of total probability we partition the interval $[0, U]$ into s equally sized sections, of width $U/s = a$, we can now write this probability as a summation:

$$P(S_Z = 1) = \sum_{i=1}^{s} P(S_Z = 1 | z \in [(i-1)a, ia]) P(z \in [(i-1)a, ia]).$$

Using the same method we can rewrite $P(S_N = 1)$:

$$P(S_N = 1) = \sum_{i=1}^{s} P(S_N = 1 | n \in [(i-1)a, ia]) P(n \in [(i-1)a, ia]).$$

Assuming initially that the probability of a norm being smooth is equal to the probability of an integer of the same size being smooth[3] the left probability value in each of the summations will be identical for every value of i and can be computed using the method of [7] (a free implementation is available at [6] courtesy of the author). From the different distributions of the norms and integers we know that the right hand values are quite different and so the sums will clearly not be equal. The probabilities on the right in the $P(S_Z = 1)$ expression are exactly the s-quantiles (s is the number of partitions); as $P(Z \leq ia) = \frac{ia}{U}$ and so for all i we have:

$$P(z \in [(i-1)a, ia]) = P(Z \leq ia) - P(Z \leq (i-1)a) = \frac{(i-1)a - ia}{U} = \frac{a}{U}.$$

That is, the right probabilities in the $P(S_Z = 1)$ expression are constant and the expression becomes

$$\begin{aligned} P(S_Z = 1) &= \frac{a}{U} \sum_{i=1}^{n} P(S_Z = 1 | z \in [(i-1)a, ia]) \\ &= \frac{a}{U} \sum_{i=1}^{n} \frac{\#\text{smooth} \in [(i-1)a, ia]}{a} \\ &= \Psi(\mathcal{S}, U)/U. \end{aligned}$$

Examining figure 5.1 we see that the probabilities $P(n \in [(i-1)a, ia])$ will be quite different. Using the law of total probability we not only highlight how the distribution of the norms affects the probability of smoothness, but present a method for computing the smoothness probability for the norms.

[3] The validity of this assumption will be discussed further in section 5.2.

5.2 Smoothness probability of norms

To compute $P(S_N = 1)$ we use the same approach as above, this alters the position of the partitions used; instead of having the partitions evenly spaced (as above) we use the s-quantiles of the distribution of the norms (that is, we fix the intervals such that the probability that n is in any interval is fixed at $\frac{1}{s}$). Fixed probability and even spacing are synonymous for the uniform distribution, but not for the distribution of the norms. Using R we easily compute the 100-quantiles of the norms, that is, we find the values $a_0, \ldots, a_{100}$, given in appendix A.2, such that $P(n \in [a_i, a_{i+1}]) = 0.01$ for $i \in [0, 99]$. It remains now, following from our assumption that $P(S_N = 1 | n \in [a_i, a_{i+1}]) = P(S_Z = 1 | z \in [a_i, a_{i+1}])$, to compute

$$0.01 \sum_{i=0}^{99} P(S_N = 1 | n \in [a_i, a_{i+1}])$$

to have the probability of smoothness of the norms. Using the implementation [6] we computed the probabilities in the summation to obtain an estimate of the smoothness probability:

$$P(S_N = 1) = 0.0001699201 \quad \text{compared to} \quad P(S_Z = 1) = 0.00005777868.$$

The probability of smoothness of the norms is higher by a factor of almost 3 (2.94). This is no surprise as, intuitively, smaller numbers are more likely to be smooth and the norms have a higher concentration of 'small' numbers than the integers do. These results also reflect the experimental results obtained by selecting integers at random, following the distribution of the norms, and testing for smoothness; almost 3 times as many smooth integers were obtained using this method compared to selecting integers uniformly at random.

The probability above is a *average-case* probability; we have made the assumption that the probability of a norm being smooth is equal to the probability that an integer of the same size is smooth. That is, for some integers b_0 and b_1 we assumed that

$$P(S_N = 1 | n \in [b_0, b_1]) = P(S_Z = 1 | z \in [b_0, b_1]).$$

This is, however, not necessarily the case. In any given interval, the integers are denser than norm values (approximately 1/6 for the MNT6 case), and the norms will not necessarily coincide with the same proportion of smooth integers; the norms may coincide with a lesser proportion of smooth integers. On the other hand, the norms may in fact fall on more smooth integers than the ratio of integers to norms in that interval; it is in this scenario that the NFS sieving stage will execute faster. In the following table are examples of polynomials found for an example MNT6 prime for which the calculated rate r_1 (from $1000/\hat{r} \approx 5$mil tests).

