Tight Security Bounds for Triple Encryption

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Abstract. In this paper, we revisit the old problem asking the exact provable security of triple encryption in the ideal cipher model. For a blockcipher with key length κ and block size n, triple encryption is known to be secure up to $2^{\kappa+\frac{1}{2}\min\{\kappa,n\}}$ queries, while the best attack requires $2^{\kappa+\min\{\kappa,\frac{n}{2}\}}$ query complexity. So there is a gap between the upper and lower bounds for the security of triple encryption. We close this gap by proving the security up to $2^{\kappa+\min\{\kappa,\frac{n}{2}\}}$ query complexity. With the DES parameters, triple encryption is secure up to $2^{82.5}$ queries, greater than the current bound of $2^{78.3}$ and comparable to $2^{83.5}$ for 2-XOR-cascade [9].
We also analyze the security of two-key triple encryption, where the first and the third keys are identical. We prove that two-key triple encryption is secure up to $2^{\kappa+\min\{\kappa,\frac{n}{2}\}}$ queries to the underlying blockcipher and $2^{\min\{\kappa,\frac{n}{2}\}}$ queries to the outer permutation. For the DES parameters, this result is interpreted as the security of two-key triple encryption up to 2^{32} plaintext-ciphertext pairs and $2^{81.7}$ blockcipher encryptions.

1 Introduction

A blockcipher is said to be secure if there is no known attack faster than exhaustive key search. On the other hand, without utilizing any weakness of a blockcipher, one can recover its secret key simply by trying all possible keys over a small number of plaintext-ciphertext pairs. So the key length of a blockcipher can be viewed as the maximum level of security that the blockcipher is able to provide. However the key length providing a sufficient level of security might change over time. For example, the Data Encryption Standard (DES) [1] using 56-bit keys was one of the most predominant algorithms for encryption of data. No feasible attacks faster than exhaustive key search have been proposed (as most of them require a huge amount of data), while the availability of increasing computational power made the brute-force attack itself practical. As a result, DES was replaced by a new standard algorithm AES [4]. On the other hand, in order to protect legacy applications based on DES, there have been considerable research on constructing DES-based encryption schemes which employ longer keys. This approach is called *key-length extension*, for which Triple-DES [2, 3, 5] and DESX (due to Rivest) are the most popular constructions.

The Triple-DES approach transforms a κ-bit key n-bit blockcipher E into an encryption scheme that accepts three κ-bit keys $k_1, k_2, k_3 \in \{0,1\}^\kappa$ and encrypts an n-bit message block u as $v = E_{k_3}(E_{k_2}(E_{k_1}(u)))$ as seen in Figure 1. Bellare and Rogaway [6] proved its security up to $2^{\kappa+\frac{1}{2}\min\{n,\kappa\}}$ query complexity assuming E is an ideal blockcipher, and later Gaži and Maurer [8] fixed some flaws of the original proof.

The DESX approach transforms a κ-bit key n-bit blockcipher E into an encryption scheme that accepts a κ-bit key $k \in \{0,1\}^\kappa$ and additional n-bit whitening keys $k_i, k_o \in \{0,1\}^n$ and encrypts an n-bit message block u as $v = k_o \oplus E_k(k_i \oplus u)$. Killan and Rogaway [10] proved its security up to $2^{\frac{\kappa+n}{2}}$ query complexity. As an efficient key-length extension, Gaži and Tessaro [9] proposed a cascade of two DESX schemes with some refinement, and proved its security up to $2^{\kappa+\frac{n}{2}}$ query complexity.

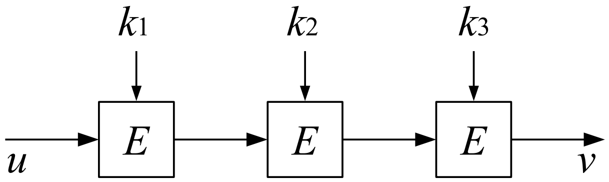


Fig. 1. Triple encryption

Our Contribution. In this paper, we revisit the old problem asking the exact provable security of triple encryption in the ideal cipher model. Since the best information theoretic attack requires $2^{\kappa+\min\{\kappa,\frac{n}{2}\}}$ query complexity [11](see also Appendix A), there has been a gap between the upper and lower bounds for the security of triple encryption. We close the gap by proving the security up to $2^{\kappa+\min\{\kappa,\frac{n}{2}\}}$ query complexity, improving over the currently known bound $2^{\kappa+\frac{1}{2}\min\{\kappa,n\}}$. With the DES parameters and the threshold distinguishing advantage $1/2$, triple encryption is secure up to $2^{82.5}$ queries, greater than the current bound $2^{78.3}$ and comparable to $2^{83.5}$ for 2-XOR-cascade.

In order to save key materials, the standards define an alternative keying option: k_1 and k_2 are independent, and $k_3 = k_1$[1]. However this variant, called *two-key triple encryption*, is vulnerable to the classic meet-in-the-middle attack using approximately 2^κ forward/backward queries to the underlying blockcipher and 2^κ queries to the outer permutation. This attack was refined in [12] into a trade-off between time and data: given q_P plaintext-ciphertext pairs one can find the secret key by making $2^{\kappa+n}/q_P$ queries to the underlying blockcipher. So these attacks naturally raise the question if the two-key triple encryption is secure with data complexity limited to a certain bound. We answer this question affirmatively, proving that two-key triple encryption is secure up to $2^{\kappa+\min\{\kappa,\frac{n}{2}\}}$ blockcipher queries and $2^{\min\{\kappa,\frac{n}{2}\}}$ construction queries. For the DES parameters, this result is interpreted as the security of two-key triple encryption up to 2^{32} plaintext-ciphertext pairs and $2^{81.7}$ blockcipher encryptions. Table 2 compares upper bounds on distinguishing advantage for three-key and two-key triple encryption with the DES parameters $\kappa = 56$ and $n = 64$.

2 Preliminaries

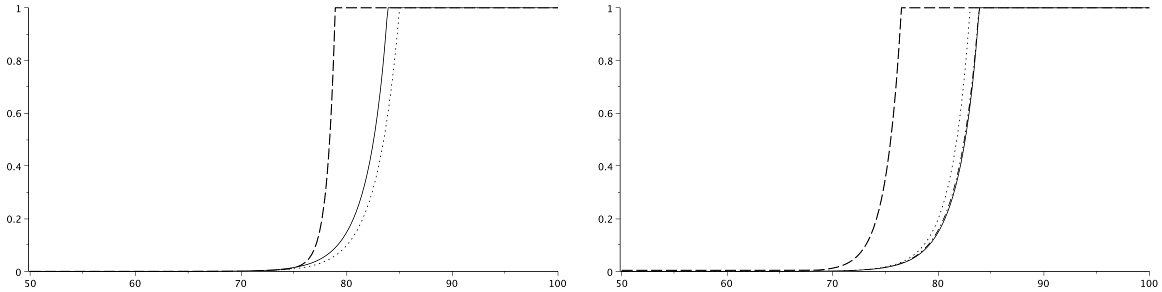
2.1 General Notation

For an integer $n \geq 1$, let $I_n = \{0,1\}^n$ be the set of binary strings of length n. The set of all permutations on I_n will be denoted $\mathcal{P}_n$. For integers $1 \leq s \leq t$, we will write $(t)_s = t(t-1)\cdots(t-s+1)$ and $(t)_0 = 1$ by convention.

2.2 The Ideal Cipher Model

A blockcipher is a function family $E : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ such that for all $k \in \mathcal{K}$ the mapping $E(k,\cdot)$ is a permutation on I_n. We write $BC(\mathcal{K}, n)$ to mean the set of all such blockciphers, shortening to $BC(\kappa, n)$ when $\mathcal{K} = \{0,1\}^\kappa$. In the ideal cipher model, a blockcipher E is chosen from $BC(\mathcal{K}, n)$ uniformly at random. It allows for two types of oracle queries $E(k,x)$

[1] In the standards, the second key is applied to the decryption algorithm, while it makes no difference in provable security in the ideal cipher model.



(a) Left to right: (1) triple encryption [6, 8] (2) triple encryption (this paper) (3) 2-XOR-cascade [9]. The number of construction queries is set to be the maximum 2^n.

(b) Left to right: (1-3) two-key triple encryption with $q_P = 2^{40}, 2^{32}, 2^{24}$, respectively (4) three-key triple encryption (this paper). Here q_P is the number of construction queries.

