Ideal Multilinear Maps Based on Ideal Lattices

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Abstract. Cryptographic multilinear maps have found many applications, such as multipartite Diffie-Hellman key exchange, general software obfuscation. However, currently only three constructions are known, and are "noisy" and bounded to polynomial degree. In this paper, we first describe basic constructions of ideal multilinear maps using ideal lattices, which support arbitrary multilinearity level. The security of our construction depends on hardness assumption over ideal lattices. Then, we describe a construction of ideal multilinear maps using unimodular matrix, which removes a usable restriction of our basic construction. Finally, we describe one-round multipartite Diffie-Hellman key exchange protocols by using our ideal constructions.

Keywords. Ideal multilinear maps, Ideal lattices, Diffie-Hellman key exchange, Zeroizing attack

1 Introduction

Constructing multilinear maps have been a long-standing open problem since 2003. Many applications on bilinear maps, such as [SOK00, Jou00, BF01, Sma03], et al, inspired the study of cryptographical multilinear maps [BS03, RS09, PTT10, Rot13]. Boneh and Silverberg [BS03] first introduced the notion of multilinear maps, which are an extension of bilinear maps. However, they suspected that such maps come from the realm of algebraic geometry.

Garg, Gentry, and Halevi recently described the first plausible construction of multilinear maps that use ideal lattices [GGH13]. Their multilinear maps, whose encodings were randomized with noise and bounded with a fixed maximum degree, were different from the ideal multilinear maps of Boneh and Silverberg. Construction security depends on the new hardness assumptions of GCDH/GDDH, which are provided an extensive cryptanalysis in [GGH13]. To reduce the public parameter size of GGH, Langlois, Stehlé, and Steinfeld [LSS14] improved the security analysis of the GGH construction re-randomization process and decreased the public parameter bit size for the GGH scheme from $O(k^3\lambda^5\log(k\lambda))$ to $O(k^3\lambda\log^2(k\lambda))$ in GGHLite, with respect to security parameter λ and multilinearity parameter k. Although the length of public parameters of the GGHLite is asymptotically close to optimal, a large hidden constant is found in $O(k^3\lambda\log^2(k\lambda))$.

Following GGH's idea, Coron, Lepoint, and Tibouchi [CLT13] described a new and relatively practical construction following the [GGH13] method. Their construction works over integers instead of ideal lattices, and multilinear maps are implemented over integers using heuristic optimization techniques. However, the CLT multilinear maps have been broken by Cheon et al. [CHL+14] using level-1 encodings of zero. To fix the construction [CLT13], Boneh, Wu, and Zimmerman [BWZ14] and Garg, Gentry, Halevi, and Zhandry [GGHZ14] proposed two independent approaches to avoid zeroizing attack, however Coron，Lepoint and Tibouchi [CTL14] show that two fixes can be defeated using extensions of the [CHL+14].

To improve the security of previous constructions, Hiromasa, Abe and Okamoto [HAO14] constructed new multilinear maps based on GSW's fully homomorphic encryption. The security of their construction is not reduced to LWE, although the security of GSW's fully homomorphic encryption is reduced to LWE. Basing same idea, Gentry，Gorbunov and

Halevi [GGH14] described graph-induced multilinear maps from lattices. Their construction encodes LWE samples on short square matrix with higher dimension, however, the security of their construction is not reduced to LWE or hard assumption of arbitrary other classic problems.

To avoid zeroizing attack in the GGH construction, Gu [Gu15] described a construction of multilinear maps without encoding of zero by designing new zero-testing parameters. Recently, Coron, Lepoint, and Tibouchi [CTL15] fixed the CLT construction by also modifying zero-testing parameters.

However, all current constructions follow the framework of the GGH construction, whose levels are in advance fixed and encodings have noisy. In this paper, we will describe a construction of ideal multilinear maps from ideal lattices.