$p = 1125909838976401$

polynomial	# smooth r = smooth rate	factor difference from $\hat{r}$	(β)
$f = 35374332x^6 + 11$	633	0.0001266	0.7451
$f = 11791444x^6 + 11$	1370	0.0002740	1.6125
$f = 17687166x^6 + 11$	652	0.0001304	0.7674
$f = x^6 + 8264689$	1379	0.0002758	1.6231
$f = 1069214x^6 + 37$	1320	0.0002640	1.5537

We see the second, fourth and fifth polynomials produce smooth norms at a rate over %50 faster than the prediction. There are, in fact, numerous polynomials for which the rate of smooth norm collection is higher than the predicted rate and we were able to find many examples, though the average case differs from $\hat{r}$ by a factor of 0.5526999 (distribution shown in Figure B in appendix B). The goal of an implementer of the NFS, however, is to find a polynomial *pair* such that the rate of smooth relation collection in each number field is as close to $\hat{r}$ as possible: that is, we want to find pairs (f_1, f_2) such that $r_1 \cdot r_2 \approx \hat{r}^2$ (r_i = smooth rate of norms computed using f_i, $i \in \{1, 2\}$); this is no straightforward task. We denote $r_1 \cdot r_2 = \beta\hat{r}^2$ and recognise that the possible values for β will be from the product distribution of B and will therefor have expected value $0.3054772 = 0.5526999^2$. This value gives us the benchmark to aim to obtain, or better, when selecting our polynomials for the NFS.

5.3 Polynomial Selection Test

For a given instance of the DLP in a finite field $\mathbb{F}_{p^k}$ we perform the following test before launching the NFS sieving stage:

1 Compute the $\hat{r}$, the upper estimate on the smoothness probability of norms as done in section 5.2.
2 Find polynomial pair f_1 and f_2 following the method given in section 4.1 or [12].
3 Generate $1000/\hat{r}$ [8] norm values randomly using both f_1 and f_2 and test for smoothness to approximate the rates of smooth norm occurrence for each number field.
4 Compare the product of the approximate rates of smooth norms to $\hat{r}^2$: if it is "large enough", proceed with sieving, otherwise return to step 1.

In step 4 by large enough we want to find $\hat{\beta} \geq \beta = 0.3054772$.

This differs to how the results in [21] would be used, we have given a benchmark to compare the rate of smooth norm collection against, whereas previous methods would have required the generation of numerous pairs of polynomials to

mutually compare the rate of smooth relation collection. Thus, our method results in fewer tests necessary to distinguish if a pair of polynomials is performing close to optimally.

To illustrate, we have generated various f_1 and f_2 pairs for our example case of MNT6 with prime around 50 bits. Our experiments resulted in polynomial pairs for which smooth norms occurred at a much higher rate than others. Some example cases of polynomial pairs for MNT6 case:

$p = 1125909838976401$

polynomial pair	# smooth	rate smooth	rate double smooth	β
$f_1 = 35374332x^6 + 11$	633	1.266E-4	7.54536E-9	0.2613
$f_2 = x^6 + 31828441$	298	5.96E-5		
$f_1 = 11791444x^6 + 11$	1370	2.740E-4	1.17272E-8	0.4062
$f_2 = x^6 + 95485323$	214	4.28E-5		
$f_1 = 17687166x^6 + 11$	652	1.304E-4	5.42464E-9	0.1879
$f_2 = x^6 + 63656882$	208	4.16E-5		
$f_1 = x^6 + 8264689$	1379	2.758E-4	1.147328E-8	0.3974
$f_2 = 136231362x^6 + 17$	208	4.16E-5		
$f_1 = 1069214x^6 + 37$	1320	2.64E-4	4.9632E-9	0.1719
$f_2 = x^6 + 1053025717$	94	1.88E-5		

Interestingly, though the same value $i = 11$ was used to construct the first three polynomial pairs the rate of smooth relation collection is vastly different. Though the rate is comparable in the number fields defined using the f_2, using the first and third pairs we found smooth relations at less than half the rate of the second pair, for which the value $\hat{\beta} = 0.4062$ exceeds the expected value.

