Delegatable Pseudorandom Functions and Applications

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Abstract

We put forth the problem of delegating the evaluation of a pseudorandom function (PRF) to an untrusted proxy. A *delegatable PRF*, or DPRF for short, is a new primitive that enables a proxy to evaluate a PRF on a strict subset of its domain using a trapdoor derived from the DPRF secret-key. PRF delegation is *policy-based*: the trapdoor is constructed with respect to a certain policy that determines the subset of input values which the proxy is allowed to compute. Interesting DPRFs should achieve *low-bandwidth delegation*: Enabling the proxy to compute the PRF values that conform to the policy should be more efficient than simply providing the proxy with the sequence of all such values precomputed. The main challenge in constructing DPRFs is in maintaining the pseudorandomness of unknown values in the face of an attacker that adaptively controls proxy servers. A DPRF may be optionally equipped with an additional property we call *policy privacy*, where any two delegation predicates remain indistinguishable in the view of a DPRF-querying proxy: achieving this raises new design challenges as policy privacy and efficiency are seemingly conflicting goals.

For the important class of policies described as (1-dimensional) *ranges*, we devise two DPRF constructions and rigorously prove their security. Built upon the well-known tree-based GGM PRF family [15], our constructions are generic and feature only logarithmic delegation size in the number of values conforming to the policy predicate. At only a constant-factor efficiency reduction, we show that our second construction is also policy private. As we finally describe, their new security and efficiency properties render our delegated PRF schemes particularly useful in numerous security applications, including RFID, symmetric searchable encryption, and broadcast encryption.

1 Introduction

Due to its practical importance the problem of securely delegating computational tasks to untrusted third parties comprises a particularly active research area. Generally speaking, secure delegation involves the design of protocols that allow the controlled authorization for an—otherwise untrusted—party to compute a given function while achieving some target security property (e.g., the verifiability of results or privacy of inputs/outputs) and also preserving the efficiency of the protocols so that the delegation itself remains meaningful. Beyond protocols for the delegation of general functionalities (e.g., [16, 31]) a variety of specific cryptographic primitives have been considered in this context (see related work in Section 2).

Quite surprisingly, pseudorandom functions (PRFs), a fundamental primitive for emulating perfect randomness via keyed functions which finds numerous applications in information security, have not been explicitly studied in the context of delegation of computations (of PRF values). Hereby, we initiate a study on this matter.

A new PRF concept. We introduce a novel cryptographic primitive, called *delegatable pseudorandom function* (DPRF) which enables the delegation of the evaluation of a pseudorandom function (PRF) to an untrusted proxy according to a given predicate that defines the inputs on which the proxy will evaluate the PRF.

Specifically, let $\mathcal{F}$ be a PRF family, and $\mathcal{P}$ a set of predicates, called the *delegation policy*, defined over the domain of $\mathcal{F}$. A DPRF is a triplet $(\mathcal{F}, T, C)$ constructed with respect to $\mathcal{P}$, which provides the elementary functionality shown in Figure 1: For any secret key k and a predicate $P \in \mathcal{P}$, the delegator computes a trapdoor τ via algorithm T, and τ is transmitted to the proxy. The latter then runs algorithm C on τ to finally derive exactly the set of PRF values

$V_P = \{f_k(x)|P(x)\}$, where $f_k \in \mathcal{F}$, i.e., the PRF value on every input x satisfying predicate P, overall correctly enabling the evaluation of PRF f_k subject to predicate P *without* explicit knowledge of secret key k (or even the input values $A_P = \{x|P(x)\}$).

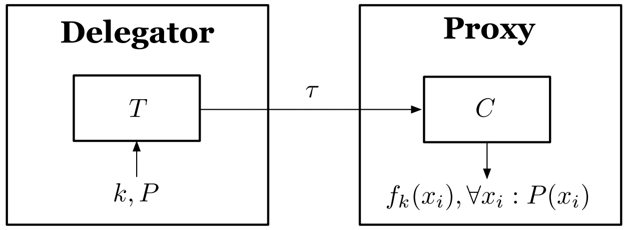


Figure 1: DPRF functionality

What motivates the above scheme is *efficiency*: As long as the trapdoor τ is sublinear in the size $n = |\{x|P(x)\}|$ of delegated PRF values, the delegation is meaningful as the delegator conserves resources (or otherwise the proxy could be provided directly with the n PRF values in the delegated set $V_P = \{f_k(x)|P(x)\}$).

At the same time, the DPRF must retain the security properties of the underlying PRF, namely, (i) pseudorandomness for any value x conforming to the delegation predicate P, i.e., $P(x)$, and (ii) unpredictability for any nonconforming value x such that $\neg P(x)$. In addition, a DPRF can optionally satisfy a *policy privacy* property which prevents the proxy from inferring information about P or the delegated set $\{x|P(x)\}$ from the trapdoor τ.

Our definitional framework. We introduce a formal definitional framework for the new primitive of DPRFs carefully capturing all the above technical requirements. We first rigorously define the *correctness* and *security* requirements that any DPRF should meet. Correctness captures the ability of the proxy to successfully evaluate the PRF on exactly those inputs specified by predicate P. Security captures the requirement that the delegation capabilities of the PRF do not compromise its core pseudorandomness property, but this condition goes beyond the standard security definition of PRFs since the pseudorandomness attacker may now adaptively query a trapdoor delegation oracle with policies of its choice.

Equally important is also a *policy privacy* property that a DPRF may optionally satisfy, intuitively capturing the inability of a malicious proxy to learn any (non-trivial) property about the delegation set $A_P = \{x|P(x)\}$ (that is not implied by set V_P). Our security notion postulates that any two policies P, P' are indistinguishable provided that $|A_P| = |A_{P'}|$ and no PRF queries are obtained in the symmetric difference of $A_P, A_{P'}$ (a necessary condition).

GGM-based realization for range predicates. We devise two *efficient and provably secure DPRF constructions* for the case where the delegation policy contains predicates described by *1-dimensional ranges*. Range-based policy predicates is an interesting use case as many applications maintain an ordering over the PRF inputs and delegation rights are defined with respect to ranges of such inputs.

Our first DPRF scheme is called *best range cover* or BRC for short, and relies on the well-known GGM PRF family [15]. This family defines a PRF based on the hierarchical application of any length-doubling pseudorandom generator (PRG) according to the structure induced by a tree, where input values are uniquely mapped to root-to-leaf paths. By exploiting the above characteristics, our BRC scheme features logarithmic delegation size in the number of values conforming to the policy predicate simply by having the trapdoor τ comprising a subset G_P of PRG values that (almost optimally) cover the target range V_P of PRF values. We provide a formal security proof for the above scheme which, interestingly, is far from being trivial as the adversary can now employ delegation queries to learn internal PRG values in the tree (not simply leaf PRF values as in the security game in [15]). (We note that although similar "range-covering" GGM-based constructions appear in the literature, e.g., [28], no formal security analysis has been given).

However, our BRC scheme does not satisfy policy privacy as the structure of intermediate PRG values in G_P leaks information about predicate P. This motivates our second construction, called *uniform range cover* or URC for short. This scheme augments BRC in a way that renders all trapdoors corresponding to ranges of the same size indistinguishable. This is achieved by carefully having τ comprising a subset G'_P of PRG values that cover the target range V_P of PRF values less optimally: G'_P contains PRG values that are descendants of those values in G_P at a tree height that depends solely on $|V_P|$ (which by definition leaks to the adversary). More interestingly, by adopting the above change, URC retains both the asymptotic logarithmic communication complexity of BRC and its DPRF security, but it crucially achieves a policy privacy notion appropriately relaxed to our setting of range predicates. Inherently, as we show, no tree-based DPRF scheme can satisfy our initial version of policy privacy, so we have to resort to a sufficient relaxation which for technical reasons we call *union policy privacy*.

Main applications. Finally, our DPRF schemes, equipped with efficiency, security and policy privacy (our second one only), lend themselves to new schemes that provide scalable solutions to a wide range of information security and applied cryptography settings that involve the controlled authorization of PRF-based computations. Generally, DPRFs are particularly useful in applications that rely on the evaluation of (secret) key-based cryptographic primitives on specific inputs (according to an underlying policy): Using a DPRF scheme then allows a cost-efficient, yet secure and private, key management for an untrusted proxy who is otherwise capable in executing a particular computational task of interest.

We outline several such applications in which DPRFs are useful, including authentication and access control in RFIDs, efficient batch querying in searchable encryption as well as broadcast encryption. Due to the underlying GGM building component, our DPRFs are extremely lightweight, as their practical implementation entails a few repeated applications of any efficient candidate instantiation of a PRG (e.g., HMAC or block ciphers).

Summary of contributions. Our main contributions are:

- We initiate the study of policy-based delegation of the task of evaluating a pseudorandom function on specific input values and introduce the concept of *delegatable* PRFs (DPRFs).
- We develop a general and rigorous definitional framework for the new DPRF primitive, capturing properties such as *efficiency, correctness, security and policy privacy*, and also offering a relaxed *union policy privacy* that is arguably necessary for a wide DPRF class with range-based policies.
- We present a framework for constructing DPRF schemes for the important case where the delegation policy is governed by range predicates over inputs; our framework augments the generic GGM construction framework of [15] to provide two concrete DPRF schemes, namely schemes BRC and URC.
- We prove the security of our framework, thus also yielding the first security analysis of similar GGM-based key-delegation schemes, and the union-policy privacy of URC.
- We describe several key applications of our DPRF primitive in the context of efficient key-delegation protocols for authentication, access control and encryption purposes.

2 Related Work

Secure delegation of computations. The notion of delegation of cryptographic operations is already mature: Starting from early work on proxy signatures [26] and proxy cryptography [5], basic primitives such as signatures (e.g., [26, 6]) and encryption (e.g.,[2, 20, 17]) have been studied in the context of an untrusted proxy who is authorized to operate on signatures or ciphertexts. Recently, there has also been an increased interest in verifiability and privacy of general outsourced computations (e.g., [8, 1, 31, 16, 7]) or specific crypto-related operations (e.g., [18, 13, 4]). To the best of our knowledge, however, no prior work explicitly and formally examines the delegation of PRFs.

PRF extensions. Closer to our DPRF new primitive are the known extensions of PRFs, namely *verifiable PRFs* (VPRFs) (e.g., [10, 25, 27]) and *oblivious PRFs* (OPRFs) (e.g., [21, 14]). A VPRF provides a PRF value along with a non-interactive proof with which anyone can verify the correctness of the PRF value. Although such proofs can be useful in third-party settings, they are not related to the delegation of the PRF evaluation without the secret key. Similarly, an OPRF is a two-party protocol that securely implements functionality $(k, x) \to (\perp, f_k(x))$—that is, a party evaluates a PRF value without knowledge of the secret key. Yet, although the party that provides the key can be viewed to preserve resources, this setting does not match our PRF delegation setting as there is no input privacy (it is the second party who provides the input x) neither communication efficiency when applied over many values. Related work includes *algebraic PRFs*, employed in [4] to achieve optimal private verification of outsourced computation.

GGM framework. This refers to the seminal work by Goldreich *et al.* [15] showing how to generically construct a PRF given black-box access to a length-doubling PRG. The approach is based on a tree structure over which hierarchical invocations of the PRG produce the PRF values at the leaves. Our constructions extend this framework in non-trivial ways: First, we support delegation of PRF evaluations; second, our security is proved in a much stronger adversarial setting where the adversary gets to adaptively learn also *intermediate* PRG values. The GGM framework as well as its tree-based key-derivation structure have been widely used in the literature; also for special/limited delegation purposes in the context of access control [28] and forward security [19]. Yet, to the best of our knowledge, no such known key-derivation method has been analyzed fully in a security model like ours that allows for adaptive PRG queries and addresses policy privacy issues.

