Obfuscation from Semantically-Secure Multi-linear Encodings

Rafael Pass[*] Karn Seth[†] Sidharth Telang[‡]

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Abstract

We define a notion of semantic security of multi-linear (a.k.a. graded) encoding schemes: roughly speaking, we require that if an algebraic attacker (obeying the multi-linear restrictions) cannot tell apart two constant-length sequences $\vec{m}_0$, $\vec{m}_1$ in the presence of some other elements $\vec{z}$, then encodings of these sequences should be indistinguishable. Assuming the existence of semantically secure multi-linear encodings and the LWE assumption, we demonstrate the existence of indistinguishability obfuscators for all polynomial-size circuits.

We rely on the beautiful candidate obfuscation constructions of Garg et al (FOCS'13), Brakerski and Rothblum (TCC'14) and Barak et al (ePrint'13) that were proven secure only in idealized generic multilinear encoding models, and develop new techniques for demonstrating security in the standard model, based only on semantic security of multi-linear encoding (which trivially holds in the generic multilinear encoding model).

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[†]Cornell University. Email: `karn@cs.cornell.edu`.

[‡]Cornell University. Email: `sidtelang@cs.cornell.edu`.

1 Introduction

The goal of *program obfuscation* is to "scramble" a computer program, hiding its implementation details (making it hard to "reverse-engineer"), while preserving the functionality (i.e, input/output behavior) of the program. Precisely defining what it means to "scramble" a program is non-trivial: on the one hand, we want a definition that can plausibly satisfied, on the other hand, we want a definition that is useful for applications.

A first formal definition of such program obfuscation was provided by Hada [Had00]: roughly speaking, Hada's definition—let us refer to it as *strongly virtual black-box*—is formalized using the simulation paradigm. It requires that anything an attacker can learn from the obfuscated code, could be simulated using just black-box access to the functionality.[1] Unfortunately, as noted by Hada, only learnable functionalities can satisfy such a strong notion of obfuscation: if the attacker simply outputs the code it is given, the simulator must be able to recover the code by simply querying the functionality and thus the functionality must be learnable.

An in-depth study of program obfuscation was initiated in the seminal work of Barak, Goldreich, Impagliazzo, Rudich, Sahai, Vadhan, and Yang [BGI$^+$01]. Their central result shows that even if we consider a more relaxed simulation-based definition of program obfuscation—called *virtual black-box (VBB)* obfuscation—where the attacker is restricted to simply outputting a single bit, impossibility can still be established (assuming the existence of one-way functions). Their result is even stronger, demonstrating the existence of families of functions such that given black-box access to f_s (for a randomly chosen s), not even a *single* bit of s can be guessed with probability significantly better than 1/2, but given the code of any program that computes f_s, the entire secret s can be recovered. Thus, even quite weak simulation-based notions of obfuscation are impossible.

But weaker notions of obfuscation may be achievable, and may still suffice for (some) applications. Indeed, Barak *et al.* [BGI$^+$01] also suggested two such notions:

- The notion of *indistinguishability obfuscation*, first defined by Barak *et al.* [BGI$^+$01] and explored by Garg, Gentry, Halevi, Raykova, Sahai, and Waters [GGH$^+$13b], roughly speaking requires that obfuscations $\mathcal{O}(C_1)$ and $\mathcal{O}(C_2)$ of any two *equivalent* circuits C_1 and C_2 (i.e., whose outputs agree on all inputs) from some class $\mathcal{C}$ are computationally indistinguishable.

- The notion of *extractability obfuscation*, first defined by Barak *et al.* [BGI$^+$01] (under the name, differing-input obfuscation) and explored by Boyle, Chung and Pass [BCP13] and by Ananth, Boneh, Garg, Sahai and Zhandry [ABG$^+$13] strengthens the notion of indistinguishability obfuscation to also require that even if C_1 and C_2 are not equivalent circuits, if an attacker can distinguish obfuscations $\mathcal{O}(C_1)$ and $\mathcal{O}(C_2)$, then the attacker must "know" an input x such that $C_1(x) \neq C_2(x)$, and this input can be efficiently "extracted" from A.

In a very recent breakthrough result, Garg, Gentry, Halevi, Raykova, Sahai, and Waters [GGH$^+$13b] provided the first candidate constructions of indistinguishability obfuscators for all polynomial-size circuits so-called, based on so-called *multi-linear (a.k.a. graded) encodings*—for which candidate constructions were recently discovered in the ground-breaking work of Garg, Gentry and Halevi [GGH13a], and more recently, alternative constructions were provided by Coron, Lepoint and Tibouchi [LCT$^+$13].

The obfuscator construction of Garg et al proceeds in two steps. They first provide a candidate construction of an indistinguishability obfuscator for NC^1 (this construction is essentially assumed to be secure); next, they demonstrate a "bootstrapping" theorem showing how to use fully homomorphic encryption (FHE) schemes [Gen09] and indistinguishability obfuscators for NC^1 to obtain indistinguishability obfuscators for all polynomial-size circuits.

[1] Hada actually considered a slight distributional weakening of this definition.

Further constructions of obfuscators for NC^1 were subsequently provided by Brakerski and Rothblum [BR13] and Barak, Garg, Kalai, Paneth and Sahai [BGK$^+$13]—in fact, these definitions achieve even stronger notions of virtual-black-box obfuscation in idealized "generic" multi-linear encoding models. Additionally, Boyle, Chung and Pass [BCP13] present an alternative bootstrapping theorem, showing how to employ on extractability obfuscation for NC^1 to obtain extractability (and thus also indistinguishability) obfuscation for both circuits and Turing machines. (Ananth et al [ABG$^+$13] also provide Turing machine extractability obfuscators but instead starting from extractability obfuscators for polynomial-size circuits.)

In parallel with the development of candidate obfuscation constructions, several surprising applications of both indistinguishability and extractability obfuscations have emerged: for instance, in the works of Garg et al [GGH$^+$13b], Sahai and Waters [SW13], Hohenberger, Sahai and Waters [HSW13], Boyle, Chung and Pass [BCP13], Boneh and Zhandry [BZ13], Garg, Gentry, Halevi and Raykova [GGHR13], Bitansky, Canetti, Paneth and Rosen [BCPR13], Boyle and Pass [BP13]. Most notable among these is the beautiful work of Sahai and Waters [SW13] (and the "punctured program" paradigm it introduces) which shows that for many of the "dream applications" of virtual black-box obfuscation (such as turning private-key encryption into public-key encryption), the weaker notion of indistinguishability obfuscation suffices.

1.1 Towards "Provably-Secure" Obfuscation

But despite these amazing developments, the following question remains wide open:

> *Can any "reasonable" notion of general-purpose obfuscation be achieved based on some "natural" intractability assumption?*

Note that while the construction of indistinguishability obfuscation of Garg et al is based on *some* intractability assumption, the assumption is very tightly tied to their scheme—in essence, the assumption stipulates that the scheme is a secure indistinguishability obfuscator. Rather, we are here concerned with the question of whether some *general* assumption (that is interesting in its own right, and isn't "tailored" to the scheme) can be used to obtain indistinguishability obfuscation.

The VBB constructions of Brakerski and Rothblum [BR13] and Barak et al [BGK$^+$13] give us more confidence in the plausible security of their obfuscators, in that they show that at least "generic" attacks—that treat multi-linear encoding as if they were "physical envelopes" on which multi-linear operations can be performed—cannot be used to break security of the obfuscators. But at the same time, non-generic attacks against their scheme are known—since general-purpose VBB obfuscation is impossible! Thus, it is not clear to what extent security arguments in the generic multi-linear encoding model should make us more confident that these constructions satisfy e.g., a notion of indistinguishability obfuscation.

In this work, we address the above question. We stipulate a new, but in our eyes natural, assumption regarding multi-linear encoding—the existence of, so-called, *semantically-secure multi-linear encodings*, and show how to construct indistinguishability obfuscators for NC^1 (which then can be bootstrapped up to general circuits) based on this assumption.

1.2 Obfuscation From Semantically-secure Multi-linear Encodings

Recall that a multi-linear (a.k.a. graded) encoding scheme [GGH13a, GGH$^+$13b] enables anyone that has access to a *public parameter* pp and *encodings* $E_S^x = \mathsf{Enc}(x, S)$, $E_S^y = \mathsf{Enc}(y, S')$ of ring elements x, y under the sets $S, S' \subset [n]$ to *efficiently*:[2]

[2]Just as [BR13, BGK$^+$13], we here rely on "set-based" graded encoding; these were originally called "generalized" graded encodings in [GGH13a]. Following [GGH$^+$13b, BGK$^+$13] (and in particular the notion of a "multi-linear jigsaw

- compute an encoding $E^{x \cdot y}_{S \cup S'}$ of $x \cdot y$ under the set $S \cup S'$, as long as $S \cap S' \neq \emptyset$;
- compute an encoding E^{x+y}_{S} of $x + y$ under the set S as long as $S = S'$;
- compute an encoding E^{x-y}_{S} of $x - y$ under the set S as long as $S = S'$.

(Given just access to the public-parameter pp, generating an encoding to a particular element x may not be efficient; however, it can be efficiently done given access to the *secret parameter* sp.) Additionally, given an encoding E^x_S where the set S is the whole universe $[n]$, we can efficiently check whether $x = 0$ (i.e., we can "zero-test" encodings under the set $[n]$.) In essence, multi-linear encodings enable computations of certain restricted set (determined by the sets S under which the elements are encoded) of arithmetic circuits, and finally determine whether the output of the circuit is 0.

Let us turn to defining a notion of semantic security for multi-linear encodings. Intuitively, given a sequence of sets $\vec{T}$ and a sequence of elements $\vec{z}$, we would want that for any two elements m_0, m_1 and set S, encodings $\mathsf{Enc}(m_0, S)$ and $\mathsf{Enc}(m_1, S)$ cannot be distinguished by any efficient attacker, that gets to see encodings $\mathsf{Enc}(\vec{z}, \vec{T})$ of $\vec{z}$ under the sets $\vec{T}$. But we can never hope that encodings of m_0 and m_1 are indistinguishable if they can already be told apart by a "algebraic" attacker that simply operates on the encodings by using the above allowed operations (in a way that respects the "set-restrictions"). Rather, our notion of *single-message* semantic security for multi-linear encodings requires that for every $S, \vec{T}$ pair, every "valid" distribution D over $m_0, m_1, \vec{z}$—where a distribution D is valid if no (even computationally unbounded) algebraic attacker (obeying the set-restrictions) can distinguish whether it gets access to $\{\mathsf{Enc}(m_0, S), \mathsf{Enc}(\vec{z}, \vec{T})\}$ or $\{\mathsf{Enc}(m_1, S), \mathsf{Enc}(\vec{z}, \vec{T})\}$—we have that encodings $\{\mathsf{Enc}(m_0, S), \mathsf{Enc}(\vec{z}, \vec{T})\}$ and $\{\mathsf{Enc}(m_1, S), \mathsf{Enc}(\vec{z}, \vec{T})\}$ are indistinguishable.

We will require a slight strengthening of the above notion to a *constant*-message settings, where m_0,m_1, and S are replaced by *constant-length* vectors $\vec{m_0}, \vec{m_1}, \vec{S}$. In the remainder of the paper, we simply refer to constant-message semantically-secure encodings as *semantically-secure multi-linear encodings.* (For our purposes, it will in fact suffice to consider an *entropic* notion of semantic security, where we only require security to hold as long as D samples $\vec{m}_0, \vec{m}_1$ and $\vec{z}$ with some high-entropy; for simplicity of exposition, we define and prove out results in the worst-case setting, but we explain in Remark 1 why our analysis also goes through if we assume the existence of just entropically secure multilinear encodings.)

Note that (randomized) multi-linear encoding scheme in the generic multi-linear encoding model of [BGK$^+$13] are trivially semantically secure. In essence, the assumption of constant-message semantic security, stipulates that for the *particular task* of distinguishing encodings of a constant number of elements generic attacks cannot be (significantly) beaten. Our central result shows how to construct indistinguishability obfuscators for NC^1 based on the existence of semantically-secure multi-linear encodings.

Theorem 1 (Informally stated). *Assume the existence of semantically secure multi-linear encodings. Then there exists indistinguishability obfuscators for NC^1.*

If additionally assuming the existence of a leveled FHE [RAD78, Gen09] with decryption in NC^1—implied e.g., by the LWE assumptions [BV11, BGV12]—this construction can be bootstrapped up to obtain indistinguishability obfuscators for all polynomial-size circuits.

On falsifiability of semantically secure multilinear encodings Let us point out that the assumption that a multilinear encoding scheme is semantically security is not necessarily "efficiently falsifiable" in the terminology of Naor [Nao03], since checking whether there exists some algebraic way of telling apart two constant-length sequences of element (in the presence of some other elements z) is not necessarily polynomial-time computable. Note, however, that the assumption that a particular scheme is

puzzles" in [GGH$^+$13b]), we additionally enable anyone with the secret parameter to encode *any* elements (as opposed to just *random* elements as in [GGH13a]).

an indistinguishability obfuscator is not an efficiently falsifiable assumption either: a presumed attacker must exhibit two functionally-equivalent circuits C_1 and C_2 that it can distinguish obfuscations of; but checking whether two circuits are functionally equivalent may not be polynomial-time computable: in fact, assuming the existence of indistinguishability obfuscation and one-way functions it is easy to come up a method to sample C_1 and C_2 that with high probability compute different functions, yet are indistinguishable.[3]

1.3 Construction Overview

Following the original work of Garg et al (as well as subsequent works), our construction proceeds in three steps:

- We view the NC^1 circuit to be obfuscated as a *branching program* BP (using Barrington's Theorem [Bar86])—that is, the program is described by m pairs of matrices, each one labelled with an input bit, and the program is evaluated computing by choosing one of the two matrices, based on the input, computing the product, and finally based on the product determining the output—there is a unique "accept" (i.e., output 1) matrix, and a unique "reject" (i.e., output 0) matrix.
- The branching program BP is *randomized* using Kilian's technique [Kil88] (roughly, each pair of matrices is appropriately multiplied with the same random matrix R while ensuring that the output is the same), and then "randomized" some more—each individual matrix is multiplied by a random *scalar* α. Let us refer to this step as Rand.
- Finally the randomized matrices are encoded using multilinear encodings with the sets selected appropriately. We here rely on the *straddling set* idea of [BGK+13] to determine the sets.[4] We refer to this step as Encode.

(The original construction as well as the subsequent works also consisted of several other steps, but for our purposes these will not be needed.) The obfuscated program is now evaluated by using the multilinear operations to evaluate the branching program and finally appropriately use the zero-test to determine the output of the program. Let us refer to this construction as the "basic obfuscator".

Roughly speaking, the idea behind the basic obfuscator is that the multi-linear encodings *intuitively* ensure that any distinguisher (attacker) getting the encoding needs to multiply matrices along paths that corresponds to some input to the branching program (the straddling sets are used to ensure that the input is used consistently in the evaluation)[5]; the scalars α ensure that a potential distinguisher without loss of generality can use a *single* "multiplication-path" and still succeed with roughly the same probability, and finally, Kilian's randomization steps ensures that if a distinguisher *only* operates on matrices along a single path that corresponds to some input x (in a consistent way), then its output can be perfectly simulated given just the output of the circuit on input x. (The final step relies on the fact that the output of the circuit uniquely determines product of the branching program along the path, and Kilian's randomization then ensures that the matrices along the path are random conditioned on the product being this unique value.) Thus, if a distinguisher can tell apart obfuscations of two programs BP_0, BP_1, there must exist some input on which they produce different outputs. The above intuitions can indeed be formalized w.r.t. *generic attackers* (that only operate on the encodings in a legal way, respecting the set restrictions), relying on arguments from [BR13, BGK+13]. However,

[3] Sample C_1 as the obfuscation of a "puncturable" PRF f_s (as in [SW13]) with the seed s hard-coded, and C_2 as the obfuscation of a "punctured" version of f_s where a random input x has been changed to a random output y. With high probability, C_1 and C_2 differ on x, yet following the hybrid argument in [SW13] it is easy to see that the circuits cannot be distinguished.

[4] Although we have not verified all the details, to achieve indistinguishability obfuscation, it seems that we could have also relied on the original encoding methods, but using straddling sets simplifies the analysis.

[5] The encodings, however, still permit an attacker to add elements within matrices.

although security w.r.t. generic attackers will be useful to us (as we shall see shortly), we are interested in proving security w.r.t. *all polynomial-size attackers*.