$p = 1126169969103937$

polynomial pair	# smooth r	=est. double smooth rate	β
$f_1 = x^6 + 4747150$	476	1.014832E-8	0.3515
$f_2 = 237230753x^6 + 13$	533		
$f_1 = x^6 + 186474550$	139	6.52744E-9	0.2261
$f_2 = 6039269x^6 + 13$	1174		
$f_1 = x^6 + 516240070$	118	5.1212E-9	0.1774
$f_2 = 2181485x^6 + 13$	1085		
$f_1 = x^6 + 20871667$	202	4.7268E-9	0.1637
$f_2 = 53956877x^6 + 22$	585		
$f_1 = x^6 + 27201934$	251	6.68664E-9	0.2316
$f_2 = 41400364x^6 + 39$	666		

Again in this table we see great variation in the rate of smooth element collection in the first three cases, though the same value $i = 13$ was used to construct the polynomial pairs. For this prime the value of $\hat{\beta} = 0.3515$ slightly exceeds β.

6 Conclusion and Open Problems

In theoretical complexity of algorithms it is common practice that many constant factors are hidden in the O and L notations. As a result, the theoretical complexity does not give a good reflection of how the algorithm will run in practice. In cryptography, the security of protocols is directly related to the *practical* complexity of solving hard mathematical problems. Until now the practical complexity of the number field sieve in the medium prime case has received little attention. Due to its relevance to the security of pairing-based protocols in particular we are examining the complexity of this algorithm in the given context; the motivation of this work was to deepen the understanding of the NFS in general, in the hope that this would help determining more precise estimates for appropriate parameter sizes for a fixed security level for PBC protocols.

In this work we presented some observations on the behaviour of the NFS in practice. We focussed on the smoothness probability of the norms of number field elements as it determines the practical complexity of the sieving stage. Our observations and analysis have resulted in a pre-sieving test that can be performed on the selected polynomials. This test ensures as efficient a set up and execution of the sieving stage as possible. The new revelation about the probability of smooth norm occurence is a step towards a more precise practical complexity estimate of the sieving stage.

This work covers the initial progress in the examination of the NFS sieving stage; in order to compute the practical complexity of this stage it remains to investigate the true cost of smoothness test (relative to computing power). Other tasks include a thorough comparison of the polynomial selection methods as given in section 4.1 and [12].

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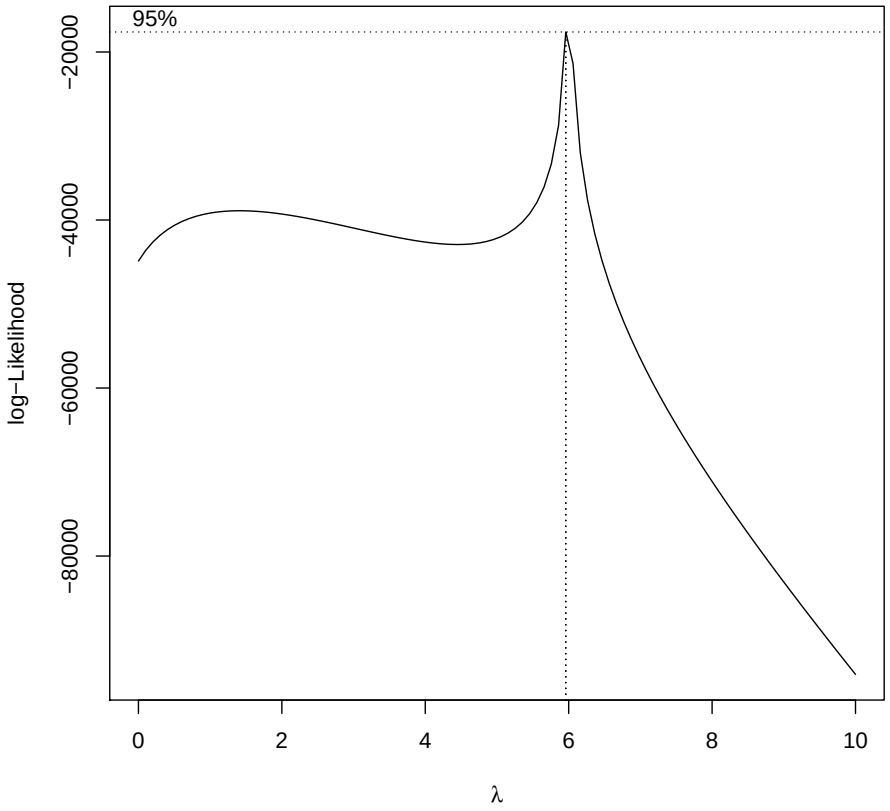
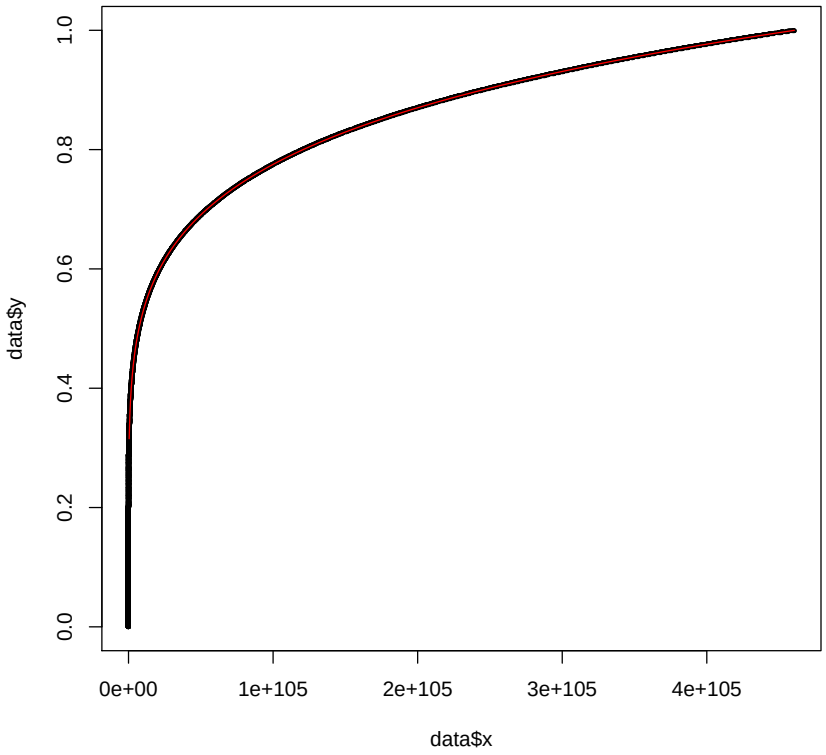
A MNT6 output