3 Definitions

A *pseudorandom function family (PRF)* $\mathcal{F}$ is a family of functions $\{f_k : A \to B \mid k \in \mathcal{K}\}$ so that $\mathcal{K}$ is efficiently samplable and all $\mathcal{F}, \mathcal{K}, A, B$ are indexed by a security parameter λ. The security property of a PRF is as follows: for any probabilistic polynomial-time (PPT) $\mathcal{A}$ running in time polynomial in λ it holds that

$$|\Pr[\mathcal{A}^{f_k(\cdot)} = 1] - \Pr[\mathcal{A}^{R(\cdot)} = 1]| = \mathrm{negl}(\lambda) ,$$

where negl denotes a negligible function and the probability above is taken over the coins of $\mathcal{A}$ and the random variables k and R which are uniform over the domains $\mathcal{K}$ and $(A \to B)$ respectively.

Delegatable PRFs. The notion of delegation for PRFs is defined with respect to a *delegation policy* $\mathcal{P}$, i.e. $\mathcal{P}$ is a set of predicates defined over A, also indexed by λ, where each predicate P has an efficient public encoding, so that the set of elements in A that satisfy P, denoted as $A_P = \{x \in A \mid P(x)\}$, is efficiently derived.

Definition 1 (Delegatable PRF) *A triple* $(\mathcal{F}, T, C)$ *is called a* delegatable PRF (DPRF) w.r.t. policy $\mathcal{P}$ *provided it satisfies two properties, correctness and security, that are defined individually below.*

Correctness. T is a PPT algorithm such that given a description of $P \in \mathcal{P}$ and a key $k \in \mathcal{K}$, it outputs a "trapdoor" τ intended to be used along with the deterministic PT algorithm C for the computation of every element of A that satisfies the predicate P. For fixed P, k, the algorithm C can be considered as a function

$$C : St_{P,k} \longrightarrow B \times St_{P,k} ,$$

where $St_{P,k}$ is a set of states and the output $C(s)$ is a pair that consists of a PRF value and a (new) state. We augment the domain and range of C with a special value $\perp$ that will be used to denote the final state. We denote $C(s) = \langle C_L(s), C_R(s)\rangle$ and define recursively the set of *reachable* states from a subset S of $St_{P,k}$ as

$$\mathcal{R}(S) \triangleq S \cup \mathcal{R}(C_R(S)).$$

The elements of the DPRF that are produced given an initial state s will be defined using the complete set of reachable states given s. We say that a set S is $\mathcal{R}$-closed provided it holds $\mathcal{R}(S) = S$. For a singleton $S = \{s\}$, we will write $\mathcal{R}(s)$ instead of $\mathcal{R}(\{s\})$, and we will also denote by $\mathcal{R}(s)$ the closure of s under $\mathcal{R}$, i.e., the fixpoint of the recursive equation for $\mathcal{R}$ that also contains s.

Definition 2 (Correctness) *The DPRF scheme* $(\mathcal{F}, T, C)$ *is* correct *for a policy* $\mathcal{P}$ *if for every* P, k *:*

1. $\{\tau \mid \tau \leftarrow T(P, k)\} \cup \{\perp\} \subseteq St_{P,k}$.
2. $C_R(\perp) = \perp$ *(*termination condition*).*
3. *There exists a polynomial* p_1 *s.t.* $|A_P| \leq p_1(\lambda)$.
4. *There exists a polynomial* p_2 *s.t. for every* $\tau \leftarrow T(P, k)$*:*
   - *(i)* $\perp \in \mathcal{R}(\tau)$ *(*termination guarantee*).*
   - *(ii)* $|\mathcal{R}(\tau)| \leq p_2(|A_P|)$ *(*efficiency*).*
   - *(iii)* $\{f_k(x) \mid P(x)\} = f_k(A_P) = \{C_L(s) \mid s \in \mathcal{R}(\tau)\}$ *(*completeness*).*

According to the above conditions, all possible trapdoors corresponding to a certain policy predicate P are valid initial inputs for C. Starting from an arbitrary trapdoor for P, the proxy can execute a number of steps, compute the DPRF image of every argument x that fulfills P, and terminate when it reaches the final state $\perp$, where no further useful information can be derived.

We note that the condition 3 of correctness stems from the fact that we consider the setting where the proxy wishes to eventually compute all the delegated PRF values. If this is not necessary (or desirable) for the DPRF application, the condition can be relaxed to any size of A_P (including super-polynomial sizes). Obviously in this case, completeness (item 4(iii) above) will have to be relaxed as well since the proxy cannot hope to be able to compute all the delegated PRF values. There are a number of ways to capture this by suitably modifying the way C works; for instance: (i) C may produce a random sample of $f_k(A_P)$, (ii) C may be given the value x as input and return $f_k(x)$ provided that $x \in A_P$, (iii) C may be given the lexicographic rank of an element x within A_P and return $f_k(x)$.

Security. For security we consider the case where the server is malicious and model DPRF security as a game $G^{\mathcal{A}}_{\mathrm{SEC}}$ between an attacker $\mathcal{A}$ and a challenger $\mathcal{C}$ indexed by parameter λ. Due to the delegation capabilities of DPRFs the security game is more elaborate than the definition of a plain PRF, as shown next.

DPRF Security Game $G^{\mathcal{A}}_{\mathsf{SEC}}(1^\lambda)$

1. The challenger $\mathcal{C}$ selects k from $\mathcal{K}$.
2. The adversary $\mathcal{A}$ is allowed to interact with $\mathcal{C}$ and ask two types of queries:
   (a) *PRF queries* for a value $x \in A$; to those queries $\mathcal{C}$ responds with $f_k(x)$ and adds the value x to a set L_{que}.
   (b) *delegation queries* for a policy predicate $P \in \mathcal{P}$; to those queries $\mathcal{C}$ responds with $\tau \leftarrow T(P, k)$ and adds P to a set L_{pol}.
3. The adversary $\mathcal{A}$ submits a challenge query x^* to which the challenger $\mathcal{C}$ responds as follows: it flips a coin b and if $b = 1$ it responds with $y^* = f_k(x^*)$, otherwise responds with a random value y^* from B.
4. The adversary $\mathcal{A}$ continues as in step 2.
5. The adversary $\mathcal{A}$ terminates by returning a single bit $\tilde{b}$. Subsequently the game returns a bit which is 1 if and only the following holds true:
$$(b = \tilde{b}) \wedge (x^* \notin L_{\mathsf{que}}) \wedge \forall P \in L_{\mathsf{pol}} : \neg P(x^*).$$

Definition 3 (Security) *The DPRF scheme $(\mathcal{F}, T, C)$ is* secure *for a policy $\mathcal{P}$ if for any PPT $\mathcal{A}$, it holds that*

$$\Pr[G^{\mathcal{A}}_{\mathsf{SEC}}(1^\lambda) = 1] \leq \frac{1}{2} + \mathrm{negl}(\lambda).$$

We make the following observations about the definition. First, it is easy to see that a delegatable PRF is indeed a PRF. Specifically, any PRF attacker $\mathcal{A}$ against f_k can be turned into an attacker $\mathcal{A}'$ that wins the DPRF game described above. We provide only a simple sketch of this which follows by a standard "walking" argument (the reader familiar with such arguments may skip to the next paragraph). Fix some PPT $\mathcal{A}$ and let α be its non-negligible distinguishing advantage. There will be some polynomial q so that, for any λ, $q(\lambda)$ is an upper bound on the number of queries submitted to $\mathcal{A}$'s oracle by $\mathcal{A}$. Given such q and for fixed λ, we define the *hybrid* oracle $(f_k/R)_j$ for any $j \in \{0, \ldots, q(\lambda)\}$ that operates as follows: $(f_k/R)_j$ responds as $f_k(\cdot)$ in the first $q(\lambda) - j$ queries and as $R(\cdot)$ in the last j queries. Taking such sequence of oracles into account, by triangular inequality, it follows that there exists some value $j \in \{0, \ldots, q(\lambda) - 1\}$ for which the distinguishing probability will be at least $\alpha/q(\lambda)$ for $\mathcal{A}$ to distinguish between two successive hybrid oracles $(f_k/R)_j$ and $(f_k/R)_{j+1}$ when R is a random function. This follows from the fact that $\mathcal{A}$ distinguishes the "extreme" hybrids $f_k(\cdot)$ and $R(\cdot)$ with probability α. We now construct $\mathcal{A}'$ as follows out of $\mathcal{A}$: $\mathcal{A}'$ plays the DPRF game and queries the DPRF function for the $q(s) - j$ first queries of $\mathcal{A}$. Then it submits the $(j + 1)$-th query of $\mathcal{A}$ as the challenge. Finally it completes the simulation of $\mathcal{A}$ by answering any remaining queries of $\mathcal{A}$ with random values drawn from B. It is easy to see that the distinguishing advantage of $\mathcal{A}'$ is $\alpha/q(\lambda)$, i.e., non-negligible in λ.

Second, we observe that there is a trivial construction of a delegatable PRF from any PRF: Consider an ordering $\leq$ over A, e.g. the lexicographical order. For fixed P, k set $T(P, k) = \langle f_k(x_1), \ldots, f_k(x_n)\rangle$, where x_i is the i-th element of A_P according to $\leq$. Given τ, the set of states is $St_{P,k} = \{\tau, (2, \tau), \ldots, (n, \tau), \perp\}$ and the reconstruction function C can be simply defined to be a table-lookup. It is straightforward to show that $(\mathcal{F}, T, C)$ is a DPRF as long as the underlying family $\mathcal{F}$ is a PRF, since any delegation query can be intepreted as a series of polynomially many PRF queries.

The existence of a trivial DPRF construction w.r.t. arbitrary policies from any given PRF motivates our primitive: Interesting DPRF constructions will be those that are communication *efficient*, i.e., they allow trapdoors with size that is sublinear in the number of elements that satisfy the corresponding policy predicate.

(General) Policy privacy. We next consider an additional policy privacy property that goes beyond the standard (D)PRF security and is quite relevant in the context of a delegatable PRF. In order to model this privacy condition we use an indistinguishability game $G^{\mathcal{A}}_{\mathsf{PP}}$ carried out between an attacker $\mathcal{A}$ and a challenger $\mathcal{C}$ indexed by a security parameter λ. The game proceeds as follows:

DPRF Policy Privacy Security Game $G^{\mathcal{A}}_{\mathsf{PP}}(1^\lambda)$

1. The challenger $\mathcal{C}$ selects k from $\mathcal{K}$.
2. The adversary $\mathcal{A}$ is allowed to interact with $\mathcal{C}$ and ask the two types of queries as in the case of the DPRF security game $G^{\mathcal{A}}_{\mathsf{SEC}}$.
3. The adversary $\mathcal{A}$ submits two policy predicates P_0, P_1 to $\mathcal{C}$. The challenger flips a bit b and computes $\tau^* \leftarrow T(P_b, k)$. It returns τ^* to the adversary.
4. The adversary continues as in step 2 and terminates returning a bit $\tilde{b}$. The game terminates with 1 provided that

$$(b = \tilde{b}) \wedge (|A_{P_0}| = |A_{P_1}|) \wedge (A_{P_0} \neq A_{P_1}) \wedge$$
$$\wedge \forall a \in L_{\mathsf{que}} : a \notin (A_{P_0} \triangle A_{P_1}) \wedge$$
$$\wedge \forall P \in L_{\mathsf{pol}} : A_P \cap (A_{P_0} \triangle A_{P_1}) = \emptyset.$$

Definition 4 (Policy privacy) *The DPRF scheme $(\mathcal{F}, T, C)$ for a policy $\mathcal{P}$ satisfies* policy privacy *if for any PPT $\mathcal{A}$, it holds that*

$$\Pr[G^{\mathcal{A}}_{\mathsf{PP}}(1^\lambda) = 1] \leq \frac{1}{2} + \mathrm{negl}(\lambda).$$

The above definition suggests that the trapdoor that corresponds to a certain policy predicate hides the predicate itself, at least among all policy predicates that enable the same number of elements and when the adversary does not know the PRF value of any element that satisfies one of them but not the other. Observe that all the restrictions stated are necessary: if $|A_{P_0}| \neq |A_{P_1}|$, then the adversary can distinguish P_0 from P_1 by counting the number of new PRF values it computes starting from state τ^* and ending in $\perp$. In addition, if the adversary learns any PRF value of an argument x within $A_{P_0} \triangle A_{P_1}$, either by making a PRF query or a delegation query, then it can quess b by computing $\{C_L(s) \mid s \in \mathcal{R}(\tau)\}$ and running the equality test $C_L(s) \stackrel{?}{=} f_k(x)$ at each execution step.

