Fine Tuning the Function Field Sieve Algorithm for the Medium Prime Case

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Abstract

This work builds on the variant of the function field sieve (FFS) algorithm for the medium prime case introduced by Joux and Lercier in 2006. We make two contributions which are particularly relevant to the descent phase of the algorithm. The first contribution introduces a divisility and smoothness technique and goes on to develop a sieving method based on the technique. This leads to significant practical efficiency improvements in the descent phase and also provides improvement to Joux's pinpointing technique. The second contribution is a detailed analysis of the degree of freedom and the use of a walk technique in the descent phase of the algorithm. Such analysis shows that it is possible to compute discrete logarithms over certain fields which are excluded by the earlier analyses performed by Joux and Lercier (2006) and Joux (2013). In concrete terms, we present record computations of discrete logs for fields with 16 and 19-bit prime characteristic. Further, we provide concrete analysis of the effectiveness of the FFS algorithm for certain fields with medium sized prime characteristic.

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1 Introduction

Let $\mathbb{F}_Q$ be a finite field and α a generator of the multiplicative group. Given an element β of $\mathbb{F}_Q$, the discrete log of β to base α, denoted as $\log_\alpha \beta$ (or simply as $\log \beta$ when α is implicit), is the integer $i \in [0, Q-2]$ such that $\beta = \alpha^i$. The discrete log problem (DLP) is the following: Given $\mathbb{F}_Q$, α and β, compute $\log_\alpha \beta$.

The function field sieve (FFS) [4, 5, 18] is an index calculus algorithm for solving the discrete log problem over finite fields. For small characteristic fields, there has been a great deal of recent research [11, 10, 9, 17, 6, 15, 12]. Barbulescu, Gaudry, Joux and Thome [6] have shown a quasi-polynomial time algorithm for extension fields with small characteristic. Further, Granger, Kleinjung and Zumbrägel [13] have computed a discrete log in the binary extension field $\mathbb{F}_{2^{9234}}$. Concrete analysis and discrete log computations for certain characteristic 3 fields have been done in [1, 2, 3]. The ideas used for small characteristic fields, however, do not apply when the characteristic of the field is larger. In this work, we will be interested in values of Q of the type p^n where p is a medium sized prime (16-bit or larger).

A variant of the FFS applicable for medium sized primes has been proposed by Joux and Lercier in [20]. Progress for this variant has been reported by Joux in [16] where the important technique of pinpointing has been introduced.

The general form of the Joux-Lercier algorithm suggests the use of a factor base consisting of irreducible polynomials of maximum degree D. For solving discrete log over $\mathbb{F}_{p^n}$, the values p and n are to be balanced in the following sense.

$$n = n(Q) = \frac{1}{\alpha}\left(\frac{\ln Q}{\ln\ln Q}\right)^{2/3} \text{ and } p = \exp\left(\alpha\sqrt[3]{\ln Q\cdot \ln^2\ln Q}\right) \tag{1}$$

where $Q = p^n$ and α are such that

$$\alpha^{3/2} \geq \tfrac{2}{3(D+1)\sqrt{D}}. \tag{2}$$

We will use ln to denote natural logarithms and lg to denote logarithms to base 2.

The algorithm is parameterised by D and as mentioned in [20], the optimal case for each algorithm happens when (2) holds with equality. The size of the factor base is about $2p^D/D$. It is suggested in [20] that by varying D it is possible to have algorithms to solve discrete log problem for a large range of Q. Let us consider the problem of computing discrete log for fields $\mathbb{F}_{p^n}$ where p is a 16 or an 18-bit prime. The numerical examples given in [20] considered primes of these sizes, so, we are not the first to consider such primes.

Suppose the value of D is chosen to be 2. Then the factor base consists of about p^2 elements. To find logarithms of the elements of the factor base, the number of relations should be a little more than the size of the factor base. Each relation will involve a constant number, say c, of elements of the factor base. For the linear algebra step, the resulting matrix will have about p^2 rows. The complexity of the block Wiedemann or the Lanczos algorithm is $O(RW)$ where R is the number of rows and W is the number of non-zero elements in the matrix. Applied to the matrix obtained after the relation collection step, the complexity will be about cp^4. If p is a 16-bit prime, the complexity is about $2^{66}c$ ring operations; if p is larger, then the difficulty of the computation grows. Also, the complexity grows if D is chosen to be greater than 2. This shows that for primes of the size 16-bit or larger, taking D to be greater than 1 makes the resulting computation very difficult to carry out. In view of this fact, in this paper we work only with $D = 1$.

For $D = 1$, the optimal value of α is $3^{-2/3}$ and the asymptotic complexity of the Joux-Lercier algorithm is $L_Q(1/3, 3^{1/3})$, where $L_Q(a, c)$ with $0 < a < 1$ and $c > 0$ denotes the sub-exponential expression

$$L_Q(a, c) \quad = \quad \exp\left((c + o(1))(\ln Q)^a(\ln\ln Q)^{1-a}\right). \tag{3}$$

The use of the pinpointing technique in [16] reduces the value of c in $L_Q(a, c)$ without affecting the value of a. The notation $L_Q(a)$ denotes $L_Q(a, c)$ for some constant c.

Suppose α is fixed to $(1/3)^{2/3}$ (corresponding to $D = 1$). Then (1) provides n as a function of Q. Let us denote this value of n as $n(Q)$. Table 1 shows the values of $\lg Q$, $n(Q)$ along with the values of n and p that have been actually been tackled in [20, 16] and here. For the previous computations shown in Table 1, the extension degree n that has been tackled is at most the value of $n(Q)$ which tallies with the analysis done in [20]. In contrast to the previous works, we perform DLP computations for $\mathbb{F}_Q$ where the extension degree n is greater than $n(Q)$. The differences are small, but, in concrete terms suggest that the asymptotic analysis in [20] may not cover all the possible tunings of the parameters that are involved. Apart from the fields for which we have actually computed discrete log, we also perform a concrete analysis of other fields. For such fields, the gap between the n that can be tackled and $n(Q)$ turns out to be significantly more. More details are provided in Section 7.

Table 1: Values of $\lg Q$, $n(Q)$ along with the values of n and p for which actual discrete log computations have been reported.

Ref.	p	$\lg p$	n	$\lg Q$	$n(Q))$	cond
[20]	65537	16	25	400	28	–
	370801	18.5	30	556	34	$n\|p-1$
[16]	33553771	25	47	1175	52	$n\|p-1$
	33341353	24.99	57	1425	57	$n\|p-1$
Here	64373	15.97	37	560	35	–
	297079	18.18	40	728	39	–

Our contributions: We revisit the FFS algorithm as described in [20, 14, 16]. There are three main phases of the algorithm, namely, relation collection, linear algebra and the descent phase. Joux [16] introduced the technique of pinpointing which improves upon the classical sieving algorithm used for relation collection. The linear algebra and descent steps in [20, 14, 16] remained unchanged.

One of our contributions is to introduce a divisibility and smoothness technique. This is then developed into a sieving method. Use of the divisibility-cum-sieving technique provides efficiency improvements to the pinpointing technique [16] for the relation collection phase. The main advantage of the technique, however, is that it provides significant practical speed-up in the descent phase of the algorithm. For the medium prime case, the descent step is more problematic and concrete improvements to this step are useful in practice. We note that the divisibility technique is similar to the special-q technique used in integer factorisation algorithms. In the context of discrete logarithms, the special-q method has been proposed earlier [19, 20]. To the best of our knowledge, the manner in which we use the divisibility technique has not been reported earlier.

Detailed consideration of the descent phase of the FFS algorithm shows that descending from degree-2 polynomials to degree-1 polynomials is the most time consuming step. The computations used in [20, 16] for such descent (which we call 2-1 descent) use one degree of freedom, resulting in at most p trials. This upper bounds the value of n for which the 2-1 descent can be carried out. To tackle this problem, the works [20, 14] briefly mention the possibility of using 3 degrees of freedom resulting in p^3 trials, but, lower smoothness probability. This idea though, is not fully explored.

A second contribution of this work is to explore in details the effect of increasing the degree of freedom for the 2-1 descent step of the algorithm. We provide a systematic framework for increasing the degree of freedom. In particular, the case of 2 degrees of freedom (not considered in [20, 14, 16]) is important. We are able to use 2 degrees of freedom to compute discrete log over a field with 19-bit characteristic and extension degree 40. An asymptotic analysis shows that using higher degrees of freedom in the 2-1 descent phase has a noticeable effect on the L-expression for the runtime of the algorithm.

There is another issue regarding a 2-1 descent which has been briefly mentioned in [16]. Suppose that the 2-1 descent fails for a certain quadratic polynomial. Then [16] suggests that one should move to another quadratic polynomial and attempt the descent. No details, however, are provided as to whether such a strategy will always succeed and the number of times it would be required to move from one quadratic polynomial to another which would be the length of the walk.

We develop the details of the walk technique by considering several options and the associated probabilities. It turns out that the walk technique will not always work and we provide explanations of why it may not work. We have conducted experiments with the walk technique for 3 fields. For the 25-bit prime, extension degree 57 field considered in [16], the walk lengths are a few steps. Apart from this, we consider two other fields. The first one has 16-bit characteristic and extension degree 37 and the walk technique works for this field. The average walk length turns out to be 17. The other field has an 19-bit characteristic and extension degree 40. For this field, the walk technique does not work due to a branching effect which we explain later. Accordingly, we used a higher degree of freedom for this field. We note that [12] also had to introduce several techniques to handle the descent phase for their record discrete log computation on characteristic two fields.

In terms of actual computations, we report two record discrete log computations for the above mentioned fields, i.e., 16-bit characteristic and extension degree 37 (592-bit field); and, 19-bit characteristic and extension degree 40 (728-bit field). Prime characteristic of these sizes were earlier considered in [20]. The extension degrees that we have tackled are currently the highest known.

In [16] discrete log computations have been reported over larger fields $\mathbb{F}_{p^n}$ where $n|p-1$. The last condition allows reducing the size of the factor base by a factor of about n considerably easing the computation of the linear algebra step. While this is useful, the property $n|p-1$ is restrictive. In the examples that we consider, this property does not hold. As a consequence, for *general* medium-characteristic fields, the 728-bit field that we consider is currently the largest over which discrete log computations have been carried out. On the other hand, the techniques that we develop are general in nature and can also be used with the special fields considered in [16].

We go beyond the actual computation of discrete log and perform a concrete analysis of the application of the algorithm to certain specific fields. For 16, 18, 20 and 25-bit primes, we indicate the maximum extension degrees that can be feasibly handled. For 32-bit primes, we show that tackling extension degree 100 can be done in about 2^{80} steps.

The values of the parameters of FFS that we consider lead to a situation where the individual descent phase (more specifically the 2-1 descent step) requires more time than the relation collection phase. It is perhaps due to this reason that we are able to tackle extension degrees which cross the value of $n(Q)$. The asymptotic analysis in [20] considered the situation where the time for the individual descent phase is at most that of the main phase.

Section 2 provides an overview of the FFS algorithm in the medium prime case. In Section 3, we introduce the notion of 'good' bivariate polynomials which allows us to describe the relation collection phase and the individual descent phase in a uniform manner. The divisibility and smoothness technique is described in Section 4. The application of this technique to the relation collection and the 2-1 descent phase is explained in Section 5. As mentioned above, the 2-1 descent phase may require a walk technique. Section 6 provides the details of this technique and the supporting analysis. A concrete analysis for different prime sizes is carried out in Section 7. In the following section, i.e., Section 8 we carry out an asymptotic analysis of the algorithm. Some record discrete log computations are presented in Section 9. Finally, we conclude in Section 10.

2 A Description of the Function Field Sieve Algorithm for the Medium Prime Case

The function field sieve algorithm [4, 5] is an index calculus technique used for computing discrete logarithm on finite fields. Our description of the algorithm is based on the two papers [20, 16] and the book [14]. The focus of our work is the medium prime case and so we do not discuss the improvements that can be made when the characteristic of the field is a small prime. There are three phases of the algorithm, namely, relation collection, linear algebra and the individual discrete log phase. There are two auxiliary phases called the preparatory phase and the final phase. Descriptions of these phases are given below.

2.1 Preparatory Phase

In the preparatory phase, a suitable representation of the finite field is chosen. Given a prime p and a positive integer n, up to isomorphism, there is exactly one finite field of order p^n. Given two different representations of the field of order p^n, it is easy to compute the isomorphism between them. Due to this, one is free to choose any convenient representation to solve the discrete log problem. The solution can later be transferred to any other given representation.

Field representation: The field $\mathbb{F}_{p^n}$ is realised as $\frac{\mathbb{F}_p[x]}{\langle f(x)\rangle}$ where $f(x)$ is an irreducible polynomial of degree n and is determined as follows: Choose two polynomials $g_1(x)$ and $g_2(x)$ in $\mathbb{F}_p[x]$ of degrees n_1 and n_2 respectively such that $f(x)$ is a factor of $x - g_2(g_1(x))$. The polynomial $x - g_2(g_1(x))$ itself may be irreducible and can be taken to be $f(x)$, otherwise, one chooses as $f(x)$ the highest degree irreducible factor of this polynomial. The values of n_1 and n_2 are crucial to determining the complexity of the algorithm.

The benefit of choosing such a representation is the following. Since $f(x)$ divides $x - g_2(g_1(x))$, if we set $y = g_1(x)$, then $x = g_2(y) \bmod f(x)$. So, the equalities

$$y = g_1(x) \text{ and } x = g_2(y) \tag{4}$$

provide two basic relations between two elements of $\mathbb{F}_{p^n}$. These relations are at the heart of the algorithm and was first suggested in [20].

The $n_1n_2{+}1$ variant: In the above representation, $n \le n_1n_2$. Joux [16] has further suggested a way to increase the extension degree while keeping n_1 and n_2 the same. Choosing $g_2(y) = ty^{-n_2}$ for some non-zero $t \in \mathbb{F}_p$, it is possible to obtain extension degree $n = n_1n_2 + 1$ with minor changes in the relation collection and the individual logarithm phases. In the examples considered in [16], $g_1(x)$ was taken to be x^{n_1} in addition to taking $g_2(y)$ in the special form. This requires working with a Kummer extension and implies the condition $n|p-1$.

It is not necessary, however, to take both $g_1(x)$ and $g_2(y)$ to be of the special form. Setting $g_2(y) = y^{-n_2}$ one can allow $g_1(x)$ to be arbitrary. The relation collection and the individual logarithm phases can be made to go through. Further, one may choose $g_1(x) = x^{-n_1}$ and $g_2(y)$ to be of general form and still obtain an appropriate field representation. Our example (later) of 16-bit prime and extension degree 37 illustrates these points.

