Security in $O(2^n)$ for the Xor of Two Random Permutations
– Proof with the standard H technique–

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Abstract

Xoring two permutations is a very simple way to construct pseudorandom functions from pseudorandom permutations. In [14], it is proved that we have security against CPA-2 attacks when $m \ll O(2^n)$, where m is the number of queries and n is the number of bits of the inputs and outputs of the bijections. In this paper, we will obtain similar (but slightly different) results by using the "standard H technique" instead of the "H_σ technique". It will be interesting to compare the two techniques, their similarities and the differences between the proofs and the results.

Key words: Pseudorandom functions, pseudorandom permutations, security beyond the birthday bound, Luby-Rackoff backwards.

1 Introduction

The problem of converting pseudorandom permutations (PRP) into pseudorandom functions (PRF) named "Luby-Rackoff backwards" was first considered in [3]. This problem is obvious if we are interested in an asymptotical polynomial versus non polynomial security model (since a PRP is then a PRF), but not if we are interested in achieving more optimal and concrete security bounds. More precisely, the loss of security when regarding a PRP as a PRF comes from the "birthday attack" which can distinguish a random permutation from a random function of n bits to n bits, in $2^{\frac{n}{2}}$ operations and $2^{\frac{n}{2}}$ queries. Therefore different ways to build PRF from PRP with a security above $2^{\frac{n}{2}}$ and by performing very few computations have been suggested (see [2, 3, 4, 6]). One of the simplest way is simply to Xor k independent pseudorandom permutations, for example with $k = 2$. In [6] (Theorem 2 p.474), it has been proved, with a simple proof, that the Xor of k independent PRP gives a PRF with security at least in $O(2^{\frac{k}{k+1}n})$. (For $k = 2$ this gives $O(2^{\frac{2}{3}n})$). In [2], a much more complex strategy (based on Azuma inequality and Chernoff bounds) is presented. It is claimed that with this strategy we may prove that the Xor of two PRP gives a PRF with security at least in $O(2^n/n^{\frac{2}{3}})$ and at most in $O(2^n)$, which is much better than the birthday bound in $O(2^{\frac{n}{2}})$. However the authors of [2] present a very general framework of proof and they do not give every details for this result. For example, page 9 they wrote "we give only a very brief summary of how this works", and page 10 they introduce O functions that are not easy to express explicitly. In this paper we will use a completely different proof strategy, based on the "standard H technique" (see Section 3 below), simple counting arguments and induction. This paper is self contained. It is nevertheless interesting to compare this paper with [14] where similar (but slightly different results, as we will explain) are obtained by using the H_σ technique instead of the standard H technique.

Related Problems. In [9] the best know attacks on the Xor of k random permutations are studied in various scenarios. For $k = 2$ the bound obtained are near our security bounds. In [7] attacks on the Xor of two **public** permutations are studied (i.e. indifferentiability instead of indistinguishibility).

Part I
From the Xor of Two Permutations to the h_i values

2 Notation and Aim of this paper

In all this paper we will denote $I_n = \{0,1\}^n$. F_n will be the set of all applications from I_n to I_n, and B_n will be the set of all permutations from I_n to I_n. Therefore $|I_n| = 2^n$, $|F_n| = 2^{n \cdot 2^n}$ and $|B_n| = (2^n)!$. $x \in_R A$ means that x is randomly chosen in A with a uniform distribution.

The aim of this paper is to prove the theorem below, with an explicit O function (to be determined).

Theorem 1 *For all CPA-2 (Adaptive chosen plaintext attack) ϕ on a function G of F_n with m chosen plaintext, we have:* $\mathrm{Adv}_\phi^{\mathrm{PRF}} \leq O(\frac{m}{2^n})$ *where* $\mathrm{Adv}_\phi^{\mathrm{PRF}}$ *denotes the advantage to distinguish $f \oplus g$, with $f, g \in_R B_n$ from $h \in_R F_n$.*

This theorem says that there is no way (with an adaptive chosen plaintext attack) to distinguish with a good probability $f \oplus g$ when $f, g \in_R B_n$ from $h \in_R F_n$ when $m \ll 2^n$ (and this even if we have access to infinite computing power, as long as we have access to only m queries). Therefore, it implies that the number λ of computations to distinguish $f \oplus g$ with $f, g \in_R B_n$ from $h \in_R F_n$ satisfies: $\lambda \geq O(2^n)$. We say also that there is no generic CPA-2 attack with less than $O(2^n)$ computations for this problem, or that the security obtained is greater than or equal to $O(2^n)$. Since we know (for example from [2] or [9]) that there is an attack in $O(2^n)$, Theorem 1 also says that $O(2^n)$ is the exact security bound for this problem.

3 The general Proof Strategy (" standard H technique")

Let $a = (a_i, 1 \leq i \leq m)$ be m pairwise distinct values of I_n.
Let $b = (b_i, 1 \leq i \leq m)$ be m values of I_n (not necessarily distinct).
• We will denote by $H(a, b)$, or by $H(b)$ since we will see that $H(a, b)$ does not depend on a, the number of $(f, g) \in B_n^2$ such that: $\forall i,\ 1 \leq i \leq m,\ (f \oplus g)(a_i) = b_i$. Often we will denote $H(b)$ by

H_m for simplicity (but $H(b)$ depends on b).

Introducing h instead of H

• We will denote by $h(b)$, or simply by h_m for simplicity (but h depends on b) the number of sequences x_i, $1 \leq i \leq m$, $x_i \in I_n$, such that:

1. The x_i are pairwise distinct, $1 \leq i \leq m$.
2. The $x_i \oplus b_i$ are pairwise distinct, $1 \leq i \leq m$.

Theorem 2 *We have*

$$H(a, b) = h(b) \cdot \frac{|B_n|^2}{(2^n(2^n-1)\dots(2^n-m+1))^2}$$

(and therefore $H(a, b)$ does not depend on a, i.e. does not depend on the pairwise distinct values a_i, $1 \leq i \leq m$).

Proof. When the x_i are fixed, f and g are fixed on exactly m pairwise distinct points by $\forall i$, $1 \leq i \leq m$, $f(a_i) = x_i$ and $g(a_i) = b_i \oplus x_i$. □

Theorem 3 *h_m is the number of $(P_1, P_2, \dots, P_m, Q_1, \dots, Q_m) \in I_n^{2m}$ such that*

1. *The P_i are pairwise distinct (i.e. $i \neq j \Rightarrow P_i \neq P_j$).*
2. *The Q_i are pairwise distinct (i.e. $i \neq j \Rightarrow Q_i \neq Q_j$).*
3. *$\forall i$, $1 \leq i \leq m$, $P_i \oplus Q_i = b_i$.*

Proof. Since Q_i is fixed when P_i is fixed, Theorem 3 is obvious from the definition of h_m, i.e. just take $P_i = x_i$ and $Q_i = x_i \oplus b_i$. □

Computation of $E(h) = \tilde{h}_m$

We will denote by $\tilde{h_m}$ the average of h_m when $b \in_R I_n^m$.

Theorem 4

$$\tilde{h_m} = \frac{(2^n(2^n-1)\dots(2^n-m+1))^2}{2^{nm}}$$

Proof. Let $b = (b_1, \dots, b_n)$, and $x = (x_1, \dots, x_n)$. For $x \in I_n^m$, let

$$\delta_x = 1 \Leftrightarrow \begin{cases} \text{The } x_i \text{ are pairwise distinct,} & 1 \leq i \leq m \\ \text{The } x_i \oplus b_i \text{ are pairwise distinct,} & 1 \leq i \leq m \end{cases}$$

and $\delta_x = 0 \Leftrightarrow \delta_x \neq 1$. Let J_n^m be the set of all sequences x_i such that all the x_i are pairwise distinct, $1 \leq i \leq m$. Then $|J_n^m| = 2^n(2^n-1)\dots(2^n-m+1)$ and $N = \sum_{x \in J_n^m} \delta_x$. So we have $E(h) = \sum_{x \in J_n^m} E(\delta_x)$. For $x \in J_n^m$,

$$E(\delta_x) = Pr_{b \in_R I_n^m}(\text{All the } x_i \oplus b_i \text{ are pairwise distinct}) = \frac{2^n(2^n-1)\dots(2^n-m+1)}{2^{nm}}$$

Therefore

$$E(h) = |J_n^m| \cdot \frac{2^n(2^n-1)\dots(2^n-m+1)}{2^{nm}} = \frac{(2^n(2^n-1)\dots(2^n-m+1))^2}{2^{nm}}$$

as expected. □

We will denote by Adv_m the best Advantage that we can get in CPA-2 with m queries when we try to distinguish $f \oplus g$, with $f, g \in_R B_n$ from $h \in_R F_n$. As we will see now, there is a very deep connection between Adv_m and the coefficients h_m. More precisely:

Theorem 5 *An exact formula for Adv.*
Let $F = \{(b_1, \dots, b_m) \in I_n^m$ *such that:* $h(b_1, \dots, b_m) \geq \tilde{h_m}\}$. *Then:*

$$\begin{array}{rcl} Adv_m & = & \frac{1}{2.[2^n(2^n-1)\dots(2^n-m+1)]^2} \sum_{b_1,\dots,b_m \in I_n} |h_m - \tilde{h_m}| \\ & = & \frac{1}{2.2^{nm}} \sum_{b_1,\dots,b_m \in I_n} |\frac{h_m}{\tilde{h_m}} - 1| \\ & = & \frac{1}{2^{nm}} \sum_{b_1,\dots,b_m \in F} (\frac{h_m}{\tilde{h_m}} - 1) \\ & = & \frac{1}{2^{nm}} \sum_{b_1,\dots,b_m \in I_n \setminus F} (1 - \frac{h_m}{\tilde{h_m}}) \end{array}$$

Proof. We have seen above that the choice of the pairwise distinct values a_i has no influence. Therefore, here the best CPA-2 is this one denoted by ϕ (ϕ is also the best KPA attack): choose m pairwise distinct values $a_1, \dots, a_m$,
$\forall i$, $1 \leq i \leq m$, ask for $f(a_i) = b_i$ and now

- If $H(b_1, \dots, b_m) \geq \tilde{H_m}$ output 1.
- If $H(b_1, \dots, b_m) < \tilde{H_m}$ output 0.

