A Study of Goldbach's conjecture and Polignac's conjecture equivalence issues

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Abstract. The famous Goldbach's conjecture and Polignac's conjecture are two of all unsolved problems in the field of number theory today. As well known, the Goldbach's conjecture and the Polignac's conjecture are equivalent. Most of the literatures does not introduce about internal equivalence in Polignac's conjecture. In this paper, we would like to discuss the internal equivalence to the Polignac's conjecture, say $T_{2k}(x)$ and $T(x)$ are equivalent. Since $T_{2k} \sim T(x) \sim 2c \cdot \frac{x}{(\ln x)^2}$, we rewrite and re-express to $T(x) \sim T_4(x) \sim T_8(x) \sim T_{16}(x) \sim T_{32}(x) \sim T_{2^n}(x) \sim 2c \cdot \frac{x}{(\ln x)^2}$. And then connected with the Goldbach's conjecture. Finally, we will point out the important prime number symmetry role of play in these two conjectures.

Keywords: Goldbach's conjecture; de Polignac's conjecture; Equivalent;

1 Introduction

As well known the Goldbach's conjecture [1, 2, 4, 5, 9, 11, 14] and the Polignac's conjecture [6] are equivalent [12]. The Goldbach's conjecture states every even integer greater than 6 can be expressed as the sum of two primes [13]. The Polignac's conjecture describes for any positive even number n, there are infinitely many prime gaps of size n. Namely there are infinitely man cases of two consecutive prime numbers with difference n. The two conjectures have common point where they are connected the twin primes [3, 10]. Recently, Zhang [15] proposed his theorem bounded gaps between primes. His work is a huge step forward in the direction of the twin prime conjecture. But, there is still a very long way to go. In this paper, we pointed out the internal equivalence of Polignac's conjecture if all the conditions are satisfied, and then connected to Goldbach's conjecture which certainly the Goldbach's conjecture and Polignac's conjecture are equivalent. In section 2, we briefly review the Polignac's conjecture, the section 3 discusses Polignac's conjecture in arithmetic progression issues. The conclusion be draw in final section.

2 Review of Polignac's Conjecture:

For all natural number k, there are infinitely many prime pairs $(p, p + 2k)$. If $k = 1$, it is the conjecture of twin primes.

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Notation:
$T(x)$: express the number of twin pair numbers in de Polignac's conjecture.
$G(x)$: express a number of prime pairs where it matches an even number of Goldbach's conjecture, also call Goldbach partition number.
c: express a constant number.
$\mathcal{O}$: mean big O notation describes the limiting behavior of a function when the argument tends towards a particular value or infinity, usually in terms of simpler functions.
$\sim$: mean equivalence relation, such as $a \sim b$, a is approximately equal to b.
$p|k$: express the p divides the k.
$p \perp x$: mean the integer x does not divide any odd prime factors where its less than $\sqrt{x}$.

Suppose $T(x)$ is not more than sufficiently large natural number x, the number of twin prime numbers. $T(x) = 2c \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}})$, or express to $T(x) \sim 2c \cdot \frac{x}{(\ln x)^2}$. Let $T_{2k}(x)$ is not more than a sufficiently large natural number x the number of prim pair $(p, p+2k)$), $T_{2k}(x) = 2c \prod_{p\geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}})$, where ($2 < p < \sqrt{x}$, $p|k$), or $T_{2k}(x) \sim 2c \prod_{p\geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}$. Since $c = \prod_{p\geq 3}(1 - \frac{1}{(p-1)^2})$, where ($2 < p < \sqrt{x}$), p is a prime number. When $k = 2^n$, where $x > (2k)^2$, between the Goldbach's and the Polignac's conjecutre are equivalent, say $T_{2k}(x)$ and $T(x)$ are equivalent. Since $T_{2k} \sim T(x) \sim 2c \cdot \frac{x}{(\ln x)^2}$, we rewrite and re-express to $T(x) \sim T_4(x) \sim T_8(x) \sim T_{16}(x) \sim T_{32}(x) \sim T_{2^n}(x) \sim 2c \cdot \frac{x}{(\ln x)^2}$.

Let $G(x)$ is large even number of x can be expressed as form $(1+1)$ number of prime numbers,

$$G(x) = 2c \prod_{p\geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}), \tag{1}$$

where ($2 < p < \sqrt{x}$, $p|k$), or

$$G(x) \sim 2c \prod_{p\geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{2}$$

Let $T(x)$ is not more than sufficiently large natural number x the numbers of twin prime pairs,

$$T(x) = 2c \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}), \tag{3}$$

or

$$T(x) \sim 2c \cdot \frac{x}{(\ln x)^2}. \tag{4}$$

Let $T_{2k}(x)$ is not more than a sufficiently large natural number x of the prime pairs $(p, p+2k)$ numbers,

$$T_{2k}(x) = 2c \prod_{p\geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}), \tag{5}$$

where ($2 < p < \sqrt{x}$, $p|k$).
Hence

$$T_{2k}(x) \sim 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{6}$$

Among which

$$c = \prod_{p \geq 3} (1 - \frac{1}{(p-1)^2}), \tag{7}$$

where $(2 < p < \sqrt{x})$, p is a prime number.
We would like to introduce some situations as follow:

Case 1. When $x = 2^n$, or $p \perp x$ where ($2 < p < \sqrt{x}$), the Goldbach's conjecture and the conjecture of twin primes are equivalent. Namely: $G(x)$ and $T(x)$ are equivalent.

$$G(x) \sim T(x) \sim 2c \cdot \frac{x}{(\ln x)^2}. \tag{8}$$

Case 2. When $k = 2^n$, and $x > (2k)^2$, between Polignac's conjecture are equivalent. Namely: $T_{2k}(x)$ and $T(x)$ are equivalent.

$$T_{2k}(x) \sim T(x) \sim 2c \cdot \frac{x}{(\ln x)^2}. \tag{9}$$

Such as the twin prime $(p, p+2)$ numbers of $T(x)$ and the cousin prime $(p, p+4)$ numbers of $T_4(x)$, the prime pair $(p, p+8)$ numbers of $T_8(x)$ are all equivalent. Namely:

$$T(x) \sim T_4(x) \sim T_8(x) \sim T_{16}(x) \sim T_{32}(x) \sim T_{2^n}(x) \sim 2c \cdot \frac{x}{(\ln x)^2}. \tag{10}$$

Case 3. When odd prime factors p of x and odd prime factors p of k are same, and $x > (2k)^2$, the Goldbach conjecture and the Polignac's conjecture are equivalent. Say

$$G(x) \sim T_{2k}(x) \sim 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{11}$$

Such as: When $x = 2^n \cdot 3$, $2k = 6$, $G(x)$ and $T_{2k}(x)$ are equivalent. $G(x)$ and prime pair ($p, p+6$) numbers of $T_6(x)$ are equivalent. This is to say

$$G(x) \sim T_6(x) \sim 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{12}$$

When $x = 2^n \cdot 5$, $2k = 10$, $G(x)$ and $T_{2k}(x)$ are equivalent. $G(x)$ and the prime pair $(p, p+10)$ numbers of $T_{10}(x)$ are equivalent, say

$$G(x) \sim T_{10}(x) \sim 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{13}$$

When $x = 2^n \cdot 3 \cdot 5$, $2k = 30$, $G(x)$ and $T_{2k}(x)$ are equivalent, say $G(x)$ and the prime pair $(p, p+30)$ numbers of T_{30} are equivalent, be expressed

$$G(x) \sim T_{30}(x) \sim 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{14}$$

Case 4. While k_m and k_n are same odd factors p where $k_n > k_m$, and $x > (2k_n)^2$, between the Polignac's conjecture are equivalent. Namely:

$$T(2k_n) \sim T(2k_m) \sim 2c \prod_{p \geq 3} \cdot \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{15}$$

For example:
When $k = 2^n \cdot 3^m$,

$$T_6(x) \sim T_{12}(x) \sim T_{18}(x) \sim 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{16}$$

When $k = 2^n \cdot 5^m$,

$$T_{10}(x) \sim T_{20}(x) \sim T_{40}(x) \sim T_{50}(x) \sim 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{17}$$

When $k = 2^n \cdot 15^m$,

$$T_{30}(x) \sim T_{60}(x) \sim T_{90}(x) \sim 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{18}$$

From above, we obtained result as follow:

$$G(x) = 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}) \quad (2 < p < \sqrt{x}, p|x), \tag{19}$$

$$T(x) = 2c \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}). \tag{20}$$

$$T_{2k}(x) = 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}) \quad (2 < p < \sqrt{x}, p|k). \tag{21}$$

Until now, the above equivalence relation is established. Therefore, the Goldbach's conjecture and the Polignac's conjecture are homologous and equivalent in some places actually.

3 The Polignac's conjecture of the arithmetic progression

In this section, we would like to discuss about Polignac's conjecture in the arithmetic sequence problems.

3.1 The Polignac's conjecture in the arithmetic sequence

In the first, the L and the tolerance q co-prime where ($L < q$) in its arithmetic sequence, the prime pair $(p, p+q)$ (q is an even number) and prime pair $(p, p+2q)$ (q is an odd number) are infinitely many pairs of prime numbers.
Let $T(x,q,L)$ is not more than a sufficiently large natural number x of the prime pair $(p, p+q)$ or $(p, p+2q)$ numbers.
When $q = 2^n$,

$$T(x,q,L) = \frac{1}{\phi(q)} \cdot 2c \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}). \tag{22}$$

Or it can be expressed as

$$T(x,q,L) \sim \frac{1}{\phi(q)} \cdot 2c \cdot \frac{x}{(\ln x)^2}, \tag{23}$$

($\phi(q)$ is Euler phi-function).
If $q = 2$, it is the conjecture of twin primes.
If $q \neq 2^n$,

$$T(x,q,L) = \frac{1}{\phi(q)} \cdot 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{X}}{\ln \sqrt{x}}), \tag{24}$$

where ($\phi(q)$ is Euler phi-function, ($2 < p < \sqrt{x}$, $p|q$).
Furthermore

$$T(x,q,L) \sim \frac{1}{\phi(q)} \cdot 2c \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2}. \tag{25}$$

When $q = 3k$, Namely: Green and Tao [7, 8] demonstrated that primes arithmetic sequence can be arbitrarily long problem.
Among which

$$c = \prod_{p \geq 3} (1 - \frac{1}{(p-1)^2}), \tag{26}$$

where $(2 < p < \sqrt{x})$, p is a prime number.

