# Practical Order-Revealing Encryption with Limited Leakage

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## Abstract

In an order-preserving encryption scheme, the encryption algorithm produces ciphertexts that preserve the order of their plaintexts. Order-preserving encryption schemes have been studied intensely in the last decade, and yet not much is known about the security of these schemes. Very recently, Boneh et al. (Eurocrypt 2015) introduced a generalization of order-preserving encryption, called order-revealing encryption, and presented a construction which achieves this notion with best-possible security. Because their construction relies on multilinear maps, it is too impractical for most applications and therefore remains a theoretical result.

In this work, we build efficiently implementable order-revealing encryption from pseudorandom functions. We present the first efficient order-revealing encryption scheme which achieves a simulation-based security notion with respect to a leakage function that precisely quantifies what is leaked by the scheme. In fact, ciphertexts in our scheme are only about 1.6 times longer than their plaintexts. Moreover, we show how composing our construction with existing order-preserving encryption schemes results in order-revealing encryption that is strictly more secure than all preceding order-preserving encryption schemes.

## 1 Introduction

A symmetric encryption scheme is order-preserving if the ciphertexts preserve the numeric ordering of their underlying plaintexts. The notion of order-preserving encryption (OPE) was introduced by Agrawal et al. [AKSX04] who showed how it could be used to efficiently answer range queries over encrypted data, as well as sorting queries, searching queries, and more. Indeed, existing OPE solutions have been implemented in practice [Sky, PRZB11] for these exact purposes. Since the introduction of OPE, there has been a plethora of work on analyzing the security of various OPE schemes, found both in the cryptography community and the database community. However, it is troubling that in spite of the numerous practical applications of OPE, the security of the best candidate OPE schemes is still not well understood.

**Prior work.** The first OPE construction by Agrawal et al. [AKSX04] relied on heuristics and lacked a formal security analysis. Subsequently, Boldyreva et al. [BCLO09] gave the first formal security definitions for OPE schemes. Boldyreva et al. introduced two primary notions for security of an OPE scheme. The first notion of security for an OPE scheme is called indistinguishability under an ordered chosen plaintext attack (IND-OCPA). The IND-OCPA definition can be viewed as

a generalization of semantic security [GM84], and effectively says that encryptions of a sequence of messages should reveal nothing about the underlying messages other than their ordering. However, in the same work, Boldyreva et al. showed that no efficient order-preserving encryption scheme can be IND-OCPA secure, even in settings where the size of the ciphertext space is exponentially larger than the size of the plaintext space.

In light of this lower bound for OPE schemes that satisfy IND-OCPA security, Boldyreva et al. introduced a weaker notion of security (POPF-CCA security) where the encryption function for the OPE scheme is compared to a random order-preserving function—that is, the encryption algorithm for an OPE scheme behaves like a truly random order-preserving function. Under this definition, an OPE scheme inherits the properties of a random order-preserving function.[1] In the same work, Boldyreva et al. gave an explicit construction of an OPE scheme that satisfies POPF-CCA security. However, the POPF-CCA security definition does not precisely specify the information that is leaked by an OPE scheme that achieves this definition. In fact, a scheme that achieves this notion of security does not even satisfy semantic security for a single encryption, and indeed, in subsequent work, Boldyreva et al. [BCO11] showed that ciphertexts in their OPE scheme leak approximately the first half of the bits of the underlying plaintexts. In addition, they introduce several new security definitions in order to better quantify the information leakage of OPE schemes that are POPF-CCA secure.

Recently, Boneh et al. [BLR$^+$15] proposed a generalization of OPE called order-revealing encryption (ORE). In an OPE scheme, the ciphertexts are numeric-valued, and the ordering of the underlying plaintexts is determined by numerically comparing the ciphertexts. In contrast, in an ORE scheme, the ciphertexts are not constrained to any particular form, and instead, there is a publicly computable comparison function which takes two ciphertexts and outputs the numeric ordering of the underlying plaintexts.[2] Although this generalization may at first seem subtle, Boneh et al. constructed an ORE scheme from multilinear maps that achieves the "best-possible" notion of security, which is equivalent to the IND-OCPA security notion for order-preserving encryption.

The main drawback of the Boneh et al. ORE construction is that it relies on complicated tools and strong assumptions on these tools, and as such, is currently impractical to implement.

## 1.1 Our Contributions

We now summarize the main contributions of this work, which include a new simulation-based security notion for ORE, along with a practical construction of an ORE scheme which achieves this security notion. We also show how our new construction can be used to achieve a strictly stronger notion of security compared to other stateless and efficiently implementable (e.g., constructions that do not rely on powerful primitives such as multilinear maps and indistinguishability obfuscation) OPE and ORE encryption schemes.

**Security model.** In our work, we take the general approach of Boneh et al. in constructing an ORE scheme, except we take a more efficient route. Our first contribution is a new security definition for order-revealing encryption schemes that both allows for and explicitly models the leakage in the scheme. Our design goals for introducing this new security model are twofold: first,

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[1] This definition is inspired by the similar definition for PRF security [GGM84], which compares the output of a keyed function to that of a truly random function.

[2] This application was also observed and independently achieved by Goldwasser et al. [GGG$^+$14] using indistinguishability obfuscation.

the security model should enable constructions that are efficiently implementable, and second, it should provide a precise quantification of any information leaked by the scheme. The two primary notions of security, IND-OCPA and POPF-CCA, introduced by Boldyreva et al. [BCLO09] each satisfy one of these two properties. In particular, all non-interactive, stateless[3] ORE schemes that achieve IND-OCPA security require strong cryptographic primitives such as multilinear maps or indistinguishability obfuscation [GGG$^+$14, BLR$^+$15], and thus, are not efficiently implementable today. At the other end of the spectrum, it is difficult to precisely quantify the leakage of schemes that satisfy POPF-CCA security. The work by Boldyreva et al. [BCO11] provides some concrete lower and upper bounds for the leakage under the strong assumption that the plaintexts are drawn from a uniform distribution. For more general distributions, the leakage remains unclear.

In our work, we give a simulation-based definition of security for ORE with respect to a leakage function $\mathcal{L}$. In other words, our definition states that whatever an adversary is able to deduce from seeing encryptions of messages $m_1, \ldots, m_t$, it could also deduce given only the leakage $\mathcal{L}(m_1, \ldots, m_t)$. The "best-possible" security for ORE would correspond to the case where the leakage function simply outputs whether $m_i < m_j$ for all pairs of messages $m_i$ and $m_j$. By allowing for the possibility of additional leakage, it becomes possible to construct practical ORE schemes from standard assumptions. Thus, our constructions provide a concrete trade-off between security and efficiency. Our security definitions are similar to the simulation-based definitions that have been considered previously in the searchable symmetric encryption literature [CM05, CGKO06].

**Constructions.** In our main construction, we show how to construct an ORE scheme from one-way functions (more precisely, from pseudorandom functions (PRFs) [GGM84]). This particular ORE scheme reveals slightly more information than just the ordering of the underlying messages. Specifically, two ciphertexts encrypting messages $m_1$ and $m_2$ also reveal the index of the first bit in $m_1$ and $m_2$ that differ. In other words, our ORE scheme leaks some information about the relative distance between the underlying messages.

We give a brief overview of our PRF-based construction. The secret key in our scheme consists of a PRF key $k$. The output space of the PRF is the set $\{0, 1, 2\}$. Each ciphertext consists of the bits of the message blinded by the outputs of the PRF evaluated on the prefixes of the message. More precisely, to encrypt an $n$-bit message $m = m_1 m_2 \cdots m_n$, the encryption algorithm effectively computes the following for each $i \in [n]$:

$$u_i = F(k, m_1 m_2 \cdots m_{i-1}) + m_i \pmod{3}.$$

Note that to support variable-length PRF inputs, we simply pad the input. We describe our construction in greater detail in Section 3. The ciphertext is then the tuple $\mathsf{ct} = (u_1, \ldots, u_n)$ of blinded values.

To compare encryptions $\mathsf{ct} = (u_1, \ldots, u_n)$ and $\mathsf{ct}' = (u'_1, \ldots, u'_n)$ of messages $m$ and $m'$, the evaluator first finds the first index $i$ for which $u_i \neq u'_i$. Since $u_i$ and $u'_i$ are functions of just the first $i$ bits of $m$ and $m'$, respectively, the first index $i$ for which $u_i \neq u'_i$ is the first bit of $m$ and $m'$ that differ. After identifying the $i^{\text{th}}$ bit that differs, the evaluator uses $u_i$ and $u'_i$ to determine which message has 0 as the $i^{\text{th}}$ bit and which message has 1.[4] Conversely, if $u_i = u'_i$ for all $i$, then $\mathsf{ct}_i = \mathsf{ct}'_i$, and so $m = m'$. Security of this construction follows from the security of the PRF (Theorems 3.2).

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[3]There are "mutable" order-preserving encryption schemes [PLZ13, KS14, Ker15] that do satisfy IND-OCPA, but they require stateful encryption, and oftentimes, an interactive protocol to "update" ciphertexts. We survey some of these constructions in Section 1.2.

[4]Either $u_i + 1 = u'_i \pmod{3}$, in which case $m < m'$, or $u_i - 1 = u'_i \pmod{3}$, in which case $m > m'$.

Ciphertexts in our candidate scheme are $\lceil n \cdot \log_2 3 \rceil \approx \lceil 1.6\,n \rceil$ bits, where $n$ is the bit-length of the message. As a point of comparison, ciphertexts in the OPE scheme of Boldyreva et al. [BCLO09] are only $n + 1$ bits long. While the ciphertexts in our scheme are longer (by a multiplicative factor $\log_2 3$), the authors of [BCO11] note that even if the size of the ciphertext space is increased beyond $n + 1$ bits in the Boldyreva et al. scheme, the security of their construction does not improve by any noticeable amount.

We then explain in Section 3.2 how to convert our ORE scheme into an OPE scheme, at the expense of longer ciphertexts. This is useful for applications where it is more convenient to have a numeric ciphertext space and for order relations to be computable without a "custom" comparison function. The transformation we describe is natural and does not reduce the security of the original ORE scheme. In particular, we note that the resulting OPE scheme does *not* behave like a random order-preserving function (the ideal object from the POPF-CCA security notion). Thus, the scheme is able to achieve stronger security than the Boldyreva et al. OPE scheme.

**Comparison with existing schemes.** First, we note in Section 2.3 that the security of any OPE scheme can be "augmented" by applying ORE encryption on top of OPE encryption. The resulting scheme is at least as secure as the underlying OPE scheme, and moreover, inherits the security properties of the ORE scheme. Hence, by composing our ORE construction with existing OPE constructions, we obtain ORE schemes that are at least as secure.

While composing an OPE scheme with an ORE scheme yields a scheme that is at least as secure as the underlying OPE scheme, we show that even without this composition, our basic ORE scheme still achieves stronger security guarantees according to the one-wayness metrics introduced by Boldyreva et al. [BCO11] for analyzing the leakage of random order-preserving functions (and by extension, any OPE scheme that is POPF-CCA secure). In our work, we introduce two generalized one-wayness notions and show that under a uniform plaintext distribution,[5] our basic ORE scheme achieves strictly stronger security compared to OPE schemes that are POPF-CCA secure. Specifically, Boldyreva et al. [BCO11] show that a random order-preserving function leaks half of the most-significant bits of the messages with probability close to 1. In contrast, under the same settings, we can show that our basic ORE scheme will not leak *any constant* fraction of the message bits with *overwhelming* probability.

## 1.2 Related Work

In recent years, there have been numerous works on order-preserving encryption and related notions [AKSX04, BCLO09, BCO11, PR12, PLZ13, TYM14, KS14, Ker15, MCO+15, RACY15]. In this section, we survey some of these works.

**Security definitions.** Though the POPF-CCA security definition introduced by Boldyreva et al. [BCLO09] is similar in flavor to PRF security, it is not immediately evident what kind of information the output of a random order-preserving function leaks about its input. In a follow-up work [BCO11], Boldyreva et al. introduce several notions (based on definitions of one-wayness [Gol01] for one-way functions) to capture the information leakage in schemes that are POPF-CCA secure. They show that a random order-preserving function leaks at least half of the bits in each message.