Generator of the field: The generator of the finite field (and the base of the discrete logarithms) is fixed as follows. If $f(x)$ is primitive polynomial, then x itself is taken as a generator and hence $\log(x) = 1$. More generally, we can take as the generator any primitive element which is smooth over the factor base. In practice it turns out that for some $a_j \in \mathbb{F}$, $x + a_j$ generates field $\mathbb{F}_{p^n}$. In that case, we have $\log(x + a_j) = 1$. For the actual computations of the FFS algorithm, the choice of the generator is not an issue.

Factor base: The factor base is the following.

$$\mathbb{B} = \{(x + a_i), (y + b_j) : a_i, b_j \in \mathbb{F}_p\}.$$

In other words, the factor base consists of all polynomials of degree one in x and y. So the size of the factor base is $2p$. More generally, one can define the factor base to consist of all irreducible polynomials in x and y of degrees at most D. This leads to a factor base of size about $2p^D$. As explained earlier, for a medium-sized p (16-bit or more in our case), having $D = 2$ makes the resulting linear algebra phase infeasible. So, we do not consider this option.

It is possible to reduce the size of the factor base by suitably choosing n and p. As explained in [20] in terms of Galois action and in [16] in terms of Kummer extensions, for certain choices of n and p, the size of the factor base can be reduced by a factor of n. This reduction can allow the computation of the linear algebra for a suitably larger prime p.

While this is interesting, achieving this loses the generality of the algorithm. A requirement for such reduction is that $\mathbb{F}_p$ contains all the n roots of unity. This in turn requires the condition that n divides $p-1$. So, to apply the reduction technique to the factor base, one has to choose n and p such that $n|p-1$. In general, one would not be allowed to choose the values of n and p; these would be provided and in such a case, it is quite unlikely that the condition $n|p-1$ will hold. In view of this, we have decided not to work with this option.

Modulus of discrete log: Since we are working over $\mathbb{F}_{p^n}$, discrete logs are defined modulo $p^n - 1$. Suppose α is a generator of $\mathbb{F}_{p^n}$ so that the order of α is $p^n - 1$. An element α^i is in $\mathbb{F}_p$ if and only if $\alpha^{ip} = \alpha^i$, i.e., if and only if, $(p^n - 1)|i(p-1)$. Let $M = (p^n - 1)/(p - 1)$. Then α^i is in $\mathbb{F}$ if and only if i is a multiple of M.

Suppose $c \in \mathbb{F}_p$ and $\beta \in \mathbb{F}_{p^n}$. Then the discrete log of $c\beta$ is $\log c + \log \beta$. By the above observation, $\log c \equiv 0 \bmod M$ and so the discrete log of $c\beta$ modulo M is simply $\log \beta \bmod M$. In other words, by working modulo M, we can ignore constants arising in the intermediate stages.

In practice, one does not even work modulo M. Instead, first the discrete log is computed modulo the large prime factors of M and then combined using the Chinese remainder theorem. At a later stage, the Pollard's rho and/or the Pohlig-Hellman algorithms are applied to compute the discrete log modulo the small factors of $p^n - 1$ to get the actual discrete log.

2.2 Relation Collection Phase

In the relation collection phase, the aim is to compute multiplicative relations amongst the elements of factor base. This is achieved as follows. Consider a field element of the following form:

$$(x + a)y + (bx + c) = xy + ay + bx + c \text{ where } a, b, c \in \mathbb{F}_p. \tag{5}$$

Expression (5) can be considered to have two equivalent representations, one in terms of x and another in terms of y, i.e.,

$$h_1(x) \overset{\Delta}{=} xg_1(x) + ag_1(x) + bx + c \text{ and } h_2(y) \overset{\Delta}{=} g_2(y)y + ay + bg_2(y) + c. \tag{6}$$

Suppose both $h_1(x)$ and $h_2(y)$ are $\mathbb{B}$-smooth, i.e., they factor into linear terms as $h_1(x) = c_1 \prod_{\alpha_i}(x + \alpha_i)$ and $h_2(y) = c_2 \prod_{\beta_j}(y + \beta_j)$, where $c_1, c_2 \in \mathbb{F}_p$. Then a relation of the following form is obtained.

$$c_1 \prod_{\alpha_i}(x + \alpha_i) = c_2 \prod_{\beta_j}(y + \beta_j). \tag{7}$$

As explained above, the constants c_1 and c_2 are ignored by computing discrete log modulo M. The relation (7) provides the following relation between the discrete logs of some of the elements of the factor base.

$$\sum_{\alpha_i} \log(x + \alpha_i) = \sum_{\beta_j} \log(y + \beta_j) \mod M \tag{8}$$

In the relation collection phase, more than $2p$ such relations are collected so that there are $2p$ linearly independent equations involving the discrete logs of the factor base elements.

Obtaining a relation of the form (7) is dependent on the degree $n_1 + 1$ of $h_1(x)$ and the degree $n_2 + 1$ of $h_2(y)$. The usual heuristic is to assume that the two polynomials behave like independent random polynomials. It is known that the probability that a random polynomial of degree m factors into linear terms is about $\frac{1}{m!}$. Under the assumption that $h_1(x)$ and $h_2(y)$ behave like independent random polynomials, the probability of obtaining a single relation is about $1/((n_1 + 1)!(n_2 + 1)!)$. The quantities a, b and c in (5) provide three degrees of freedom leading to a total of p^3 choices. Among these, about $p^3/((n_1 + 1)!(n_2 + 1)!)$ relations are to be expected. The relation collection phase succeeds if

$$\frac{p^3}{(n_1 + 1)!(n_2 + 1)!} > 2p. \tag{9}$$

The quantity on the left side is maximised when n_1 and n_2 are roughly equal.

Pinpointing: Joux [16] introduced the pinpointing technique to speed up the relation collection phase. The technique works for any given p and n. The idea is to choose $g_1(x) = x^{n_1}$. This choice of $g_1(x)$ is not restrictive. Experiments show that it is possible to set $g_1(x) = x^{n_1}$ and then obtain a suitable $g_2(y)$ such that the polynomial $x - g_2(g_1(x))$ is irreducible and can be taken to be the field defining polynomial $f(x)$.

It is required to factor $xg_1(x) + ag_1(x) + bx + c$ into linear terms. With $g_1(x) = x^{n_1}$, the polynomial $h_1(x) = xg_1(x) + ag_1(x) + bx + c$ becomes $x^{n_1+1} + ax^{n_1} + bx + c$. Suppose that $x^{n_1+1} + ax^{n_1} + bx + c$ factors into linear terms, i.e.,

$$x^{n_1+1} + ax^{n_1} + bx + c = \prod_{\alpha_i}(x + \alpha_i). \tag{10}$$

The nice idea of Joux is to observe that using the transformation $x \to tx$, $t \in \mathbb{F}_p$, ensures that the right side of (10) remains smooth while the left side transforms into a polynomial of a similar form with t^{n_1+1} as the leading coefficient. By working modulo M, it is possible to divide the whole of left side by t^{n_1+1} to get a monic polynomial of the same form with different a, b and c. The polynomial $h_2(y)$ is then still of degree n_2+1.

As a result, once after $(n_1+1)!$ trials, one obtains a single set of values for a, b and c with x-side smooth, then by varying t over all non-zero elements of $\mathbb{F}_p$, it is possible to obtain $p-1$ x-side smooth polynomials at very little extra cost. Using pinpointing, the amortised number of trials for obtaining one relation which has both sides smooth, is [16]

$$\frac{(n_1+1)!+(p-1)}{(p-1)/(n_2+1)!} = \frac{(n_1+1)!(n_2+1)!}{p-1} + (n_2+1)!. \tag{11}$$

Pinpointing can also be applied from the y-side and in that case, the amortised number of trials for obtaining one relation which has both sides smooth is

$$\frac{(n_2+1)!+(p-1)}{(p-1)/(n_1+1)!} = \frac{(n_1+1)!(n_2+1)!}{p-1} + (n_1+1)!. \tag{12}$$

The pinpointing side will be determined based on which of the above two expressions is smaller.

Pinpointing versus sieving: Joux [16] mentions that the cost of pinpointing "is clearly better than the cost of classical sieving which, in this case, amounts to $(n_2+1)!(n_1+1)!$." Later we show how to combine pinpointing with a sieving method based on the divisibility technique.

The choice of $g_1(x) = x^{n_1}$ helps in speeding up the x-side computation of the relation collection phase. This corresponds to applying pinpointing from the x-side. Similarly, choosing $g_2(y)$ to be y^{n_2} one can apply pinpointing from the y-side. The choice of which side to apply pinpointing from depends on the relative values of n_1 and n_2. If $n_1 \geq n_2$, then it is advantageous to apply pinpointing from the x-side and if not, then it is advantageous to apply the technique from the y-side.

The question arises as to whether it is possible to simultaneously choose $g_1(x) = x^{n_1}$ and $g_2(y) = ty^{n_2}$ for some $t \in \mathbb{F}_p$ and whether such a choice indeed speeds up both sides of the computation of the relation collection phase. The answer to both questions is yes and the technique has been called advanced pinpointing [16]. However, advanced pinpointing does not apply in general; to apply this technique, one has to carefully choose n and p. Due to this reason, we do not discuss the technique any further in this work.

2.3 Linear Algebra

The size of the factor base is $2p$ and so a little more than $2p$ relations are collected between the discrete logs of the elements of the factor base. The linear algebra phase computes the discrete logs (modulo large prime factors of p^n-1) of the elements of the factor base.

Once a sufficient number of relations is obtained, the algorithm proceeds to the linear algebra phase. The system of linear equations that is obtained is highly sparse. The works [20, 16] first reduce the number of unknowns using structured Gaussian elimination and then apply either the Lanczos algorithm or the block Wiedemann algorithm to solve the resulting system of linear equations. The computations are separately done modulo the large prime factors of p^n-1 and then combined using the CRT.

For our computations, we have used MAGMA [7] to perform the linear algebra step. MAGMA provides options for using either the structured Gaussian elimination or the Lanczos algorithm. There does not appear to be a method by which the structured Gaussian elimination is used up to some extent and then the Lanczos algorithm is applied. As a result, we used the Lanczos option. The computation was done on a single core without any parallelisation.

2.4 Individual logarithm phase

The aim of the individual discrete log phase is to express a given element $\phi(x)$ of the field as a ratio of products of elements of the factor base thereby expressing the discrete log of $\phi(x)$ as a sum and difference of the discrete logs of the elements of the factor base. This is achieved recursively. First express $\phi(x)$ as a ratio of products of polynomials of degrees less than the degree of $\phi(x)$. Next, the same algorithm is applied to each factor of $\phi(x)$ recursively until the process descends to linear factors. This phase is also known as the descent phase as in every step the degree keeps on descending until linear terms are reached.

A simple way to do this is the following. Randomly choose a polynomial $D(x)$ having low degree factors (the degrees of these factors are to be less than the degree of $\phi(x)$; $D(x)$ can itself be smooth) and let $N(x) = \phi(x)D(x)$. If the factors of $N(x)$ are of degree lower than that of $\phi(x)$, then we have a descent from $\phi(x) = N(x)/D(x)$ to the lower degree factors of $N(x)$ and $D(x)$. In the initial stages this simple randomisation strategy works well. A more systematic approach is the following.

Suppose $\phi(x)$ is a factor of degree d and it is desired to reduce the problem of finding the log of $\phi(x)$ to that of finding logs of lower degree polynomials. Let $T(x, y)$ be a bi-variate polynomial where the degree of x is t_1 and the degree of y is t_2. Then $T(x, y)$ has $(t_1+1)(t_2+1)$ monomials of the form $x^i y^j$ with $0 \le i \le t_1$ and $0 \le j \le t_2$. Write $T(x, y)$ into two equivalent forms $F(x) \triangleq T(x, g_1(x))$ and $C(y) \triangleq T(g_2(y), y)$ in the variables x and y respectively. Three things are to be ensured: $\phi(x)$ divides $F(x)$; $G(x) \triangleq F(x)/\phi(x)$ is $(d-1)$-smooth; and $C(y)$ is also $(d-1)$-smooth. One option is to try random choices of $T(x, y)$ until a suitable one is found.

Instead of trying random $T(x, y)$, Joux and Lercier [20] consider $\phi(x)$ as a special-q element and propose to sieve over all polynomials $T(x, y)$ such that a descent is possible. More concrete details are provided in Section 15.2.1.1 of [14]. It is possible to generate the set of all $T(x, y)$ such that the corresponding $F(x)$ are divisible by $\phi(x)$. Let $F_{i,j}(x)$, $0 \le i \le t_1, 0 \le j \le t_2$, be defined as $F_{i,j}(x) \triangleq x^i(g_1(x))^j \bmod \phi(x)$. Since the degree of $\phi(x)$ is d, the degree of each $F_{i,j}(x)$ is at most $d-1$ and hence $F_{i,j}(x)$ can be expressed by a column vector of length d. Arrange these column vectors in a $d \times (t_1 + 1)(t_2 + 1)$ matrix M. Assuming that the rank of M is d (which will be true in practice), the dimension of the kernel (null space) of M is $(t_1+1)(t_2+1)-d$. Then it is easy to argue that a column vector $(a_{i,j})^T$ $(0 \le i \le t_1, 0 \le j \le t_2)$ is in the kernel of M if and only if $T(x, y) \triangleq \sum a_{i,j}x^i y^j$ is such that $\phi(x)$ divides $F(x) = T(x, g_1(x))$. So, obtaining a basis for the kernel of M allows the generation of all $T(x, y)$ such that the divisibility condition by $\phi(x)$ is satisfied. Note that since we are interested in only monic polynomials, the degree of freedom in generating the $T(x, y)$ is actually one less than the dimension of the kernel, i.e., it is $(t_1 + 1)(t_2 + 1) - d - 1$. So, $p^{(t_1+1)(t_2+1)-d-1}$ polynomials are obtained by this method. Note that the method ensures that $F(x) = T(x, g_1(x))$ is divisible by $\phi(x)$ but, it does not ensure the smoothness of either $F(x)$ or $G(y)$. These have to be ensured using repeated trials.

This defines the descent from a degree d polynomial in x to degree $d-1$ polynomials in x and also polynomials in y. As a result, in subsequent steps, it is also required to apply the descent method to the polynomials in y. This is a one-step descent. In fact for higher values of d, it is possible to get several steps of descent in one round, but for the lower values of d, the descent proceeds one step at a time. It is the lower side of descent, particularly the 2-1 descent, which takes most of the time.

The idea of the descent from a polynomial in y is the same, except that one starts with $A(y)x + B(y)$. The descent from y-side will also involve lower degree polynomials in y as well as polynomials in x. The method is continued until descent to degree 1 is achieved for both polynomials in x and polynomials in y whence the elements of $\mathbb{B}$ are involved. Since the logs of these elements have been computed in Phase 2, it is possible to use these values and retrace the descent steps to compute the log of $r(x)$.

2.5 Restrictions on the Extension Degree

For a given p, Equation (9) which determines the possibility of relation collection also provides an upper bound on the values of n_1 and n_2 (and so on the value of n) that can be tackled using the FFS algorithms in [16, 20].