Here $\tilde{H_m}$ denotes the average of $H(b_1, \dots, b_m)$ when $(b_1, \dots, b_m) \in I_n^m$, i.e. $\tilde{H_m} = \frac{|B_n|^2}{2^{nm}}$.
Let p_1^* be the probability that ϕ outputs 1 when $f \in_R F_n$. p_1^* is also the probability that $H(b_1, \dots, b_m) \geq \tilde{H_m}$ when $(b_1, \dots, b_m) \in_R I_n^m$. Therefore $p_1^* = \frac{|F_n|}{2^{nm}}$. Let p_1 be the probability that ϕ outputs 1 when $f = g \oplus h$ with $(g, h) \in_R B_n^2$. Then: $Adv = Adv(\phi) = |p_1 - p_1^*|$. $p_1 = \sum_{(b_1,\dots,b_m) \in F} \frac{H(b_1,\dots,b_m)}{|B_n|^2}$. We know that $H_m = \frac{h_m |B_n|^2}{[2^n(2^n-1)\dots(2^n-m+1)]^2}$ (cf (3.2)). Therefore,

$$p_1 - p_1^* = \sum_{b_1,\dots b_m \in F} (\frac{h_m(b_1, \dots, b_m)}{[2^n(2^n-1)\dots(2^n-m+1)]^2} - \frac{1}{2^{nm}})$$

$$p_1 - p_1^* = \sum_{b_1,\dots b_m \in F} (\frac{h_m - \tilde{h_m}}{[2^n(2^n-1)\dots(2^n-m+1)]^2})$$

Therefore from Theorem 4:

$$Adv_m = p_1 - p_1^* = \frac{1}{2^{nm}} \sum_{b_1,\dots b_m \in F} (\frac{h_m}{\tilde{h_m}} - 1)$$

Now from $\frac{1}{2^{nm}} \sum_{b_1,\dots b_m \in F} h_m = \frac{\tilde{h_m}}{2}$, we obtain the other equality of Theorem 5. □

As a direct corollary of this Theorem 5 we get:

Theorem 6 *("Standard H technique theorem")*
Let α and β be real numbers, $\alpha > 0$ and $\beta > 0$. Let $\mathcal{E}$ be a subset of I_n^m such that $|\mathcal{E}| \geq (1-\beta) \cdot 2^{nm}$. If
1. For all sequences b_i, $1 \leq i \leq m$ of $\mathcal{E}$ we have $h_m(b) \geq \tilde{h_m}(1-\alpha)$.
Then
2. $Adv_m \leq \alpha + \beta$.

Proof From Theorem 5

$$Adv_m = \frac{1}{2^{nm}} \sum_{b_1,\ldots,b_m \in I_n \setminus F} (1 - \frac{h_m}{\tilde{h_m}})$$

$I_n \setminus F \subset (I_n \setminus E) \cup (E \setminus F)$, so

$$Adv_m \leq \frac{1}{2^{nm}}(\beta \cdot 2^{nm} + \alpha \cdot 2^{nm}) \leq \alpha + \beta$$

as claimed. □

Theorem 4 and theorem 5 show the proof strategy that we will follow in this paper: we will study and evaluate the values h_m, and try to show that "most of the time" $h_m \gtrsim \tilde{h_m}$ where $a \gtrsim b$ means $a \geq b$ or $a \simeq b$.

Remarks

1. In [14] a slightly different strategy is used, by studying $\sigma(h_m)$, the standard deviation on the h_m values.
2. Theorem 4 and theorem 5 are specific of this problem. However Theorem 6 is a very classical "coefficient H theorem" and can also be proved independently of Theorem 5 with more general conditions (see for example [14]).
3. The probability to distinguish is $\frac{1}{2} + Adv \cdot \frac{1}{2}$, as usual.

Theorem 7 *($H_{worse\,case}$ theorem)*
Let $\alpha \geq 0$. If
1. For all sequences b_i, $1 \leq b_i \leq m$, of I_n^m we have $h_m(b) \geq \tilde{h_m}(1-\alpha)$
Then
2. $Adv_m \leq 2\alpha$.

Proof. This follows immediately from Theorem 6 with $\beta = 0$. □

Part II

Analysis of the h_i values

4 Orange equations, security in $O(\frac{m^3}{2^{2n}})$

Let $\epsilon \geq 0$. From Theorem 7, (i.e. coefficients H technique) we know that if for all $b_1, \ldots, b_\alpha \in I_n$ we have $h_\alpha(b_1, b_2, \ldots, b_\alpha) \geq \tilde{h}_\alpha(1-\epsilon)$, then: $Avd^{PRF} \leq 2\epsilon$ (where Avd^{PRF} is as before the advantage to distinguish $f \oplus g$ with $f, g \in_R B_n$ from $h \in_R F_n$ with a CPA-2 attack). Therefore we want to study $\frac{h_\alpha}{\tilde{h}_\alpha}$.

$$\tilde{h}_{\alpha+1} = \tilde{h}_\alpha \frac{(2^n - \alpha)^2}{2^n}$$

$$\tilde{h}_{\alpha+1} = \tilde{h}_\alpha (2^n - 2\alpha + \frac{\alpha^2}{2^n}) \quad (4.1)$$

Now we want to evaluate $h_{\alpha+1}$ from h_α and compare the result with (4.1). In $h_{\alpha+1}$, we have:

1. The previous conditions on h_α.
2. Two new variables $P_{\alpha+1}$ and $Q_{\alpha+1}$.
3. One more equation $P_{\alpha+1} \oplus Q_{\alpha+1} = b_{\alpha+1}$. We call X this equation.
4. 2α new non equalities: $P_{\alpha+1} \neq P_i$, $\forall i$, $1 \leq i \leq \alpha$, and $Q_{\alpha+1} \neq Q_i$, $\forall i$, $1 \leq i \leq \alpha$. We will denote by $\beta_1, \beta_2, \ldots, \beta_{2\alpha}$, the 2α equalities that should not be satisfied here (for example $P_{\alpha+1} = P_1$).

Let $B_i = \{(P_1, P_2, \ldots, P_{\alpha+1}, Q_1, Q_2, \ldots, Q_{\alpha+1}) \in I_n^{2\alpha+2}$ that satisfy the conditions on h_α, the equation X, and the equalitites $\beta_i\}$.

Remark. We use here the notations β_i and β_j as in sections 6 and 7 (for other values) in order to illustrate the deep similarities between our analysis of h_α and our previous analysis of λ_α.

We have

$$h_{\alpha+1} = 2^n h_\alpha - |\cup_{i=1}^{2\alpha} B_i|$$

Moreover, since 3 equalities β_i are necessarily not compatible with the conditions on h_α, we have:

$$h_{\alpha+1} = 2^n h_\alpha - \sum_{i=1}^{2\alpha} |B_i| + \sum_{i<j} |B_i \cap B_j| \quad (4.2)$$

- $X + 1$ **equations**.

We have $|B_i| = h_\alpha$ (since X and β_i will fix $P_{\alpha+1}$ and $Q_{\alpha+1}$), and $-\sum_{i=1}^{2\alpha} |B_i| = -2\alpha h_\alpha$.

- $X + 2$ **equations**.

X is : $P_{\alpha+1} \oplus Q_{\alpha+1} = b_{\alpha+1}$. To be compatible with the conditions on h_α the 2 new equalities should be of the type: $P_{\alpha+1} = P_i$ and $Q_{\alpha+1} = Q_j$, with $i \leq \alpha$ and $j \leq \alpha$. Therefore $P_i \oplus Q_j = b_{\alpha+1}$ We will denote by $h'_\alpha(b_1, \ldots, b_\alpha)(i,j)$ or simply by $h'_\alpha(i,j)$ for simplicity, the number of $(P_1, \ldots P_\alpha, Q_1, \ldots, Q_\alpha) \in I_n^{2\alpha}$ such that

1. We have the conditions on h_α (i.e. the P_i are pairwise distinct, the Q_i are pairwise distinct, and $\forall i$, $1 \leq i \leq \alpha$, $p_i \oplus Q_i = b_i$).
2. $P_i \oplus Q_j = b_{\alpha+1}$ (this is one more affine equality).