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[5]This is the only distribution for which we have concrete analysis of the leakage in any POPF-CCA secure scheme.

Teranishi et al. [TYM14] also introduce a stronger indistinguishability-based notion (stronger than the one-wayness definitions from [BCO11], but weaker than IND-OCPA) for OPE schemes, as well as a construction that achieves these stronger notions. Notably, their definition ensures that under a uniform message distribution, any fraction of the low-order bits of the messages being encrypted are hidden.

Recently, Naveed et al. [NKW15] analyzed the information leaked by order-preserving encryption used in practical scenarios.

**Modular OPE.** Boldyreva et al. also introduced the notion of modular OPE as a possible extension of standard OPE [BCO11]. In modular OPE, a modular shift is applied to each plaintext before applying OPE—so the scheme is not order-preserving, but naturally supports "wrap-around" range queries. Their modular OPE scheme adds an extra layer of security to vanilla OPE, but it is worth noting that leakage of a small amount of information (say, a single plaintext-ciphertext pair) reveals the shift value and nullifies this added security. Subsequently, Mavroforakis et al. [MCO+15] designed several protocols to avoid leaking the shift value while using modular OPE schemes in practice.

**Mutable OPE.** Popa et al. [PLZ13] introduced a related notion of a mutable order-preserving encoding scheme which can be viewed as a two-party protocol that allows a user to insert and store encrypted values in a database such that the database is able to perform comparisons and range queries on the encrypted values without learning anything more about the values. Their construction is interactive and leverages stateful encryption. By working in this setting, the authors are able to circumvent the Boldyreva et al. [BCLO09] lower bound for order-preserving encryption and show that their scheme is IND-OCPA secure.

In subsequent work, Kerschbaum and Schröpfer [KS14] improved on the communication complexity of the Popa et al. construction at the expense of increasing the amount of client-side state. Specifically, in their construction, the amount of persistent state the client has to maintain increases linearly in the number of elements inserted into the database. More recently, Kerschbaum [Ker15] introduced a new notion of frequency-hiding OPE that introduces additional randomness to hide whether multiple ciphertexts encrypt the same value. Their notions provide a strictly stronger guarantee than IND-OCPA.

Very recently, Roche et al. [RACY15] introduced the notion of partial order-preserving encodings, which optimizes for the setting where there are a huge number of insertion queries but only a moderate number of range queries. Their protocol improves upon the round-complexity for insertions compared to the Popa et al. protocol [PLZ13], and requires the client to maintain less state than the Kerschbaum-Schröpfer construction [KS14]. All of the schemes described here require stateful encryption and employ an interactive encryption procedure.

**ORE.** Order-revealing encryption schemes, as introduced by Boneh et al. [BLR+15] provide another method of circumventing the Boldyreva et al. lower bound [BCLO09]. In an ORE scheme, the public comparison operation is not required to correspond to numerically comparing the ciphertexts, and in fact, the ciphertexts themselves need not be elements of a numeric, well-ordered set. This type of relaxation was previously considered by Pandey and Rouselakis [PR12] in the context of property-preserving encryption. In a property-preserving encryption scheme, there is a publicly computable function that can be evaluated on ciphertexts to determine the value of

some property on the underlying plaintexts. Order-revealing encryption can thus be viewed as a property-preserving encryption scheme for the comparison operation. Pandey and Rouselakis introduce and explore several indistinguishability-based notions of security for property-preserving encryption; however, they do not construct an order-revealing encryption scheme.

To the best of our knowledge, all existing ORE schemes that provide IND-OCPA security either rely on very strong (and currently impractical) cryptographic primitives such as indistinguishability obfuscation [GGG+14] and cryptographic multilinear maps [BLR+15], or only achieve a weaker notion of security [AJ15, BKS15] when instantiated with simple cryptographic primitives such as public key cryptography. For the constructions based on indistinguishability obfuscation or multilinear maps [GGG+14, BLR+15], security of the ORE scheme is conditional on the conjectured security of cryptographic multilinear maps [BS03, GGH13a, CLT13, LSS14, GGH15, CLT15, AFH+16].[6] However, in the last few months, numerous attacks [CHL+15, BWZ14, CGH+15, HJ15, CLR15, MF15, Cor15] on these multilinear maps have emerged, raising some doubts about the security of constructions that leverage them.

To avoid multilinear maps in favor of more well-studied number-theoretic or lattice-based assumptions, one can apply arity-amplification techniques [AJ15, BKS15] to a single-input functional encryption scheme based on simpler assumptions such as learning with errors [GKP+13] or semantically-secure public-key encryption [SS10, GVW12]. However, due to limitations of the underlying functional encryption schemes, the resulting ORE scheme only provides "bounded-message" security—that is, security only holds if there is an *a priori* (polynomial) bound on the maximum number of messages that will be encrypted. Moreover, the length of the ciphertexts in this scheme grows *polynomially* in the bound on the number of messages that will be encrypted. These constraints severely limit the practicality of the resulting ORE scheme. To obtain full semantic security, it would be necessary to apply the arity-amplification transformation to a more powerful functional encryption scheme, but to date, the only known candidates of such schemes rely again on indistinguishability obfuscation [GGH+13b] or multilinear maps [GGHZ14].

Recently, Bun and Zhandry [BZ15] investigated the connection between order-revealing encryption and problems in learning theory.

**Other schemes.**  Numerous ad hoc or heuristic order-preserving encryption schemes [BHF09, KAK10, XYH12] have been proposed in the literature, but most lack formal security analysis.

## 2   Order-Revealing Encryption

In this section, we establish and review some conventions that we use in this work, and also formally define our security notions for our encryption schemes.

**Preliminaries.**  For $n \in \mathbb{N}$, we write $[n]$ to denote the set of integers $\{1, \ldots, n\}$, and $\mathbb{Z}_n$ to denote the additive group of integers modulo $n$. If $\mathcal{P}(x)$ is a predicate on $x$, we write $\mathbf{1}(\mathcal{P}(x))$ to denote the indicator function for $\mathcal{P}$: that is, $\mathbf{1}(\mathcal{P}(x)) = 1$ if and only if $\mathcal{P}(x) = 1$, and 0 otherwise. If $x, y \in \{0, 1\}^*$ are bit-strings, we write $x \| y$ to denote the concatenation of $x$ and $y$. For a finite set $S$, we write $\mathsf{Unif}(S)$ to denote the uniform distribution on $S$. We say a function $f(\lambda)$ is negligible in a security parameter $\lambda$ if $f = o(1/\lambda^c)$ for all $c \in \mathbb{N}$. We write $\mathsf{negl}(\lambda)$ to denote a negligible

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[6]To date, the only concrete instantiations of indistinguishability obfuscation [GGH+13b, BGK+14, AB15, Zim15] leverage multilinear maps.

function in $\lambda$ and $\mathrm{poly}(\lambda)$ to denote a polynomial in $\lambda$. We say an event occurs with negligible probability if the probability of the event is $\mathrm{negl}(\lambda)$, and it occurs with overwhelming probability if the complement of the event occurs with negligible probability. Finally, we review the definition of a pseudorandom function (PRF) [GGM84]. Let $\mathsf{Funs}[\mathcal{D}, \mathcal{R}]$ denote the set of all functions from a domain $\mathcal{D}$ to a range $\mathcal{R}$. In this paper, we specialize the domain of our PRFs to $\{0,1\}^n$.

**Definition 2.1** (Pseudorandom Function [GGM84]). Fix a security parameter $\lambda$. A PRF $F : \mathcal{K} \times \{0,1\}^n \to \mathcal{R}$ with key space $\mathcal{K}$, domain $\{0,1\}^n$, and range $\mathcal{R}$ is secure if for all efficient adversaries $\mathcal{A}$,

$$\left| \Pr\left[ k \xleftarrow{\mathrm{R}} \mathcal{K} : \mathcal{A}^{F(k,\cdot)}(1^\lambda) = 1 \right] - \Pr\left[ f \xleftarrow{\mathrm{R}} \mathsf{Funs}[\{0,1\}^n, \mathcal{R}] : \mathcal{A}^{f(\cdot)}(1^\lambda) = 1 \right] \right| = \mathrm{negl}(\lambda).$$

## 2.1 Order-Revealing Encryption

An order-revealing encryption (ORE) scheme is a tuple of algorithms $\Pi = (\mathsf{ORE.Setup}, \mathsf{ORE.Encrypt}, \mathsf{ORE.Compare})$ defined over a well-ordered domain $\mathcal{D}$ with the following properties:

- $\mathsf{ORE.Setup}(1^\lambda) \to \mathsf{sk}$. On input a security parameter $\lambda$, the setup algorithm $\mathsf{ORE.Setup}$ outputs a secret key $\mathsf{sk}$.

- $\mathsf{ORE.Encrypt}(\mathsf{sk}, m) \to \mathsf{ct}$. On input the secret key $\mathsf{sk}$ and a message $m \in \mathcal{D}$, the encrypt algorithm $\mathsf{ORE.Encrypt}$ outputs a ciphertext $\mathsf{ct}$.

- $\mathsf{ORE.Compare}(\mathsf{ct}_1, \mathsf{ct}_2) \to b$. On input two ciphertexts $\mathsf{ct}_1$, $\mathsf{ct}_2$, the compare algorithm $\mathsf{ORE.Compare}$ outputs a bit $b \in \{0,1\}$.

**Remark 2.2** (Public Parameters). In general, the setup algorithm of an ORE scheme can also output public parameters $\mathsf{pp}$ which are then passed as an additional input to the comparison algorithm, as is done in Boneh et al. [BLR+15]. However, none of our constructions require these public parameters, so we omit them in this work for simplicity.

**Remark 2.3** (Support for Decryption). As described, our definition of an order-revealing encryption scheme does not include a "decryption" function. However, this omission is without loss of generality. To decrypt a message, the holder of the secret key can use the secret key to encrypt messages of her choosing, apply the comparison algorithm, and perform binary search to recover the message. An alternative method that avoids the need for binary search is to augment each ORE encryption of a message $m$ with an encryption of $m$ under a CPA-secure symmetric encryption scheme. The secret key of the ORE scheme would also include the key for the symmetric encryption scheme. As long as the underlying encryption scheme is CPA-secure, including this additional ciphertext does not compromise security. For the remainder of this work, we use the schema described above that does not explicitly specify a decryption function.

**Correctness.** Fix a security parameter $\lambda$. An ORE scheme $\Pi = (\mathsf{ORE.Setup}, \mathsf{ORE.Encrypt}, \mathsf{ORE.Compare})$ over a well-ordered domain $\mathcal{D}$ is correct if for $\mathsf{sk} \leftarrow \mathsf{ORE.Setup}(1^\lambda)$, and all messages $m_1, m_2 \in \mathcal{D}$,

$$\Pr[\mathsf{ORE.Compare}(\mathsf{ct}_1, \mathsf{ct}_2) = \mathbf{1}(m_1 < m_2)] = 1 - \mathrm{negl}(\lambda),$$

where $\mathsf{ct}_1 \leftarrow \mathsf{ORE.Encrypt}(\mathsf{sk}, m_1)$ and $\mathsf{ct}_2 \leftarrow \mathsf{ORE.Encrypt}(\mathsf{sk}, m_2)$, and the probability is taken over the random coins in $\mathsf{ORE.Setup}$ and $\mathsf{ORE.Encrypt}$.

**Security.** We now give our simulation-based notion of security for an ORE scheme. As described in Section 1.1, our security definition is parameterized by a leakage function $\mathcal{L}$, which exactly specifies what is leaked by an ORE scheme.