Our Results. Our main contribution is to describe a basic construction of ideal multilinear maps that use ideal lattices. Construction security depends on a new hardness assumption. Our construction works in a polynomial ring $R = \mathbb{Z}[x]/(x^n + 1)$, where n is a positive integer. Given secret ring elements $\mathbf{f}_j, \mathbf{g}_j \in R, j = 1,...,m$, we denote $\mathbf{f} = \prod_{j=1}^{m} \mathbf{f}_j$. A level-$1$ encoding of level-0 element $\mathbf{a} \in R$ is $\mathbf{c} = (\mathbf{a} \cdot \mathbf{g}) \bmod \mathbf{f}$, where $\mathbf{g} = \mathbf{g}_j \bmod \mathbf{f}_j$, $\mathbf{a} = \mathbf{a}_j \bmod \mathbf{f}_j$, $j = 1,...,m$. If only given $\mathbf{a}$ and $\mathbf{c}$, then one can compute $\mathbf{g} = (\mathbf{c}/\mathbf{a}) \bmod \mathbf{f}$. So, we provide the multiplier $\mathbf{q} = \mathbf{q}_0 \cdot \mathbf{f}$ of $\mathbf{f}$ in the public parameters to avoid this simple attack. Now, we transform the encoding $\mathbf{c}$ to a new encoding $\mathbf{u} = (\mathbf{c} + \mathbf{r} \cdot \mathbf{f}) \bmod \mathbf{q}$. To decide whether or not $\mathbf{u}$ is an encoding of zero, we provide a zero-testing parameter $\mathbf{p}_{zt} = (\sum_{j=1}^{m} \mathbf{h}_j \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q$ in the public parameters. If the norm of $[\mathbf{p}_{zt} \cdot \mathbf{u}]_q$ is small, then $\mathbf{u}$ is the encoding of zero; otherwise, it is the encoding of non-zero. To use level-0 encoding of level-1 encoding, we generate a list of level-1 encodings and zero-testing parameters $(\mathbf{y}_i, \mathbf{p}_{zt,i})$ such that the level-0 encoding of $\mathbf{y}_i$ is hidden in the corresponding zero-testing parameter $\mathbf{p}_{zt,i}$. This defines an arbitrary degree multilinear map.

Our second contribution is to describe a variant of our basic ideal construction to avoid zeroizing attack. This is because $\mathbf{q}$ is an encoding of zero. To support arbitrary multilinearity levels, we must provide this encoding of zero. So in the variant, we take a large enough multiplier $\mathbf{q}_0$ so that a non-reduced quantity over the modulo q cannot be directly obtained multiplying $\mathbf{q}$ by $\mathbf{p}_{zt}$. In the public parameters, we provide a list of non-zero encodings and its corresponding zero-testing parameters to gradually reduce encoding. We have used this method of constructing new zero-testing parameters in [Gu15] to improve the GGH construction [GGH13].

Our third contribution is to describe a construction of ideal multilinear maps using unimodular matrix. We choose two unimodular matrices $\mathbf{T}, \mathbf{S}$. A level-0 and level-1 encoding $\mathbf{a}$, $\mathbf{c} = \mathbf{a} \cdot \mathbf{g}$ are transformed into $\mathbf{X} = \mathbf{S}Rot(\mathbf{a} + \mathbf{ef})\mathbf{S}^{-1}$, $\mathbf{Y} = \mathbf{T}^{-1}Rot(\mathbf{ag} + \mathbf{tf})\mathbf{T}$, respectively, where $Rot(\mathbf{a})$ is anti-cyclic matrix of $\mathbf{a} \in R$. The zero-testing parameter is modified as $\mathbf{P}_{zt} = [\mathbf{S}Rot(\mathbf{p}_{zt})\mathbf{T}]_q$. In addition, we provide some encodings of zero $\mathbf{Q}_{x,t} = \mathbf{S}Rot(\mathbf{b}_{x,t}\mathbf{q})\mathbf{S}^{-1}$, $\mathbf{Q}_{y,t} = \mathbf{T}^{-1}Rot(\mathbf{b}_{y,t}\mathbf{q})\mathbf{T}$, where $\mathbf{q} = \mathbf{q}_0 \cdot \mathbf{f}$. This defines a construction of ideal multilinear maps.