Fig. 2. Upper bounds on distinguishing advantage for three-key and two-key triple encryption. Given as functions of $\log_2 q$ where q is the number of queries made to the underlying blockcipher.

and $E^{-1}(k, y)$ for $x, y \in \{0,1\}^n$ and $k \in \mathcal{K}$.[2] The response to an inverse query $E^{-1}(k, y)$ is $x \in \{0,1\}^n$ such that $E(k, x) = y$. For simplicity of notation, we will write $K = 2^\kappa$ and $N = 2^n$.

2.3 Indistinguishability

Let C be an n-bit encryption scheme that employs λ-bit keys and makes oracle queries to a blockcipher $E \in BC(\kappa, n)$. So each key $\mathbf{k} \in \{0,1\}^\lambda$ and a blockcipher $E \in BC(\kappa, n)$ define a permutation $\mathsf{C}_\mathbf{k}[E]$ on I_n. In the *indistinguishability* framework (in the ideal cipher model), $\mathsf{C}_\mathbf{k}[E]$ uses a random secret key $\mathbf{k}$ and makes oracle queries to an ideal blockcipher E, while a permutation P is chosen uniformly at random from $\mathcal{P}_n$. A distinguisher $\mathcal{A}$ would like to tell apart two worlds $(\mathsf{C}_\mathbf{k}[E], E)$ and (P, E) by adaptively making forward and backward queries to the permutation and the blockcipher. Formally, $\mathcal{A}$'s distinguishing advantage is defined by

$$\begin{aligned}\mathbf{Adv}_{\mathsf{C}}^{\mathsf{PRP}}(\mathcal{A}) = \mathbf{Pr}\left[P \xleftarrow{\$} \mathcal{P}_n, E \xleftarrow{\$} BC(\kappa, n) : \mathcal{A}[P, E] = 1\right] \\ - \mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} \{0,1\}^\lambda, E \xleftarrow{\$} BC(\kappa, n) : \mathcal{A}[\mathsf{C}_\mathbf{k}[E], E] = 1\right].\end{aligned}$$

Here we might assume that the distinguishing advantage is always nonnegative. For q_P, $q_E > 0$, we define

$$\mathbf{Adv}_{\mathsf{C}}^{\mathsf{PRP}}(q_P, q_E) = \max_{\mathcal{A}} \mathbf{Adv}_{\mathsf{C}}^{\mathsf{PRP}}(\mathcal{A})$$

where the maximum is taken over all distinguishers $\mathcal{A}$ making at most q_P queries to the outer permutation and at most q_E queries to the underlying blockcipher.

COMBINATORIAL FRAMEWORK. We assume that a distinguisher $\mathcal{A}$ making q_P forward and/or backward queries to the permutation oracle records a query history

$$\mathcal{Q}_P = (u^j, v^j)_{1 \le j \le q_P}$$

where (u^j, v^j) represents the evaluation obtained by the j-th query to the permutation oracle. So according to the instantiation, it implies either $\mathsf{C}_\mathbf{k}[E](u^j) = v^j$ or $P(u^j) = v^j$. By making

[2] We interchangeably use both representations $E(k, x)$ and $E_k(x)$, and similarly $E^{-1}(k, y)$ and $E_k^{-1}(y)$.

q_E queries to the underlying blockcipher E, $\mathcal{A}$ also records the second query history

$$\mathcal{Q}_E = (x^j, k^j, y^j)_{1 \le j \le q_E}$$

where (x^j, k^j, y^j) represents the evaluation $E(k^j, x^j) = y^j$ obtained by the j-th query to the blockcipher. Sometimes we need to record the direction in which a blockcipher query has been made. If the j-th query has been made in a forward direction, the evaluation might be denoted as $(x^j, k^j, y^j, +)$. If it is obtained by a backward query, it is denoted as $(x^j, k^j, y^j, -)$. The pair of the query histories

$$\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E)$$

is called the *transcript* of the attack; it contains all the information that $\mathcal{A}$ has obtained at the end of the attack. In this work, we will only consider information theoretic distinguishers. Therefore we can assume that a distinguisher is deterministic without making any redundant queries, and hence the output of $\mathcal{A}$ can be regarded as a function of $\mathcal{T}$, denoted $\mathcal{A}(\mathcal{T})$ or $\mathcal{A}(\mathcal{Q}_P, \mathcal{Q}_E)$.

If a permutation $\mathsf{C}_{\mathbf{k}}[E]$(resp. P) is consistent with $\mathcal{Q}_P$, i.e., $\mathsf{C}_{\mathbf{k}}[E](u^j) = v^j$(resp. $P(u^j) = v^j$) for every $j = 1, \ldots, q_P$, then we will write $\mathsf{C}_{\mathbf{k}}[E] \vdash \mathcal{Q}_P$(resp. $P \vdash \mathcal{Q}_P$). Similarly, if a blockcipher $E \in BC(\kappa, n)$ is consistent with $\mathcal{Q}_E$ (i.e., $E(k^j, x^j) = y^j$ for $j = 1, \ldots, q_E$), then we will write $E \vdash \mathcal{Q}_E$. Using these notations, we have

$$\begin{aligned}\mathbf{Adv}^{\mathsf{PRP}}_{\mathsf{C}}(\mathcal{A}) = &\sum_{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1} \mathbf{Pr}\left[P \xleftarrow{\$} \mathcal{P}_n, E \xleftarrow{\$} BC(\kappa, n) : P \vdash \mathcal{Q}_P \wedge E \vdash \mathcal{Q}_E\right] \\ &- \sum_{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1} \mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} \{0,1\}^{\lambda}, E \xleftarrow{\$} BC(\kappa, n) : \mathsf{C}_{\mathbf{k}}[E] \vdash \mathcal{Q}_P \wedge E \vdash \mathcal{Q}_E\right]\end{aligned}$$

where the sum is taken over all the possible transcripts $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E)$ such that $\mathcal{A}(\mathcal{Q}_P, \mathcal{Q}_E) = 1$. Here we only consider "valid" transcripts that $\mathcal{A}$ might produce by communicating with a permutation $P \in \mathcal{P}_n$ and a blockcpher $E \in BC(\kappa, n)$. For example, in a valid transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E)$, query-response pairs (x, k, y) and (x', k, y) with $x \neq x'$ could not be both contained in $\mathcal{Q}_E$. If $\mathcal{A}$'s first query is supposed to be a forward query made to the construction with u^*, then we will not consider $\mathcal{Q}_P = (u^j, v^j)_{1 \le j \le q_P}$ such that $u^1 \neq u^*$.

3 Security of Triple Encryption

In this section, we prove the security of triple encryption TE using κ-bit key n-bit blockcipher E. Our goal is to prove the security of TE far beyond N queries, so we will assume that a distinguisher makes all possible N queries to the outer permutation. Let q denote the number of queries made to the underlying blockcipher E.

3.1 Bad Transcripts

We first define certain "bad" transcripts. Typically we expect the probability of the bad transcripts being produced in the ideal world to be small, while excluding such transcripts would make the analysis easier. Precisely, a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E)$ is defined to be *bad* if either

$$\max_{y^* \in I_n} |\{(x, k, y^*, +) \in \mathcal{Q}_E\}| > L$$

or

$$\max_{x^* \in I_n} |\{(x^*, k, y, -) \in \mathcal{Q}_E\}| > L$$

for a certain parameter $L > 0$. So a bad transcript means an L-multi-collision on the blockcipher obtained by only forward queries or only backward queries. We will denote the set of bad transcripts by $\mathsf{BadT}(L)$. Let a distinguisher $\mathcal{A}$ interact with a truly random permutation P. If $L \leq K$, then

$$\mathbf{Pr}\left[P \overset{\$}{\leftarrow} \mathcal{P}_n, E \overset{\$}{\leftarrow} BC(\kappa, n) : \mathcal{A} \text{ produces } (\mathcal{Q}_P, \mathcal{Q}_E) \in \mathsf{BadT}(L)\right]$$
$$\leq N \binom{q}{L} \left(\frac{1}{N}\right)^L \leq N \left(\frac{eq}{LN}\right)^L \overset{\text{def}}{=} \epsilon_1. \quad (1)$$

If $L > K$, then we have

$$\mathbf{Pr}\left[P \overset{\$}{\leftarrow} \mathcal{P}_n, E \overset{\$}{\leftarrow} BC(\kappa, n) : \mathcal{A} \text{ produces } (\mathcal{Q}_P, \mathcal{Q}_E) \in \mathsf{BadT}(L)\right] = 0.$$

We will define a directed graph $\mathcal{G}$ on I_n using a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E)$, where $(u, v) \in \mathcal{Q}_P$ if and only if $\mathcal{G}$ contains an edge $v \to u$ (with no label and the direction inversed) and $(x, k, y, \sigma) \in \mathcal{Q}_E$ if and only if $\mathcal{G}$ contains an edge $x \xrightarrow{(k,\sigma)} y$, where $\sigma \in \{+, -\}$ denotes the sign. Sometimes we will drop the sign for simplicity. In this graph, we define certain types of paths.