**Definition 2.4** (Security of ORE with Leakage). Fix a security parameter $\lambda \in \mathbb{N}$. Let $\Pi_{\mathsf{ore}} = (\mathsf{ORE.Setup}, \mathsf{ORE.Encrypt}, \mathsf{ORE.Compare})$ be an ORE scheme. Let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_q)$ be an adversary for some $q \in \mathbb{N}$. Let $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_q)$ be a simulator, and let $\mathcal{L}(\cdot)$ be a leakage function. We define the experiments $\mathsf{REAL}_{\mathcal{A}}^{\mathrm{ORE}}(\lambda)$ and $\mathsf{SIM}_{\mathcal{A},\mathcal{S},\mathcal{L}}^{\mathrm{ORE}}(\lambda)$ as follows:

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$\mathsf{REAL}_{\mathcal{A}}^{\mathrm{ORE}}(\lambda)$:

1. $\mathsf{sk} \leftarrow \mathsf{ORE.Setup}(1^\lambda)$
2. $(m_1, \mathsf{st}_\mathcal{A}) \leftarrow \mathcal{A}_1(1^\lambda)$
3. $c_1 \leftarrow \mathsf{ORE.Encrypt}(\mathsf{sk}, m_1)$
4. for $2 \leq i \leq q$:
   - (a) $(m_i, \mathsf{st}_\mathcal{A}) \leftarrow \mathcal{A}_i(\mathsf{st}_\mathcal{A}, c_1, \ldots, c_{i-1})$
   - (b) $c_i \leftarrow \mathsf{ORE.Encrypt}(\mathsf{sk}, m_i)$
5. output $(c_1, \ldots, c_q)$ and $\mathsf{st}_\mathcal{A}$

$\mathsf{SIM}_{\mathcal{A},\mathcal{S},\mathcal{L}}^{\mathrm{ORE}}(\lambda)$:

1. $\mathsf{st}_\mathcal{S} \leftarrow \mathcal{S}_0(1^\lambda)$
2. $(m_1, \mathsf{st}_\mathcal{A}) \leftarrow \mathcal{A}_1(1^\lambda)$
3. $(c_1, \mathsf{st}_\mathcal{S}) \leftarrow \mathcal{S}_1(\mathsf{st}_\mathcal{S}, \mathcal{L}(m_1))$
4. for $2 \leq i \leq q$:
   - (a) $(m_i, \mathsf{st}_\mathcal{A}) \leftarrow \mathcal{A}_i(\mathsf{st}_\mathcal{A}, c_1, \ldots, c_{i-1})$
   - (b) $(c_i, \mathsf{st}_\mathcal{S}) \leftarrow \mathcal{S}_i(\mathsf{st}_\mathcal{S}, \mathcal{L}(m_1, \ldots, m_i))$
5. output $(c_1, \ldots, c_q)$ and $\mathsf{st}_\mathcal{A}$

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We say that $\Pi_{\mathsf{ore}}$ is a secure ORE scheme with leakage function $\mathcal{L}(\cdot)$ if for all polynomial-size adversaries $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_q)$ where $q = \mathrm{poly}(\lambda)$, there exists a polynomial-size simulator $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_q)$ such that the outputs of the two distributions $\mathsf{REAL}_{\mathcal{A}}^{\mathrm{ORE}}(\lambda)$ and $\mathsf{SIM}_{\mathcal{A},\mathcal{S},\mathcal{L}}^{\mathrm{ORE}}(\lambda)$ are computationally indistinguishable.

**Remark 2.5** (IND-OCPA Security). We briefly note how the IND-OCPA definition of security is captured by this definition. Let $\mathcal{L}$ be the following leakage function:

$$\mathcal{L}(m_1, \ldots, m_t) = \{\mathbf{1}(m_i < m_j) : 1 \leq i < j \leq t\}.$$

If an ORE scheme is secure with leakage $\mathcal{L}$, then it is IND-OCPA secure.

## 2.2 Order-Preserving Encryption (OPE)

An OPE scheme [AKSX04, BCLO09] is a special case of an ORE scheme, where the ciphertext space is required to be a well-ordered range $\mathcal{R}$ and moreover, for two ciphertexts $\mathsf{ct}_1, \mathsf{ct}_2 \in \mathcal{R}$, the comparison algorithm outputs 1 if $\mathsf{ct}_1 < \mathsf{ct}_2$. For simplicity, we can write an OPE scheme as a tuple of algorithms $\Pi = (\mathsf{OPE.Setup}, \mathsf{OPE.Encrypt})$ defined over a well-ordered domain $\mathcal{D}$ and well-ordered range $\mathcal{R}$ with the following properties:

- $\mathsf{ORE.Setup}(1^\lambda) \to \mathsf{sk}$. On input a security parameter $\lambda$, the setup algorithm $\mathsf{ORE.Setup}$ outputs a secret key $\mathsf{sk}$.

- $\mathsf{ORE.Encrypt}(\mathsf{sk}, m) \to \mathsf{ct}$. On input the secret key $\mathsf{sk}$ and a message $m \in \mathcal{D}$, the encrypt algorithm $\mathsf{OPE.Encrypt}$ outputs a ciphertext $\mathsf{ct} \in \mathcal{R}$.

**Correctness.** An OPE scheme $\Pi = (\mathsf{OPE.Setup}, \mathsf{OPE.Encrypt})$ over a well-ordered domain $\mathcal{D}$ and well-ordered range $\mathcal{R}$ is correct if $\mathsf{sk} \leftarrow \mathsf{OPE.Setup}(1^\lambda)$, and all messages $m_1, m_2 \in \mathcal{D}$,

$$m_1 < m_2 \qquad \Longleftrightarrow \qquad \mathsf{OPE.Encrypt}(\mathsf{sk}, m_1) < \mathsf{OPE.Encrypt}(\mathsf{sk}, m_2)$$

with overwhelming probability.

## 2.3 Composing OPE with ORE

By composing an ORE scheme with an OPE scheme, we obtain an ORE scheme whose security is at least as strong as the security of the underlying OPE scheme. Let $\Pi_{\mathsf{ope}} = (\mathsf{OPE.Setup}, \mathsf{OPE.Encrypt})$ be an OPE scheme and $\Pi_{\mathsf{ore}}^{\mathsf{in}} = (\mathsf{ORE^{in}.Setup}, \mathsf{ORE^{in}.Encrypt}, \mathsf{ORE^{in}.Compare})$ be an ORE scheme. Consider the following composed construction $\Pi_{\mathsf{ore}} = (\mathsf{ORE.Setup}, \mathsf{ORE.Encrypt}, \mathsf{ORE.Compare})$ of an ORE scheme with an OPE scheme:

- $\mathsf{ORE.Setup}(1^\lambda)$. The setup algorithm runs $\mathsf{sk}_1 \leftarrow \mathsf{OPE.Setup}(1^\lambda)$ and $\mathsf{sk}_2 \leftarrow \mathsf{ORE^{in}.Setup}(1^\lambda)$. The secret key is $\mathsf{sk} = (\mathsf{sk}_1, \mathsf{sk}_2)$.

- $\mathsf{ORE.Encrypt}(\mathsf{sk}, m)$. The encryption algorithm outputs $\mathsf{ORE^{in}.Encrypt}(\mathsf{sk}_2, \mathsf{OPE.Encrypt}(\mathsf{sk}_1, m))$.

- $\mathsf{ORE.Compare}(\mathsf{ct}_1, \mathsf{ct}_2)$. The compare algorithm outputs $\mathsf{ORE^{in}.Compare}(\mathsf{ct}_1, \mathsf{ct}_2)$.

Correctness of $\Pi_{\mathsf{ore}}$ follows immediately from the correctness of $\Pi_{\mathsf{ore}}^{\mathsf{in}}$ and $\Pi_{\mathsf{ope}}$. Furthermore, we note that under our simulation-based definition of security, the composed scheme $\Pi_{\mathsf{ore}}$ is at least as secure as $\Pi_{\mathsf{ope}}$. This intuition is formalized in the following remark, whose proof follows immediately by construction.

**Remark 2.6** (Security of Composed Scheme). For any leakage function $\mathcal{L}(\cdot)$, if the OPE scheme $\Pi_{\mathsf{ope}}$ is secure with leakage function $\mathcal{L}(\cdot)$, then the ORE scheme $\Pi_{\mathsf{ore}}$ is also secure with leakage function $\mathcal{L}(\cdot)$.

# 3 Main Construction

In this section, we give a construction of an ORE scheme for the set of $n$-bit positive integers with the following leakage function:

$$\mathcal{L}_{\mathsf{f}}(m_1, \ldots, m_t) := \{(\mathsf{ind}_{\mathsf{diff}}(m_i, m_j), \mathbf{1}(m_i < m_j)) : 1 \leq i < j \leq t\}, \tag{3.1}$$

where $\mathsf{ind}_{\mathsf{diff}}(x, y)$ gives the index of the first bit where $x$ and $y$ differ. If $x = y$, we set $\mathsf{ind}_{\mathsf{diff}}(x, y) = n + 1$. In other words, for $x \neq y$, if $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$, then $\mathsf{ind}_{\mathsf{diff}}(x, y)$ is the smallest index $\ell \in [n]$ for which $x_\ell \neq y_\ell$.

**Construction.** Fix a security parameter $\lambda \in \mathbb{N}$, and take an integer $M \geq 3$. Let $F : \mathcal{K} \times ([n] \times \{0,1\}^{n-1}) \to \mathbb{Z}_M$ be a secure PRF. We define our ORE scheme $\Pi_{\mathsf{ore}} = (\mathsf{ORE.Setup}, \mathsf{ORE.Encrypt}, \mathsf{ORE.Compare})$ as follows:

- $\mathsf{ORE.Setup}(1^\lambda)$. The setup algorithm chooses a uniformly random PRF key $k$ for $F$. The secret key is $\mathsf{sk} = k$.

- ORE.Encrypt($\mathsf{sk}, m$). Let $b_1 \cdots b_n$ be the binary representation of $m$ and let $\mathsf{sk} = k$. For each $i \in [n]$, the encryption algorithm computes

$$u_i = F(k, (i, b_1 b_2 \cdots b_{i-1} \| 0^{n-i})) + b_i \pmod{M},$$

and outputs the tuple $(u_1, u_2 \ldots, u_n)$.

- ORE.Compare($\mathsf{ct}_1, \mathsf{ct}_2$). The compare algorithm first parses

$$\mathsf{ct}_1 = (u_1, u_2, \ldots, u_n)$$
$$\mathsf{ct}_2 = (u'_1, u'_2, \ldots, u'_n),$$

where $u_1, \ldots, u_n, u'_1, \ldots, u'_n \in \mathbb{Z}_M$. Let $i$ be the smallest index where $u_i \neq u'_i$. If no such index exists, output 0. If such an index exists, output 1 if $u'_i = u_i + 1 \pmod{M}$, and 0 otherwise.

## 3.1 Correctness and Security

We now show that the above ORE scheme $\Pi_{\mathsf{ore}}$ is correct and secure against the leakage function $\mathcal{L}_{\mathsf{f}}$ from Equation (3.1). We give the proof of the following theorem in Appendix A.

**Theorem 3.1.** *The ORE scheme $\Pi_{\mathsf{ore}}$ is correct.*

Next, we state and prove the security theorem for $\Pi_{\mathsf{ore}}$.

**Theorem 3.2.** *The order-revealing encryption scheme $\Pi_{\mathsf{ore}}$ is secure with respect to leakage function $\mathcal{L}_{\mathsf{f}}$ (Definition 2.4) under the PRF security of $F$.*

*Proof.* Fix a security parameter $\lambda$ and let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_q)$ where $q = \mathrm{poly}(\lambda)$ be an efficient adversary for the ORE security game (Definition 2.4). To prove security, we give an efficient simulator $\mathcal{S} = (\mathcal{S}_0, \ldots, \mathcal{S}_q)$ for which the outputs of the distributions $\mathsf{REAL}^{\mathrm{ORE}}_{\mathcal{A}}(\lambda)$ and $\mathsf{SIM}^{\mathrm{ORE}}_{\mathcal{A}, \mathcal{S}, \mathcal{L}_{\mathsf{f}}}(\lambda)$ are computationally indistinguishable.

We use a hybrid argument. We begin by defining the hybrid experiments:

- **Hybrid** $\mathsf{H}_0$: This is the real experiment $\mathsf{REAL}^{\mathrm{ORE}}_{\mathcal{A}}(\lambda)$.

- **Hybrid** $\mathsf{H}_1$: Same as $\mathsf{H}_0$, except during ORE.Setup, a random function $f \xleftarrow{\mathrm{R}} \mathsf{Funs}[([n] \times \{0,1\}^{n-1}), \mathbb{Z}_M]$ is chosen. In all invocations of ORE.Encrypt, the function $F(k, \cdot)$ is replaced by $f(\cdot)$.