Our final contribution is presenting a one-round multipartite Diffie-Hellman key exchange protocol, which supports arbitrary multilinearity levels. The integer modulo in our construction does not increase by multilinearity levels. Thus, our ideal multilinear maps using ideal lattices are practical.

The remainder of this paper is organized as follows: we recall some preliminaries in Section 2. We describe a basic construction of ideal multilinear maps that use ideal lattices in

Section 3, and a variant of our basic ideal construction in Section 4. We describe a construction of ideal multilinear maps using unimodular matrix in Section 5, and extend its asymmetric ideal variant in Section 6, and commutative variant in Section 7. Finally, we construct one-round multipartite Diffie-Hellman key exchange protocol in Section 8.

2 Preliminaries

2.1 Notations

We denote $\mathbb{Z},\mathbb{Q},\mathbb{R}$ the integer ring, rational number field and real number field. Vectors and matrices are denoted in bold. We denote $[\![k]\!]=\{1,2,\cdots,k\}$ for $k\in\mathbb{N}$. We take n as a power of two, the polynomial ring $R=\mathbb{Z}[x]/<x^n+1>$ and polynomial field $\mathbb{k}=\mathbb{Q}[x]/<x^n+1>$. For $\mathbf{a}\in R$, $\|\mathbf{a}\|_\infty$ ($\|\mathbf{a}\|$ for short) denotes the infinity norm of the vector corresponding to $\mathbf{a}$.
In this paper, we use the absolute minimum residual system, that is $[a]_q=a \bmod q\in(-q/2,q/2]$. Similarly, notation $[\mathbf{a}]_q$ denotes each entry (or each coefficient) $a_i\in(-p/2,p/2]$.

2.2 Lattices and Ideal Lattices

An *n*-dimension full-rank lattice $L\subset\mathbb{R}^n$ is the set of all integer linear combinations $\sum_{i=1}^n y_i\mathbf{b}_i$ of *n* linearly independent vectors $\mathbf{b}_i\in\mathbb{R}^n$. If we arrange the vectors $\mathbf{b}_i$ as the columns of matrix $\mathbf{B}\in\mathbb{R}^{n\times n}$, then $L=\{\mathbf{By}:\mathbf{y}\in\mathbb{Z}^n\}$. We say that $\mathbf{B}$ spans L if $\mathbf{B}$ is a basis for L. For any lattice basis, we define $P(\mathbf{B})=\{\mathbf{Bz}\mid\mathbf{z}\in\mathbb{R}^n,\forall i:-1/2\le z_i<1/2\}$. Let $\det(\mathbf{B})$ denote the determinant of the matrix $\mathbf{B}$.

Given $\mathbf{a},\mathbf{g}\in R$, we let the principal ideal $I=<\mathbf{g}>$ with the R_1-basis $Rot(\mathbf{g})=(\mathbf{g},x_2\cdot\mathbf{g},...,x_2^{n-1}\cdot\mathbf{g})$ and $[\mathbf{a}]_\mathbf{g}$ denote the modulo reduction of $I=<\mathbf{g}>$, namely, $[\mathbf{a}]_\mathbf{g}\in P(Rot(\mathbf{g}))$ and $(\mathbf{a}-[\mathbf{a}]_\mathbf{g})\in L(Rot(\mathbf{g}))$.