Towards this, we will add an additional program transformation steps before the Rand and Encode steps. Roughly speaking, we would like to have a method $\mathsf{Merge}(BP_0, BP_1, b)$ that "merges" BP_0 and BP_1 into a single branching program that evaluates BP_b; additionally, we require that $\mathsf{Merge}(BP_0, BP_1, 0)$ and $\mathsf{Merge}(BP_0, BP_1, 1)$ only differ in a constant (four will suffice) number of matrices. We achieve this merge procedure by connecting together BP_0, BP_1 into a branching program of double width and adding two "switch" matrices in the beginning and the end, determining if we should go "up" or "down". Thus, to switch between $\mathsf{Merge}(BP_0, BP_1, 0)$ (which is functionally equivalent to BP_0) and $\mathsf{Merge}(BP_0, BP_1, 1)$ (which is functionally equivalent to BP_1) we just need to switch the "switch matrices". Our candidate obfuscator is now defined as $i\mathcal{O}(B) = \mathsf{Encode}(\mathsf{Rand}(\mathsf{Merge}(BP, I, 0)))$, where I is simply a "dummy" program of the same size as BP.[6]

The idea behind the merge procedure is that to prove that obfuscations of two programs BP_0, BP_1 are indistinguishable, we can come up with a sequence of hybrid experiments that start with $i\mathcal{O}(BP_0)$ and end with $i\mathcal{O}(BP_1)$, but between any two hybrid only change a constant number of encodings, and thus intuitively we may rely on semantic security of multilinear encodings to formalize the above intuitions. At a high level, our strategy will be to matrix-by-matrix, replace the dummy branching program in the obfuscation of BP_0 with the branching program for BP_1. Once the entire dummy branching program has been replaced by BP_1, we flip the "switch" so that the composite branching program now computes the branching program for BP_1. We then replace the branching program for BP_0 with BP_1, matrix by matrix, so that we have two copies of the branching program for BP_1. We now flip the "switch" again, and finally restore the dummy branching program, so that we end up with one copy of BP_1 and one copy of the dummy, which is now a valid obfuscation of BP_1. In this way, we transition from an obfuscation of BP_0 to an obfuscation of BP_1, while only changing a small piece of the obfuscation in each step.

More precisely, consider the following sequence of hybrids.

- We start off with $i\mathcal{O}(BP_0) = \mathsf{Enc}(\mathsf{Rand}(\mathsf{Merge}(BP_0, I, 0)))$
- We consider a sequence of hybrids where we gradually change the dummy program I to become BP_1; that is, we consider $\mathsf{Encode}(\mathsf{Rand}(\mathsf{Merge}(BP_0, BP', 0)))$, where BP' is "step-wise" being populated with elements from BP_1.
- We reach $\mathsf{Encode}(\mathsf{Rand}(\mathsf{Merge}(BP_0, BP_1, 0)))$.
- We turn the "switch" : $\mathsf{Encode}(\mathsf{Rand}(\mathsf{Merge}(BP_0, BP_1, 1)))$.
- We consider a sequence of hybrids where we gradually change the BP_0 to become BP_1; that is, we consider $\mathsf{Encode}(\mathsf{Rand}(\mathsf{Merge}(BP', BP_1, 1)))$, where BP' is "step-wise" being populated with elements from BP_1.
- We reach $\mathsf{Encode}(\mathsf{Rand}(\mathsf{Merge}(BP_1, BP_1, 1)))$.
- We turn the "switch" back: $\mathsf{Encode}(\mathsf{Rand}(\mathsf{Merge}(BP_1, BP_1, 0)))$.
- We consider a sequence of hybrids where we gradually change the second BP_1 to become I; that is, we consider $\mathsf{Encode}(\mathsf{Rand}(\mathsf{Merge}(BP_1, BP', 0)))$, where BP' is "step-wise" being populated with elements from I.
- We reach $\mathsf{Encode}(\mathsf{Rand}(\mathsf{Merge}(BP_1, I, 0))) = i\mathcal{O}(BP_1)$.

[6]This description oversimplifies a bit. Formally, the Rand step needs to depends on the field size used in the Encode steps, and thus in our formal treatment we combine these two steps together.

By construction we have that if BP_0 and BP_1 are functionally equivalent, then so will all the hybrid programs–the key point is that we only "morph" between two branching programs on the "inactive" part of the merged branching program. Furthermore, by construction, between any two hybrids we only change a constant number of elements. Thus, if some distinguisher can tell apart $i\mathcal{O}(BP_0)$ and $i\mathcal{O}(BP_1)$, it must be able to tell apart two consecutive hybrids. But, by semantic security it then follows that some *algebraic* attacker can tell apart the encodings in the two hybrids. Roughly speaking, we can now rely on indistinguishable security of the basic obfuscator w.r.t. to just *generic* attackers to complete the argument.

There is a catch with the final step though. Recall that to rely on Kilian's simulation argument it was crucial that there are unique accept and reject matrices. For our "merged" programs, this is no longer the case (the output matrix is also a function of the second "dummy" program). We overcome this issue by noting that the first column of the output matrix actually is unique, and this is all we need to determine the output of the branching program. Consequently it suffices to release encodings of the *just* first column (as opposed to the whole matrices) of the last matrix pair in the branching program, and we can still determine the output of the branching program. As we show, for such a modified scheme, we can also simulate the (randomized) matrices along an "input-path" given just the first column of the output matrix. This concludes the description of our indistinguishability obfuscator.

2 Preliminaries

Let $\mathbb{N}$ denote the set of positive integers, and $[n]$ denote the set $\{1, 2, \ldots, n\}$. Let $\mathbb{Z}$ denote the integers, and $\mathbb{Z}_p$ the integers modulo p. Given a string x, we let $x[i]$, or equivalently x_i, denote the i-th bit of x. For a matrix M, we let $M[i, j]$ denote the entry of M in the ith row and jth column. We use $\mathbf{e}_k$ to denote the vector that is 1 in position k, and 0 in all other positions. The length of $\mathbf{e}_k$ is generally clear from the context. We use $I_{w\times w}$ to denote the identity matrix with dimension $w \times w$.

By a probabilistic algorithm we mean a Turing machine that receives an auxiliary random tape as input. If M is a probabilistic algorithm, then for any input x, $M(x)$ represents the distribution of outputs of $M(x)$ when the random tape is chosen uniformly. An oracle algorithm M^O is a machine M that gets oracle access to another machine O, that is, it can access O's functionality as a black-box.

By $x \leftarrow S$, we denote an element x is sampled from a distribution S. If F is a finite set, then $x \leftarrow F$ means x is sampled uniformly from the set F. To denote the ordered sequence in which the experiments happen we use semicolon, e.g. $(x \leftarrow S; (y, z) \leftarrow A(x))$. Using this notation we can describe probability of events. For example, if $p(\cdot, \cdot)$ denotes a predicate, then $\Pr[x \leftarrow S; (y, z) \leftarrow A(x) : p(y, z)]$ is the probability that the predicate $p(y, z)$ is true in the ordered sequence of experiments $(x \leftarrow S; (y, z) \leftarrow A(x))$. The notation $\{(x \leftarrow S; (y, z) \leftarrow A(x) : (y, z))\}$ denotes the resulting probability distribution $\{(y, z)\}$ generated by the ordered sequence of experiments $(x \leftarrow S; (y, z) \leftarrow A(x))$. We define the support of a distribution $\mathsf{supp}(S)$ to be $\{y : \Pr[x \leftarrow S : x = y] > 0\}$.

2.1 Obfuscation

We recall the definition of indistinguishability obfuscation due to [BGI+01].

Definition 1 (Indistinguishability Obfuscator)**.** *A uniform PPT machine $i\mathcal{O}$ is an* indistinguishability obfuscator *for a class of circuits $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ if the following conditions are satisfied*

- **Correctness:** *There exists a negligible function ε such that for every $n \in \mathbb{N}$, for all $C \in \mathcal{C}_n$, for all inputs $x \in \{0, 1\}^n$, we have*

$$\Pr[C' \leftarrow i\mathcal{O}(C_n) : C'(x) = C_n(x)] = 1 - \varepsilon(n).$$

- **Security:** *For every pair of circuit ensembles $\{C_n^0\}_{n\in\mathbb{N}}$ and $\{C_n^1\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, for every pair of circuits $C_n^0, C_n^1 \in \mathcal{C}_n$ such that $C_n^0(x) = C_n^1(x)$ for all $x \in \{0,1\}^n$ the following holds: For every nuPPT adversary A there exists a negligible function ε such that for all $n \in \mathbb{N}$,*

$$|Pr[C' \leftarrow i\mathcal{O}(C_n^0) : A(C') = 1] - Pr[C' \leftarrow i\mathcal{O}(C_n^1) : A(C') = 1]| \leq \varepsilon(n)$$

2.2 Graded Encoding Schemes

We proceed to defining graded (multilinear) encoding schemes, originally introduced by Garg, Gentry and Halevi [GGH13a]. Just as [BR13, BGK$^+$13], we here rely on "set-based" graded encoding; these were originally called "generalized" graded encodings in [GGH13a]. Following [GGH$^+$13b, BGK$^+$13] and the notion of "multi-linear jigsaw puzzles" from [GGH$^+$13b], we additionally enable anyone with the secret parameter to encode *any* elements (as opposed to just *random* elements as in [GGH13a]).

Definition 2 ((k, R)-Graded Encoding Scheme). *A (k, R)-graded encoding scheme for $k \in \mathbb{N}$ and ring R is a collection of sets $\{E_S^\alpha : \alpha \in R, S \subseteq [k]\}$ with the following properties*

- *For every $S \subseteq [k]$ the sets $\{E_S^\alpha : a \in R\}$ are disjoint.*
- *There are associative binary operations $\oplus$ and $\ominus$ such that for every $\alpha_1, \alpha_2 \in R$, $S \subseteq [k]$, $u_1 \in E_S^{\alpha_1}$ and $u_2 \in E_S^{\alpha_2}$ it holds that $u_1 \oplus u_2 \in E_S^{\alpha_1+\alpha_2}$ and $u_1 \ominus u_2 \in E_S^{\alpha_1-\alpha_2}$ where '$+$' and '$-$' are the addition and subtraction operations in R.*
- *There is an associative binary operation $\otimes$ such that for every $\alpha_1, \alpha_2 \in R$, $S_1, S_2 \subseteq [k]$ such that $S_1 \cap S_2 = \emptyset$, $u_1 \in E_{S_1}^{\alpha_1}$ and $u_2 \in E_{S_2}^{\alpha_2}$ it holds that $u_1 \otimes u_2 \in E_{S_1 \cup S_2}^{\alpha_1 \cdot \alpha_2}$ where '$\cdot$' is multiplication in R.*

Definition 3 (Graded Encoded Scheme). *A graded encoding scheme $\mathcal{E}$ is associated with a tuple of PPT algorithms,* $(\mathsf{InstGen}_\mathcal{E}, \mathsf{Enc}_\mathcal{E}, \mathsf{Add}_\mathcal{E}, \mathsf{Neg}_\mathcal{E}, \mathsf{Mult}_\mathcal{E}, \mathsf{isZero})_\mathcal{E}$ *which behave as follows:*

- *Instance Generation:* $\mathsf{InstGen}_\mathcal{E}$ *takes as input the security parameter 1^n and multi-linearity parameter 1^k, and outputs secret parameters* sp *and public parameters* pp *which describe a (k, R)-graded encoding scheme $\{E_S^\alpha : \alpha \in R, S \subseteq [k]\}$. We refer to E_S^α as the set of encodings of the pair (α, S). In this work we consider graded encoding schemes where R is $\mathbb{Z}_p$ and p is a prime exponential in n.*
- *Encoding:* $\mathsf{Enc}_\mathcal{E}$ *takes as input the secret parameters* sp*, an element $\alpha \in R$ and set $S \subseteq [k]$, and outputs a random encoding of the pair (α, S).*
- *Addition:* $\mathsf{Add}_\mathcal{E}$ *takes as input the public parameters* pp *and encodings $u_1 \in E_{S_1}^{\alpha_1}, u_2 \in E_{S_2}^{\alpha_2}$, and outputs an encoding of the pair $(\alpha_1 + \alpha_2, S)$ if $S_1 = S_2 = S$ and outputs $\perp$ otherwise.*
- *Negation:* $\mathsf{Neg}_\mathcal{E}$ *takes as input the public parameters* pp *and encodings $u_1 \in E_{S_1}^{\alpha_1}, u_2 \in E_{S_2}^{\alpha_2}$, and outputs an encoding of the pair $(\alpha_1 - \alpha_2, S)$ if $S_1 = S_2 = S$ and outputs $\perp$ otherwise.*
- *Multiplication:* $\mathsf{Mult}_\mathcal{E}$ *takes as input the the public parameters* pp *and encodings $u_1 \in E_{S_1}^{\alpha_1}, u_2 \in E_{S_2}^{\alpha_2}$, and outputs an encoding of the pair $(\alpha_1 \cdot \alpha_2, S_1 \cup S_2)$ if $S_1 \cap S_2 = \emptyset$ and outputs $\perp$ otherwise.*
- *Zero testing:* $\mathsf{isZero}_\mathcal{E}$ *takes as input the public parameters* pp *and an encoding $u \in E_S(\alpha)$, and outputs 1 if and only if $\alpha = 0$ and S is the universe set $[k]$.*

Whenever it is clear from the context, to simplify notation we drop the subscript $\mathcal{E}$ when we refer to the above procedures (and simply call them $\mathsf{InstGen}, \mathsf{Enc}, \ldots$*).*

Note that the above procedures allow algebraic operations on the encodings in a restricted way. Given the public parameters and encodings made under the sets $\vec{S}$, one can only perform algebraic operations that are allowed by the structure of the sets in $\vec{S}$. We call such operations $\vec{S}$-respecting and formalize this notion as follows:

Definition 4 (Set Respecting Arithmetic Circuits)**.** *For any ring R, $k \in \mathbb{N}$ and $\vec{S} \in (2^{[k]})^n$, we say that an arithmetic circuit C (i.e. gates perform only ring operations $\{+, -, \cdot\}$) of input size n is $\vec{S}$-*respecting *if it holds that*

- *We tag every input wire of C with the corresponding set in $\vec{S}$. The i^{th} input wire is tagged with $\vec{S}[i]$.*
- *For every $+$ and $-$ gate in C, if the tags of the two input wires are the same set S then the output wire of the gate is tagged with S. Otherwise the output wire is tagged with $\perp$.*
- *For every $\cdot$ gate in C, if the tags of the two input wires are sets S_1 and S_2 and $S_1 \cap S_2 = \emptyset$ then the output wire of the gate is tagged with $S_1 \cup S_2$. Otherwise the output wire is tagged with $\perp$.*
- *It holds that the output wire is tagged with the universe set $[k]$.*

The following lemma is a simple corollary of the efficient procedures described in Definition 3. It states that given the public parameters and some encodings made under the sets $\vec{S}$, one can efficiently zero test the result of any $\vec{S}$ respecting arithmetic circuit on the elements underlying the encodings.

Lemma 2 (Correctness)**.** *Let $\mathcal{E}$ be a graded encoding scheme. There exists a PPT* Eval *such that for any $k, n, m \in \mathbb{N}$,* Eval *takes as input the public parameters $\mathsf{pp} \in \mathsf{InstGen}(1^n, 1^k)$ that describe a (k, R)-graded encoding scheme, a sequence of encodings of some ring elements under some sets $\{u_i\}_{i=1}^m$ where $u_i \in E_{S_i}^{\alpha_i}$, α_i is a ring element and $S_i \subseteq [k]$, and any $\vec{S} = \{S_i\}_{i=1}^m$-respecting arithmetic circuit C, and outputs 1 if and only if $C(\{\alpha_i\}_{i=1}^m) = 0$.*

2.3 Branching programs for NC^1

Definition 5 (Oblivious Matrix Branching Program)**.** *A* branching program *of width w and length m for n-bit inputs is given by a sequence:*

$$BP = \{\mathsf{inp}(i), B_{i,0}, B_{i,1})\}_{i=1}^m,$$

where each $B_{i,b}$ is a permutation matrix in $\{0,1\}^{w \times w}$ and $\mathsf{inp}(i) \in [n]$ is the input bit position examined in step i. Then the output of the branching program on input $x \in \{0,1\}^n$ is as follows:

$$BP(x) \overset{def}{=} \begin{cases} 1, & \text{if } \left(\prod_{i=1}^m B_{i,x[\mathsf{inp}(i)]}\right) \cdot \mathbf{e}_1 = \mathbf{e}_1. \\ 0, & \text{otherwise} \end{cases}$$

The branching program is said to be oblivious *if $\mathsf{inp} : [m] \to [n]$ is a fixed function, independent of the function being evaluated. The branching program is said to have fixed accept and reject matrices $\mathsf{P}_{\mathsf{accept}}$ and $\mathsf{P}_{\mathsf{reject}}$ if, for all $x \in \{0,1\}^n$,*

$$\prod_{i=1}^m B_{i,x[\mathsf{inp}(i)]} = \begin{cases} \mathsf{P}_{\mathsf{accept}} & \text{when } BP(x) = 1 \\ \mathsf{P}_{\mathsf{reject}} & \text{when } BP(x) = 0 \end{cases}$$

In other words, the branching program accepts x whenever $\prod_{i=1}^m B_{i,x[\mathsf{inp}(i)]}$ is a matrix for a permutation that fixes the first element, and rejects x otherwise.
In particular, we still have the following theorem due to Barrington:

Theorem 3. *([Bar86]) For any depth d and input length n, there exists a length $m = 4^d$, a labeling function $\mathsf{inp} : [m] \to [n]$, an accepting permutation $\mathsf{P}_{\mathsf{accept}}$ with $\mathsf{P}_{\mathsf{accept}} \cdot \mathbf{e}_1 = \mathbf{e}_1$, and a rejecting permutation $\mathsf{P}_{\mathsf{reject}}$ with $\mathsf{P}_{\mathsf{reject}} \cdot \mathbf{e}_1 = \mathbf{e}_k$ where $k \neq 1$ such that, for every fan-in 2 boolean circuit C of depth d and n input bits, there exists an oblivious matrix branching program $BP = \{\mathsf{inp}(i), B_{i,0}, B_{i,1}\}_{i=1}^m$, of width 5 and length m that computes the same function as the circuit C.*

In particular, every circuit in NC^1 has a polynomial length branching program of width 5. Further, two circuits of the same depth d will have fixed accepting and rejecting permutations $\mathsf{P}_{\mathsf{accept}}$ and $\mathsf{P}_{\mathsf{reject}}$, and a fixed labelling function $\mathsf{inp} : [m] \to [n]$.

