A LINEAR ATTACK ON KAHROBAEI-LAM-SHPILRAIN KEY EXCHANGE PROTOCOL

JINTAI DING, ALEXEI MIASNIKOV, AND ALEXANDER USHAKOV

Abstract. In this paper we analyze the Kahrobaei-Lam-Shpilrain (KLS) key exchange protocols that use extensions by endomorpisms of matrices over a Galois field proposed in [2]. We show that both protocols are vulnerable to a simple linear algebra attack.
Keywords. Group-based cryptography, semidirect product, Galois field.
2010 Mathematics Subject Classification. 94A60, 68W30.

1. Introduction

The key-exchange protocol proposed by Habeeb, Kahrobaei, Koupparis, and Shpilrain (HKKS) in [1] uses exponentiation in general semidirect products of (semi)groups. When used with an appropriate finite field, it gives the standard Diffie-Hellman protocol based on cyclic groups. The authors of [1] claimed that "when the protocol is used with non-commutative (semi)groups, it acquires several useful features" and proposed a particular platform semigroup which is the extension of the semigroup of 3×3 matrices over the group ring $\mathbb{F}_7[A_5]$ (where A_5 is the alternating group) using inner automorphisms of $\mathbf{GL}_3(\mathbb{F}_7[A_5])$. It was shown in [3] that the protocol is susceptible to a simple linear algebra attack.

Later, Kahrobaei, Lam, and Shpilrain in [2] (see also [patent]) proposed two other instantiations of the HKKS protocol that use certain extension of the semigroup of 2×2 matrices over the field $\mathbb{GF}(2^{127})$ and claim that the new protocols are safe for the linear attack described in [3]. In this paper we discuss security properties of the new protocols and show that they are susceptible to attacks similar to those of [3].

2. HKKS key exchange protocol

Let G and H be groups, let $\mathbf{Aut}(G)$ be the group of automorphisms of G, and let $\rho : H \to \mathbf{Aut}(G)$ be a group homomorphism. The *semidirect product* of G and H with respect to ρ is the set of pairs $\{(g, h) \mid g \in G,\ h \in H\}$ equipped with the binary operation given by

$$(g, h) \cdot (g', h') = (g^{\rho(h')} g', h \circ h').$$

for $g \in G$ and $h \in H$. It is denoted by $G \rtimes_\rho H$. Here $g^{\rho(h')}$ denotes the image of g under the automorphism $\rho(h')$, and $h \circ h'$ denotes a composition of automorphisms with h acting first.

Some specific semidirect products can be constructed as follows. First choose your favorite group G. Then let $H = \mathbf{Aut}(G)$ and $\rho = \mathrm{id}_G$. In which this case the semidirect product $G \rtimes_\rho H$ is called the *holomorph* of G. More generally, the

The work was partially supported by NSF grant DMS-1318716.

group H can be chosen as a subgroup of $\mathbf{Aut}(G)$. Using this construction, the authors of [1] propose the following key exchange protocol.

Algorithm 1. HKKS key exchange protocol

Initial Setup: Fix the platform group G, an element $g \in G$, and $\varphi \in \mathbf{Aut}(G)$. All this information is made public.

Alice's Private Key: A randomly chosen $m \in \mathbb{N}$.

Bob's Private Key: A randomly chosen $n \in \mathbb{N}$.

Alice's Public Key: Alice computes $(g, \varphi)^m = (\varphi^{m-1}(g) \ldots \varphi^2(g)\varphi(g)g, \varphi^m)$ and publishes the first component $a = \varphi^{m-1}(g) \ldots \varphi^2(g)\varphi(g)g$ of the pair.

Bob's Public Key: Bob computes $(g, \varphi)^n = (\varphi^{n-1}(g) \ldots \varphi^2(g)\varphi(g)g, \varphi^n)$ and publishes the first component $b = \varphi^{n-1}(g) \ldots \varphi^2(g)\varphi(g)g$ of the pair.

Alice's Shared Key: Alice computes the key $K_A = \varphi^m(b)a$ taking the first component of the product $(b, \varphi^n) \cdot (a, \varphi^m) = (\varphi^m(b)a, \varphi^n\varphi^m)$. (She cannot compute the second component since she does not know φ^n.)