We will next argue that even though desirable, the above property conflicts with efficiency (sublinear trapdoors) for a wide class of schemes. This will motivate relaxing the property as we will see in the end of the section; the reader unwilling to go over the details of the lower bound type of argument we sketch below may skip directly to the policy privacy relaxation in the end of the section.

Assume that the policy predicates are ranges $[a, b]$ that lie in an interval $[0, 2^\lambda - 1]$. Then, no efficient and policy private DPRF scheme exists, if the trapdoor generation algorithm $T(P, k)$ is deterministic and public and the delegated computation is *tree-wise*, i.e., for each range the trapdoor is a specific array of keys that enable in a deterministic way the calculation of the final set of values through a tree-like derivation. Indeed, for every $0 < j \leq b - a$ the delegated computation of the intersection $[a + j, b]$ must be identical for $[a, b]$ and $[a + j, b + j]$. Otherwise, an adversary can make PRF queries for all the values in $[a + j, b]$ and, since it knows the way that each unique trapdoor of $[a, b]$ and $[a + j, b + j]$ computes the PRF values of the range $[a + j, b]$, can thus distinguish the two range predicates by making the corresponding equality tests. By induction, this implies that for any $j < a - b$, in the trapdoor of $[a, b]$ there exist keys $d_0, \ldots, d_j$ that allow the computation of $f_k(a), \ldots, f_k(a + j)$ respectively. Thus, the size of the trapdoor of the range $[a, b]$ consists of $r = b - a + 1 = \#[a, b]$ keys which means that the DPRF cannot be efficient (i.e., delegate the range with a trapdoor size less than the set of values being delegated).

The above argument suggests that if policy privacy is to be attained we need trapdoors at least as long as the values we delegate. However the trivial construction for ranges, i.e., when the trapdoor of $[a, b]$ is the $\langle f_k(a), f_k(a + 1), \ldots, f_k(b)\rangle$ does not satisfy policy privacy. For instance, when delegating $[1, 3]$ and $[3, 5]$ the attacker can easily distinguish them by obtaining the value $f_k(3)$ (it belongs in the intersection hence the attacker is allowed to have it) and checking its location within the trapdoor vector. Nevertheless, arranging the PRF values of $[a, b]$ as so that $f_k(x)$ is placed in position $x \bmod (r)$ is a solution that works. Given the inefficiency of this construction we omit a detailed proof and instead provide a toy example for the ranges $[0,3]$, $[1,4]$, $[2,5]$, $[3,6]$, $[4,7]$ of size 4 below. Policy privacy follows from the fact that elements that belong in two overlapping ranges will be placed in the same fixed location in the two corresponding vector trapdoors, as shown below.

$$\begin{array}{cccc} f_k(0) & f_k(1) & f_k(2) & f_k(3) \\ f_k(4) & f_k(1) & f_k(2) & f_k(3) \\ f_k(4) & f_k(5) & f_k(2) & f_k(3) \\ f_k(4) & f_k(5) & f_k(6) & f_k(3) \\ f_k(4) & f_k(5) & f_k(6) & f_k(7) \end{array}$$

In our oncoming constructions, we obtain efficient DPRFs out of tree-based PRFs where the values computed are at the leaves of a full binary tree, and the policy predicates are ranges covered by proper subtrees. In this case, even allowing a probabilistic trapdoor generator does not suffice to provide policy privacy for a very efficient scheme. To see this, let $[a,b]$ be a range of size r and τ be a trapdoor for $[a,b]$ of size $g(r) = O(\log^c r)$, i.e., $g(r)$ is polylogarithmic in r, hence also in λ. By the Pigeonhole principle, there exists a delegation key y in τ that computes all the values corresponding to a set $T \subseteq [a,b]$ of size at least $r/g(r)$, that is, a subtree T of depth at least $d \geq \log(r) - \log(g(r)) = \omega(1)$. Assuming w.l.o.g. that $r/g(r)$ is an integer, the latter implies that there exists a value $x \in [a,b]$ with a zero suffix of length d that is computed by this key y. Since, there are $g(r)$ such possibles values $x \in [a,b]$, an adversary has significant probability $1/g(r)$ of guessing x. Then it can make a query x, receive $f_k(x)$ and submit as the challenge the two policies $P_0 = [a,b]$ and $P_1 = [x, b + (x-a)]$. After it receives the challenge trapdoor τ^*, it can locate all keys that correspond to subtrees of size $\geq d$ and check if there is minimum leaf value (an all-zero path) in some of these subtrees that equals to $f_k(x)$. Then, the adversary can distinguish effectively P_0, P_1 since x cannot be covered by any subtree of size d in a trapdoor for $[a+x, b+(x-a)]$. This argument can be extended to more general tree-like delegation schemes but we omit further details as it already demonstrates that efficient tree-like constructions require a somewhat more relaxed definition of policy privacy (which we introduce below).

Union policy privacy. We finally introduce the notion of *union policy privacy*, where the adversary is restricted from making queries in the intersection of the challenge policy predicates (but is allowed to query at arbitrary locations outside the targeted policy set). We model this property by a game $G^{\mathcal{A}}_{\mathsf{UPP}}(1^\lambda)$ that proceeds *identically* as $G^{\mathcal{A}}_{\mathsf{PP}}(1^\lambda)$ but terminates with 1 provided that all the following *relaxed* conditions are met: $b = \tilde{b}$, $|A_{P_0}| = |A_{P_1}|$, $A_{P_0} \neq A_{P_1}$, $\forall a \in L_{\mathsf{que}} : a \notin (A_{P_0} \cup A_{P_1})$ and $\wedge \forall P \in L_{\mathsf{pol}} : A_P \cap (A_{P_0} \cup A_{P_1}) = \emptyset$. Obviously, for policies consisting of disjoint predicates, games $G^{\mathcal{A}}_{\mathsf{PP}}(1^\lambda)$ and $G^{\mathcal{A}}_{\mathsf{UPP}}(1^\lambda)$ are equivalent.

4 Constructions

In this section we present DPRF schemes for *range* policy predicates. In Section 4.1 we describe a first construction, called *best range cover* (BRC), which satisfies the correctness and security properties of DPRFs, achieving trapdoor size logarithmic in the range size. However, BRC lacks the policy privacy property. In Section 4.2 we build upon BRC and obtain a policy-private DPRF scheme, called *uniform range cover* (URC), which retains the trapdoor size complexity of BRC. In Section 4.3 we include the security proofs of the two schemes. In Section 4.4 we prove the policy privacy property of URC.

4.1 The BRC Construction

Let $G : \{0,1\}^\lambda \to \{0,1\}^{2\lambda}$ be a pseudorandom generator and $G_0(k), G_1(k)$ be the first and second half of the string $G(k)$, where the specification of G is public and k is a secret random seed. The GGM pseudorandom function family [15] is defined as $\mathcal{F} = \{f_k : \{0,1\}^n \to \{0,1\}^\lambda\}_{k \in \{0,1\}^\lambda}$, such that $f_k(x_{n-1} \cdots x_0) = G_{x_0}(\cdots(G_{x_{n-1}}(k)))$, where n is polynomial in λ and $x_{n-1} \cdots x_0$ is the input bitstring of size n.

The GGM construction defines a binary tree on the PRF domain. We illustrate this using Figure 2, which depicts a binary tree with 4 levels. The leaves are labeled with a decimal number from 0 to 15, sorted in ascending order. Every edge is labeled with 0 (1) if it connects a left (resp. right) child. We label every *internal* node with the binary string determined by the labels of the edges along the path from the root to this node. Suppose that the PRF domain is $\{0,1\}^4$. Then, the PRF value of 0010 is $f_k(0010) = G_0(G_1(G_0(G_0(k))))$. Observe that the composition of G is performed according to the edge labels in the path from the root to leaf $2 = (0010)_2$, selecting the first (second) half of the output of G when the label of the visited edge is 0 (resp. 1). Based on the above, the binary representation of the leaf labels constitute the PRF domain, and every leaf is associated with the PRF value of its label.

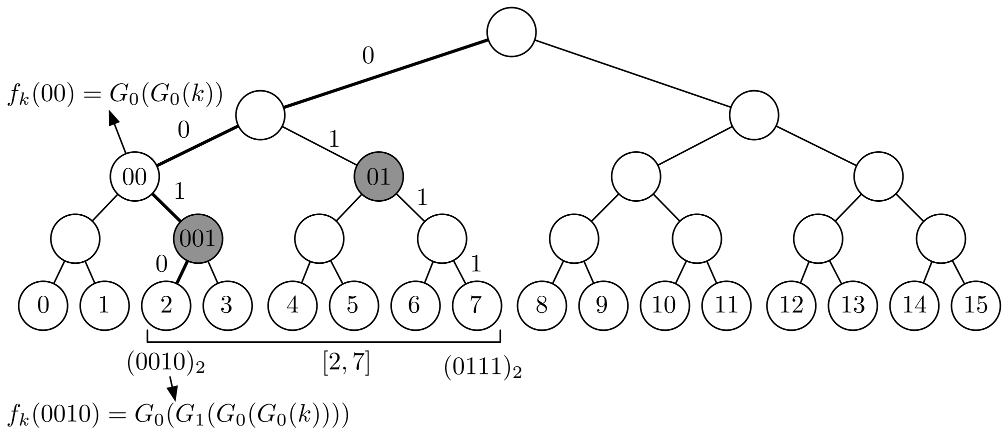


Figure 2: A GGM tree example

Note that we can also associate every *internal* node of the GGM tree with a *partial* PRF value, by performing the composition of G as determined by the path from the root to that node. For example, node 00 in Figure 2 is associated with partial PRF $G_0(G_0(k))$. Henceforth, for simplicity, we denote by $f_k(x_{n-1} \cdots x_j)$ the partial PRF $G_{x_j}(\cdots(G_{x_{n-1}}(k)))$. Observe that if a party has the partial PRF $f_k(x_{n-1} \cdots x_j)$, then it can compute the PRF values of all 2^j inputs that have prefix $x_{n-1} \cdots x_j$, simply by following a DFS traversal in the subtree with root $x_{n-1} \cdots x_j$ and composing with seed $f_k(x_{n-1} \cdots x_j)$. In our running example, using the partial PRF value at node 00, we can derive the PRF values of the inputs in (decimal) range $[0, 3]$ as $f_k(0000) = G_0(G_0(f_k(00)))$, $f_k(0001) = G_1(G_0(f_k(00)))$, $f_k(0010) = G_0(G_1(f_k(00)))$, and $f_k(0011) = G_1(G_1(f_k(00)))$.

For any range $[a, b]$ of leaf labels, there is a (non-unique) set of subtrees in the GGM tree that *cover* exactly the corresponding leaves. For instance, $[2, 7]$ is covered by the subtrees rooted at nodes 001 and 01 (colored in grey). According to our discussion above, a party having the partial PRF values of these subtree roots and the subtree depths, it can derive all the PRF values of the leaves with labels in $[a, b]$. In our example, having $(f_k(001), 1)$ and $(f_k(01), 2)$, it can derive the PRF values of the leaves with labels in $[2, 7]$. Our first construction is based on the above observations. In particular, given a range policy predicate $[a, b] \in \mathcal{P}$ with size $|A_P| = b - a + 1$, it finds the *minimum* number of subtrees that cover $[a, b]$, which is similar to finding the canonical subsets for 1-dimensional range queries. As such, we call this scheme as *best range cover* (BRC).