3 Semantically Secure Graded Encoding Schemes

In this section we define what it means for a graded encoding scheme to be semantically secure. Intuitively, we want to say that such an encoding scheme is like semantically secure encryption; the encodings of any two messages m_0 and m_1 should be indistinguishable. However, as previously mentioned, graded encoding schemes enable certain "set-restricted" computations on the encodings, and zero-testing. Given this ability it may be trivial to distinguish encodings of m_0 and m_1 given some auxiliary encodings of additional elements. Rather, we require that for every $S, \vec{T}$ pair, every "valid" distribution D over $m_0, m_1, \vec{z}$—where a distribution D is valid if no algebraic attacker, even computionally unbounded ones, obeying the set-restrictions, can distinguish whether it gets access to $\{\mathsf{Enc}(m_0, S), \mathsf{Enc}(\vec{z}, \vec{T})\}$ or $\{\mathsf{Enc}(m_1, S), \mathsf{Enc}(\vec{z}, \vec{T})\}$—we have that encodings $\{\mathsf{Enc}(m_0, S), \mathsf{Enc}(\vec{z}, \vec{T})\}$ and $\{\mathsf{Enc}(m_1, S), \mathsf{Enc}(\vec{z}, \vec{T})\}$ are indistinguishable. What this assumption guarantees is that no adversary can gain an advantage by acting non-algebraically.

As mentioned before, we focus on a *constant*-message settings, where m_0,m_1, and S are replaced by *constant-length* vectors $\vec{m_0}, \vec{m_1}, \vec{S}$. On the other hand, for our purposes, it will suffice to consider an *entropic* notion of semantic security, where we only require security of the graded encoded scheme to hold as long as D samples $\vec{m}_0, \vec{m}_1$ and $\vec{z}$ with some high-entropy.

We start by formally defining an *algebraic adversary.* Such an adversary, when given a set of encodings, is restricted to only the public efficient procedures of the graded encoding scheme. That is, it can only homomorphically evaluate certain algebraic operations on the encoded elements (restricted by the sets under which the encodings are made), and can check whether an element encoded under the universe set is zero or not. We formalize this restriction by considering adversaries that interact with the following oracle.

Definition 6 (Oracle $\mathcal{M}$)**.** *Let $\mathcal{M}$ be an oracle which operates as follows:*

- *$\mathcal{M}$ gets as initial input a ring R, $k \in \mathbb{N}$ and list L of m pairs $\{(\alpha_i, S_i)\}_{i=1}^m$, $\alpha \in R$ and $S \subseteq [k]$.*
- *Every oracle query to $\mathcal{M}$ is an arithmetic circuit $C : R^m \to R$. When queried with C, $\mathcal{M}$ checks whether C is a $\vec{S}$-respecting arithmetic circuit where $\vec{S} = \{S_i\}_{i=1}^m$. If not, $\mathcal{M}$ outputs $\perp$. Otherwise, $\mathcal{M}$ computes C on $\{\alpha_i\}_{i=1}^m$ and outputs 1 if and only if the output of C is zero, and outputs 0 otherwise.*

We next formalize what it means for a distribution over $(\vec{m_0}, \vec{m_1}, \vec{z})$ to be *valid* with respect to the sets $(\vec{S}, \vec{T})$. Intuitively, we say that a distribution is valid if no algebraic adversary can distinguish the encodings of $(\vec{m_0}, \vec{z})$ and $(\vec{m_1}, \vec{z})$, under the sets $(\vec{S}, \vec{T})$. We define such a distribution through the notion of a $(\vec{S}, \vec{T})$-respecting message sampler.

Definition 7 (Respecting Message Sampler)**.** *Let $\mathcal{E}$ be a graded encoding scheme, $q(\cdot)$, $k(\cdot)$ and $\mu(\cdot)$ be polynomials, $c \in \mathbb{N}$ be a constant and $\{(\vec{S_n}, \vec{T_n})\}_{n\in\mathbb{N}}$ be an ensemble where $\vec{S_n}$ is a sequence of c sets and $\vec{T_n}$ is a sequence of $q(n)$ sets $\subseteq [k]$. We say that a nuPPT M is a $(q, k, c, \{(\vec{S_n}, \vec{T_n})\}_{n\in\mathbb{N}})$*-respecting message sampler *if*

- *M takes as input the public parameters of a graded encoding scheme* $\mathsf{pp} \in \mathsf{InstGen}(1^n, 1^{k(n)})$ *which describes a ring R and outputs*
  - *a pair of sequences of c ring elements, $\vec{m}_0$ and $\vec{m}_1$ and*
  - *a sequence of $q(n)$ ring elements $\vec{z}$.*
- *For every (computationally unbounded) oracle machine A that makes at most polynomially many oracle queries*[7] *(called the* algebraic adversary*) there exists a negligible function ε such that for every security parameter $n \in \mathbb{N}$,*

$$|Pr[(\mathsf{sp}, \mathsf{pp}) \leftarrow \mathsf{InstGen}(1^n, 1^{k(n)}), (\vec{m}_0, \vec{m}_1, \vec{z}) \leftarrow M(\mathsf{pp}) : A^{\mathcal{M}(\mathsf{pp}, \vec{p_0})}(1^n) = 1] -$$
$$Pr[(\mathsf{sp}, \mathsf{pp}) \leftarrow \mathsf{InstGen}(1^n, 1^{k(n)}), (\vec{m}_0, \vec{m}_1, \vec{z}) \leftarrow M(\mathsf{pp}) : A^{\mathcal{M}(\mathsf{pp}, \vec{p_1})}(1^n) = 1]| \leq \varepsilon(n)$$

where $\vec{p_b} = \{(m_b[i], S_i)\}_{i=1}^{c}, \{(z[i], T_i)\}_{i=1}^{q(n)}$.

We now define what it means for a graded encoding scheme to be semantically secure. Roughly speaking, we require that for any sets $(\vec{S}, \vec{T})$, and any $(\vec{S}, \vec{T})$-respecting message sampler, encodings of $(\vec{m}_0, \vec{z})$ and $(\vec{m}_1, \vec{z})$ under the sets $(\vec{S}, \vec{T})$ are indistinguishable, when $(\vec{m}_0, \vec{m}_1, \vec{z})$ is sampled by the message sampler.

Definition 8 (Semantic Security)**.** *We say a graded encoding scheme $\mathcal{E}$ is* semantically secure *if for every every polynomials $q(\cdot)$ and $k(\cdot)$, constant $c \in \mathbb{N}$, every ensemble $\{(\vec{S}_n, \vec{T}_n)\}_{n \in \mathbb{N}}$ where $\vec{S}_n \subseteq [k(n)]^c$ and $\vec{T}_n \subseteq [k(n)]^{q(n)}$, every $(q, k, c, \{\vec{S}_n\}_{n \in \mathbb{N}})$-respecting message sampler M and nuPPT adversary A, there exists a negligible function ϵ such that for every security parameter $n \in \mathbb{N}$,*

$$|Pr[\boldsymbol{Output}_0(q, k, c, A, M, n, (\vec{S}_n, \vec{T}_n)) = 1] - Pr[\boldsymbol{Output}_1(q, k, c, A, M, n, (\vec{S}_n, \vec{T}_n)) = 1]| \leq \epsilon(n)$$

where $\boldsymbol{Output}_b(q, k, c, A, M, n, (\vec{S}_n, \vec{T}_n))$ is A's output in the following game.

- *Run* $(\mathsf{sp}, \mathsf{pp}) \leftarrow \mathsf{InstGen}(1^n, 1^{k(n)})$.
- *M takes as input the public parameters* pp *and outputs $\vec{m}_0$, $\vec{m}_1$ and $\vec{z}$.*
- *Encode each element of $\vec{m}_b$ and $\vec{z}$ with the corresponding set in $\vec{S}_n$ and $\vec{T}_n$. That is, compute the following encodings*

$$\vec{u_b} \leftarrow \{\mathsf{Enc}(\mathsf{sp}, \vec{m}_0[i], \vec{S}_n[i])\}_{i=1}^{c}, \{\mathsf{Enc}(\mathsf{sp}, \vec{z}[i], \vec{T}_n[i])\}_{i=1}^{q(n)}$$

- *A takes as input $\vec{u_b}$ and outputs a bit $b' \in \{0, 1\}$*

4 Construction of an Indistinguishability Obfuscator

In this section, we describe our construction of an indistinguishability obfuscator $i\mathcal{O}$. We will prove its security in Section 5, based on the security notions defined above.

As in previous works [GGH+13b, BR13, BGK+13], the strategy for our construction will be to convert an NC^1 circuit into an oblivious matrix branching program, apply Kilian's randomization technique to the matrices, and then encode these matrices using the graded encoding scheme. The encoding will be using a so-called "straddling set system" (as in [BGK+13]) that will enforce that any arithmetic circuit operating on these encodings can be decomposed into a sum of terms such that each term can be expressed using only encodings that come from one branch of the branching program (more specifically,

[7]Our proofs work even if the algebraic adversary A makes only subexponentially many oracle queries.

from the path through the branching program corresponding to evaluating a particular input x to the branching program).

The biggest change from previous work is that before randomizing and encoding the branching program, we double its width by chaining a dummy branching program to it that computes the constant 1, and then add a branch at the very start that chooses whether to use the true program or the dummy, based on a "switch".

At a high level, to show indistinguishability of obfuscations of C_1 and C_2, our strategy will be to obfuscate the branching program for C_1 together with the dummy, and then, matrix by matrix, replace the dummy branching program with the branching program for C_2. Once the entire dummy branching program has been replaced by C_2, we flip the "switch" so that the composite branching program now computes the branching program for C_2. We then replace the branching program for C_1 with C_2, matrix by matrix, so that we have two copies of the branching program for C_2. We now flip the "switch" again, and finally restore the dummy branching program, so that we end up with one copy of C_2 and one copy of the dummy.

In this way, we transition from an obfuscation of C_1 to an obfuscation of C_2, while only changing a small piece of the obfuscation in each step, namely a single level of the underlying branching program. We will later show, in the following section, that each step of the transitions must be indistinguishable based on our hardness assumption. In particular, we show that no algebraic adversary can distinguish between two hybrids, and thus the two distributions should be computationally indistinguishable based on our assumption.

4.1 Merging Branching Programs

We now describe a method Merge for combining two branching programs together to create a composite branching program of double width, in a way that enables switching by changing only a small number of matrices.

Construction 1 (Merging branching programs)**.** *Let $BP_0 = \{\mathsf{inp}(i), B^0_{i,0}, B^0_{i,1}\}_{i=1}^m$ and $BP_1 = \{\mathsf{inp}(i), B^1_{i,0}, B^1_{i,1}\}_{i=1}^m$ be oblivious matrix branching programs, each of width w and length m for n input bits. (We assume that the same labelling function $\mathsf{inp} : [m] \to [n]$ is used for each of BP_0 and BP_1.)*
Define branching programs $\hat{BP}_0 = \{\mathsf{inp}'(i), \hat{B}^0_{i,0}, \hat{B}^0_{i,1}\}_{i=1}^{m+2}$ and $\hat{BP}_1 = \{\mathsf{inp}'(i), \hat{B}^1_{i,0}, \hat{B}^1_{i,1}\}_{i=1}^{m+2}$, each of width $2w$ and length $m+2$ on l input bits, where:

$$\mathsf{inp}'(i) \overset{def}{=} \begin{cases} 1, & \text{when } i = 1 \\ \mathsf{inp}(i-1), & \text{when } 2 \le i \le m+1 \\ 1, & \text{when } i = m+2 \end{cases}$$

and, for all levels except the first and the last, $\hat{BP}_0$ and $\hat{BP}_1$ agree, given by:

$$\hat{B}^0_{i,b} = \hat{B}^1_{i,b} \overset{def}{=} \begin{pmatrix} B^0_{(i-1),b} & 0 \\ 0 & B^1_{(i-1),b} \end{pmatrix} \quad \text{for all } 2 \le i \le m+1 \text{ and } b \in \{0,1\}$$

and the first and last levels are given by:

$$\hat{B}^0_{1,b} = \hat{B}^0_{m+2,b} = I_{2w\times 2w} \qquad \text{for } b \in \{0,1\}$$

$$\hat{B}^1_{1,b} = \hat{B}^1_{m+2,b} = \begin{pmatrix} 0 & I_{w\times w} \\ I_{w\times w} & 0 \end{pmatrix} \qquad \text{for } b \in \{0,1\}$$

We define Merge so that $\mathsf{Merge}(BP_0, BP_1, 0) = \hat{BP}_0$ and $\mathsf{Merge}(BP_0, BP_1, 1) = \hat{BP}_1$.

We will show that $\hat{BP}_0$ and $\hat{BP}_1$ are matrix branching programs that compute the same functions as BP_0 and BP_1 respectively, with the additional feature that $\hat{BP}_0$ and $\hat{BP}_1$ differ from each other in only two levels, namely the first and the last. Further, since inp' does not depend on the function being computed, $\hat{BP}_0$ and $\hat{BP}_1$ are *oblivious* matrix branching programs.

Accordingly, with respect to $\mathsf{Merge}(BP_0, BP_1, b)$ we will often use the phrase *active branching program* to refer to BP_b.

Claim 4. *For $BP_0 = \{\mathsf{inp}(i),\ B^0_{i,0}, B^0_{i,1}\}_{i=1}^m$ and $BP_1 = \{\mathsf{inp}(i), B^1_{i,0}, B^1_{i,1}\}_{i=1}^m$ each of width w and length m on n input bits, define $\hat{BP}_0$ and $\hat{BP}_1$ as above. Then, for each $b \in \{0,1\}$, $x \in \{0,1\}^n$,*

$$\prod_{i=1}^{m+2} \widehat{B}^b_{i,x[\mathsf{inp}'(i)]} = \begin{pmatrix} \prod_{i=1}^m B^b_{i,x[\mathsf{inp}(i)]} & 0 \\ 0 & \prod_{i=1}^m B^{1-b}_{i,x[\mathsf{inp}(i)]} \end{pmatrix}$$

Proof. We observe that $\hat{BP}_0$ and $\hat{BP}_1$ agree on each level except the first and last, that is,

$$\widehat{B}^0_{i,b} = \widehat{B}^1_{i,b} = \begin{pmatrix} B^0_{(i-1),b} & 0 \\ 0 & B^1_{(i-1),b} \end{pmatrix} \qquad \forall \quad i : 2 \le i \le m+1, \quad b \in \{0,1\}$$

Then we have, for any $x \in \{0,1\}^n$,

$$\begin{aligned}\prod_{i=2}^{m+1} \widehat{B}^0_{i,x[\mathsf{inp}'(i)]} = \prod_{i=2}^{m+1} \widehat{B}^1_{i,x[\mathsf{inp}'(i)]} &= \prod_{i=2}^{m+1} \begin{pmatrix} B^0_{(i-1),x[\mathsf{inp}'(i)]} & 0 \\ 0 & B^1_{(i-1),x[\mathsf{inp}'(i)]} \end{pmatrix} \\ &= \prod_{i=1}^{m} \begin{pmatrix} B^0_{i,x[\mathsf{inp}(i)]} & 0 \\ 0 & B^1_{i,x[\mathsf{inp}(i)]} \end{pmatrix} \\ &= \begin{pmatrix} \prod_{i=1}^m B^0_{i,x[\mathsf{inp}(i)]} & 0 \\ 0 & \prod_{i=1}^m B^1_{i,x[\mathsf{inp}(i)]} \end{pmatrix}\end{aligned}$$

Where the change of indices in the second step follows because $\mathsf{inp}'(i) = \mathsf{inp}(i-1)$ when $2 \le i \le m+1$. We now consider the two case for $b \in \{0,1\}$.

Case 1: (b = 0)

In this case,

$$\begin{aligned}\prod_{i=1}^{m+2} \widehat{B}^0_{i,x[\mathsf{inp}'(i)]} &= I_{2w\times 2w} \cdot \begin{pmatrix} \prod_{i=1}^m B^0_{i,x[\mathsf{inp}(i)]} & 0 \\ 0 & \prod_{i=1}^m B^1_{i,x[\mathsf{inp}(i)]} \end{pmatrix} \cdot I_{2w\times 2w} \\ &= \begin{pmatrix} \prod_{i=1}^m B^0_{i,x[\mathsf{inp}(i)]} & 0 \\ 0 & \prod_{i=1}^m B^1_{i,x[\mathsf{inp}(i)]} \end{pmatrix}\end{aligned}$$

as required.