The values of n_1 and n_2 are further restricted by the 2-1 descent. For this descent, [20, 16] work with the form $T(x,y) = A(x)y + B(x)$ where both $A(x)$ and $B(x)$ are linear polynomials with $A(x)$ monic, so that $T(x,y)$ has three undetermined coefficients. The degree of $\phi(x)$ for which descent is attempted has degree $d = 2$. As a result, the degree of freedom is $3 - 2 = 1$. So, there is a single degree of freedom which allows for p trials. In these many trials, it is desired to obtain $G(x)$ and $C(y)$ to be smooth. The degrees of $G(x)$ and $C(y)$ are $n_1 - 1$ and $n_2 + 1$ respectively. Heuristically, the probability that both are smooth is $1/((n_1-1)!(n_2+1)!)$. So, for the descent to be possible, it is required that

$$\frac{p}{(n_1-1)!(n_2+1)!} \geq 1. \tag{13}$$

It is clear that if p is such that (13) holds, then certainly (9) is also satisfied. So, given p, (13) determines the maximum values of n_1 and n_2 and hence of n for which the method can be successful. What happens if the condition in (13) does not hold? Two methods are briefly indicated in [20, 16, 14].

Walk: Suppose $\phi(x)$ is a quadratic polynomial for which the algorithm is unable to descend to linear polynomials. To tackle such a scenario, the following has been mentioned in [16]. "When not possible, we use a relation that also includes another degree 2 polynomial and restart from that polynomial." This is very brief and does not address the questions about whether the walk technique will always succceed and the number of steps in the walk that would be required before a descent is obtained.

Additional degrees of freedom: Both [20, 14] mention that one way to tackle the problem is to increase the degrees of $A(x)$ and $B(x)$ to 2. This increases the degree of freedom to 3 and lowers the simultaneous smoothness probability of the resulting $G(x)$ and $C(y)$ to $1/((n_1 + 2)!(2n_2 + 1)!)$. Note that the degree of freedom jumps from 1 to 3 and the intermediate two degrees of freedom is not considered in [20, 14]. These works also do not report any computations using additional degrees of freedom.

3 Good Bivariate Polynomials

Recall that $\mathbb{F}_{p^n}$ is represented as $\mathbb{F}_p[x]/\langle f(x)\rangle$, where $f(x)$ is an irreducible polynomial of degree n such that $f(x)$ divides $x - g_2(g_1(x))$. The polynomials $g_1(x)$ and $g_2(x)$ are of degrees n_1 and n_2 respectively.

From the description of FFS in Section 2, we note that the basic computation required in the relation collection and the descent phases have a similarity. The similarity is observed most clearly when we consider the 2-1 descent. In both the phases, the idea is to factorise two equivalent forms of the expression $A(x)y + B(x)$. For relation collection, all the factors have to be linear while for the descent phase, there is one fixed quadratic factor $\phi(x)$ (whose descent is to be worked out) and rest are to be linear. So, if we consider $\phi(x)$ to be 1 in the relation collection step, then the two phases are almost the same.

"Good" $T(x, y)$**:** Given $\phi(x)$ of degree $d \geq 0$, we say that a bivariate polynomial $T(x, y)$ is *good for* $\phi(x)$ if the following conditions hold.

1. $\phi(x)$ divides $F(x) = T(x, g_1(x))$ (**Divisibility**);

2. Both $G(x) = F(x)/\phi(x)$ and $C(y) = T(g_2(y), y)$ factor into linear terms (**Smoothness**).

The above definition does not assume a special form for $T(x, y)$. When $\phi(x)$ is clear from the context, we will simply write $T(x, y)$ is good. The notion of good $T(x, y)$ that we consider starts with $\phi(x)$ which is a polynomial in x. A similar definition can be provided if one wishes to start with a polynomial in y.

Suppose $\phi(x)$ is a monic polynomial of degree d and $T(x, y)$ is any bi-variate polynomial having a total of $\rho + 1$ monomials out of which the coefficient of any one monomial (usually the leading monomial) is 1 and the other ρ monomials are undetermined. We will assume $d < \rho$. Further, suppose that the degree of $F(x) = T(x, g_1(x))$ is ρ_1 and the degree of $C(y) = T(g_2(y), y)$ is ρ_2.

We assume that $F(x)$ is monic (which is easy to ensure since we are interested in discrete log modulo M). There are ρ_1 undetermined coefficients of $F(x)$. The degree of $G(x) = F(x)/\phi(x)$ is $\rho_1 - d$. Then $G(x)$ is also monic and there are $\rho_1 - d$ unknown coefficients of $G(x)$. Let $e = \rho - d$. We say that the degree of freedom is e. The reason for this will become clear in the next section. Before proceeding, for ease of reference, we summarise the different quantities below.

$$\left.\begin{array}{lcl} n & : & \text{extension degree;} \\ n_1 & : & \text{degree of } g_1(x); \\ n_2 & : & \text{degree of } g_2(x); \\ \rho & : & \text{the number of undetermined coefficients in } T(x,y); \\ \rho_1 & : & \text{the degree of } F(x) = T(x, g_1(x)); \\ \rho_2 & : & \text{the degree of } C(y) = T(g_2(y), y); \\ d & : & \text{the degree of } \phi(x); \\ \rho_1 - d & : & \text{the degree of } G(x) = F(x)/\phi(x); \\ e & : & \text{equals } \rho - d \text{ which is the degree of freedom.} \end{array}\right\} \qquad (14)$$

In anticipation of the divisibility technique we describe later, we define

$$h \quad = \quad \deg(G(x)) - e = \rho_1 - d - e. \tag{15}$$

Proposition 1 *Suppose $T(x,y) = a_0(x) + a_1(x)y + \cdots + a_\ell(x)y^\ell$ and let $F(x) = T(x, g_1(x))$ and $C(y) = T(g_2(y), y)$. Given $F(x)$, it is possible to recover the $a_i(x)$'s modulo $g_1(x)$. Consequently, if the degrees of $a_i(x)$'s are less than that of $g_1(x)$, then it is possible to recover $T(x,y)$ and hence $C(y)$.*

Proof: It is possible to obtain $a_i(x) \bmod g_1(x)$ in the following manner. Note that $F(x) = T(x, g_1(x)) = a_0(x) + a_1(x)g_1(x) + \cdots + a_\ell(x)g_1^\ell(x)$. Then $a_0(x) = F(x) \bmod g_1(x)$; $a_1(x) = (F(x) - a_0(x))/g_1(x) \bmod g_1(x)$; and more generally for $i \geq 1$;

$$a_i(x) = \frac{F(x) - a_0(x) - a_1(x)g_1(x) - \cdots - a_{i-1}g_1^{i-1}(x)}{g_1^i(x)} \bmod g_1(x).$$

If the degrees of the a_i's are all less than n_1, then this procedure recovers the $a_i(x)$'s and so the polynomial $T(x,y)$. So, given $F(x)$ it is possible to obtain $T(x,y)$ and hence $C(y)$ and vice versa. ■

This simple result turns out to be useful in the sieving algorithm we describe later.

3.1 Probability of Obtaining Good $T(x,y)$

Given $\phi(x)$ of degree d, we wish to obtain an estimate of the probability of obtaining a $T(x,y)$ which is good for $\phi(x)$. In general, ensuring that $\phi(x)$ divides $T(x, g_1(x))$ uses up d degrees of freedom and we are left with $\rho - d$ degrees of freedom. So, a maximum of $p^{\rho-d}$ trials can be carried out to ensure that both $G(x) = T(x, g_1(x))/\phi(x) = F(x)/\phi(x)$ and $C(y) = T(g_2(y), y)$ factor into linear terms. Heuristically, the probability of getting both these polynomials to be smooth in a single trial is $1/((\rho_1 - d)!\rho_2!)$.

Let $\mathbf{E}$ be the event of obtaining in $p^{\rho-d}$ trials a $T(x,y)$ such that $\phi(x)|T(x,y)$ and both $F(x)$ and $C(y)$ are smooth. Again, heuristically,

$$\Pr[\mathbf{E}] \quad = \quad 1 - \left(1 - \frac{1}{(\rho_1 - d)!\rho_2!}\right)^{p^{\rho-d}}. \tag{16}$$

Note that the above is a heuristic expression, since it may turn out that $\mathbf{E}$ is an impossible event and so its probability is 0. Nevertheless, (16) turns out to be a good approximation in practice. In $p^{\rho-d}$ trials, the expected number of good $T(x,y)$ is

$$\frac{p^{\rho-d}}{(((\rho_1 - d))!(\rho_2)!)} \quad = \quad \frac{p^e}{(e+h)!\rho_2!}. \tag{17}$$

The value of this expression turns out to be good indicator of the probability of $\mathbf{E}$. If the value of (17) is at least 1, then $\Pr[E]$ is high and it is likely that we will obtain a desirable $T(x,y)$. On the other hand, if the value of (17) is very low, then $\Pr[\mathbf{E}]$ is close to zero and it is unlikely that a suitable $T(x,y)$ will be obtained. This issue is discussed later in connection with the walk technique.

4 Divisibility and Smoothness Technique

We start by considering the special-q technique due to Davis and Holdridge [8]. The technique was originally proposed in the context of the quadratic field sieve (QFS) algorithm and was later suggested for use with the number field sieve algorithm by Pollard. We briefly describe the technique as it is used in the QFS algorithm. Let N be the number to be factored. In the QFS algorithm, one starts with a factor base FB of "small" primes. Let $g(x) = x^2 - N$. The basic idea is to obtain values of x close to $\sqrt{N}$ such that $g(x)$ factors over FB. A modification of this idea was suggested in [8]. Let q be a "large" prime which is outside FB and suppose it is possible to obtain an r such that $r^2 \equiv N \bmod q$. Then q divides $g(tq+r)$ for all $t \geq 0$. Instead of looking at all integers close to $\sqrt{N}$, this modified method looks at only integers of the form $tq + r$ such that $g(tq+r)$ factors over $FB \cup \{q\}$. This leads to practical savings. The core idea is to ensure that $g(tq+r)$ has q as a factor and then try out choices for t in an attempt to factor $g(tq+r)/q$ over FB.

We suggest a similar technique for speeding up discrete log computation[1]. For one thing, this requires us to work with polynomials. Also, there are some differences from the special-q method. In our method there is no "special" q. This role is played by products of elements of the factor base. The second difference arises from this one. In the special-q method, only integers of the form $g(tq+r)$ are considered and so, some integers which may be smooth over FB may be omitted. For our method, since there is no "special" q, we do not omit any polynomial which may be smooth over the factor base. There are other differences in how the technique is actually used.

4.1 A Special Case

First consider the *special* case when $g_1(x) = x^{n_1}$. Let $A(x)$ be a monic polynomial of degree d_1 and $B(x)$ be a polynomial of degree d_2. The number of undetermined coefficients of $T(x,y) = A(x)y + B(x)$ is $\rho = d_1 + d_2 + 1$ and the degree of freedom $e = \rho - d = d_1 + d_2 + 1 - d$.

The key observation is that certain powers of x are missing in the expression $T(x, g_1(x)) = A(x)g_1(x) + B(x)$. Since $g_1(x) = x^{n_1}$, the lowest degree term in $A(x)g_1(x)$ is x^{n_1} while the highest degree term in $B(x)$ is d_2. If $n_1 > d_2 + 1$, then the coefficients of $x^{d_2+1}, \ldots, x^{n_1-1}$ in $A(x)g_1(x) + B(x)$ are zero. These missing powers of x create a gap and the number of such missing powers is the size of the gap which is equal to $n_1 - d_2 - 1$.

Supposing $\phi(x)$ divides $A(x)g_1(x) + B(x)$, the degree of the quotient is $d_1 + n_1 - d$. Let e be the difference between the degree of this quotient and the gap in $A(x)g_1(x) + B(x)$, i.e., $e = (d_1 + n_1 - d) - (n_1 - d_2 - 1) = d_1 + d_2 - d + 1$. For undetermined elements $a_1, \ldots, a_e$ in $\mathbb{F}_p$, write

$$A(x)g_1(x) + B(x) \quad = \quad (x - a_1) \cdots (x - a_e) H(x) \phi(x) \qquad (18)$$

where $H(x) = b_0 + \cdots + b_{m-1}x^{m-1} + x^m$ is a monic polynomial of degree $m = n_1 - d_2 - 1$. There are m undetermined coefficients of $H(x)$ and the size of the gap in $A(x)g_1(x) + B(x)$ is also m. This leads to a system of m linear equations in these many undetermined coefficients of $H(x)$.

[1] We had obtained our technique without reference to the special-q technique. Reviewers of the previous version of this paper had pointed out the connection.

By *symbolically* solving this system of linear equations, we obtain $b_i = h_i(a_1, \ldots, a_e)$ for some easily computed functions h_i, where $0 \leq i \leq m-1$. Note that solutions for b_i are symbolically expressed in terms of $a_1, \ldots, a_e$ using the function h_i. By independently choosing values for $a_1, \ldots, a_e$, we obtain solutions for $H(x)$ and hence for $A(x)$ and $B(x)$. As a result, after symbolically solving the small linear system once, by varying $a_1, \ldots, a_e$, we obtain the different choices of $H(x)$ leading to the different choices of $A(x)$ and $B(x)$ such that the following three conditions hold.

1. $\phi(x)$ divides $A(x)g_1(x) + B(x)$;
2. the quotient $G(x)$ when $F(x) = A(x)g_1(x) + B(x)$ is divided by $\phi(x)$ is $(x-a_1)\cdots(x-a_e)H(x)\phi(x)$ and is directly obtained without any further computation;
3. $G(x)$ has e linear factors and hence satisfies a partial smoothness condition.

The cost for this method is to symbolically solve once a system of m linear equations in m variables and to obtain the $H(x)$ for a particular choice of $a_1, \ldots, a_e$, one needs to evaluate the functions $h_0, \ldots, h_{m-1}$.

Let us consider the conditions under which the method is applicable. First, we need m to be positive to ensure that there is a non-empty system of linear equations and second, we need e to be positive to ensure that there is at least one degree of freedom. So, a set of necessary and sufficient conditions for the method to be applicable is the following.

$$\left.\begin{array}{rcll} e &=& d_1 + d_2 - d + 1 & > 0 \\ m &=& n_1 - d_2 - 1 & > 0 \end{array}\right\} \tag{19}$$

Thus, every choice of $a_1, \ldots, a_e$ ensures divisibility and at the same time yields e linear factors, providing partial smoothness. We provide examples to illustrate the method.