Hybrids $\mathsf{H}_0$ and $\mathsf{H}_1$ are computationally indistinguishable under the PRF security of $F$. Thus, it suffices to show that there exists a simulator $\mathcal{S}$ such that the distribution of outputs in $\mathsf{H}_1$ is computationally indistinguishable from $\mathsf{SIM}^{\mathrm{ORE}}_{\mathcal{A}, \mathcal{S}, \mathcal{L}_{\mathsf{f}}}(\lambda)$.

**Description of the simulator.** We now describe the simulator $\mathcal{S} = (\mathcal{S}_0, \ldots, \mathcal{S}_q)$. First, $\mathcal{S}_0$ initializes an empty lookup tables $\mathsf{L} : [q] \times [n] \rightarrow \mathbb{Z}_M$. It then outputs $\mathsf{st}_{\mathcal{S}} = \mathsf{L}$. Then, for each $t \in [q]$, after the adversary outputs a query $m_t$, the simulation algorithm $\mathcal{S}_t$ is invoked on input $\mathsf{st}_{\mathcal{S}} = \mathsf{L}$ and $\mathcal{L}_{\mathsf{f}}(m_1, \ldots, m_t)$. In particular, $\mathcal{L}_{\mathsf{f}}(m_1, \ldots, m_t)$ contains the values $\mathbf{1}(m_j < m_t)$ and $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t)$ for all $j \in [t-1]$, where $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t)$ is the index of the first bit in $m_j$ and $m_t$ that differ. For each $s \in [n]$, there are three cases to consider:

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- **Case 1:** There exists a $j \in [t-1]$ such that $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t) > s$. If there are multiple $j$ for which $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t) > s$, let $j$ be the smallest one. Then, the simulator sets $\overline{u}_s = \mathsf{L}(j, s)$.

- **Case 2:** For each $\ell \in [t-1]$, $\mathsf{ind}_{\mathsf{diff}}(m_\ell, m_t) \leq s$, and there exists a $j \in [t-1]$ for which $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t) = s$. If there are multiple $j$ for which $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t) = s$, let $j$ be the smallest one. Then, the simulator sets $\overline{u}_s = \mathsf{L}(j, s) - (1 - 2 \cdot \mathbf{1}(m_j < m_t)) \pmod{M}$.

- **Case 3:** For each $\ell \in [t-1]$, $\mathsf{ind}_{\mathsf{diff}}(m_\ell, m_t) < s$. In this case, the simulator samples $y \xleftarrow{\text{R}} \mathbb{Z}_M$ and sets $\overline{u}_s = y$.

For each $s \in [n]$, the simulator adds the mapping $(t, s) \mapsto \overline{u}_s$ to $\mathsf{L}$. Finally, the simulator $\mathcal{S}_t$ outputs the ciphertext $\overline{\mathsf{ct}}_t = (\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n)$ and the updated state $\mathsf{st}_{\mathcal{S}} = \mathsf{L}$. This completes the description of the simulator $\mathcal{S}$.

**Correctness of the simulation.** We show that the simulator $\mathcal{S} = (\mathcal{S}_0, \ldots, \mathcal{S}_q)$ perfectly simulates the distribution in hybrid $\mathsf{H}_1$. Let $(\mathsf{ct}_1, \ldots, \mathsf{ct}_q)$ be the joint distribution of the ciphertexts output in hybrid $\mathsf{H}_1$, and let $(\overline{\mathsf{ct}}_1, \ldots, \overline{\mathsf{ct}}_q)$ be the joint distribution of the ciphertexts output by the simulator. We proceed inductively in the number of queries $q$. The base case ($q = 0$) follows trivially.

Suppose now that $(\mathsf{ct}_1, \ldots, \mathsf{ct}_{t-1}) \equiv (\overline{\mathsf{ct}}_1, \ldots, \overline{\mathsf{ct}}_{t-1})$ for some $t \in [q]$. We show that the statement holds for $t + 1$. Consider the distributions of $\mathsf{ct}_t$ and $\overline{\mathsf{ct}}_t$. First, for any $j \in [t]$, write ciphertext $\mathsf{ct}_j$ as $(u_{j,1}, u_{j,2}, \ldots, u_{j,n})$ and $\overline{\mathsf{ct}}_j$ as $(\overline{u}_{j,1}, \overline{u}_{j,2}, \ldots, \overline{u}_{j,n})$. In addition, for $j \in [t]$, we write $b_{j,s}$ to denote the $s^{\text{th}}$ bit of $m_j$. For each $s \in [n]$, we consider three cases:

- **Case 1:** There exists a $j \in [t-1]$ such that $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t) > s$. If there are multiple $j$ for which $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t) > s$, let $j$ be the smallest one. This means that $m_j$ and $m_t$ share a prefix of length at least $s$. Let $p \in \{0, 1\}^{s-1}$ be the first $s - 1$ bits of this common prefix. Then, in hybrid $\mathsf{H}_1$, we have
$$u_{t,s} = f(s, p\|0^{n-s}) + b_{t,s} = u_{j,s}.$$
In the simulation, $\overline{u}_{t,s} = \mathsf{L}(j, s) = \overline{u}_{j,s}$. Since $j < t$, we conclude from the induction hypothesis that $u_{t,s}$ and $\overline{u}_{t,s}$ are identically distributed.

- **Case 2:** For each $\ell \in [t-1]$, $\mathsf{ind}_{\mathsf{diff}}(m_\ell, m_t) \leq s$, and there exists a $j \in [t-1]$ such that $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t) = s$. If there are multiple $j$ for which $\mathsf{ind}_{\mathsf{diff}}(m_j, m_t) = s$, let $j$ be the smallest one. This means that $m_j$ and $m_t$ share a prefix $p \in \{0, 1\}^{s-1}$ of length $s - 1$. Then, in hybrid $\mathsf{H}_1$, we have
$$u_{t,s} = f(s, p\|0^{n-s}) + b_{t,s} \pmod{M}.$$
In the simulation,
$$\overline{u}_{t,s} = \mathsf{L}(j, s) - (1 - 2 \cdot \mathbf{1}(m_j < m_t)) = \overline{u}_{j,s} - (1 - 2 \cdot \mathbf{1}(m_j < m_t)) \pmod{M}.$$
In hybrid $\mathsf{H}_1$, $u_{j,s} = f(s, p\|0^{n-s}) + b_{j,s}$. By assumption, $b_{j,s} \neq b_{t,s}$, so we can write $b_{t,s} = b_{j,s} - (1 - 2 \cdot \mathbf{1}(m_j < m_t))$. Thus, in hybrid $\mathsf{H}_1$, we have
$$u_{t,s} = f(s, p\|0^{n-s+1}) + b_{t,s} = u_{j,s} - (1 - 2 \cdot \mathbf{1}(m_j < m_t)) \pmod{M}.$$
By the inductive hypothesis, $u_{j,s}$ and $\overline{u}_{j,s}$ are identically distributed, so we conclude that $u_{t,s}$ and $\overline{u}_{t,s}$ are identically distributed.

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- **Case 3:** For each $\ell \in [t-1]$, $\mathsf{ind}_{\mathsf{diff}}(m_\ell, m_t) < s$. Let $p \in \{0,1\}^{s-1}$ be the first $s - 1$ bits of $m_t$. In hybrid $\mathsf{H}_1$, we have

$$u_{t,s} = f(s, p\|0^{n-s}) + b_{t,s} \pmod{M},$$

  while in the simulation $\bar{u}_{t,s}$ is a uniformly random string. By assumption, none of the messages $m_1, \dots, m_{t-1}$ begin with the prefix $p$. Since $f$ is a truly random function, the value of $f(s, p\|0^{n-s})$ is uniform in $\mathbb{Z}_M$ and independent of all other ciphertexts. Thus, $u_{t,s}$ and $\bar{u}_{t,s}$ are identically distributed.

We conclude that for all $s \in [n]$, $u_{t,s} \equiv \bar{u}_{t,s}$. Since the components of each ciphertext are constructed independently in both hybrid $\mathsf{H}_1$ and in the simulation, this suffices to show that $\mathsf{ct}_t$ and $\overline{\mathsf{ct}}_t$ are identically distributed. The claim then follows by induction on $t$. $\qquad\square$

**Space usage.** The order-revealing encryption scheme $\Pi_{\mathsf{ore}}$ on $n$-bit inputs produces encryptions of size $\lceil n \cdot \log_2 M \rceil$. By setting $M = 3$, an encryption of an $n$-bit message under $\Pi_{\mathsf{ore}}$ consists of only $\lceil n \cdot \log_2 3 \rceil \approx 1.59\, n$ bits. In Appendix B, we describe a "$d$-ary" generalization of $\Pi_{\mathsf{ore}}$ that further reduces the size of the ciphertexts in the ORE scheme, but with a slight loss in security. Specifically, we construct an ORE scheme where an encryption of an $n$-bit message has length approximately $n \cdot \log_d(2d - 1)$ for any integer $d \geq 2$. Since $\log_d(2d - 1)$ is a monotonically decreasing function in $d$, larger values of $d$ yield shorter ciphertexts, but increased leakage.

## 3.2  Conversion to OPE

In this section, we explain how to convert $\Pi_{\mathsf{ore}}$, an ORE scheme, into an OPE scheme. This means that ciphertexts of the resulting OPE scheme can be compared using the normal comparison function on numbers. To do this, we apply a simple transformation of any ciphertext $\mathsf{ct}$ of $\Pi_{\mathsf{ore}}$ into a number $c$ that lies in the range $[0, M^n - 1]$ for which direct numeric comparisons of two numbers $c_1$ and $c_2$ reveal the order relation of the underlying plaintexts.

Recall that in $\Pi_{\mathsf{ore}}$, ciphertexts are of the form $\mathsf{ct} = (u_1, u_2 \dots, u_n)$, where for each $i \in [n]$, $u_i$ lies in the range $\mathbb{Z}_M$. The ciphertext in the resulting OPE scheme is taken to be the $\lceil n \cdot \log_2 M \rceil$-bit number

$$c = \sum_{i=1}^{n} u_i \cdot M^{n-i} \qquad \in [0, M^n - 1]. \tag{3.2}$$

Intuitively, we view $u_1 u_2 \cdots u_n$ as a base-$M$ representation of the OPE ciphertext. Correctness follows similarly to $\Pi_{\mathsf{ore}}$, except here, there is a non-zero probability of error (as opposed to $\Pi_{\mathsf{ore}}$ where correctness held with probability 1). We claim that for any two messages $m_1, m_2 \in [0, 2^n - 1]$,

$$m_1 < m_2 \qquad \Longleftrightarrow \qquad c_1 < c_2,$$

with probability $1 - 1/M$, where $c_1, c_2 \in [0, M^n - 1]$ are the ciphertexts obtained by first invoking ORE.Encrypt on $m_1, m_2$, respectively, and then applying the transformation in Equation (3.2). To see this, let $i \in [n]$ be the first bit position on which $m_1$ and $m_2$ differ. Observe that the numeric comparison of the OPE ciphertexts behaves identically as the ORE comparison procedure, except when the output of the PRF on the first $i - 1$ bits of the messages is the value $M - 1$.[7] However, by

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[7] If no reduction modulo $M$ occurs in the ORE.Encrypt encryption, then numerically comparing the transformed ciphertexts is identical to evaluating the ORE.Compare procedure (since all relations hold over the integers).

PRF security, this event happens with probability $1/M$, and thus, correctness holds with probability $1 - 1/M$. For instance, if $M = 2^\lambda$ (that is, $\lambda$ bits), correctness holds with overwhelming probability. For practical scenarios, it may be suitable to only take $M \approx 2^{40}$ (the failure probability in this case is $2^{-40}$).

Security of the resulting OPE scheme follows identically from security of $\Pi_{\mathsf{ore}}$, as the transformation from ciphertexts $\mathsf{ct}$ to numbers $c$ is bijective. We note that while this scheme is order-preserving, it does not behave like a random order-preserving function, and thus, does not inherit the security limitations associated with such OPE schemes [BCO11]. In fact, our simulation-based security model and associated security theorem (Theorem 3.1) enables us to precisely specify the information leakage in this order-preserving encryption scheme.

