Reducing Pairing Inversion to Exponentiation Inversion using Non-degenerate Auxiliary Pairing

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Abstract

The security of pairing-based cryptosystems is closely related to the difficulty of the pairing inversion problem. Building on previous works, we provide further contributions on the difficulty of pairing inversion. In particular, we revisit the approach of Kanayama-Okamoto who modified exponentiation inversion and Miller inversion by considering an "auxiliary" pairing. First, by generalizing and simplifying Kanayama-Okamoto's approach, we provide a simpler approach for inverting generalized ate pairings of Vercauteren. Then we provide a complexity of the modified Miller inversion, showing that the complexity depends on the sum-norm of the integer vector defining the auxiliary pairing. Next, we observe that the auxiliary pairings (choice of integer vectors) suggested by Kanayama-Okamoto are degenerate and thus the modified exponentiation inversion is expected to be harder than the original exponentiation inversion. We provide a sufficient condition on the integer vector, in terms of its max norm, so that the corresponding auxiliary paring is non-degenerate. Finally, we define an infinite set of curve parameters, which includes those of typical pairing friendly curves, and we show that, within those parameters, pairing inversion of arbitrarily given generalized ate pairing can be reduced to exponentiation inversion in polynomial time.

keywords: Ate pairing, elliptic curve, exponentiation inversion, Miller inversion, pairing inversion

1 Introduction

Pairings on elliptic curves [1, 9, 12, 13, 17, 24, 28] play an important role in cryptography [2, 3, 4, 14, 26]. The security of pairing-based cryptosystems is closely related to the pairing inversion problem (PI). Thus it is important to investigate the difficult of PI. In this paper, inspired by significant previous works [25, 21, 22, 23, 11, 19, 27, 15, 7], we provide further contributions toward understanding the difficulty of pairing inversion.

In order to provide the context and the motivation for the main contributions of this paper, we review some of the previous works [11, 15] on PI by recasting them for the generalized ate pairing of Vercauteren [24], which currently is one of the most general constructions of cryptographic pairings. For a given integer vector ε, the generalized ate pairing $a_\varepsilon(\cdot,\cdot)$ takes two points P, Q and produces a value z. It is carried out in two steps: Miller step (M) [18] and Exponentiation step (E).

1. $[\mathsf{M}_\varepsilon]$ $\quad \gamma_\varepsilon = Z_\varepsilon(Q, P)$
2. $[\mathsf{E}_\varepsilon]$ $\quad z = \gamma_\varepsilon^L$

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where Z_ε is a certain rational function depending on the integer vector ε and L is a certain natural number. Depending on the choice of ε, one gets a different pairing (see Section 2.2 for details).

Pairing inversion takes Q, z and produces P. A natural approach for PI is first to invert the exponentiation step (EI) and then to invert the Miller step (MI).

1. [EI_ε] Find the "right" γ_ε from the set $\{\gamma : z = \gamma^L\}$
2. [MI_ε] Find P from $\gamma_\varepsilon = Z_\varepsilon(Q, P)$

By the "right" γ_ε, we mean the one satisfying the condition $\gamma_\varepsilon = Z_\varepsilon(Q, P)$. This approach has been carefully investigated in [11] for ate pairings.

In [15], Kanayama-Okamoto proposed an interesting modification of the natural approach for PI, which amounts to the following:

1. [Choice] Choose an integer vector e (which might be different from ε), giving rise to another generalized ate pairing, which we will call *an auxiliary pairing*, which may or may not be non-degenerate.
2. [$\mathsf{EI}_{\varepsilon,e}$] Find the "right" γ_e from a certain set defined by exponential relations (See Section 2.3)
3. [MI_e] Find P from $\gamma_e = Z_e(Q, P)$

Again by the "right" γ_e, we mean the one satisfying the condition $\gamma_e = Z_e(Q, P)$. From now on, we will call $\mathsf{EI}_{\varepsilon,e}$ and MI_e as the *modified* exponentiation inversion and the *modified* Miller inversion, respectively. If $e = \varepsilon$, then $\mathsf{EI}_{\varepsilon,e}$ and MI_e are exactly same as EI_ε and MI_ε. The key idea is to choose an integer vector e which may be different from ε, but which may be better for PI. Specifically, Kanayama-Okamoto suggested that the integer vector e is chosen from either coefficients of cyclotomic polynomials or $(1, \ldots, 1)$, because such e yields Z_e of low degree, making MI_e easy.

Building upon the previous works, we provide the following contributions toward better understanding of the difficulty of pairing inversion.

1. In Section 3, we provide another approach for pairing inversion (Approach 1), by simplifying the step $\mathsf{EI}_{\varepsilon,e}$ of Kanayama-Okamoto's approach. The simplicity of the proposed approach significantly facilitates the subsequent investigation. We prove its correctness (Theorem 1), and then prove that the simpler approach is equivalent to Kanayama-Okamoto's original approach (Theorem 2).
2. In Section 4, we provide a complexity analysis of MI_e (Theorem 3). It essentially says that the complexity is bounded by $||e||_1^2$ where $||e||_1$ stands for the sum norm of the chosen integer vector e. Hence, in order to reduce the complexity of MI_e, one needs to choose e with small sum norm.
3. In Section 5, we provide an incremental result toward the understanding of the complexity of $\mathsf{EI}_{\varepsilon,e}$. We begin by observing that the degeneracy of the auxiliary pairing has a potential impact on the difficulty of $\mathsf{EI}_{\varepsilon,e}$ (Proposition 6 and Remark 3). More precisely, if the auxiliary paring defined by the choice of e is degenerate, then the exponential relation in $\mathsf{EI}_{\varepsilon,e}$ step becomes independent of the input z, that is, the exponential relation does not capture any information about the input. As a result, $\mathsf{EI}_{\varepsilon,e}$ is expected to be harder than EI_ε, when such e is chosen. If the auxiliary pairing corresponding to e is non-degenerate, then $\mathsf{EI}_{\varepsilon,e}$ is likely as hard as EI_ε. Hence, in order to reduce the complexity of $\mathsf{EI}_{\varepsilon,e}$, one better choose e such that the auxiliary paring defined by e is non-degenerate. We provide a sufficient condition on e, in terms of the max norm of e, so that the pairing corresponding to e is non-degenerate (Theorem 7).
4. In Section 6, we discuss when pairing inversion can be reduced to exponentiation inversion. The question was originally addressed by Kanayama-Okamoto [15]. They showed that, if the integer vector e is chosen from either coefficients of cyclotomic polynomials or $(1, \ldots, 1)$, then MI_e can be carried out in polynomial time, reducing PI to the modified exponentiation inversion $\mathsf{EI}_{\varepsilon,e}$. However according to Corollary 6 of Vercauteren [24], such e makes the corresponding auxiliary pairing degenerate. Hence the

modified exponentiation inversion $\mathsf{EI}_{\varepsilon,e}$ is expected to be harder than the exponentiation inversion $\mathsf{EI}_{\varepsilon}$ and thus it is not clear that such choices of e allows the reduction of pairing inversion to exponentiation inversion. In order to reduce pairing inversion to exponentiation inversion, it is safer to find e such that it is *small* and the corresponding auxiliary pairing is *non-degenerate*. In this section, we investigate the existence of such e in various cases. In particular, we define an infinite set of curve parameters (Definition 1), which includes those of typical pairing friendly curves as in Table 1 of [10] and show that, within those parameters, pairing inversion of an arbitrarily given pairing can be reduced to exponentiation inversion in polynomial time (Theorem 9). We furthermore provide tighter upper bounds on the number of bit operations needed by such reductions for several concrete cases (Table 1).