Example 1: Suppose $A(x) = x + b$, $B(x) = ax + c$, $n_1 = 4$ and so $g_1(x) = x^4$. Further, suppose $\phi(x) = 1$, i.e., we are only interested in ensuring partial smoothness. Then we can write $A(x)x^4 + B(x) = (x-a_1)(x-a_2)(x-a_3)(x^2 + b_1 x + b_0)$. Equating the coefficients of x^2 and x^3 of the right side to 0, we obtain the following two equations.

$$\begin{array}{rcl} -a_1a_2a_3 - b_0a_1 - b_0a_2 - a_3b_0 + a_1a_2b_1 + a_1a_3b_1 + a_2a_3b_1 &=& 0 \\ a_1a_2 + a_1a_3 + a_2a_3 + b_0 - a_2b_1 - a_3b_1 &=& 0. \end{array}$$

Solving for b_0 and b_1 in terms of a_1, a_2 and a_3 gives the following expressions.

$$\begin{array}{rcl} b_0 &=& h_0(a_1,a_2,a_3) = \dfrac{-a_1a_2(-a_1-a_2-a_3)a_3 - (a_1a_2 + a_1a_3 + a_2a_3)^2}{a_1^2 + a_1a_2 + a_2^2 + a_1a_3 + a_2a_3 + a_3^2} \\ b_1 &=& h_1(a_1,a_2,a_3) = \dfrac{-a_1^2a_2 - a_1a_2^2 - a_1^2a_3 - 2a_1a_2a_3 - a_2^2a_3 - a_1a_3^2 - a_2a_3^2}{a_1^2 + a_1a_2 + a_2^2 + a_1a_3 + a_2a_3 + a_3^2}. \end{array}$$

Example 2: With $A(x)$, $B(x)$ and $g_1(x)$ as above, suppose that $\phi(x) = x^2 + x + 1$. Then we can write $A(x)x^4 + B(x) = \phi(x)(x-a_1)(x^2 + b_1x + b_0)$. Again equating the coefficients of x^2 and x^3 of the right side to 0, we obtain the following two equations.

$$\begin{array}{rcl} -a_1 + b_0 - a_1b_0 + b_1 - a_1b_1 &=& 0 \\ 1 - a_1 + b_0 + b_1 - a_1b_1 &=& 0. \end{array}$$

Solving for b_0 and b_1 in terms of a_1 gives the following expressions.

$$\begin{aligned} b_0 &= h_0(a_1) = -\frac{1}{a_1} \\ b_1 &= h_1(a_1) = \frac{1 - a_1 + a_1^2}{(a_1 - 1)a_1}. \end{aligned}$$

4.2 A General Description

The above discussion is for $T(x, y)$ to be of the form $T(x, y) = A(x)y + B(x)$ and $g_1(x)$ to be of the form x^{n_1}. We consider the more general case. Suppose $\phi(x)$ is a monic polynomial of degree d and $T(x, y)$ is any bi-variate polynomial having a total of $\rho + 1$ monomials out of which the coefficient of any one monomial (usually the leading monomial) is 1 and the other ρ monomials are undetermined. We will assume $d < \rho$.

Since $T(x, y)$ has ρ monomials whose coefficients are unknown, the coefficients of $F(x)$ can be written as linear functions of these ρ unknowns. The degree of $G(x) = F(x)/\phi(x)$ is $\rho_1 - d$. Then $G(x)$ is also monic and there are $\rho_1 - d$ unknown coefficients of $G(x)$. Note $e = \rho - d$ and write $G(x)$ as $G(x) = (x - a_1) \cdots (x - a_e)H(x)$.

Let the degree of $H(x)$ be h and so $h = (\rho_1 - d) - (\rho - d) = \rho_1 - \rho = \rho_1 - d - e$ which is the expression for h defined in (15). $H(x)$ is monic and so there are a total of $\rho_1 - \rho$ unknown coefficients of $H(x)$. We now consider the identity

$$F(x) \quad = \quad \phi(x)(x - a_1) \cdots (x - a_e)H(x). \tag{20}$$

The left side of (20) is a monic polynomial of degree ρ_1 and the coefficients are linear functions of ρ unknowns. The right side is also a monic polynomial of degree ρ_1. Treating $a_1, \ldots, a_e$ as constants, the unknowns on the right side consist of the $\rho_1 - \rho$ coefficients of $H(x)$. These together with the ρ unknowns on the left side of (20) give a total of ρ_1 unknowns. Equating the coefficients of both sides give a system of ρ_1 linear equations in the ρ_1 unknowns. Symbolically solving this equations gives the coefficients of $H(x)$ and the unknown coefficients of $T(x, y)$ as functions of $a_1, \ldots, a_e$. As a result, the coefficients of $C(y)$ can also be determined as functions of $a_1, \ldots, a_e$.

At this point, it has been ensured that $\phi(x)$ divides $F(x)$ and that the quotient has e linear factors. Since the coefficients of $H(x)$ and $C(y)$ are expressed as functions of $a_1, \ldots, a_e$, it is possible to obtain different $H(x)$ and $C(y)$ by varying $a_1, \ldots, a_e$ over the elements of $\mathbb{F}$. These can then be checked for smoothness.

For $\mathbf{a} = (a_1, \ldots, a_e)$, denote by $H_{\mathbf{a}}(x)$, $F_{\mathbf{a}}(x)$, $T_{\mathbf{a}}(x, y)$ and $C_{\mathbf{a}}(x)$ the polynomials corresponding to $\mathbf{a}$ obtained using the divisibility technique.

Our description of the divisibility technique starts with $\phi(x)$ which is a polynomial in x. A similar description can be provided if one wishes to start with a polynomial in y.

Working with a smaller system of linear equations: In the above description, we have a system of ρ_1 linear equations in ρ_1 variables. Depending upon the structure of $T(x, y)$, it may be possible to reduce the size of the linear system. This can be useful in practice, since the linear system is solved symbolically. We illustrate the idea in the following example where for the sake of illustration we consider the problem from the y-side.

Let $T(x, y) = A'(y)x + B'(y)$ and consider $\phi(y)$ to be the polynomial for which we need to ensure that $\phi(y)$ divides $T(g_2(y), y)$. Let the degrees of $A'(y)$ and $B'(y)$ be d_1 and d_2

respectively. Further, let $d_1 = d_2 = 2, d = 3$ and $g_2(y) = x^7 + c_6y^6 + c_5y^5 + c_4y^4 + c_3y^3 + c_2y^2 + c_1y + c_0$ be given. Here, $c_0, \ldots, c_6$ are known constants. Suppose $\phi(y) = l_0 + l_1y + l_2y^2 + y^3$ (so that $d = 3$), where l_0, l_1 and l_2 are known. Write

$$A'(y)g_2(y) + B'(y) = (y^2 + ay + b)g_2(y) + (\alpha y^2 + \beta y + \gamma) \tag{21}$$

as

$$(y + a_1)(y + a_2)\underbrace{(b_0 + b_1y + b_2y^2 + b_3y^3 + y^4)}_{H'(y)}\underbrace{(l_0 + l_1y + l_2y^2 + y^3)}_{\phi(y)} \tag{22}$$

As per the general description, we need to compare the coefficients of y^0 to y^8 of (21) and (22). Symbolically solving the resulting system of equations will provide all the unknowns, i.e., $b_0, \ldots, b_3, a, b, \alpha, \beta, \gamma$ as functions of a_1 and a_2. Due to the particular form of $T(x, y)$, we can work with a smaller system of equations. Note that α, β and γ are involved only in determining the coefficients of y^2, y and the constant term. If we leave out these 3 powers of y, then we get the following 6 linear equations in b_0, b_1, b_2, b_3, a, b by comparing the coefficients of $y^3, \ldots, y^8$ of (21) and (22). Note that in these equations a_1 and a_2 are to be regarded as constants.

$$\begin{aligned}
0 &= l_2a_1a_2b_1 + l_1a_1a_2b_2 + l_0a_1a_2b_3 + l_2a_1b_0 + l_2a_2b_0 + a_1a_2b_0 + l_1a_1b_1 + l_1a_2b_1 \\
&\quad + l_0a_1b_2 + l_0a_2b_2 - ac_2 - bc_3 + l_1b_0 + l_0b_1 - c_1; \\
0 &= l_2a_1a_2b_2 + l_1a_1a_2b_3 + l_0a_1a_2 + l_2a_1b_1 + l_2a_2b_1 + a_1a_2b_1 + l_1a_1b_2 + l_1a_2b_2 \\
&\quad + l_0a_1b_3 + l_0a_2b_3 - ac_3 - bc_4 + l_2b_0 + a_1b_0 + a_2b_0 + l_1b_1 + l_0b_2 - c_2; \\
0 &= l_2a_1a_2b_3 + l_1a_1a_2 + l_2a_1b_2 + l_2a_2b_2 + a_1a_2b_2 + l_1a_1b_3 + l_1a_2b_3 - ac_4 - bc_5 \\
&\quad + l_0a_1 + l_0a_2 + l_2b_1 + a_1b_1 + a_2b_1 + l_1b_2 + l_0b_3 - c_3 + b_0; \\
0 &= l_2a_1a_2 + l_2a_1b_3 + l_2a_2b_3 + a_1a_2b_3 - ac_5 - bc_6 + l_1a_1 + l_1a_2 + l_2b_2 + a_1b_2 \\
&\quad + a_2b_2 + l_1b_3 - c_4 + l_0 + b_1; \\
0 &= -ac_6 + l_2a_1 + l_2a_2 + a_1a_2 + l_2b_3 + a_1b_3 + a_2b_3 - b - c_5 + l_1 + b_2; \\
0 &= -a - c_6 + l_2 + a_1 + a_2 + b_3.
\end{aligned}$$

This system can be symbolically solved to obtain solutions for b_0, b_1, b_2, b_3, a, b in terms of a_1 and a_2. Once b_0, b_1, b_2, b_3, a, b are fixed there is no more freedom left and α, β and γ are determined. So, given $g_2(y)$ and $\phi(y)$, by varying a_1 and a_2, it is possible to generate $A'(y)$ and $B'(y)$ such that $A'(y)g_2(y) + B'(y) = (y - a_1)(y - a_2)H'(y)\phi(y)$. So, we have ensured that $\phi(y)$ divides $A'(y)g_2(y) + B'(y)$ and that the quotient has two linear factors, namely $(y - a_1)$ and $(y - a_2)$.

Completeness: Consider once more the problem from the x-side, i.e., to find a good $T(x, y)$ for a polynomial $\phi(x)$. We would like to argue that if there is indeed such a good $T(x, y)$, then this will not be missed by the divisibility technique. Suppose that $F(x) = T(x, g_1(x))$ is divisible by $\phi(x)$ and that $G(x) = F(x)/\phi(x)$ is smooth. Then we can write $F(x) = (x - \alpha_1) \cdots (x - \alpha_{\rho_1 - d})\phi(x)$. In (20), if we choose $a_1 = \alpha_1, \ldots, a_e = \alpha_e$, then the resulting $H(x)$ will be $(x - \alpha_{e+1}) \cdots (x - \alpha_{\rho_1 - d})$. So, the divisibility technique will not miss this particular $F(x)$.

4.3 Repetitions

One problematic issue is that each possible $G(x)$ may occur several times. Let

$$\mathcal{S}_p \;=\; \{(\alpha_1,\ldots,\alpha_e) : \alpha_1 \le \alpha_2 \le \cdots \le \alpha_e,\ \alpha_i \in \mathbb{F}_p, i = 1,\ldots,e\}. \tag{23}$$

The size of $\mathcal{S}_p$ is approximately $p^e/e!$.

A basic observation is that if a polynomial $a(x)$ equals $(x-\alpha_1)\ldots(x-\alpha_\ell)$, then the coefficients of $a(x)$ are given by symmetric functions of $\alpha_1,\ldots,\alpha_\ell$. The following simple generalisation of this observation will be useful.

Proposition 2 *Let $\phi(x)$ be a fixed polynomial whose coefficients are known and suppose for some integer $\ell \ge$, there is a solution for $a(x)$ and $b(x)$ such that*

$$a(x) = \phi(x)(x-\alpha_1)\cdots(x-\alpha_\ell)b(x). \tag{24}$$

Then the coefficients of $a(x)$ and $b(x)$ are given by symmetric functions of $\alpha_1,\ldots,\alpha_\ell$.

Proof: Write $(x-\alpha_1)\cdots(x-\alpha_\ell) = x^\ell + s_{\ell-1}x^{\ell-1} + \cdots + s_1x + s_0$ where the s_i's are symmetric functions of $\alpha_1,\ldots,\alpha_\ell$. Expanding the right side of (24) and comparing coefficients on both sides gives a system of linear equations for the unknown coefficients of $a(x)$ and $b(x)$ where the matrix is formed from the known coefficients of $\phi(x)$ and the s_i's. Since we are assuming that there is an $a(x)$ and $b(x)$ satisfying (24), the system of equations has a solution. Then the coefficients of $a(x)$ and $b(x)$ can be expressed as functions of the known coefficients of $\phi(x)$ and the s_i's. Since the s_i's are symmetric functions of $\alpha_1,\ldots,\alpha_\ell$, we get the desired result. ■

We now apply Proposition 2 to the divisibility technique. Suppose it is possible to write $F(x) = \phi(x)(x-\alpha_1)\cdots(x-\alpha_e)H(x)$ where $H(x) = (x-\beta_1)\cdots(x-\beta_k)b(x)$. Then

$$F(x) = \phi(x)(x-\alpha_1)\cdots(x-\alpha_e)(x-\beta_1)\cdots(x-\beta_k)b(x)$$

and the coefficients of $F(x)$ and $b(x)$ are symmetric functions of $\alpha_1,\ldots,\alpha_\ell,\beta_1,\ldots,\beta_k$. Let $\alpha'_1,\ldots,\alpha'_e$ be a choice of e of the quantities $\alpha_1,\ldots,\alpha_\ell,\beta_1,\ldots,\beta_k$. Suppose we write

$$F_1(x) = \phi(x)(x-\alpha'_1)\cdots(x-\alpha'_e)\left(\prod(x-\gamma)\right)b(x)$$

where the product is over values of γ from $\alpha_1,\ldots,\alpha_\ell,\beta_1,\ldots,\beta_k$ which are different from $\alpha'_1,\ldots,\alpha'_e$. Then the coefficients of $F_1(x)$ are still symmetric functions of $\alpha_1,\ldots,\alpha_\ell,\beta_1,\ldots,\beta_k$ and are determined in the same way as the coefficients of $F(x)$ are determined from $\alpha_1,\ldots,\alpha_\ell$, $\beta_1,\ldots,\beta_k$. So, $F_1(x) = F(x)$.

Considering (20), we see that the coefficients of $F(x)$ are symmetric functions of $\alpha_1,\ldots,\alpha_e$. So, while applying the divisibility technique we should ensure that $\alpha_1 \le \cdots \le \alpha_e$. This will avoid permutations of $\alpha_1,\ldots,\alpha_e$ giving rise to the same $F(x)$. Repetitions, however, can still occur. Suppose that $H(x)$ in (20) has k linear factors. Then $F(x)/\phi(x)$ has a total of $e+k$ linear factors. Applying the divisility technique with $\phi(x)$ will result in the same $F(x)$ occurring $\binom{e+k}{e}$ times if we allow all possible choices of $\alpha_1,\ldots,\alpha_e$ with $\alpha_1 \le \cdots \le \alpha_e$. When $H(x)$ is smooth, the value of k is $h = \rho_1 - \rho$ and the number of repetitions of $F(x)$ (and $G(x) = F(x)/\phi(x)$) is $\binom{h+e}{e}$. Our experiments confirm that such repetitions indeed occur.