Bob's Shared Key: Bob computes the key $K_B = \varphi^n(a)b$ taking the first component of the product $(a, \varphi^m) \cdot (b, \varphi^n) = (\varphi^n(a)b, \varphi^m\varphi^n)$. (He cannot compute the second component since he does not know φ^m.)

Note that $K_A = K_B$ since $(b, \varphi^n) \cdot (a, \varphi^m) = (a, \varphi^m) \cdot (b, \varphi^n) = (g, \varphi)^n$. The general protocol described above can be used with any non-abelian group G and an inner automorphism φ (conjugation by a fixed non-central element of G). Furthermore, since all formulas used in the description of this protocol hold if G is a semigroup and φ is a semigroup automorphism of G, the protocol can be used with semigroups. The private keys m, n can be chosen smaller than the order of (g, ϕ). For a finite group G, this can be bounded by $(\#G) \cdot (\# \mathbf{Aut}(G))$.

2.1. **Proposed parameters for the HKKS key exchange protocol.** In the original paper [1], the authors propose and extensively analyze the following specific instance of their key exchange protocol. Consider the alternating group A_5, i.e. the group of even permutations on five symbols (a simple group of order 60) and the field $\mathbb{F}_7 = \mathbb{GF}(7)$. Let $G = \mathrm{Mat}_3(\mathbb{F}_7[A_5])$ be the monoid of all 3×3 matrices over the ring $\mathbb{F}_7[A_5]$ equipped with multiplication. As usual, by $\mathbf{GL}_3(\mathbb{F}_7[A_5])$ we denote the group of invertible 3×3 matrices over the ring $\mathbb{F}_7[A_5]$. Fix an inner automorphism of G, i.e., a map $\varphi = \varphi_H : G \to G$ for some $H \in \mathbf{GL}_3(\mathbb{F}_7[A_5])$ defined by:

$$M \mapsto H^{-1}MH.$$

Clearly, we have $(\varphi_H)^m = \varphi_{H^m}$ and

$$\begin{aligned} & \varphi_H^{m-1}(M) \ldots \varphi_H^2(M)\varphi_H(M)M \\ = \;& H^{-(m-1)}MH^{m-1} \ldots H^{-2}MH^2 \cdot H^{-1}MH^1 \cdot M \\ = \;& H^{-m}(HM)^m. \end{aligned}$$

This way we obtain the following specific instance of the HKKS key exchange protocol.

Algorithm 2. HKKS key exchange protocol using $\mathrm{Mat}_3(\mathbb{F}_7[A_5])$

Initial Setup: Fix matrices $M \in \mathrm{Mat}_3(\mathbb{F}_7[A_5])$ and $H \in \mathbf{GL}_3(\mathbb{F}_7[A_5])$. They are made public.
Alice's Private Key: A randomly chosen $m \in \mathbb{N}$.
Bob's Private Key: A randomly chosen $n \in \mathbb{N}$.
Alice's Public Key: Alice computes $A = H^{-m}(HM)^m$ and makes A public.
Bob's Public Key: Bob computes $B = H^{-n}(HM)^n$ and makes B public.
Shared Key: $K_A = K_B = H^{-n-m}(HM)^{n+m}$.

The security of this protocol is based on the assumption that, given the matrices $M \in \mathrm{Mat}_3(\mathbb{F}_7[A_5])$, $H \in \mathbf{GL}_3(\mathbb{F}_7[A_5])$, $A = H^{-m}(HM)^m$, and $B = H^{-n}(HM)^n$, it is hard to compute the matrix $H^{-n-m}(HM)^{n+m}$.

In [3] it was shown that the problem above can be easily solved using the fact that H is invertible. Indeed, any solution of the system:

$$\begin{cases} LA = R, \\ LH = HL, \\ RHM = HMR, \\ L \text{ is invertible}, \end{cases}$$

with unknown matrices L, R immediately gives the shared key as the product $L^{-1}BR$. To solve the system above we describe the set of all solutions to the linear system:

$$\begin{cases} LA = R, \\ LH = HL, \\ RHM = HMR, \end{cases}$$

and try-and-check if L is invertible for randomly chosen solutions. With high probability a required solution will be found in a few tries.