2 Preliminaries

In this section, we briefly review elliptic curves, the generalized ate pairings due to Vercauteren [24] and an approach to pairing inversion due to Kanayama-Okamoto [15]. We encourage all the readers to skim through them, as the notations and the assumptions therein will be extensively used throughout the subsequent sections.

2.1 Elliptic curves

We fix the basic notations for elliptic curves. Let q be a power of a prime and let r be a prime such that $\gcd(q,r) = 1$. Let k be the embedding degree defined as the multiplicative order of q in $\mathbb{F}_r^*$, denoted by $k = \mathrm{ord}_r(q)$, and $L = (q^k - 1)/r$. Let E be an elliptic curve defined over $\mathbb{F}_q$ such that $r \mid \#E(\mathbb{F}_q)$. Let $G_1 = E[r] \cap \ker(\pi_q - [1])$ and $G_2 = E[r] \cap \ker(\pi_q - [q])$ where $\pi_q : E \to E$ denotes the q-power Frobenius endomorphism.

2.2 Vercauteren's generalized ate pairings

We review the generalized ate pairings due to Vercauteren [24]. Let $\mu_r = \left\{u \in \mathbb{F}_{q^k}^{\times} : u^r = 1\right\}$. Let $f_{n,Q}, l_{P,Q}$ and v_P be the normalized functions with divisors $n(Q) - ([n]Q) - (n-1)(O)$, $(P) + (Q) + (-(P+Q)) - 3(O)$ and $(P) + (-P) - 2(O)$ respectively, where O denotes the identity element of the group E. Let

$$g(X) = X^k - 1$$
$$\lambda_\varepsilon(X) = \sum_{j=0}^{k-1} \varepsilon_j X^j$$
$$W_\varepsilon(X) = \det \begin{pmatrix} g(X) & \lambda_\varepsilon(X) \\ g'(X) & \lambda'_\varepsilon(X) \end{pmatrix}$$

for $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{k-1}) \in \mathbb{Z}^k$. Vercauteren [24] defined a map $a_\varepsilon : G_2 \times G_1 \to \mu_r$ such that, for all $P \in G_1, Q \in G_2$,

$$a_\varepsilon(Q,P) = Z_\varepsilon(Q,P)^L, \quad \text{where}$$
$$Z_\varepsilon(Q,P) = \prod_{j=0}^{k-1} f_{\varepsilon_j, q^j Q}(P) \prod_{j=0}^{k-2} \frac{l_{\varepsilon_j q^j Q,\ (\varepsilon_{j+1}q^{j+1} + \cdots + \varepsilon_{k-1}q^{k-1})Q}}{v_{(\varepsilon_j q^j + \cdots + \varepsilon_{k-1}q^{k-1})Q}}(P)$$

and showed that it is a well-defined bilinear map if $r \mid \lambda_\varepsilon(q)$, $r^2 \nmid \lambda_\varepsilon(q)$ and $r^2 \nmid g(q)$. He also showed that a_ε is non-degenerate if and only if $r^2 \nmid W_\varepsilon(q)$.

From now on, we will assume $r \mid \lambda_\varepsilon(q)$, $r^2 \nmid \lambda_\varepsilon(q)$, $r^2 \nmid g(q)$ and $r^2 \nmid W_\varepsilon(q)$, so that a_ε is a non-degenerate pairing. We will also assume, without losing generality, that $\gcd(\varepsilon_0, \ldots, \varepsilon_{k-1}) = 1$ because the vector ε is selected as small as possible for faster pairing computation. In summary, Vercauteren proposed the following approach for pairings.

In: $P \in G_1, Q \in G_2$

Out: $z = a_\varepsilon(Q, P)$

1. $[\mathsf{M}_\varepsilon]$ $\gamma_\varepsilon \leftarrow Z_\epsilon(P, Q)$
2. $[\mathsf{E}_\varepsilon]$ $z \leftarrow \gamma_\epsilon^L$

2.3 Kanayama-Okamoto's approach to pairing inversion

We review an approach for pairing inversion due to Kanayama-Okamoto [15]. They proposed the following approach and proved its correctness.

In: $Q \in G_2$, $z \in \mu_r$

Out: $P \in G_1$ such that $z = a_\epsilon(Q, P)$.

1. $[\mathsf{Choice}]$ Choose $e \in \mathbb{Z}^k$ such that $r \mid \lambda_e(q)$ and $\gcd(e_0, \ldots, e_{k-1}) = 1$.
2. $[\mathsf{EI}_{\varepsilon,e}]$ Find γ_e by carrying out the following.
   (a) $T_j \leftarrow \operatorname{rem}\left(q^j, r\right)$, the remainder of q^j modulo r
   (b) $a_j \leftarrow \operatorname{ord}_r(T_j)$
   (c) $n_j \leftarrow \frac{T_j^{a_j} - 1}{r}$
   (d) $N_j \leftarrow \gcd(T_j^{a_j} - 1, q^k - 1)$
   (e) $d_j \leftarrow \sum_{h=0}^{a_j - 1} T_j^{a_j - 1 - h} q^{jh}$
   (f) $c_j \leftarrow \operatorname{rem}(d_j, N_j)$
   (g) $c'_j \leftarrow c_j^{-1} \bmod r$.
   (h) $U_e \leftarrow \frac{1}{r} \sum_{j=0}^{k-1} e_j T_j$
   (i) $\psi_\varepsilon \leftarrow U_\varepsilon - \sum_{j=0}^{k-1} \varepsilon_j c'_j n_j$
   (j) Find the "right" γ_e from the set $\Theta_{\varepsilon,e,z} = \left\{ \frac{\tau^{U_e}}{\prod_{j=0}^{k-1} \alpha_j^{e_j}} \ : \ \exists \tau, \alpha_j \in \mathbb{F}_{q^k}^\times \quad \alpha_j^{Lc_j} = \tau^{Ln_j} \ \wedge \tau^{L\psi_\varepsilon} = z \right\}$
3. $[\mathsf{MI}_e]$ Find P from $\gamma_e = Z_e(P, Q)$.

By the "right" γ_e, we mean the one satisfying the condition $\gamma_e = Z_e(Q, P)$.

Remark 1. The above description is a bit different from the original one by Kanayama-Okamoto [15] in three ways.

- They used the quantity $\frac{\prod_{j=0}^{k-1} \alpha_j^{e_j}}{\tau^{U_e}}$ for γ_e, which is the reciprocal of the quantity shown above. We changed it in the current form, because it is more consistent with the notation used in Vercauteren's generalized pairings [24].
- They elaborated their idea for ate_i pairing (corresponding to a particular class of ε) and indicated that it could be extended to the generalized ate pairing of Vercauteren [24] (corresponding to a general class of ε). Indeed, such an extension is straightforward. The above description allows arbitrary ε.
- They elaborated their idea for particular choices of e such as coefficients of cyclotomic polynomials or $(1, \ldots, 1)$. The extension to arbitrary e is also straightforward. The above description allows arbitrary e.

3 A Simpler Approach for Paring Inversion

In this section, we describe an approach for inverting the generalized ate pairing of Vacauteren (Approach 1). We will use the notations introduced in Section 2.2. Comparing to Kanayama-Okamoto's approach (See Section 2.3), one sees that the modified exponentiation inversion step $\mathsf{EI}_{\varepsilon,e}$ is simplified. The simplicity of the proposed approach facilitates the subsequent investigation. We prove its correctness (Theorem 1). Then we prove that the simpler approach is equivalent to Kanayama-Okamoto's original approach (Theorem 2). We let $a \equiv_n b$ abbreviate $a \equiv b \pmod n$ for simplicity.

Approach 1. Pairing Inversion

In: $Q \in G_2$, $z \in \mu_r$

Out: $P \in G_1$ such that $z = a_\varepsilon(Q, P)$.