Case 2: (b = 1)

In this case,

$$\begin{aligned}\prod_{i=1}^{m+2} \widehat{B}^1_{i,x[\mathsf{inp}'(i)]} &= \begin{pmatrix} 0 & I_{w\times w} \\ I_{w\times w} & 0 \end{pmatrix} \cdot \begin{pmatrix} \prod_{i=1}^m B^0_{i,x[\mathsf{inp}(i)]} & 0 \\ 0 & \prod_{i=1}^m B^1_{i,x[\mathsf{inp}(i)]} \end{pmatrix} \cdot \begin{pmatrix} 0 & I_{w\times w} \\ I_{w\times w} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \prod_{i=1}^m B^1_{i,x[\mathsf{inp}(i)]} \\ \prod_{i=1}^m B^0_{i,x[\mathsf{inp}(i)]} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & I_{w\times w} \\ I_{w\times w} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \prod_{i=1}^m B^1_{i,x[\mathsf{inp}(i)]} & 0 \\ 0 & \prod_{i=1}^m B^0_{i,x[\mathsf{inp}(i)]} \end{pmatrix}\end{aligned}$$

as required. □

Claim 5. *For all BP_0 and BP_1 each of width w and length m on n input bits, for each $b \in \{0,1\}$, for all $x \in \{0,1\}^n$,*

$$\mathsf{Merge}(BP_0, BP_1, b)(x) = BP_b(x)$$

Proof. Let $BP_0 = \{\mathsf{inp}(i), B^0_{i,0}, B^0_{i,1}\}_{i=1}^m$ and $BP_1 = \{\mathsf{inp}(i), B^1_{i,0}, B^1_{i,1}\}_{i=1}^m$. Define $\hat{BP}_0 = \mathsf{Merge}(BP_0, BP_1, 0)$ and $\hat{BP}_1 = \mathsf{Merge}(BP_0, BP_1, 1)$ as above. We observe that for any $x \in \{0,1\}^n$,

$$\begin{aligned}
&\mathsf{Merge}(BP_0, BP_1, b)(x) = 1 \\
\iff &(\prod_{i=1}^{m+2} \widehat{B}^b_{i,x[\mathsf{inp}'(i)]}) \cdot \mathbf{e}_1 = \mathbf{e}_1 \\
\iff &\begin{pmatrix} \prod_{i=1}^m B^b_{i,x[\mathsf{inp}(i)]} & 0 \\ 0 & \prod_{i=1}^m B^{1-b}_{i,x[\mathsf{inp}(i)]} \end{pmatrix} \cdot \mathbf{e}_1 = \mathbf{e}_1 && \text{(from Claim 4)} \\
\iff &(\prod_{i=1}^{m} B^b_{i,x[\mathsf{inp}(i)]}) \cdot \mathbf{e}_1 = \mathbf{e}_1 \\
\iff &BP_b(x) = 1
\end{aligned}$$

Thus $\mathsf{Merge}(BP_0, BP_1, b)(x) = BP_b(x)$. □

The following claim illustrates some useful properties of the Merge procedure that we will use later. Firstly it notes that changing the bit Merge gets as input changes only the "switch" matrices in the first and last level of the program Merge outputs. Secondly, changing one level of a program Merge gets as input changes the output program in one level only. Finally, the first column of the output matrix of the widened program output by Merge depends only on the first column of the output matrix of the active program. The claim follows by observing the definition of Merge.

Claim 6. *Let BP_0 andBP_1 be length m, width w branching programs, with input length n.*

- $\mathsf{Merge}(BP_0, BP_1, 0)$ *and* $\mathsf{Merge}(BP_0, BP_1, 1)$ *differ in only 4 matrices, the matrices corresponding to the first and last level.*
- *Let BP_1' be a length m branching program that differs from BP_1 in only the i^{th} level for some $i \in [m]$. Then for both $b \in \{0,1\}$,* $\mathsf{Merge}(BP_0, BP_1, b)$ *and* $\mathsf{Merge}(BP_0, BP_1', b)$ *also differ only in the i^{th} level. A similar statement holds for branching programs BP_0' that differ from BP_0 in only one level.*
- *For any $b \in \{0,1\}$, let $BP = \mathsf{Merge}(BP_0, BP_1, b)$, and ${\mathsf{P}_{\mathsf{out}}}^{BP}(\cdot)$ and ${\mathsf{P}_{\mathsf{out}}}^{BP_b}(\cdot)$ be the functions computing the output matrices on a given input for BP and BP_b respectively. Then for every input $x \in \{0,1\}^n$,*

  $$\mathsf{col}_1({\mathsf{P}_{\mathsf{out}}}^{BP}(x)) = \mathsf{extend}(\mathsf{col}_1({\mathsf{P}_{\mathsf{out}}}^{BP_b}(x)))$$

  where extend *extends a length w vector by appending w zeroes to the end.*

4.2 Randomizing Branching Programs

We now describe Kilian's randomization technique [Kil88] for a branching program, adapted to our setting, by defining a process Rand that randomizes the matrices of a branching program BP. We will decompose the randomization into two parts, Rand^B and Rand^α, defined below, and define Rand as their composition.

Definition 9 (Rand^B)**.** *Let $BP = \{\mathsf{inp}(i), B_{i,0}, B_{i,1}\}_{i=1}^m$ be a branching program of width w and length m, with length-n inputs. Let p be a prime exponential in n. Then the process $\mathsf{Rand}^B(BP, p)$ samples m random invertible matrices $R_1, R_2, \ldots, R_m \in Z_p^{w\times w}$ uniformly and independently, and computes*

$$\tilde{B}_{i,b} = R_{(i-1)} \cdot B_{i,b} \cdot R_i^{-1} \quad \textit{for every } i \in [m], \textit{ and } b \in \{0,1\}$$

where R_0 is defined as $I_{w\times w}$, and

$$\mathbf{t} = R_m \cdot \mathbf{e}_1$$

Rand^B *then outputs*

$$(\{\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}, p)$$

Definition 10 (Rand^α)**.** *Let $(\{\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}, p)$ be the output of $\mathsf{Rand}^B(BP, p)$ as defined above. On this input, $\mathsf{Rand}^\alpha(\{\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, p)$ samples $2m$ non-zero scalars $\{\alpha_{i,b} \in \mathbb{Z}_p : i \in [m], b \in \{0,1\}\}$ uniformly and independently, and outputs*

$$(\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t})$$

Definition 11 (Rand)**.** *Let $BP = \{\mathsf{inp}(i), B_{i,0}, B_{i,1}\}_{i=1}^m$ be a branching program of width w and length m, with length-n inputs. Let p be a prime exponential in n. Then we define $\mathsf{Rand}(BP, p)$ to be:*

$$\begin{aligned}\mathsf{Rand}(BP, p) &= (\mathsf{Rand}^\alpha(\mathsf{Rand}^B(BP, p)))\\ &= (\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t})\end{aligned}$$

Where $(\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t})$ are as computed in the executions of Rand^α and Rand^B.

Execution of a randomized branching program: To compute $BP(x)$ from the output of $\mathsf{Rand}(BP, p)$, given some input labelling function $\mathsf{inp} : [m] \to [n]$, and $x \in \{0,1\}^n$, we compute

$$\mathsf{Out}(x) = (\prod_{i=1}^m \alpha_{i,x[\mathsf{inp}(i)]} \cdot \tilde{B}_{i,x[\mathsf{inp}(i)]}) \cdot \mathbf{t}$$

Where $\mathsf{Out} \in Z_P^w$ is a $w \times 1$ vector. The intermediate multiplications cause each R_i^{-1} to cancel each R_i, and $R_0 = I_{w\times w}$, so the above computation can also be expressed as:

$$\mathsf{Out}(x) = (\prod_{i=1}^m \alpha_{i,x[\mathsf{inp}(i)]} \cdot B_{i,x[\mathsf{inp}(i)]}) \cdot \mathbf{e}_1$$

When $BP(x) = 1$, we have that

$$\prod_{i=1}^m \alpha_{i,x[\mathsf{inp}(i)]} \cdot B_{i,x[\mathsf{inp}(i)]} \cdot \mathbf{e}_1 = (\prod_{i=1}^m \alpha_{i,x[\mathsf{inp}(i)]}) \cdot \mathbf{e}_1$$

When $BP(x) = 0$, we have that

$$\prod_{i=1}^m \alpha_{i,x[\mathsf{inp}(i)]} \cdot B_{i,x[\mathsf{inp}(i)]} \cdot \mathbf{e}_1 = (\prod_{i=1}^m \alpha_{i,x[\mathsf{inp}(i)]}) \cdot \mathbf{e}_k$$

for $k \neq 1$. Hence, to compute $BP(x)$, we compute $\mathsf{Out}(x)$ and output 0 if $\mathsf{Out}(x)[1] = 0$, and 1 otherwise.

Simulating a randomized branching program: Previous works ([BGK$^+$13, BR13]) followed [Kil88] to show how to simulate the distribution of any single path corresponding to an input x using just $BP(x)$. However, the simulator required that branching programs have unique accept and reject matrices $\mathsf{P}_{\mathsf{accept}}$ and $\mathsf{P}_{\mathsf{reject}}$.

We would also like a theorem, along the lines of [Kil88], that shows that any single path through a randomized branching program BP corresponding to an input x can be simulated knowing just the accept/reject behavior of BP on x (i.e. by knowing whether $BP(x) = 1$).

In our setting, however, branching programs only meet the relaxed requirement that the output matrix $\mathsf{P}_{\mathsf{out}}(x)$ computed by evaluating BP on input x satisfies $\mathsf{P}_{\mathsf{out}}(x) \cdot \mathbf{e}_1 = \mathbf{e}_1 \iff BP(x) = 1$. There can thus be multiple accept and reject matrices, and the particular accept or reject matrix output by BP can depend both on x and on the specific implementation of BP (and not simply its accept/reject behavior). In contrast, in previous works, because $\mathsf{P}_{\mathsf{accept}}$ and $\mathsf{P}_{\mathsf{reject}}$ were unique, knowing just the accept/reject behavior of BP on x also determines $\mathsf{P}_{\mathsf{out}}(x)$.

What we will show is that, for the particular randomization scheme chosen above, we can simulate any single path through a randomized branching program BP corresponding to an input x without knowing the exact accept/reject matrix $\mathsf{P}_{\mathsf{out}}(x)$, but rather just knowing the first column $\mathsf{p}_{\mathsf{out}}(x) = \mathsf{col}_1(\mathsf{P}_{\mathsf{out}}(x))$.

This will be sufficient for our applications, because the class of branching programs we randomize will have the property that there are fixed columns $\mathsf{p}_{\mathsf{accept}}$ and $\mathsf{p}_{\mathsf{reject}} \in \mathbb{Z}_p^w$ such that for all $x \in \{0,1\}^n$, if $BP(x) = 1$ then $\mathsf{col}_1(\mathsf{P}_{\mathsf{out}}(x)) = \mathsf{p}_{\mathsf{accept}}$, and if $BP(x) = 0$ then $\mathsf{col}_1(\mathsf{P}_{\mathsf{out}}(x)) = \mathsf{p}_{\mathsf{reject}}$. In the case of such programs, $\mathsf{col}_1(\mathsf{P}_{\mathsf{out}}(x))$ is determined solely by $BP(x)$, and not the particular implementation of BP. Thus, for these programs, we can simulate given only $BP(x)$.

Before we show this theorem, we define notation for a path through a branching program corresponding to an input x.

Definition 12 (proj_x). *Let* $\mathsf{inp} : [m] \to [n]$ *be an input labelling function, and, for any* $x \in \{0,1\}^n$, *define* proj_x, *relative to* inp, *such that for any branching program* BP *with labelling function* inp, *for any prime* $p \in \mathbb{N}$, *and for any* $(\{\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}) \leftarrow \mathsf{Rand}^B(BP, p)$

$$\mathsf{proj}_x(\{\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}) = (\{\tilde{B}_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]}, \mathbf{t}),$$

that is, proj_x *selects the elements from* $(\{\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t})$ *used when evaluating input* x.

We now show a version of Kilian's theorem, adapted to our construction.

Theorem 7. *There exists an efficient simulator* KSim *such that the following holds. Let* $BP = \{\mathsf{inp}(i), B_{i,0}, B_{i,1}\}_{i\in[m]}$ *be a width-*w *branching program of length* m *on* n *bit inputs, and* p *a prime exponential in* n. *Let* $x \in \{0,1\}^n$ *be an input to* BP, *and let* $b_i = x[\mathsf{inp}(i)]$ *for each* $i \in [m]$. *Let* $\mathsf{P}_{\mathsf{out}}(x) = \prod_{i=1}^m B_{i,b_i}$ *denote the matrix obtained by evaluating* BP *on* x, *and let* $\mathsf{p}_{\mathsf{out}}(x) = \mathsf{col}_1(\mathsf{P}_{\mathsf{out}}(x))$ *denote the first column of this output. Let* $\mathsf{proj}_x(\mathsf{Rand}^B(BP, p))$ *be defined respecting the labelling function* inp. *Then* $\mathsf{KSim}(1^m, p, \mathsf{p}_{\mathsf{out}}(x))$ *is identically distributed to* $\mathsf{proj}_x(\mathsf{Rand}^B(BP, p))$.

Proof. We begin by defining $\mathsf{KSim}(1^n, p, BP(x))$ as follows:

- For each i, KSim selects $\tilde{B}_{i,b_i}$ to be a uniformly random invertible matrix in $Z_p^{w\times w}$.
- KSim selects $\mathbf{t} \in \mathbb{Z}_p^w$ solving

$$(\prod_{i\in[m]} \tilde{B}_{i,b_i}) \cdot \mathbf{t} = \mathsf{p}_{\mathsf{out}}(x) \tag{1}$$

where $b_i = x[\mathsf{inp}(i)]$ for each i.

- KSim outputs $\{\{\tilde{B}_{i,b_i}\}_{i\in[m]}, \mathbf{t}\}$

We want to show that the distribution output by KSim matches the real distribution of $\{\{\tilde{B}_{i,b_i}\}_{i\in[m]}, \mathbf{t}\}$ in the output of $\mathsf{Rand}^B(BP, p)$. But from [Kil88], we have the following:

Claim 8. *The distribution of $\{\{\tilde{B}_{i,b_i}\}_{i\in[m]}, R_m\}$ can be exactly sampled given $\mathsf{P}_{\mathsf{out}}(x)$, by sampling $\{\tilde{B}_{i,b_i}\}_{i\in[m]}$, R_m to be uniformly random and independent invertible matrices in $\mathbb{Z}_p^{w\times w}$ subject to*

$$(\prod_{i\in[m]} \tilde{B}_{i,b_i}) \cdot R_m = \mathsf{P}_{\mathsf{out}}(x) \tag{2}$$

The above claim implies the following:

Claim 9. *The distribution of $\{\{\tilde{B}_{i,b_i}\}_{i\in[m]}, R_m\}$ can be sampled by independently choosing each $\tilde{B}_{i,b_i}$ uniform and invertible, and fixing R_m solving equation (2).*

Proof. This follows because for every choice of invertible $\tilde{B}_{i,b_i}$, there exists R_m solving equation (2) given by

$$R_m = (\prod_{i\in[m]} \tilde{B}_{i,b_i}))^{-1} \cdot \mathsf{P}_{\mathsf{out}}(x) \tag{3}$$

Further, every solution to equation (2) can be represented as invertible $\tilde{B}_{i,b_i}$, and an R_m solving equation (3). Thus choosing a random solution to equation (2) corresponds to independently choosing each $\tilde{B}_{i,b_i}$ uniformly and invertible, and fixing R_m solving equation (3). □

From the above argument, we have that the distribution of $\mathsf{proj}_x(\mathsf{Rand}(BP, p))$ is exactly the same as the distribution produced by independently choosing each $\tilde{B}_{i,b_i}$ uniform and invertible, fixing R_m solving equation (3), setting $\mathbf{t}$ to be the first column of R_m, and outputting $\{\{\tilde{B}_{i,b_i}\}_{i\in[m]}, \mathbf{t}\}$. But note that each column $\mathsf{col}_i(R_m), i \in [w]$ is the unique solution to

$$(\prod_{i\in[m]} \tilde{B}_{i,b_i}) \cdot \mathsf{col}_i(R_m) = \mathsf{col}_i(\mathsf{P}_{\mathsf{out}}(x))$$

Thus we have that each $\tilde{B}_{i,b_i}$ is independent, uniform, and invertible, and, using $i = 1$, $\mathbf{t}$ is the unique solution to

$$(\prod_{i\in[m]} \tilde{B}_{i,b_i}) \cdot \mathbf{t} = \mathsf{p}_{\mathsf{out}}(x)$$

and, in particular, that $\mathbf{t}$ is determined by *only* the first column of $\mathsf{P}_{\mathsf{out}}(x)$. Thus, we see that the distribution of $\mathsf{proj}_x(\mathsf{Rand}^B(BP, p))$ is exactly the same as that output by KSim. □

4.3 Choosing a Set System

In this section we will describe how to choose a collection of sets under which to encode a randomized branching program using the graded encoding scheme. Our selection of sets will closely follow [BGK+13], in that we use straddling set systems. However, one difference is that while they use dual input branching programs, we restrict our attention to single-input schemes.