(1) 3-paths of type $(0, +)$: $u \xrightarrow{(k,\sigma)} x \xrightarrow{(k',+)} y \xrightarrow{(k'',\sigma'')} v$
(2) 3-paths of type $(0, -)$: $u \xrightarrow{(k,\sigma)} x \xrightarrow{(k',-)} y \xrightarrow{(k'',\sigma'')} v$
(3) 4-paths of type $(1, +)$: $x \xrightarrow{(k,\sigma)} y \xrightarrow{(k',+)} v \longrightarrow u \xrightarrow{(k'',\sigma'')} z$
(4) 4-paths of type $(1, -)$: $x \xrightarrow{(k,\sigma)} y \xrightarrow{(k',-)} v \longrightarrow u \xrightarrow{(k'',\sigma'')} z$
(5) 4-paths of type $(2, +)$: $y \xrightarrow{(k,\sigma)} v \longrightarrow u \xrightarrow{(k',+)} z \xrightarrow{(k'',\sigma'')} w$
(6) 4-paths of type $(2, -)$: $y \xrightarrow{(k,\sigma)} v \longrightarrow u \xrightarrow{(k',-)} z \xrightarrow{(k'',\sigma'')} w$

All these paths will be called *bad* and we would like to restrict the number of bad paths by excluding bad transcripts. For a node $x \in I_n$, let $d_{in}(x)$ and $d_{out}(x)$ denote the in-degree and the out-degree of x, respectively, with respect to the edges defined by $\mathcal{Q}_E$. If a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E)$ is not bad, then the number of 3-paths of type $(0, +)$ is upper bounded by

$$KL \sum_{y \in I_n} d_{out}(y) \leq KLq$$

since for each $y \in I_n$ the number of $(k, +)$-labeled edges coming into y is at most L, and for each $x \in I_n$ such that there exists an edge $x \xrightarrow{(k,+)} y$ in $\mathcal{G}$, we have $d_{in}(x) \leq K$. Applying similar arguments to the other types of paths, we conclude the number of bad paths are upper bounded by $6KLq$.

3.2 Bad Keys

Given a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)$, we define three types of "bad" keys.

Colliding Keys. Let

$$\mathsf{Col} = \{(k_1, k_2, k_3) \in I_\kappa^3 : \text{either } k_1 = k_2 \text{ or } k_1 = k_3 \text{ or } k_2 = k_3\}$$

denote the set of "colliding" keys. We have

$$\mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} I_\kappa^3 : \mathbf{k} \in \mathsf{Col}\right] \le \frac{3}{K}.$$

Heavy Keys. For a fixed parameter $M > 0$, we say a key $\mathbf{k} = (k_1, k_2, k_3) \in I_\kappa^3$ is *heavy* if

$$|\{k_i : (x, k_i, y) \in \mathcal{Q}_E\}| > M,$$

for some $i = 1, 2, 3$. Let $\mathsf{Hv}(M)$ denote the set of heavy keys. Since the number of keys that are queried more than M times is at most q/M, we have

$$\mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} I_\kappa^3 : \mathbf{k} \in \mathsf{Hv}(M)\right] \le \frac{3q}{KM}.$$

Keys Making Relative Bad Paths. The set of keys $\mathbf{k} = (k_1, k_2, k_3)$ such that there is a path $u \overset{(k_1,\sigma)}{\longrightarrow} x \overset{(k_2,+)}{\longrightarrow} y \overset{(k_3,\sigma'')}{\longrightarrow} v$ of type $(0, +)$ in $\mathcal{G}$ is denoted by $\mathsf{Chn}_{(0,+)}$. With similar definitions for other types of bad paths, let

$$\mathsf{Chn} = \mathsf{Chn}_{(0,+)} \cup \mathsf{Chn}_{(0,-)} \cup \mathsf{Chn}_{(1,+)} \cup \mathsf{Chn}_{(1,-)} \cup \mathsf{Chn}_{(2,+)} \cup \mathsf{Chn}_{(2,-)}.$$

Since $\mathsf{Chn} \le 6KLq$, we have

$$\mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} I_\kappa^3 : \mathbf{k} \in \mathsf{Chn}\right] \le \frac{6Lq}{K^2}.$$

Summary. Let $\mathsf{BadK}(M) = \mathsf{Col} \cup \mathsf{Hv} \cup \mathsf{Chn}$ be the set of bad keys. Then

$$\mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} I_\kappa^3 : \mathbf{k} \in \mathsf{BadK}(M)\right] \le \frac{3}{K} + \frac{3q}{KM} + \frac{6Lq}{K^2} \overset{\text{def}}{=} \epsilon_2. \tag{2}$$

By abuse of notation, we will sometimes use $\mathsf{BadK}(M)$ to denote the event that a random key $\mathbf{k}$ is contained in $\mathsf{BadK}(M)$.

3.3 Main Lemma

The security proof of triple encryption TE is based on the following lemma.

Lemma 1. *Let q, L, M, $\delta > 0$ and let $\mathcal{A}$ be an distinguisher making all possible N queries to the outer permutation and at most q queries to the underlying blockcipher. Assume that for any transcript $(\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)$ such that $|\mathcal{Q}_P| = N$ and $|\mathcal{Q}_E| = q$,*

$$\mathsf{p}_1(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) \ge (1-\delta)\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M))$$

where

$$\mathsf{p}_1(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) = \mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} I_\kappa^3, E \xleftarrow{\$} BC(\kappa, n) : E \vdash \mathcal{Q}_E \;\middle|\; \mathsf{TE}_{\mathbf{k}}[E] \vdash \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)\right],$$

$$\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) = \mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} I_\kappa^3, P \xleftarrow{\$} \mathcal{P}_n, E \xleftarrow{\$} BC(\kappa, n) : E \vdash \mathcal{Q}_E \;\middle|\; P \vdash \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)\right].$$

Then we have

$$\mathbf{Adv}_{\mathsf{TE}}^{\mathsf{PRP}}(\mathcal{A}) \le \delta + \epsilon_1 + \epsilon_2$$

where ϵ_1 and ϵ_2 are defined as in (1) and (2), respectively.

Proof. For a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)$, define

$$\begin{aligned}
\mathsf{p}_1(\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) &= \mathbf{Pr}\left[\mathbf{k} \overset{\$}{\leftarrow} I_\kappa^3, E \overset{\$}{\leftarrow} BC(\kappa, n) : \mathsf{TE}_{\mathbf{k}}[E] \vdash \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)\right], \\
\mathsf{p}_2(\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) &= \mathbf{Pr}\left[\mathbf{k} \overset{\$}{\leftarrow} I_\kappa^3, P \overset{\$}{\leftarrow} \mathcal{P}_n : P \vdash \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)\right], \\
\mathsf{p}_1(\mathcal{Q}_P \wedge \mathcal{Q}_E \wedge \neg\mathsf{BadK}(M)) &= \mathbf{Pr}\left[\mathbf{k} \overset{\$}{\leftarrow} I_\kappa^3, E \overset{\$}{\leftarrow} BC(\kappa, n) : \mathsf{TE}_{\mathbf{k}}[E] \vdash \mathcal{Q}_P \wedge E \vdash \mathcal{Q}_E \wedge \neg\mathsf{BadK}(M)\right] \\
&= \mathsf{p}_1(\mathcal{Q}_E | \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M))\mathsf{p}_1(\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)), \\
\mathsf{p}_2(\mathcal{Q}_P \wedge \mathcal{Q}_E \wedge \neg\mathsf{BadK}(M)) &= \mathbf{Pr}\left[\mathbf{k} \overset{\$}{\leftarrow} I_\kappa^3, P \overset{\$}{\leftarrow} \mathcal{P}_n, E \overset{\$}{\leftarrow} BC(\kappa, n) : P \vdash \mathcal{Q}_P \wedge E \vdash \mathcal{Q}_E \wedge \neg\mathsf{BadK}(M)\right] \\
&= \mathsf{p}_2(\mathcal{Q}_E | \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M))\mathsf{p}_2(\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)).
\end{aligned}$$