Given $\mathbf{c}\in\mathbb{R}^n$, $\sigma>0$, we define $D_{L,\sigma,\mathbf{c}}=\rho_{\sigma,\mathbf{c}}(\mathbf{x})/\rho_{\sigma,\mathbf{c}}(L)$ the Gaussian distribution of a lattice L, where $\mathbf{x}\in L$, $\rho_{\sigma,\mathbf{c}}(\mathbf{x})=\exp(-\pi\|\mathbf{x}-\mathbf{c}\|^2/\sigma^2)$, $\rho_{\sigma,\mathbf{c}}(L)=\sum_{x\in L}\rho_{\sigma,\mathbf{c}}(\mathbf{x})$. In the following, we will write $D_{\mathbb{Z}^n,\sigma,0}$ as $D_{\mathbb{Z}^n,\sigma}$. We denote a Gaussian sample as $\mathbf{x}\leftarrow D_{L,\sigma}$ (or $\mathbf{d}\leftarrow D_{I,\sigma}$) over the lattice L (or ideal lattice I).

2.3 Multilinear Maps

Definition 2.1 (Multilinear Map [BS03]). For $k+1$ cyclic groups $G_1,...G_k,G_T$ of the same order p, a k-multilinear map $e:G_1\times\cdots\times G_k\to G_T$ has the following properties:

(1)Elements $\{g_j\in G_j\}_{j=1,...,k}$, index $j\in[\![k]\!]$, and integer $a\in Z_p$ hold that

$$e(g_1,\cdots,a\cdot g_j,\cdots,g_k)=a\cdot e(g_1,\cdots,g_k).$$

(2)Map e is non-degenerate in the following sense: if elements $\left\{g_j \in G_j\right\}_{j=1,\ldots,k}$ are generators of their respective groups, then $e(g_1,\cdots,g_k)$ is a generator of G_T.

Definition 2.2 (k-Graded Encoding System [GGH13]). A k-graded encoding system over R is a set system of $S=\left\{S_j^{(\alpha)} \subset R : \alpha \in R, j \in [\![k]\!]\right\}$ with the following properties:

(1) For every index $j \in [\![k]\!]$, the sets $\left\{S_j^{(\alpha)} : \alpha \in R\right\}$ are disjoint.

(2) Binary operations '$+$' and '$-$' exist, such that every α_1, α_2, every index $j \in [\![k]\!]$, and every $u_1 \in S_j^{(\alpha_1)}$ and $u_2 \in S_j^{(\alpha_2)}$ hold that $u_1+u_2 \in S_j^{(\alpha_1+\alpha_2)}$ and $u_1-u_2 \in S_j^{(\alpha_1-\alpha_2)}$, where $\alpha_1+\alpha_2$ and $\alpha_1-\alpha_2$ are the addition and subtraction operations in R respectively.

(3) Binary operation '$\times$' exists, such that every α_1, α_2, every index $j_1, j_2 \in [\![k]\!]$ with $j_1+j_2 \le k$, and every $u_1 \in S_{j_1}^{(\alpha_1)}$ and $u_2 \in S_{j_2}^{(\alpha_2)}$ hold that $u_1 \times u_2 \in S_{j_1+j_2}^{(\alpha_1 \times \alpha_2)}$, where $\alpha_1 \times \alpha_2$ is the multiplication operation in R and j_1+j_2 is the integer addition.

3 Basic ideal multilinear maps

In this section, we first construct symmetric multilinear maps over ideal lattices. Then we show the correctness of our construction. Next, we show the security of our construction. Finally, we give known cryptanalysis for our construction.

3.1 Construction

Setting the parameters. Let λ be the security parameter, n the dimension of R. Concrete parameters are set as $\sigma=\sqrt{\lambda n}$, $\sigma'=\lambda n^{1.5}$, $q \ge 2^{\eta+O(\lambda)} n^{O(1)}$, $n=O(\lambda^2)$, $m=O(\lambda)$, $l=O(n^2)$, $\tau=O(n^2)$.

Instance generation: $(\mathrm{par}_1) \leftarrow \mathrm{InstGen}_1(1^\lambda)$.

(1) Choose a large enough prime q.

(2) Sample $\mathbf{f}_j \leftarrow D_{\mathbb{Z}^n,\sigma}$, $j \in [\![m]\!]$ and $\mathbf{p}_0 \leftarrow D_{\mathbb{Z}^n,\sigma'}$ such that ideal lattices $\mathbf{f}_j$'s are coprime, $\mathbf{f}_j^{-1} \in \mathbb{k}$ and $\left\|\mathbf{f}_j^{-1}\right\| \le l$.