3. Defense against the linear attack

The attack described in Section 2.1 splits the public key A into a product of two "appropriate" matrices L, R that act as H^{-m} and $(HM)^m$, respectively. The following countermeasure was proposed in [2, Section 5] to prevent the attack. If M is not invertible, then M is not invertible and the annihilator of HM:

$$\mathrm{Ann}(HM) = \{K \in \mathrm{Mat}_3(\mathbb{F}_7[A_3]) \mid K \cdot HM = O\}$$

(where O is the zero matrix) is not trivial. Since in addition we have $m, n > 0$, then adding $O_A, O_B \in \mathrm{Ann}(HM)$ to the public keys A and B changes the keys, but does not change the deduced shared key. This gives the following scheme.

Algorithm 3. Modified HKKS key exchange protocol using $\mathrm{Mat}_3(\mathbb{F}_7[A_5])$

Initial Setup: Fix matrices $M \in \mathrm{Mat}_3(\mathbb{F}_7[A_5])$ and $H \in \mathbf{GL}_3(\mathbb{F}_7[A_5])$. They are made public.
Alice's Private Key: A randomly chosen $m \in \mathbb{N}$ and $O_A \in \mathrm{Ann}(HM)$.
Bob's Private Key: A randomly chosen $n \in \mathbb{N}$ and $O_B \in \mathrm{Ann}(HM)$.
Alice's Public Key: Alice computes $A = H^{-m}(HM)^m + O_A$ and makes A public.
Bob's Public Key: Bob computes $B = H^{-n}(HM)^n + O_B$ and makes B public.
Shared Key: $K_A = K_B = H^{-n-m}(HM)^{n+m}$.

The idea behind this modification is that one can not simply split A into a product of two matrices and move one of them to the left hand side. Below, using the property that annihilator is a left ideal and H is invertible, we show that this is incorrect and the same attack applies. Indeed, it is easy to see that any solution of the system of equations:

$$\begin{cases} LA = R + Z \\ LH = HL \\ R \cdot HM = HM \cdot R \\ Z \cdot HM = O, \\ L \text{ is invertible.} \end{cases}$$

with unknown matrices L, R and Z, immediately gives the shared key as the product $L^{-1}BR$. It is important that H is invertible.

4. HKKS protocol using an extension of the semigroup of matrices over a Galois field by an endomorphism

Another countermeasure suggested in [2, Section 4] is to replace the inner automorphism φ_H with a more complex endomorphism. That requires change of the platform semigroup. Consider the semigroup $G = \mathrm{Mat}_2(\mathbb{GF}(2^{127}))$ of 2×2 matrices over a finite field $\mathbb{GF}(2^{127})$. Let ψ be the endomorphism of G which raises every entry of a given matrix to the 4th power:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{\psi}{\mapsto} \begin{bmatrix} a^4 & b^4 \\ c^4 & d^4 \end{bmatrix}.$$

Fix $H \in \mathbf{GL}_2(\mathbb{GF}(2^{127}))$ and the corresponding inner automorphism φ_H. Now, $\varphi = \psi \circ \varphi_H$ with ψ acting first. This choices give us another instance of the HKKS protocol.

4.1. **Analysis of the protocol.** The map $x \overset{\tau}{\mapsto} x^4$ defined on $\mathbb{GF}(2^{127})$ can be recognized as a square of the Frobenius automorphism and, in particular, $\tau \in \mathbf{Aut}(\mathbb{GF}(2^{127}))$. It induces an automorphism ψ of $\mathrm{Mat}_2(\mathbb{GF}(2^{127}))$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{\psi}{\mapsto} \begin{bmatrix} a^4 & b^4 \\ c^4 & d^4 \end{bmatrix}.$$

Lemma 4.1. *$|\tau| = 127$ in $\mathbf{Aut}(M_2(\mathbb{GF}(2^{127})))$. Therefore, $|\psi| = 127$ in $\mathbf{Aut}(\mathrm{Mat}_3(\mathbb{GF}(7)))$.*

Proof. Consider the Frobenius automorphism ρ which squares elements of $\mathbb{GF}(2^{127})$. Then $\rho^{127}(x) = x^{2^{127}} = x$ for every $x \in \mathbb{GF}(2^{127})$. On the other hand, since $x^{2^k} - x = 0$ can not have more than 2^k solutions in a field, we can deduce that $|\rho| = 127$. Now $|\tau| = |\rho^2| = 127$. $\square$