Then:

$$\sum_{1 \leq i < j \leq 2\alpha} |B_i \cap B_j| = \sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} h'_\alpha(i,j)$$

and from (4.2), we get:

$$h_{\alpha+1} = (2^n - 2\alpha)h_\alpha + \sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha} h'_\alpha(i,j) \quad (4.3)$$

Let $M = \{i,\, 1 \le i \le \alpha,\, b_i = b_{\alpha+1}\}$. Let $Y(i,j)$ be the equation added in h'_α (i.e. $Y(i,j)$ is $P_i \oplus Q_j = b_{\alpha+1}$). If $i \in M$, then $h'_\alpha(i,i) = h_\alpha$, and if $i \notin M$, then $h'_\alpha(i,i) = 0$. (This is because $Y(i,i)$ is $P_i \oplus Q_i = b_{\alpha+1}$ and we have $P_i \oplus Q_i = b_i$). Moreover, if $i \in M$, then $\forall j,\, 1 \le j \le \alpha,\ j \ne i$, we have $h'_\alpha(i,j) = 0$, and $h'_\alpha(j,i) = 0$ (*).
(Proof: This is because $Y(i,j)$ is $P_i \oplus Q_j = b_{\alpha+1}$. Moreover $b_{\alpha+1} = b_i$, since $i \in M$, and $P_i \oplus Q_i = b_i$. So we would have $Q_i = Q_j$. Similarly, $Y(j,i)$ is $P_j \oplus Q_i = b_{\alpha+1} = b_i$ and from $P_i \oplus Q_i = b_i$, we would have $P_j = P_i$). Therefore, from these results and (4.3), we have obtained:

Theorem 8 *("Orange equations")*
With $M = \{i,\, 1 \le i \le \alpha,\ b_i = b_{\alpha+1}\}$, we have:

$$h_{\alpha+1} = (2^n - 2\alpha + |M|)h_\alpha + \sum_{i \notin M}\ \sum_{j \notin M, j \ne i} h'_\alpha(i,j)$$

Theorem 9 *("First stabilisation formula")*

$$\sum_{b_{\alpha+1} \in I_n} h_{\alpha+1} = (2^n - \alpha)^2 h_\alpha$$

Proof. This comes immediately from the fact that in $h_{\alpha+1}$ we have $P_{\alpha+1}$ and $Q_{\alpha+1}$ as new variables, with $P_{\alpha+1} \notin \{P_1, \ldots, P_\alpha\}$ and $Q_{\alpha+1} \notin \{Q_1, \ldots, Q_\alpha\}$. □

Theorem 10 *("Second stabilisation formula")*
$\forall i, j,\ i \ne j,\ \sum_{b_{\alpha+1} \notin \{b_1,\ldots,b_\alpha\}} h_\alpha(i,j) = h_\alpha$.

Proof. Theorem 10 follows immediately from $(*)$ above (just as before Theorem 8). □

First Approximation: Security in $O(\frac{m^3}{2^{2n}})$

From (4.2) we have: $h_{\alpha+1} \ge (2^n - 2\alpha)h_\alpha$. Then from (4.1)

$$\frac{h_{\alpha+1}}{\tilde{h}_{\alpha+1}} = \frac{h_\alpha}{\tilde{h}_\alpha}\frac{(2^n - 2\alpha)}{2^n - 2\alpha + \frac{\alpha^2}{2^n}}$$

$$\frac{h_{\alpha+1}}{\tilde{h}_{\alpha+1}} = \frac{h_\alpha}{\tilde{h}_\alpha}\Big(1 - \frac{\frac{\alpha^2}{2^n}}{2^n - 2\alpha + \frac{\alpha^2}{2^n}}\Big)$$

Now since $h_1 = 2^n$ and $\tilde{h}_1 = 2^n$,

$$h_\alpha \ge \tilde{h}_\alpha\Big(1 - \frac{\alpha^2}{2^{2n} - 2\alpha \cdot 2^n + \alpha^2}\Big)^\alpha$$

First step result:

$$\underline{h_\alpha \geq \tilde{h}_\alpha \Big(1 - \frac{\alpha^3}{2^{2n} - 2\alpha \cdot 2^n + \alpha^2}\Big)} \quad (4.4)$$

Therefore (from Theorem 7):

Theorem 11

$$Adv_\alpha^{PRF} \leq \frac{2\alpha^3}{2^{2n} - 2\alpha \cdot 2^n + \alpha^2} \quad (4.5)$$

(and the probability to distinguish is $\frac{1}{2} \cdot Adv_m$ as usual).

We have proved security in $O(\frac{\alpha^3}{2^{2n}})$.
Remark. the fact that we have so far proved security when $\alpha \ll 2^{\frac{2n}{3}}$ is not very impressive compared with we have previously obtained with the H_σ technique (i.e. with the λ_α values). However, the fact that Adv^{PRF} decreases in 2^{2n} when α is fixed is interesting.

5 Second Approximation: Security in $O(\frac{m^4}{2^{3n}} + \frac{m^2}{2^{2n}})$

Lemma 1 *(Simple Approximation of h'_α)*
If $i \notin M$, $j \notin M$, and $i \neq j$, we always have:

$$\frac{h_\alpha}{2^n}(1 - \frac{4\alpha}{2^n}) \leq h'_\alpha(i,j) \leq \frac{h_\alpha}{2^n(1 - \frac{4\alpha}{2^n})}$$

Proof. Without loss of generality, just by changing the order of the indices, we can assume that $i = \alpha - 1$ and $j = \alpha$, i.e. that the new equation Y is: $P_{\alpha-1} \oplus Q_\alpha = b_{\alpha+1}$. We will now evaluate h_α and h'_α from $h_{\alpha-2}$. When we go from $h_{\alpha-2}$ to h_α, we have 4 new variables $P_\alpha, Q_\alpha, P_{\alpha-1}, Q_{\alpha-1}$ such that $P_\alpha \oplus Q_\alpha = b_\alpha$, $P_{\alpha-1} \oplus Q_{\alpha-1} = b_{\alpha-1}$,
$\forall i,\ 1 \leq i \leq \alpha - 2,\ P_{\alpha-1} \neq P_i$
$\forall i,\ 1 \leq i \leq \alpha - 2,\ Q_{\alpha-1} \neq Q_i$
$\forall i,\ 1 \leq i \leq \alpha - 1,\ P_\alpha \neq P_i$
$\forall i,\ 1 \leq i \leq \alpha - 1,\ Q_\alpha \neq Q_i$
For $P_{\alpha-1}$, we have between $2^n - (\alpha - 2)$ and $2^n - 2(\alpha - 2)$ possibilities. Now, when $P_{\alpha-1}$ is fixed, for P_α, we have between $2^n - (\alpha - 1)$ and $2^n - 2(\alpha - 1)$ possibilities.
Therefore:

$$(2^n - 2(\alpha-1))(2^n - 2(\alpha-2))h_{\alpha-2} \leq h_\alpha \leq (2^n - (\alpha-1))(2^n - (\alpha-2))h_{\alpha-2}$$

So

$$(2^{2n} - 4\alpha \cdot 2^n)h_{\alpha-2} \leq h_\alpha \leq 2^{2n}h_{\alpha-2} \quad (5.1)$$

Similarly, when we go from $h_{\alpha-2}$ to h'_α, we have 4 new variables $P_\alpha, Q_\alpha, P_{\alpha-1}, Q_{\alpha-1}$ such that: $P_\alpha \oplus Q_\alpha = b_\alpha$, $P_{\alpha-1} \oplus Q_{\alpha-1} = b_{\alpha-1}$, $P_{\alpha-1} \oplus Q_\alpha = b_{\alpha+1}$, and $\forall i, 1 \leq i \leq \alpha - 2 :\ P_{\alpha-1} \neq P_i$, $Q_{\alpha-1} \neq Q_i$, $P_\alpha \neq P_i$, and $Q_\alpha \neq Q_i$. (we necessarily have $P_\alpha \neq P_{\alpha-1}$ and $Q_\alpha \neq Q_{\alpha-1}$ since $P_\alpha \oplus P_{\alpha-1} = b_\alpha \oplus b_{\alpha+1}$ and $Q_\alpha \oplus Q_{\alpha-1} = b_\alpha \oplus b_{\alpha+1}$ and these values are $\neq 0$ since $i \notin M$ and $j \notin M$).
Therefore, for P_α we have between $2^n - (\alpha - 2)$ and $2^n - 4(\alpha - 2)$ possibilities.

$$(2^n - 4(\alpha-2))h_{\alpha-2} \leq h_\alpha \leq (2^n - (\alpha-2))h_{\alpha-2} \quad (5.2)$$

From (5.1) and 15.2), we obtain lemma 1, as claimed. □

Lemma 2 *(A simple way to get ride of $|M|$)*
If $\alpha \leq \frac{2^n}{6}$, then there exists a value h'_α such that:

$$h_{\alpha+1} \geq (2^n - 2\alpha)h_\alpha + \alpha(\alpha - 1)h'_\alpha$$

(The condition $\alpha \leq \frac{2^n}{6}$ could be improved with further analysis).