The BRC DPRF construction is a triplet $(\mathcal{F}, T, C)$, where $\mathcal{F}$ is the GGM PRF family described above with tree depth n. The delegation policy is $\mathcal{P} = \{[a, b] \mid 0 \le a < b \le a + \lambda^\gamma \le 2^n - 1\}$, where γ is a constant integer. The trapdoor generation algorithm T of BRC is given below. This algorithm takes as input secret key k and a range predicate $[a, b] \in \mathcal{P}$. It outputs a delegation trapdoor τ that enables the computation of $f_k(x)$ for every x whose decimal representation is in $[a, b]$ (i.e., the PRF values of the leaves in the GGM tree with labels in $[a, b]$). T initially finds the first bit in which a and b differ (Line 2), which determines the common path from the root to leaves a and b. Suppose that this common path ends at node u. T then traverses the left and right subtree of u separately (Lines 3-10 and 11-18, respectively). We will describe only the left traversal (the right one is performed symmetrically). T continues the path from the left child of u, denoted by v, to a. It checks whether a is the leftmost leaf of v's subtree. In this case the PRF value of v is included in τ along with the depth of v's subtree, and the traversal terminates (Lines 3-4). Otherwise, it includes the PRF value of the right child of v and the depth of the subtree rooted at that child, and continues the traversal in the left child of v iteratively (Lines 5-10).

The Trapdoor Generation Algorithm T in BRC

Input: $a, b : 0 \leq a < b \leq a + \lambda^{\gamma} \leq 2^n - 1$ and $k \in \{0,1\}^{\lambda}$
Output: Trapdoor τ for computing the PRFs for $[a, b]$

$\tau \leftarrow \langle\rangle$
$t \leftarrow \max\{i \mid a_i \neq b_i\}$
if $(\forall i \leq t : \; a_i = 0)$ **then**
  Append $(f_k(a_{n-1} \cdots a_t), t)$ to τ
else
  $\mu \leftarrow \min\{i \mid i < t \wedge a_i = 1\}$
  for $i = t - 1$ to $\mu + 1$
    if $a_i = 0$ **then**
      Append $(f_k(a_{n-1} \cdots a_{i+1} 1), i)$ to τ
  Append $(f_k(a_{n-1} \cdots a_{\mu}), \mu)$ to τ
if $(\forall i \leq t : b_i = 1)$ **then**
  Append $(f_k(b_{\nu-1} \cdots b_t), t)$ to τ
else
  $\nu \leftarrow \min\{i \mid i < t \wedge b_i = 0\}$
  for $i = t - 1$ to $\nu + i$
    if $b_i = 1$ **then**
      Append $(f_k(b_{n-1} \cdots b_{i+1} 0), i)$ to τ
  Append $(f_k(b_{n-1} \cdots b_{\nu}), \nu)$ to τ
return τ

In the example of Figure 2, for input $[2, 7]$, T outputs trapdoor $\tau = \langle (f_k(001), 1), (f_k(01), 2) \rangle$. By its description, it is easy to see that the algorithm covers the input range with *maximal* subtrees. However, there is one exception; if the input range is exactly covered by a single subtree (e.g., $[4, 7]$), then T covers it with *two* subtrees (rooted at 010 and 011, respectively). In general, T in BRC always covers the input range with *at least two* subtrees. This is due to a technical requirement in the proof of correctness of our second DPRF construction (described in Section 4.2), whose trapdoor generation algorithm builds upon that of BRC.

We next describe the PRF computation algorithm C of BRC. For fixed $P = [a, b]$ of size r and key k, we define the set of states as $St_{P,k} = \{\tau, (1, \tau), \ldots, (r-1, \tau), \perp\}$, where $\tau = \langle (y_1, d_1), \ldots, (y_m, d_m) \rangle$ is the trapdoor produced by algorithm T. Note that, every y value corresponds to a partial PRF value associated with the root of a GGM subtree. Therefore, there is a natural ordering of PRF values for a given τ, starting from the leftmost leaf of the subtree of y_1 to the rightmost leaf of the subtree of y_m. Note that this order is not necessarily the same as the order of the leaves in the GGM tree. Starting from the PRF value of the leftmost leaf of the y_1 subtree, C computes in every next step the PRF value of the next leaf in the ordering discussed above. Specifically, C starts with state τ and computes $C(\tau) = \langle G_0(\cdots(G_0(y_1))), (1, \tau) \rangle$, where the composition is performed d_1 times. Next, given state (i, τ), C locates the unique subtree that covers the $(i+1)$-th leaf x in the ordering of τ. Specifically, it finds the pair (y_t, d_t) in τ, where t is such that $\sum_{\rho=1}^{t-1} 2^{d_\rho} \leq i < \sum_{\rho=1}^{t} 2^{d_\rho}$. Then, the suffix $x_{d_t} \cdots x_0$ is the binary representation of $i - \sum_{\rho=1}^{t-1} 2^{d_\rho} \in [0, 2^{d_t} - 1]$. Given this suffix, C can compute $f_k(x) = G_{x_0}(\cdots(G_{x_{d_t}}(y_t)))$ and output $C((i, \tau)) = \langle G_{x_0}(\cdots(G_{x_{d_t}}(y_t))), (i+1, \tau) \rangle$. Note that, for input state $(r-1, \tau)$, C outputs $C((r-1, \tau)) = \langle G_1(\cdots(G_1(y_m))), \perp \rangle$, where the composition is performed d_m times.

In the example of Figure 2, the trapdoor for range $[2, 14]$ is $\langle (f_k(01), 2), (f_k(001), 1), (f_k(10), 2), (f_k(110), 1), (f_k(1110), 0) \rangle$. Algorithm C computes the PRF values for leaves 4, 5, 6, 7, 2, 3, 8, 9, 10, 11, 12, 13, 14 in this order, i.e.,

$$
\begin{aligned}
&C(\tau) = \langle G_0(G_0(f_k(01))), (1, \tau) \rangle = \langle f_k(0100), (1, \tau) \rangle \rightarrow \\
&C((1, \tau)) = \langle G_1(G_0(f_k(01))), (2, \tau) \rangle = \langle f_k(0101), (2, \tau) \rangle \rightarrow \\
&\vdots \\
&C((12, \tau)) = \langle f_k(1110), \perp \rangle.
\end{aligned}
$$

Based on the above description, some partial PRF values are computed *multiple* times in C, e.g., $f_k(011)$ is computed twice; once during calculating $f_k(0110)$ and once for $f_k(0111)$. Note that this can be easily avoided by employing an external data structure of size $O(r)$; every PRF value is computed once, stored in the data structure, and retrieved in a next step if necessary. In this manner, the computational cost of C can become $O(r)$, since we compute *one* PRF value for every node in the subtrees covering the range (of size r).

The correctness of BRC is stated in the following theorem.

Theorem 1 *The BRC DPRF construction* $(\mathcal{F}, T, C)$ *w.r.t delegation policy* $\mathcal{P} = \{[a,b] \mid 0 \leq a < b \leq a + \lambda^\gamma \leq 2^n - 1\}$*, where* n *is the depth of the underlying GGM tree,* γ *is a constant integer, and* λ^γ *is the maximum range size, is correct.*

Proof. We will prove that all conditions of Definition 2 are satisfied. Condition 1 is met by the definition of the set of states $St_{P,k}$ in algorithm C. Moreover, since $\mathcal{R}(\tau) = St_{P,k}$, conditions 2, 4(i) and 4(ii) are directly satisfied by the description of algorithm C. Condition 3 holds because the size of range r (which determines the number of values conforming to the range policy predicate) is upper bounded by λ^γ which is a polynomial. Therefore, it remains to prove that condition 4(iii) is satisfied. Based on our description on C, it is apparent that the algorithm computes exactly the PRF values of the leaves covered by the subtrees corresponding to the y values included in τ by algorithm T. Hence, it suffices prove that the labels of these leaves are *exactly* $[a, b]$.

Let $t = \max\{i \mid a_i \neq b_i\}$. Also, let V_1, V_2 be the sets of arguments of the PRF values computed by Lines 3-10 and 11-18 of algorithm T, respectively. It holds that

$$\begin{aligned} V_1 &= \{x \in [0, 2^n - 1] \mid x_{n-1} \cdots x_t = a_{n-1} \cdots a_t \wedge \\ &\quad \wedge (\exists i < t : [x_i = 1 \wedge a_i = 0] \vee x = a)\} = \\ &= \{x \in [0, 2^n - 1] \mid a \leq x \leq a_{n-1} \cdots a_t 1 \cdots 1\} \\ &= [a, a_{n-1} \cdots a_t 1 \cdots 1] \quad . \end{aligned}$$

Similarly, we get that $V_2 = [b_{n-1} \cdots b_t 0 \cdots 0, b]$. Observe that, by the definition of t, $a_t = 0$ and $b_t = 1$. Thus, it holds that $a_{n-1} \cdots a_t 1 \cdots 1 + 1 = b_{n-1} \cdots b_t 0 \cdots 0$. This means that $[a, b] = V_1 \cup V_2$, which concludes our proof.

We next discuss the trapdoor size complexity in BRC. Let V_1, V_2 be the sets of arguments of the PRF values computed by Lines 3-10 and 11-18 of algorithm T, respectively. Then, $|V_1| + |V_2| = r$. We will analyze $|V_1|$ (the analysis for V_2 is similar). Observe that, in every step in Lines 3-10, the algorithm covers more than $\lfloor |V_1|/2 \rfloor$ values of V_1 with a *maximal* subtree of the sub-range defined by V_1. Iteratively, this means that the algorithm needs no more than $\log(r)$ maximal subtrees to cover the entire sub-range of V_1. Consequently, the total number of elements in τ is $O(\log(r))$.

We have explained the correctness of BRC and its efficient trapdoor size. We also prove its security in Section 4.3. However, BRC does not satisfy policy privacy, even for non-intersecting policy predicates. We illustrate with a simple example in Figure 2. Consider ranges $[2, 7]$ and $[9, 14]$, both with size 6. The trapdoors generated for these ranges are $\langle((f_k(001), 1), (f_k(01), 2)\rangle$ and $\langle(f_k(101), 1), (f_k(1001), 0), (f_k(110), 1), (f_k(1110), 0)\rangle$, respectively. Clearly, these trapdoors are distinguishable due to their different sizes. This motivates our second DPRF construction presented in the next sub section.

4.2 The URC Construction

Consider again the ranges $[2, 7]$ and $[9, 14]$, for which BRC generates two distinguishable trapdoors, $\langle((f_k(001), 1), (f_k(01), 2)\rangle$ and $\langle(f_k(101), 1), (f_k(1001), 0), (f_k(110), 1), (f_k(1110), 0)\rangle$, respectively. Instead of computing the trapdoor of $[2, 7]$ as above, assume that we generate an alternative trapdoor equal to

$$\langle(f_k(010), 1), (f_k(0010), 0), (f_k(011), 1), (f_k(0011), 0)\rangle$$

Observe that this trapdoor appears to be indistinguishable to that of $[9, 14]$; the two trapdoors have the same number of elements, the first parts of their elements are all partial PRFs, whereas their second parts (i.e., the depths) are pairwise equal. This suggests that, we could achieve policy privacy, if we could devise a trapdoor algorithm T such that, for *any* range predicate of a fixed size r it *always* generates a trapdoor with a *fixed* number of elements and a *fixed* sequence of depths. More simply stated, the algorithm should produce *uniform* trapdoors for ranges of the same size. The challenge is to design such an algorithm retaining the logarithmic trapdoor size of BRC. Next, we present our second DPRF construction, called *uniform range cover* (URC), which enjoys the efficiency of BRC and the union policy privacy property.

URC builds upon BRC. In particular, it starts by producing a trapdoor as in BRC, and then modifies it to generate a uniform trapdoor for the given range r. Before embarking on its detailed description, we must investigate some interesting properties of the trapdoors of BRC, and provide some important definitions. Recall that a trapdoor in BRC is a sequence of elements, where the first part is a (full or partial) PRF value, and the second is a depth value. Moreover, the depths have some useful structure, which we call *decomposition* and formalize as follows:

Definition 5 *Let r be an integer greater than 1. A pair of non-negative integral sequences $D = ((k_1, \dots, k_c), (l_1, \dots, l_d))$ is called a* decomposition *of r if the following hold:*

(i) $\sum_{i=1}^{c} 2^{k_i} + \sum_{j=1}^{d} 2^{l_j} = r$

(ii) $k_1 > \cdots > k_c$ *and* $l_1 > \cdots > l_d$

A decomposition of r $D = ((k_1, \dots, k_c), (l_1, \dots, l_d))$ is a worst-case decomposition *(w.c.d.) of r if it is of maximum size, i.e., for every decomposition of r $D' = ((k'_1, \dots, k'_{c'}), (l'_1, \dots, l'_{d'}))$, we have that $c' + d' \le c + d$. We define $M_D \triangleq \max\{k_1, l_1\}$ as the maximum integer that appears in D.*

By the description of algorithm T in BRC, for fixed range size r, the depths in the trapdoor can be separated into two sequences that form a decomposition of r. Each sequence corresponds to a set of full binary subtrees of *decreasing* size that cover leaves in the range predicate. The usage of the worst case decomposition will become clear soon.