Now, φ is the composition of the endomorphism ψ and conjugation by H:

$$\varphi(M) = H^{-1}\psi(M)H$$

for every $M \in M_2(\mathbb{GF}(2^{127}))$. For every $k \in \mathbb{N}$ we have:

$$\varphi^k(M) = \prod_{i=0}^{k-1} \psi^i(H^{-1}) \cdot \psi^k(M) \cdot \prod_{i=k-1}^{0} \psi^i(H).$$

With so defined φ, the Alice's public key $A = \varphi^{m-1}(M)\dots\varphi(M)M$ is of the form:

$$\left(\prod_{i=0}^{m-1}\psi^i(H^{-1})\cdot\psi^m(M)\cdot\prod_{i=m-1}^{0}\psi^i(H)\right)\cdot\left(\prod_{i=0}^{m-2}\psi^i(H^{-1})\cdot\psi^{m-1}(M)\cdot\prod_{i=m-2}^{0}\psi^i(H)\right)\dots H^{-1}\psi(M)H\cdot M$$

$$= \left(\prod_{i=0}^{m-1}\psi^i(H^{-1})\cdot\psi^m(M)\right)\psi^{m-1}(H)\psi^{m-1}(M)\cdot\psi^{m-2}(H)\psi^{m-2}(M)\cdot\ldots\cdot\psi(H)\psi(M)\cdot HM$$

$$= \left(\prod_{i=0}^{m}\psi^i(H^{-1})\right)\cdot\left(\prod_{i=m}^{0}\psi^i(HM)\right)$$

Since $|\psi| = 127$ we can divide $m = 127\cdot q + r$ and write the key as follows:

$$A = \left(\prod_{i=0}^{126}\psi^i(H^{-1})\right)^q\cdot\left(\prod_{i=0}^{r}\psi^i(H^{-1})\right)\cdot\left(\prod_{i=r}^{0}\psi^i(HM)\right)\cdot\left(\prod_{i=126}^{0}\psi^i(HM)\right)^q.$$

The Bob's public key B is has a similar form (with $n = 127\cdot s + t$):

$$B = \left(\prod_{i=0}^{126}\psi^i(H^{-1})\right)^s\cdot\left(\prod_{i=0}^{t}\psi^i(H^{-1})\right)\cdot\left(\prod_{i=t}^{0}\psi^i(HM)\right)\cdot\left(\prod_{i=126}^{0}\psi^i(HM)\right)^s.$$

Now we can use the "old trick". For each $0 \le r \le 126$ try to solve the system of equations:

$$\begin{cases} L\cdot A = \left(\prod_{i=0}^{r}\psi^i(H^{-1})\right)\cdot\left(\prod_{i=r}^{0}\psi^i(HM)\right)\cdot R, \\ L\cdot\prod_{i=0}^{126}\psi^i(H^{-1}) = \prod_{i=0}^{126}\psi^i(H^{-1})\cdot L, \\ R\cdot\prod_{i=126}^{0}\psi^i(HM) = \prod_{i=126}^{0}\psi^i(HM)\cdot R, \\ L \text{ is invertible.}\end{cases}$$

If the pair (L, R) satisfies the system above, then $L^{-1}BR$ is the shared key.

5. Conclusion

In this paper we analyzed two modifications of the HKKS protocol proposed in [2] and proved that both protocols can be easily broken by simple linear algebra attacks.

References

[1] M. Habeeb, K. Kahrobaei, C. Koupparis, and V. Shpilrain. Public key exchange using semidirect product of (semi)groups. In *Applied Cryptography and Network Security – ACNS 2013*, volume 7954 of *Lecture Notes Comp. Sc.*, pages 475–486. Springer, 2013.

[2] D. Kahrobaei, H. Lam, and V. Shpilrain. Public key exchange using extensions by endomorphisms and matrices over a Galois field. Preprint, available at `http://www.sci.ccny.cuny.edu/~shpil/semi_galois.pdf`.

[3] A. D. Myasnikov and A. Ushakov. A linear algebra attack to group-ring-based key exchange protocols. In *Applied Cryptography and Network Security – ACNS 2014*, volume 8479 of *Lecture Notes Comp. Sc.*, pages 37–43. Springer, 2014.

Department of Mathematics, University of Cincinnati, OH, USA
E-mail address: `jintai.ding@gmail.com`

Department of Mathematics, Stevens Institute of Technology, Hoboken, NJ, USA
E-mail address: `amiasnik,aushakov@stevens.edu`