Proof of Lemma 2. From Theorem 8, we have:

$$h_{\alpha+1} \geq (2^n - 2\alpha + |M|)h_\alpha + [(\alpha - |M|)(\alpha - |M|) - \alpha]h'_\alpha$$

So a sufficient condition for $h_{\alpha+1} \geq (2^n-2\alpha)h_\alpha+[\alpha^2-\alpha]h'_\alpha$ is to have $|M|h_\alpha \geq (2\alpha|M|-|M|^2)h'_\alpha$ $(*)$. From Lemma 1, we have $h'_\alpha \leq \frac{h_\alpha}{2^n(1-\frac{4\alpha}{2^n})}$. Therefore a sufficient condition for $(*)$ is to have: $2^n - 4\alpha \geq 2\alpha - |M|$ i.e. $\alpha \leq \frac{2^n+|M|}{6}$. This condition is satisfied if $\alpha \leq \frac{2^n}{6}$ as claimed. □

Security in $O(\frac{m^2}{2^{2n}} + \frac{m^4}{2^{3n}})$
From Theorem 8 and Lemma 1, we have:

$$h_{\alpha+1} \geq (2^n - 2\alpha + |M|)h_\alpha + [(\alpha - |M|)(\alpha - |M|) - \alpha]\frac{h_\alpha}{2^n}(1 - \frac{4\alpha}{2^n})$$

$$h_{\alpha+1} \geq (2^n - 2\alpha + |M| + \frac{\alpha^2 - 2|M|\alpha + |M|^2 - \alpha}{2^n})h_\alpha - \frac{4\alpha^3}{2^{2n}}h_\alpha$$

We have

$$|M| + \frac{-2|M|\alpha + |M|^2}{2^n} \geq 0 \Leftrightarrow \alpha \leq \frac{2^n + |M|}{2}$$

We will assume that $\alpha \leq \frac{2^n}{2}$ (this condition could be improved with further analysis). Then

$$h_{\alpha+1} \geq (2^n - 2\alpha + \frac{\alpha^2 - \alpha}{2^n} - \frac{4\alpha^3}{2^{2n}})h_\alpha$$

Remark. We can also get this directly from Lemma 2 and Lemma 1 but with $\alpha \leq \frac{2^n}{6}$ instead of $\alpha \leq \frac{2^n}{2}$

$$\frac{h_{\alpha+1}}{\tilde{h}_{\alpha+1}} \geq \frac{2^n - 2\alpha + \frac{\alpha^2-\alpha}{2^n} - \frac{4\alpha^3}{2^{2n}}}{2^n - 2\alpha + \frac{\alpha^2}{2^n}} \frac{h_\alpha}{\tilde{h}_\alpha}$$

$$\frac{h_{\alpha+1}}{\tilde{h}_{\alpha+1}} \geq (1 - \frac{\alpha}{(2^n - \alpha)^2} - \frac{4\alpha^3}{2^n(2^n - \alpha)^2})\frac{h_\alpha}{\tilde{h_\alpha}}$$

Therefore

$$h_\alpha \geq (1 - \frac{\alpha}{(2^n - \alpha)^2} - \frac{4\alpha^3}{2^n(2^n - \alpha)^2})^\alpha \tilde{h_\alpha}$$

Second Step result:

$$\underline{h_\alpha \geq (1 - \frac{\alpha^2}{(2^n - \alpha)^2} - \frac{4\alpha^4}{2^n(2^n - \alpha)^2})\tilde{h_\alpha}} \quad (5.3)$$

Now from (5.3) we have for all CPA-2 attacks with m queries:

$$Adv^{PRF} \leq \frac{m^2}{(2^n-m)^2} + \frac{4m^4}{2^n(2^n-m)^2} \quad (5.4)$$

(here we do not need to say "when $m \leq \frac{2^n}{2}$" since for larger m, this value is larger than 1).
Remark (5.4) gives security in $O(\frac{m^2}{2^{2n}} + \frac{m^4}{2^{3n}})$ with m queries as wanted in this section. In (5.4), we have two terms. The first term in $\frac{m^2}{2^{2n}}$ is consistent with the fact that when $m = 2$ for example we know that we must have a term in 2^{2n} (see Appendix A: the exact value for the advantage with 2 queries is exactly $\frac{1}{2^n(2^n-1)}$). The second term gives security only when $m \ll 2^{\frac{3n}{4}}$ and we will see in the next sections how to improve this term.

6 An induction formula on h'_α ("First purple equations")

Theorem 12 *("first purple equations"):*

$$h'_{\alpha+1} = h_\alpha + (-2\alpha + 2 + \nu + \xi + \mu)h'_\alpha + \sum_{i,j \in \mathcal{N}} h''_\alpha$$

with

- $\mathcal{N} = \{(i,j),\ i \neq j,\ 2 \leq i \leq \alpha,\ 2 \leq j \leq \alpha,$ *such that none of the 4 equalitites* $\mathcal{S}$ *are satified*$\}$
  The $\mathcal{S}$ *equalities are*
  . $b_{\alpha+1} = b_j$.
  . $b_{\alpha+2} = b_j$.
  . $b_{\alpha+1} = b_i$.
  . $b_1 \oplus b_{\alpha+1} \oplus b_{\alpha+2} \oplus b_i = 0$.
- μ *is the number of* i, $2 \leq i \leq \alpha,\ b_{\alpha+1} = b_i$.
- ξ *is the number of* i, $2 \leq i \leq \alpha,\ b_{\alpha+2} = b_i$.
- ν *is the number of* i, $2 \leq i \leq \alpha,\ b_1 \oplus b_{\alpha+1} \oplus b_{\alpha+2} \oplus b_i = 0$.

Proof. We have that $h'_{\alpha+1}$ is the number of $P_1, \ldots, P_{\alpha+1}, Q_1, \ldots, Q_{\alpha+1}$ such that:

1. The P_i values are pairwise distinct.
2. The Q_i values are pairwise distinct.
3. We have all the equalities of $h_{\alpha+1}$: $P_1 \oplus Q_1 = b_1, \ldots, P_{\alpha+1} \oplus Q_{\alpha+1} = b_{\alpha+1}$.
4. We have the extra equation X: $Q_{\alpha+1} \oplus P_1 = b_{\alpha+1}$, with $b_{\alpha+1} \neq b_{\alpha+2}$ (in order to have $P_{\alpha+1} \neq P_1$) and $b_1 \neq b_{\alpha+2}$ (in order to have $Q_{\alpha+1} \neq Q_1$).
   $\forall i,\ 1 \leq i \leq 2\alpha$, we define by B'_i the set of all $(P_1, \ldots, P_{\alpha+1}, Q_1, \ldots, Q_{\alpha+1})$ that satisfy:
   (a) All the conditions h_α.

(b) The equation X.

(c) The equation $P_{\alpha+1} \oplus Q_{\alpha+1} = b_{\alpha+1}$.

(d) The equations β_i (these equations have been defined in section 4, for example β_1 is $P_{\alpha+1} = P_1$, β_2 is $P_{\alpha+1} = P_2$, ..., β_α is $P_{\alpha+1} = P_\alpha$, $\beta_{\alpha+1}$ is $Q_{\alpha+1} = Q_1$, ..., $Q_{\alpha+1} = Q_\alpha$).

In $h'_{\alpha+1}$ we have 2 new variables $P_{\alpha+1}$ and $Q_{\alpha+1}$. These variables are fixed by:

$$\begin{cases} P_{\alpha+1} \oplus Q_{\alpha+1} = b_{\alpha+1} \\ X : Q_{\alpha+1} \oplus P_1 = b_{\alpha+2} \end{cases}$$

We have:

$$h'_{\alpha+1} = h_\alpha - |\bigcup_{\substack{i=2 \\ i \neq \alpha+1}}^{2\alpha} B'_i|$$

(here $i = 1$ is excluded since $P_{\alpha+1} \neq P_1$ and similarly $i = \alpha+1$ is excluded since $Q_{\alpha+1} \neq Q_1$).

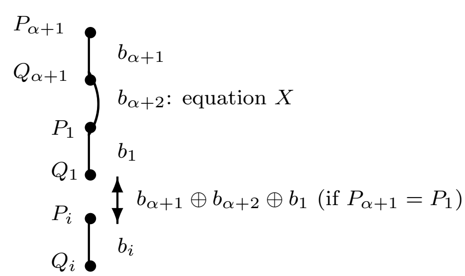

Figure 1: "6-point figure" for X +(1 equation $P_{\alpha+1} = P_i$).

Since 3 equations β_i are not compatible with the conditions h_α, we have:

$$h'_{\alpha+1} = h_\alpha - \sum_{\substack{i=2 \\ i \neq \alpha+1}}^{2\alpha} |B'_i| + \sum_{i=2}^{\alpha} \sum_{j=\alpha+2}^{2\alpha} |B'_i \cap B'_j|$$

$X + 1$ **equations**
We want to evaluate $|B'_i|$. These values are denoted as a value h'_α (since we have one more equation than in h_α), except when $|B'_i| = 0$.
Case 1. If β_i is $P_{\alpha+1} = P_i$ ($2 \leq i \leq \alpha$) we will have $|B'_i| = 0$ if β_i generates a collision on Q_i, this means here the collision $Q_i = Q_1$ (see Figure 1).
$P_{\alpha+1} = P_i$ means $P_1 = P_i \oplus b_{\alpha+1} \oplus b_{\alpha+1}$.