3.2 The smallest prime pairs problem of the arithmetic sequence

In the first, the L and the tolerance q co-prime ($L < q$) of the arithmetic sequence, there exists at a least upper bound of the prime pair $(p, p+q)$ (q is an even number) and prime pair $(p, p+2q)$ (q is an odd number): $\phi(q) \cdot q^2$, ($\phi(q)$ is Euler phi-function), namely: $p < \phi(q) \cdot q^2$.

3.3 The Generalized Goldbach's conjecture: The corollary of symmetry primes

In this section, we would like to introduce about generalized Goldbach's conjecture issue, say the prime numbers symmetrical situation in the arithmetic sequence. The detail follow:

Corollary 1. *In the former L and tolerances q ($L < q, q \geq 2$) co-prime to arithmetic sequence, for any one item of this sequence such as $\frac{x}{2}$ where $\frac{x}{2} > \phi(q) \cdot q^2$, ($q$ is the Euler phi-function). There exists at least one pair of primes about $\frac{x}{2}$, namely x can be expressed for the sum of two primes in its series.*

Proof.
Let $G(x, q, L)$ are numbers of prime in its arithmetic sequence which it can be presented to $(1 + 1)$ form by variable integer x.

1.) If $q = 2^m$, $G(x, q, L) = \frac{1}{\phi(q)} \cdot 2c \cdot \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}})$, where $2 < p < \sqrt{x}, p|x$.
2.) If $q = 2^m$ and $p \perp x$ where $(2 < p < \sqrt{x})$, say

$$G(x, q, L) = \frac{1}{\phi(q)} \cdot 2c \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}). \tag{27}$$

3.) If q is an odd number,

$$G(x, q, L) = \frac{1}{(\phi(q) - 1)} \cdot 2c \cdot \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}), \tag{28}$$

where $2 < p < \sqrt{x}$, and $p|x$.
4.) If q is an odd number, and $x = 2^n$ or $p \perp x$ where $(2 < p < \sqrt{x})$,

$$G(x, q, L) = \frac{1}{(\phi(q) - 1)} \cdot 2c \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}). \tag{29}$$

5.) If $q = 2^m \cdot j$ (j is odd number),

$$G(x, q, L) = \frac{1}{(\phi(2^m))} \cdot \frac{1}{(\phi(j) - 1)} \cdot 2c \cdot \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}), \tag{30}$$

where $2 < p < \sqrt{x}$, and $p|x$.
6.) If $q = 2^m \cdot j$ (j is an odd number), and $p \perp x$ where $(2 < p < \sqrt{x})$,

$$G(x, q, L) = \frac{1}{\phi(2^m)} \cdot \frac{1}{(\phi(j) - 1)} \cdot 2c \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}) \tag{31}$$

where $(2 < p < \sqrt{x}, p|x)$.

It can be seen, we classify two situations.

1. While $q = 2$; $q = 3$ or $q = 6$ alternative,

$$G(x) = G(x, q, L) = 2c \cdot \prod_{p \geq 3} \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}), \tag{32}$$

where $(2 < p < \sqrt{x}, p|x)$, this formula prompts the expression of Goldbach conjecture.

2. While $q = 2$; $q = 3$ or $q = 6$ alternative, and $x = 2^n$ or $p \perp x$ where $(2 < p < \sqrt{x})$,

$$G(x) = G(x, q, L) = 2c \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}}). \tag{33}$$

We say, the Goldbach's conjecture expression is same with twin prime conjecture. Since

$$c = \prod_{p \geq 3} (1 - \frac{1}{(p-1)^2}), \tag{34}$$

where $(2 < p < \sqrt{x})$, p is a prime number.

4 Conclusion

In this paper, we clearly give the expression of Polignac's conjecture, say $T_{2k}(x) = 2c \prod \frac{(p-1)}{(p-2)} \cdot \frac{x}{(\ln x)^2} + \mathcal{O}(\frac{\sqrt{x}}{\ln \sqrt{x}})$ where $2 < p < \sqrt{x}$, while $k = 1$, it matches twin-prime conjecture. Previously, there have some references discussed the equivalent of conjectures between Goldbach and Polignac. But, there is no one to discuss about internal equivalence in Polignac's conjecture. We clearly describe the equivalent situations in this topic, and also give the detailed proofs. Although we have not found any regularity to Polignac's conjecture. We suppose upon on this equivalence point view, we found the orderliness in disorder of prime number distribution. To prime numbers symmetry in Generalized Goldbach conjecture, is our future research goal.

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