Since $\mathsf{TE}_{\mathbf{k}}[E]$ becomes a truly random permutation without any condition on the underlying blockcipher E, we have

$$\mathsf{p}_1(\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) = \mathsf{p}_2(\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)).$$

In the following estimation, we will also use inequalities

$$\begin{aligned}
&\sum_{\substack{\mathcal{A}(\mathcal{Q}_P, \mathcal{Q}_E)=1 \\ (\mathcal{Q}_P, \mathcal{Q}_E) \in \mathsf{BadT}(L)}} \mathbf{Pr}\left[\mathbf{k} \overset{\$}{\leftarrow} I_\kappa^3, P \overset{\$}{\leftarrow} \mathcal{P}_n, E \overset{\$}{\leftarrow} BC(\kappa, n) : P \vdash \mathcal{Q}_P \wedge E \vdash \mathcal{Q}_E\right] \\
&\qquad \leq \mathbf{Pr}\left[P \overset{\$}{\leftarrow} \mathcal{P}_n, E \overset{\$}{\leftarrow} BC(\kappa, n) : \mathcal{A} \text{ produces } (\mathcal{Q}_P, \mathcal{Q}_E) \in \mathsf{BadT}(L)\right] \leq \epsilon_1
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\substack{\mathcal{A}(\mathcal{Q}_P, \mathcal{Q}_E)=1 \\ (\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)}} \mathbf{Pr}\left[\mathbf{k} \overset{\$}{\leftarrow} I_\kappa^3, P \overset{\$}{\leftarrow} \mathcal{P}_n, E \overset{\$}{\leftarrow} BC(\kappa, n) : P \vdash \mathcal{Q}_P \wedge E \vdash \mathcal{Q}_E \wedge \mathsf{BadK}(M)\right] \\
&\leq \sum_{\substack{\mathcal{A}(\mathcal{Q}_P, \mathcal{Q}_E)=1 \\ (\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)}} \mathbf{Pr}\left[\mathbf{k} \overset{\$}{\leftarrow} I_\kappa^3, P \overset{\$}{\leftarrow} \mathcal{P}_n, E \overset{\$}{\leftarrow} BC(\kappa, n) : \mathsf{BadK}(M) \;\middle|\; P \vdash \mathcal{Q}_P \wedge E \vdash \mathcal{Q}_E\right] \\
&\qquad \times \mathbf{Pr}\left[P \overset{\$}{\leftarrow} \mathcal{P}_n, E \overset{\$}{\leftarrow} BC(\kappa, n) : P \vdash \mathcal{Q}_P \wedge E \vdash \mathcal{Q}_E\right] \\
&\qquad \leq \epsilon_2 \sum_{\substack{\mathcal{A}(\mathcal{Q}_P, \mathcal{Q}_E)=1 \\ (\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)}} \mathbf{Pr}\left[P \overset{\$}{\leftarrow} \mathcal{P}_n, E \overset{\$}{\leftarrow} BC(\kappa, n) : P \vdash \mathcal{Q}_P \wedge E \vdash \mathcal{Q}_E\right] \leq \epsilon_2.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathbf{Adv}^{\mathsf{PRP}}_{\mathsf{TE}}(\mathcal{A}) \le & \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathbf{Pr}\left[\mathbf{k}\overset{\$}{\leftarrow} I^3_\kappa, P\overset{\$}{\leftarrow}\mathcal{P}_n, E\overset{\$}{\leftarrow} BC(\kappa,n) : P\vdash\mathcal{Q}_P\wedge E\vdash\mathcal{Q}_E\right]\\
& - \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathbf{Pr}\left[\mathbf{k}\overset{\$}{\leftarrow} I^3_\kappa, E\overset{\$}{\leftarrow} BC(\kappa,n) : \mathsf{TE}_{\mathbf{k}}[E]\vdash\mathcal{Q}_P\wedge E\vdash\mathcal{Q}_E\right]\\
& + \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\in\mathsf{BadT}(L)}} \mathbf{Pr}\left[\mathbf{k}\overset{\$}{\leftarrow} I^3_\kappa, P\overset{\$}{\leftarrow}\mathcal{P}_n, E\overset{\$}{\leftarrow} BC(\kappa,n) : P\vdash\mathcal{Q}_P\wedge E\vdash\mathcal{Q}_E\right]\\
& \le \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathsf{p}_2(\mathcal{Q}_P\wedge\mathcal{Q}_E\wedge\neg\mathsf{BadK}(M)) - \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathsf{p}_1(\mathcal{Q}_P\wedge\mathcal{Q}_E\wedge\neg\mathsf{BadK}(M))\\
& + \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathbf{Pr}\left[\mathbf{k}\overset{\$}{\leftarrow} I^3_\kappa, P\overset{\$}{\leftarrow}\mathcal{P}_n, E\overset{\$}{\leftarrow} BC(\kappa,n) : P\vdash\mathcal{Q}_P\wedge E\vdash\mathcal{Q}_E\wedge\mathsf{BadK}(M)\right] + \epsilon_1\\
& \le \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M))\mathsf{p}_2(\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M))\\
& - \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathsf{p}_1(\mathcal{Q}_E|\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M))\mathsf{p}_1(\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M)) + \epsilon_2 + \epsilon_1\\
& \le \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M))\mathsf{p}_2(\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M))\\
& - (1-\delta) \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M))\mathsf{p}_2(\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M)) + \epsilon_2 + \epsilon_1\\
& \le \delta \sum_{\substack{\mathcal{A}(\mathcal{Q}_P,\mathcal{Q}_E)=1\\(\mathcal{Q}_P,\mathcal{Q}_E)\notin\mathsf{BadT}(L)}} \mathsf{p}_2(\mathcal{Q}_P\wedge\mathcal{Q}_E\wedge\neg\mathsf{BadK}(M)) + \epsilon_2 + \epsilon_1 \le \delta + \epsilon_1 + \epsilon_2. \qquad \square
\end{aligned}$$

3.4 Comparing $\mathsf{p}_1(\mathcal{Q}_E|\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M))$ and $\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M))$

In this section, we compute a small δ satisfying the condition of Lemma 1. First, we fix a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)$ and a key $\mathbf{k} = (k_1, k_2, k_3) \notin \mathsf{BadK}(M)$. Then we decompose the blockcipher query history $\mathcal{Q}_E$ as

$$\mathcal{Q}_E = \mathcal{Q}_E^{k_1} \cup \mathcal{Q}_E^{k_2} \cup \mathcal{Q}_E^{k_3} \cup \mathcal{Q}_E^{*}$$

where

$$\mathcal{Q}_E^{k_i} = \{(x, k, y) \in \mathcal{Q}_E : k = k_i\}$$

for $i = 1, 2, 3$, and $\mathcal{Q}_E^*$ is the set of the remaining queries. Let $h_i = |\mathcal{Q}_E^{k_i}|$ for $i = 1, 2, 3$. Since $\mathbf{k}$ is not a bad key, h_i's are not greater than M. Since the choice of $\mathbf{k}$ and P is independent of E, we have

$$\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P\wedge\neg\mathsf{BadK}(M)) = \mathbf{Pr}\left[P\overset{\$}{\leftarrow}\mathcal{P}_n, E\overset{\$}{\leftarrow} BC(\kappa,n) : E\vdash\mathcal{Q}_E\right] = \mathsf{p}^*\cdot\frac{1}{(N)_{h_1}(N)_{h_2}(N)_{h_3}}$$

where

$$\mathsf{p}^* = \mathbf{Pr}\left[E \overset{\$}{\leftarrow} BC(\kappa, n) : E \vdash \mathcal{Q}_E^*\right].$$

On the other hand, let

$$\mathsf{p}_1(\mathbf{k}) = \mathbf{Pr}\left[E \overset{\$}{\leftarrow} BC(\kappa, n) : E \vdash \mathcal{Q}_E \;\middle|\, \mathsf{TE}_{\mathbf{k}}[E] \vdash \mathcal{Q}_P\right]$$

for each $\mathbf{k} \in I_\kappa^3 \setminus \mathsf{BadK}(M)$. Since $\mathbf{Pr}\left[E \overset{\$}{\leftarrow} BC(\kappa, n) : \mathsf{TE}_{\mathbf{k}}[E] \vdash \mathcal{Q}_P\right]$ is the same for every $\mathbf{k} \notin \mathsf{BadK}(M)$, we have

$$\mathsf{p}_1(\mathcal{Q}_E | \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) = \frac{1}{|I_\kappa^3 \setminus \mathsf{BadK}(M)|} \sum_{\mathbf{k} \notin \mathsf{BadK}(M)} \mathsf{p}_1(\mathbf{k}).$$