(3) Set $\mathbf{f}=\prod_{j=1}^m \mathbf{f}_j$, $\mathbf{q}=\mathbf{p}_0 \cdot \mathbf{f}$ and $\mu=\|\mathbf{f}\|$.

(4) Sample $\mathbf{g} \leftarrow D_{\mathbb{Z}^n,\mu}, \mathbf{a}_i \leftarrow D_{\mathbb{Z}^n,\mu}, \mathbf{t}_i \leftarrow D_{\mathbb{Z}^n,\sigma'}, i \in [\![\tau]\!]$.

(5) Set $\mathbf{y}_i=(\mathbf{a}_i\mathbf{g}+\mathbf{t}_i\mathbf{f}) \bmod \mathbf{q}, i \in [\![\tau]\!]$.

(6) Sample $\mathbf{h}_j \leftarrow D_{\mathbb{Z}^n,\sigma}$, $\mathbf{s}_{i,j} \leftarrow D_{\mathbb{Z}^n,\sigma'}$, $j \in [\![m]\!], i \in [\![\tau]\!]$.

(7) Set $\mathbf{p}_{zt,i}=(\sum_{j=1}^m \mathbf{h}_j(\mathbf{a}_i+\mathbf{s}_{i,j}\cdot\mathbf{f}_j)\cdot(\mathbf{f}_j^{-1} \bmod q)) \bmod q$.

(8) Output the public parameters $\mathrm{par}_1=\left\{q, \mathbf{q}, \left\{\mathbf{y}_i, \mathbf{p}_{zt,i}\right\}_{i\in[\![\tau]\!]}\right\}$.

Generating a random level-k encoding: $\mathbf{u} \leftarrow \mathrm{Enc}_1(\mathrm{par}_1, k, \{\mathbf{w}_i\}_{i\in[\![\tau]\!]})$.

Choose $\mathbf{w}_i \leftarrow D_{\mathbb{Z}^n,\sigma}$, $i \in [\![\tau]\!]$, generate a level-k encoding $\mathbf{u}=\sum_{i=1}^\tau \mathbf{w}_i \cdot (\mathbf{y}_i)^k \bmod \mathbf{q}$.

Adding encodings: $\mathbf{u} \leftarrow \mathrm{Add}_1(\mathrm{par}_1, k, \mathbf{u}_1, \cdots, \mathbf{u}_s)$.

Given s level-k encodings $\mathbf{u}_t, t \in [\![s]\!]$, their sum $\mathbf{u} = \sum_{t=1}^{s} \mathbf{u}_t \bmod \mathbf{q}$ is a level-k encoding.

Multiplying encodings: $\mathbf{u} \leftarrow \mathrm{Mul}_1(\mathrm{par}_1, 1, \mathbf{u}_1, \cdots, \mathbf{u}_k)$.

Given k level-1 encodings $\mathbf{u}_t, t \in [\![k]\!]$, their product $\mathbf{u} = \prod_{t=1}^{k} \mathbf{u}_t \bmod \mathbf{q}$ is a level-k encoding.

Zero Testing: $\mathrm{isZero}_1(\mathrm{par}_1, \mathbf{u}, \{\mathbf{r}_i\}_{i=1}^{\tau})$.