Diagram from Box-Cox method clearly indicating that for the MNT6 case that $\lambda = 6$.



Comparison of the experimental data and the fitted line.



A.1 Output of the linear model

Output from R:

Residuals:

Min	1Q	Median	3Q	Max
-8.589e-04	-7.149e-05	3.580e-06	1.868e-05	7.928e-04

Coefficients:

	Estimate	Std. Error	t value	Pr(¿—t—)	
(Intercept)	-1.868e-05	1.701e-07	-109.8	¡ 2e-16	***
x	2.171e-106	1.331e-112	1630932.3	¡ 2e-16	***

Residual standard error: 0.0002061 on 1999998 degrees of freedom
Multiple R-squared: 1, Adjusted R-squared: 1
F-statistic: 2.66e+12 on 1 and 1999998 DF, p-value: ¡ 2.2e-16

A.2 MNT6 100-quantiles

a_0 =71058 96343 98717 100100 101069 101816 102434 103022 103837 104597
a_{10} =105289 105918 106496 107028 107524 107984 108412 108814 109192 109551
a_{20} =109893 110222 110534 110829 111113 111383 111644 111894 112137 112371
a_{30} =112596 112816 113027 113235 113434 113629 113817 114000 114180 114353
a_{40} =114522 114687 114848 115002 115154 115304 115449 115594 115735 115871
a_{50} =116004 116136 116265 116392 116517 116641 116762 116881 116997 117110
a_{60} =117221 117331 117440 117546 117650 117754 117856 117956 118055 118152
a_{70} =118249 118342 118435 118527 118618 118707 118795 118884 118971 119056
a_{80} =119140 119224 119306 119386 119465 119544 119622 119698 119775 119850
a_{90} =119925 119999 120073 120144 120215 120286 120356 120424 120493 120561
a_{100} =120628
When N is a random value takes values from the range of the function $n(X_0, X_1, X_2, X_3) = A^2X_0^6 - 2ABX_0^3X_2^3 + 9ABX_0^2X_1^2X_2^2 - 6ABX_0X_1^4X_2 + ABX_1^6 + B^2X_2^6$ where X_i are uniformly distributed discrete random variables in the interval $[0, 349879]$ then the above values are such that for $N = n$, $P(n \in [a_i, a_{i+1}]) = 0.01$ for $i \in [0, 99]$.

B Rate multiple distribution