1. [Choice] Choose $e \in \mathbb{Z}^k$ such that $r \mid \lambda_e(q)$ and $\gcd(e_0, \ldots, e_{k-1}) = 1$.
2. [$\mathsf{EI}_{\varepsilon,e}$] Find the "right" γ_e from $\Gamma_{\varepsilon,e,z} = \left\{\gamma \in \mathbb{F}_{q^k}^\times : \gamma^L = z^{\delta_{\varepsilon,e}}\right\}$, where $\delta_{\varepsilon,e} \equiv_r w_e/w_\varepsilon$ and $w_\eta = \frac{1}{r} W_\eta(q)$.
3. [MI_e] Find P from $\gamma_e = Z_e(P, Q)$.

Theorem 1 (Correctness)**.** *If $\gamma_e = Z_e(Q, P)$, then $\gamma_e^L = z^{\delta_{\varepsilon,e}}$.*

Proof. Recall that $\gamma_e^L = a_e(Q, P)$ and $z = a_\varepsilon(Q, P)$. Hence we need to show that

$$a_e(Q, P) = a_\varepsilon(Q, P)^{\delta_{\varepsilon,e}}.$$

Recall, from the proof of Theorem 4 in [24], that

$$f_{q,Q}(P)^{L\frac{\lambda_e(q)}{r}g'(q)\left(\frac{g(q)}{r}\right)^{-1}} = f_{q,Q}(P)^{L\lambda'_e(q)} \cdot a_e(Q, P),$$

and thus

$$a_e(Q, P) = f_{q,Q}(P)^{L\left(\frac{\lambda_e(q)}{r}g'(q)\left(\frac{g(q)}{r}\right)^{-1} - \lambda'_e(q)\right)} = f_{q,Q}(P)^{L\left(-\left(\frac{g(q)}{r}\right)^{-1}w_e\right)}.$$

Similarly, one gets

$$a_\varepsilon(Q, P) = f_{q,Q}(P)^{L\left(-\left(\frac{g(q)}{r}\right)^{-1}w_\varepsilon\right)}.$$

Thus,

$$a_e(Q, P) = f_{q,Q}(P)^{L\left(-\left(\frac{g(q)}{r}\right)^{-1}w_e\right)} = a_\varepsilon(Q, P)^{w_e w_\varepsilon^{-1}} = a_\varepsilon(Q, P)^{\delta_{\varepsilon,e}}.$$

□

We claim that the above approach is equivalent to that of Kanayama-Okamoto. Since the only difference is in $\mathsf{EI}_{\varepsilon,e}$ step, we only need to show the equivalence for the step. Since $\mathsf{EI}_{\varepsilon,e}$ is essentially a search problem, we need to show that the search spaces $\Gamma_{\varepsilon,e,z}$ and $\Theta_{\varepsilon,e,z}$ are the same.

Theorem 2 (Equivalence to Kanayama-Okamoto's approach)**.** *We have*

$$\Gamma_{\varepsilon,e,z} = \Theta_{\varepsilon,e,z}.$$

Proof. We will prove the inclusion in both directions.

Claim 1: $\Theta_{\varepsilon,e,z} \subset \Gamma_{\varepsilon,e,z}$

Let $\tau \in \mathbb{F}_{q^k}^{\times}$ and $\alpha_j \in \mathbb{F}_{q^k}^{\times}$ be such that $\alpha_j^{Lc_j} = \tau^{Ln_j}$ and $\tau^{L\psi_\varepsilon} = z$. Let $\theta = \frac{\tau^{U_e}}{\prod_{j=0}^{k-1} \alpha_j^{e_j}}$. We need to show that $\theta^L = z^{\delta_{\varepsilon,e}}$. Note

$$\theta^L = \left(\frac{\tau^{U_e}}{\prod_{j=0}^{k-1} \alpha_j^{e_j}} \right)^L = \frac{\tau^{LU_e}}{\prod_{j=0}^{k-1} \alpha_j^{Le_j}} = \frac{\tau^{LU_e}}{\prod_{j=0}^{k-1} \tau^{Le_j c'_j n_j}} = \tau^{L\left(U_e - \sum_{j=0}^{k-1} e_j c'_j n_j\right)} = \tau^{L\psi_e}$$

As $z = \tau^{L\psi_\varepsilon}$, we have $\theta^L = z^{\psi_e \psi'_\varepsilon}$ where $\psi'_\varepsilon \equiv_r 1/\psi_\varepsilon$. Since $Z_e(Q,P) \in \Theta_{e,z}$ as [15] showed, we also have $Z_e(Q,P)^L = z^{\psi_e \psi'_\varepsilon}$. Recall $Z_e(Q,P)^L = a_\varepsilon(Q,P)^{w_e w'_\varepsilon} = z^{w_e w'_\varepsilon}$. Thus,

$$\theta^L = z^{\psi_e \psi'_\varepsilon} = Z_e(Q,P)^L = a_\varepsilon(Q,P)^{w_e w'_\varepsilon} = z^{w_e w'_\varepsilon} = z^{\delta_{\varepsilon,e}}.$$

Claim 2: $\Gamma_{\varepsilon,e,z} \subset \Theta_{\varepsilon,e,z}$

Let $\gamma \in \mathbb{F}_{q^k}^{\times}$ be such that $\gamma^L = z^{\delta_{\varepsilon,e}}$. We need to find τ and α_j such that $\alpha_j^{Lc_j} = \tau^{Ln_j}$, $\tau^{L\psi_\varepsilon} = z$ and $\gamma = \frac{\tau^{U_e}}{\prod_{j=0}^{k-1} \alpha_j^{e_j}}$. Let $P \in G_1$ and $Q \in G_2$ be such that $z = a_\varepsilon(Q,P)$. Such P, Q exist because the map $G_1 \to \mu_r, P \mapsto a_\varepsilon(Q,P)$ is bijective if $Q \in G_2 - \{O\}$. Let $\tilde{\tau} = f_{r,Q}(P)$ and $\tilde{\alpha}_j = f_{T_j,Q}(P)$ and $\tilde{\gamma} = \frac{\tilde{\tau}^{U_e}}{\prod_{j=1}^{k-1} \tilde{\alpha}_j^{e_j}}$. Let $h \in \mathbb{Z}^k$ be such that $\sum_{j=0}^{k-1} h_j e_j = 1$. Such h exists because $\gcd(e_0, \ldots, e_{k-1}) = 1$. Let

$$\tau = \tilde{\tau}$$
$$\alpha_j = \tilde{\alpha}_j \left(\frac{\tilde{\gamma}}{\gamma} \right)^{h_j}$$

Then we have

- $\tau^{L\psi_\varepsilon} = z$: Note
$$\tau^{L\psi_\varepsilon} = \tilde{\tau}^{L\psi_\varepsilon} = z$$
- $\alpha_j^{Lc_j} = \tau^{Ln_j}$: Note
$$\alpha_j^{Lc_j} = \left(\tilde{\alpha}_j \left(\frac{\tilde{\gamma}}{\gamma} \right)^{h_j} \right)^{Lc_j} = \tilde{\alpha}_j^{Lc_j} \left(\frac{\tilde{\gamma}}{\gamma} \right)^{Lh_j c_j} = \tilde{\alpha}_j^{Lc_j} \left(\frac{z^{\delta_{\varepsilon,e}}}{z^{\delta_{\varepsilon,e}}} \right)^{h_j c_j} = \tilde{\tau}^{Ln_j} = \tau^{Ln_j}.$$
- $\gamma = \frac{\tau^{U_e}}{\prod_{j=0}^{k-1} \alpha_j^{e_j}}$: Note
$$\gamma = \tilde{\gamma} \frac{\gamma}{\tilde{\gamma}} = \frac{\tilde{\tau}^{U_e}}{\prod_{j=0}^{k-1} \tilde{\alpha}_j^{e_j}} \prod_{j=0}^{k-1} \left(\frac{\gamma}{\tilde{\gamma}} \right)^{h_j e_j} = \frac{\tilde{\tau}^{U_e}}{\prod_{j=0}^{k-1} \left(\tilde{\alpha}_j \left(\frac{\tilde{\gamma}}{\gamma} \right)^{h_j} \right)^{e_j}} = \frac{\tau^{U_e}}{\prod_{j=1}^{k-1} \alpha_j^{e_j}}.$$

□

4 Complexity of Modified Miller Inversion

In this section, we provide a bit-complexity of the modified Miller inversion step MI_e. It essentially says that, when q and k are fixed, the complexity is bounded by $||e||_1^2$ where $||e||_1$ stands for the sum norm of the integer vector e. Hence in order to reduce the complexity of MI_e, one needs to choose e with small sum norm.