We first define straddling set systems.

Definition 13 (Straddling Set Systems [BGK+13]). *A* straddling set system *with n entries is a collection of sets $\mathbb{S}_n = \{S_{i,b} : i \in [n], b \in \{0,1\}\}$ over a universe U, such that:*

$$\bigcup_{i\in[n]} S_{i,0} = \bigcup_{i\in[n]} S_{i,1} = U$$

and for every distinct non-empty sets $C, D \subseteq \mathbb{S}_n$, we have that if:

1. *(Disjoint Sets:) C contains only disjoint sets. D contains only disjoint sets.*
2. *(Collision:) $\bigcup_{S\in C} S = \bigcup_{S\in D} S$*

Then it must be that $\exists b \in \{0,1\}$ such that:

$$C = \{S_{j,b}\}_{j\in[n]} \quad , \quad D = \{S_{j,(1-b)}\}_{j\in[n]}$$

Informally, the guarantee provided by a straddling set system is that only way to exactly cover U using elements from $\mathbb{S}_n$ is to use either all sets $\{S_{i,0}\}_{i\in n}$ or all sets $\{S_{i,1}\}_{i\in n}$. [BGK+13] give a construction for straddling set systems, choosing U to be $[2n-1]$, each $S_{i,0}$ to be one of $\{1\}, \{2,3\}, \ldots, \{2n-2, 2n-1\}$, and each $S_{i,1}$ to be one of $\{1,2\}, \{3,4\}, \ldots, \{2n-1\}$. They further show that this construction is a straddling set system.

Theorem 10 (From Construction 1 in [BGK+13]). *For every $n \in N$, there exists a straddling set system $\mathbb{S}_n$ with n entries, over a universe U of $2n-1$ elements.*

We now define the process SetSystem which takes as input the length m of a branching program, the number of input bits n, and the input labelling function $\mathsf{inp} : [m] \to [n]$ for a branching program. SetSystem will output the collection of straddling set systems that we will use to encode any branching program of length m on n input bits, with labelling function inp.

Execution of SetSystem(m, n, inp):
We let n_j denote the number of levels that inspect the jth input bit in inp. That is,

$$n_j = |\{i \in [m] : \mathsf{inp}(i) = j\}|$$

For every $j \in [n]$, SetSystem chooses $\mathbb{S}^j$ to be a straddling set system with n_j entries over a set U_j, such that the sets $U_1, \ldots, U_n$ are disjoint. Let $U = \bigcup_{j\in[n]} U_j$. SetSystem then chooses S_t be a set disjoint from U. We associate the set system $\mathbb{S}^j$ with the j'th input bit of the branching program corresponding to inp. SetSystem then re-indexes the elements of $\mathbb{S}^j$ to match the steps of the branching program as described by inp, so that:

$$\mathbb{S}^j = \{S_{i,b} : \mathsf{inp}(i) = j, b \in \{0,1\}\}$$

By this indexing, we also have that $S_{i,b} \in \mathbb{S}^{\mathsf{inp}(i)}$ for every $i \in [m]$, for every $b \in \{0,1\}$.

Let $k = |U \cup S_t|$, and WLOG, assume that the U^js and S_t are disjoint subsets of $[k]$ (otherwise SetSystem relabels the elements to satisfy this property).

SetSystem then outputs

$$k, \quad \{S_{i,b}\}_{i\in[m], b\in\{0,1\}}, \quad S_t$$

4.4 Obfuscating Branching Programs

In this section, we will describe a process Obf that obfuscates a given branching program BP. This process will use Rand and SetSystem as subroutines. The output of Obf will be a randomized width-10 oblivious matrix branching program, encoded under the graded encoding scheme.

Description of Obf(BP) :

Input. Obf takes as input an oblivious permutation branching program $BP = \{\mathsf{inp}(i), B_{i,0}, B_{i,1}\}_{i=1}^m$ of width w and length m on n input bits.

Choosing sets. Obf runs $\mathsf{SetSystem}(m, n, \mathsf{inp})$ and receives $k, \{S_{i,b}\}_{i\in[m+2],b\in\{0,1\}}, S_t$.

Initializing the GES. Obf runs $\mathsf{InstGen}(1^n, 1^k)$, and receives secret parameters sp and public parameters pp which describe a (k, R)-graded encoding scheme. We assume the ring R is equal to $\mathbb{Z}_p$ for some p exponential in n.

Randomizing BP. Obf executes $\mathsf{Rand}(BP, p)$, and obtains its output, $\{\{\mathsf{inp}(i), \alpha_{i,0}\cdot\tilde{B}_{i,0}, \alpha_{i,1}\cdot\tilde{B}_{i,1}\}_{i\in[m]}, \mathbf{t}\}$

Output. Obf outputs:

$$\mathsf{pp}, \quad \{\mathsf{inp}(i), \quad \mathsf{Enc}(\mathsf{sp}, \alpha_{i,0}\cdot\tilde{B}_{i,0}, S_{i,0}), \quad \mathsf{Enc}(\mathsf{sp}, \alpha_{i,0}\cdot\tilde{B}_{i,0}, S_{i,1})\}_{i\in[m]}, \quad \mathsf{Enc}(\mathsf{sp}, \mathbf{t}, S_t)$$

We also define a generic version of Obf, which we refer to as GObf. It's output will be used to initialize an oracle $\mathcal{M}$ for the idealized version of the graded encoded scheme. $\mathsf{GObf}(BP)$ acts exactly as $\mathsf{Obf}(BP)$, except in the **Output** step, GObf outputs

$$\mathsf{pp}, \quad \{\mathsf{inp}(i), (\alpha_{i,0}\cdot\tilde{B}_{i,0}, S_{i,0}), (\alpha_{i,1}\cdot\tilde{B}_{i,1}, S_{i,1})\}_{i\in[m]}, \quad (\mathbf{t}, S_t)$$

4.5 Putting it all together: Obfuscating NC^1 circuits

We now define our indistinguishability obfuscator $i\mathcal{O}$ for NC^1, as follows:

Description of $i\mathcal{O}(C)$:

1. $i\mathcal{O}$ takes as input $C \in \mathsf{NC}^1$, a fan-in 2 circuit with depth d on n input bits. $i\mathcal{O}$ uses Barrington's Theorem to convert C into an oblivious width 5 permutation branching program $BP = \{\mathsf{inp}(i), B_{i,0}, B_{i,1}\}_{i=1}^m$ of length $m = 4^d$ on n input bits.

2. $i\mathcal{O}$ generates a dummy width-5 branching program $I = \{\mathsf{inp}(i), I_{5\times5}, I_{5\times5}\}_{i=1}^m$ of length m, where each permutation matrix at each level is the identity matrix. $i\mathcal{O}$ then computes $\hat{BP} = \mathsf{Merge}(BP, I, 0)$.

3. $i\mathcal{O}$ outputs $\mathsf{Obf}(\hat{BP})$, which yields the public parameter pp for the graded encoding scheme, together with the encoded branching program $\{\mathsf{inp}(i), \mathsf{Enc}(\alpha_{i,0}\cdot\tilde{B}_{i,0}, S_{i,0}), \mathsf{Enc}(\alpha_{i,1}\cdot\tilde{B}_{i,1}, S_{i,1})\}_{i\in[m+2]}, \mathsf{Enc}(\mathbf{t}, S_t)$.

Correctness of $i\mathcal{O}$: In order to compute the output of $C(x)$ given its obfuscation $i\mathcal{O}(C)$, we perform matrix multiplication on the encoded matrices using the functions Add and Mult of the graded encoding scheme. That is, letting $b_i = x[\mathsf{inp}(i)]$ for each $i \in [m+2]$, using the Eval function guaranteed by Lemma 2, we compute the encoding of

$$\mathsf{Out}(x) = (\prod_{i=1}^{m+2} \alpha_{i,b_i}\cdot\tilde{B}_{i,b_i})\cdot\mathbf{t}$$

and perform isZero on the encoding of $\mathsf{Out}(x)[1]$ (Note we can only apply Eval and Lemma 2 if the above computation is $\vec{S}$-respecting, but we will show that it is momentarily). From the correctness of the underlying randomized branching program, we have that $C(x) = 0 \iff \mathsf{Out}(x)[1] = 0$. Thus, $i\mathcal{O}$ is correct as long as the above computation is a $\vec{S}$-respecting circuit.

Note that when multiplying two matrices M_1 and M_2 encoded under S_1 and S_2 respectively, the multiplication is $\vec{S}$-respecting as long as $S_1 \cap S_2 = \emptyset$. Thus is suffices to show that the sets encoding the matrices being multiplied, namely:

$$S_{1,b_1}, \quad S_{2,b_2}, \quad \ldots, \quad S_{m+2,b_{m+2}}, \quad S_t$$

are all disjoint, and that their union is $[k]$.

Disjointness follows by observing that each of $U_1, U_2, \ldots, U_n, B_t$ is disjoint, and further that for each $j \in [n]$, for any i, i' such that $\mathsf{inp}(i) = \mathsf{inp}(i') = j$, we have that $b_i = b_{i'} = x[\mathsf{inp}(i)]$ and S_{i,b_i} and $S_{i',b_{i'}}$ are both elements of the straddling set system $\mathbb{S}^{\mathsf{inp}(i)}$, so $S_{i,b_i} \cap S_{i',b_{i'}} = \emptyset$.

To show that the union of the sets is $[k]$, we note that

$$(\bigcup_{i=1}^{m+2} S_{i,b_i}) \cup S_t = (\bigcup_{j=1}^{n} \bigcup_{i:\mathsf{inp}(i)=j} S_{i,x[j]}) \cup S_t = (\bigcup_{j=1}^{n} U_j) \cup S_t = [k]$$

by construction. Thus we have that $i\mathcal{O}$ is correct.

5 Proof of Indistinguishability Obfuscation

Theorem 11. *Assume the existence of semantically-secure multilinear encoding schemes. Then there exists indistinguishability obfuscators the the class of NC^1 circuits.*

Proof. We show that the obfuscator defined in Section 4.5 is an indistinguishability obfuscator for NC^1 circuits. Consider two NC^1 circuit ensembles $\{C_n^0\}_{n\in\mathbb{N}}$ and $\{C_n^1\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ and $x \in \{0,1\}^n$, $C_n^0(x) = C_n^1(x)$. Assume for contradiction there exists a nuPPT distinguisher D and polynomial p such that for infinitely many n, D distinguishes $i\mathcal{O}(C_n^0)$ and $i\mathcal{O}(C_n^1)$ with advantage $1/p(n)$. For any $n \in \mathbb{N}$ let BP_0 and BP_1 be the branching programs of length $m = poly(n)$ obtained by applying Theorem 3 to the circuits C_n^0 and C_n^1 respectively.

We organize the proof in three parts. In the first part we show that if D distinguishes between obfuscations of C_n^0 and C_n^1 then there exists widened branching programs BP and BP' that differ in only few matrices and evaluate the same function such that D distinguishes between $\mathsf{Obf}(BP)$ and $\mathsf{Obf}(BP')$. Furthermore, the first column of the output matrix is the same for BP and BP', and depends only on the output of the program $BP(x) = BP'(x)$. More concretely, there exist vectors v_0 and v_1 such that for all inputs x the first column of the output matrix for both BP and BP' is always $v_{BP(x)}$.

In the second part, we apply the semantic security of the graded encoding scheme used to argue that if D distinguishes $\mathsf{Obf}(BP)$ and $\mathsf{Obf}(BP')$ then there exists an algebraic adversary that does the same. In particular, this adversary can distinguish between the oracles $\mathcal{M}(\mathsf{GObf}(BP))$ and $\mathcal{M}(\mathsf{GObf}(BP'))$. Finally, in the third part we show that these oracles can be simulated given oracle access to BP (resp. BP') and input v_0 and v_1. This, together with the fact that BP and BP' agree on all inputs, will imply a contradiction and hence prove the theorem.

5.1 Setting up BP and BP' via a hybrid argument

Let Hyb_i be a procedure that takes an input two length m branching programs P_0 and P_1 (with the same labeling function) and outputs a "hybrid" length m branching program whose first i levels are identical to the first i levels of P_0 and all the other levels are identical to those of P_1. Formally, let $P_0 = \{\mathsf{inp}(j), B_{j,0}, B_{j,1}\}_{j\in[m]}$ and $P_1 = \{\mathsf{inp}(j), B'_{j,0}, B'_{j,1}\}_{j\in[m]}$.

$$\mathsf{Hyb}_i(P_0, P_1) = \{\mathsf{inp}(j), B_{j,0}, B_{j,1}\}_{j=1}^{i}, \{\mathsf{inp}(j), B'_{j,0}, B'_{j,1}\}_{j=i+1}^{m}$$

For every $n \in \mathbb{N}$ we define hybrid distributions in the following way.

- We start with H_0 which is the obfuscation of the circuit C_n^0.
$$H_0 = i\mathcal{O}(C_n^0) = \mathsf{Obf}(\mathsf{Merge}(BP_0, I, 0))$$
- For $i = 1, 2 \dots m$, let
$$H_i = \mathsf{Obf}(\mathsf{Merge}(BP_0, \mathsf{Hyb}_i(BP_1, I), 0))$$
We change, one level at a time, the second branching program Merge takes as input from I to BP_1.
- We have that $H_m = \mathsf{Obf}(\mathsf{Merge}(BP_0, BP_1, 0))$. We change the "switch" input to Merge so that the second branching program BP_1 is active.
$$H_{m+1} = \mathsf{Obf}(\mathsf{Merge}(BP_0, BP_1, 1))$$
- For $i = 1, 2 \dots m$, let
$$H_{m+i+1} = \mathsf{Obf}(\mathsf{Merge}(\mathsf{Hyb}_i(BP_1, BP_0), BP_1, 1))$$
We change the first program Merge takes as input from BP_0 to BP_1, one level at a time as before.
- We have that $H_{2m+1} = \mathsf{Obf}(\mathsf{Merge}(BP_1, BP_1, 1))$. We switch back so that the first program is active (which in this case is the same as the second program BP_1)
$$H_{2m+2} = \mathsf{Obf}(\mathsf{Merge}(BP_1, BP_1, 0))$$
- For $i = 1, 2 \dots m$, let
$$H_{2m+i+2} = \mathsf{Obf}(\mathsf{Merge}(BP_1, \mathsf{Hyb}_i(I, BP_1), 0))$$
We change the second program Merge takes as input from BP_1 to I, one level at a time as before. Finally we get
$$H_{3m+2} = i\mathcal{O}(C_n^1) = \mathsf{Obf}(\mathsf{Merge}(BP_1, I, 0))$$
which is the obfuscation of the circuit C_n^1.

Recall that by assumption D distinguishes between $\{i\mathcal{O}(C_n^0)\}_{n\in\mathbb{N}}$ and $\{i\mathcal{O}(C_n^1)\}_{n\in\mathbb{N}}$. That is, there is a polynomial p such that for infinitely many n
$$|Pr[D(H_0) = 1] - Pr[D(H_{3m+2})]| > 1/p(n)$$
By the above hybrid argument, D must distinguish between a pair of consecutive hybrids. That is, there exists some $i \in \{0, 1, \dots 3m+1\}$ such that
$$|Pr[D(H_i) = 1] - Pr[D(H_{i+1})]| > 1/4mp(n)$$
We now show that H_i and H_{i+1} can be expressed as the $\mathsf{Obf}(BP)$ and $\mathsf{Obf}(BP')$ respectively where BP and BP' are (widened) branching programs that differ in only two levels and agree on all inputs. Furthermore, both BP and BP' have the property that for all inputs x the first column of the output matrix $\mathsf{col}_1(\mathsf{P}_{\mathsf{out}}(x))$ is the same for BP and BP', and depends only on the output of these programs on x. More formally,

Claim 12. *There exist branching programs BP and BP' of length $m' = m + 2$ and width 10 such that*

- *$H_i = \mathsf{Obf}(BP)$ and $H_{i+1} = \mathsf{Obf}(BP')$.*

- *BP and BP' differ in at most 2 levels.*
- *For all $x \in \{0,1\}^n$, $BP(x) = BP'(x)$.*
- *Let ${\mathsf{P_{out}}}^{BP}(\cdot)$ and ${\mathsf{P_{out}}}^{BP'}(\cdot)$ be the functions computing the output matrices for BP and BP' respectively. There exist length 10 vectors v_0 and v_1 such that for every $x \in \{0,1\}^n$, $\mathsf{col}_1({\mathsf{P_{out}}}^{BP}(x)) = \mathsf{col}_1({\mathsf{P_{out}}}^{BP'}(x)) = v_{BP(x)}$*

Proof. Let $v_1 = \mathsf{extend}(\mathsf{col}_1(\mathsf{P_{accept}}))$ and $v_0 = \mathsf{extend}(\mathsf{col}_1(\mathsf{P_{reject}}))$ where $\mathsf{P_{accept}}$ and $\mathsf{P_{reject}}$ are the accepting and rejecting matrices from Theorem 3 for branching programs of input lengths n, and extend extends a length w vector by appending w zeroes. We consider three cases: when i is equal to m, $2m+1$ and otherwise.