$Q_i = Q_1$ means $b_i = b_1 \oplus b_{\alpha+1} \oplus b_{\alpha+2}$ (ν values i are like this).
Remark. $Q_i \neq Q_{\alpha+1}$ (unlike $Q_i \neq Q_1$) is not one of the conditions of B'_i. (we want to evaluate $h'_{\alpha+1}$ from solutions h_α and in h_α we do not have the variables $P_{\alpha+1}$ and $Q_{\alpha+1}$). Here $Q_i = Q_{\alpha+1}$ means $b_i = b_{\alpha+1}$, and we have μ values i like this but we do not need this here.
Case 2. If β_j is $Q_{\alpha+1} = Q_i$ ($2 \leq i \leq \alpha$, $j = \alpha + i$) we will have $|B'_i| = 0$ if β_j generates a collision $P_i = P_1$.
$Q_{\alpha+1} = Q_i$ means $P_1 = P_i \oplus b_i \oplus b_{\alpha+2}$
$P_i = P_1$ means $b_i = b_{\alpha+2}$ (ξ values i are like this).
Remark. $P_i \neq P_{\alpha+1}$ (unlike $P_i \neq P_1$) is not one of the conditions of B'_i. Here $P_i = P_{\alpha+1}$ means $b_i = b_{\alpha+1}$, and we have μ values i like this, but we do not need this here.
From Case 1 and Case 2, we have:

$$\sum_{\substack{i=2\\ i\neq\alpha+1}}^{2\alpha} |B'_i| = (2\alpha - 2 - \xi - \nu)h'_\alpha$$

(Remember: h'_α is generally not a constant, but, as we will see, all the values h'_α will have about the same value).
$X + 2$ **equations**
We want to evaluate here $|B'_i \cap B'_j$. These values are denoted as a value h''_α (since we have two more equations than in h_α), except when these two equations are not compatible with the conditions h_α (because they generate a collision $P_i = P_j$ or $Q_i = Q_j$, $i \neq j$) or when these two equations are not independent (and create a term in h'_α).

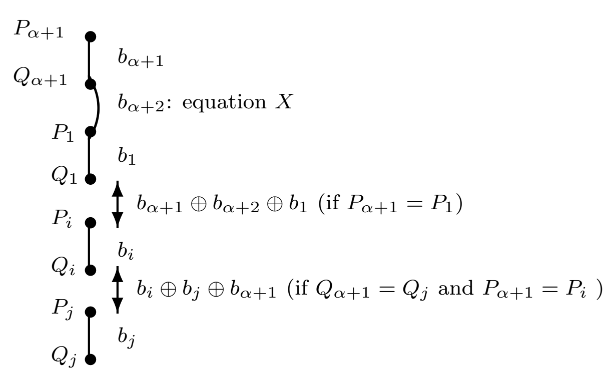

Figure 2: "8-point figure" for $X + 2$ equations

Case 1.
The two equations $P_{\alpha+1} = P_i$ and $Q_{\alpha+1} = Q_j$ are not independent and create a term in h'_α. This implies that $i = j$ (because if $i \neq j$ then Q_i and Q_j are not connected by $P_{\alpha+1} = P_i$) and then

$Q_i = Q_{\alpha+1}$ means $b_i = b_{\alpha+1}$ as seen in the section $X+1$ equation above. Therefore, we have here a term in $\mu\, h'_\alpha$. If $i = j$ and $b_i \neq b_{\alpha+1}$ then $Q_{\alpha+1} \neq Q_j$. Therefore we will now assume (in Case 2) that $i \neq j$.

Case 2.

$i \neq j$ and the two equations $P_{\alpha+1} = P_i$ and $Q_{\alpha+1} = Q_j$ generate a collision. We have seen in the section $X+1$ equation that from $P_{\alpha+1} = P_i$,

$Q_i = Q_1$ means $b_i = b_1 \oplus b_{\alpha+1} \oplus b_{\alpha+2}$ (ν values)

($Q_i = Q_{\alpha+1}$ means $b_i = b_{\alpha+1}$, μ values, this will occur below on $Q_i = Q_j$)

and from $Q_{\alpha+1} = Q_j$,

$P_j = P_1$ means $b_j = b_{\alpha+2}$ (ξ values)

($P_j = P_{\alpha+1}$ means $b_j = b_{\alpha+1}$, μ values, this will occur below on $P_i = P_j$)

Moreover: $P_i = P_j$ means here $b_j = b_{\alpha+1}$ (the same as $P_j = P_{\alpha+1}$ since we have $P_{\alpha+1} = P_i$)

$Q_i = Q_j$ means here $b_i = b_{\alpha+1}$ (the same as $Q_i = Q_{\alpha+1}$ since we have $Q_{\alpha+1} = Q_j$)

With the definitions of $\mathcal{N}$ and $\mathcal{S}$ given above and from Case 1 and Case 2, we see that

$$\sum_{i=2}^{\alpha} \sum_{j=\alpha+2}^{2\alpha} |B'_i \cap B'_j| = \mu h'_\alpha + \sum_{(i,j)\in\mathcal{N}} h''_\alpha$$

Therefore

$$h'_{\alpha+1} = h_\alpha + (-2\alpha + 2 + \nu + \xi + \mu)h'_\alpha + \sum_{i,j\in\mathcal{N}} h''_\alpha$$

as claimed. □

Example

Let compute h'_3 when $b_2 = b_3$ is the only exceptional relation on the b_i values. We have: $P_1 \oplus Q_1 = b_1$, $P_2 \oplus Q_2 = b_2$, $P_3 \oplus Q_3 = b_3 = b_2$ and $X : Q_3 \oplus P_1 = b_4$. Here $\alpha = 2$, $\mu = 1$, $\nu = \xi = 0$. Theorem 6 gives:

$$h'_3 = h_2 + (-4 + 2 + 1)h'_2 + \sum_{(i,j)\in\mathcal{N}} h''_2$$

Since $h''_2 = 0$ (because all the variables are already linked in h'_2) and since $h'_2 = 2^n$ and $h_2 = 2^n(2^n - 1)$ (because $b_1 \neq b_2$), this gives: $h'_3 = 2^n(2^n - 1) - 2^n = 2^n(2^n - 3)$. This value can also be verified directly.

It is also interesting to see how the proof of Theorem 6 proceeds on this example.

We have:

$\beta_1 : P_3 = P_1$ (impossible since $b_2 \neq b_3$)

$\beta_2 : P_3 = P_2$

$\beta_3 : Q_3 = Q_1$ (impossible since $b_1 \neq b_4$)

$\beta_4 : Q_3 = Q_2$

In h'_3 we have 2 new variables P_3 and Q_3. These variables are fixed from P_1; however all the solutions for h_2 do not necessary give a solution h'_3 since in h'_3 we must have $P_3 \neq P_2$ and $Q_3 \neq Q_2$. More precisely: $h'_3 = h_2 - |B'_2 \cup B'_4|$. This gives:

$$h'_3 = h_2 - |B'_2| - |B'_4| + |B'_2 \cap B'_4|$$

We have: $|B'_2| = h'_3 = 2^n$ because $P_3 = P_2$ do not generate $Q_2 = Q_1$ (since $b_1 \neq b_4$). Similarly $|B'_4| = h'_3 = 2^n$. Moreover $|B'_2 \cap B'_4| = 2^n$ since here $P_3 = P_2 \Leftrightarrow Q_3 = Q_2$ (since $b_2 = b_3$). Therefore $h'_3 = 2^n(2^n - 2) - 2^n = 2^n(2^n - 3)$.

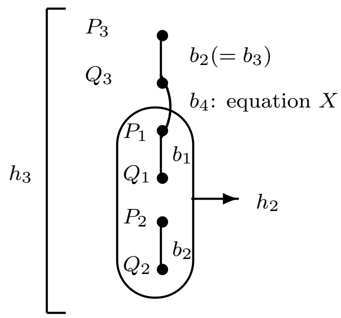

Figure 3: Computation of h_3' from h_2 on this example.

Theorem 13 *With the same notation of Theorem 6, we have:*

$$|\mathcal{N}| = (\alpha - 1)(\alpha - 2) - (\alpha - 2)(2\mu + \nu + \xi) + \mu(\mu - 1) + \mu\xi + \nu\xi + \xi(2\nu - \epsilon_1)$$

with

$$\epsilon_1 = 1 \Leftrightarrow b_1 = b_{\alpha+1}$$

and

$$\epsilon_1 = 1 \Leftrightarrow b_1 \neq b_{\alpha+1}$$

Proof. $\mathcal{N}$ is the set of all (i, j), $i \neq j$, $2 \leq i \leq \alpha$, $2 \leq j \leq \alpha$ such that we have these 4 equalities:
$b_i = b_{\alpha+1}$ (1) (μ values) $b_i = b_1 \oplus b_{\alpha+1} \oplus b_{\alpha+2}$ (2) (ν values)
$b_j = b_{\alpha+1}$ (3) (μ values)
$b_j = b_{\alpha+2}$ (4) (ξ values)
(1) and (2) are not compatible since $b_1 \neq b_{\alpha+2}$
(3) and (4) are not compatible since $b_{\alpha+1} \neq b_{\alpha+2}$
We have $(\alpha - 1)(\alpha - 2)$ values (i, j), $i \neq j$, $2 \leq i \leq \alpha$, $2 \leq j \leq \alpha$. We have $(\mu + \nu)(\alpha - 2)$ values (i, j) such that i satisfies (1) or (2) and $j \neq i$. We have $(\mu + \xi)(\alpha - 2)$ values (i, j) such that i satisfies (3) or (4) and $j \neq i$. Therefore

$$|\mathcal{N}| = (\alpha - 1)(\alpha - 2) - (\mu + \nu)(\alpha - 2) - (\mu + \xi)(\alpha - 2) + |\mathcal{P}|$$

where $\mathcal{P} = \{(i, j)$ such that i satisfies (1) or (2) and j satisfies (3) or (4)$\}$
(1) and (3) : we have $\mu(\mu - 2)$ values.
(2) and (4): we have $\mu.\xi$ values ($i \neq j$ since $b_{\alpha+1} \neq b_{\alpha+2}$).
(2) and (3) : we have $\nu\xi$ values. (because $b_1 \neq b_{\alpha+2}$).
(3) and (4): $\begin{cases} b_i = b_1 \oplus b_{\alpha+1} \oplus b_{\alpha+2} \ (\nu \text{ values}) \\ b_j = b_{\alpha+2} \ (\xi \text{ values}) \end{cases}$
We must have $i \neq j$. However $i = j$ gives $b_1 = b_{\alpha+1}$.
Case 1. $b_1 = b_{\alpha+1}$. Then for (2) and (4) we have $\nu\xi$ possibilities.