Given a level-k encoding $\mathbf{u}$ and $\mathbf{r}_i \leftarrow D_{\mathbb{Z}^n,\sigma}, i \in [\![\tau]\!]$, determine whether $\mathbf{u}$ is an encoding of zero, $\mathbf{v} = [\mathbf{p}_{zt} \cdot \mathbf{u}]_q$ is computed in R_q and checked whether $\|\mathbf{v}\|$ is short:

$$\mathrm{isZero}_1(\mathrm{par}_1, \mathbf{u}, \{\mathbf{r}_i\}_{i=1}^{\tau}) = \begin{cases} 1 & \text{if } \left\| [\mathbf{p}_{zt} \cdot \mathbf{u}]_q \right\| < q/2^{\eta} \\ 0 & \text{otherwise} \end{cases}, \text{ where } \mathbf{p}_{zt} = \sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,i}.$$

Extract: $sk \leftarrow \mathrm{Ext}_1\left(\mathrm{par}_1, \mathbf{u}, \{\mathbf{r}_i\}_{i=1}^{\tau}\right)$.

Given a level-k encoding $\mathbf{u}$ and $\mathbf{r}_i \leftarrow D_{\mathbb{Z}^n,\sigma}, i \in [\![\tau]\!]$,

$$\mathrm{Ext}_1\left(\mathrm{par}_1, \mathbf{u}, \{\mathbf{r}_i\}_{i=1}^{\tau}\right) = \mathrm{Extract}_s(\mathrm{msbs}_{\eta}([\mathbf{p}_{zt} \cdot \mathbf{u}]_q)),$$

where $\mathbf{p}_{zt} = \sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,i}$.

In this paper, we omit the seed s and concrete extraction algorithm $\mathrm{Extract}$.

Remark 3.1 (1) In the above construction, one can generate a level-k encoding, whose level-0 encoding is hidden. However, one can use hidden level-0 encoding $\mathbf{d} = \sum_{i=1}^{\tau} (\mathbf{w}_i \cdot \mathbf{a}_i)$ corresponding to level-1 encoding $\mathbf{u} = \sum_{i=1}^{\tau} \mathbf{w}_i \cdot \mathbf{y}_i \bmod \mathbf{q}$ by zero-testing parameter $\mathbf{p}_{zt} = \sum_{i=1}^{\tau} \mathbf{w}_i \mathbf{p}_{zt,i}$. (2) Our construction supports arbitrary level encoding.

3.2 Correctness

Lemma 3.2 $\mathrm{InstGen}_1(1^{\lambda})$ is a probabilistic polynomial time algorithm.

Proof. A prime q can be efficiently generated. By [GGH13], $\mathbf{p}_0 \leftarrow D_{\mathbb{Z}^n,\sigma'}$, $\mathbf{f}_j \leftarrow D_{\mathbb{Z}^n,\sigma}$, $j \in [\![m]\!]$ can be sampled and are satisfied to that $\mathbf{f}_j$'s are coprime, $\mathbf{f}_j^{-1} \in \mathbb{k}$ and $\left\|\mathbf{f}_j^{-1}\right\| \le l$. It is easy to solve $\mathbf{f}_j^{-1} \bmod q$ when $\gcd(\det(Rot(\mathbf{f}_j)), q) = 1$. So, the public parameters par_1 can be generated in polynomial time. ■

Lemma 3.3 $\mathbf{u} \leftarrow \mathrm{Enc}_1(\mathrm{par}_1, k, \mathbf{d})$ is a level-k encoding**.**

Proof. Since $\mathbf{u} = \sum_{i=1}^{\tau} \mathbf{w}_i \cdot (\mathbf{y}_i)^k \bmod \mathbf{q}$, we have

$$
\begin{aligned}
&\mathbf{u} \bmod \mathbf{f}_j \\
&= (\sum_{i=1}^{\tau} \mathbf{w}_i \cdot (\mathbf{y}_i)^k \bmod \mathbf{q}) \bmod \mathbf{f}_j \\
&= (\sum_{i=1}^{\tau} \mathbf{w}_i \cdot ((\mathbf{a}_i \mathbf{g} + \mathbf{t}_i \mathbf{f}) \bmod \mathbf{q})^k \bmod \mathbf{q}) \bmod \mathbf{f}_j \\
&= (\sum_{i=1}^{\tau} \mathbf{w}_i \cdot ((\mathbf{a}_i \mathbf{g} + \mathbf{t}_i \mathbf{f}) \bmod \mathbf{f}_j)^k) \bmod \mathbf{f}_j \quad , \\
&= (\sum_{i=1}^{\tau} \mathbf{w}_i \cdot (\mathbf{a}_i \mathbf{g}_j)^k) \bmod \mathbf{f}_j \\
&= (\sum_{i=1}^{\tau} \mathbf{w}_i \cdot (\mathbf{a}_{i,j})^k) \cdot (\mathbf{g}_j)^k \bmod \mathbf{f}_j \\
&= \mathbf{v}_j \cdot (\mathbf{g}_j)^k \bmod \mathbf{f}_j
\end{aligned}
$$