Theorem 3 (Complexity of MI_e)**.** *There exists an algorithm for* MI_e *requiring at most*

$$2^8 \; ||e||_1^2 \; k^2 \; (\log_2 q)^3$$

bit operations.

Remark 2. Even though the above theorem is stated for the modified Miller inversion, it is in fact the complexity of the Miller inversion for the generalized ate paring a_η defined by arbitrary given integer vector η.

In the remainder of this section, we will prove Theorem 3. We will divide the proof into several lemmas that are interesting on their own. We begin with a slight reformulation of the expression for the generalized ate pairing [24], because it greatly simplifies the derivation of the above upper bound.

Lemma 4. *Let* $e^{(+)}, e^{(-)} \in \mathbb{Z}^k$ *be*

$$e_i^{(+)} = \begin{cases} e_i & \text{if } e_i > 0 \\ 0 & \text{else} \end{cases}$$
$$e_j^{(-)} = \begin{cases} e_j & \text{if } e_j < 0 \\ 0 & \text{else} \end{cases}$$

Then, for all $Q \in G_2$ *and all* $P \in G_1$*, we have*

$$Z_e\,(Q,P) = \frac{Z_{e^{(+)}}\,(Q,P)}{Z_{-e^{(-)}}\,(Q,P)}$$

Proof. Let $e_{m_1}, \ldots, e_{m_s} > 0$ and $e_{n_1}, \ldots, e_{n_t} < 0$ and all other components of e are zero. Then we have

$$e_{m_i}^{(+)} = e_{m_i}$$
$$e_{n_j}^{(-)} = e_{n_j}$$

and all other components of $e^{(+)}$ and $e^{(-)}$ are zero. Note

$$U_e r - e_{n_1} q^{n_1} - \cdots - e_{n_t} q^{n_t} = e_{m_1} q^{m_1} + \cdots + e_{m_s} q^{m_s}$$

Thus

$$\begin{aligned}
& f_{e_{m_1} q^{m_1} + \cdots + e_{m_s} q^{m_s}, Q} \\
&= \prod_{i=1}^{s} f_{e_{m_i} q^{m_i}, Q} \; \prod_{i=1}^{s-1} \frac{l_{e_{m_i} q^{m_i} Q, \left(e_{m_{i+1}} q^{m_{i+1}} + \cdots + e_{m_s} q^{m_s}\right) Q}}{v_{\left(e_{m_i} q^{m_i} + \cdots + e_{m_s} q^{m_s}\right) Q}} \\
&= \prod_{i=1}^{s} f_{q^{m_i}, Q}^{e_{m_i}} \prod_{i=1}^{s} f_{e_{m_i}, q^{m_i} Q} \; \prod_{i=1}^{s-1} \frac{l_{e_{m_i} q^{m_i} Q, \left(e_{m_{i+1}} q^{m_{i+1}} + \cdots + e_{m_s} q^{m_s}\right) Q}}{v_{\left(e_{m_i} q^{m_i} + \cdots + e_{m_s} q^{m_s}\right) Q}} \\
&= \prod_{i=1}^{s} f_{q^{m_i}, Q}^{e_{m_i}}(P) \cdot Z_{e^{(+)}}(Q,P)
\end{aligned}$$

$$
\begin{aligned}
&f_{U_e r - e_{n_1} q^{n_1} - \cdots - e_{n_t} q^{n_t}, Q} \\
&= f_{U_e r, Q} \quad \frac{l_{U_e r Q, \left(-e_{n_1} q^{n_1} - \cdots - e_{n_t} q^{n_t}\right) Q}}{v_{\left(U_e r - e_{n_1} q^{n_1} - \cdots - e_{n_t} q^{n_t}\right) Q}} \quad \prod_{j=1}^{t} f_{-e_{n_j} q^{n_j}, Q} \prod_{j=1}^{t-1} \frac{l_{-e_{n_j} q^{n_j} Q, \left(-e_{n_{j+1}} q^{n_{j+1}} - \cdots - e_{n_t} q^{n_t}\right) Q}}{v_{\left(-e_{n_{j+1}} q^{n_{j+1}} - \cdots - e_{n_t} q^{n_t}\right) Q}} \\
&= f_{U_e r, Q} \quad \prod_{j=1}^{t} f_{-e_{n_j} q^{n_j}, Q} \prod_{j=1}^{t-1} \frac{l_{-e_{n_j} q^{n_j} Q, \left(-e_{n_{j+1}} q^{n_{j+1}} - \cdots - e_{n_t} q^{n_t}\right) Q}}{v_{\left(-e_{n_{j+1}} q^{n_{j+1}} - \cdots - e_{n_t} q^{n_t}\right) Q}} \\
&= f_{r,Q}^{U_e} f_{U_e, rQ} \quad \prod_{j=1}^{t} f_{q^{n_j}, Q}^{-e_{n_j}} \prod_{j=1}^{t} f_{-e_{n_j}, q^{n_j} Q} \quad \prod_{j=1}^{t-1} \frac{l_{-e_{n_j} q^{n_j} Q, \left(-e_{n_{j+1}} q^{n_{j+1}} - \cdots - e_{n_t} q^{n_t}\right) Q}}{v_{\left(-e_{n_{j+1}} q^{n_{j+1}} - \cdots - e_{n_t} q^{n_t}\right) Q}} \\
&= f_{r,Q}^{U_e}(P) \quad \prod_{j=1}^{t} f_{q^{n_j}, Q}^{-e_{n_j}}(P) \cdot Z_{-e^{(-)}}(Q, P)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&f_{r,Q}^{U_e}(P) \quad \prod_{j=1}^{t} f_{q^{n_j}, Q}^{-e_{n_j}}(P) \cdot Z_{-e^{(-)}}(Q, P) \\
&= \prod_{i=1}^{s} f_{q^{m_i}, Q}^{e_{m_i}} \cdot Z_{e^{(+)}}(Q, P)
\end{aligned}
$$

equivalently,

$$
\frac{f_{r,Q}^{U_e}(P)}{\prod_{i=0}^{k-1} f_{q^i, Q}^{e_i}(P)} = \frac{Z_{e^{(+)}}(Q, P)}{Z_{-e^{(-)}}(Q, P)}
$$

From [24], we have

$$
Z_e(Q, P) = \frac{f_{r,Q}^{U_e}(P)}{\prod_{i=0}^{k-1} f_{q^i, Q}^{e_i}(P)},
$$

Hence we have

$$
Z_e(Q, P) = \frac{Z_{e^{(+)}}(Q, P)}{Z_{-e^{(-)}}(Q, P)}
$$

□

Lemma 5. *For every $Q \in G_2$, $\theta \in \mathbb{F}_{q^k}^*$ and $e \in \mathbb{Z}^\ell$, there exists a bivariate polynomial h over $\mathbb{F}_{q^k}$ such that*