Since each key defines an independent random permutation in the ideal cipher model, we have

$$\begin{aligned}
\mathsf{p}_1(\mathbf{k}) &= \mathsf{p}^* \cdot \mathbf{Pr}\left[E \overset{\$}{\leftarrow} BC(\kappa, n) : E \vdash \mathcal{Q}_E^{k_1} \cup \mathcal{Q}_E^{k_2} \cup \mathcal{Q}_E^{k_3} \;\middle|\, \mathsf{TE}_{\mathbf{k}}[E] \vdash \mathcal{Q}_P\right] \\
&= \mathsf{p}^* \cdot \mathbf{Pr}\left[P_1, P_2, P_3 \overset{\$}{\leftarrow} \mathcal{P}_n : P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P_2 \vdash \overline{\mathcal{Q}}_E^{k_2} \wedge P_3 \vdash \overline{\mathcal{Q}}_E^{k_3} \;\middle|\, P_3 \circ P_2 \circ P_1 \vdash \mathcal{Q}_P\right] \\
&= \mathsf{p}^* \cdot \mathbf{Pr}\left[P_1, P_2, P \overset{\$}{\leftarrow} \mathcal{P}_n : P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P_2 \vdash \overline{\mathcal{Q}}_E^{k_2} \wedge P \circ P_1^{-1} \circ P_2^{-1} \vdash \overline{\mathcal{Q}}_E^{k_3} \;\middle|\, P \vdash \mathcal{Q}_P\right]
\end{aligned}$$

where $\overline{\mathcal{Q}}_E^{k_i} = \left\{(x, y) : (x, k_i, y) \in \mathcal{Q}_E^{k_i}\right\}$ for $i = 1, 2, 3$. The conditional probability appearing in the last line is the probability of event

$$\mathsf{E}_{P_*} : \; P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P_2 \vdash \overline{\mathcal{Q}}_E^{k_2} \wedge P_* \circ P_1^{-1} \circ P_2^{-1} \vdash \overline{\mathcal{Q}}_E^{k_3}$$

over random choice of P_1 and P_2, where P_* is the unique permutation that is consistent with $\mathcal{Q}_P$. Let

$$\begin{aligned}
V &= \left\{v \in I_n : \text{there exists } y \xrightarrow{k_3} v \text{ in } \mathcal{G}\right\} \\
V' &= \left\{v \in I_n : \text{there exists } x \xrightarrow{k_2} y \xrightarrow{k_3} v \text{ in } \mathcal{G}\right\}.
\end{aligned}$$

Then event E_{P_*} requires that P_1 satisfy the following.

1. $P_1(u) = x$ for $(u, x) \in \overline{\mathcal{Q}}_E^{k_1}$.
2. $P_1(P_*^{-1}(v)) = x$ for $v \in V'$ and x such that $x \xrightarrow{k_2} y \xrightarrow{k_3} v$ in $\mathcal{G}$.
3. $P_1(P_*^{-1}(v)) \neq x$ for $v \in V \setminus V'$ and x such that $x \xrightarrow{k_2} y$ in $\mathcal{G}$.

When we consider the lazy sampling of P_1, we note the following properties.

1. For any $v \in V'$ and x such that $x \xrightarrow{k_2} y \xrightarrow{k_3} v$ in $\mathcal{G}$, neither $P_1(P_*^{-1}(v))$ nor $P_1^{-1}(x)$ is determined by $\overline{\mathcal{Q}}_E^{k_1}$ since $\mathbf{k} \notin \mathsf{Chn}_{(0,+)} \cup \mathsf{Chn}_{(0,-)} \cup \mathsf{Chn}_{(1,+)} \cup \mathsf{Chn}_{(1,-)}$.
2. For any $v \in V \setminus V'$, if $x = P_1(P_*^{-1}(v))$ is determined by $\overline{\mathcal{Q}}_E^{k_1}$, then there is no edge $x \xrightarrow{k_2} y$ in $\mathcal{G}$ since $\mathbf{k} \notin \mathsf{Chn}_{(2,+)} \cup \mathsf{Chn}_{(2,-)}$.

By these properties, the probability that a random permutation P_1 satisfies the above three conditions is lower bounded by

$$\left(1-\frac{\alpha_1 h_2}{N-h_1-\alpha_1}\right)\frac{1}{(N)_{h_1+\alpha_1}}$$

where $\alpha_1 = |V'|$. Once P_1 is determined satisfying the three conditions, E_{P_*} requires that P_2 satisfy the following.

1. $P_2(x) = y$ for $(x,y) \in \overline{\mathcal{Q}}_E^{k_2}$.
2. $P_2(P_1(P_*^{-1}(v))) = y$ for $v \in V \setminus V'$ and y such that $y \xrightarrow{k_3} v$ in $\mathcal{G}$.

For any $v \in V \setminus V'$, $P_2(P_1(P_*^{-1}(v)))$ is not determined by $\overline{\mathcal{Q}}_E^{k_2}$ since $\mathbf{k} \notin \mathsf{Chn}_{(2,+)} \cup \mathsf{Chn}_{(2,-)}$ and by the third condition on the choice of P_1. Therefore the probability that a random permutation P_2 satisfies the above two conditions is given by $\frac{1}{(N)_{h_2+\alpha_2}}$ where $\alpha_2 = |V \setminus V'| = h_3 - \alpha_1$. To summarize, we have

$$\begin{aligned}\mathsf{p}_1(\mathbf{k}) = \mathsf{p}^*\mathbf{Pr}\left[\mathsf{E}_{P_*}\right] &\geq \mathsf{p}^*\left(1-\frac{\alpha_1 h_2}{N-h_1-\alpha_1}\right)\frac{1}{(N)_{h_1+\alpha_1}(N)_{h_2+\alpha_2}}\\ &\geq \mathsf{p}^*\left(1-\frac{M^2}{N-2M}\right)\frac{1}{(N)_{h_1+\alpha_1}(N)_{h_2+\alpha_2}}.\end{aligned}$$

Since

$$\begin{aligned}\frac{(N)_{h_1}(N)_{h_2}(N)_{h_3}}{(N)_{h_1+\alpha_1}\cdot(N)_{h_2+\alpha_2}} &= \frac{(N)_{h_3}}{(N-h_1)_{\alpha_1}\cdot(N-h_2)_{\alpha_2}} = \frac{(N)_{\alpha_1}}{(N-h_1)_{\alpha_1}}\cdot\frac{(N-\alpha_1)_{\alpha_2}}{(N-h_2)_{\alpha_2}}\\ &\geq \frac{(N-\alpha_1)_{\alpha_2}}{(N)_{\alpha_2}} \geq \left(1-\frac{\alpha_1}{N-\alpha_2+1}\right)^{\alpha_2} \geq 1-\frac{M^2}{N-M+1}\end{aligned}$$

we have

$$\begin{aligned}\mathsf{p}_1(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) &= \frac{1}{|I_\kappa^3 \setminus \mathsf{BadK}(M)|}\sum_{\mathbf{k}\notin\mathsf{BadK}(M)}\mathsf{p}_1(\mathbf{k})\\ &\geq \left(1-\frac{M^2}{N-2M}\right)\frac{\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M))}{|I_\kappa^3 \setminus \mathsf{BadK}(M)|}\sum_{\mathbf{k}\notin\mathsf{BadK}(M)}\frac{(N)_{h_1}(N)_{h_2}(N)_{h_3}}{(N)_{h_1+\alpha_1}(N)_{h_2+\alpha_2}}\\ &\geq \left(1-\frac{M^2}{N-2M}\right)\left(1-\frac{M^2}{N-M+1}\right)\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M))\\ &\geq \left(1-\frac{2M^2}{N-2M}\right)\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)).\end{aligned}$$

3.5 Putting the Pieces Together

By applying Lemma 1 with

$$\delta = \frac{2M^2}{N-2M}$$

we obtain the following theorem.

Theorem 1. *For q, L,$M > 0$, we have*

$$\mathbf{Adv}_{\mathsf{TE}}^{\mathsf{PRP}}(N,q) \leq \frac{2M^2}{N-2M} + N\left(\frac{eq}{LN}\right)^L + \frac{3}{K} + \frac{3q}{KM} + \frac{6Lq}{K^2}.$$

Optimizing Parameters. By setting $\frac{M^2}{N} = \frac{q}{KM}$, let

$$M = \left(\frac{Nq}{K}\right)^{\frac{1}{3}}.$$

Furthermore, if we let $L = \max\{\frac{2eq}{N}, 2n\}$, then we have

$$N\left(\frac{eq}{LN}\right)^L \leq \frac{1}{N}.$$

Assuming $N - 2M \geq \frac{2N}{3}$ or equivalently $q \leq \frac{KN^2}{216}$, we have our final result.