The following lemma shows that the maximum integer that appears in *any* decomposition of r and, hence, the maximum depth of a subtree in a cover of a range of size r, can have just one of two consecutive values that depend only on r.

Lemma 1 *Let $D = ((k_1, \dots, k_c), (l_1, \dots, l_d))$ be a decomposition of r. Define $B(r) \triangleq \lceil \log(r + 2) \rceil - 2$. Then $M_D \in \{B(r), B(r) + 1\}$. In addition, if $M_D = B(r) + 1$ then the second largest value is less than M_D.*

Proof. Observe that $r > 1 \Rightarrow B(r) \ge 0$. By the two properties of D, we have that

$$r = \sum_{i=1}^{c} 2^{k_i} + \sum_{j=1}^{d} 2^{l_j} \le 2^{k_1+1} + 2^{l_1+1} - 2 \le$$
$$\le 2^{M_D+2} - 2 \Leftrightarrow 2^{M_D+2} \ge r + 2 \Rightarrow M_D \ge B(r).$$

Since $2^{B(r)+2} \ge 2^{\log(r+2)} > r \ge 2^{k_1} + 2^{l_1}$, we have that the maximum value $M_D \in \{k_1, l_1\}$ is less than $B(r) + 2$ and k_1, l_1 cannot be both equal to $B(r) + 1$.

By Lemma 1, the trapdoor that is generated by BRC for a range $P = [a, b]$ of size $|A_P| = r$, even if this trapdoor corresponds to a w.c.d. of r, consists of at most $|\{0, \dots, B(r)\}| + |\{0, \dots, B(r) + 1\}| = 2B(r) + 3$ pairs. Hence, the trapdoor size is $O(\log(r))$, which complies with the bound we described in Section 4.1. Moreover, since $|A_P| \le \lambda^\gamma$, every trapdoor has no more than $2\lceil \log(\lambda^\gamma + 2) \rceil - 1$ pairs. In our security and privacy proofs, for simplicity, we will use the upper bound $3\gamma \log(\lambda)$ that holds for $\lambda^\gamma > 1$.

Observe that two trapdoors that correspond to the same w.c.d. (i.e., the two sequences in the decomposition are identical pairwise) appear indistinguishable. Moreover, we have proven that a trapdoor following a w.c.d. retains the logarithmic size in r. Therefore, our goal for the trapdoor algorithm T in URC is to construct a converter that takes as input a BRC trapdoor, and produces an alternative trapdoor that complies with a *fixed* w.c.d. Before proceeding to the details of T, we must prove the following vital theorem.

Theorem 2 *Let $D = ((k_1, \dots, k_c), (l_1, \dots, l_d))$ be a decomposition of r. Then all the elements in $\{0, \dots, M_D\}$ appear in D iff D is a w.c.d. of r.*

Proof. Assume that the implication does not hold and let x be the maximum integer in $\{0, \dots, M_D\}$ that does not appear in D. Since $x < M_D$, $x + 1$ is in $\{0, \dots, M_D\}$. Assume w.l.o.g. that $l_j = x + 1$. If $x > k_1$, then the decomposition of r, $((x, k_1, \dots, k_c), (l_1, \dots, l_{j-1}, x, l_{j+1}, \dots, l_d))$, is of greater size than that of D. If $i = \min\{i \mid x < k_i\}$, then the decomposition of r, $((k_1, \dots, k_i, x, k_{i+1}, \dots, k_c), (l_1, \dots, l_{j-1}, x, l_{j+1}, \dots, l_d))$, is of greater size than that of D. Both cases contradict to the hypothesis, hence, x must appear in D.

For the converse, let $D' = ((k'_1, \dots, k'_{c'}), (l'_1, \dots, l'_{d'}))$ be a w.c.d. of r. Then the integers $0, \dots, M_{D'}$ appear in D'. By Lemma 1, all integers $0, \dots, B(r)$ appear in D and D'. By removing the integers $0, \dots, B(r)$ from D and D', the remaining integers are $y_1 \ge \ldots \ge y_s$ and $z_1 \ge \ldots \ge z_t$, respectively. Since an integer cannot appear more than twice in a decomposition of r and, by Lemma 1, the maximum possible value $B(r) + 1$ cannot appear more than once, we have that $y_1, \dots, y_s$ and $z_1, \dots, z_t$ are sequences of distinct integers s.t. $\sum_{i=1}^{s} 2^{y_i} = \sum_{j=1}^{t} 2^{z_j}$. Assume that there exists a minimum index $\rho \le s$ s.t. $y_\rho \ne z_\rho$ and that w.l.o.g $y_\rho > z_\rho$. Then we have the contradiction

$$\sum_{i=1}^{s} 2^{y_i} \geq \sum_{i<\rho} 2^{y_i} + 2^{y_\rho} > \sum_{i<\rho} 2^{y_i} + 2^{z_\rho+1} - 1 \geq$$

$$\geq \sum_{i<\rho} 2^{z_i} + \sum_{i\geq\rho} 2^{z_i} = \sum_{j=1}^{t} 2^{z_j} = \sum_{i=1}^{s} 2^{y_i}.$$

Thus, $\{y_1, \ldots, y_s\}$ and $\{z_1, \ldots, z_t\}$ are equal, and therefore D is a w.c.d. of r.

A consequence of Theorem 2 and Lemma 1 is that, for every integer $r > 1$, a w.c.d. of r is a proper rearrangement of the integers that appear in the w.c.d. r where the first sequence is $(B(r), \ldots, 0)$ and the second sequence is the remaining integers in the decomposition in decreasing order. We term this unique w.c.d. as *uniform decomposition* of r. The main idea in URC is to always generate a trapdoor that complies with the uniform decomposition of r.

We are ready to describe algorithm T in URC, whose pseudo code is provided below.

The Trapdoor Generation Algorithm T in URC

Input: $a, b : 0 \leq a < b \leq a + \lambda^\gamma \leq 2^n - 1$ and $k \in \{0,1\}^\lambda$
Output: Trapdoor τ for computing the PRFs for $[a, b]$

1. Invoke $\tau = T(a, b, k) = \langle (y_1, d_1), \ldots, (y_n, d_m) \rangle$ from BRC
2. Let $D = ((d_1, \ldots, d_c), (d_{c+1}, \ldots, d_m))$ be the corredponding decomposition of $r = b - a + 1$
3. **while** there is a maximum integer x in $\{0, \ldots, M_D\}$ that does not appear in D:
4. Find the rightmost pair $(y_i, x+1)$ and compute values $y_i^0 = G_0(y_i)$, $y_i^1 = G_1(y_i)$
5. Remove $(y_i, x+1)$ from τ, properly insert the pairs (y_i^0, x) and (y_i^1, x) in τ respecting the definition of the decomposition and update D accordingly
6. **if** the leftmost sequence of D is not $(B(r), \ldots, 0)$ **then**
7. Arrange τ according to the uniform decomposition of r
8. **return** τ

The process starts with invoking the T algorithm of BRC to get an initial trapdoor τ (Line 1). Let D be the decomposition implied by τ (Line 2). The loop in Lines 3-5 utilizes Theorem 2 and works as follows. It finds the highest depth x that does not exist in D, and "splits" the partial PRF value y_i in the rightmost pair in τ with depth $x+1$, producing two new partial PRF values with depth x, i.e., $y_i^0 = G_0(y_i)$ and $y_i^1 = G_1(y_i)$. (Line 4). Note that these values correspond to the two children of the subtree of y_i. Thus, if we substitute element $(y_i, x+1)$ by $(y_i^0, x), (y_i^1, x)$ in τ, then the trapdoor can still produce the same leaf PRF values as before. However, at all times we wish the sequence of depths in τ to form a decomposition. Hence, after removing $(y_i, x+1)$, we appropriately insert (y_i^0, x) in the left pair sequence and (y_i^1, x) in the right pair sequence of τ (Line 5). Upon termination of the loop, all the values in $\{0, \ldots, M_D\}$ appear in D and, thus, we have reached a w.c.d. according to Theorem 2. The process concludes, after properly re-arranging the elements of τ, such that they comply with the unique uniform decomposition of r (Line 7). This is done deterministically, by simply filling the missing depths from $\{0, \ldots, B(r)\}$ in the left sequence with the unique appropriate pair that exists (by Theorem 2) in the right sequence.

In our running example, for the range $[2, 7]$, the T algorithm in URC converts the original token retrieved by the trapdoor algorithm of BRC, $\tau = \langle (f_k(001), 1), (f_k(01), 2) \rangle$, as follows (we underline a newly inserted element to the left sequence, and depict as bold a newly inserted element to the right sequence):

$$\langle (f_k(001), 1), (f_k(01), 2) \rangle$$
$$\downarrow$$
$$\langle \underline{(f_k(0010), 0)}, (f_k(01), 2), (\mathbf{f_k(0011), 0}) \rangle$$
$$\downarrow$$
$$\langle \underline{(f_k(010), 1)}, (f_k(0010), 0), (\mathbf{f_k(011), 1}), (f_k(0011), 0) \rangle$$

The C algorithm of URC is identical to BRC, because the trapdoor in URC has exactly the format expected by this algorithm, i.e., pairs of PRF values corresponding to GGM subtrees along with their depths. Moreover, during the evolution of the initial BRC trapdoor into one of uniform decomposition in the T algorithm of URC, a partial PRF value y is substituted by two new PRF values that can generate the same leaf PRF values as y. As such, the correctness of the BRC scheme is inherited in URC. Finally, due to Lemma 1, the size of a w.c.d. (and, hence, also a uniform decomposition) of r is $O(\log(r))$, which means that the trapdoor size in URC is also $O(\log(r))$.

4.3 Security

In this section we prove the security of BRC and URC. Both proofs rely on the security of the pseudorandom generator of the underlying GGM construction. However, DPRFs entail a more involved security game than GGM and, hence, the constructions require more complex proofs. The reason is that, contrary to the case of GGM where the adversary obtains only *leaf* PRFs, the adversary in a DPRF can obtain also *partial* PRF values in the GGM tree through the trapdoors.

Note that the formation of a trapdoor (which is independent of k) for a range predicate P of size r is deterministic and public in both BRC and URC. Thus, when querying the oracle in the security game, the adversary can map the partial or full PRF values appearing in τ for a selected P to subtrees in the GGM tree. This implies that it can reduce its delegation queries to *prefixes* for strings in $\{0,1\}^n$, where n is the depth of the GGM tree. Hence, any queried prefix of the adversary is denoted as $x_{n-1}\cdots x_t$, where the maximum value of t depends on the maximum possible range size λ^γ. Since the size of every trapdoor is $O(\log(\lambda))$, a PPT adversary makes polynomially many such queries. The formation of a trapdoor does not play any role in the security of a DPRF scheme, since the challenge will always be a single PRF value. Therefore, there will be no separation in proving the security of the BRC and the URC constructions.

In order to prove the security property of the DPRF schemes, we insure and exploit the security of two special cases. First, we consider the case that the adversary is non-adaptive, i.e., it poses all its queries in advance.