Case 2. $b_1 b_{\alpha+1}$. Then $\xi = \nu$ and for (2) and (4) we have $\xi(\nu-1)$ possibilities.
Therefore

$$|\mathcal{N}| = (\alpha-1)(\alpha-2) - (\alpha-2)(2\mu+\nu+\xi) + \mu(\mu-1) + \mu\xi + \nu\xi + \xi(2\nu-\epsilon_1)$$

as claimed. □

7 A simple variant of the schemes with only one permutation

Instead of $G = f_1 \oplus f_2, f_1, f_2 \in_R B_n$, we can study $G'(x) = f(x\|0) \oplus f(x\|1)$, with $f \in_R B_n$ and $x \in I_{n-1}$. This variant was already introduced in [2] and it is for this that in [2] p.9 the security in $\frac{m}{2^n} + O(n)(\frac{m}{2^n})^{3/2}$ is presented. In fact, from a theoretical point of view, this variant G' is very similar to G, and it is possible to prove that our analysis can be modified to obtain a similar proof of security for G'.

8 A simple property about the Xor of two permutations and a new conjecture

I have conjectured this property:

$$\forall f \in F_n, \text{ if } \bigoplus_{x \in I_n} f(x) = 0, \text{ then } \exists (g,h) \in B_n^2, \text{ such that } f = g \oplus h.$$

Just one day after this paper was put on eprint, J.F. Dillon pointed to us that in fact this was proved in 1952 in [5]. We thank him a lot for this information. (This property was proved again independently in 1979 in [15]).

A new conjecture. However I conjecture a stronger property. Conjecture:

$$\forall f \in F_n, \text{ if } \bigoplus_{x \in I_n} f(x) = 0, \text{ then the number } H \text{ of } (g,h) \in B_n^2,$$

$$\text{such that } f = g \oplus h \text{ satisfies } H \geq \frac{|B_n|^2}{2^{n2^n}}.$$

Variant: I also conjecture that this property is true in any group, not only with Xor.

Remark: in this paper, I have proved weaker results involving m equations with $m \ll O(2^n)$ instead of all the 2^n equations. These weaker results were sufficient for the cryptographic security wanted.

9 Conclusion

The results in this paper improve our understanding of the PRF-security of the Xor of two random permutations. More precisely in this paper we have proved that the Adaptive Chosen Plaintext security for this problem is in $O(2^n)$, and we have obtained an explicit O function. These results belong to the field of finding security proofs for cryptographic designs above the "birthday bound". (In [1, 8, 12], some results "above the birthday bound" on completely different cryptographic designs are also given). Since building PRF from PRP has many practical applications,we believe that these results are of real interest both from a theoretical point of view and a practical point of view. Our proofs need a few pages, so are a bit hard to read, but the results obtained are very easy to use and the mathematics used are elementary (essentially combinatorial and induction arguments). Moreover, we have proved (in Section 5) that this cryptographic problem of security is directly related to a very simple to describe and purely combinatorial problem. We have obtained this transformation by using the "H_σ technique", i.e. combining the "coefficient H technique" of [11, 12] and a specific computation of the standard deviation of H. (In a way, from a cryptographic point of view, this is maybe the most important result, and all the analysis after Section 5 can be seen as combinatorial mathematics and not cryptography anymore). It is also interesting to notice that in our proof with have proceeded with "necessary and sufficient" conditions, i.e. that the H_σ property that we proved is exactly equivalent to the cryptographic property that we wanted. Moreover, as we have seen, less strong results of security are quickly obtained.

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Appendices

A Examples of h_m with $m = 1, 2$ or 3

As examples, we present here the exact values for h_m and h'_m when $m = 1, 2$ or 3. The values that we will obtain are summarized in Table 1.

$(*)$ h'_3 denotes the condition h_3 plus $X : P_1 \oplus Q_3 = b_4$ with $b_1 \neq b_4$ and $b_3 \neq b_4$.
$\mathcal{S}$ denotes these 4 equalities: $b_2 = b_3,\ b_2 = b_1 \oplus b_3 \oplus b_4,\ b_2 = b_4$ and $b_1 = b_4$.
From h_m we get the exact value for Adv_m by using Theorem 5 (and Theorem 4 to get the value of $\tilde{h}_m$).

A.1 $m = 1$

By definition, h_1 is the number of $P_1, Q_1 \in I_n$ such that $P_1 \oplus Q_1 = b_1$. Therefore, $h_1 = 2^n$. Now from $Adv_1 = \frac{1}{2^{2n}} \sum_{b_1 \in I_n} |h_1 - \tilde{h_1}|$ and $\tilde{h_1} = 2^n$, we get: $Adv_1 = 0$.

Table 1: Summary of the results on h_m for $m = 1, 2, 3$

$h_1 = 2^n$	• If $b_1 \neq b_2$: $h_2 = 2^n(2^n - 2)$ • If $b_1 = b_2$: $h_2 = 2^n(2^n - 1)$	• If b_1, b_2, b_3 are pairwise distinct : $h_3 = 2^n(2^{2n} - 6.2^n + 10)$ • If $b_1 = b_2 \neq b_3$: $h_3 = 2^n(2^{2n} - 5.2^n + 6)$ • If $b_1 = b_2 = b_3$: $h_3 = 2^n(2^{2n} - 3.2^n + 2)$
$\downarrow$ $Adv_1 = 0$	$h'_2 = 2^n$	• If we have no equality in $\mathcal{S}$ $(*)$: $h'_3 = 2^n(2^n - 4)$ • If we have 1 equality in $\mathcal{S}$: $h'_3 = 2^n(2^n - 3)$ • If we have 2 equalities in $\mathcal{S}$: $h'_3 = 2^n(2^n - 2)$
	$\downarrow$ $Adv_2 = \frac{1}{2^n(2^n-1)}$ $Adv_2 \simeq \frac{1}{2^{2n}}$	$h''_3 = 2^n$
		$\downarrow$ If $n \geq 3$, $Adv_3 = \frac{1}{2^{2n}}\left(\frac{3.2^{2n}-12.2^n+4}{(2^n-1)(2^n-2)}\right)$ $Adv_3 \simeq \frac{3}{2^{2n}}$

A.2 $m = 2$

By definition, h_2 is the number of $P_1, P_2, Q_1, Q_2 \in I_n$ such that: $P_1 \neq P_2$, $Q_1 \neq Q_2$, $P_1 \oplus Q_1 = b_1$ and $P_2 \oplus Q_2 = b_2$. We have $Q_1 \neq Q_2 \Leftrightarrow P_1 \oplus P_2 \neq b_1 \oplus b_2$.
Case 1. $b_1 \neq b_2$. Then $h_2 = 2^n(2^n - 2)$ (because for P_1 we have 2^n possibilities, and then for P_2, we have $2^n - 2$ possibilities).
Case 2. $b_1 = b_2$. Then $h_2 = 2^n(2^n - 1)$ (because for P_1 we have 2^n possibilities, and then for P_2, we have $2^n - 1$ possibilities).
Now from $Adv_2 = \frac{1}{2.[2^n(2^n-1)]^2} \sum_{b_1, b_2 \in I_n} |h_2 - \tilde{h_2}|$ and $\tilde{h_2} = \frac{[2^n(2^n-1)]^2}{2^{2n}} = (2^n - 1)^2$, we get: $Adv_2 = \frac{1}{2^n(2^n-1)} \simeq \frac{1}{2^{2n}}$.
Standard deviation for $m = 2$
Les σ be the standard deviation of h_2 when $b_1, b_2 \in_R I_n$. $\sigma = \sqrt{V(h_2)} = \sqrt{E(h_2 - \tilde{h_2})^2}$. Let σ' be the average deviation of h_2 when $b_1, b_2 \in_R I_n$. $\sigma' = E(|h_2 - \tilde{h_2}|)$.

$$V(h_2) = \frac{1}{2^{2n}}[2^n(2^n - 1)^2 + 2^n(2^n - 1)] = 2^n - 1$$

Therefore $\sigma = \sqrt{2^n - 1} \simeq \frac{\tilde{h_2}}{2^{1.5n}}$.

$$\sigma' = \frac{1}{2^{2n}}[2^n(2^n - 1) + 2^n(2^n - 1).1]$$

Therefore $\sigma' = \frac{2(2^n-1)}{2^n} \simeq \frac{2\tilde{h_2}}{2^{2n}}$. We see that here $\sigma' \simeq \frac{2\sigma}{\sqrt{2^n}}$.
So σ is much larger than σ' when n is large. This is one of the reasons that explains that when m is fixed and small the approximation of Adv obtained by Bienaymé-Tchebichev from σ (used in [14]) gives when m is fixed and small only $Adv \leq O(\frac{1}{2^n})$ while the real Advantage is in $O(\frac{1}{2^{2n}})$.