- *(a)* $\forall (x, y) \in G_1 \quad \theta = Z_e(Q, (x, y)) \implies h(x, y) = 0$
- *(b)* $\deg_X(h) \leq ||e||_1$
- *(c)* $\deg_Y(h) \leq 2 \max\{s, t\}$*, where* $s := \#\{j : e_j > 0\}$ *and* $t := \#\{j : e_j < 0\}$*.*

Proof. Let $Q \in G_2$, $\theta \in \mathbb{F}_{q^k}^*$ and $e \in \mathbb{Z}^\ell$. We will construct a witness for the existentially quantified h. From Lemma 14 of [11], we have

$$
f_{\mu,\ \nu Q}(X, Y) = \begin{cases} 1 & \mu = 1 \\ \frac{f_{\mu,\nu,1}(X) + Y f_{\mu,\nu,2}(X)}{v_{\mu\nu Q}} & \mu > 1 \end{cases}
$$

where $f_{\mu,\nu,1}, f_{\mu,\nu,2} \in \mathbb{F}_{q^k}[X]$ such that

$$
\begin{aligned}
\deg(f_{\mu,\nu,1}) &\leq \left\lfloor \frac{\mu + 1}{2} \right\rfloor \\
\deg(f_{\mu,\nu,2}) &\leq \left\lfloor \frac{\mu}{2} - 1 \right\rfloor
\end{aligned}
$$

From Lemma 4, we have

$$Z_e\left(Q,(x,y)\right) = \frac{Z_{e^{(+)}}(x,y)}{Z_{-e^{(-)}}(x,y)} =: \frac{A(x,y)}{B(x,y)} \quad \text{for all} \quad (x,y) \in G_1$$

where

$$\begin{aligned}
A = & \prod_{\substack{1\le i\le s \\ e_{m_i}\ge 2}} \left(f_{e_{m_i},q^{m_i},1} + Y f_{e_{m_i},q^{m_i},2}\right) \prod_{\substack{1\le j\le t \\ e_{n_j}\le -2}} v_{-e_{n_j}q^{n_j}Q} \\
& \prod_{i=1}^{s-1} l_{e_{m_i}q^{m_i}Q,\left(e_{m_{i+1}}q^{m_{i+1}}+\cdots+e_{m_s}q^{m_s}\right)Q} \prod_{j=1}^{t-1} v_{\left(-e_{n_{j+1}}q^{n_{j+1}}-\cdots-e_{n_t}q^{n_t}\right)Q} \\
B = & \prod_{\substack{1\le j\le t \\ e_{n_j}\le -2}} \left(f_{-e_{n_j},q^{n_j},1} + Y f_{-e_{n_j},q^{n_j},2}\right) \prod_{\substack{1\le i\le s \\ e_{m_i}\ge 2}} v_{e_{m_i}q^{m_i}Q} \\
& \prod_{j=1}^{t-1} l_{-e_{n_j}q^{n_j}Q,\left(-e_{n_{j+1}}q^{n_{j+1}}-\cdots-e_{n_t}q^{n_t}\right)Q} \prod_{i=1}^{s-1} v_{\left(e_{m_i}q^{m_i}+\cdots+e_{m_s}q^{m_s}\right)Q}
\end{aligned}$$

Finally, we propose the following h as a witness for the existential quantification:

$$h = A - \theta B.$$

We will show that h is indeed a witness satisfying the three conditions.

(a) $\forall (x,y) \in G_1, \quad Z_e(Q,(x,y)) = \theta \implies h(x,y) = 0.$
proof: Let $(x,y) \in G_1$. Assume that $\theta = Z_e(Q,(x,y))$. Then Obviously $\theta = \frac{A(x,y)}{B(x,y)}$. Thus $h(x,y) = A(x,y) - \theta B(x,y) = 0$.

(b) $\deg_X(h) \le ||e||_1$
Proof: Note

$$\begin{aligned}
\deg_X(A) &\le \sum_{e_i\ge 2}\left\lfloor\frac{e_i+1}{2}\right\rfloor + \sum_{e_i\le -2} 1 + \sum_{e_i\ge 1} 1 + \sum_{e_i\le -1} 1 \\
&= \sum_{e_i\ge 2}\left\lfloor\frac{e_i+1}{2}\right\rfloor + \sum_{e_i\le -2} 1 + \sum_{e_i\ge 2} 1 + \sum_{e_i=1} 1 + \sum_{e_i=-1} 1 + \sum_{e_i\le -2} 1 \\
&= \sum_{e_i\ge 2}\left\lfloor\frac{e_i+3}{2}\right\rfloor + \sum_{e_i\le -2} 2 + \sum_{e_i=1} 1 + \sum_{e_i=-1} 1 \\
&\le \sum_{e_i\ge 2}|e_i| + \sum_{e_i\le -2}|e_i| + \sum_{e_i=1}|e_i| + \sum_{e_i=-1}|e_i| \\
&= ||e||_1 \\
\deg_X(B) &\le \sum_{e_i\le -2}\left\lfloor\frac{-e_i+1}{2}\right\rfloor + \sum_{e_i\ge 2} 1 + \sum_{e_i\le -1} 1 + \sum_{e_i\ge 1} 1 \\
&= \sum_{e_i\le -2}\left\lfloor\frac{-e_i+1}{2}\right\rfloor + \sum_{e_i\ge 2} 1 + \sum_{e_i\le -2} 1 + \sum_{e_i=-1} 1 + \sum_{e_i\ge 2} 1 + \sum_{e_i=1} 1 \\
&= \sum_{e_i\le -2}\left\lfloor\frac{-e_i+3}{2}\right\rfloor + \sum_{e_i\ge 2} 2 + \sum_{e_i=-1} 1 + \sum_{e_i=1} 1 \\
&\le \sum_{e_i\le -2}|e_i| + \sum_{e_i\ge 2}|e_i| + \sum_{e_i=-1}|e_i| + \sum_{e_i=1}|e_i| \\
&= ||e||_1
\end{aligned}$$

Hence $\deg_X(h) \le ||e||_1$.

(c) $\deg_Y(h) \le 2\max\{s,t\}$.
proof: Note

$$\begin{aligned}\deg_Y(A) &\le s+s \le 2s\\ \deg_Y(B) &\le t+t \le 2t\end{aligned}$$

Hence $\deg_Y(h) \le 2\max\{s,t\}$.

□

Proof of Theorem 3. To solve MI_e for given $Q \in G_2$ and $e \in \mathbb{Z}^\ell$, we have to find $P = (x,y) \in G_1$ such that

$$\begin{aligned}\theta &= Z_e(Q,(x,y))\\ y^2 &= x^3+ax+b\end{aligned} \tag{1}$$

From Lemma 5, there exists a bivariate polynomial h over $\mathbb{F}_{q^k}$ such that

$$\begin{aligned}\forall (x,y)\in G_1 \quad \theta = Z_e(Q,(x,y)) &\implies h(x,y)=0\\ \deg_X(h) &\le ||e||_1\\ \deg_Y(h) &\le 2\max\{s,t\} \le 2||e||_1.\end{aligned}$$

Let

$$F(X,Y) = Y^2 - X^3 - aX - b$$

and let

$$u(X) = \mathrm{res}_Y(h(X,Y), F(X,Y)).$$

Then for all $(x,y)\in G_1$, if $\theta = Z_e(Q,(x,y))$ then $u(x)=0$ and

$$\begin{aligned}\deg u &\le \deg_Y F \deg_X h + \deg_Y h \deg_X F\\ &\le 2\cdot ||e||_1 + 2||e||_1\cdot 3\\ &= 8\,||e||_1.\end{aligned}$$

From [11], there exists an algorithm for solving a polynomial of degree d in $\mathbb{F}_q$ whose complexity is $O(d^2k^2(\log q)^3)$. In fact, a more detailed analysis shows that the algorithm requires at most

$$4\,d^2\,k^2\,(\log_2 q)^3$$

bit operations. Since solving $u(X)=0$ is enough to solve the system of equations (1), we see that MI_e can be solved within

$$4\,(8\,||e||_1)^2\,k^2\,(\log_2 q)^3 = 2^8\,||e||_1^2\,k^2\,(\log_2 q)^3.$$

bit operations. □

5 Toward Complexity of Modified Exponentiation Inversion

It would be nice to have a complexity estimate for the modified exponentiation inversion $\mathsf{EI}_{\varepsilon,e}$, just as for the modified Miller inversion MI_e (Theorem 3). Unfortunately, we do *not* have a result on it. We are not aware of any results in the literature either. We expect it to be a very non-trivial task, most likely requiring patient and long arduous efforts of many researchers, each making an incremental contribution. In this section, we report on an incremental finding toward complexity of $\mathsf{EI}_{\varepsilon,e}$.