In Appendix B, we describe a "$d$-ary" generalization of $\Pi_{\mathsf{ore}}$. While this generalization does not reduce the size of the resulting ciphertexts in the ORE scheme, it does yield shorter ciphertexts in the OPE instantiation (by approximately a $\log_2 d$ multiplicative factor), with a slight loss in security. Correctness in this generalized scheme holds with probability $1 - d/M$.

# 4 Comparison to Existing OPE Schemes

We now compare the leakage of our order-revealing encryption scheme to that of existing order-preserving encryption schemes by Boldyreva et al. [BCLO09, BCO11]. As explained in Section 2.3, composing any existing OPE scheme with an ORE scheme results in a new ORE scheme which is at least as secure as the underlying OPE scheme.[8] In this section, we show that even *without* the composition, our construction still achieves stronger security according to the metrics proposed by Boldyreva et al.

The security definition achieved by an order-preserving encryption scheme is that the encryption function behaves like a random order-preserving function (ROPF) from the plaintext space to the ciphertext space. While this definition has the same flavor as that for PRFs, the behavior of a truly random function is very different from that of a random order-preserving function. In particular, the output of an order-preserving function is not independent of its input, and thus, reveals some information about the input. It turns out that quantifying the exact information leakage is a non-trivial task in general. However, under certain assumptions (for example, if the messages are drawn from a uniform distribution), it is possible to obtain concrete upper bounds on the information leakage [BCO11]. In particular, Boldyreva et al. propose two security notions, window one-wayness and window distance one-wayness, to analyze the security of an OPE scheme. In our setting, the nature of our security definition allows us to analyze the construction under a more generalized set of definitions compared to [BCO11]. We present our analysis for window one-wayness here, and defer the analysis of window distance one-wayness to Appendix D.

## 4.1 One-Wayness

One of the most basic requirements of an encryption scheme is that it is one-way. Given a ciphertext, an adversary that does not have the secret key should not be able to recover the underlying message. In the standard definition of one-wayness [Gol01], the adversary is given the encryption of a random message, and its goal is to guess the message. This is a very weak notion of security, and even if

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[8]In most cases, the security of the composed scheme is strictly greater than that of the base OPE scheme since our ORE construction provides semantic security for a single ciphertext, whereas existing OPE schemes generally do not.

an encryption is one-way, the adversary might still be able to deduce nontrivial information about the message given only the ciphertext. To address this, Boldyreva et al. [BCLO09] introduce a more general notion of one-wayness where the adversary is allowed to guess a contiguous interval (a window) in the one-wayness challenge. The adversary succeeds if the message is contained within the interval. Moreover, the adversary is given multiple encryptions (of random messages) and succeeds if it outputs an interval that contains at least one of the messages.

The notion of window one-wayness is useful for arguing that an adversary does not learn many of the *most significant* bits of the message, but if all bits of the message are equally sensitive, then this definition is less useful. In our work, we present a more general definition of one-wayness, where instead of outputting an interval, the adversary is allowed to specify a set of guesses. To allow the adversary to specify a super-polynomially-sized set of guesses, we instead require the adversary to submit a circuit $C$ that encodes its set ($C(x) = 1$ if and only if $x$ is in the set). By requiring that the circuit encodes a contiguous interval, we recover the window one-wayness definition by Boldyreva et al. [BCO11]. We now give our generalized definition.

**Definition 4.1** (Generalized One-Wayness)**.** Fix a plaintext space $\mathcal{D}$ and let $\Pi = (\mathsf{ORE.Setup},$ $\mathsf{ORE.Encrypt}, \mathsf{ORE.Compare})$ be an ORE over $\mathcal{D}$. The $(r, z)$-*generalized one-wayness advantage* of an adversary $\mathcal{A}$ against $\Pi$ is given by

$$\mathsf{Adv}^{\mathsf{gow}}_{r,z,\Pi}(\mathcal{A}) \stackrel{\mathsf{def}}{=} \Pr[\mathsf{Expt}^{\mathsf{gow}}_{r,z,\Pi,\mathcal{A}}(1^\lambda) = 1],$$

where the $(r, z)$-generalized one-wayness experiment $\mathsf{Expt}^{\mathsf{gow}}_{r,z,\Pi,\mathcal{A}}(1^\lambda)$ is defined as follows:

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**Experiment** $\mathsf{Expt}^{\mathsf{gow}}_{r,z,\Pi,\mathcal{A}}(1^\lambda)$**:**

1. $\mathsf{sk} \leftarrow \mathsf{ORE.Setup}(1^\lambda)$
2. sample $m_1, \ldots, m_z$ uniformly from $\mathcal{D}$ without replacement
3. for $i \in [z]$, $\mathsf{ct}_i \leftarrow \mathsf{ORE.Encrypt}(\mathsf{sk}, m_i)$
4. $C \leftarrow \mathcal{A}(\mathsf{ct}_1, \ldots, \mathsf{ct}_z)$, where $C : \mathcal{D} \rightarrow \{0, 1\}$ is a circuit of size $\mathrm{poly}(\lambda)$
5. output 1 if $C(m_i) = 1$ for some $i \in [z]$ and $|\{x \in \mathcal{D} : C(x) = 1\}| \leq r$; otherwise, output 0

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**Remark 4.2** (Comparison with Existing One-Wayness Notions)**.** By restricting the parameters $(r, z)$ and the classes of circuits the adversary is allowed to output, Definition 4.1 captures many existing notions of one-wayness. For example, when $r = z = 1$, we recover the usual notion of one-wayness [Gol01]. When the underlying plaintext space is the ring $\mathbb{Z}_M$ for some integer $M$ and we require that the circuit output by the adversary encodes a contiguous interval of length at most $r$ in $\mathbb{Z}_M$, our definition corresponds to the notion of window one-wayness introduced by Boldyreva et al. [BCO11].

We now state our security theorem, but defer the proof to Appendix C.

**Theorem 4.3.** *Fix a security parameter $\lambda$ and a plaintext space $\{0, 1\}^n$ where $n = \omega(\log \lambda)$. Let $\Pi_{\mathsf{ore}}$ be the ORE scheme given at the beginning of Section 3. Then, for any constant $\varepsilon \in (0, 1]$, any $z = \mathrm{poly}(\lambda)$, and all efficient adversaries $\mathcal{A}$,*

$$\mathsf{Adv}^{\mathsf{gow}}_{r,z,\Pi_{\mathsf{ore}},\mathcal{A}}(1^\lambda) = \mathrm{negl}(\lambda),$$

*where $r = 2^{n(1-\varepsilon)}$.*

**Comparison to existing schemes.** When discussing the notion of one-wayness, we will always assume that the message-space is super-polynomial in the security parameter. Otherwise, the trivial adversary that just guesses a random point in the message space will succeed with non-negligible probability.

In [BCO11], Boldyreva et al. give an upper bound on the one-wayness advantage of any (possibly computationally unbounded) adversary $\mathcal{A}$ against a random order-preserving function ROPF. This corresponds to setting $r = 1$ in our definition. They show [BCO11, Theorem 4.1] that for $z = \text{poly}(\lambda)$, $\mathsf{Adv}^{\mathsf{gow}}_{1,z,\mathsf{ROPF},\mathcal{A}} = \text{negl}(\lambda)$. The same statement holds for our ORE construction assuming a computationally bounded adversary: simply instantiate Theorem 4.3 with $\varepsilon = 1$.

In addition to giving an upper bound on an adversary's ability to guess the plaintext from the ciphertext, Boldyreva et al. also give a lower bound on the advantage for the case when $r$ is large. In particular, they exhibit an efficient adversary $\mathcal{A}$ against an ROPF such that $\mathsf{Adv}^{\mathsf{gow}}_{r,z,\mathsf{ROPF},\mathcal{A}}(1^\lambda) = 1 - 2e^{-b^2/2}$ for a constant $b$ when $r = O(\sqrt{2^n})$ and for any $z$ [BCO11, Theorem 4.2].[9] In other words, the authors describe a concrete adversary that is able to break the generalized one-wayness of any POPF-CCA-secure scheme (with probability close to 1) if the adversary is allowed to specify a set with $r = O(\sqrt{2^n})$ elements, even when $z = 1$. An intuitive way to understand this result is that given the output of an ROPF, an adversary can deduce roughly half of the bits of the associated input. In contrast, in our ORE scheme, if the adversary only sees a polynomial number of ciphertexts ($z = \text{poly}(\lambda)$), then invoking Theorem 4.3 with $\varepsilon = 1/2$, we have that for all efficient adversaries $\mathcal{A}$, $\mathsf{Adv}^{\mathsf{gow}}_{r,z,\Pi_{\mathsf{ore}},\mathcal{A}}(1^\lambda) = \text{negl}(\lambda)$ where $r = \sqrt{2^n}$. In fact, as Theorem 4.3 demonstrates, the adversary's advantage remains negligible even if we further increase the size of the sets the adversary is allowed to submit.

Intuitively, our results show that if the adversary only sees a polynomial number of ciphertexts, then it does not learn any constant fraction $\varepsilon$ of the bits in the underlying plaintext from each ciphertext. In contrast, with an ROPF, and correspondingly, any OPE scheme that realizes a ROPF, each ciphertext alone leaks *half* of the most-significant bits of the underlying plaintext.

Similarly, while the OPE scheme by Teranishi et al. [TYM14] can be shown to hide any constant fraction of the least significant bits of the plaintext, no such guarantee exists for the other bits of the plaintext. Note though that the security notion proposed in [TYM14] is indistinguishability-based and hence, stronger than the one-wayness security notions. In fact, our basic ORE construction (by itself) does not achieve their indistinguishability-based definition. However, by composing our ORE construction with their OPE construction, we obtain a resulting ORE scheme which is strictly more secure, since it inherits the security properties of the underlying OPE scheme as well as semantic security for a single ciphertext (Section 2.3, Remark 2.6).

# 5    Conclusions

In this work, we introduced a new notion of security for order-preserving, and more generally, order-revealing encryption. Our simulation-based security notion is defined with respect to a leakage function which precisely characterizes what the ciphertexts in the scheme leak about the underlying messages. We then give a practical order-revealing encryption scheme which achieves this security notion for a specific leakage function. By composing our ORE construction with existing OPE

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[9]Strictly speaking, the adversary they describe is for the window one-wayness experiment, but any adversary that succeeds in the window one-wayness experiment also succeeds in the generalized one-wayness experiment (Definition 4.1).

schemes, we obtain an ORE scheme with increased security. It is our hope that having a concrete leakage model will enable practitioners to make better-informed decisions on whether an ORE scheme is appropriate for their particular application. We conclude with several open problems:

1. Can we construct a practical ORE scheme with stronger security guarantees?

2. Can we reduce the ciphertext length of our ORE scheme while still maintaining a similar level of security?

3. Is it possible to build a practical ORE scheme with best-possible security from standard assumptions?

# Acknowledgments

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# A    Proof of Theorem 3.1

Fix any two messages $m_1, m_2 \in \{0,1\}^n$. To show correctness, we show that for $\mathsf{sk} \leftarrow \mathsf{ORE.Setup}(1^\lambda)$, and $\mathsf{ct}_i \leftarrow \mathsf{ORE.Encrypt}(\mathsf{sk}, m_i)$ for $i \in \{1, 2\}$, then

$$\Pr[\mathsf{ORE.Compare}(\mathsf{ct}_1, \mathsf{ct}_2) = \mathbf{1}(m_1 < m_2)] = 1. \tag{A.1}$$

Let $k = \mathsf{sk}$ be the PRF key output by the setup algorithm. Let $b_1 \cdots b_n$ be the binary representation of $m_1$ and let $b'_1 \cdots b'_n$ be the binary representation of $m_2$. Next, write $\mathsf{ct}_1 = (u_1, u_2, \ldots u_n)$ and $\mathsf{ct}_2 = (u'_1, u'_2, \ldots, u'_n)$ We now show that $m_1 < m_2$ if and only if $\mathsf{ORE.Compare}(\mathsf{ct}_1, \mathsf{ct}_2) = 1$.