Corollary 1. *For $q > 0$, we have*

$$\mathbf{Adv}_{\mathsf{TE}}^{\mathsf{PRP}}(N,q) \leq 6\left(\frac{q^2}{K^2N}\right)^{\frac{1}{3}} + \frac{1}{N} + \frac{3}{K} + \max\left\{\frac{12eq^2}{K^2N}, \frac{12nq}{K^2}\right\}.$$

In other words, triple encryption is secure if

$$q \ll \min\left\{\frac{KN^{\frac{1}{2}}}{\sqrt{12e}}, \frac{K^2}{12n}\right\}.$$

4 Security of Two-Key Triple Encryption

In this section, we prove the security of triple encryption where the first and the third keys are identical. The two-key triple encryption is denoted as TE^*. Suppose that a distinguisher $\mathcal{A}$ makes q_P queries to the outer permutation and q_E queries to the underlying blockcipher. The proof strategy is similar to the three-key triple encryption, based on the same graph representation $\mathcal{G}$ defined by a query history.

Bad Transcripts and Bad Keys. Bad transcripts are defined as for the three-key triple encryption: a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E)$ is defined to be *bad* if either

$$\max_{y^* \in I_n} |\{(x, k, y^*, +) \in \mathcal{Q}_E\}| > L \text{ or } \max_{x^* \in I_n} |\{(x^*, k, y, -) \in \mathcal{Q}_E\}| > L$$

for a certain parameter $L > 0$. Let $\mathsf{BadT}(L)$ denote the set of bad transcripts. Given a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)$, we can upper bound the number of paths of length 3. We classify the 3-paths as follows.

(1) 3-paths of type $(0,+)$: $u \overset{(k,\sigma)}{\longrightarrow} x \overset{(k',+)}{\longrightarrow} y \overset{(k,\sigma'')}{\longrightarrow} v$

(2) 3-paths of type $(0,-)$: $u \overset{(k,\sigma)}{\longrightarrow} x \overset{(k',-)}{\longrightarrow} y \overset{(k,\sigma'')}{\longrightarrow} v$

(3) 3-paths of type $(1,+)$: $x \overset{(k',\sigma')}{\longrightarrow} y \overset{(k,+)}{\longrightarrow} v \longrightarrow u$

(4) 3-paths of type $(1,-)$: $x \overset{(k',\sigma')}{\longrightarrow} y \overset{(k,-)}{\longrightarrow} v \longrightarrow u$

(5) 3-paths of type $(2,+)$: $v \longrightarrow u \overset{(k,+)}{\longrightarrow} x \overset{(k',\sigma')}{\longrightarrow} y$

(6) 3-paths of type $(2,-)$: $v \longrightarrow u \overset{(k,-)}{\longrightarrow} x \overset{(k',\sigma')}{\longrightarrow} y$

First, consider a path of type $(0,+)$. The number of 2-paths of form $x \overset{(k',+)}{\longrightarrow} y \overset{(k,\sigma'')}{\longrightarrow} v$ is upper bounded by

$$L \sum_{y \in I_n} d_{out}(y) \leq Lq_E$$

since the number of nodes coming into y by forward queries is at most L. Each of such 2-paths is uniquely extended to a 3-path $u \overset{(k,\sigma)}{\longrightarrow} x \overset{(k',+)}{\longrightarrow} y \overset{(k,\sigma'')}{\longrightarrow} v$ since the first and the third keys are identical. A similar analysis applies to 3-paths of type $(0,-)$, $(1,-)$ and $(2,+)$.

On the other hand, in order to restrict the number of 3-paths of type $(1,+)$, consider 2-paths of form $y \overset{(k,+)}{\longrightarrow} v \longrightarrow u$. The number of 2-paths of this form is at most Lq_P. Each of these paths is extended to $x \overset{(k',\sigma')}{\longrightarrow} y \overset{(k,+)}{\longrightarrow} v \longrightarrow u$ with K possible keys k'. Therefore the number of 3-paths of type $(1,+)$ is upper bounded by Lq_PK, and a similar analysis applies to 3-paths of type $(2,-)$.

Given a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)$, sets of bad keys Col, $\mathsf{Hv}(M)$ and Chn are also defined as similar to the three-key triple encryption. For example, the set of keys $\mathbf{k} = (k_1, k_2)$ such that there is a path $u \overset{(k_1,\sigma)}{\longrightarrow} x \overset{(k_2,+)}{\longrightarrow} y \overset{(k_1,\sigma'')}{\longrightarrow} v$ of type $(0,+)$ in $\mathcal{G}$ is denoted by $\mathsf{Chn}_{(0,+)}$. With similar definitions for other types of 3-paths, we define

$$\mathsf{Chn} = \mathsf{Chn}_{(0,+)} \cup \mathsf{Chn}_{(0,-)} \cup \mathsf{Chn}_{(1,+)} \cup \mathsf{Chn}_{(1,-)} \cup \mathsf{Chn}_{(2,+)} \cup \mathsf{Chn}_{(2,-)}.$$

We also define

$$\begin{aligned} \mathsf{Col} &= \{(k_1, k_2) \in I_\kappa^2 : k_1 = k_2\} \\ \mathsf{Hv}(M) &= \{(k_1, k_2) \in I_\kappa^2 : |\{k_1 : (x, k_1, y) \in \mathcal{Q}_E\}| > M \vee |\{k_2 : (x, k_2, y) \in \mathcal{Q}_E\}| > M\} \end{aligned}$$

for a parameter $M > 0$. Then for $\mathsf{BadK}(M) = \mathsf{Col} \cup \mathsf{Hv} \cup \mathsf{Chn}$, we have

$$\mathbf{Pr}\left[\mathbf{k} \overset{\$}{\leftarrow} I_\kappa^2 : \mathbf{k} \in \mathsf{BadK}(M)\right] \leq \frac{1}{K} + \frac{2q_E}{KM} + \frac{4Lq_E}{K^2} + \frac{2Lq_P}{K} \overset{\text{def}}{=} \epsilon_2. \tag{3}$$

Comparing $\mathsf{p}_1(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M))$ and $\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M))$. Since Lemma 1 also holds for TE^*, we need to lower bound the ratio of $\mathsf{p}_1(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M))$ to $\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M))$. First, we fix a transcript $\mathcal{T} = (\mathcal{Q}_P, \mathcal{Q}_E) \notin \mathsf{BadT}(L)$ and a key $\mathbf{k} = (k_1, k_2) \notin \mathsf{BadK}(M)$. Then we decompose the blockcipher query history $\mathcal{Q}_E$ as

$$\mathcal{Q}_E = \mathcal{Q}_E^{k_1} \cup \mathcal{Q}_E^{k_2} \cup \mathcal{Q}_E^*$$

where $\mathcal{Q}_E^{k_i} = \{(x, k, y) \in \mathcal{Q}_E : k = k_i\}$ for $i = 1, 2$, and $\mathcal{Q}_E^*$ is the set of the remaining queries. Then we have

$$\mathsf{p}_2(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) = \mathsf{p}^* \cdot \frac{1}{(N)_{h_1}(N)_{h_2}}$$

where $h_1 = |\mathcal{Q}_E^{k_1}|$, $h_2 = |\mathcal{Q}_E^{k_2}|$ and $\mathsf{p}^* = \mathbf{Pr}\left[E \overset{\$}{\leftarrow} BC(\kappa, n) : E \vdash \mathcal{Q}_E^*\right]$. On the other hand, let

$$\mathsf{p}_1(\mathbf{k}) = \mathbf{Pr}\left[E \overset{\$}{\leftarrow} BC(\kappa, n) : E \vdash \mathcal{Q}_E \;\middle|\; \mathsf{TE}_{\mathbf{k}}^*[E] \vdash \mathcal{Q}_P\right]$$

for each $\mathbf{k} \in I_\kappa^2 \setminus \mathsf{BadK}(M)$. Then we have

$$\mathsf{p}_1(\mathcal{Q}_E|\mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) = \frac{1}{|I_\kappa^2 \setminus \mathsf{BadK}(M)|} \sum_{\mathbf{k} \notin \mathsf{BadK}(M)} \mathsf{p}_1(\mathbf{k}).$$