Lemma 2 *The BRC and URC DPRF schemes with depth n and maximum range size λ^γ are secure against PPT adversaries that make all their queries, even the challenge query, non-adaptively.*

Proof. Let $\mathcal{A}$ be such an adversary against a DPRF scheme of depth n and maximum range size λ^γ. We define recursively two sequences of non-adaptive PPT adversaries $\mathcal{A}=\mathcal{A}_n,\ldots,\mathcal{A}_1$ and PPT algorithms $S_n,\ldots,S_1$ as follows. For $i=n-1,\ldots,1$, $\mathcal{A}_i$ on input 1^λ, initially invokes $\mathcal{A}_{i+1}$ receiving all of its non-adaptive queries, and chooses a random value k' in $\{0,1\}^\lambda$. If a query $x_i\cdots x_t$ has the same most significant bit (MSB) as the challenge query, $\mathcal{A}_i$ makes the query $x_{i-1}\cdots x_t$, and responds with the received value y. Otherwise, it responds with the value $G_{x_t}(\cdots(G_{x_{i-1}}(k')))$. It returns $\mathcal{A}_{i+1}$'s output.

For $i=n,\ldots,1$, on input $(z_0,z_1)\in\{0,1\}^{2\lambda}$, S_i invokes $\mathcal{A}_i$ and receives all of its queries. For every query $x_{i-1}\cdots x_t$, it responds with $G_{x_t}(\cdots(G_{x_{i-2}}(z_{x_{i-1}})))$ (or z_{x_0} for $i=1$). On the challenge phase, it flips a coin b and acts as the challenger in the DPRF security game. It returns 1 iff $\mathcal{A}_i$ returns b.

We denote by q_i and p_i the probability that S_i outputs 1 when it receives its input from the uniform distribution $U_{2\lambda}$ in $\{0,1\}^{2\lambda}$ and the pseudorandom distribution $G(U_\lambda)$, respectively. By the definition of S_i, $p_n=\Pr[G^{\mathcal{A}}_{\mathsf{SEC}}(1^\lambda)=1]$ while $q_1\le 1/2$, since it corresponds to a totally random game.

We observe that $\mathcal{A}_{n-1},\ldots,\mathcal{A}_1$ behave like attackers against the security of DPRF schemes with respective depths $n-1,\ldots,1$, enhanced with the abort option. The behavior of A_i as an attacker is the same as A_{i+1}'s, when the latter interacts with a modified challenger that replaces the two possible partial PRF values for the MSB of a prefix, with two random values. Following the previous notation, we have that $p_i=q_{i+1}$. Since $G(U_\lambda)$ and $U_{2\lambda}$ are indistinguishable, it holds that $|p_i-q_i|\le\epsilon_i(\lambda)$, where $\epsilon_i(\cdot)$ is a negligible function. We also have

$$|p_n-q_1|=|\sum_{i=1}^{n}(p_i-q_i)|\le\sum_{i=1}^{n}|p_i-q_i|\le\sum_{i=1}^{n}\epsilon_i(\lambda),$$

hence $\Pr[G^{\mathcal{A}}_{\mathsf{SEC}}(1^\lambda)=1]=p_n\le q_1+n\cdot\epsilon(\lambda)\le 1/2+n\cdot\epsilon(\lambda)$, where $\epsilon(\lambda)=\max_i\{\epsilon_i(\lambda)\}$.

We use the above lemma to prove the security of a special category of DPRF schemes, where the maximum range size is at least half of $[0,2^n-1]$, which represents the biggest interval where range predicates are defined. This will serve as a stepping stone for designing our final security proof.

Lemma 3 *The BRC and URC DPRF schemes with depth n and maximum range size λ^γ are secure if $2^{n-1} \leq \lambda^\gamma < 2^n$.*

Proof. Let $\mathcal{A}$ be a PPT adversary. We construct a non-adaptive adversary $\mathcal{B}$ for the DPRF security game that on input 1^λ chooses randomly a challenge x^* in $[0, 2^n - 1]$ and makes the $\log(n)$ queries that cover all the possible values except from x^*. Namely, $\mathcal{B}$ makes queries $x^*_{n-1} \oplus 1, \ldots, x^*_{n-1} \cdots x^*_i \oplus 1, \ldots, x^*_{n-1} \cdots x^*_0 \oplus 1$ and submits challenge x^*. It receives responses $y_{n-1}, \ldots, y_0$ respectively, along with y^* which is the response to x^*. Then, it invokes $\mathcal{A}$ and plays the security game with $\mathcal{A}$, as a challenger that can respond appropriety for every value that is not x^* or a range that does not contain x^*. If $\mathcal{A}$ sets x^* as a challenge, then $\mathcal{B}$ responds with y^*, and returns $\mathcal{A}$'s guess. Otherwise, $\mathcal{A}$ has either made a query which is a prefix of x^*, or it has submitted a challenge different than x^*, so $\mathcal{B}$ terminates the game it plays with $\mathcal{A}$ and returns a random bit ,as guess to its challenge.

Let E be the event that $\mathcal{B}$ guesses $\mathcal{A}$'s challenge, i.e. $\mathcal{A}$'s challenge is x^*. By the description of $\mathcal{B}$ we have that

$$\Pr[G^{\mathcal{B}}_{\mathsf{SEC}}(1^\lambda) = 1 \wedge \neg E] = (1 - 1/2^n) \cdot 1/2 \quad \text{and}$$
$$\Pr[G^{\mathcal{B}}_{\mathsf{SEC}}(1^\lambda) = 1 \wedge E] = 1/2^n \cdot \Pr[G^{\mathcal{A}}_{\mathsf{SEC}}(1^\lambda) = 1].$$

Since $\mathcal{B}$ is non-adaptive, by Lemma 2 we get that for some negligible function $\epsilon(\cdot)$, $\Pr[G^{\mathcal{B}}_{\mathsf{SEC}}(1^\lambda) = 1] \leq 1/2 + \epsilon(\lambda)$. By adding the above equations we have that

$$1/2^n \cdot (\Pr[G^{\mathcal{A}}_{\mathsf{SEC}}(1^\lambda) = 1] - 1/2) + 1/2 \leq 1/2 + \epsilon(\lambda),$$

so, $\Pr[G^{\mathcal{A}}_{\mathsf{SEC}}(1^\lambda) = 1] \leq 1/2 + 2^n \cdot \epsilon(\lambda) \leq 1/2 + 2\lambda^\gamma \cdot \epsilon(\lambda)$.

We can now apply Lemma 3 to prove the security of our constructions.

Theorem 3 *The BRC and URC DPRF schemes with depth n and maximum range size λ^γ are secure.*

Proof. See Appendix A.

4.4 Policy Privacy

This section analyzes the policy privacy of URC. According to Section 3, URC cannot satisfy the general policy privacy property, because it is efficient. We illustrate with a toy example. For challenge ranges $[2, 5]$ and $[4, 7]$, the trapdoors will contain PRF values corresponding to subtrees covering the ranges as $[2, 3], \{4\}, \{5\}$ and $[4, 5], \{6\}, \{7\}$, respectively. Therefore, the adversary can issue query for leaf 4 and receive a PRF value y. Having $\langle (y_1, 1), (y_2, 0), (y_3, 0) \rangle$ as challenge trapdoor, it can check whether $y_2 = y$, which happens only when $[2, 5]$ was chosen by the challenger.

Nevertheless, in the theorem below we prove that URC achieves *union policy privacy*. The main idea can be observed on the above example. In the union policy privacy game, the adversary cannot obtain a PRF value for a leaf in the *intersection* of the challenge ranges, i.e., for 4 and 5. This prevents the adversary from launching the attack described above.

Theorem 4 *The URC scheme with depth n and maximum range size λ^γ is a DPRF scheme with union policy privacy.*

Proof. See Appendix A.

5 Applications

In this section we discuss interesting applications of the general DPRF primitive and our specialized range constructions. We stress, though, that their applicability is not limited to these scenarios; we are confident that they can capture a much wider set of applications.

Authentication and access control in RFID. *Radio Frequency Identification* (RFID) is a popular technology that is expected to become ubiquitous in the near future. An RFID *tag* is a small chip with an antenna. It typically stores a unique ID along with other data, which can be transmitted to a *reading device* lying within a certain range from the tag. Suppose that a *trusted center* (TC) possesses a set of RFID tags (attached to books, clothes, etc), and distributes RFID readers to specified locations (e.g., libraries, campuses, restaurants, etc.). Whenever a person or object carrying a tag lies in proximity with a reader, it transmits its data (e.g., the title of a book, the brand of a jacket, etc.). The TC can then retrieve these data from the RFID readers, and mine useful information (e.g., *hotlist* books, clothes, etc.).

Despite its merits, RFID technology is challenged by security and privacy issues. For example, due to the availability and low cost of the RFID tags, one can easily create tags with arbitrary information. As such, an adversary may *impersonate* other tags, and provide falsified data to legitimate readers. On the other hand, a reader can receive data from any tag in its vicinity. Therefore, sensitive information may be leaked to a reader controlled by an adversary. For example, the adversary may learn the ID and the title of a book stored in a tag, match it with public library records, and discover the identity and reading habits of an individual.

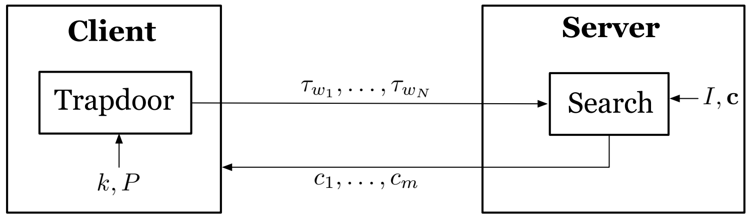
Motivated by the above, the literature has addressed *authentication* and *access control* in RFID. A notable paradigm was introduced in [28], which can be directly benefited by DPRFs. At a high level, every tag is associated with a key, and the TC *delegates* to a reader a set of these keys (i.e., the reader is authorized to authenticate and access data from only a subset of the tags). The goal is for the TC to reduce certain costs, e.g., the size of the delegation information required to *derive* the tag keys.

Observe that a DPRF $(\mathcal{F}, T, C)$ is directly applicable to the above setting. $\mathcal{F}$ is defined on the domain of the tag IDs, and its range is the tag keys. Given a delegation predicate on the tag IDs, the TC generates a trapdoor via algorithm T, and sends it to the reader. The latter runs C on the trapdoor to retrieve the tag keys. In fact, for the special case where the access policy is a *range* of IDs, the delegation protocol suggested in [28] is identical to the non-private BRC scheme (we should stress though that [28] lacks rigorous definitions and proofs). Range policies are meaningful, since tag IDs may be sequenced according to some common theme (e.g., books on the same topic are assigned consecutive tag IDs). In this case, a range policy *concisely* describes a set of tags (e.g., books about a certain religion) and, hence, the system can enjoy the logarithmic delegation size of BRC. However, as explained in Section 4, BRC leaks the position of the IDs in the tree, which may further leak information about the tags. Although [28] addresses tag privacy, it provides no privacy formulation, and overlooks the above structural leakage. This can be mitigated by directly applying our policy-private URC construction for delegating tag keys to the readers. To sum up, DPRFs find excellent application in authentication and access control in RFID, enabling communication-efficient tag key delegation from the TC to the reader. Moreover, policy-private DPRFs provide a higher level of protection for the tag IDs against the readers.

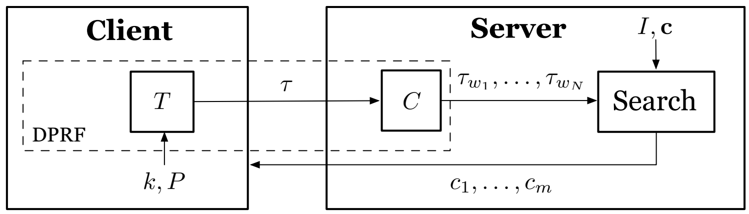
Batch queries in symmetric searchable encryption. Symmetric searchable encryption (SSE) [22, 9] enables queries to be processed directly on ciphertexts generated with symmetric encryption. Although SSE is a general paradigm, here we focus on the definitions and schemes of [9]. These works support the special case of *keyword* queries, and provide an acceptable level of provable security. The general framework underlying [9] is as follows. In an offline stage, a client encrypts his data with his secret key k, and uploads the ciphertexts $\mathbf{c}$ to an untrusted server. He also creates and sends a *secure index* I on the data for efficient keyword search, which is essentially an encrypted lookup table or inverted index. Given a keyword w, the client generates a query *token* τ_w using k, and forwards it to the server. It is important to stress that this trapdoor is merely comprised of one or more PRF values computed on w with k, which were used as keys to encrypt I. For simplicity and w.l.o.g., we assume that τ_w is a single PRF value. The server uses τ_w on I and retrieves the IDs of the ciphertexts associated with w. The results $c_1, \ldots, c_m$ are retrieved and transmitted back to the client, who eventually decrypts them with his key. The security goal is to protect *both* the data and the keyword from the server.