Case 1: $i = m$: By definition of H_i and H_{i+1}, the branching programs BP and BP' are $\mathsf{Merge}(BP_0, BP_1, 0)$ and $\mathsf{Merge}(BP_0, BP_1, 1)$ respectively. By Claim 6, BP and BP' differ in the "switch" matrices, which make up the first and last level. Furthermore, BP and BP' compute BP_0 and BP_1 respectively which are equivalent programs by assumption. It remains to show the fourth condition. By Claim 6, the first column of the output matrix for a widened branching program only depends on the first column of the output matrix of the active program. Hence, for every input x, $\mathsf{col}_1({\mathsf{P_{out}}}^{BP}(x)) = \mathsf{extend}(\mathsf{col}_1({\mathsf{P_{out}}}^{BP_0}(x)))$. By Theorem 3, ${\mathsf{P_{out}}}^{BP_0}(x)$ is either $\mathsf{P_{accept}}$ or $\mathsf{P_{reject}}$ depending on the output $BP_0(x)$. Therefore, for all inputs x such that $BP(x) = 0$,

$$\mathsf{col}_1({\mathsf{P_{out}}}^{BP}(x)) = \mathsf{extend}(\mathsf{col}_1(\mathsf{P_{reject}})) = v_0$$

Similarly, for all inputs x such that $BP(x) = 1$,

$$\mathsf{col}_1({\mathsf{P_{out}}}^{BP}(x)) = \mathsf{extend}(\mathsf{col}_1(\mathsf{P_{accept}})) = v_1$$

The same argument holds for BP' too, in which case BP_1 is active and has the same accepting and rejecting permutations $\mathsf{P_{accept}}$ and $\mathsf{P_{reject}}$ by Theorem 3. Therefore, for all inputs x,

$$\mathsf{col}_1({\mathsf{P_{out}}}^{BP'}(x)) = v_{BP_1(x)}$$

Since $BP_0(x) = BP_1(x) = BP(x)$ for all x, the claim follows.

Case 2: $i = 2m+1$: By definition of H_i and H_{i+1}, the branching programs BP and BP' are $\mathsf{Merge}(BP_1, BP_1, 0)$ and $\mathsf{Merge}(BP_1, BP_1, 1)$ respectively. As before, these programs differ in the first and level only. Furthermore, both BP and BP' compute the same function, as the active program is the same (BP_1). Also, directly from Claim 6 and Theorem 3 we have that for all inputs $x \in \{0,1\}^n$,

$$\mathsf{col}_1({\mathsf{P_{out}}}^{BP}(x)) = \mathsf{col}_1({\mathsf{P_{out}}}^{BP'}(x)) = \mathsf{extend}(\mathsf{col}_1({\mathsf{P_{out}}}^{BP_1}(x))) = v_{BP_1(x)} = v_{BP(x)}$$

Case 3: $i \neq m$ and $i \neq 2m+1$: First, consider the subcase when $i < m$ or $i > 2m+1$. The programs BP and BP' are of the form $\mathsf{Merge}(BP_0, P_i)$ and $\mathsf{Merge}(BP_0, P_{i+1})$ respectively where P_i and P_{i+1} are branching programs that differ only in the $i+1^{th}$ level. By Claim 6, BP and BP' differ only in the $i+1^{th}$ level too. Furthermore, in both BP and BP', the active program is BP_0. Hence BP and BP' compute the same function and similarly as the previous case, we have that for all inputs $x \in \{0,1\}^n$,

$$\mathsf{col}_1({\mathsf{P_{out}}}^{BP}(x)) = \mathsf{col}_1({\mathsf{P_{out}}}^{BP'}(x)) = \mathsf{extend}(\mathsf{col}_1({\mathsf{P_{out}}}^{BP_0}(x))) = v_{BP_0(x)} = v_{BP(x)}$$

The case when $m < i < 2m+1$ follows similarly. This concludes the proof of the claim. ☐

This concludes the first part of the proof. At this point we have that there is a polynomial p such that for infinitely many n there exist branching programs BP and BP' with the properties described in Claim 12 such that

$$|Pr[D(\mathsf{Obf}(BP)) = 1] - Pr[D(\mathsf{Obf}(BP'))]| > 1/4mp(n)$$

In the next part we show that the distinguisher D can be used to break the semantic security game of the graded encoding scheme used by Obf.

5.2 Applying Semantic Security

Fix $n \in \mathbb{N}$, and let $BP = \{\mathsf{inp}(i), B_{i,0}, B_{i,1}\}_{i\in[m']}$ and $BP' = \{\mathsf{inp}(i), B'_{i,0}, B'_{i,1}\}_{i\in[m']}$. Let $l_1, l_2 \in [m]$ be the levels in which BP and BP' differ. All other matrices of BP and BP' are the same. That is, for every $i \notin \{l_1, l_2\}, b \in \{0,1\}$ we have that $B_{i,b} = B'_{i,b}$.

Define nuPPT M that gets BP and BP' as non-uniform advice and on input the public parameters pp that describe a $(\mathbb{Z}_p, k)$-graded encoding scheme samples m' random invertible 10×10 matrices over $\mathbb{Z}_p$, $\{R_i\}_{i\in[m']}$ and $2m'$ random scalars from $\mathbb{Z}_p$, $\{\alpha_{i,b}\}_{i\in[m'],b\in\{0,1\}}$. M then uses these matrices and scalars to randomize BP and BP' as described by $\mathsf{Rand}(\cdot, p)$ to obtain $\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in[m'],b\in\{0,1\}}$, $\{\alpha_{i,b} \cdot \tilde{B}'_{i,b}\}_{i\in[m'],b\in\{0,1\}}$ and $\mathbf{t}$. M outputs

$$\vec{m_0} = (\{\alpha_{l_1,b} \cdot \tilde{B}_{l_1,b}\}_{b\in\{0,1\}}, \{\alpha_{l_2,b} \cdot \tilde{B}_{l_2,b}\}_{b\in\{0,1\}})$$

$$\vec{m_1} = (\{\alpha_{l_1,b} \cdot \tilde{B'}_{l_1,b}\}_{b\in\{0,1\}}, \{\alpha_{l_2,b} \cdot \tilde{B'}_{l_2,b}\}_{b\in\{0,1\}})$$

$$\vec{z} = (\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in[m']/\{l_1,l_2\},b\in\{0,1\}}, \mathbf{t})$$

We observe that $D(\mathsf{Obf}(BP))$ (resp. $D(\mathsf{Obf}(BP'))$) is simply the output of D when playing the semantic security game with the message sampler M and parameterized by the bit $b = 0$ (resp. $b = 1$). Formally, there exist polynomials q, k constant c and set ensembles $\{(\vec{S}_n, \vec{T}_n)\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$

$$D(\mathsf{Obf}(BP)) \equiv \mathbf{Output}_0(q, k, c, D, M, n, (\vec{S}_n, \vec{T}_n))$$

and

$$D(\mathsf{Obf}(BP')) \equiv \mathbf{Output}_1(q, k, c, D, M, n, (\vec{S}_n, \vec{T}_n))$$

where $\vec{S}_n$, $\vec{T}_n$ contain sets from $\mathsf{SetSystem}(m', n, \mathsf{inp})$ and $\mathbf{Output}_b$ is as defined in Definition 8.

To see this, observe that $\vec{m_0}$ and $\vec{m_1}$ consist of a constant number of ring elements while $\vec{z}$ contains polynomially many ring elements. Note that the distribution of $(\vec{m_0}, \vec{z})$ is identical to $\mathsf{Rand}(BP, p)$ and the distribution of $(\vec{m_1}, \vec{z})$ is identical to $\mathsf{Rand}(BP', p)$. When these elements are encoded under sets in $\mathsf{SetSystem}(m', n, \mathsf{inp})$ [8] then we obtain the distributions $\mathsf{Obf}(BP)$ and $\mathsf{Obf}(BP')$ respectively.

Recall that for infinitely many n,

$$|Pr[D(\mathsf{Obf}(BP)) = 1] - Pr[D(\mathsf{Obf}(BP'))]| > 1/4mp(n)$$

Since the graded encoding scheme is semantically secure, it must be that M is not a $(q, k, c, \{(\vec{S}_n, \vec{T}_n)\}_{n\in\mathbb{N}})$-respecting message sampler. Therefore, there exists a polynomial p' and algebraic adversary nuPPT A such that for infinitely many $n \in \mathbb{N}$,

$$Pr[A^{\mathcal{M}(\mathsf{GObf}(BP))}(1^n) = 1] - Pr[A^{\mathcal{M}(\mathsf{GObf}(BP'))}(1^n) = 1] > 1/p'(n)$$

In the remainder of the proof we show that if BP and BP' agree on all inputs then such algebraic adversary A cannot exist. More precisely, we will show that the oracles $\mathcal{M}(\mathsf{GObf}(BP))$ and $\mathcal{M}(\mathsf{GObf}(BP'))$ can be simulated with only oracle access to BP and BP'.

Similar statements were shown in [BGK+13] and [BR13]. In particular, GObf is a simplified version of the obfuscator of [BGK+13], which [BGK+13] shows is VBB secure against algebraic adversaries. We will follow the structure of the proof in [BGK+13], but cannot use it in a black-box way due to the differences in the construction and the fact that their proof only works for branching programs that have unique accepting and rejecting output matrices. The branching programs we consider may not have this property.

[8]Every element is encoded with the corresponding set in $\mathsf{SetSystem}(m', n, \mathsf{inp})$. For example, elements from $\alpha_{i,b}\tilde{B}_{i,b}$ are encoded under the set $S_{i,b}$ in $\mathsf{SetSystem}(m', n, \mathsf{inp})$

5.3 Simulating an algebraic adversary

We show the following claim.

Claim 13. *There exists Turing machine* Sim *such that for every nuPPT A there exists a negligible function ε such that the following holds. For every $n \in \mathbb{N}$, every length $m = poly(n)$ and width 10 branching program BP with labeling function* $\mathsf{inp} : [m] \to [n]$ *and for which there exist $v_0, v_1 \in \{0,1\}^{10}$ such that on every input $x \in \{0,1\}^n$, $\mathsf{col}_1(\mathsf{P_{out}}(x)) = v_{BP(x)}$, it holds that*

$$\Delta(A^{\mathcal{M}(\mathsf{GObf}(BP))}(1^n), \mathsf{Sim}^{BP}(1^n, 1^m, A, \mathsf{inp}, v_0, v_1)) \leq \varepsilon(n)$$

Sim's strategy will be to run A and simulate the oracle $\mathcal{M}(\mathsf{GObf}(BP))$ for A. Recall that $\mathsf{GObf}(BP)$ contains the public parameters of the encoding scheme pp and a list of the ring elements in $\mathsf{Rand}(BP)$ paired with the corresponding set in $\vec{S} = \mathsf{SetSystem}(n, m, \mathsf{inp})$. $\mathcal{M}(\mathsf{GObf}(BP))$ when queried with an arithmetic circuit C, first checks if C is $\vec{S}$-respecting and then outputs the result of C on the ring elements in $\mathsf{Rand}(BP)$.

Sim only has oracle access to BP and can not run $\mathcal{M}$ directly on $\mathsf{GObf}(BP)$. However, we show in the following lemma that Sim can simulate the output of $\mathcal{M}$ on a single query. In particular, except with negligible probability (over $\mathsf{GObf}(BP)$ and the simulation) the simulated query response will be *identical* to the actual query response. Since A makes only polynomially many queries, by the Union Bound, it follows that except with negligible probability Sim succeeds in correctly simulating all queries. Therefore, it suffices to show the following lemma.

Lemma 14. *There exists a Turing machine* CSim *such that for every $m, n, w \in \mathbb{N}$, $v_0, v_1 \in \{0,1\}^w$, labeling function* $\mathsf{inp} : [m] \to [n]$*, prime number p, and $\vec{S}$-respecting arithmetic circuit C where $\vec{S} =$* $\mathsf{SetSystem}(m, n, \mathsf{inp})$*, the following holds. For every branching program BP of length m, width w and labeling function* inp *for which on every input x, $\mathsf{col}_1(\mathsf{P_{out}}(x)) = v_{BP(x)}$ it holds that*

$$Pr[\mathsf{isZero}(C(\mathsf{Rand}(BP, p))) \neq \mathsf{CSim}^{BP}(1^m, p, C, v_0, v_1)] \leq 32wm/p$$

The proof of the lemma follows the structure of the VBB simulation in [BGK$^+$13], appropriately adapted to deal with the fact that our branching programs do not have a unique output by relying on Theorem 7.

Proof. Roughly speaking the lemma follows from the the property that $\vec{S}$-respecting arithmetic circuits, due to the straddling set systems in $\vec{S}$, can only evaluate expressions that are "consistent" with some inputs. In particular, following [BGK$^+$13], the polynomial C evaluates can be expressed as the sum of *single-input terms* where each *single-input term* is a function of those elements of that are consistent with some input to the branching program. Next, we rely on Theorem 7 to show that the sum of these single-input terms will depend only on the value of the branching program on these inputs.

The following proposition states that the function a $\vec{S}$-respecting arithmetic circuit computes can be expressed as the sum of several *single-input* terms. This decomposition is very similar to the one shown in [BGK$^+$13].[9]

Proposition 1. *Fix $m, n, w \in \mathbb{N}$ and* $\mathsf{inp} : [m] \to [n]$*. Let $\vec{S} =$* $\mathsf{SetSystem}(m, n, \mathsf{inp}) = (\{S_{i,b}\}_{i \in [m], b \in \{0,1\}}, S_t)$*, and let C be any $\vec{S}$-respecting arithmetic circuit. There exists a set $X \subseteq \{0,1\}^n$ of inputs such that*

(i)

$$C \equiv \sum_{x \in X} C_x$$

[9]The key difference is that [BGK$^+$13] proves such a decomposition for "dual-input" branching program. Another minor difference is that since our scheme is slightly different, the terms of the decomposition are slightly different as well.

where each C_x is a $\vec{S}$-respecting arithmetic circuit, whose input wires are labelled only with sets respecting a single input $x \in \{0,1\}^n$, that is, only with sets $\in \{S_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]} \cup \{S_t\}$.

(ii) *For each C_x above, for every branching program BP of width w and length m on n input bits, with input labelling function inp, every prime p, and every $(\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in m, b\in\{0,1\}}, \mathbf{t}) \leftarrow \mathsf{Rand}(BP, p)$*

$$C_x(\{\alpha_{i,b}\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}) = \alpha_x \cdot p_x(\{\tilde{B}_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]}, \mathbf{t})$$

where p_x is some polynomial, and $\alpha_x = (\prod_{i\in[m]} \alpha_{i,x[\mathsf{inp}(i)]})$. Furthermore, when p_x is viewed as a sum of monomials, each monomial contains exactly one entry from each $\tilde{B}_{i,x[\mathsf{inp}(i)]}$, and one entry from $\mathbf{t}$.

The proof of Proposition 1 closely follows [BGK+13]; for completeness we provide a complete proof in Appendix B.

Now we are ready to describe the simulator CSim. CSim gets as input 1^m, prime p, a $\vec{S}$-respecting circuit C, vectors v_0, v_1 and has oracle access to a length m branching program BP. Let X be the set of inputs and $\{p_x\}_{x\in X}$ be the single-input polynomials corresponding to the decomposition of C. For every $x \in X$, CSim queries BP on x, samples $d_x \leftarrow \mathsf{KSim}(1^m, p, v_{BP(x)})$ and checks whether $p_x(d_x) = 0$. CSim outputs 1 if and only if for every input $x \in X$, $p_x(d_x) = 0$.

Now we prove correctness of our simulation. First, we prove some claims that will be useful. In each of these claims, let proj_x be defined with respect to the labeling function inp of the branching program BP. The following claim states that if $C(\mathsf{Rand}(BP, p))$ is always zero, then every single-input term is always zero.

Claim 15. *If $Pr[C(\mathsf{Rand}(BP, p) = 0] = 1$ then for every input $x \in X$,*

$$Pr[p_x(\mathsf{proj}_x(\mathsf{Rand}^B(BP, p))) = 0] = 1$$

Proof. Consider a fixed $d = (\{\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t})$ in the support of $\mathsf{Rand}^B(BP, p)$ and let $C_d(\{\alpha_{i,b}\}_{i\in[m],b\in\{0,1\}}) = C(\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t})$. By Proposition 1, we know that

$$C_d(\{\alpha_{i,b}\}) = \sum_{x\in X} (\prod_{i\in[m]} \alpha_{i,x[\mathsf{inp}(i)]}) p_x(\mathsf{proj}_x(d))$$

and C_d is a degree $m + 2$ polynomial. By assumption, $C(\mathsf{Rand}(BP, p))$ is always zero (over the support of $\mathsf{Rand}(BP, p)$); hence, $C_d(\{\alpha_{i,b}\}) = 0$ for all non-zero $\{\alpha_{i,b}\}$. By the Schwartz-Zippel lemma, this can happen only if C_d is the zero polynomial. By the structure of C_d, this implies that for every $x \in X$, $p_x(\mathsf{proj}_x(d)) = 0$. This argument works for every fixed value of d, hence we have that for every $x \in X$, $\Pr[p_x(\mathsf{proj}_x(\mathsf{Rand}^B(BP, p))) = 0] = 1$. □

The next claim states that if $C(\mathsf{Rand}(BP, p))$ is not always zero, then it is zero with small probability. Furthermore, there exists a single-input term that is zero with small probability.