A.3 $m = 3$

In section 4 we have sen that (orange equation):

$$h_{\alpha+1} = (2^n - 2\alpha + |M|)h_\alpha + \sum_{i \notin M} \sum_{j \notin M, j \neq i} h'_\alpha(i,j)$$

with $M = \{i,\ 1 \leq i \leq \alpha,\ b_i = b_{\alpha+1}\}$.
With $\alpha = 2$, this formula will give us h_3 from h_2 and h'_2.
$M = \{i,\ 1 \leq i \leq 2,\ b_i = b_3\}$.
Case 1. b_1, b_2, b_3 are pairwise distinct. Then $|M| = 0$ and $h_3 = (2^n - 4)h_2 + 2h'_2$. $h_3 = (2^n - 4).2^n.(2^n - 2) + 2.2^n$.
$h_3 = 2^n(2^{2n} - 6.2^n + 10)$ and since $\tilde{h_3} = \frac{[2^n(2^n-1)(2^n-2)]^2}{2^{3n}} = 2^{3n} - 6.2^{2n} + 13.2^n - 12 + \frac{4}{2^n}$, we have $h_3 - \tilde{h_3} = -3.2^n + 12 - \frac{4}{2^n}$. Therefore, when $n \geq 2$, we have $h_3 < \tilde{h_3}$ in this case 1 (and without loss of generality, we can assume $n \geq 2$ since for $n = 1$ we have only two values in I_n but here the number m of queries is $m = 3$).
Case 2. We have $b_1 = b_3 \neq b_2$. Then $|M| = 1$, $h_3 = (2^n - 3)h_2$, $h_3 = (2^n - 3).2^n(2^n - 2)$, $h_3 = 2^n(2^{2n} - 5.2^n + 6)$. Here $h_3 - \tilde{h_3} = 2^{2n} - 7.2^n + 12 - \frac{4}{2^n} = (2^n - 2)(2^n - 5 + \frac{2}{2^n})$. Therefore, when $n \geq 3$, we have $h_3 > \tilde{h_3}$, and when $n = 3$, we have $h_3 < \tilde{h_3}$.
Case 2 bis. We can check that when $b_1 = b_2 \neq b_3$ we obtain the same value (this is obvious by symmetry of the hypothesis but not obvious from the orange equation).
Here $|M| = 0$ and $h_3 = (2^n - 4)h_2 + 2h'_2$.
$h_3 = (2^n - 4).2^{\cdot}(2^n - 1) + 2.2^n$
$h_3 = 2^n(2^{2n} - 5.2^n + 6)$ as in Case 2.
Case 3. $b_1 = b_2 = b_3$. Here $|M| = 2$ and $h_3 = (2^n - 2)h_2 = (2^n - 2)2^n(2^n - 1)$ So $h_3 = 2^n(2^{2n} - 3.2^n + 2)$ and $h_3 - \tilde{h_3} = 3.2^{2n} - 11.2^n + 12 - \frac{4}{2^n}$ and it is easy to see that this is always ≥ 0 if $n \geq 0$. (We can also say that we have

$$\begin{aligned} h_3 \geq \tilde{h_3} \quad &\Leftrightarrow \quad 2^n(2^n-1)(2^n-2) \geq \tfrac{[2^n(2^n-1)(2^n-2)]^2}{2^{3n}} \\ &\Leftrightarrow \quad 2^{2n} \geq (2^n-1)(2^n-2) \end{aligned}$$

since $n \geq 2$ since we have $m = 3$ queries). Therefore h_3 is always $\geq \tilde{h_3}$ in Case 3.
Finally, from

$$Adv_3 = \frac{1}{2.[2^n(2^n-1)(2^n-2)]^2} \sum_{b_1,b_2,b_3 \in I_n} |h_3 - \tilde{h_3}|$$

or from

$$Adv_3 = \frac{1}{[2^n(2^n-1)(2^n-2)]^2} \sum_{b_1,b_2,b_3 / h_3 < \tilde{h}_3} (\tilde{h}_3 - h_3)$$

we obtain, if $n \geq 3$

$$Adv_3 = \frac{1}{[2^n(2^n-1)(2^n-2)]^2} 2^n(2^n-1)(2^n-2)(3.2^n - 12 + \frac{4}{2^n})$$

$$Adv_3 = \frac{1}{2^{2n}(2^n-1)(2^n-2)}(3.2^{2n} - 12.2^n + 4) \simeq \frac{3}{2^{2n}}$$

(We did not need the value h'_3 to compute h_3. However these values are directly given from section 6 (i.e. the “first purple equations”).

B Example of unusual values for h_m

h_m; or more precisely, $h_m(b)$,is the number of $(P_1, P_2, \dots, P_m, Q_1, \dots, Q_m) \in I_n^{2m}$ such that

1. The P_i are pairwise distinct.
2. The Q_i are pairwise distinct.
3. $\forall i,\ 1 \leq i \leq m,\ P_i \oplus Q_i = b_i$.

The average value of h_m, when $(b_1, \dots, b_m) \in I_n^m$ is:

$$\tilde{h_m} = \frac{(2^n(2^n-1)\dots(2^n-m+1))^2}{2^{nm}} \quad \text{(cf Theorem 4)}$$

Theorem 14 *When b_i is a constant, i.e. $\forall i,\ 1 \leq i \leq m,\ b_i = b_1$, we have:*

$$h_m = 2^n(2^n-1)\dots(2^n-m+1)$$

Proof. We have to choose the P_i pairwise distinct, and then the values Q_i are fixed and pairwise distinct by: $\forall i,\ 1 \leq i \leq m, Q_i = b_1 \oplus P_i$. □

This value $2^n(2^n-1)\dots(2^n-m+1)$ is the maximum possible value for h_m, since when $P_1, \dots, P_m$ are fixed, there is at most one possibility for $Q_1, \dots, Q_m$.
Remark. Il is conjectured that the minimum value for h_m is obtained when the values $b_1, \dots, b_m$ are pairwise distinct. When m is small (for example $m \leq \sqrt{2^n}$), this is proven, but when $m = 2^n$ for example, no proof of this conjecture is known.
From the results above, when b_i is a constant, we have:

$$h_m/\tilde{h_m} = \frac{2^{nm}}{2^n(2^n-1)\dots(2^n-m+1)} = \frac{1}{(1-\frac{1}{2^n})(1-\frac{2}{2^n})\dots(1-\frac{m-1}{2^n})}$$

It is easy to see that this expression can tend to infinity when m is large and $\sqrt{2^n} \ll m \leq 2^n$ (by taking the log of $h_m/\tilde{h_m}$ for example). Therefore, we see that $h_m/\tilde{h_m}$ is not bounded in general. Unlike this result, h_m is generally $\geq \tilde{h_m}(1-\epsilon)$ where ϵ is small (see the results of this paper, when $m \ll 2^{\frac{2n}{3}}$ for example).

Figure 4 illustrate these results. (This figure is a classical figure in "Mirror Theory", i.e. it appears often when we deal with sets of linear equalities and linear non equalities).
It is also interesting to notice that very large values h_m exist, but do not occur often, and that very large values h_m will affect more the standard deviation $\sigma(h_m)$ of h_m than the average deviation $\sigma'(h_m)$ of h_m. ($\sigma(h_m) = \sqrt{E(h-h_m)^2}$ and $\sigma'(h_m) = E(|h-h_m|)$).

C Summary of our notation and of our General Proof Strategy

In this Appendix C we will summarize the proof strategy and the main notations used in this paper.

- m and n are two integers. $I_n = \{0,1\}^n$. (From a cryptographic point of view, m will be the number of queries, and n is the number of bits of the inputs and outputs of each query).

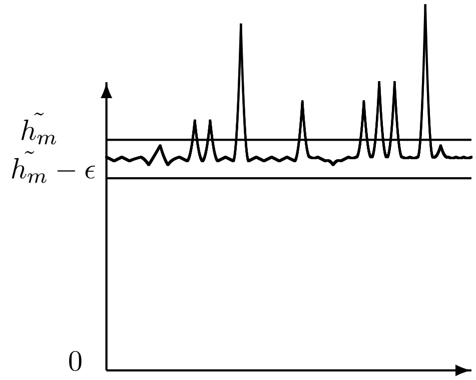


Figure 4: The different values h_m

- H_m (cf section 3) denotes the number of $(f, g) \in B_n^2$ such that $\forall i,\ 1 \leq i \leq m,\ (f \oplus g)(a_i) = b_i$. H_m is a compact notation for $H_m(b_1, b_2, \ldots, b_m)$.
- h_m (cf section 3) denotes the number of $(P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_m) \in I_n^{2m}$ such that: the P_i are pairwise distinct, the Q_i are pairwise distinct, and: $\forall i,\ 1 \leq i \leq m,\ P_i \oplus Q_i = b_i$. h_m is a compact notation for $h_m(b_1, b_2, \ldots, b_m)$. ($H_m$ and h_m are equal up to a multiplicative constant: $H_m = h_m \cdot \frac{|B_n|^2}{(2^n(2^n-1)\ldots(2^n-m+1))^2}$, cf Theorem 2 of section 3).
- $\tilde{h}_m$ denotes $\frac{(2^n(2^n-1)\ldots(2^n-m+1))^2}{2^{nm}}$. $\tilde{h}_m$ is the average value of h_m when $b \in_R I_n^m$.