- Suppose $m_1 < m_2$. Then, there exists some index $i \in [n]$ such that $b_i = 0$ and $b'_i = 1$ and for all $j < i$, $b_j = b'_j$. By construction of $\mathsf{ct}_1$ and $\mathsf{ct}_2$, for all $j < i$,

$$u_j = F(k, (j, b_1 b_2 \cdots b_{j-1} \| 0^{n-j})) + b_j$$
$$= F(k, (j, b'_1 b'_2 \cdots b'_{j-1} \| 0^{n-j})) + b'_j = u'_j \pmod M.$$

  Since $b_i = 0$ and $b'_i = 1$, we have that

$$u'_i = F(k, (i, b'_1 b'_2 \cdots b'_{i-1} \| 0^{n-i})) + b'_i = u_i + 1 \pmod M,$$

  in which case, $\mathsf{ORE.Compare}(\mathsf{ct}_1, \mathsf{ct}_2)$ outputs 1.

- Suppose that $\mathsf{ORE.Compare}(\mathsf{ct}_1, \mathsf{ct}_2) = 1$. This means that there exists an index $i \in [n]$ such that $u'_i = u_i + 1 \pmod{M}$ and for all $j < i$, $u_j = u'_j$. We first argue that for all $1 \le j < i$, $b_j = b'_j$. We proceed inductively on $j$. For the base case $j = 1$, we have that

$$F(k, (1, 0^{n-1})) + b_1 = u_1 = u'_1 = F(k, (1, 0^{n-1})) + b'_1 \pmod{M},$$

from which we conclude that $b_1 = b'_1$. Next, suppose that $b_1 b_2 \cdots b_j = b'_1 b'_2 \cdots b'_j$. Then,

$$u_{j+1} = F(k, (j+1, b_1 b_2 \cdots b_j \| 0^{n-j-1})) + b_{j+1} \pmod{M}$$
$$u'_{j+1} = F(k, (j+1, b'_1 b'_2 \cdots b'_j \| 0^{n-j-1})) + b'_{j+1} \pmod{M}.$$

By the inductive hypothesis, $b_1 b_2 \cdots b_j = b'_1 b'_2 \cdots b'_j$, and by assumption $u_{j+1} = u'_{j+1}$, so we conclude that $b_{j+1} = b'_{j+1}$. By induction on $j$, we have that for all $j < i$, $b_j = b'_j$. To complete the proof, we argue that $b_i = 0$ and $b'_i = 1$. By assumption, $u'_i = u_i + 1 \pmod{M}$. Since $b_1 b_2 \cdots b_{i-1} = b'_1 b'_2 \cdots b'_{i-1}$, it follows that $b'_i = b_i + 1 \pmod{M}$. Because $M \ge 3$ and $b_i, b'_i \in \{0, 1\}$ the only solution is for $b_i = 0$ and $b'_i = 1$. The claim follows. $\qquad\square$

# B  Generalizing the Conversion to OPE

In this section, we consider the "$d$-ary" generalization of $\Pi_{\mathsf{ore}}$, which on $\ell$-bit inputs produces ciphertexts of size approximately $\ell \cdot \log_d M$, for any $M \ge 2d - 1$. For ORE, this generalization yields only a modest decrease in the size of the ciphertexts, but if we apply the conversion to OPE from Section 3.2 to the generalized scheme, we obtain a $\log_2 d$ multiplicative factor reduction in ciphertext size, but with a slight loss in security. In particular, this $d$-ary generalization of $\Pi_{\mathsf{ore}}$ is secure with respect to the following leakage function:

$$\mathcal{L}_{\mathsf{f}}^d(m_1, \ldots, m_t) := \left\{ \left( \mathsf{ind}_{\mathsf{diff}}^{(d)}(m_i, m_j), \mathbf{1}(m_i < m_j) \right) : 1 \le i < j \le t \right\}, \tag{B.1}$$

where $\mathsf{ind}_{\mathsf{diff}}^{(d)}(x, y)$ is defined as follows. For two $d$-ary inputs $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$ where $x \ne y$, we define

$$\mathsf{ind}_{\mathsf{diff}}^{(d)}(x, y) = (k, x_k - y_k),$$

where $k$ is the smallest index where $x_k \ne y_k$. If $x = y$, then $\mathsf{ind}_{\mathsf{diff}}^{(d)}(x, y) = (n+1, 0)$.

**Generalized construction.**  Fix a security parameter $\lambda \in \mathbb{N}$, and take an integer $M \ge 2d-1$. Let $F : \mathcal{K} \times ([n] \times [0, d-1]^{n-1}) \to \mathbb{Z}_M$ be a secure PRF. We define our ORE scheme $\Pi_{\mathsf{gen\text{-}ore}} = (\mathsf{ORE.Setup}, \mathsf{ORE.Encrypt}, \mathsf{ORE.Compare})$ as follows:

- $\mathsf{ORE.Setup}(1^\lambda)$. The setup algorithm chooses a uniformly random PRF key $k$ for $F$. The secret key is $\mathsf{sk} = k$.

- $\mathsf{ORE.Encrypt}(\mathsf{sk}, m)$. Let $b_1 \cdots b_n$ be the $d$-ary representation of $m$ and let $\mathsf{sk} = k$. For each $i \in [n]$, the encryption algorithm computes

$$u_i = F(k, (i, b_1 b_2 \cdots b_{i-1} \| 0^{n-i})) + b_i \pmod{M},$$

and outputs $(u_1, u_2 \ldots, u_n)$.

- ORE.Compare($\mathsf{ct}_1, \mathsf{ct}_2$). The compare algorithm first parses

$$\mathsf{ct}_1 = (u_1, u_2, \ldots, u_n)$$
$$\mathsf{ct}_2 = (u'_1, u'_2, \ldots, u'_n),$$

where $u_1, \ldots, u_n, u'_1, \ldots, u'_n \in \mathbb{Z}_M$. Let $i$ be the smallest index where $u_i \neq u'_i$. If no such index exists, output 0. If such an index exists, output 1 if there exists $k \in [d-1]$ such that $u'_i = u_i + k \pmod{M}$, and 0 otherwise.

Correctness of the ORE scheme $\Pi_{\mathsf{gen\text{-}ore}}$ follows almost identically as the correctness of $\Pi_{\mathsf{ore}}$ by adapting the correctness proof to deal with the $d$-ary representation of the message as opposed to the binary representation. The proof of security for $\Pi_{\mathsf{gen\text{-}ore}}$ is also similar to that for Theorem 3.2.

**Theorem B.1.** *The order-revealing encryption scheme $\Pi_{\mathsf{gen\text{-}ore}}$ is secure with respect to leakage function $\mathcal{L}_{\mathsf{f}}^d$ under the PRF security of $F$.*

*Proof.* Fix a security parameter $\lambda$ and let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_q)$ where $q = \mathrm{poly}(\lambda)$ be an efficient adversary for the ORE security game (Definition 2.4). To prove security, we give an efficient simulator $\mathcal{S} = (\mathcal{S}_0, \ldots, \mathcal{S}_q)$ for which the outputs of the distributions $\mathsf{REAL}_{\mathcal{A}}^{\mathrm{ORE}}(\lambda)$ and $\mathsf{SIM}_{\mathcal{A}, \mathcal{S}, \mathcal{L}_{\mathsf{f}}^d}^{\mathrm{ORE}}(\lambda)$ are computationally indistinguishable.

We use a hybrid argument similar to the argument used in the proof of Theorem 3.2. We begin by defining the hybrid experiments:

- **Hybrid $\mathsf{H}_0$:** This is the real experiment $\mathsf{REAL}_{\mathcal{A}}^{\mathrm{ORE}}(\lambda)$.

- **Hybrid $\mathsf{H}_1$:** Same as $\mathsf{H}_1$, except during ORE.Setup, a random function $f \xleftarrow{\mathrm{R}} \mathsf{Funs}[([n] \times [0, d-1]^{n-1}), \mathbb{Z}_M]$ is chosen. In all invocations of ORE.Encrypt, the function $F(k, \cdot)$ is replaced by $f(\cdot)$.

By the same argument as in the proof of Theorem 3.2, hybrids $\mathsf{H}_0, \mathsf{H}_1$ are computationally indistinguishable under the PRF security of $F$. It suffices to show that there exists a simulator $\mathcal{S}$ such that the distribution of outputs in $\mathsf{H}_1$ is computationally indistinguishable from $\mathsf{SIM}_{\mathcal{A}, \mathcal{S}, \mathcal{L}_{\mathsf{f}}^d}^{\mathrm{ORE}}(\lambda)$.

**Description of the simulator.** We now describe the simulator $\mathcal{S} = (\mathcal{S}_0, \ldots, \mathcal{S}_q)$. First, $\mathcal{S}_0$ initializes an empty lookup tables $\mathsf{L} : [q] \times [n] \to \mathbb{Z}_M$. It then outputs $\mathsf{st}_{\mathcal{S}} = \mathsf{L}$. Then, for each $t \in [q]$, after the adversary outputs a query $m_t$, the simulation algorithm $\mathcal{S}_t$ is invoked on input $\mathsf{st}_{\mathcal{S}} = \mathsf{L}$ and $\mathcal{L}_{\mathsf{f}}^d(m_1, \ldots, m_t)$. In particular, $\mathcal{L}_{\mathsf{f}}^d(m_1, \ldots, m_t)$ contains the values $\mathbf{1}(m_j < m_t)$ and $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_j, m_t)$ for all $j \in [t-1]$, where $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_j, m_t)$ consists of a pair $(k, w)$ where $k \in [n]$ is the index of the first $d$-ary chunk in $m_j$ and $m_t$ that differ, $w = m_{j,k} - m_{t,k}$, and $m_{j,k}$ and $m_{t,k}$ are the $k^{\text{th}}$ $d$-ary chunks of $m_j$ and $m_t$, respectively. For each $s \in [n]$, there are three cases to consider:

- **Case 1:** There exists a $j \in [t-1]$ and $(k, w) \in [n] \times [-d, d]$ such that $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_j, m_t) = (k, w)$ and $k > s$. If there are multiple $j$ for which the property holds, let $j$ be the smallest one. Then, the simulator sets $\overline{u}_s = \mathsf{L}(j, s)$.

- **Case 2:** For each $\ell \in [t-1]$, there exists a $(k, z) \in [n] \times [-d, d]$ for which $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_\ell, m_t) = (k, z)$ and $k \leq s$, and there exists a $j \in [t-1]$ and $w \in [-d, d]$ for which $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_j, m_t) = (s, w)$. If there are multiple $j$ for which $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_j, m_t) = (s, w)$ for some $w$, let $j$ be the smallest one. Then, the simulator sets $\overline{u}_s = \mathsf{L}(j, s) - w \pmod{M}$.

- **Case 3:** For each $\ell \in [t-1]$, there exist a $(k, z) \in [n] \times [-d, d]$ for which $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_\ell, m_t) = (k, z)$ and $k < s$. In this case, the simulator samples $y \xleftarrow{\mathrm{R}} \mathbb{Z}_M$ and sets $\overline{u}_s = y$.

For each $s \in [n]$, the simulator adds the mapping $(t, s) \mapsto \overline{u}_s$ to $\mathsf{L}$. Finally, the simulator $\mathcal{S}_t$ outputs the ciphertext $\overline{\mathsf{ct}}_t = (\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n)$ and the updated state $\mathsf{st}_\mathcal{S} = \mathsf{L}$. This completes the description of the simulator $\mathcal{S}$.

**Correctness of the simulation.** We show that the simulator $\mathcal{S} = (\mathcal{S}_0, \ldots, \mathcal{S}_q)$ perfectly simulates the distribution in hybrid $\mathsf{H}_1$. Let $(\mathsf{ct}_1, \ldots, \mathsf{ct}_q)$ be the joint distribution of the ciphertexts output in hybrid $\mathsf{H}_1$, and let $(\overline{\mathsf{ct}}_1, \ldots, \overline{\mathsf{ct}}_q)$ be the joint distribution of the ciphertexts output by the simulator. We proceed inductively in the number of queries $q$. The base case $(q = 0)$ follows trivially.