Claim 16. *For any $\vec{S}$-respecting circuit C, if $Pr[C(\mathsf{Rand}(BP, p)) = 0] < 1$ then the following holds.*

1. *$Pr[C(\mathsf{Rand}(BP, p)) = 0] \le 16wm/p$*
2. *There exists $x \in X$ such that $Pr[p_x(\mathsf{proj}_x(\mathsf{Rand}^B(BP, p))) = 0] \le 16wm/p$, where X is obtained from the decomposition of C by Proposition 1.*

Proof. We start by showing part 1.

Part 1: If $\mathsf{Rand}(BP, p) = \mathsf{Rand}^\alpha(\mathsf{Rand}^B(BP, p))$ can be expressed as a low-degree ($\le 2w$) polynomial on uniformly random values in $\mathbb{Z}_p$—namely, the α's and the randomization matrices R_i's—then by the Schwartz-Zippel lemma the first part of the claim directly follows. However, there are two barriers to applying this argument:

- Rand^B does not sample uniformly random matrices $\{R_i\}_{i\in[m]}$; rather, it chooses uniformly random *invertible* matrices R_i. Similarly, Rand^α does not sample uniformly random $\{\alpha_{i,b}\}_{i\in[m],b\in\{0,1\}}$; rather, it chooses uniformly random *non-zero* $\alpha_{i,b}$.
- Rand^B also needs to compute inverses R_i^{-1} to R_i for every $i \in [m]$ (which may no longer be expressed as low degree polynomials in the matrices $\{R_i\}_{i\in[m]}$).

To handle the second issue, consider the distribution $\mathsf{Rand}^B_{adj}(BP,p)$ that is defined exactly as $\mathsf{Rand}^B(BP,p)$ except that for every $i \in [m]$ it uses $adj(R_i) = R_i^{-1}det(R_i)$ instead of R_i^{-1}. Note that every entry of the adjoint of a $w \times w$ matrix M is some cofactor of M (obtained by the determinant of the $w-1 \times w-1$ matrix obtained by deleting some row and column of A). Hence every entry of $adj(R_i)$ can be expressed as a degree w polynomial in R_i. Let $\mathsf{Rand}_{adj}(BP,p) = \mathsf{Rand}^\alpha(\mathsf{Rand}^B_{adj}(BP,p))$. It follows that $\mathsf{Rand}_{adj}(BP,p)$ is computed by degree (at most) $2w$ polynomial in the matrices $\{R_i\}_{i\in[m]}$ and scalars $\{\alpha_{i,b}\}_{i\in[m],b\in\{0,1\}}$.

Furthermore, we show that $Pr[C(\mathsf{Rand}_{adj}(BP,p)) = 0] = Pr[C(\mathsf{Rand}(BP,p)) = 0]$. Recall that by Proposition 1,

$$C \equiv \sum_{x\in X} C_x$$

and for each C_x above and every $(\{\alpha_{i,b}\cdot\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}) \leftarrow \mathsf{Rand}(BP,p)$,

$$C_x(\{\alpha_{i,b}\cdot\tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}) = \alpha_x \cdot p_x(\{\tilde{B}_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]}, \mathbf{t})$$

where $\alpha_x = (\prod_{i\in[m]} \alpha_{i,x[\mathsf{inp}(i)]})$ and p_x is a polynomial such that, when viewed as a sum of monomials, each monomial contains exactly one entry from each $\tilde{B}_{i,x[\mathsf{inp}(i)]}$, and one entry from $\mathbf{t}$. Recall that for every $i \in [m]$,

$$\tilde{B}_{i,x[\mathsf{inp}(i)]} = R_{i-1}B_{i,x[\mathsf{inp}(i)]}R_i^{-1}$$

For every $i \in [m]$, replacing R_i^{-1} with $adj(R_i)$ has the effect of multiplying each monomial in p_x with the scalar $det(R_i)$. Hence

$$C_x(\mathsf{Rand}_{adj}(BP,p)) = (\prod_{i\in[m]} det(R_i)) \cdot C_x(\mathsf{Rand}(BP,p))$$

Since C is the sum of such C_x terms, it holds that $C(\mathsf{Rand}_{adj}(BP,p)) = (\prod_{i\in[m]} det(R_i))C(\mathsf{Rand}(BP,p))$. For every $i \in [m]$, by invertibility, $det(R_i) \neq 0$ and hence

$$Pr[C(\mathsf{Rand}_{adj}(BP,p)) = 0] = Pr[C(\mathsf{Rand}(BP,p)) = 0]$$

So far, we have that $\mathsf{Rand}_{adj}(BP,p)$ is computed by a degree $2w$ polynomial in the matrices $\{R_i\}_{i\in[m]}$ and scalars $\{\alpha_{i,b}\}_{i\in[m],b\in\{0,1\}}$. However the first issue remains: each R_i is uniformly random invertible and each $\alpha_{i,b}$ is uniformly random non-zero, whereas we need them to be uniformly random. Consider the distribution $\mathsf{Rand}_{adj,U}(BP,p)$ that is obtained by the computing the same polynomial on uniformly random matrices $\{R_i\}_{i\in[m]}$ and scalars $\{\alpha_{i,b}\}_{i\in[m],b\in\{0,1\}}$ over $\mathbb{Z}_p$. In Claim 19, we show that the statistical distance between $\mathsf{Rand}_{adj}(BP,p)$ and $\mathsf{Rand}_{adj,U}(BP,p)$ is at most $8wm/p$. Furthermore, the support of $\mathsf{Rand}_{adj,U}(BP,p)$ contains the support of $\mathsf{Rand}_{adj}(BP,p)$. This implies that if $Pr[C(\mathsf{Rand}_{adj}(BP,p)) = 0] < 1$ then $Pr[C(\mathsf{Rand}_{adj,U}(BP,p)) = 0] < 1$.

We now turn to proving the statement of the claim. Using facts shown above, we have that

$$Pr[C(\mathsf{Rand}(BP,p)) = 0] < 1 \implies Pr[C(\mathsf{Rand}_{adj}(BP,p)) = 0] < 1 \implies Pr[C(\mathsf{Rand}_{adj,U}(BP,p)) = 0] < 1$$

By Proposition 1, C evaluates a $m+1$ degree polynomial, and $\mathsf{Rand}_{adj,U}(BP,p)$ is computed by a degree $2w$ polynomial in uniformly random values in $\mathbb{Z}_p$. By the Schwartz-Zippel lemma,

$$Pr[C(\mathsf{Rand}_{adj,U}(BP,p)) = 0] < 1 \implies Pr[C(\mathsf{Rand}_{adj,U}(BP,p) = 0] \leq 2w(m+1)/p \leq 8wm/p$$

We have that the statistical distance between $\mathsf{Rand}_{adj,U}(BP,p)$ and $\mathsf{Rand}_{adj}(BP,p)$ is at most $8wm/p$. Therefore, $Pr[C(\mathsf{Rand}(BP,p)) = 0] = Pr[C(\mathsf{Rand}_{adj}(BP,p)) = 0] \leq 16wm/p$ thus proving the first part of the claim. We proceed to show part 2.

Part 2: By Proposition 1, for every $x \in X$, there exists a $\vec{S}$-respecting arithmetic circuit C_x such that for every $(\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}) \leftarrow \mathsf{Rand}(BP,p)$,

$$C_x(\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}) = \alpha_x \cdot p_x(\{\tilde{B}_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]}, \mathbf{t})$$

where $\alpha_x = (\prod_{i\in[m]} \alpha_{i,x[\mathsf{inp}(i)]})$ and $C = \sum_{x\in X} C_x$. In particular, $p_x(\{\tilde{B}_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]}, \mathbf{t}) = 0$ iff $C_x(\{\alpha_{i,b} \cdot \tilde{B}_{i,b}\}_{i\in[m],b\in\{0,1\}}, \mathbf{t}) = 0$ (since $\alpha_{i,b}$ is non-zero).

Thus, we have that

$$Pr[C(\mathsf{Rand}(BP,p))) = 0] = Pr[C_x(\mathsf{Rand}^{\alpha}(\mathsf{Rand}^{B}(BP,p))) = 0] = Pr[p_x(\mathsf{proj}_x(\mathsf{Rand}^{B}(BP,p))) = 0]$$

There must exist an input $x \in X$ such that $Pr[C_x(\mathsf{Rand}(BP,p))) = 0] < 1$ or else $Pr[C(\mathsf{Rand}(BP,p))) = 0] = 1$. By the first part of the claim, it follows that

$$Pr[C(\mathsf{Rand}(BP,p))) = 0] \leq 16wm/p,$$

which concludes the proof. □

Now we analyze the correctness of the simulator CSim. We consider the following two cases: when $C(\mathsf{Rand}(BP,p))$ is always zero, and otherwise.

Case 1: $Pr[C(\mathsf{Rand}(BP,p)) = 0] = 1$**:** In this case we will show that the simulation always succeeds. If $Pr[C(\mathsf{Rand}(BP,p)) = 0] = 1$ then by Claim 15, for every $x \in X$, $Pr[p_x(\mathsf{proj}_x(\mathsf{Rand}^B(BP,p))) = 0] = 1$. Recall that $\mathsf{KSim}(1^m, p, v_{BP(x)})$ simulates $\mathsf{proj}_x(\mathsf{Rand}^B(BP,p))$ perfectly. Therefore, CSim always outputs 1 and hence succeeds.

Case 2: $Pr[C(\mathsf{Rand}(BP,p)) = 0] < 1$**:** In this case, by the first part of Claim 16 we have that

$$Pr[\mathsf{isZero}(C(\mathsf{Rand}(BP,p))) = 1] \leq 16wm/p$$

By the perfect simulation of KSim, we have that

$$Pr[\mathsf{CSim}^{BP} = 1] = Pr[\forall x\ (d_x \leftarrow \mathsf{proj}_x(\mathsf{Rand}^B(BP,p)) : p_x(d_x) = 0)]$$

By second part of Claim 16 there exists input x_C such that $Pr[p_{x_C}(\mathsf{proj}_{x_C}(\mathsf{Rand}^B(BP,p))) = 0] \leq 16wm/p$. Therefore,

$$Pr[\mathsf{CSim}^{BP} = 1] \leq Pr[p_{x_C}(\mathsf{proj}_{x_C}(\mathsf{Rand}^B(BP,p))) = 0] \leq 16wm/p$$

Therefore, by a union bound we have that

$$Pr[\mathsf{isZero}(C(\mathcal{D})) = \mathsf{CSim}^{BP} = 0] > 1 - 32wm/p$$

This concludes the proof of the lemma. □

□

Remark 1. In the above proof, we can rely on a weaker *entropic* notion of semantic security, where security holds only for message samplers that sample $\vec{m}_0$, $\vec{m}_1$ and $\vec{z}$ with high entropy. In particular, for our proof to go through it suffices to restrict to message samplers that ensure that the the entropy of $\vec{m}_0$ and $\vec{m}_1$, conditioned on $\vec{z}$ is "very high". This follows from the observation that the message sampler considered in the above proof has this property: Recall that M outputs $\vec{m}_0 = \{(\alpha_{i,0}\cdot\tilde{B}_{i,0}, \alpha_{i,1}\cdot\tilde{B}_{i,1})\}_{i\in l_1,l_2}$ for two levels l_1, l_2, where $(\{\alpha_{i,b}\cdot\tilde{B}_{i,b}\}_{i\in[m'],b\in\{0,1\}}, \mathbf{t}) \leftarrow \mathsf{Rand}(BP, p)$ for some length m' branching program BP. $\vec{z}$ contains $(\alpha_{i,0}\cdot\tilde{B}_{i,0}, \alpha_{i,1}\cdot\tilde{B}_{i,1})$, for all other levels $i \notin \{l_1, l_2\}$, and $\mathbf{t}$. Also recall that for every $i \in [m']$ and $b \in \{0,1\}$, $\tilde{B}_{i,b} = R_{i-1}B_{i,b}R_i^{-1}$ where $\{R_i\}_{i\in[m']}$ are random invertible matrices (and $R_0 = I_{5\times5}$) and $\{B_{i,b}\}_{i\in[m'],b\in\{0,1\}}$ are the matrices of BP. It easily follows (using the same argument as [Kil88]) that even conditioned on $\{\alpha_{i,b}, \tilde{B}_{i,b}\}_{i\notin\{l_1,l_2\},b\in\{0,1\}}$ and the whole of $R_{m'}$ (as opposed to just $\mathbf{t}$), $\tilde{B}_{l_1,0}$ is a random invertible matrix, and thus $\vec{m}_0$ has at least the entropy of a random invertible matrix, conditioned on $\vec{z}$. The same argument applies to $\vec{m}_1$.

5.4 Achieving Obfuscation for Arbitrary Programs

[GGH+13b] show that any indistinguishability obfuscation scheme for NC^1 can be bootstrapped into an indistinguishability obfuscation scheme for all poly-sized circuits using FHE. That is, they prove the following theorem.

Theorem 17 ([GGH+13b]—informally stated)**.** *Assume the existence of indistinguishability obfuscators $i\mathcal{O}$ for NC^1 and a leveled Fully Homomorphic Encryption scheme with decryption in NC^1. Then there exists an indistinguishability obfuscator $i\mathcal{O}'$ for arbitrary poly-sized circuits.*

Applying their construction to our indisinguishability obfuscator yields an indistinguishability obfuscator for arbitrary polynomial size circuits:

Theorem 18. *Assume the existence of semantically secure multi-linear encodings and a leveled Fully Homomorphic Encryption scheme with decryption in NC^1. Then there exists an indistinguishability obfuscators for arbitrary poly-sized circuits.*

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References

[ABG+13] Prabhanjan Ananth, Dan Boneh, Sanjam Garg, Amit Sahai, and Mark Zhandry. Differing-inputs obfuscation and applications. 2013.

[Bar86] David A. Mix Barrington. Bounded-width polynomial-size branching programs recognize exactly those languages in nc^1. In *STOC*, pages 1–5, 1986.

[BCP13] Elette Boyle, Kai-Min Chung, and Rafael Pass. On extractability obfuscation. Technical report, Cryptology ePrint Archive, Report 2013/650, 2013. http://eprint. iacr. org, 2013.

[BCPR13] Nir Bitansky, Ran Canetti, Omer Paneth, and Alon Rosen. Indistinguishability obfuscation vs. auxiliary-input extractable functions: One must fall. Technical report, Cryptology ePrint Archive, Report 2013/641, 2013. 5, 2013.

[BGI+01] Boaz Barak, Oded Goldreich, Rusell Impagliazzo, Steven Rudich, Amit Sahai, Salil Vadhan, and Ke Yang. On the (im) possibility of obfuscating programs. In *Advances in CryptologyâĂŤCRYPTO 2001*, pages 1–18. Springer, 2001.

[BGK$^+$13] Boaz Barak, Sanjam Garg, Yael Tauman Kalai, Omer Paneth, and Amit Sahai. Protecting obfuscation against algebraic attacks. Cryptology ePrint Archive, Report 2013/631, 2013.

[BGV12] Zvika Brakerski, Craig Gentry, and Vinod Vaikuntanathan. (leveled) fully homomorphic encryption without bootstrapping. In *ITCS*, pages 309–325, 2012.

[BP13] Elette Boyle and Rafael Pass. Limits of extractability assumptions with distributional auxiliary input. 2013.

[BR13] Zvika Brakerski and Guy N Rothblum. Virtual black-box obfuscation for all circuits via generic graded encoding. Technical report, Cryptology ePrint Archive, Report 2013/563, 2013. http://eprint. iacr. org, 2013.

[BV11] Zvika Brakerski and Vinod Vaikuntanathan. Efficient fully homomorphic encryption from (standard) lwe. In *FOCS*, pages 97–106, 2011.

[BZ13] Dan Boneh and Mark Zhandry. Multiparty key exchange, efficient traitor tracing, and more from indistinguishability obfuscation. Technical report, Cryptology ePrint Archive, Report 2013/642, 2013. http://eprint. iacr. org, 2013.

[Gen09] Craig Gentry. *A fully homomorphic encryption scheme*. PhD thesis, Stanford University, 2009.

[GGH13a] Sanjam Garg, Craig Gentry, and Shai Halevi. Candidate multilinear maps from ideal lattices. In *Advances in Cryptology–EUROCRYPT 2013*, pages 1–17. Springer, 2013.