Recall that $\mathsf{EI}_{\varepsilon,e}$ asks to find the "right" γ_e from the search space $\Gamma_{\varepsilon,e,z}$. Hence it is reasonable to begin with the study of the relationship between the search space $\Gamma_{\varepsilon,e,z}$ and the chosen vector e.

Proposition 6. *We have*

1. *If the auxiliary pairing a_e is degenerate, then $\Gamma_{\varepsilon,e,z} = \Gamma_{\varepsilon,\varepsilon,1} = \mu_L$.*
2. *If the auxiliary pairing a_e is non-degenerate, then $\Gamma_{\varepsilon,e,z} = \Gamma_{\varepsilon,\varepsilon,z^{\delta_{\varepsilon,e}}}$.*

Proof. Note that $\delta_{\varepsilon,\varepsilon} = 1$. Recall that $\delta_{\varepsilon,e} \equiv_r w_e/w_\varepsilon$ and $w_e = \frac{1}{r}W_e(q) \in \mathbb{Z}$. Therefore we have

$$a_e \text{ is degenerate } \iff r^2|W_e(q) \iff w_e \equiv_r 0 \iff \delta_{\varepsilon,e} \equiv_r 0$$

If a_e is degenerate, then we have

$$\Gamma_{\varepsilon,e,z} = \left\{\gamma \in \mathbb{F}_{q^k}^\times : \gamma^L = z^0\right\} = \left\{\gamma \in \mathbb{F}_{q^k}^\times : \gamma^L = 1\right\} = \left\{\gamma \in \mathbb{F}_{q^k}^\times : \gamma^L = 1^{\delta_{\varepsilon,\varepsilon}}\right\} = \Gamma_{\varepsilon,\varepsilon,1} = \mu_L$$

If a_e is non-degenerate, then we have

$$\Gamma_{\varepsilon,e,z} = \left\{\gamma \in \mathbb{F}_{q^k}^\times : \gamma^L = z^{\delta_{\varepsilon,e}}\right\} = \left\{\gamma \in \mathbb{F}_{q^k}^\times : \gamma^L = \left(z^{\delta_{\varepsilon,e}}\right)^{\delta_{\varepsilon,\varepsilon}}\right\} = \Gamma_{\varepsilon,\varepsilon,z^{\delta_{\varepsilon,e}}}$$

□

Remark 3. From the above proposition, we observe the followings:

- If a_e is degenerate then the search space of $\mathsf{EI}_{\varepsilon,e}$ is *independent* of the input z, that is, the exponential relation in $\mathsf{EI}_{\varepsilon,e}$ does not capture any information about the input. Thus the modified exponentiation inversion $\mathsf{EI}_{\varepsilon,e}$ will be most likely *harder* when a_e is degenerate than when a_e is non-degenerate.
- If a_e is non-degenerate then the search space of $\mathsf{EI}_{\varepsilon,e}$ for an input z is the same as that of EI_ε for *another* input $z^{\delta_{\varepsilon,e}}$. Thus the modified exponentiation inversion $\mathsf{EI}_{\varepsilon,e}$ is likely as hard as the original exponentiation inversion EI_ε.

Therefore, as a first step toward finding an efficient method for $\mathsf{EI}_{\varepsilon,e}$, we better ensure that a_e is non-degenerate. The following theorem gives a sufficient condition on e, in terms of the max norm of e, for the non-degeneracy of a_e.

Theorem 7. *Let $e \in \mathbb{Z}^k$ be such that $r \mid \lambda_e(q)$ and $\lambda_e(X) \nmid \Phi_k(X)$. Let $m_e = [\mathbb{Q}(\zeta_k) : \mathbb{Q}(\lambda_e(\zeta_k))]$. If*

$$||e||_\infty < \frac{r^{2m_e/\varphi(k)}}{\varphi(k)}$$

then a_e is non-degenerate.

Proof. We will prove the contra-positive. Assume that a_e is degenerate. We need to prove

$$||e||_\infty \geq \frac{r^{2m_e/\varphi(k)}}{\varphi(k)}.$$

Let $s \in \mathbb{Z}$ be such that $s \equiv q \pmod{r}$ and $\mathrm{ord}_{r^2}(s) = k$. If we let $s = q + \iota r$ where $\iota = \frac{q^k-1}{r} \cdot (-kq^{k-1})'$, then we have the desired s:

$$\begin{aligned}
(q+\iota r)^k &\equiv_{r^2} q^k + kq^{k-1}\iota r \\
&= q^k + kq^{k-1}\frac{q^k-1}{r}(-kq^{k-1})'r \\
&= q^k + kq^{k-1}(-kq^{k-1})'(q^k-1) \\
&= q^k + (-1+rD)(q^k-1) \quad \text{for some } D \in \mathbb{Z} \\
&= 1 + r^2 D\frac{q^k-1}{r} \\
&\equiv_{r^2} 1
\end{aligned}$$

If $(q+\iota r)^d \equiv_{r^2} 1$ for some $d < k$, then

$$\begin{aligned} r^2 &\mid q^d + dq^{d-1}\iota r - 1 \\ \Rightarrow r &\mid q^d + dq^{d-1}\iota r - 1 \\ \Rightarrow r &\mid q^d - 1 \end{aligned}$$

This contradicts the fact $k = \mathrm{ord}_r(q)$. Thus we have $\mathrm{ord}_{r^2}(s) = k$.

Now, to prove our claim, we will use the fact that a_e is degenerate if and only if $r^2 \mid \lambda_e(s)$; see [12]. Note $r^2 \mid (s^k - 1) = \prod_{d|k} \Phi_d(s)$. Since $r \mid \Phi_d(s) = \Phi_d(q+\iota r)$ implies $r \mid \Phi_d(q)$, r divides only $\Phi_k(s)$ and $r \nmid \Phi_d(s)$ for all $d < k$. Therefore, $r^2 \mid \Phi_k(s)$.