Suppose that the client wishes to search for a *batch* of N keywords $w_1, \ldots, w_N$. For instance, the client may ask for documents that contain *multiple* keywords of his choice (instead of just a single one). As another example, assume that the client's data are employee records, and each record contains a salary attribute that takes as values intervals of the form [iK, $(i+1)$K] (i.e., instead of exact salaries). These intervals can serve as keywords in SSE search. Suppose that the client wishes to retrieve records with salaries in a *range* of intervals, e.g., [1K, 10K]. Observe that this cannot be processed with a single keyword query (no record is associated with [1K, 10K]). To overcome this while utilizing the SSE functionality, the client can ask 9 *distinct* queries with keywords "[1K, 2K]", "[2K, 3K]", . . ., "[9K, 10K]", which cover the query range [1K, 10K]. Such scenarios are handled with "traditional" SSE as shown in Figure 3(a). Given a predicate P that describes the keywords $w_1, \ldots, w_N$ in the batch query, the client generates N trapdoors $\tau_{w_1}, \ldots, \tau_{w_N}$ using the standard SSE trapdoor algorithm, and sends them to the server. The server then searches I with every τ_{w_i} following the SSE protocol. Apparently, for large N, the computational and communication cost at the client is greatly impacted.

We can augment the discussed SSE framework with DPRF functionality, in order to support batch queries with *sublinear* (in N) processing and communication cost at the client, while providing provable security along the lines of [22]. Figure 3(b) illustrates our enhanced SSE architecture. Recall that τ_{w_i} is practically a PRF on w_i produced



(a) “Traditional” SSE



(b) SSE augmented with DPRFs

Figure 3: Batch keyword query processing in SSE

with key k. Therefore, instead of computing a PRF value for every w_i himself, the client *delegates* the computation of these PRF values to the server by employing a DPRF $(\mathcal{F}, T, C)$, where $\mathcal{F}$ is defined over the domain of the keywords. Given predicate P and k, the client runs T and generates trapdoor τ, which is sent to the server. The latter executes C on τ to produce $\tau_{w_1}, \ldots, \tau_{w_N}$. Execution then proceeds in the same manner as in “traditional” SSE. Observe that, for the special case of ranges, if URC is used as the DPRF, the computational and communication cost at the client decreases from $O(N)$ to $O(\log(N))$.

The above framework can be proven secure against *adaptive* adversaries along the lines of [9]. The most important alteration in the security game and proof is the formulation of the *information leakage* of the C algorithm of the DPRF. Although a policy-private DPRF provides predicate indistinguishability, there may still be some information leaked from C about the computed PRFs, which may be critical in SSE. Specifically, C may disclose some relationships between two PRF values satisfying the predicate P. For instance, in our URC scheme, C may disclose that two PRF values correspond to *adjacent* domain values (e.g., when the two values are produced as left and right children in a subtree). Through this leakage, the adversary may infer that, e.g., two ciphertexts are employee records with adjacent salary intervals (but these values will remain encrypted obviously). If we incorporate this leakage information to the simulator in the security game, then slight modifications in the proof methodology of [9] suffice in order to perfectly simulate the real execution DPRF-enhanced SSE framework; despite this moderate privacy loss the scheme offers an exponential improvements in terms of efficiency which from a practical point of view is a more deciding factor. We defer the detailed definitions and proofs to the long version of this paper.

Broadcast encryption. In a broadcast encryption scheme [12, 29, 30] a sender wishes to transmit data to a set of receivers so that at each transmission the set of receivers excluded from the recipient set can be chosen on the fly by the sender. In particular this means that the sender has to be able to make an initial key assignment to the recipients and then use the suitable key material so that only the specific set of users of its choosing can receive the message. In such schemes it was early on observed that there is a natural tradeoff between receiver memory storage and ciphertext length (e.g., see lower bounds in [24]). The intuition behind this is that if the receivers have more keys this gives to the sender more flexibility in the way it can encrypt a message to be transmitted.

In the above sense one can think of the key assignment step of a broadcast encryption scheme as a PRF defined over the set Φ which contains all distinct subsets that the broadcast encryption scheme assigns a distinct key. Given this configuration the user u will have to obtain all the keys corresponding to subsets $S \in \Phi$ for which it holds that $u \in S$ (we denote those subsets by $S_u \subseteq \Phi$). In DPRF language this would correspond to a delegation policy for a PRF: users will need to store the trapdoor that enables the evaluation of any value in the delegated key set S_u.

Seen in this way, *any* DPRF is a key assignment mechanism for a broadcast encryption scheme that saves space on receiver storage. For example our construction for ranges gives rise to the following broadcast encryption scheme:

receivers $[n] = \{1, \ldots, n\}$ are placed in sequence; each receiver $u \in [n]$ is initialized with the trapdoor for a range $[u - t, u + t]$ for some predetermined parameter $t \in \mathbb{Z}$. In this broadcast encryption scheme the sender can very efficiently enable any range of receivers that is positioned at distance at most t from a fixed location $v \in [n]$. This is done with a single ciphertext (encrypted with the key of location v). Any receiver of sufficient proximity t to location v can derive the corresponding decryption key from its trapdoor. Furthermore, given the efficiency of our construction, storage on receivers is only logarithmic on t. While the semantics of this scheme are more restricted than a full-fledged broadcast encryption scheme (which enables the sender to exclude any subset of users on demand) it succinctly illustrates the relation between broadcast encryption and DPRF; further investigation in the relation between the two primitives from a construction point of view will be motivated by our notion.

Regarding policy privacy, it is interesting to point out that this security property is yet unstudied in the domain of broadcast encryption. A different privacy notion [3, 23, 11] was in fact put forth that deals with the structure of ciphertext. Our policy privacy on the other hand deals with the privacy of memory contents from the receiver point of view. Maintaining the indistinguishability of the storage contents is a useful security measure in broadcast encryption schemes and our DPRF primitive will motivate the study of this security property in the context of braodcast encryption (note that none of the tree-like key-delegation methods used in broadcast encryption schemes prior to our work satisfy policy privacy).

6 Conclusion

We have introduced the new concept of *delegatable pseudorandom functions* (DPRFs), a new cryptographic primitive that allows for policy-based computation at an untrusted proxy of PRF values without knowledge of a secret or even the input values. We provided formal definitions of the core properties of DPRFs for (1) correctness, the ability of the proxy to compute PRF values only for inputs that satisfy a given predicate, (2) security, the standard pseudorandomness guarantee but against a stronger adversary that also issues delegation queries, and (3) policy privacy, preventing leakage of the secret preimages of the computed PRF values. Moreover, we presented two DPRF constructions along with a comprehensive analysis in terms of their security and privacy guarantees and some inherent trade-offs with efficiency. Our proposed DPRFs are generic, yet practical, based on the well-understood and widely-adopted GGM design framework for PRFs and, as we showed, they find direct application in many key delegation or derivation settings providing interesting new results. Further exploration of DPRFs holds promise for new interesting results. Open problems include: Designing DPRFs for other classes of predicates, establishing upper and lower bounds on the connection between efficiency and policy privacy, and studying applications in other settings.

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Proof of Theorem 3

By Lemma 3, it suffices to show for the case that $\lambda^\gamma < 2^{n-1}$. This is definitely the interesting case, since 2^n is normally superpolynomial in λ. Let $\mathcal{A}$ be a PPT adversary against the DPRF scheme that makes at most $q(\lambda)$ queries and d be the minimum integer that $\lambda^\gamma < 2^d$. We construct a PPT PRF distinguisher $\mathcal{B}$ that makes oracle queries of fixed size $n' = n - d \geq 1$. On input 1^λ, $\mathcal{B}$ flips a coin b, invokes $\mathcal{A}$ and initializes a security game, itself being the challenger. By the bound on the size of the ranges, all queries of $\mathcal{A}$ have length greater than n'[1]. Thus, for every query $x_{n-1}\cdots x_t$, we have that $t < d$ and $\mathcal{B}$ responds by making query $x_{n-1}\cdots x_d$, which is of length n', receiving a value y and answering to $\mathcal{A}$ as $G_{x_t}(\cdots(G_{x_{d-1}}(y)))$. If $\mathcal{A}$ submits a challenge, $\mathcal{B}$ acts as a normal challenger would do according to option b. Finally, $\mathcal{B}$ returns 1 iff $\mathcal{A}$ returns b. Clearly, when $\mathcal{B}$'s oracle is a PRF F_k of length n' $\mathcal{B}$ returns 1 iff $\mathcal{A}$ wins $G^{\mathcal{A}}_{\mathsf{SEC}}(1^\lambda)$.

Consequently, we construct a PPT adversary $\tilde{\mathcal{A}}$ against the security of a DPRF of depth d and maximum range size λ^γ as follows: $\tilde{\mathcal{A}}$ invokes $\mathcal{A}$ and chooses a random index $j \in [q(\lambda)]$ and $q(\lambda)-1$ random values $k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_{q(\lambda)} \in \{0,1\}^\lambda$. Index j reflects $\tilde{\mathcal{A}}$'s worst-case attempt to guess which of all of possible different prefixes of length n' that will appear in $\mathcal{A}$'s queries will be the one that a prospective challenge query will have. Then $\tilde{\mathcal{A}}$ keeps count of the different prefixes of length n' that gradually appear and reacts to a query $x_{n-1}\cdots x_t$ according to the following checks:

- If $x_{n-1}\cdots x_d$ is the i-th prefix and $i \neq j$, then it responds with $G_{x_t}(\cdots(G_{x_{d-1}}(k_i)))$.
- If $x_{n-1}\cdots x_d$ is the j-th prefix, then if all of queries made that have the j-th prefix, along with $x_{n-1}\cdots x_t$, cover the whole subtree of prefix $x_{n-1}\cdots x_d$, then $\tilde{\mathcal{A}}$ terminates the game with $\mathcal{A}$, locates an element

  $$x_{n-1}\cdots x_d z_{d-1}\cdots z_0$$

  that has not been covered yet, submits $z_{d-1}\cdots z_0$ as its challenge and returns a random bit. Otherwise, it makes query $x_{d-1}\cdots x_0$ and responds with the received value y.
- If $t = 0$ and $x_{n-1}\cdots x_0$ is $\mathcal{A}$'s challenge query, then if $x_{n-1}\cdots x_d$ is not the j-th prefix, then $\tilde{\mathcal{A}}$ terminates the game with $\mathcal{A}$, locates an element $z_{n-1}\cdots z_0$ that has the j-th prefix and has not been covered yet, submits $z_{d-1}\cdots z_0$ as its challenge and returns a random bit. Otherwise, it submits challenge $x_{d-1}\cdots x_0$, gets value y^* and responds with y^*.

If $\mathcal{A}$ ever aborts or returns a value not in $\{0,1\}$, $\tilde{\mathcal{A}}$ submits a valid challenge and returns a random guess as above. Thus, $\tilde{\mathcal{A}}$ follows its own security game in any case. If $\mathcal{A}$ returns a value in $\{0,1\}$, after it has followed the phases of a proper security game, then $\tilde{\mathcal{A}}$ returns $\mathcal{A}$'s output. By the choice of d, $2^{d-1} \leq \lambda^\gamma < 2^d$, hence Lemma 3 implies that $\tilde{\mathcal{A}}$ has negligible distinguishing advantage.