where $\mathbf{a}_{i,j} = \mathbf{a}_i \bmod \mathbf{f}_j$, $\mathbf{g}_j = \mathbf{g} \bmod \mathbf{f}_j$, $\mathbf{v}_j = \sum_{i=1}^{\tau} \mathbf{w}_i \cdot (\mathbf{a}_{i,j})^k \bmod \mathbf{f}_j$. So, $\mathbf{u}$ is a level-k encoding of $\mathbf{d}$. ■

Lemma 3.4 $\mathbf{u} \leftarrow \mathrm{Add}_1(\mathrm{par}_1, k, \mathbf{u}_1, \cdots, \mathbf{u}_s)$ is a level-k encoding.

Proof. Using modulo operation, it is easy to verify that $\mathbf{u}$ is a level-k encoding. ■

Lemma 3.5 $\mathbf{u} \leftarrow \mathrm{Mul}_1(\mathrm{par}_1, 1, \mathbf{u}_1, \cdots, \mathbf{u}_k)$ is a level-k encoding.

Proof. Since $\mathbf{u}_t, t \in [\![k]\!]$ are level-1 encodings, we have $\mathbf{u}_t \bmod \mathbf{f}_j = \mathbf{u}_{t,j} \mathbf{g}_j$. Then, we have

$$
\begin{aligned}
&\mathbf{u} \bmod \mathbf{f}_j \\
&= (\prod_{t=1}^{k} \mathbf{u}_t \bmod \mathbf{q}) \bmod \mathbf{f}_j \\
&= (\prod_{t=1}^{k} \mathbf{u}_t \bmod \mathbf{f}_j) \bmod \mathbf{f}_j \\
&= (\prod_{t=1}^{k} (\mathbf{u}_t \bmod \mathbf{f}_j)) \bmod \mathbf{f}_j \\
&= (\prod_{t=1}^{k} \mathbf{u}_{t,j} \mathbf{g}_j) \bmod \mathbf{f}_j \\
&= (\prod_{t=1}^{k} \mathbf{u}_{t,j}) (\mathbf{g}_j)^k \bmod \mathbf{f}_j
\end{aligned}
$$

So, $\mathbf{u}$ is a level-k encoding. ■

Lemma 3.6 For an arbitrary integer $k > 0$, the zero-testing algorithm $\mathrm{isZero}_1(\mathrm{par}_1, \mathbf{u})$ correctly determines whether a level-k encoding $\mathbf{u}$ is an encoding of zero.

Proof. Given an arbitrary level-k encoding $\mathbf{u}$, we have $\mathbf{u} = \mathbf{d} + \mathbf{r} \cdot \mathbf{f}$ and $\|\mathbf{u}\| < \|\mathbf{q}\|$ with $\|\mathbf{d}\| < \|\mathbf{f}\|$.