Let $\mu_e(X) = \mathrm{rem}(\lambda_e(X), \Phi_k(X))$ and $\zeta_k \in \mathbb{C}$ be a primitive k-th root of unity. Note that $\mu_e \neq 0$ from the assumption. Let $v(X) \in \mathbb{Q}[X]$ be the minimal polynomial of $\mu_e(\zeta_k)$ over $\mathbb{Q}$. Note that $v(x) \in \mathbb{Z}[x]$ as $\mu_e(\zeta_k) \in \mathbb{Z}[\zeta_k]$, the ring of integers of $\mathbb{Q}(\zeta_k)$. Since $v(\mu_e(X))$ is zero at ζ_k and $\Phi_k(x)$ is monic, we have

$$v(\mu_e(X)) = \Phi_k(X)h(X) \quad \text{for some} \quad h(X) \in \mathbb{Z}[X].$$

From $r^2 \mid \lambda_e(s)$ and $r^2 \mid \Phi_k(s)$, we have $r^2 \mid \mu_e(s)$ and

$$\begin{aligned} v(0) &\equiv_{r^2} v(\mu_e(s)) \\ &\equiv_{r^2} \Phi_k(s)h(s) \\ &\equiv_{r^2} 0 \end{aligned}$$

Therefore, we have either $v(0) = 0$ or $|v(0)| \geq r^2$. Noting that, by [6, Proposition 4.3.2] and the fact that v is monic,

$$|v(0)| = |\mathrm{Norm}(\mu_e(\zeta_k))| = \left|\mathrm{Norm}_{\mathbb{Q}(\zeta_k)/\mathbb{Q}}(\mu_e(\zeta_k))\right|^{1/m_e} = \left|\prod_{\gcd(j,k)=1} \mu_e(\zeta_k^j)\right|^{1/m_e},$$

we conclude that $v(0) \neq 0$. Indeed if $v(0) = 0$, then $\Phi_k \mid \lambda_e$, a contradiction to $\mu_e \neq 0$. Thus, we have

$$\begin{aligned} r^2 &\leq |v(0)| \\ &= \left|\prod_{\gcd(j,k)=1} \mu_e(\zeta_k^j)\right|^{1/m_e} \\ &\leq \left(\prod_{\gcd(j,k)=1} \varphi(k)||e||_\infty\right)^{1/m_e} \\ &= (\varphi(k)||e||_\infty)^{\varphi(k)/m_e}, \end{aligned}$$

Therefore, we finally have

$$\frac{r^{2m_e/\varphi(k)}}{\varphi(k)} \leq ||e||_\infty.$$

□

6 Reducing Paring Inversion to Exponentiation Inversion

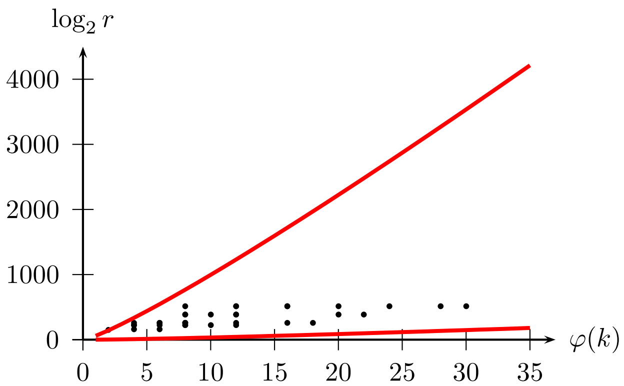
In this section, we discuss when pairing inversion can be reduced to exponentiation inversion. The question was initiated and addressed by Kanayama-Okamoto [15]. They showed that, if the integer vector e is chosen from either coefficients of cyclotomic polynomials or $(1, \ldots, 1)$, then MI_e can be carried out in polynomial time in $\log_2 r$ and PI is reduced to the modified exponentiation inversion $\mathsf{EI}_{\varepsilon,e}$. However according to Corollary 6 of Vercauteren [24], such e makes the corresponding auxiliary pairing degenerate. Hence, from Proposition 6, the modified exponentiation inversion $\mathsf{EI}_{\varepsilon,e}$ is expected to be harder than the exponentiation inversion $\mathsf{EI}_{\varepsilon}$ and thus it is not clear that such choices of e allows the reduction of pairing inversion to exponentiation inversion. In order to reduce pairing inversion to exponentiation inversion, it is safer to find e such that it is *small* and the corresponding auxiliary pairing is *non-degenerate*. In this section, we investigate the existence of such e in various cases (Theorem 9 and the subsequent examples in Table 1).

Definition 1. *Let C_α be the set of all $(r,k) \in \mathbb{Z}^2_{>0}$ satisfying*

C1: $r^{1/\varphi(k)} > \varphi(k)$

C2: $r^{1/\varphi(k)} \leq (\log_2 r)^\alpha$

Remark 4. In the following figure, the bottom curve is from the condition C1 in Definition 1 and the top curve is from the condition C2 when $\alpha = 10$. Thus, the regions between the two curves is the set C_{10}, The black dots represent typical pairing friendly curves from Table 1 in [10]. Note that the parameters for the typical pairing friendly curves belong to C_{10}.



Lemma 8. *If $\alpha > 1$, then C_α is an infinite set.*

Proof. We first observe that $r = 9$ and $\varphi(k) = 2$ satisfy the above two conditions. We will show that the two curves defined by

$$r^{1/\varphi(k)} = \varphi(k)$$
$$r^{1/\varphi(k)} = (\log_2 r)^\alpha$$

do not meet when $\varphi(k) > 2$. The above system is equivalent to

$$r^{1/\varphi(k)} = \varphi(k)$$
$$(\log_2 r)^\alpha = \varphi(k)$$

The first equation is equivalent to

$$\log_2 r = \varphi(k) \log_2 \varphi(k)$$

By substituting it into the second equation, we have

$$\varphi(k)^\alpha (\log_2 \varphi(k))^\alpha = \varphi(k),$$

which does not have a solution when $\varphi(k) > 2$. Thus the above two curves do not meet when $\varphi(k) > 2$. Therefore, we conclude that C_α is an infinite set. □

Theorem 9. *Let $\alpha > 1$, $(r,k) \in C_\alpha$ and $r \geq \sqrt{q}$. Then the inversion of every generalized ate pairing can be reduced to exponentiation inversion in polynomial time in $\log_2 r$. Specifically, there exists e such that the auxiliary pairing a_e is non-degenerate and MI_e can be carried out in at most*

$$2^{13} \quad (\log_2 r)^{8\alpha+3}$$

bit operations.

Proof. Let $(q,r) \in C_\alpha$ and $r \geq \sqrt{q}$. We need to find a "witness" e such that a_e is non-degenerate and MI_e can be carried out in the claimed number of bit operations.. From Minkowski's theorem (see III.C of [24]), there exists $e \in \mathbb{Z}^k$ with $r \mid \lambda_e(q)$ such that the last $k - \varphi(k)$ elements of e are zero and

$$||e||_\infty \leq r^{1/\varphi(k)}$$

We will take it as the witness.

First we show that a_e is non-degenerate. Since the last $k - \varphi(k)$ elements of e are zero, we have $\lambda_e(X) \nmid \Phi_k(X)$. From the condition that $r^{1/\varphi(k)} > \varphi(k)$, we have

$$\frac{r^{(2m_e-1)/\varphi(k)}}{\varphi(k)} \geq \frac{r^{1/\varphi(k)}}{\varphi(k)} > 1$$

and thus

$$||e||_\infty \leq r^{1/\varphi(k)} < r^{1/\varphi(k)} \frac{r^{(2m_e-1)/\varphi(k)}}{\varphi(k)} = \frac{r^{2m_e/\varphi(k)}}{\varphi(k)}$$

Therefore, by Theorem 7, a_e is non-degenerate.