Our aim is to prove that: when $m \ll 2^n$ then for most values b, $h_m \gtrsim \tilde{h}_m$ $(C1)$(where $a \gtrsim b$ means $a \geq b$ or $a \simeq b$), because then, from Theorem 6 (standard "H technique theorem") we will get CPA-2 security. In fact we will often prove a stronger property: that: when $m \ll 2^n$ then for **all** values b, $h_m \gtrsim \tilde{h}_m$ $(C2)$.
In order to prove $(C1)$ or $(C2)$, we proceed in this paper with what we call the "usual proof strategy in Mirror Theory" or the "colored proof strategy". ("Mirror Theory" is the theory that analyses the number of solutions of sets of affine equalities ($=$) and affine non equalities ($\neq$) in finite fields). Essentially the two main ideas of this "colored proof strategy" are:

1. To compare $\frac{h_{\alpha+1}}{h_\alpha}$ with $\frac{\tilde{h}_{\alpha+1}}{\tilde{h}_\alpha}$ and to use

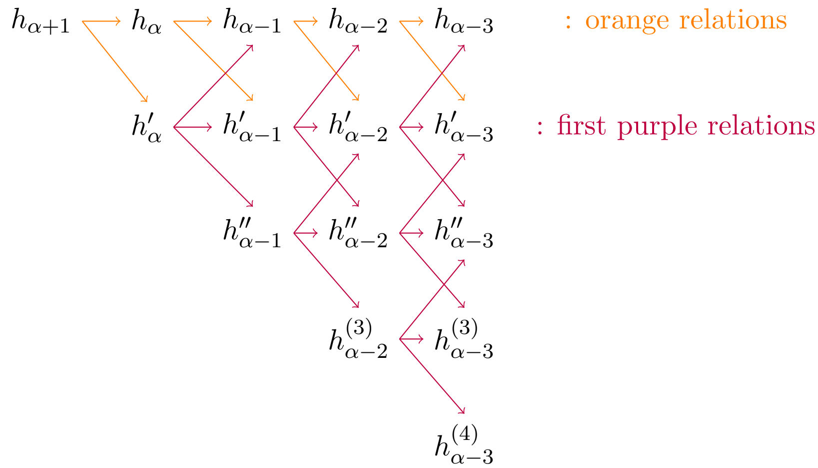
$$h_\alpha = \frac{h_\alpha}{h_{\alpha-1}} \cdot \frac{h_{\alpha-1}}{h_{\alpha-2}} \cdot \frac{h_{\alpha-2}}{h_{\alpha-3}} \cdots \frac{h_2}{h_1} h_1$$

   instead of studying h_α globally.

2. To look carefully if the affine equations that will appear in the analysis of $\frac{h_{\alpha+1}}{h_\alpha}$ are independent, consequences, or in contradiction with the linear equalities in h_α.

More precisely, here, with h_α values, this "colored proof strategy" is this one:

Figure 5: General view of the “colored proof strategy” used in this paper



1. We get an equation (called the “orange equation”) that evaluates $h_{\alpha+1}$ from h_α and h'_α where $h'_\alpha(X)$ denotes the number of solutions that satisfy the conditions h_α plus one equation X: $Q_{\alpha+1} \oplus P_1 = b_{\alpha+1}$, when this equation X is linearly independent with the non equalities of h_α. h'_α denotes any such value $h'_\alpha(X)$. This was done in section 4 of this paper.

2. We get an equation (called the “first purple equation”) that evaluates h'_α from $h_{\alpha-1}$, $h'_{\alpha-1}$ and $h''_{\alpha-1}$ (where in $h''_{\alpha-1}$ we have introduced two extra and independent affine equations from the $\lambda_{\alpha-1}$ conditions). This was done in section 6 of this paper.

3. We get the equations (called “all purple equations”) that evaluate $h^{(d)}_\alpha$ from $h^{(d-1)}_{\alpha-1}$, $h^{(d)}_{\alpha-1}$, and $h^{(d+1)}_{\alpha-1}$, (where in $h^{(d)}_{\alpha-1}$, we have introduced d extra and independent affine equations from the $h_{\alpha-1}$ equations).

4. Now, from these evaluations we are able to compare $\frac{h_{\alpha+1}}{h_\alpha}$ with $\frac{\tilde{h}_{\alpha+1}}{\tilde{h}_\alpha}$ and therefore h_α from $\tilde{h}_\alpha$.

D About my Conjecture on H_{2^n}

In [5] in 1952 (and independently in [14] in 1979) it was proved that:

$$\forall f \in F_n, \text{ if } \oplus_{x \in I_n} f(x) = 0, \text{ then } \exists (g,h) \in B_n^2 \text{ such that } f = g \oplus h$$

([5] was pointed to me by J.F. Dillon).

A new conjecture

Since 2008, I conjectured a stronger property.

Conjecture: $\forall f \in F_n$, if $\oplus_{x \in I_n} f(x) = 0$, then the number H of $(g,h) \in B_n^2$ such that $f = g \oplus h$ satisfies $H \geq \frac{|B_n|^2}{2^{n.2^n}}$.

Variant: I also conjectures that this property is true in any group (commutative or not), not only

with Xor.
In this paper I have proved results involving m equations with $m \ll O(2^n)$ instead of all the 2^n equations. These results were sufficient for the cryptographic security wanted (cf Figure 6).

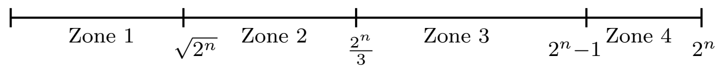


Figure 6: The different cases for the values m

Zone 1: (i.e. "below the birthday bound"): when $1 \leq m \ll \sqrt{2^n}$.
Zone 2: (i.e. the cryptographic zone "above the birthday bound"): when $\sqrt{2^n} \leq m \leq \frac{2^n}{3}$: the properties of this zone are the main subject of this paper.
Zone 3: $\frac{2^n}{3} \leq m \leq 2^n - 1$: this zone was not studied carefully in this paper. Our proof technique may also give some results in this zone, but this was not studied.
Zone 4: $m = 2^n - 1$ and $m = 2^n$: the zone of the new conjecture, and of [5] and [14].

Equivalent Conjectures
Let $\tilde{H}_\alpha = \frac{|B_n|^2}{2^{n\alpha}}$ be the average value of H_α.

Theorem 15 *The new conjecture given above is equivalent to each of these (not proved properties):*

1. $\forall f \in F_n$, *if* $\oplus_{x \in I_n} f(x) = 0$, *then* $H_{2^n}(f) \geq \frac{|B_n|^2}{2^{n.2^n}} (= \tilde{H}_{2^n})$
2. $\forall f \in F_n,\ H_{2^n-1}(f) \geq \frac{\tilde{H}_{2^n-1}}{2^n} (= \frac{|B_n|^2}{2^{n.2^n}})$
3. $\forall f \in F_n,\ \forall \alpha,\ 1 \leq \alpha \leq 2^n - 1,\ H_\alpha(f) \geq \frac{\tilde{H}_\alpha}{2^n}$
4. $\forall \alpha,\ 1 \leq \alpha \leq 2^n - 1,\ \forall b_1, \ldots, b_\alpha,\ h_\alpha(b_1, \ldots, b_\alpha) \geq \frac{\tilde{h}_\alpha}{2^n}$

Proof of Theorem 15.

- (1) is the conjecture given above.
- $\tilde{H}_{2^n-1} = \frac{\tilde{H}_{2^n}}{2^n}$ and if $\bigoplus_{x \in I_n} f(x) = 0$ then the last value of f is fixed when all the other values of f are given, and therefore if $\bigoplus_{x \in I_n} f(x) = 0$, then $H_{2^n}(f) = H_{2^n-1}(f)$. So (1) $\Leftrightarrow$ (2).
- (3) $\Rightarrow$ (2) is obvious (just take $\alpha = 2^n - 1$).
- Let assume that (2) is true. Then $\forall \alpha,\ 1 \leq \alpha \leq 2^n - 1,\ \forall f \in F_n$,

$$H_\alpha(f) = \sum_{b_{\alpha+1}, \ldots, b_{2^n-1}} [\text{Number of } (g, h) \in B_n^2 \,/\, g \oplus h = f']$$

where $\forall i,\ 1 \leq i \leq \alpha,\ f'(a_i) = f(a_i)$ and $\forall i,\ \alpha + 1 \leq i \leq 2^n - 1,\ f'(a_i) = b_i$.
So from (2): $H_\alpha \geq (2^n)^{2^n - \alpha - 1} \frac{\tilde{H}_{2^n-2}}{2^n} = \frac{2^{n.2^n}}{2^n.2^{\alpha n}} \cdot \frac{|B_n|^2}{2^{n.2^n}} = \frac{\tilde{H}_\alpha}{2^n}$. Therefore (2) $\Rightarrow$ (3).

- Finally since $H_\alpha = h_\alpha \cdot \frac{|B_n|^2}{[2^n(2^n-1)\ldots(2^n-\alpha)]^2}$, we have: (3) $\Leftrightarrow$ (4)

Remark 1. The coefficient 2^n in (3) looks a bit artificial, and even stronger properties may be true (as suggested by our simulations done in [16]).

Remark 2. If we compare the conjecture above with the results of section 6, we see that:

. Section 6 is stronger when $m \ll 2^{\frac{3n}{4}}$ since it shows that $H_\alpha \simeq \tilde{H}_\alpha$ (instead of $\geq \frac{\tilde{H}_\alpha}{2^n}$).

. The conjecture is stronger when $m \gg 2^{\frac{3n}{4}}$ since $H_\alpha \geq \frac{\tilde{H}_\alpha}{2^n}$ implies that $H_\alpha \neq 0$ for example.