Suppose now that $(\mathsf{ct}_1, \ldots, \mathsf{ct}_{t-1}) \equiv (\overline{\mathsf{ct}}_1, \ldots, \overline{\mathsf{ct}}_{t-1})$ for some $t \in [q]$. We show that the statement holds for $t + 1$. Consider the distributions of $\mathsf{ct}_t$ and $\overline{\mathsf{ct}}_t$. First, for any $j \in [t]$, write ciphertext $\mathsf{ct}_j$ as $(u_{j,1}, u_{j,2}, \ldots, u_{j,n})$ and $\overline{\mathsf{ct}}_j$ as $(\overline{u}_{j,1}, \overline{u}_{j,2}, \ldots, \overline{u}_{j,n})$. In addition, we write $b_{j,s}$ to denote the $s^{\text{th}}$ $d$-ary chunk of message $m_j$. For each $s \in [n]$, we again consider three cases:

- **Case 1:** There exists a $j \in [t-1]$ and $(k, w) \in [n] \times [-d, d]$ such that $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_j, m_t) = (k, w)$ and $k > s$. If there are multiple $j$ for which this property holds, let $j$ be the smallest one. This means that $m_j$ and $m_t$ share a prefix of length at least $s$. Let $p \in [0, d-1]^{s-1}$ be the first $s - 1$ $d$-ary chunks of this common prefix. Then, in hybrid $\mathsf{H}_1$, we have

$$u_{t,s} = f(s, p \| 0^{n-s}) + b_{t,s} = u_{j,s}.$$

  In the simulation, $\overline{u}_{t,s} = \mathsf{L}(j, s) = \overline{u}_{j,s}$. Since $j < t$, we conclude from the induction hypothesis that $u_{t,s}$ and $\overline{u}_{t,s}$ are identically distributed.

- **Case 2:** For each $\ell \in [t-1]$, there exists a $(k, z) \in [n] \times [-d, d]$ for which $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_\ell, m_t) = (k, z)$ and $k \leq s$, and there exists a $j \in [t-1]$ and $w \in [-d, d]$ for which $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_j, m_t) = (s, w)$. If there are multiple $j$ for which $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_j, m_t) = (s, w)$ for some $w$, let $j$ be the smallest one. This means that $m_j$ and $m_t$ share a prefix $p \in \{0, 1\}^{s-1}$ of length $s - 1$. In hybrid $\mathsf{H}_1$, we have

$$u_{t,s} = f(s, p \| 0^{n-s}) + b_{t,s} \pmod{M}.$$

  In the simulation,
$$\overline{u}_{t,s} = \mathsf{L}(j, s) - w = \overline{u}_{j,s} - w \pmod{M}.$$

  In hybrid $\mathsf{H}_1$, $u_{j,s} = f(s, p \| 0^{n-s}) + b_{j,s}$. By definition of $\mathsf{ind}_{\mathsf{diff}}^{(d)}$, we have that $w = b_{j,s} - b_{t,s}$, so we can write $b_{t,s} = b_{j,s} - w$. Thus, in hybrid $\mathsf{H}_1$, we have

$$u_{t,s} = f(s, p \| 0^{n-s}) + b_{t,s} = u_{j,s} - w \pmod{M}.$$

  By the inductive hypothesis, $u_{j,s}$ and $\overline{u}_{j,s}$ are identically distributed, and the claim follows.

- **Case 3:** For each $\ell \in [t-1]$, there exist a $(k, w) \in [n] \times [-d, d]$ for which $\mathsf{ind}_{\mathsf{diff}}^{(d)}(m_\ell, m_t) = (k, w)$ and $k < s$. Let $p \in \{0, 1\}^{s-1}$ be the first $s - 1$ $d$-ary chunks of $m_t$. In hybrid $\mathsf{H}_1$, we have

$$u_{t,s} = f(s, p \| 0^{n-s}) + b_{t,s} \pmod{M},$$

while in the simulation $\bar{u}_{t,s}$ is a uniformly random string. By assumption, none of the messages $m_1, \ldots, m_{t-1}$ begin with the prefix $p$. Since $f$ is a truly random function, the value of $f(s, p \| 0^{n-s})$ is uniform in $\mathbb{Z}_M$ and independent of all other ciphertexts. Thus, $u_{t,s}$ and $\bar{u}_{t,s}$ are identically distributed.

We conclude that for all $s \in [n]$, $u_{t,s} \equiv \bar{u}_{t,s}$. Since the components of each ciphertext are constructed independently in both hybrid $\mathsf{H}_1$ and in the simulation, this suffices to show that $\mathsf{ct}_t$ and $\overline{\mathsf{ct}}_t$ are identically distributed. The claim then follows by induction on $t$.

**Space usage and conversion to OPE.** The $d$-ary generalized ORE scheme $\Pi_{\mathsf{gen\text{-}ore}}$ on $n$-bit inputs produces ciphertexts of length $\lceil \log_d 2^n \rceil \cdot \log_2 M \approx n \cdot \log_d M$, where the first term is the number of blocks in each ciphertext (equal to the number of base-$d$ digits needed to represent an $n$-bit value) and the second term is the number of bits needed to represent each block in the ciphertext. Since we can set $M = 2d - 1$, an encryption of an $n$-bit message under $\Pi_{\mathsf{gen\text{-}ore}}$ is approximately $n \cdot \log_d(2d - 1)$ bits. The ciphertext expansion factor $\log_d(2d - 1)$ is a monotonically decreasing function in $d$, so generalizing to higher arity allows for shorter ciphertexts. However, larger values of $d$ also translate to increased leakage.

The reduction in ciphertext size is much more significant if we apply our ORE-to-OPE conversion from Section 3.2 to the generalized scheme $\Pi_{\mathsf{gen\text{-}ore}}$. When we apply our OPE conversion to $\Pi_{\mathsf{gen\text{-}ore}}$, the resulting OPE scheme is correct with probability $1 - d/M$. Thus, for correctness to hold with overwhelming probability, we require $M = 2^{\omega(\log \lambda)}$. In these scenarios where $M$ is very large (say, $M = 2^{40}$ to obtain a failure probability of $2^{-40}$), applying the OPE conversion to the $d$-ary generalized ORE scheme $\Pi_{\mathsf{gen\text{-}ore}}$ provides a $(\log_2 d)$-factor reduction in the size of the ciphertexts.

## C  Proof of Theorem 4.3

We first define a sequence of hybrid experiments:

- Hybrid $\mathsf{H}_0$: This corresponds to experiment $\mathsf{Expt}^{\mathsf{gow}}_{r,z,\Pi_{\mathsf{ore}},\mathcal{A}}(1^\lambda)$.

- Hybrid $\mathsf{H}_1$: Same as $\mathsf{H}_0$, except $m_1, \ldots, m_z \xleftarrow{\mathrm{R}} \{0,1\}^n$ (rather than sampled without replacement).

- Hybrid $\mathsf{H}_2$: Same as $\mathsf{H}_1$, except the ciphertexts $\mathsf{ct}_1, \ldots, \mathsf{ct}_z$ are constructed by invoking the simulator (Theorem 3.2) on $\mathcal{L}_{\mathsf{f}}(m_1, \ldots, m_z)$ where $\mathcal{L}_{\mathsf{f}}$ is the leakage function from Equation (3.1).

- Hybrid $\mathsf{H}_3$: Fix a parameter $k = \omega(\log \lambda)$ where $k \leq n$. In hybrid $\mathsf{H}_3$, after sampling the messages $m_1, \ldots, m_z \xleftarrow{\mathrm{R}} \{0,1\}^n$, the experiment also samples messages $m'_i \xleftarrow{\mathrm{R}} \{0,1\}^n$ for all $i \in [z]$ subject to the restriction that the first $k$ bits of $m_i$ and $m'_i$ are equal. The ciphertexts are constructed by invoking the simulator on $\mathcal{L}_{\mathsf{f}}(m'_1, \ldots, m'_z)$, but the success criterion at the end of the experiment is still checked using $m_1, \ldots, m_z$.

**Lemma C.1.** *Hybrids $\mathsf{H}_0$ and $\mathsf{H}_1$ are statistically indistinguishable.*

*Proof.* In $\mathsf{H}_0$, the messages are sampled without replacement from $\{0,1\}^n$. In particular, this means that for $i \in [z]$, the message $m_i$ is sampled from $\mathsf{Unif}(\{0,1\}^n \setminus \{m_1, \ldots, m_{i-1}\})$. But since $z = \mathrm{poly}(\lambda)$ and $n = \omega(\log \lambda)$ (so the size of the message space $\{0,1\}^n$ is super-polynomial in $\lambda$),

the distribution $\mathsf{Unif}(\{0,1\}^n)$ is statistically close to $\mathsf{Unif}(\{0,1\}^n \setminus \{m_1, \ldots, m_{i-1}\})$ for all $i \in [z]$. Thus, $\mathsf{H}_0$ and $\mathsf{H}_1$ are statistically indistinguishable. $\square$

**Lemma C.2.** *Hybrids $\mathsf{H}_1$ and $\mathsf{H}_2$ are computationally indistinguishable.*

*Proof.* Follows directly by simulation security of $\Pi_{\mathsf{ore}}$ (Theorem 3.2). $\square$

**Lemma C.3.** *Hybrids $\mathsf{H}_2$ and $\mathsf{H}_3$ are statistically indistinguishable.*

*Proof.* It suffices to argue that $\mathcal{L}_{\mathsf{f}}(m_1, \ldots, m_z) = \mathcal{L}_{\mathsf{f}}(m'_1, \ldots, m'_z)$ with overwhelming probability. Take any distinct pair of indices $i, j$. Consider the probability $\Pr[\mathsf{ind}_{\mathsf{diff}}(m_i, m_j) = k]$. Since $\mathsf{ind}_{\mathsf{diff}}(m_i, m_j)$ gives the first bit on which messages $m_i$ and $m_j$ differ, $\mathsf{ind}_{\mathsf{diff}}(m_i, m_j) = k$ only if the first $k-1$ bits of $m_i$ and $m_j$ match and the $k^{\mathrm{th}}$ bit of $m_i$ and $m_j$ differ. Since $m_i$ and $m_j$ are both sampled uniformly and independently from $\{0,1\}^n$, we conclude that $\Pr[\mathsf{ind}_{\mathsf{diff}}(m_i, m_j) = k] = 2^{-k}$. Moreover, $\Pr[\mathsf{ind}_{\mathsf{diff}}(m_i, m_j) \geq k] = \sum_{s=k}^{n} 2^{-s} \leq 2^{-k+1}$. By a union bound, we have that for each $i \in [z]$,

$$\Pr[\forall j \neq i : \mathsf{ind}_{\mathsf{diff}}(m_i, m_j) < k] \geq 1 - \frac{z}{2^{k-1}}.$$

Since $z = \mathrm{poly}(\lambda)$ and $k = \omega(\log \lambda)$ we have that $\Pr[\forall j \neq i : \mathsf{ind}_{\mathsf{diff}}(m_i, m_j) < k] = 1 - \mathrm{negl}(\lambda)$. By another union bound, we conclude that $\mathsf{ind}_{\mathsf{diff}}(m_i, m_j) < k$ for all distinct $i, j$ with probability $1 - \mathrm{negl}(\lambda)$. This means that all pairs of messages $m_i, m_j$ differ on at least one of the first $k$ bits. In $\mathsf{H}_3$, the messages $m_i$ and $m'_i$ all agree on the first $k$ bits for all $i \in [q]$, and so the first bit on which each pair of messages $m'_i, m'_j$ differ is unaffected. We conclude that $\mathcal{L}_{\mathsf{f}}(m_1, \ldots, m_z) = \mathcal{L}_{\mathsf{f}}(m'_1, \ldots, m'_z)$, and the claim follows. $\square$

To complete the proof we bound the adversary's advantage in hybrid $\mathsf{H}_3$. Let $C$ be the circuit output by the adversary in $\mathsf{H}_3$, and let $S = \{x \in \{0,1\}^n : C(x) = 1\}$. Without loss of generality, assume that $|S| \leq r$ (otherwise, the adversary's advantage is 0). Take any message $m_i$ and consider the probability $\Pr[m_i \in S]$. In hybrid $\mathsf{H}_3$, the view of the adversary depends only on the first $k$ bits of each message $m_1, \ldots, m_z$, and in fact, it is equivalent to sample the remaining $n - k$ bits of the test messages $m_1, \ldots, m_z$ *after* the adversary has submitted its circuit.[10] If the latter $n - k$ bits of $m_i$ are chosen after the adversary has committed to the circuit $C$, then $\Pr[m_i \in S] \leq |S|/2^{n-k}$. By a union bound over each message, we have

$$\mathsf{Adv}_{r,z,\Pi_{\mathsf{ore}},\mathcal{A}}^{\mathsf{gow}}(1^\lambda) = \Pr[\exists i \in [z] : m_i \in S] \leq \frac{z\,|S|}{2^{n-k}} = \frac{z \cdot r}{2^{n-k}}.$$

We can set $k = \varepsilon n/2 = \omega(\log \lambda)$ since $n = \omega(\log \lambda)$. Finally, since $r = 2^{n(1-\varepsilon)}$, the advantage is bounded by

$$\mathsf{Adv}_{r,z,\Pi_{\mathsf{ore}},\mathcal{A}}^{\mathsf{gow}}(1^\lambda) \leq \frac{z \cdot 2^{n(1-\varepsilon)}}{2^{n(1-\varepsilon/2)}} = \frac{z}{2^{n\varepsilon/2}} = \mathrm{negl}(\lambda),$$

since $z = \mathrm{poly}(\lambda)$. $\square$

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[10] This is equivalent since $m \xleftarrow{\mathrm{R}} \{0,1\}^n$, so each bit of $m$ can be viewed as an independent and uniform value in $\{0,1\}$.