By replacing $\mathsf{TE}^*_{\mathbf{k}}[E]$ by a truly random permutation P, we have

$$\mathsf{p}_1(\mathbf{k}) = \mathsf{p}^* \cdot \mathbf{Pr}\left[P_1, P \xleftarrow{\$} \mathcal{P}_n : P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P_1^{-1} \circ P \circ P_1^{-1} \vdash \overline{\mathcal{Q}}_E^{k_2} \,\middle|\, P \vdash \mathcal{Q}_P\right]$$

where $\overline{\mathcal{Q}}_E^{k_i} = \left\{(x, y) : (x, k_i, y) \in \mathcal{Q}_E^{k_i}\right\}$ for $i = 1, 2$. Let

$$X = \{x \in I_n : x \xrightarrow{k_2} y \in \mathcal{G} \text{ for some } y \in I_n\},$$
$$Y = \{y \in I_n : x \xrightarrow{k_2} y \in \mathcal{G} \text{ for some } x \in I_n\}$$

be the sets of end nodes of k_2-labeled edges. We decompose X as a disjoint union of X_1, X_2 and X_3, where

$$X_2 = \{x \in I_n : x \xrightarrow{k_2} y \xrightarrow{k_1} z \in \mathcal{G} \text{ for some } y, z \in I_n\}$$
$$X_3 = \{x \in I_n : w \xrightarrow{k_1} x \xrightarrow{k_2} y \in \mathcal{G} \text{ for some } w, y \in I_n\}$$

and $X_1 = X \setminus (X_1 \cup X_2)$. Accordingly, we define

$$Y_i = \{y \in I_n : x \xrightarrow{k_2} y \in \mathcal{G} \text{ for some } x \in X_i\}$$

for $i = 1, 2, 3$. Assuming $P_1 \vdash \overline{\mathcal{Q}}_E^{k_1}$ and $P \vdash \mathcal{Q}_P$, we will determine $v = P_1(y)$ for $y \in Y_1$ by lazy sampling, where we would like to avoid the following conditions denoted E_1.

1. $\mathcal{G}$ contains an edge $v \xrightarrow{k_2} y$ for some $y \in I_n$.
2. $\mathcal{G}$ contains a 2-path $v \longrightarrow u \xrightarrow{k_1} y$ for some $u, y \in I_n$.
3. $\mathcal{G}$ contains an edge $v \longrightarrow u$ for some $u \in I_n$ and $x \xrightarrow{k_2} u$ for some $u \in I_n$.

The probability of E_1 is upper bounded as follows.

$$\mathbf{Pr}\left[P_1, P \xleftarrow{\$} \mathcal{P}_n : \mathsf{E}_1 \,\middle|\, P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P \vdash \mathcal{Q}_P\right] \leq \frac{(h_1 + 2h_2)h_2}{N - h_1} \leq \frac{3M^2}{N - M}. \tag{4}$$

By avoiding the first condition, each evaluation $P_1(y)$ does not generate an edge coming into $x \in X_1 \cup X_2$. By avoiding the second condition, $P_1(y)$ does not generate any 4-path. We allow the node v to be connected with some node u by $\mathcal{Q}_P$, while $P_1(y)$ will not determine $P_1(u)$ for any other node y in $Y_1 \cup Y_3$ since we exclude the third condition.

Assuming that $P_1(y)$ has been determined for every $y \in Y_1$ avoiding the above conditions, and under the conditions $P_1 \vdash \overline{\mathcal{Q}}_E^{k_1}$ and $P \vdash \mathcal{Q}_P$, we evaluate P^{-1} at $P_1(y)$ for $y \in Y_1 \cup Y_2$ if not determined, and evaluate P at $P_1^{-1}(x)$ for $x \in X_3$, where $P_1^{-1}(x)$ is determined by $\overline{\mathcal{Q}}_E^{k_1}$. In this evaluation, we would like to avoid the following conditions denoted E_2.

1. $P^{-1}(P_1(y)) \in Y_3$ for some $y \in Y_1 \cup Y_2$.
2. $P(P_1^{-1}(x)) \in X_1 \cup X_2$ for some $x \in X_3$.

The probability of E_2 under condition $\neg\mathsf{E}_1 \wedge P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P \vdash \mathcal{Q}_P$ is upper bounded as follows.

$$\mathbf{Pr}\left[P_1, P \xleftarrow{\$} \mathcal{P}_n : \mathsf{E}_2 \;\middle|\; \neg\mathsf{E}_1 \wedge P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P \vdash \mathcal{Q}_P\right] \leq \frac{h_2^2}{N - q_P} \leq \frac{M^2}{N - q_P}. \tag{5}$$

Finally, assuming $P^{-1}(P_1(y)) \in Y_3$ and $P(P_1^{-1}(x)) \in X_1 \cup X_2$ have been determined for $y \in Y_1 \cup Y_2$ and $x \in X_3$ respectively, and under condition $\neg\mathsf{E}_2 \wedge \neg\mathsf{E}_1 \wedge P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P \vdash \mathcal{Q}_P$, we would like to upper bound the probability of $P_1^{-1} \circ P \circ P_1^{-1} \vdash \overline{\mathcal{Q}}_E^{k_2}$. The event $P_1^{-1} \circ P \circ P_1^{-1} \vdash \overline{\mathcal{Q}}_E^{k_2}$ implies the evaluations

1. $P_1(P^{-1}(P_1(y)) = x$ for each $y \in Y_1 \cup Y_2$ and x such that $x \xrightarrow{k_2} y \in \mathcal{G}$,
2. $P_1(y) = P(P_1^{-1}(x))$ for each $y \in Y_3$ and x such that $x \xrightarrow{k_2} y \in \mathcal{G}$,

where $P^{-1}(P_1(y))$ and $P(P_1^{-1}(x))$ are all determined from the previous steps while P_1-evaluations at these points are all free and independent. Therefore we have

$$\mathbf{Pr}\left[P_1, P \xleftarrow{\$} \mathcal{P}_n : P_1^{-1} \circ P \circ P_1^{-1} \vdash \overline{\mathcal{Q}}_E^{k_2} \;\middle|\; \neg\mathsf{E}_2 \wedge \neg\mathsf{E}_1 \wedge P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P \vdash \mathcal{Q}_P\right] \geq \frac{1}{(N)_{h_2}}. \tag{6}$$

By (4), (5), (6), we have

$$\begin{aligned}
\mathsf{p}_1(\mathbf{k}) &\geq \mathsf{p}^* \cdot \mathbf{Pr}\left[P_1, P \xleftarrow{\$} \mathcal{P}_n : \neg\mathsf{E}_2 \wedge \neg\mathsf{E}_1 \wedge P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P_1^{-1} \circ P \circ P_1^{-1} \vdash \overline{\mathcal{Q}}_E^{k_2} \;\middle|\; P \vdash \mathcal{Q}_P\right] \\
&= \mathsf{p}^* \cdot \mathbf{Pr}\left[P_1, P \xleftarrow{\$} \mathcal{P}_n : P_1^{-1} \circ P \circ P_1^{-1} \vdash \overline{\mathcal{Q}}_E^{k_2} \;\middle|\; \neg\mathsf{E}_2 \wedge \neg\mathsf{E}_1 \wedge P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P \vdash \mathcal{Q}_P\right] \\
&\quad \times \mathbf{Pr}\left[P_1, P \xleftarrow{\$} \mathcal{P}_n : \neg\mathsf{E}_2 \;\middle|\; \neg\mathsf{E}_1 \wedge P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P \vdash \mathcal{Q}_P\right] \\
&\quad \times \mathbf{Pr}\left[P_1, P \xleftarrow{\$} \mathcal{P}_n : \neg\mathsf{E}_1 \;\middle|\; P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \wedge P \vdash \mathcal{Q}_P\right] \\
&\quad \times \mathbf{Pr}\left[P_1, P \xleftarrow{\$} \mathcal{P}_n : P_1 \vdash \overline{\mathcal{Q}}_E^{k_1} \;\middle|\; P \vdash \mathcal{Q}_P\right] \geq \frac{\mathsf{p}^*}{(N)_{h_2}} \cdot \left(1 - \frac{M^2}{N - q_P}\right) \cdot \left(1 - \frac{3M^2}{N - M}\right) \cdot \frac{1}{(N)_{h_1}},
\end{aligned}$$

and then

$$\begin{aligned}
\mathsf{p}_1(\mathcal{Q}_E | \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) &= \frac{1}{|V_\kappa^3 \setminus \mathsf{BadK}(M)|} \sum_{\mathbf{k} \notin \mathsf{BadK}(M)} \mathsf{p}_1(\mathbf{k}) \\
&\geq \frac{1}{|V_\kappa^3 \setminus \mathsf{BadK}(M)|} \sum_{\mathbf{k} \notin \mathsf{BadK}(M)} \frac{\mathsf{p}^*}{(N)_{h_1}(N)_{h_2}} \left(1 - \frac{M^2}{N - q_P}\right)\left(1 - \frac{3M^2}{N - M}\right) \\
&= \left(1 - \frac{M^2}{N - q_P}\right)\left(1 - \frac{3M^2}{N - M}\right) \mathsf{p}_2(\mathcal{Q}_E | \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)) \\
&\geq \left(1 - \frac{M^2}{N - q_P} - \frac{3M^2}{N - M}\right) \mathsf{p}_2(\mathcal{Q}_E | \mathcal{Q}_P \wedge \neg\mathsf{BadK}(M)).
\end{aligned}$$

Applying Lemma 1 with (1) and (3), we have the following theorem.