Suppose $k = 0$, i.e., $H(x)$ does not have any linear factors. Then there is no repetition of $F(x)$. We argue that there are an overwhelming number of $\boldsymbol{\alpha} \in \mathcal{S}_p$ such that $H_{\boldsymbol{\alpha}}(x)$ does not have any linear factor. The number of polynomials of degree h having no linear factors is $\sum_{j\geq 0}(-1)^j\binom{p}{j}p^{n-j}$ [21, Page 678]. For large p, the probability that a polynomial of degree h has no linear factors is approximately $1/\boldsymbol{e}$ (which is independent of the degree h) where $\boldsymbol{e}$ is the base of natural logarithms. So, for about $p^e/(\boldsymbol{e}\times e!)$ of the $\boldsymbol{\alpha}$'s the corresponding $H_{\boldsymbol{\alpha}}(x)$'s do not have any linear factor. So, an algorithm which considers all $\boldsymbol{\alpha}$ in $\mathcal{S}_p$ has to consider at least $p^e/(\boldsymbol{e}\times e!)$ many $H(x)$'s even if it manages to avoid considering two different $H(x)$'s which lead to the same $F(x)$. This shows that trying to avoid repetitions will not be helpful if all elements of $\mathcal{S}_p$ are considered.

Number of Good $T(x,y)$: An estimate of the number of good $T(x,y)$ that can be obtained by considering all elements of $\mathcal{S}_p$ is given as follows. Recall that for each element of $\mathcal{S}_p$, the divisibility technique defines $H_{\boldsymbol{\alpha}}(x)$, $G_{\boldsymbol{\alpha}}(x)$ and $F_{\boldsymbol{\alpha}}(x)$. As $\boldsymbol{\alpha}$ varies over $\mathcal{S}_p$, there are a total of about $p^e/e!$ $H(x)$'s that are considered. Out of these, a fraction of about $1/h!$ of the $H(x)$'s are smooth and so the number of smooth $H(x)$'s is about $p^e/(e!h!)$. As discussed above, $\binom{e+h}{e}$ smooth $H(x)$'s give rise to the same $G(x)$ (and hence the same $F(x)$). So, varying $\boldsymbol{\alpha}$ over $\mathcal{S}_p$, about

$$\frac{1}{\binom{e+h}{e}}\times\frac{p^e}{e!h!}=\frac{p^e}{(e+h)!}$$

smooth $F(x)$'s are obtained. For $T(x,y)$ to be good, both $G(x)$ and $C(y)$ have to be smooth. So, the number of good $T(x,y)$'s that are obtained by varying $\boldsymbol{\alpha}$ over all elements of $\mathcal{S}_p$ is

$$\frac{p^e}{(e+h)!\rho_2!} \quad = \quad \frac{p^e}{(\rho_1-d)!\rho_2!}. \tag{25}$$

Repetitions in random trials: Let $\mathcal{L}$ be a list of elements from $\mathcal{S}_p$ constructed in the following manner. A random choice of $\boldsymbol{\alpha}$ is made from $\mathcal{S}_e$ and the divisibility technique is applied with this tuple to obtain the corresponding $H(x)$ and $F(x)$. Suppose that $H(x)$ has $k\geq 0$ linear factors. Then there are a total of $\binom{e+k}{e}$ tuples from $\mathcal{S}_p$ all of which give rise to the same $F(x)$. All these tuples are added to $\mathcal{L}$. The method is repeated for different random choices for $\boldsymbol{\alpha}$ and the list $\mathcal{L}$ grows. Each trial adds a maximum of $\binom{h+e}{e}$ tuples to $\mathcal{L}$. By the birthday bound, repetitions begin to take place when the size of $\mathcal{L}$ is about the square root of the size of $\mathcal{S}_p$. If κ random choices of $\boldsymbol{\alpha}$ have been made, then there are a negligible number of repetitions in $\mathcal{L}$ if $\kappa < p^{e/2}/((e!)^{1/2}\binom{h+e}{e})$. In this case, the number of $G(x)$ (and $F(x)$) that are obtained is about κ and so the number of good $T(x,y)$ that will be obtained in this case is about $\kappa/(h!\rho_2!)$.

4.4 Sieving

In this section, we describe how a sieving method can be developed based on the divisibility technique. First consider $e=1$, i.e., the degree of freedom is 1. In this case, we can write

$$F(x) \quad = \quad \phi(x)(x-\alpha_1)H(x). \tag{26}$$

The coefficients of $H(x)$ are determined by α_1 and to emphasise this let us write $H_{\alpha_1}(x)$. Note that $G_{\alpha_1}(x)=(x-\alpha_1)H_{\alpha_1}(x)$ and $F_{\alpha_1}(x)=\phi(x)(x-\alpha_1)H_{\alpha_1}(x)$. Recall that given $F(x)$ it is possible to recover the bi-variate polynomial $T(x,y)$ and hence the y-side polynomial $C(y)$. Denote the polynomials corresponding to $F_{\alpha_1}(x)$ by $T_{\alpha_1}(x,y)$ and $C_{\alpha_1}(y)$.

The sieving occurs as follows. For each $\alpha_1 \in \mathbb{F}_p$, generate the corresponding $H_{\alpha_1}(x)$ and then compute $G_{\alpha_1}(x) = (x - \alpha_1)H_{\alpha_1}(x)$ and store all of these $G_{\alpha_1}(x)$ in a list. After all the $G_{\alpha_1}(x)$'s have been generated, sort the list. If some $G_{\alpha_1}(x)$ occurs a total of $h + 1$ times in the list, then it is smooth. For each such $G_{\alpha_1}(x)$, obtain $F_{\alpha_1}(x)$ and then using Proposition 1 generate the corresponding $C_{\alpha_1}(y)$ and check it for smoothness. If $C_{\alpha_1}(y)$ is found to be smooth, then factor the corresponding $G_{\alpha_1}(x)$.

Heuristically, at the end of sieving about $p/(h+1)!$ smooth $G_{\alpha_1}(x)$'s will be obtained and this number of $C_{\alpha_1}(y)$'s will have to be tested for smoothness. Note that only one $G_{\alpha_1}(x)$ is actually factored. So, the total cost of sieving is the generation, storing and sorting of p $G(x)$'s; smoothness testing of $p/(\rho_1 - d)!$ $C(y)$'s; and the factoring of one $G(x)$ and one $C(y)$.

The extension to higher degrees of freedom is simple. The idea is to use sieving in the innermost loop. For a fixed value of $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{e-1})$ use sieving over all possible values of $\alpha_e \in \{\alpha_{e-1}, \ldots, p-1\}$. For each such choice of α_e, generate $H_{\boldsymbol{\alpha}}(x)$ and then $G_{\boldsymbol{\alpha}}(x) = (x - \alpha_1)\cdots(x - \alpha_e)H_{\boldsymbol{\alpha}}(x)$ and store these $G(x)$'s in a list. After completing the iteration over α_e, sort the list and identify the polynomials $G(x)$'s which occur a total of $h+1$ times. These $G(x)$'s are smooth. To see this note that by construction $G(x)$ has the roots $\alpha_1, \ldots, \alpha_{e-1}$ and further the sieving over α_e has encountered $h + 1$ additional roots of $G(x)$; a total of $e - 1 + h + 1 = e + h$ roots of $G(x)$ are accounted for and the degree of $G(x)$ is $e + h$. For each $G(x)$ found to be smooth, generate the corresponding $C(y)$ and check for smoothness. If a $C(y)$ is found to be smooth, then factor the corresponding $G(x)$. Again, only one $G(x)$ is factored.

In the sieving method if a polynomial $G(x)$ is found to be smooth in some iteration of the outer loop variables (i.e., $\alpha_1, \ldots, \alpha_{e-1}$), then no subsequent iteration of these variables will declare $G(x)$ to be smooth and so there will be no repetitions. To see this suppose that $G(x) = (x - \beta_1)\cdots(x - \beta_{e-1})(x - \beta_e)\cdots(x - \beta_{h+e})$ and assume that $\beta_1 \leq \cdots \leq \beta_{h+e}$. When the outer loop variables $\alpha_1, \ldots, \alpha_{e-1}$ take the values $\beta_1, \ldots, \beta_{e-1}$ respectively and sieving with α_e varied over α_{e-1} to $p - 1$ is carried, the roots $\beta_e, \ldots, \beta_{h+e}$ are encountered and so $G(x)$ is generated a total of $h + 1$ times. In every subsequent sieving step the values of $\alpha_1, \ldots, \alpha_{e-1}$ will miss at least one of the roots $\beta_1, \ldots, \beta_{e-1}$ and then sieving over α_e will visit less than $h+1$ roots. So, $G(x)$ will be generated less than $h+1$ times and will not be reported in such sieving steps.

For each $\boldsymbol{\alpha} \in \mathcal{S}$, sieving considers the polynomial $G_{\boldsymbol{\alpha}}(x)$. So, the total number of polynomials considered by sieving is $p^e/e!$. The total number of good $T(x, y)$ in $\mathcal{S}$ is given by (25). As a consequence, the number of $G(x)$'s to consider for obtaining a single good $T(x, y)$ is about

$$\left(\frac{p^e}{e!}\right) \div \left(\frac{p^e}{(e+h)!\rho_2!}\right) = \frac{(e+h)!\rho_2!}{e!} = \frac{(\rho_1 - d)!\rho_2!}{e!}. \tag{27}$$

The total cost of obtaining a single good $T(x, y)$ is constructing about $((e + h)!\rho_2!)/e!$ $G(x)$'s; testing $\rho_2!$ $C(y)$'s for smoothness; and factoring a single $G(x)$ and a single $C(y)$. In addition, list operations are required. The list of $G(x)$'s is at most p. During the construction stage, $G(x)$'s are appended to the list and then the list is sorted. The same space is used for each sieving step and so the space requirement is to store at most p polynomials of degree $e+h$ each. The total time for the list operations including sorting is negligible in comparison to the algebraic operations. Our experiments indeed confirm this behaviour.

Note: Suppose that in addition to checking for smooth $G(x)$'s we are also interested in checking for $G(x)$'s which have one irreducible quadratic factor and the rest of the factors are linear. This information can be obtained from sieving in the following manner. The main sieving loop and the subsequent sorting of the list of $G(x)$'s are done as usual. To check for smoothness we check if a $G(x)$ has occurred $h+1$ times. To check for a single quadratic factor and the rest linear factors, we check whether a $G(x)$ has occurred $h-1$ times. If it has occurred $h-1$ times, then $e-1+h-1=e+h-2$ linear roots of $G(x)$ has been encountered and so the other factor is quadratic. Such a $G(x)$ will be required when we consider the walk technique for the 2-1 descent.

5 Application to Relation Collection and 2-1 Descent

Below, we separately discuss the application of the divisibility and smoothness technique to the relation collection phase and the 2-1 descent step.

5.1 Relation Collection

Recall from (9) that the relation collection phase succeeds if $p^3/((n_1+1)!(n_2+1)!) > 2p$. For the relation collection phase, $T(x,y)=A(x)y+B(x)$ where both $A(x)$ and $B(x)$ are linear polynomials with $A(x)$ being monic. (The degrees d_1 and d_2 of $A(x)$ and $B(x)$ are both 1.) So, there are 3 undetermined coefficients in $T(x,y)$, i.e., $\rho=3$. Here $\phi(x)=1$ and so $d=0$.

The degree of $F(x)$ is $\rho_1=n_1+1$ and that of $C(y)$ is $\rho_2=n_2+1$. The divisibility technique is applied with $\phi(x)=1$ and then $e=d_1+d_2-d+1=3=\rho-d$. So, $F(x)=\phi(x)(x-a_1)(x-a_2)(x-a_3)H(x)=(x-a_1)(x-a_2)(x-a_3)H(x)$ and the degree of $H(x)$ is $\rho_1-e=n_1-2$.

Let us now consider this in combination with the pinpointing technique. Assume that $g_1(x)=x^{n_1}$ and pinpointing is applied from the x-side. The sieving technique of Section 4.4 is applied. Since $e=3$, there are two outer loop variables and one inner loop variable. For fixed values of the outer loop variables, sieving over the inner loop variable is carried out to obtain a set of smooth $F(x)$'s. (Note that since $\phi(x)=1$, $F(x)=G(x)$.) From each one of these smooth $F(x)$, the pinpointing technique generates $(p-1)$ additional smooth $F(x)$'s and the corresponding $C(y)$'s are generated resulting in about $(p-1)/(n_2+1)!$ smooth $C(y)$'s. So, each smooth $F(x)$ discovered in the sieving step provides about $(p-1)/(n_2+1)!$ good $T(x,y)$'s (i.e., both $F(x)$ and $C(y)$ are smooth) and hence these many relations. Sieving is carried out until a little more than $2p$ relations have been obtained. This shows how to combine sieving based on the divisibility technique with the pinpointing method.

Examples: Later we report computations of discrete log in two cases. Here we compare the timings of relation collection for these two cases using only pinpointing and pinpointing along with sieving based on the divisibility technique. The implementations were done in Magma and timings are for a single core on Xeon @2.3 GHz.

Case I: $p=64373$, $n=37=6\times 6+1$ and $n_1=n_2=6$. A total of 130000 relations were collected. Using pinpointing the reported time was 22927 seconds and using pinpointing along with the above mentioned sieving the reported time was 21111 seconds.

Case II: $p = 297079$, $n = 40 = 5 \times 8$ and $n_1 = 8, n_2 = 5$. A total of 600000 relations were collected. Using pinpointing the reported time was 27537 seconds and using pinpointing along with the above mentioned sieving the reported time was 14253 seconds.

In the second case, the value of n_2 is less than that in the first case. Also, the value of p in the second case is larger. Due to the use of pinpointing, for one smooth polynomial on the x-side the number of relations obtained is $(p-1)/(n_2+1)!$. So, in the second case, for one smooth x-side polynomial, the number of relations obtained is more than in the first case. This shows that compared to the first case, pinpointing is more effective in the second case.

In both cases, the timing reported by sieving is better while the improvement is more marked in the second case. In the second case, pinpointing requires to check the smoothness of polynomials of degree 9 as compared to the smoothness check of polynomials of degree 7 in the first case. Since sieving eliminates these smoothness checks, the speed-up is more pronounced in the second case than in the first case.