# D   Distance One-Wayness

Boldyreva et al. [BCO11] also introduce an additional metric they use to analyze the security of an OPE scheme called "window distance one-wayness." This metric helps quantify the extent to which an OPE scheme reveals information about the distance between the underlying plaintexts. In a semantically-secure ORE scheme, recall that no distance information between ciphertexts is revealed aside from information that can be directly inferred from the ordering of the ciphertexts. As was the case for one-wayness, we present here a more general notion of window distance one-wayness, and then show that our proposed ORE scheme is robust under this definition.

**Definition D.1** (Generalized Distance One-Wayness)**.** Fix a plaintext space $\mathbb{Z}_M$ and let $\Pi = (\mathsf{ORE.Setup}, \mathsf{ORE.Encrypt}, \mathsf{ORE.Compare})$ be an ORE over $\mathbb{Z}_M$. The $(r, z)$-*generalized distance one-wayness advantage* of an adversary $\mathcal{A}$ against $\Pi$ is given by

$$\mathsf{Adv}^{\mathsf{gdow}}_{r,z,\Pi}(\mathcal{A}) \stackrel{\mathsf{def}}{=} \Pr[\mathsf{Expt}^{\mathsf{gdow}}_{r,z,\Pi,\mathcal{A}}(1^\lambda) = 1],$$

where the $(r, z)$-generalized distance one-wayness experiment $\mathsf{Expt}^{\mathsf{gdow}}_{r,z,\Pi,\mathcal{A}}(1^\lambda)$ is defined as follows:

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**Experiment** $\mathsf{Expt}^{\mathsf{gdow}}_{r,z,\Pi,\mathcal{A}}(1^\lambda)$**:**

1. $\mathsf{sk} \leftarrow \mathsf{ORE.Setup}(1^\lambda)$
2. sample $m_1, \ldots, m_z$ uniformly from $\mathbb{Z}_M$ without replacement
3. for $i \in [z]$, $\mathsf{ct}_i \leftarrow \mathsf{ORE.Encrypt}(\mathsf{sk}, m_i)$
4. $C \leftarrow \mathcal{A}(\mathsf{ct}_1, \ldots, \mathsf{ct}_z)$, where $C : \mathbb{Z}_M \to \{0, 1\}$ is a circuit of size $\mathrm{poly}(\lambda)$
5. output 1 if there exists distinct $i, j \in [z]$ such that $C(d_{ij}) = 1$ where $d_{ij} = m_j - m_i$ $(\mathrm{mod}\ M)$ and $|\{x \in \mathbb{Z}_M : C(x) = 1\}| \leq r$; otherwise, output 0

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**Remark D.2** (Comparison with Window Distance One-Wayness [BCO11])**.** In Definition D.1, if we restrict the adversary to outputting circuits $C$ that represent a contiguous interval of length at most $r$ in $\mathbb{Z}_M$, then we recover the notion of window distance one-wayness from [BCO11, §3].

**Theorem D.3.** *Fix a security parameter $\lambda$ and a plaintext space $\{0, 1\}^n$ (or equivalently, the ring $\mathbb{Z}_{2^n}$) where $n = \omega(\log \lambda)$. Let $\Pi_{\mathsf{ore}}$ be the ORE scheme given at the beginning of Section 3. Then, for any constant $\varepsilon \in (0, 1]$, any $z = \mathrm{poly}(\lambda)$, and all efficient adversaries $\mathcal{A}$,*

$$\mathsf{Adv}^{\mathsf{gdow}}_{r,z,\Pi_{\mathsf{ore}},\mathcal{A}}(1^\lambda) = \mathrm{negl}(\lambda),$$

*where $r = 2^{n(1-\varepsilon)}$.*

*Proof.* The proof of this statement proceeds very similarly to that of Theorem 4.3. For completeness, we describe the set of hybrid arguments we use:

- Hybrid $\mathsf{H}_0$: This corresponds to experiment $\mathsf{Expt}^{\mathsf{gdow}}_{r,z,\Pi_{\mathsf{ore}},\mathcal{A}}(1^\lambda)$.

- Hybrid $\mathsf{H}_1$: Same as $\mathsf{H}_0$, except $m_1, \ldots, m_z \xleftarrow{\mathrm{R}} \{0, 1\}^n$.

- Hybrid $\mathsf{H}_2$: Same as $\mathsf{H}_1$, except the ciphertexts $\mathsf{ct}_1, \ldots, \mathsf{ct}_z$ are constructed by invoking the simulator (Theorem 3.2) on $\mathcal{L}_{\mathsf{f}}(m_1, \ldots, m_z)$.

- Hybrid $\mathsf{H}_3$: Fix a parameter $k = \omega(\log \lambda)$ where $k \leq n$. In hybrid $\mathsf{H}_3$, after sampling the messages $m_1, \ldots, m_z \xleftarrow{\text{R}} \{0,1\}^n$, the experiment also samples messages $m_i' \xleftarrow{\text{R}} \{0,1\}^n$ subject to the restriction that the first $k$ bits of $m_i$ and $m_i'$ are equal. The ciphertexts are constructed by invoking the simulator on $\mathcal{L}_{\mathsf{f}}(m_1', \ldots, m_z')$, but the success criterion at the end of the experiment is still checked using $m_1, \ldots, m_z$.

Invoking Lemmas C.1 through C.3 from the proof of Theorem 4.3, we have that hybrids $\mathsf{H}_0$ and $\mathsf{H}_3$ are computationally indistinguishable. Consider the probability that the experiment $\mathsf{H}_3$ outputs 1. Let $C$ be the circuit output by the adversary and let $S = \{x \in \{0,1\}^n : C(x) = 1\}$. It suffices to assume that $|S| \leq r$ since otherwise, the output of $\mathsf{H}_3$ is 0. Take any distinct pairs of indices $i, j$ and let $d_{ij} = m_j - m_i \pmod{2^n}$. Take any $z \in \{0,1\}^n$. Since the view of the adversary is *independent* of the last $n - k$ bits of $m_j$ and $m_j$ is sampled uniformly and independently from $\{0,1\}^n$, we conclude that

$$\Pr[d_{ij} = z] = \Pr[m_j = z + m_i \pmod{2^n}] \leq \frac{1}{2^{n-k}}.$$

By a union bound over the elements of $S$, we conclude that $\Pr[d_{ij} \in S] \leq r/2^{n-k}$. Applying another union bound to all pairs $i, j$, we conclude that in $\mathsf{H}_3$,

$$\Pr[\forall i \neq j : d_{ij} \notin S] \geq 1 - \frac{r \cdot z^2}{2^{n-k}}.$$

As in the proof of Theorem 4.3, the theorem now follows by setting $k = \varepsilon n/2 = \omega(\log \lambda)$. $\qquad\square$

**Comparison to existing schemes.**  As was done with one-wayness, Boldyreva et al. [BCO11] also give a lower and upper bound for the advantage of an adversary for a random order-preserving function in the distance one-wayness game. Specifically, they show [BCO11, Theorem 4.3] that when $r = 1$ and $z = \text{poly}(\lambda)$, for all (not necessarily efficient) adversaries $\mathcal{A}$, $\mathsf{Adv}^{\mathsf{gdow}}_{r,z,\mathsf{ROPF},\mathcal{A}} = \text{negl}(\lambda)$. A matching upper bound holds for the advantage of all computationally bounded adversaries for our ORE scheme (simply instantiate Theorem D.3 with $\varepsilon = 1$). Intuitively, this statement says that given encryptions of several uniformly sampled ciphertexts, the adversary cannot guess the *exact* distance between any pair of ciphertexts. This does not rule out learning *partial* information about the distances between ciphertexts, and as it turns out, some partial information is in fact leaked.

Similar to the case of one-wayness, when $r$ is large, there exists an efficient adversary against any ROPF that can win the generalized distance one-wayness game with advantage close to 1. Specifically, when $r = O(\sqrt{2^n})$, and any $z \geq 2$, there exists an efficient adversary $\mathcal{A}$ where $\mathsf{Adv}^{\mathsf{gdow}}_{r,z,\mathsf{ROPF},\mathcal{A}}(1^\lambda) = 1 - c$ where $c$ is a constant (independent of $\lambda$) [BCO11, Theorem 4.4]. Note that the results in [BCO11] are with respect to adversaries restricted to submitting circuits that encode a contiguous interval in the ring $\mathbb{Z}_{2^n}$. Conceptually, this means that two ciphertexts encrypted under an OPE scheme that implements an ROPF reveal roughly half the bits of the distance between the underlying plaintext values.

In contrast, instantiating Theorem D.3 with $\varepsilon = 1/2$, we have that for $r = \sqrt{2^n}$, and for all efficient adversaries $\mathcal{A}$, the advantage $\mathsf{Adv}^{\mathsf{gdow}}_{r,z,\Pi_{\mathsf{ore}},\mathcal{A}}$ is negligible. In fact, Theorem D.3 states that given polynomially many ciphertexts, no efficient adversary is able to even learn a *constant* fraction

of the bits in the distance between any pair of encrypted messages. Thus, in this setting, our ORE scheme provides a provably stronger security guarantee.

In the same work [BCO11], Boldyreva et al. also introduced the notion of a modular OPE (MOPE) scheme to obtain stronger security guarantees over a standard OPE scheme. At a high level, in an MOPE scheme, a secret and fixed modular offset is added to the plaintext value before encryption with an OPE scheme. On the one hand, an MOPE scheme substantially improves the one-wayness security of the underlying encryption scheme. Specifically, Boldyreva et al. show that the $(r, w)$ window one-wayness advantage (Remark 4.2) of any adversary is optimal (no better than the trivial adversary that outputs a random window of size $r$). However, this security guarantee is unstable in the sense that a small piece of information (such as a single plaintext-ciphertext pair) reveals the secret offset, reverting the one-wayness security to that of the underlying OPE scheme. Moreover, information about the secret offset can be inferred from range query distribution in a naïve practical implementation [BCO11].

Subsequent work [MCO+15] proposed strategies to avoid leaking the MOPE secret offset when the distribution of range query inputs is known, through the use of "dummy queries." However, if the distribution is far from uniform, the number of necessary dummy queries can be prohibitive. In addition, Boldyreva et al. [BCO11] show that MOPE schemes do not help to hide the distances between plaintext values. Like an OPE scheme, an MOPE scheme reveals roughly half of the bits of the distance between two plaintext values. Thus, for a uniform message distribution, we conclude that our ORE scheme achieves better hiding properties on the distances between plaintexts, while retaining direct support for efficiently answering range queries.