By the description of $\tilde{\mathcal{A}}$, the interaction between $\mathcal{A}$ and $\mathcal{B}$ in the case that $\mathcal{B}$'s oracle is random and $\mathcal{A}$ submits a challenge and a value in $\{0,1\}$ is fully simulated by $\tilde{\mathcal{A}}$ when the latter's guess for the prefix of $\mathcal{A}$'s challenge is correct. Formally, let C be the event that $\mathcal{A}$ submits a challenge, E be the event that $\tilde{\mathcal{A}}$ guesses this challenge and B be the event that $\mathcal{A}$ returns a value in $\{0,1\}$. For brevity, we denote $\tilde{C} := C \wedge B$. It holds that

$$\Pr[\mathcal{B}^R(1^\lambda) = 1|\tilde{C}] = \Pr[G^{\tilde{\mathcal{A}}}_{\mathsf{SEC}}(1^\lambda) = 1|\tilde{C} \wedge E]. \tag{1}$$

Assume that $\mathcal{A}$ wins $G^{\mathcal{A}}_{\mathsf{SEC}}(1^\lambda)$ with a non-negligible distinguishing advantage $\epsilon(\cdot)$, so $\Pr[\mathcal{B}^{F_k}(1^\lambda) = 1] > 1/2 + \epsilon(\lambda)$. But F_k is a PRF, so $\Pr[\mathcal{B}^R(1^\lambda) = 1] > 1/2 + \delta(\lambda)$ for some non-negligible function $\delta(\cdot)$. If $\mathcal{A}$ does not submit a challenge, then its output is independent from the coin-flip of $\mathcal{B}$, hence $\Pr[\mathcal{B}^R(1^\lambda) = 1|\neg C] \leq 1/2$. We have that $\Pr[\mathcal{B}^R(1^\lambda) \wedge \neg\tilde{C}] = \Pr[\mathcal{B}^R(1^\lambda) = 1 \wedge \neg C]$ and $\Pr[\mathcal{B}^R(1^\lambda) = 1 \wedge \tilde{C}] + \Pr[\mathcal{B}^R(1^\lambda) = 1 \wedge \neg\tilde{C}] > 1/2 + \delta(\lambda)$, thus we get that $\Pr[\mathcal{B}^R(1^\lambda) = 1 \wedge \tilde{C}] > \delta(\lambda)$ and $Pr[\tilde{C}] > \delta(\lambda)$. Moreover, if $\Pr[\mathcal{B}^R(1^\lambda) = 1|\tilde{C}] - 1/2 \leq \zeta(\lambda)$, for some negligible function $\zeta(\cdot)$, then

$$\begin{aligned}\Pr[\mathcal{B}^R(1^\lambda) = 1] = \Pr[\tilde{C}] \cdot \Pr[\mathcal{B}^R(1^\lambda) = 1|\tilde{C}] + \\ + \Pr[\neg\tilde{C}] \cdot \Pr[\mathcal{B}^R(1^\lambda) = 1|\neg\tilde{C}] \leq 1/2 + \zeta(\lambda),\end{aligned}$$

[1]Recall that even a range of maximum size 2^d will be covered by subtrees of depth $< d$.

so $\Pr[\mathcal{B}^R(1^\lambda) = 1|\tilde{C}] - 1/2 > \delta'(\lambda)$ for some non-negligible function $\delta'(\cdot)$. Applying (1), we get,

$$\begin{aligned}&\Pr[G^{\tilde{\mathcal{A}}}_{\mathsf{SEC}}(1^\lambda) = 1 \wedge \tilde{C}] = \\ &= \Pr[\tilde{C}] \cdot (\Pr[E] \cdot \Pr[\mathcal{B}^R(1^\lambda) = 1|\tilde{C}] + 1/2 \cdot (1 - \Pr[E])).\end{aligned}$$

Moreover, $\Pr[G^{\tilde{\mathcal{A}}}_{\mathsf{SEC}}(1^\lambda) = 1 \wedge \neg\tilde{C}] = 1/2 \cdot (1 - \Pr[\tilde{C}])$, so by adding the two equations we have that

$$\begin{aligned}&\Pr[G^{\tilde{\mathcal{A}}}_{\mathsf{SEC}}(1^\lambda) = 1] = \\ &\quad = 1/2 + \Pr[\tilde{C}] \cdot \Pr[E] \cdot (\Pr[\mathcal{B}^R(1^\lambda) = 1|\tilde{C}] - 1/2) > \\ &\quad > 1/2 + (\delta(\lambda) \cdot \delta'(\lambda))/q(\lambda),\end{aligned}$$

which contradicts Lemma 3.

Proof of Theorem 4

Let $\mathcal{A}$ be a PPT adversary that wins the union policy privacy game with non-negligible advantage $\epsilon(\cdot)$, i.e., $\Pr[G^{\mathcal{A}}_{\mathsf{PP}}(1^\lambda) = 1] > 1/2 + \epsilon(\lambda)$. Let P_0, P_1 be the two challenge ranges and b the chosen random bit, hence $\mathcal{A}$ receives the challenge trapdoor τ_b for P_b. We define a sequence of hybrid games $G^{\mathcal{A}}_0(1^\lambda), \ldots, G^{\mathcal{A}}_{\lambda^\gamma}(1^\lambda)$, where G_i is executed as the original game $G^{\mathcal{A}}_{\mathsf{UPP}}(1^\lambda)$ during the pre-challenge and the post-challenge phase. Define the ordering $<_D$ over the distinct elements of a decomposition D as follows: $x <_D y$ iff $x < y$ or $x = y$ and x is in the left sequence of D. The only modification in $G^{\mathcal{A}}_i(1^\lambda)$ is that the first i elements in the challenge trapdoor, arranged according to the decomposition ordering $<_D$ of the depths, are the same as τ_b's, and all the other elements are replaced by random values in $\{0,1\}^\lambda$. Since $G^{\mathcal{A}}_{\lambda^\gamma}(1^\lambda)$ is the original game, and in $G^{\mathcal{A}}_0(1^\lambda)$ the trapdoor is totally random, it holds that $\Pr[G^{\mathcal{A}}_{\lambda^\gamma}(1^\lambda) = 1] - \Pr[G^{\mathcal{A}}_0(1^\lambda) = 1] > \epsilon(\lambda)$.

Let E_r be the event that the size of the challenge ranges $|A_{P_0}|, |A_{P_1}|$ that $\mathcal{A}$ submits is r. Then for some $b \in \{0,1\}$ and $r \in [\lambda^\gamma]$

$$\Pr[G^{\mathcal{A}}_{\lambda^\gamma}(1^\lambda) = 1 \wedge b \wedge E_r] - \Pr[G^{\mathcal{A}}_0(1^\lambda) = 1 \wedge b \wedge E_r] > \epsilon(\lambda)/2\lambda^\gamma$$

Consider the uniform decomposition of r, $D = ((k_1, \ldots, k_c), (l_1, \ldots, l_d))$. The different elements of D are arranged as $z_1 <_D \cdots <_D z_{c+d}$. We observe that for $j \geq c+d$ the games G_j and G_{λ^γ} are identical. So $\Pr[G^{\mathcal{A}}_{c+d}(1^\lambda) = 1 \wedge b \wedge E_r] - \Pr[G^{\mathcal{A}}_0(1^\lambda) = 1 \wedge b \wedge E_r] > \epsilon(\lambda)/2\lambda^\gamma$, which implies that there exists an $i \in [c+d] \subseteq [3\gamma \log(\lambda)]$ such that

$$\Pr[G^{\mathcal{A}}_i(1^\lambda) = 1 \wedge b \wedge E_r] - \Pr[G^{\mathcal{A}}_{i-1}(1^\lambda) = 1 \wedge b \wedge E_r] > \epsilon(\lambda)/6\gamma\lambda^\gamma \log(\lambda)$$

We will show that for these fixed b, r, i, we can construct an adversary $\mathcal{A}_i$ for the security game of a DPRF construction of depth $n - z_i$ that has non-negligible winning advantage. The main idea is that $\mathcal{A}_i$ invokes $\mathcal{A}$ and simulates either G_i or G_{i-1} on selected challenge P_b of size r depending on the value of the challenge bit b_i for the security game $G^{\mathcal{A}_i}_{\mathsf{SEC}}(1^\lambda)$.

On input 1^λ, $\mathcal{A}_i$ computes D and arranges its elements as $z_1 <_D \cdots <_D z_{c+d}$. It invokes $\mathcal{A}$ and answers all of its pre-challenge queries as follows: for each query $x_{n-1} \cdots x_t$, if $t \geq z_i$, it just transfers the query, receives value y, and responds with y. Otherwise it makes query $x_{n-1} \cdots x_{z_i}$, receives value y, and responds with $G_{x_t}(\cdots(G_{x_{z_i-1}}(y)))$. In the challenge phase, if $|A_{P_b}| \neq r$, then $\mathcal{A}_i$ terminates the game with $\mathcal{A}$, chooses a valid random challenge, and returns a random bit. Otherwise, it makes $i-1$ queries and computes the $<_D$-first $i-1$ partial delegation keys $y_1, \ldots, y_{i-1}$ of τ_b as in the pre-challenge phase, and sets the string $x^*_{n-1} \cdots x^*_{z_i}$ that corresponds to the i-th partial key as its challenge receiving y^*. It arranges the values $y_1, \ldots, y_{i-1}, y^*$ according to the order that τ_b imposes and "fills" the $c+d-i$ remaining positions of a trapdoor-like array τ_i with $c+d-i$ random values in $\{0,1\}^\lambda$. It returns τ_i to $\mathcal{A}$ and answers to $\mathcal{A}$'s post-challenge queries as in the pre-challenge phase. If $\mathcal{A}$ returns 0, then $\mathcal{A}_i$ returns 1, otherwise it returns a random bit.

We compute the probability that $\mathcal{A}_i$ wins the security game as

$$\begin{aligned}\Pr[G^{\mathcal{A}_i}_{\mathsf{SEC}}(1^\lambda) = 1] = &\Pr[G^{\mathcal{A}_i}_{\mathsf{SEC}}(1^\lambda) = 1 \wedge \neg E_r] + \\ &+ \Pr[G^{\mathcal{A}_i}_{\mathsf{SEC}}(1^\lambda) = 1 \wedge E_r \wedge b_i = 1] + \\ &+ \Pr[G^{\mathcal{A}_i}_{\mathsf{SEC}}(1^\lambda) = 1 \wedge E_r \wedge b_i = 0].\end{aligned} \tag{2}$$

It holds that $\Pr[G^{\mathcal{A}_i}_{\mathsf{SEC}}(1^\lambda) = 1 \wedge \neg E_r] = 1/2 \cdot (1 - \Pr[E_r])$.

For the other two terms, we observe that when $b_i = 1$, $\mathcal{A}_i$ simulates G_i when b and E_r occur, whereas when $b_i = 0$ $\mathcal{A}_i$ simulates G_{i-1} when b and E_r occur. Therefore,

$$\begin{aligned}&\Pr[G_{\mathsf{SEC}}^{\mathcal{A}_i}(1^\lambda) = 1 \wedge E_r \wedge b_i = 1] = \\ &\quad = 1/4 \cdot \Pr[E_r] \cdot (1 + \Pr[G_i^{\mathcal{A}}(1^\lambda) = 1 | b \wedge E_r])\end{aligned}$$

$$\begin{aligned}&\Pr[G_{\mathsf{SEC}}^{\mathcal{A}_i}(1^\lambda) = 1 \wedge E_r \wedge b_i = 0] = \\ &\qquad = 1/4 \cdot \Pr[E_r] \cdot (1 - \Pr[G_{i-1}^{\mathcal{A}}(1^\lambda) = 1 | b \wedge E_r])\end{aligned}$$

By adding $\Pr[G_{\mathsf{SEC}}^{\mathcal{A}_i}(1^\lambda) = 1]$ as evaluated by Eq. 2

$$\begin{aligned}&\Pr[G_{\mathsf{SEC}}^{\mathcal{A}_i}(1^\lambda) = 1] = 1/2 + 1/2 \cdot (\Pr[G_i^{\mathcal{A}}(1^\lambda) = 1 \wedge b \wedge E_r] - \\ &\quad - \Pr[G_{i-1}^{\mathcal{A}}(1^\lambda) = 1 \wedge b \wedge E_r]).\end{aligned}$$

Therefore, $\Pr[G_{\mathsf{SEC}}^{\mathcal{A}_i}(1^\lambda) = 1] > \frac{1}{2} + \epsilon(\lambda)/12\gamma\lambda^\gamma \log(\lambda)$, which contradicts Theorem 3.