Theorem 2. *For q_P, q_E, L, $M > 0$, we have*

$$\mathbf{Adv}^{\mathsf{PRP}}_{\mathsf{TE}^*}(q_P, q_E) \leq \frac{M^2}{N - q_P} + \frac{3M^2}{N - M} + N\left(\frac{eq_E}{LN}\right)^L + \frac{1}{K} + \frac{2q_E}{KM} + \frac{4Lq_E}{K^2} + \frac{2Lq_P}{K}.$$

Let $M = \left(\frac{Nq_E}{4K}\right)^{\frac{1}{3}}$ and let $L = \max\{\frac{2eq_E}{N}, 2n\}$. Assuming M, $q_P \le \frac{N}{2}$, we have the following corollary.

Corollary 2. *For q_P, $q_E > 0$, we have*

$$\mathbf{Adv}^{\mathsf{PRP}}_{\mathsf{TE}^*}(q_P, q_E) \le 16\left(\frac{q_E^2}{16K^2N}\right)^{\frac{1}{3}} + \frac{1}{N} + \frac{1}{K} + \max\left\{\frac{8eq_E^2}{K^2N} + \frac{4eq_Pq_E}{KN}, \frac{8nq_E}{K^2} + \frac{4nq_P}{K}\right\}.$$

We can interpret this result in two ways.

1. Two-key triple encryption is secure if $q_P \ll \frac{K}{4n}$, $q_E \ll \min\left\{\frac{KN^{\frac{1}{2}}}{\sqrt{8e}}, \frac{K^2}{8n}\right\}$ and $q_Pq_E \ll \frac{KN}{4e}$.
2. Two-key triple encryption is secure if $q_P \ll \min\left\{\frac{K}{4n}, \frac{N^{\frac{1}{2}}}{\sqrt{2e}}\right\}$ and $q_E \ll \min\left\{\frac{KN^{\frac{1}{2}}}{\sqrt{8e}}, \frac{K^2}{8n}\right\}$.

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A Matching Attacks on Triple Encryption

A.1 An Attack of $2^{\kappa+\frac{n}{2}}$ Query Complexity

This attack has been proposed by Lucks [11] and later extended by Gazi [7]. Let S denote the outer permutation instantiated with either $\mathsf{TE}_{\mathbf{k}}[E]$ using a random key $\mathbf{k} \in I_\kappa^3$ or a truly random permutation P. A distinguisher $\mathcal{A}$, parameterized by $r > 0$, executes the following steps.

1. Fix two sets S_0, $S_1 \subset I_n$ such that $|S_0| = |S_1| = rN^{\frac{1}{2}}$.
2. For each key $k \in I_\kappa$, find a subset $U_k \subset S_0$ such that $|U_k| = \frac{r^2}{2}$ and $E_k(x) \in S_1$ for each $x \in U_k$. If there are a multiple number of such subsets, fix any of them. If U_k is not found for any key $k \in I_\kappa$, then output 1. Otherwise, proceed to the next step.

3. For each key k for which U_k exists, check if there are k', $k'' \in I_\kappa$ such that $E_{k'}(E_k(x)) = E^{-1}_{k''}(\mathsf{S}(x))$ for every $x \in U_k$. If there exists such a key, then output 0. Otherwise, output 1.

Analysis. Let $\mathsf{S} = \mathsf{TE}_{\mathbf{k}}[E]$ with a random key $\mathbf{k} = (k_1, k_2, k_3) \in I^3_\kappa$. In the ideal cipher model, $|E_{k_1}(S_0) \cap S_1|$ becomes a random variable that follows the hypergeometric distribution of mean r^2 and variance not greater than r^2. Therefore by Chevishev's inequality, the probability of $|E_{k_1}(S_0) \cap S_1| < \frac{r^2}{2}$ is at most $\frac{4}{r^2}$. Once $|E_{k_1}(S_0) \cap S_1| \geq \frac{r^2}{2}$, $\mathcal{A}$ moves to the next step, where $\mathcal{A}$ checks that $E_{k_2}(E_{k_1}(x)) = E^{-1}_{k_3}(\mathsf{S}(x))$ for every $x \in U_{k_1}$, and outputs 0. Therefore we have

$$\mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} I^3_\kappa, E \xleftarrow{\$} BC(\kappa, n) : \mathcal{A}[\mathsf{TE}_{\mathbf{k}}[E], E] = 1\right] \leq \frac{4}{r^2}.$$

On the other hand, let $\mathsf{S} = P$ be a truly random permutation on I_n. For each key $\mathbf{k} = (k_1, k_2, k_3)$, the probability that $E_{k_2}(E_{k_1}(x)) = E^{-1}_{k_3}(\mathsf{S}(x))$ for every $x \in U_{k_1}$, assuming $|E_{k_1}(S_0) \cap S_1| \geq \frac{r^2}{2}$, is upper bounded by $1/(N)_{rN^{\frac{1}{2}}}$. Therefore we have

$$\mathbf{Pr}\left[P \xleftarrow{\$} \mathcal{P}_n, E \xleftarrow{\$} BC(\kappa, n) : \mathcal{A}[P, E] = 1\right] \geq 1 - \frac{1}{(N)_{rN^{\frac{1}{2}}}}.$$

We might set $S_0 = S_1$. Then $\mathcal{A}$ would make $2rKN^{\frac{1}{2}}$ queries to the underlying blockcipher and $rN^{\frac{1}{2}}$ queries to the outer permutation.

A.2 A Meet-in-the-Middle Attack of $2^{2\kappa}$ Query Complexity

A distinguisher $\mathcal{A}$, parameterized by $r > 0$, executes the following steps.

1. Fix a set $S_0 \subset I_n$ such that $|S_0| = r$.
2. For each $(k, k') \in I^2_\kappa$, compute
$$S_1(k, k') = \{E_{k'}(E_k(x)) : x \in S_0\}.$$
3. For each $k'' \in I_\kappa$, compute
$$S_2(k'') = \{E^{-1}_{k''}(\mathsf{S}(x)) : x \in S_0\}.$$
4. If there is a key $\mathbf{k} = (k, k', k'')$ such that $S_1(k, k') = S_2(k'')$, then output 0. Otherwise, output 1.

Analysis. Let $\mathsf{S} = \mathsf{TE}_{\mathbf{k}}[E]$ with a random key $\mathbf{k} = (k_1, k_2, k_3) \in I^3_\kappa$. Since $E_{k_2}(E_{k_1}(x)) = E^{-1}_{k_3}(\mathsf{S}(x))$ for every $x \in S_0$, we have

$$\mathbf{Pr}\left[\mathbf{k} \xleftarrow{\$} I^3_\kappa, E \xleftarrow{\$} BC(\kappa, n) : \mathcal{A}[\mathsf{TE}_{\mathbf{k}}[E], E] = 1\right] = 0.$$

On the other hand, let $\mathsf{S} = P$ be a truly random permutation on I_n. For each key $\mathbf{k} = (k, k', k'')$, the probability that $E_{k'}(E_k(x)) = E^{-1}_{k''}(\mathsf{S}(x))$ for every $x \in S_0$ is upper bounded by $1/(N)_r$. Therefore we have

$$\mathbf{Pr}\left[P \xleftarrow{\$} \mathcal{P}_n, E \xleftarrow{\$} BC(\kappa, n) : \mathcal{A}[P, E] = 1\right] \geq 1 - \frac{K^3}{(N)_r}.$$

In the second and third steps, $\mathcal{A}$ makes $rK + rK^2$ queries to the underlying blockcipher and r queries to the outer permutation.