[GGH$^+$13b] Sanjam Garg, Craig Gentry, Shai Halevi, Mariana Raykova, Amit Sahai, and Brent Waters. Candidate indistinguishability obfuscation and functional encryption for all circuits. *Proc. of FOCS 2013*, 2013.

[GGHR13] Sanjam Garg, Craig Gentry, Shai Halevi, and Mariana Raykova. Two-round secure mpc from indistinguishability obfuscation. Technical report, IACR Cryptology ePrint Archive, 2013: 601, 2013.

[Had00] Satoshi Hada. Zero-knowledge and code obfuscation. In *Advances in Cryptology–ASIACRYPT 2000*, pages 443–457. Springer, 2000.

[HSW13] Susan Hohenberger, Amit Sahai, and Brent Waters. Replacing a random oracle: Full domain hash from indistinguishability obfuscation. Technical report, Cryptology ePrint Archive, Report 2013/509, 2013. http://eprint. iacr. org, 2013.

[Kil88] Joe Kilian. Founding crytpography on oblivious transfer. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, pages 20–31. ACM, 1988.

[LCT$^+$13] Tancrède Lepoint, Jean-Sébastien Coron, Mehdi Tibouchi, et al. Practical multilinear maps over the integers. In *CRYPTO 2013-33rd Annual Cryptology Conference Advances in Cryptology*, volume 8042, pages 476–493, 2013.

[Nao03] Moni Naor. On cryptographic assumptions and challenges. In *CRYPTO*, pages 96–109, 2003.

[RAD78] R L Rivest, L Adleman, and M L Dertouzos. On data banks and privacy homomorphisms. *Foundations of Secure Computation, Academia Press*, pages 169–179, 1978.

[SW13] Amit Sahai and Brent Waters. How to use indistinguishability obfuscation: Deniable encryption, and more. Technical report, IACR Cryptology ePrint Archive, 2013: 454, 2013.

A Technical Lemma

Claim 19. *Fix $m, w \in \mathbb{N}$, and let $p \in \mathbb{N}$ be a prime. Let $\mathcal{D}_0$ be the following distribution:*

$$\mathcal{D}_0 = \{\{R_i\}_{i\in[m]}, \{\alpha_{i,b}\}_{i\in[m],b\in\{0,1\}}\}$$

where each R_i is a uniformly random invertible matrix in $\mathbb{Z}_p^{w\times w}$ (i.e $det(R_i) \neq 0$, and each $\alpha_{i,b}$ is a uniformly random non-zero scalar in $\mathbb{Z}_p$.

Let $\mathcal{D}_1$ be a distribution defined identically to $\mathcal{D}_0$, except with each R_i being a uniformly random (not necessarily invertible) matrix in $\mathbb{Z}_p^{w\times w}$, and each $\alpha_{i,b}$ a uniformly random (not necessarily non-zero) scalar in $\mathbb{Z}_p$.
Then:

$$\Delta(\mathcal{D}_0, \mathcal{D}_1) \leq 8wm/p$$

where $\Delta(\mathcal{D}_0, \mathcal{D}_1)$ denotes the statistical distance between distributions $\mathcal{D}_0$ and $\mathcal{D}_1$.

Proof. Note that $\mathcal{D}_0$ and $\mathcal{D}_1$ are each uniformly distributed on their respective supports, and that $\mathsf{supp}(\mathcal{D}_0) \subseteq \mathsf{supp}(\mathcal{D}_1)$. Then the statistical distance between $\mathcal{D}_0$ and $\mathcal{D}_1$ can be computed as follows:

$$\begin{aligned}
\Delta(\mathcal{D}_0, \mathcal{D}_1) &= \sum_{d\in\mathsf{supp}(\mathcal{D}_0)\cup\mathsf{supp}(D_1)} |\Pr[\mathcal{D}_0 = d] - \Pr[\mathcal{D}_1 = d]| \\
&= \sum_{d\in\mathsf{supp}(\mathcal{D}_0)} |\Pr[\mathcal{D}_0 = d] - \Pr[\mathcal{D}_1 = d]| + \sum_{d\in\mathsf{supp}(\mathcal{D}_1)\setminus\mathsf{supp}(\mathcal{D}_0)} |\Pr[\mathcal{D}_1 = d]| \\
&= \sum_{d\in\mathsf{supp}(\mathcal{D}_0)} |\frac{1}{|\mathsf{supp}(\mathcal{D}_0)|} - \frac{1}{|\mathsf{supp}(\mathcal{D}_1)|}| + \sum_{d\in\mathsf{supp}(\mathcal{D}_1)\setminus\mathsf{supp}(\mathcal{D}_0)} |\frac{1}{|\mathsf{supp}(\mathcal{D}_1)|}| \\
&= (|\mathsf{supp}(\mathcal{D}_0)| \cdot |\frac{1}{|\mathsf{supp}(\mathcal{D}_0)|} - \frac{1}{|\mathsf{supp}(\mathcal{D}_1)|}|) + (|\mathsf{supp}(\mathcal{D}_1) \setminus \mathsf{supp}(\mathcal{D}_0)| \cdot |\frac{1}{|\mathsf{supp}(\mathcal{D}_1)|}|) \\
&= 2 \cdot (1 - \frac{|\mathsf{supp}(\mathcal{D}_0)|}{|\mathsf{supp}(\mathcal{D}_1)|})
\end{aligned}$$

But notice that $(1 - \frac{|\mathsf{supp}(\mathcal{D}_0)|}{|\mathsf{supp}(\mathcal{D}_1)|})$ can be interpreted as $\Pr[\exists i \in [m], b \in \{0,1\} : \det(R_i) = 0 \vee \alpha_{i,b} = 0]$. For each $i \in [m]$, the probability $det(R_i) = 0$ can be bounded by applying the Schwartz-Zippel lemma to the $det(\cdot)$, which is a polynomial of degree w. Thus we have that $\Pr[det(R_i) = 0] \leq w/p$. Further, each $\alpha_{i,b}$ is zero with probability $1/p$. Hence, applying a union bound, we have that

$$\begin{aligned}
\Delta(\mathcal{D}_0, \mathcal{D}_1) &= 2 \cdot (1 - \frac{|\mathsf{supp}(\mathcal{D}_0)|}{|\mathsf{supp}(\mathcal{D}_1)|}) \\
&\leq 2 \cdot (2m/p + mw/p) \\
&\leq 8wm/p
\end{aligned}$$

□

B Proof of Proposition 1

(i) Our proof closely follows the corresponding decomposition algorithms in [BGK+13], with the following differences:

- [BGK+13]'s decomposition additionally shows that X is polynomial-sized, relying on some extra features in their construction, namely dual-input branching programs, and a modified collection of sets chosen by SetSystem.

- We use the definition of set-respecting circuits to slightly simplify the proof.

In order to prove the decomposition of C into the sum of C_x, we will define a recursive algorithm Decomp that can compute this decomposition. Before we do so, however, we define some notation to help us denote that a circuit respects a single input $x \in \{0,1\}^n$. For any circuit C that has its input wires labelled with sets in $\vec{S}$, let the *profile* of C, denoted $\mathsf{prof}(C)$, be a string $x \in \{0,1,*\}^n \cup \{\bot\}$, defined as follows:

- $x = \bot$ if for some $j \in [n]$, C contains both an input wire labelled with with a set from $\{S_{i,0} : \mathsf{inp}(i) = j\}$, and an input wire labelled with a set from $\{S_{i,1} : \mathsf{inp}(i) = j\}$.
- otherwise:
  - $x[j] = b$ if some input wire of C is labelled with with a set from $\{S_{i,b} : \mathsf{inp}(i) = j\}$, and no input wire is labelled with a set from $\{S_{i,(1-b)} : \mathsf{inp}(i) = j\}$.
  - $x[j] = *$ if no input wire of C is labelled with a set from either of $\{S_{i,b} : \mathsf{inp}(i) = j\}$ or $\{S_{i,(1-b)} : \mathsf{inp}(i) = j\}$.

We say that $\mathsf{prof}(C)$ is *consistent* if $\mathsf{prof}(C) \neq \bot$. We say that $\mathsf{prof}(C)$ is *complete and consistent* if $\mathsf{prof}(C)$ is not $\bot$, and it does not contain any $*$ characters, that is, $\mathsf{prof}(C) \in \{0,1\}^n$. Notice that if $\mathsf{prof}(C)$ is complete and consistent, then C's input wires are labelled only with sets respecting a single input $x = \mathsf{prof}(C) \in \{0,1\}^n$, that is, only with sets $\in \{S_{i,x[\mathsf{inp}(i)]}\}_{i \in [m]} \cup \{S_t\}$.

We further define $S(C)$ to be the label on the output wire of C.

The algorithm Decomp will take as input an arithmetic circuit C, and return a set $X \subseteq \{0,1,*\}^n \cup \{\bot\}$, together with a set of circuits $L = \{C_x)\}_{x \in X}$ satisfying

- $\forall x \in X$, $\mathsf{prof}(C_x) = x$
- $C = \sum_{x \in X} C_x$

We will later show that if C is $\vec{S}$-respecting, then each C_x is also $\vec{S}$-respecting, and further, that each C_x has a complete and consistent profile. This implies that, when C is $\vec{S}$-respecting, that $X \subseteq \{0,1\}^n$.

We define $\mathsf{Decomp}(C)$ recursively, as follows:

- When C is a single input wire, if C is labelled with $S_{i,b}$, Decomp outputs $X = \{\mathsf{prof}(C)\}$ and the singleton set $\{C\}$.
- When C is of the form $C_1 + C_2$, then Decomp computes $X_1, L_1 = \mathsf{Decomp}(C_1)$ and $X_2, L_2 = \mathsf{Decomp}(C_2)$, and sets $X = X_1 \cup X_2$ and $L = L_1 \cup L_2$. If L contains C_x and $C_{x'}$ with the same profile, that is, $x = x'$, then Decomp replaces the two circuits with the single circuit $(C_x + C_{x'})$, repeating this process until all the circuits in L have distinct profiles, and outputs X and L. The case for $C = C_1 - C_2$ follows analogously, except subtracting two circuits with the same profile rather than adding them.
- When C is of the form $C_1 \cdot C_2$, then Decomp recursively obtains sets $X_1, L_1 = \mathsf{Decomp}(C_1)$ and $X_2, L_2 = \mathsf{Decomp}(C_2)$. For each element $C_x \in L_1$ and $C_{x'} \in L_2$, Decomp determines $C_{x''} = C_x \cdot C_{x'}$, where $x'' = \mathsf{prof}(C_x \cdot C_{x'})$ and adds x'' to X and $C_{x''}$ to L. Decomp combines circuits with the same profile in L as described above, so that each circuit in L has a distinct profile, and outputs X and L.

By examining the description of Decomp, we see that the following properties hold inductively, at each recursive level of Decomp:

- $C = \sum_{x \in X} C_x$
- The output wire of each C_x has the same label as the output wire of C, that is, $S(C_x) = S(C)$
- If C only makes additions and multiplications respecting $\vec{S}$, then each C_x also only makes additions and multiplications respecting $\vec{S}$.

From the last two properties, we infer that if C is $\vec{S}$-respecting, then each C_x is also $\vec{S}$-respecting. Further, if C is $\vec{S}$-respecting, then no $x \in X$ contains '$*$', since in order to be $\vec{S}$-respecting, each C_x must have its output wire labelled with $[k]$, and thus must have at least one input wire labelled with some entry of $\mathbb{S}^j$, for each $j \in [n]$.

We finally argue that if C is $\vec{S}$-respecting, then $\perp \notin X$. Assume for contradiction that $\perp \in X$, that is, there is some $\vec{S}$-respecting $C_\perp$ output by $\mathsf{Decomp}(C)$. We will show that such a $C_\perp$ cannot exist, using the properties of the straddling set system.

Let $C'_\perp$ be the first sub-circuit of $C_\perp$ that has profile $\perp$. That is, all subcircuits of $C'_\perp$ have profiles $\neq \perp$, but $C'_\perp$ has profile $\perp$. Since $C_\perp$ is output by Decomp, at some recursive step of Decomp, $C'_\perp$ must have been added to the set of circuits L output by Decomp at that step.

Then this step must be a multiplication step, since addition (or subtraction) steps never introduce new profiles because they only add (or subtract) circuits with the same profiles, and in the base case, single input wires never have $\perp$ as their profile. Thus, at this step, Decomp must multiply together two circuits C_x and $C_{x'}$ such that, for some j, C_x has an input wire labelled with $S_{i,b}$, and $C_{x'}$ has an input wire labelled with $S_{i',1-b}$ for i, i' with $\mathsf{inp}(i) = \mathsf{inp}(i') = j$.

We now use the following claim, along the lines of Claim 4 in [BGK+13]:

Claim 20. *Let C be $\vec{S}$-respecting. If C' is a sub-circuit of C and $\mathcal{T}' \subseteq \mathbb{S}^j$ is an exact cover of $S(C') \cap U_j$, then there exists an exact cover $\mathcal{T}$ of $S(C) \cap U_j$ such that $\mathcal{T}' \subseteq \mathcal{T}$.*

Proof. The proof is by induction. If C is of the form $C_1 + C_2$, and $\mathcal{T}_1 \subseteq \mathbb{S}^j$ is an exact cover of $S(C_1) \cap U_j$, then $\mathcal{T}_1$ is also an exact cover of $S(C) \cap U_j$, since $S(C) = S(C_1)$, and similarly for C_2.

If C is of the form $C_1 \cdot C_2$, and $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathbb{S}^j$ are exact covers of $S(C_1) \cap U_j$ and $S(C_2) \cap U_j$ respectively, then since $S(C_1) \cap S(C_2) = \emptyset$ and since $S(C) = S(C_1) \cup S(C_2)$ then $\mathcal{T}_1 \cup \mathcal{T}_2$ is an exact cover of $S(C) \cap U_j$. □

Applying the above claim, we see that since C_x has an input wire labelled with $S_{i,b}$ as a "sub-circuit", then there exists an exact cover of $S(C_x) \cap U_j$ that contains $S_{i,b}$. Similarly, there exists an exact cover of $S(C_{x'}) \cap U_j$ that contains $S_{i',1-b}$. Since $C'_\perp = C_x \cdot C_{x'}$ and this multiplication is $\vec{S}$-respecting, there exists an exact cover of $S(C'_\perp) \cap U_j$ that contains both $S_{i,b}$ and $S_{i',1-b}$. But then there exists an exact cover of $S(C_\perp) \cap U_j$ that also contains both $S_{i,b}$ and $S_{i',1-b}$. But since $C_\perp$ is $\vec{S}$-respecting, $S(C_\perp) \cap U_j = U_j$. However, since $\mathbb{S}^j$ is a straddling set system, we know that no exact cover of U_j can contain both $S_{i,b}$ and $S_{i',1-b}$. This contradicts the existence of $C_\perp$, and so $\perp \notin X$.

We thus have that each $x \in X$ must be such that $x \in \{0,1\}^n$, implying that each C_x has a complete and consistent profile. Hence we have that that $X \subseteq \{0,1\}^n$, that $C \equiv \sum_{x \in X} C_x$, that each C_x is $\vec{S}$-respecting, and, from the earlier discussion, that each C_x has input wires using only sets $\in \{S_{i,x[\mathsf{inp}(i)]}\}_{i \in [m]} \cup \{S_t\}$.

(ii) Consider each circuit C_x in the decomposition described above. We will show that each C_x can be decomposed into α_x and p_x as defined in the proposition statement. Since C_x is an arithmetic circuit, we can write C_x as a polynomial, and in fact, as the sum of monomials s_x (possibly exponentially many), so that $C_x = \sum s_x$. Since C_x is $\vec{S}$-respecting, each s_x is $\vec{S}$-respecting also. Further, since s_x is a monomial, it can be represented as an arithmetic circuit consisting only of multiplication gates. Further, it must have exactly $m + 1$ input wires, 1 corresponding to each level of the branching program, and 1 corresponding to t, since this is the only way to obtain an output wire with label $[k]$ using only multiplication gates.

But since C_x only has input labels from sets $\in \{S_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]} \cup \{S_t\}$, so must each s_x, and thus the inputs to s_x must consist of a choice of a single element from each of $\{\alpha_{i,x[\mathsf{inp}(i)]} \cdot \tilde{B}_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]} \cup \{\mathbf{t}\}$. But then each s_x can be written as $\alpha_x \cdot \tilde{s}_x(\{\tilde{B}_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]}, \mathbf{t})$, where $\tilde{s}_x$ contains exactly one term from each $\tilde{B}_{i,x[\mathsf{inp}(i)]}$, and one term from $\mathbf{t}$. Hence $C_x = \sum s_x$ can also be written in the form $C_x = \alpha_x \cdot p_x(\{\tilde{B}_{i,x[\mathsf{inp}(i)]}\}_{i\in[m]}, \mathbf{t})$, where $p_x = \sum \tilde{s}_x$. Further, when p_x is viewed as a sum of monomials, each monomial contains exactly one term from each $\tilde{B}_{i,x[\mathsf{inp}(i)]}$, and from $\mathbf{t}$.