5.2 2-1 Descent

Suppose we wish to descend from a quadratic polynomial $\phi(x)$ so that $d = 2$. Let $T(x, y) = A(y)x + B(y)$ and the degrees of $A(y)$ and $B(y)$ are d_1 and d_2 respectively with $A(y)$ being monic. The polynomials $A(y)$ and $B(y)$ have $d_1 + 1$ and $d_2 + 1$ terms respectively and so the total number of monomials in $T(x, y)$ is $d_1 + d_2 + 2$. Since $A(y)$ is constrained to be monic the number of undetermined coefficients in $T(x, y)$ is $\rho = d_1 + d_2 + 1$.

The degree of $F(x) = T(x, g_1(x))$ is $\rho_1 = \max(n_1 d_1 + 1, n_1 d_2)$ and the degree of $C(y) = F(g_2(y), y)$ is $\rho_2 = \max(n_2 + d_1, d_2)$. Typically, the choice of d_1 and d_2 will be such that $\rho_2 = n_2 + d_1$. The degree of freedom $e = \rho - d = \rho - 2 = d_1 + d_2 - d + 1 = d_1 + d_2 - 1$. For different values of d_1 and d_2, Table (2) provides the values of ρ, the expected number of good $T(x, y)$ predicted by (17) and the number of polynomials (given by (27)) to be considered to obtain one good $T(x, y)$.

A table similar to (2) can be derived if one wishes to descend from a quadratic polynomial in y. If n_1 and n_2 are not equal or if the degree of freedom is more than 1, then the number of trials required for descending from the x-side and the y-side are not equal. Hence, it is advisable to make the 2-1 descend either from x-side or from the y-side but, not both. If the descent from the y-side is faster, then from the quadratic x-polynomials one should move to quadratic y-polynomials and then descend from the y-side. The converse strategy should be applied if the descent from the x-side is faster.

As mentioned earlier, degrees of freedom one and three have been considered in [20, 14]. The degree of freedom two (corresponding to $d_1 = 1$ and $d_2 = 2$) does not seem to have been considered earlier although it turns out to be important in practice.

Examples (continued): We consider the timings for the 2-1 descent for the example in Case I above after discussing the walk technique. For the example mentioned in Case II above (i.e., $p = 297079$, $n = 40 = 5 \times 8$ and $n_1 = 8, n_2 = 5$), we have compared average timings for a 2-1 descent required by the divisibility technique based sieving method and the technique described in Section 2.4 (which we call the kernel method for the sake of convenience). The kernel method requires smoothness check of polynomials. For this we used Swan's test [23] to perform an initial filtering and then used the check $x^p \bmod u(x) \overset{?}{=} x$ to determine whether the polynomial $u(x)$ is smooth. Factoring of $u(x)$ was attempted only after confirming that

Table 2: Estimates of the expected number of good $T(x, y)$ given by (17) and the number of trials for a single 2-1 descent from the x-side given by (27).

d_1	d_2	ρ	e	ρ_1	ρ_2	# good $T(x, y)$ from (17)	# trials from (27)
1	1	3	1	$n_1 + 1$	$n_2 + 1$	$p/((n_1 - 1)!(n_2 + 1)!)$	$(n_1 - 1)!(n_2 + 1)!$
1	2	4	2	$2n_1$	$n_2 + 1$	$p^2/((2n_1 - 2)!(n_2 + 1)!)$	$((2n_1 - 2)!(n_2 + 1)!)/2!$
2	2	5	3	$2n_1 + 1$	$n_2 + 2$	$p^3/((2n_1 - 1)!(n_2 + 2)!)$	$((2n_1 - 1)!(n_2 + 2)!)/3!$

$u(x)$ is smooth. On an average a 2-1 descent using the kernel method took about 100 hours while it took about 8 hours using the divisibility-cum-sieving method. This clearly shows that practical improvements in timing that can be obtained using the new method.

6 The Walk Technique for a 2-1 Descent

Suppose that we wish to descend from a quadratic polynomial $\phi(x)$ and there are e degrees of freedom. With e degrees of freedom, there can be at most $p^e/e!$ trials. The probability of obtaining a good $T(x, y)$ in these number of trials is given by (17) and the expected number of good $T(x, y)$ is given by (27).

Let us assume that in none of these trials it is possible to obtain a $T(x, y)$ which is good for $\phi(x)$. Typically, the divisibility of $F(x) = T(x, g_1(x))$ by $\phi(x)$ can be ensured, but, the simultaneous smoothness for $G(x) = T(x, g_1(x))/\phi(x)$ and $C(y) = T(g_2(y), y)$ will not be achieved. At this point, the descent can get stuck. The way around is to try to move to another quadratic polynomial and try to descend from that. There are two ways to do this.

1. Given $\phi(x)$, try to obtain $T(x, y)$ such that $G(x)$ has one quadratic factor $\phi_1(x)$ and the other factors are linear and $C(y)$ has only linear factors. Then $\phi(x) = C(y)/G(x)$. If a descent from $\phi_1(x)$ is possible, then this descent can be combined with the factorisation of $G(x)/\phi_1(x)$ and $C(y)$ to get a descent for $\phi(x)$.

2. Alternatively, given $\phi(x)$, one can try to obtain $T(x, y)$ such that $G(x)$ is smooth and $C(y)$ has one quadratic factor $\psi(y)$ and the others are linear factors. If a descent from $\psi(y)$ is possible, then so is a descent from $\phi(x)$.

In the descent phase we will get quadratic polynomials in both x and y. So there are 4 kinds of step one may consider during the 2-1 descent viz. x-x, x-y, y-y and y-x. Let ζ_{ab} represent the probability of an a-b step. This probability depends on the available degree of freedom for the 2-1 descent. We provide heuristics estimates of ζ_{ab} when the degree of freedom is 1. Similar estimates can be obtained for higher degrees of freedom.

Consider ζ_{xy} which is the probability of an x-y step. The x-side is a polynomial of degree $n_1 + 1$ of which $\phi(x)$ is a quadratic factor and it is required for the other factor of degree $n_1 - 1$ to factorise into linear terms. The probability of this happening is $1/(n_1 - 1)!$. The y-side is a polynomial of degree $n_2 + 1$. For a complete 2-1 descent, it is required for this polynomial to factorise completely which occurs with probability $1/(n_2 + 1)!$. On the other hand, for an x-y

step, the requirement on the y-side is a quadratic factor and the other factor of degree $n_2 - 1$ to completely factorise into linear terms. This probability is about $1/(2(n_2 - 1)!)$ (using the fact that a quadratic polynomial is irreducible with probability about $1/2$). Using the heuristic assumption of independence of factorisation on the x and the y sides, the probability of an x-y step is about $1/(2(n_1 - 1)!(n_2 - 1)!)$. The probabilities of the other a-b steps can be similarly worked out and these are given below.

$$\left.\begin{array}{lllllll} \zeta_{xx} & = & \frac{1}{2(n_1-3)!(n_2+1)!}; & & \zeta_{xy} & = & \frac{1}{2(n_1-1)!(n_2-1)!} \\ \zeta_{yy} & = & \frac{1}{2(n_1+1)!(n_2-3)!}; & & \zeta_{yx} & = & \frac{1}{2(n_1-1)!(n_2-1)!}. \end{array}\right\} \tag{28}$$

Experimentally, we have found that the probabilities ζ_{ab} are close to the stated values. As a consequence, $\zeta_{xy} \approx \zeta_{yx}$ and this value is greater than ζ_{xx} or ζ_{yy}. So, for one degree of freedom, it is advisable to have an alternating walk, i.e., if the descent fails for $\phi_1(x)$, then take an x-y step to obtain $\psi_1(y)$; if the descent fails for $\psi_1(y)$, then take a y-x step to $\phi_2(x)$; and so on. If the degree of freedom is more than 1, then the values of ζ_{ab} will change and the walk will have to be appropriately designed.

Branching: As described above, the walk technique converts a quadratic polynomial to another quadratic polynomial. This, however, may not always be possible. In such a situation, one has to try and move from a quadratic polynomial to a pair of quadratic polynomials. This creates a branching and increases the number of quadratic polynomials for which descent is required. If the number of such branchings is too much, then one may end up considering descent from almost all the quadratic polynomials. In such a scenario, the method will not succeed. On the other hand, a few branchings do not affect convergence.

For the actual implementation, there can be several ways of realising branchings. Suppose, we are trying to descend from an x-polynomial $\phi(x)$ and it turns out that for this polynomial descent is not possible and neither are x-x or x-y steps of the walk possible. Then, from $\phi(x)$ one needs to move to two quadratic polynomials. There are three options for this:

- from $\phi(x)$ move to $\phi_1(x), \phi_2(x)$;
- from $\phi(x)$ move to $\phi_1(x), \psi_1(y)$;
- from $\phi(x)$ move to $\psi_1(y), \psi_2(y)$.

These options have different probabilities which also depend on the degree of freedom. For the actual implementation, one should choose the option with the highest probability first.

Cycling: There is another way in which the walk may fail. Suppose the walk starts from $\phi(x)$ and alternates between x and y-polynomials. If one of the x-polynomials turn out to be $\phi(x)$, then the walk may cycle. Appropriately using randomisation can usually avoid this possibility. An alternative simple method is to maintain a list of polynomials encountered so far in the walk and check in the list whenever a new polynomial is obtained. If the length of the walk is small, this works well enough. For larger walks more sophisticated cycle detection algorithms may be used.

Probability of making a successful step: The probability that an a-b step is successful in a single trial is ζ_{ab} and so the probability that such a step is not possible in a single trial is $(1-\zeta_{ab})$. Due to the single degree of freedom, it is possible to make p trials. The probability that all the trials will fail is $(1-\zeta_{ab})^p$ (under the heuristic assumption that the trials are independent) and so the probability that an a-b step is successful is $1-(1-\zeta_{ab})^p$. If this probability is close to one, i.e., if $(1-\zeta_{ab})^p \approx 0$, then the a-b step will go through.

Length of the walk: Let η_x (resp. η_y) be the probability that given a fixed quadratic irreducible x-polynomial (resp. y-polynomial) it is possible to descend into linear factors using one degree of freedom. So, $\eta_x = 1/((n_1-1)!(n_2+1)!)$ (resp. $\eta_y = 1/((n_1+1)!(n_2-1)!)$) and with probability $1-\eta_x$ (resp. $1-\eta_y$) the descent fails at the x-polynomial (resp. y-polynomial). Due to one degree of freedom, we can make about p trials and the probability that the descent fails in all the trials is $(1-\eta_x)^p$ (resp. $(1-\eta_y)^p$). This is the probability that the walk at an x-polynomial (resp. y-polynomial) has to continue further. The walk alternates between x and y-polynomials. Suppose there are t_1 x-polynomials and t_2 y-polynomials in the walk. We make the simplifiying assumption that $t_1 = t_2 = t$.

The probability that descent is not successful at a particular x-polynomial is $(1-\eta_x)^p$. Heuristically assuming independence, the probability that descent is not successful at all the t x-polynomials is $(1-\eta_x)^{pt}$. Similarly, the probability that descent is not successful at all the t y-polynomials is $(1-\eta_y)^{pt}$. So, the total probability that the descent is not successful is $((1-\eta_x)(1-\eta_y))^{pt}$. From this we get that for the descent to be successful with high probability, we must have $((1-\eta_x)(1-\eta_y))^{pt} \approx 0$. In this expression, the quantities η_x, η_y and p are fixed. So, the value of t must be chosen large enough so that $((1-\eta_x)(1-\eta_y))^{pt}$ becomes close to zero. Then the number of steps in the walk is about $2t$.

Examples (continued): We now report on the walk technique for some examples that we have considered.

Case I: For $p = 64373$, $n_1 = 6$ and $n_2 = 6$, using (28) we have $(1-\zeta_{xy})^p \approx 0.01$. This is small and suggests that the walk technique should work in this case and as we report later, it indeed does. There were very few branchings and the average walk length turned out to be 17. We made use of the note mentioned at the end of Section 4.4 on the divisibility based sieving method. This allowed us to check for smooth polynomials and also for polynomials which have one quadratic irreducible factor and the rest are linear factors.

For the other side where there is no sieving, we still need to check whether the polynomial is smooth or whether it has one quadratic factor and other linear factors. Let us call this a modified smoothness check. Swan's test returns the parity of the number of factors and by comparing to the parity of the polynomial degree it is possible to quickly filter out some non-smooth polynomials. The two cases of the modified smoothness check has different parities and so Swan's test is not helpful in this case. For the modified smoothness test, we have done the following. To test a polynomial $u(x)$, compute $u_1(x) = x^p \bmod u(x)$ and then $v(x) = \gcd(u_1(x) - x, u(x))$. If $v(x) = u(x)$, then $u(x)$ is smooth; and if $\deg(v) = \deg(u) - 2$, then $u(x)$ has one quadratic irreducible factor.

We have implemented the kernel method described in Section 2.4 for 2-1 descent using the walk technique for this case. This also requires the modified smoothness check. Timing comparison between the divisibility-cum-sieving and the kernel method is the following. The complete 2-1 descent using the first method is about 14 minutes while it is about 25 minutes using the second one. (Further details are mentioned in Section 9.1.) While both the timings are well within the range of practicality, the improvement is noticeable and for bigger examples will become significant.

Case II: For $p = 297079$, $n_1 = 8$ and $n_2 = 5$, using (28), we have $(1 - \zeta_{xy})^p \approx 0.09$. This value is not sufficiently close to 0 for the walk technique to work. We observed this in our experiments. For this case, we are not always able to move from one quadratic polynomial to another. The number of times when we have to move from one quadratic polynomial to two quadratic polynomials turns out to be significantly high. As a result of such branching, the walk technique does not converge. We provide some results for the experiment that we have run. For the element given by (35) in Section 9.2, we have carried out the descent up to degree 2 polynomials. This resulted in 59 quadratic polynomials. Out of these for 28 polynomials, the walk terminated and it was possible to make the descent. In these cases, the average walk length was 12. Some very few branchings also take place, but, these terminate very quickly.

The walks for the other 31 polynomials did not terminate. For each polynomial, we ran the walk for about 600 steps and observed that on an average there were about 80 branchings. These branchings create a tree like structure resulting in more and more quadratic polynomials from which descents are to be made. As a result, the walks for these polynomials do not terminate. Due to this reason, we decided to switch to a different representation of the field where we can make use of 2 degrees of freedom to perform the 2-1 descent.

Case-III: Consider the 25-bit prime and extension degree 57 example used in [16]. In this case, $p = 33341353$, $n_1 = 8$ and $n_2 = 7$. Using (28) we have $(1 - \zeta_{xy})^p \approx 0.0001$ which is close to 0. This suggests that the walk technique works well in this case. The work [16] does not mention the walk length. To get an idea, we have run some experiments with the 2-1 descent for this example and have observed the the average walk length is about 5. This indicates that the application of the walk technique for this example is not very difficult. Note that this walk length is less than the walk length for Case I which corresponds to the difference in the values of ζ_{xy} in the two cases.

7 Concrete Analysis

We perform the following analysis. Fix the size of the underlying prime p to be δ bits. Then we consider different possible extension degrees and the associated costs for the different phases of the FFS algorithm.