Next we show that MI_e can be carried out in the claimed number of bit operations. Let N be the number of bit operations for MI_e. Note that $||e||_1 \leq \varphi(k)\, ||e||_\infty$. Hence $||e||_1 \leq \varphi(k) r^{1/\varphi(k)}$. Therefore, from Theorem 3, we have

$$N \leq 2^8 \left(\varphi(k) r^{1/\varphi(k)}\right)^2 \; k^2 \; (\log_2 q)^3$$

From the condition $r \geq \sqrt{q}$, we have

$$N \leq 2^8 \left(\varphi(k) r^{1/\varphi(k)}\right)^2 \; k^2 \; (2\log_2 r)^3 = 2^{11} \quad \varphi(k)^2 \quad r^{2/\varphi(k)} \quad k^2 \quad (\log_2 r)^3$$

Since $\sqrt{k} \leq \sqrt{2}\varphi(k)$, we have

$$N \leq 2^{11} \quad \varphi(k)^2 \quad r^{2/\varphi(k)} \quad 4\,\varphi(k)^2 \quad (\log_2 r)^3$$

Since $r^{1/\varphi(k)} > \varphi(k)$, we have

$$N < 2^{11} \quad r^{2/\varphi(k)} \quad r^{2/\varphi(k)} \quad 4\, r^{4/\varphi(k)} \quad (\log_2 r)^3 = 2^{13} \quad r^{8/\varphi(k)} \quad (\log_2 r)^3$$

Since $r^{1/\varphi(k)} \leq (\log_2 r)^\alpha$, we have

$$N < 2^{13} \quad (\log_2 r)^{8\alpha} \quad (\log_2 r)^3 = 2^{13} \quad (\log_2 r)^{8\alpha+3}$$

□

The upper bound in Theorem 9 is not tight. In Table 1, we provide tighter upper bounds for several examples. For each example, the first row of the table shows $k, \varphi(k), \log_2 r, \alpha$ with which we can estimate an upper bound of the bit complexity for reducing PI to EI, using Theorem 9. The next rows show actual parameters q, r and a vector $e \in \mathbb{Z}^{\varphi(k)}$. The vector e is the one with smallest sum norm among the LLL reduced vectors for the lattice with respect to q, r, k [24]. The vector e is verified to yield non-degenerate a_e. For the vector e, the last row has been calculated using Theorem 3, which estimates the bit complexity of

Table 1: Estimates on time needed for reducing pairing inversion to exponentiation inversion

BN1	$k,\ \varphi(k),\ \log_2 r,\ \alpha$	12, 4, 638, 18
	q	64159320946300023828492322868916880111762978904323835687136071698951558449723949405178199179425361909648131547026236743201969864263165015207506792223195135492530183970874045708346979371712522 3
	r	64159320946300023828492322868916880111762978904323835687136071698951558449723949405178199179425281810134433709869000390627222138759939120166637880796058352523383264556559295512203435263079228 9
	e	[730750817984886725259965488841096484605724196867, 0, 730750817984886725259965488841096484605724196866, 1]
	$\|\|e_1\|\|$	$\approx 2^{160}$
	bit ops	$< 2^{364} \approx 3.67 \times 10^{82}$ years
BN2	$k,\ \varphi(k),\ \log_2 r,\ \alpha$	12, 4, 158, 6
	q	206327671360737302491015800744139033450591027219
	r	206327671360737302491015346511080613560608358413
	e	[−550292684801, 0, −550292684802, 1]
	$\|\|e_1\|\|$	$\approx 2^{41}$
	bit ops	$< 2^{118} \approx 3.13 \times 10^{8}$ years
KSS1	$k,\ \varphi(k),\ \log_2 r,\ \alpha$	40, 16, 270, 3
	q	178326709713245217260627572968724387343855332468581993976702897877627553783690964596151952604627173843429620177224588 89
	r	1033360998958592639176333946764816221704441278553743659647994766150169434118209921
	e	[−89353, −1, 0, 0, 0, 0, 0, 0, 0, 0, −178706]
	$\|\|e_1\|\|$	$\approx 2^{19}$
	bit ops	$< 2^{81} \approx 2$ days
KSS2	$k,\ \varphi(k),\ \log_2 r,\ \alpha$	36, 12, 169, 2
	q	2751543160631368260054651194751592326705827593927804159297383466 9
	r	705708527028528420873135632253194587092728456673193
	e	[644, 966, 2899, −2255, 8697, 10307, 12562, −2577, 5798, 0, 6120, 2577]
	$\|\|e_1\|\|$	$\approx 2^{16}$
	bit ops	$< 2^{74} \approx 10$ minutes
CP1	$k,\ \varphi(k),\ \log_2 r,\ \alpha$	23, 22, 257, 2
	q	1458119576027446083401734045920739711311839096227166687615127623004851268003597885800006313754045399948707280439848940248906689382680399441035897388657793
	r	171162823577658908923577123057263396229244166914410717458536445501121285956693
	e	[−196, −527, −851, −89, −648, 115, 1086, −14, 547, −1053, 409, −611, 680, −1368, −891, −1808, −3226, −1664, 577, 22, 213, 15, 0]
	$\|\|e_1\|\|$	$\approx 2^{15}$
	bit ops	$< 2^{73} \approx 5$ minutes
C6.6	$k,\ \varphi(k),\ \log_2 r,\ \alpha$	33, 20, 265, 2
	q	17156052909325454315949246639981773625245302303247192926147820123623493657479532833935571050205 9
	r	57482237782367522519498203534411140773179333661921340353191781776555783843120129
	e	[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, −9727, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
	$\|\|e_1\|\|$	$\approx 2^{14}$
	bit ops	$< 2^{70} \approx 48$ seconds
CP2	$k,\ \varphi(k),\ \log_2 r,\ \alpha$	37, 36, 180, 1
	q	680636298409565134719382266884377703639421309150423676134085574745224265663903130936241446716295581618258151
	r	899466048605063172720901741349859664476910393125914353
	e	[12, −1, −26, 8, 2, 15, 15, 17, 7, −6, 31, −6, −5, 21, 4, 4, 14, 4, 3, 23, −12, 6, −9, 0, 4, 2, 15, −8, 0, −3, −2, 11, 17, 7, 1, 1, 0]
	$\|\|e_1\|\|$	$\approx 2^{9}$
	bit ops	$< 2^{61} \approx 1$ seconds

MI_e on the curve more precisely. The estimated upper bounds on the computing times are based on the assumption that one uses the currently fastest super-computer [8], which can perform about

$$17.59 \cdot 10^{15} \text{ flops } \times 1000 \frac{\text{bops}}{\text{flops}} = 2^{64} \text{bops}$$

(bit operations per second).

First two examples BN1 and BN2 are the biggest and the smallest values respectively taken from Table 1 in [20]. Since $\varphi(k)$ for the BN curves [5] are small ($\varphi(k) = 4$), they easily satisfy the condition $\mathsf{C1}$ in Definition 1 but large α values are needed to satisfy $\mathsf{C2}$. Therefore, from Theorem 9, we expect that it will be difficult to reduce PI to EI for BN curves. The tighter upper bound on the bit operations on the last row, based on Theorem 3, supports the observation.

Next two examples are the KSS curves described in Example 4.6 and Example 4.7 in [16]. The parameters are obtained by evaluating the polynomials in the Examples in [16] at $x_0 = -188$ for KSS1 and $x_0 = 107$ for KSS2. The example CP1 is constructed by Cocks-Pinch method to have small α and "typical" parameters $(k, \log_2 r)$ in Table 1 in [10]. The example C6.6 is obtained from evaluating the polynomials in Construction 6.6 with $k = 33$ in [10] at $x_0 = -9727$, which is also a pairing-friendly curve (Definition 2.3 in [10]). The $\varphi(k)$ for these curves are small enough to satisfy $\mathsf{C1}$, and big enough for small α values to satisfy $\mathsf{C2}$. Therefore, from Theorem 9, we expect that it will be relatively easy to reduce PI to EI for these curves. The tighter upper bound on the bit operations on the last row, based on Theorem 3, supports the observation.

The last example CP2 is constructed by Cocks-Pinch method for big $\varphi(k)$ and $\alpha = 1$. The curve does not satisfy the condition $\mathsf{C1}$ and thus we cannot use Theorem 9. However the tighter upper bound on the bit ops on the last row, based on Theorem 3, shows that it will be easy to reduce PI to EI for the curve.

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