The extension degree n is taken to be $n_1 n_2$ or $n_1 n_2 + 1$. Any relation obtained in the relation collection phase involves $n_1 + n_2$ elements of the factor base. The number of relations required is slightly more than $2p$. Consider the matrix which is provided as input to the linear algebra phase. The number of rows R in the matrix is about $2p$ and each row has $n_1 + n_2$ non-zero entries. So, the total weight W of the matrix is around $2p(n_1 + n_2)$. As mentioned earlier, the cost of the Lanczos or the block Wiedemann algorithm is proportional to $R \times W$ which in the present case is about $4(n_1 + n_2)p^2$. This determines the time taken by the linear algebra step.

For the relation collection phase, we need to consider two things. The first is whether the relation collection step is indeed feasible which is determined by the inequality in (9).

The second consideration is the number of trials required to obtain a single relation which is determined by either (11) or (12) depending on whether pinpointing is applied from the x or the y-side. The total number of trials for relation collection is obtained by multiplying the expression in (11) or (11) by $2p$.

In the descent phase, the most time consuming step is the 2-1 descent. Again, there are two considerations for this – feasiblity and the number of trials. This depends on the values of d_1 and d_2 and the degree of freedom e as indicated in Table 2. The number of trials is for a single 2-1 descent. Typically, the descent to degree 2 polynomials will result in several polynomials. In the cases for which we have done the complete discrete log computations, the number of such polynomials is less than 100. For larger fields, there may be few hundred quadratic polynomials. To get the total number of trials for the complete 2-1 descent it is required to multiply the number of trials for a single descent with the number of polynomials for which descent is required.

Concrete estimates of the above quantities for different values of δ are provided in Tables 3 to 7. In these tables, the value of $Q = p^n$ and $n(Q)$ are given by (1) with $\alpha = (1/3)^{2/3}$ for $D = 1$ (i.e., the factor base consists only of linear polynomials). In later sections, we report on the actual computation of discrete log for 16-bit and 19-bit primes with extension degrees 37 and 40 respectively. Here we note the following points.

1. In all cases (with the exception of $\delta = 25$ and $n = 57$), we find $n > n(Q)$, where $n(Q)$ is as defined in (1). This shows that (contrary to the analysis in [20]) it is feasible to compute discrete log on $\mathbb{F}_Q$ for $n > n(Q)$.

2. As n increases, the expected number of good $T(x, y)$ for the 2-1 descent step decreases. If this number goes below 1, then the walk technique is required. Further, if this number becomes very low, then the walk technique leads to a lot of branchings and does not converge. At this point, it is required to move to a higher degree of freedom.

3. As the degrees of freedom increas, the number of trials required for a 2-1 descent also increases. So, the shift to a higher degree of freedom is to be made only after ascertaining that descent is not possible for a lower degree of freedom even with the walk technique.

4. As n increases beyond $n(Q)$, the time for 2-1 descent becomes more than the time for the relation collection phase and also grows at a faster rate. To a certain extent this is to be expected, since the bound $n(Q)$ in [20] was obtained assuming that the individual logarithm phase takes at most as much time as the relation collection phase.

5. The number of trials for obtaining one relation is based on the pinpointing technique. We have not considered the faster two-sided (or advanced) pinpointing since it is not generally applicable.

6. For certain cases $((\delta, n) = (16, 40), (18, 45), (20, 64))$, the expected number of good $T(x, y)$ required for descent is less than 1 even though the degree of freedom is greater than 1. This suggests that for the corresponding computations, the walk technique will be required. Our probability estimates and the alternating x-y walk is for a single degree of freedom. For higher degrees of freedom, appropriate estimates can be developed and a suitable walk designed.

Based on the tables, we can conclude that access to sufficiently powerful computational resources will enable the computation of discrete log for the following parameter choices:

$$(\delta, n) = (16, 49), (18, 56), (20, 64), (25, 64).$$

The concrete analysis for 32-bit primes shows that it is unlikely that these computations will be carried out anytime soon. For this size primes, the most striking point is perhaps the fact that for $n = 100$ (corresponding to $\log Q = 3200$), the discrete log computation should be possible in about 2^{80} time.

8 Asymptotic Complexity

In this section, we consider the asymptotic complexity analysis of the algorithm which is expressed in terms of the L-notation defined in (3). The asymptotic analysis shows that the complexity is $L_Q(1/3)$ as has been earlier obtained by Joux-Lercier [20] and Joux [16].

One issue is that the analysis in [20] and [16] did not take the descent step into consideration. When the degree of freedom for the 2-1 descent is one, ignoring the descent step is justified. On the other hand, when the degree of freedom is more than one, it turns out that the asymptotic complexity of the algorithm can actually be higher than that reported in [20, 16].

8.1 Previous Analysis

We start by revisiting the asymptotic complexity analysis carried out in Joux [16] which is an improvement of the earlier analysis provided by Joux and Lercier [20]. Recall that $Q = p^n$. For $0 < a < 1$, define

$$\alpha = \frac{1}{n}\left(\frac{\ln Q}{\ln\ln Q}\right)^{1-a} \tag{29}$$

so that $p = L_Q(a, \alpha)$. The value of a is fixed later. The size of the factor basis is $2p$ and linear algebra takes time proportional to a constant multiple of p^2. Ignoring the constant, the time for the linear algebra step is

$$L_Q(a, 2\alpha). \tag{30}$$

The total time for relation collection is $2(n_1+1)!(p+(n_2+1)!)$. In the asymptotic analysis the factor of 2 is ignored. Note $(n_1+1)! = \exp(\ln((n_1+1)!)) \approx \exp((n_1+1)\ln(n_1+1))$. Assuming n_1+1 to be approximately $\sqrt{n}$, $(n_1+1)! \approx \exp((\sqrt{n}/2)\ln n)$. From (29), it follows that $\sqrt{n} = \alpha^{-1/2}(\ln Q/\ln\ln Q)^{(1-a)/2}$. From $Q = p^n$, we obtain $\ln n = \ln\ln Q - \ln\ln p$. Using $p = L_Q(a, \alpha)$ and ignoring the $o(1)$ term, we get $\ln\ln p = a\ln\ln Q + (1-a)\ln\ln\ln Q + \ln\alpha$. From this, we obtain $\ln n = (1-a)\ln\ln Q - (1-a)\ln\ln\ln Q - \ln\alpha$ which is approximated as $(1-a)\ln\ln Q$. So,

$$\begin{aligned}(n_1+1)! &\approx \exp((\sqrt{n}/2)\ln n)\\ &\approx \exp(1/(2\sqrt{\alpha})(\ln Q/\ln\ln Q)^{(1-a)/2}(1-a)\ln\ln Q)\\ &= \exp((1-a)/(2\sqrt{\alpha})(\ln Q)^{(1-a)/2}(\ln\ln Q)^{(1+a)/2})\\ &= L_Q((1-a)/2, (1-a)/(2\sqrt{\alpha})).\end{aligned} \tag{31}$$

Table 3: A concrete analysis for a 16-bit prime p.

parameters			2-1 descent (from Tab 2)		relation collection		
$(n, \lg Q, n(Q))$	(n_1, n_2)	(d_1, d_2, e)	good $T(x, y)$	no of trials	lhs of (9)	pinpt side	no of trials (11) or (12)
(35, 560, 34)	(7, 5)	(1, 1, 1)	0.13	$2^{18.98}$	$2^{23.21}$	x	$2^{10.18}$
(37, 592, 35)	(6, 6)	(1, 1, 1)	0.11	$2^{19.21}$	$2^{23.40}$	x or y	$2^{12.41}$
(40, 640, 37)	(5, 8)	(1, 2, 2)	0.29	$2^{32.77}$	$2^{20.04}$	y	$2^{12.20}$
	(5, 8)	(2, 2, 3)	213.75	$2^{34.35}$	$2^{20.04}$	y	$2^{12.20}$
(42, 672, 38)	(6, 7)	(2, 2, 3)	19.43	$2^{37.96}$	$2^{20.40}$	y	$2^{12.99}$
(45, 720, 39)	(5, 9)	(2, 2, 3)	19.43	$2^{37.67}$	$2^{16.72}$	y	$2^{15.30}$
(48, 768, 41)	(6, 8)	(2, 2, 3)	1.94	$2^{41.13}$	$2^{17.23}$	y	$2^{15.00}$
(49, 784, 41)	(7, 7)	(2, 2, 3)	0.12	$2^{45.25}$	$2^{17.40}$	x or y	$2^{15.99}$

Table 4: A concrete analysis for an 18-bit prime p.

parameters			2-1 descent (from Tab 2)		relation collection		
$(n, \lg Q, n(Q))$	(n_1, n_2)	(d_1, d_2, e)	good $T(x, y)$	no of trials	lhs of (9)	pinpt side	no of trials (11) or (12)
(40, 720, 39)	(8, 5)	(1, 1, 1)	0.07	$2^{21.79}$	$2^{26.04}$	x	$2^{10.74}$
	(5, 8)	(1, 2, 2)	4.70	$2^{32.77}$	$2^{26.04}$	y	$2^{10.74}$
(42, 756, 40)	(6, 7)	(1, 2, 2)	0.50	$2^{36.09}$	$2^{26.40}$	y	$2^{12.51}$
	(6, 7)	(2, 2, 3)	1243.66	$2^{37.96}$	$2^{26.40}$	y	$2^{12.51}$
(45, 810, 42)	(5, 9)	(1, 2, 2)	0.50	$2^{36.09}$	$2^{22.72}$	y	$2^{13.38}$
	(5, 9)	(2, 2, 3)	1243.66	$2^{37.67}$	$2^{22.72}$	y	$2^{13.38}$
(48, 864, 43)	(6, 8)	(2, 2, 3)	124.37	$2^{41.13}$	$2^{23.23}$	y	$2^{13.55}$
(49, 882, 44)	(7, 7)	(2, 2, 3)	7.97	$2^{45.25}$	$2^{23.40}$	x or y	$2^{15.51}$
(56, 1008, 47)	(7, 8)	(2, 2, 3)	0.80	$2^{48.42}$	$2^{20.23}$	y	$2^{16.55}$

Table 5: A concrete analysis for a 20-bit prime p.

parameters			2-1 descent (from Tab 2)		relation collection		
$(n, \lg Q, n(Q))$	(n_1, n_2)	(d_1, d_2, e)	good $T(x, y)$	no of trials	lhs of (9)	pinpt side	no of trials (11) or (12)
(45, 900, 44)	(5, 9)	(1, 2, 2)	7.51	$2^{36.09}$	$2^{28.72}$	y	$2^{11.65}$
(48, 960, 46)	(6, 8)	(1, 2, 2)	0.83	$2^{39.26}$	$2^{29.23}$	y	$2^{12.72}$
(49, 980, 47)	(7, 7)	(2, 2, 3)	510.21	$2^{45.25}$	$2^{29.40}$	x or y	$2^{15.35}$
(56, 1120, 50)	(7, 8)	(2, 2, 3)	51.02	$2^{48.42}$	$2^{26.23}$	y	$2^{15.72}$
(64, 1280, 54)	(8, 8)	(2, 2, 3)	0.24	$2^{56.13}$	$2^{23.06}$	x or y	$2^{18.90}$

Table 6: A concrete analysis for a 25-bit prime p.

parameters			2-1 descent (from Tab 2)		relation collection		
$(n,\, \lg Q,\, n(Q))$	$(n_1,\, n_2)$	$(d_1,\, d_2,\, e)$	good $T(x,y)$	no of trials	lhs of (9)	pinpt side	no of trials (11) or (12)
(57, 1425, 57)	(8, 7)	(1, 1, 1)	0.17	$2^{27.60}$	$2^{41.23}$	x	$2^{15.31}$
(64, 1600, 61)	(8, 8)	(1, 1, 1)	0.02	$2^{30.77}$	$2^{38.06}$	x or y	$2^{18.48}$
	(8, 8)	(1, 2, 2)	0.04	$2^{53.82}$	$2^{38.06}$	x or y	$2^{18.48}$
	(8, 8)	(2, 2, 3)	7961.36	$2^{56.13}$	$2^{38.06}$	x or y	$2^{18.48}$

Table 7: A concrete analysis for a 32-bit prime p.

parameters			2-1 descent (from Tab 2)		relation collection		
$(n,\, \lg Q,\, n(Q))$	$(n_1,\, n_2)$	$(d_1,\, d_2,\, e)$	good $T(x,y)$	no of trials	lhs of (9)	pinpt side	no of trials (11) or (12)
(81, 2592, 81)	(9, 9)	(1, 2, 2)	0.24	$2^{65.04}$	$2^{52.72}$	x or y	$2^{21.79}$
(90, 2880, 86)	(9, 10)	(2, 2, 3)	465023	$2^{71.00}$	$2^{48.96}$	y	$2^{21.80}$
(100, 3200, 91)	(10, 10)	(2, 2, 3)	1359.72	$2^{79.42}$	$2^{45.5}$	x or y	$2^{25.26}$

Taking $a = 1/3$ makes the first arguments of the L-expressions in (30) and (31) equal. So, the time for linear algebra is $L_Q(1/3, 2\alpha)$ and $(n_1 + 1)!$ is $L_Q(1/3, 1/(3\sqrt{\alpha}))$. Note that putting $a = 1/3$ in (29) makes it equivalent to the expression for n given by (1).

The term $p + (n_2 + 1)!$ is at most $2 \max(p, (n_2 + 1)!)$. Ignoring the factor of 2 and assuming $n_2 + 1$ to be approximately $\sqrt{n}$, we have $\max(p, (n_2 + 1)!) = L_Q(1/3, \max(\alpha, 1/(3\sqrt{\alpha})))$. From this the total time for relation collection comes to $L_Q(1/3, 1/(3\sqrt{\alpha}) + \max(\alpha, 1/(3\sqrt{\alpha}))$ which is the expression reported by Joux in [16].

The medium prime condition: The feasibility condition for obtaining sufficiently many relations is $p^3/((n_1 + 1)!(n_2 + 1)!) > 2p$ (see (9)). Ignoring the factor of 2 and translating into L-notation as above, this gives the relation $L_Q(1/3, 2\alpha) \geq L_Q(1/3, 2/(3\sqrt{\alpha}))$ which is equivalent to $2\alpha \geq 2/(3\sqrt{\alpha})$, i.e. $\alpha \geq 3^{-2/3}$. This is exactly the condition obtained by Joux and Lercier [20] for the feasibility of the algorithm (when the factor basis consists of only linear polynomials).

Recall from (1) the expression for $n(Q)$, which for a fixed α is the maximum n that can be achieved when Q is given. As discussed in the introduction and supported by the above analysis, fixing the value of α to be $3^{-2/3}$ determines the maximum value of n that can be achieved when Q is given. Writing $Q = p^n$, relates the value of p and n via the expression for $n(Q)$ showing that the algorithm works for primes which are larger than what could be tackled previously to [20]. This leads to the 'medium prime' case as named in [20].

It has been mentioned in the introduction and substantiated later, that the bound given by $n(Q)$ is not exact; it is indeed possible to use the algorithm for values of n greater than $n(Q)$.

This is due to the fact that the asymptotic analysis makes a lot of approximations which leads to the non-exactness of the bound $n(Q)$. In our view, the condition (9) should be taken as the relation which determines the feasibility of the algorithm rather than the asymptotically obtained expression for $n(Q)$.

8.2 Asymptotic Analysis of Descent

Both the analyses by Joux and Lercier in [20] and by Joux in [16] ignore the cost of descent assuming it to be less than that of relation collection. On the other hand, the concrete analysis shown in Tables 3 to 7 shows that in certain cases, the time for 2-1 descent can be more than the time for relation collection. To see this, we refer to the times for a single 2-1 descent using different degrees of freedom given in Table 2.

In practice, there will be several quadratic polynomials on which 2-1 descent has to be applied. Further, a descent may not be possible in a single step and the walk technique would have to be employed. For the sake of asymptotic analysis, let us assume that all these factors lead to a constant multiple of the time required for a 2-1 descent. So, we only perform an asymptotic analysis on the time for a single 2-1 descent.

When a single degree of freedom is used, the cost of a single 2-1 descent from Table 2 is $(n_1-1)!(n_2+1)!$ which is indeed less than the cost of relation collection. Since both the papers [20] and [16] perform computations using a single degree of freedom, they ignored the time for descent step. On the other hand, suppose two degrees of freedom are used. Then the time for a single 2-1 descent is $(2n_1-2)!(n_2+1)!/2$.

As noted earlier, we need to choose n_1 and n_2 to be roughly equal to maximise the left hand side of (9). This means that for a fixed p, to maximise the value of n that can be tackled we should try to make n_1 and n_2 to be more or less equal.

So, as in the earlier analysis, we have $(n_2+1)!$ to be approximately $L_Q(1/3, 1/(3\sqrt{\alpha}))$. Further, taking $(2(n_1-1))!$ to be approximately $(2\sqrt{n})!$, we have

$$\begin{aligned}(2(n_1-1))! \approx (2\sqrt{n})! = \exp(\ln((2\sqrt{n})!)) &\approx \exp(2\sqrt{n}\ln(2\sqrt{n}))\\ &= \exp(2\sqrt{n}(\ln 2 + \ln\sqrt{n}))\\ &= \exp(\sqrt{n}\ln 4 + \sqrt{n}\ln n)\\ &\approx \exp(2\sqrt{n}\ln n).\end{aligned}$$

From this it follows that $(2n_1-2)!$ is approximately $L_Q(1/3, 4/(3\sqrt{\alpha}))$ and so $(2n_1-2)!(n_2+1)!/2$ is $L_Q(1/3, 5/(3\sqrt{\alpha}))$. In this case, the L-expressions for relation collection, linear algebra and 2-1 descent are respectively the following:

$$L_Q(1/3, 1/(3\sqrt{\alpha}) + \max(\alpha, 1/(3\sqrt{\alpha})), \quad L_Q(1/3, 2\alpha), \quad L_Q(1/3, 5/(3\sqrt{\alpha})).$$

Suppose $\alpha < 1/(3\sqrt{\alpha})$, then the time for relation collection is $L_Q(1/3, 2/(3\sqrt{\alpha}))$ which is smaller than the time for a 2-1 descent. On the other hand, if $\alpha \geq 1/(3\sqrt{\alpha})$, the time for relation collection is $L_Q(1/3, \alpha + 1/(3\sqrt{\alpha}))$ which is smaller than that for 2-1 descent for $\alpha < (4/3)^{2/3}$.

A similar analysis can be routinely carried out for three and higher degrees of freedom. We do not present these details since they do not seem to provide any additional insights into the algorithm. This shows that taking higher degrees of freedom into consideration has an effect on the asymptotic complexity of the algorithm.

As can be seen from the above, the whole asymptotic analysis is based on approximations and ignores factors which turn out to be important in practice. One such example was mentioned by Joux in [16] where he noted that a reduction by a factor of n of the factor basis is very significant in practice but, is not reflected in the asymptotic analysis. Another such issue is the effect of the sieving technique that we have introduced. This avoids all smoothness tests which leads to a marked improvement in practical runtime but, does not show up in the asymptotic analysis. We believe that while asymptotic analysis is useful in providing a rough idea of the performance of an algorithm, it is the detailed concrete analysis of the type that we have done which is really required to understand how an algorithm actually behaves in practice.

9 Actual Discrete Log Computations

We report the actual computations of discrete logs over some fields. For these computations, we have used Magma [7] and SAGE [22] computer algebra systems.

9.1 A Record on a Field of Size p^{37}, $p = 64373$

We have carried out the discrete log computation over a field $Q = p^n$ for a 16-bit prime $p = 64373$ and extension degree $n = 37 = 6 \times 6 + 1$ with $n_1 = 6$ and $n_2 = 6$. This uses the $n_1 n_1 + 1$ variant of the FFS algorithm as mentioned in Section 2.

The defining polynomial $f(x)$ of this field is generated by $g_1(x) = x^{-6}$ and $g_2(x) = x^6 + 14833x^5 + 50952x^4 + 62125x^3 + 6269x^2 + 35223x + 53172$ and $f(x)$ is

$$f(x) = x^{37} + 11201x^{36} + 29150x^{30} + 58104x^{24} + 2248x^{18} + 13421x^{12} + 49540x^6 + 64372.$$

The polynomial $f(x)$ generates the field $\mathbb{F}_{p^n}$ (which is a 592-bit field). This polynomial is not primitive. So x will not generate the multiplicative group of the field. We found that $x + 4$ is primitive, and so, we took this to be the generator of the multiplicative group and hence the base of the discrete log problem.

We have used Magma Computer Algebra Software for entire computation. The hardware platform consists of Intel Xeon(R) CPU X5675 @ 3.07GHz and Intel(R) Xeon(R) CPU E5-2630 @ 2.30GHz. We have used the divisibility technique along with pinpointing in the relation collection phase. In around 6 CPU (@ 2.30GHz) hours we have generated the matrix M for the linear algebra step. This phase provides us a sparse matrix with 138837 rows and 128746 columns. We have

$$\begin{aligned} \frac{p^n - 1}{p - 1} &= 149 \times 71 \times 295683503 \times 1389373163 \times 423185284367 \\ &\quad \times \underbrace{68025462649322545375322695825650761}_{p_2} \times p_1; \end{aligned} \tag{32}$$

$$\begin{aligned} p_1 &= 7367283583673663628070887860770003291427347784355798\overline{1} \\ &\quad 28494828009201911092413871736009947545119473873918456\overline{3} \end{aligned}$$

where p_1 is a 356-bit prime number. So clearly Pollard's Rho and Pohlig-Hellman algorithms will not work here. We have done the linear algebra using Lanczos Algorithm on the Magma

Computer Algebra Software with respect to the two largest primes p_1 and p_2 in the product (32). It took 17 and 7 CPU (@ 3.07GHz) hours to complete the linear algebra with respect to primes p_1 and p_2 respectively. After completion of linear algebra we have got the discrete log of factor base elements modulo primes p_1 and p_2.

As target for computing the discrete log, we have chosen the following field element.

$$\Pi(x) := \text{Normalize}\left(\sum_{i=0}^{n-1} \lfloor \pi p^{i+1} \mod p \rfloor x^i\right), \text{ i.e.,}$$

$$\begin{aligned}\Pi(x) \;=\; & x^{36}+841x^{35}+8875x^{34}+2723x^{33}+58297x^{32}+37489x^{31}\\ & +14681x^{30}+33725x^{29}+27283x^{28}+23704x^{27}+54974x^{26}+45806x^{25}\\ & +8606x^{24}+30661x^{23}+16779x^{22}+46411x^{21}+9333x^{20}+32131x^{19}+\\ & 14681x^{18}+47095x^{17}+51019x^{16}+2810x^{15}+12343x^{14}+12447x^{13}\\ & +16645x^{12}+12487x^{11}+12856x^{10}+51853x^{9}+59500x^{8}+3681x^{7}+\\ & 4146x^{6}+56159x^{5}+9055x^{4}+57991x^{3}+1660x^{2}+48553x+9983. \qquad (33)\end{aligned}$$

The initial descent was quite fast and in few minutes we got $\Pi(x) = \frac{N(x)}{D(x)}$ as follows.

$$\begin{aligned}N(x) = \;& (x+49376)\times(x^2+6357x+10310)\times\\ & (x^2+54990x+10403)\times(x^2+58305x+29395)\times\\ & (x^3+53070x^2+32425x+31087)\times\\ & (x^3+60334x^2+48969x+34559)\times\\ & (x^4+46865x^3+13867x^2+5909x+49679)\times\\ & (x^4+53913x^3+48795x^2+9304x+28222)\times\\ & (x^5+13129x^4+45787x^3+11181x^2+14710x+3003)\times\\ & (x^5+33678x^4+41480x^3+31467x^2+42544x+31415)\times\\ & (x^5+53759x^4+34226x^3+44358x^2+772x+14985);\\ D(x) = \;& (x^2+16528x+61907). \qquad (34)\end{aligned}$$

We have descended all the factors into degree two polynomials in x and in y within 6 minutes. This resulted in 92 quadratic polynomials. The 2-1 descent for these polynomials were completed using the walk technique with 1 degree of freedom. It took about 14 minutes on a single core of the CPU (@ 3.07GHz) to get the complete 2-1 descent. The average walk length of the 2-1 descent is 17.

Using this descent and discrete logs of factor base elements we got the discrete log of $\Pi(x)$ modulo p_1 and p_2 respectively. Discrete logs modulo other small prime factors were computed using Pollard's Rho method and the results were combined using the CRT. We finally got the discrete log of $\Pi(x)$ as follows.

$$\begin{aligned}\log(\Pi(x)) \;=\; & 770654269744664411364422170900833384393002927833357493\\ & 898475477568896179995725185460938555074307834593294 7255\\ & 052244377820217461230522621222197519835055557916394 4401\\ & 19840511924082.\end{aligned}$$

9.2 A Record on a Field of Size p^{40}, $p = 297079$ (728-bits)

In the second case we have worked with $p = 297079$ and extension degree $n = 40$ with $n_1 = 8$ and $n_2 = 5$. The defining polynomial $f(x)$ of this field is generated by $g_1(x) = x^8$ and $g_2(x) = x^5 + 44024x^4 + 224924x^3 + 77320x^2 + 291141x + 80867$ with $f(x) = x^{40} + 44024x^{32} + 224924x^{24} + 77320x^{16} + 291141x^8 + 297078x + 80867$. This $f(x)$ generates a field of size p^{40} and $(x + 3)$ is a primitive element of the field.

Since the value of n_2 is 5, pinpointing fits very well here and we are able to complete the relation collection phase in 4 CPU (@ 2.30Ghz) hours. This provided a sparse matrix A with 624159 rows and 594158 columns. We solved this matrix modulo the largest prime factor p_1 of $p^n - 1$, which is a 286-bit prime number. It took 504 CPU (@3.07 GHz) hours using the in-built Lanczos algorithm of Magma. The discrete logs modulo other small prime factors of $p^n - 1$ were obtained using Pollard's rho and Pohlig Hellman algorithm.

We have chosen our target element $\Pi(x)$ as follows.

$$\Pi(x) = \text{Normalize}\left(\sum_{i=0}^{n-1} \lfloor \pi p^{i+1} \mod p \rfloor x^i\right) \text{ i.e.,}$$

$$\begin{aligned}
\Pi(x) \quad = \quad & x^{39} + 154424x^{38} + 219291x^{37} + 2288x^{36} + 290227x^{35} + 295582x^{34} \\
& +27398x^{33} + 200403x^{32} + 6836x^{31} + 123295x^{30} + 94923x^{29} + 89389x^{28} \\
& +239023x^{27} + 115439x^{26} + 249309x^{25} + 196503x^{24} + 87998x^{23} + 240098x^{22} \\
& +136326x^{21} + 191206x^{20} + 9602x^{19} + 53215x^{18} + 25787x^{17} + 17954x^{16} \\
& +880x^{15} + 158602x^{14} + 241303x^{13} + 246920x^{12} + 52944x^{11} + 212605x^{10} \\
& +234395x^{9} + 196868x^{8} + 106113x^{7} + 207883x^{6} + 198491x^{5} + 106250x^{4} \\
& +165294x^{3} + 28548x^{2} + 76555x + 241986. \qquad (35)
\end{aligned}$$

It is the descent phase which took most of the time. The initial descent was quite fast and in few seconds we got $\Pi(x) = \frac{N(x)}{D(x)}$ as follows.

$$\begin{aligned}
N(x) = \quad & (x^2 + 129346x + 178289) \\
& \times(x^2 + 129508x + 284926) \\
& \times(x^3 + 280690x^2 + 73103x + 113966) \\
& \times(x^4 + 45948x^3 + 55848x^2 + 84717x + 225935) \\
& \times(x^4 + 192446x^3 + 209512x^2 + 71975x + 24689) \\
& \times(x^4 + 214578x^3 + 134299x^2 + 75777x + 185981) \\
& \times(x^5 + 99215x^4 + 101352x^3 + 277735x^2 + 249592x + 166624) \\
& \times(x^5 + 149426x^4 + 93110x^3 + 223082x^2 + 29091x + 16179) \\
& \times(x^5 + 171408x^4 + 31726x^3 + 58503x^2 + 190122x + 213530) \\
& \times(x^5 + 209418x^4 + 233661x^3 + 279042x^2 + 64930x + 238441) \\
D(x) = \quad & (x^2 + 164221x + 32560).
\end{aligned}$$

In around one hour we further converted them to degree 1 polynomials in x and degree at most 2 polynomials in y. This resulted in a total of 59 quadratic polynomials. In around

500 CPU (@ 3.07 GHz) hours, we were able to get the complete descent of $\Pi(x)$ into degree 1 polynomials in x as well as y. Finally we used the CRT to get the discrete log of $\Pi(x)$ as follows.

$$\begin{aligned}\log(\Pi(x)) \quad = \quad & 7301937027753043840467459472283135963464 80\\ & 803400240950763141174029187190517313409792 5\\ & 342153702522654039372608184558507369137933 7\\ & 832616768741252142993539044632260376087765 9\\ & 740520962963146604000921389665780564632839\\ & 420364.\end{aligned}$$

10 Conclusion

In this paper, we have described a technique which ensures divisibility and partial smoothness which leads to concrete speed-ups in the relation collection and individual descent phases of the FFS algorithm. A second contribution has been to systematically develop the walk technique and the technique of using additional degrees of freedom for the 2-1 descent step. As a consequence, we report record computations of discrete logs over fields of 16 and 19-bit characteristic. A concrete analysis of the algorithm shows that tackling fields of greater sizes may be feasible for organisations having access to sufficiently powerful computational resources.

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