(1) If $\mathbf{u}$ is an encoding of zero, then $\mathbf{u} \bmod \mathbf{f}_j = 0, j \in [\![m]\!]$. Since $\mathbf{f}_j, j \in [\![m]\!]$ are coprime, $\mathbf{u} \bmod \mathbf{f} = 0$. That is, $\mathbf{d} \bmod \mathbf{f} = 0$ and $\mathbf{d} = 0$ according to $\|\mathbf{d}\| < \|\mathbf{f}\|$. So, we have

$$
\begin{aligned}
\mathbf{v} &= \left\| \left[\mathbf{p}_{zt} \cdot \mathbf{u} \right]_q \right\| \\
&= \left\| \left[(\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,i}) \cdot \mathbf{u} \right]_q \right\| \\
&= \left\| \left[\sum_{j=1}^{m} \mathbf{h}_j (\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{a}_i + \sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{s}_{i,j} \cdot \mathbf{f}_j) \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q \cdot \mathbf{u} \right]_q \right\| \\
&= \left\| ((\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q \cdot \mathbf{u}) \bmod q \right\| \\
&= \left\| ((\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{u} \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q) \bmod q \right\| \\
&= \left\| ((\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{r} \cdot \mathbf{f} \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q) \bmod q \right\| \\
&= \left\| (\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{r} \cdot \mathbf{f} / \mathbf{f}_j) \bmod q \right\| \\
&\le \left\| \sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{r} \cdot \mathbf{f} / \mathbf{f}_j \right\| \\
&\le \sum_{j=1}^{m} \left\| \mathbf{h}_j^{'} \cdot \mathbf{r} \cdot \mathbf{f} / \mathbf{f}_j \right\| \\
&\le \sum_{j=1}^{m} \left\| \mathbf{h}_j^{'} \right\| \cdot \left\| \mathbf{r} \right\| \cdot \left\| \mathbf{f} / \mathbf{f}_j \right\| \\
&\le q / 2^{\eta} \qquad ,
\end{aligned}
$$

where $\mathbf{h}_j^{'} = \mathbf{h}_j (\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{a}_i + \sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{s}_{i,j} \cdot \mathbf{f}_j)$.

(2) If $\mathbf{u}$ is not an encoding of zero, then $\mathbf{u} \bmod \mathbf{f} \neq 0$. That is, $\exists j \in [\![m]\!], \mathbf{d} \bmod \mathbf{f}_j \neq 0$. So,

$$
\begin{aligned}
\mathbf{v} &= \left\| \left[\mathbf{p}_{zt} \cdot \mathbf{u} \right]_q \right\| \\
&= \left\| \left[(\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,i}) \cdot \mathbf{u} \right]_q \right\| \\
&= \left\| \left[\sum_{j=1}^{m} \mathbf{h}_j (\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{a}_i + \sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{s}_{i,j} \cdot \mathbf{f}_j) \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q \cdot \mathbf{u} \right]_q \right\| \\
&= \left\| ((\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q \cdot \mathbf{u}) \bmod q \right\| \\
&= \left\| ((\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{u} \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q) \bmod q \right\| \\
&= \left\| ((\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot (\mathbf{d} + \mathbf{r} \cdot \mathbf{f}) \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q) \bmod q \right\| \\
&= \left\| (\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{r} \cdot \mathbf{f} / \mathbf{f}_j) + (\sum_{j=1}^{m} (\mathbf{h}_j^{'} \cdot \mathbf{d}) \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q \right\| \\
&\ge \left\| (\sum_{j=1}^{m} (\mathbf{h}_j^{'} \cdot \mathbf{d}) \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q \right\| - \sum_{j=1}^{m} \left\| \mathbf{h}_j^{'} \right\| \cdot \left\| \mathbf{r} \right\| \cdot \left\| \mathbf{f} / \mathbf{f}_j \right\| \\
&\ge q^{1-\varepsilon} - q / 2^{\eta} \\
&\ge q^{1-\varepsilon'}
\end{aligned}
$$

So, $\mathrm{isZero}_1(\mathrm{par}_1, \mathbf{u})$ correctly decides the encoding of $\mathbf{u}$. ■

Lemma 3.7 Suppose two level-k encodings $\mathbf{u}_1, \mathbf{u}_2$ encode same level-0 encodings, that is, $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{a}_j (\mathbf{g}_j)^k \bmod \mathbf{f}_j$, $\forall j \in [\![m]\!]$, then

$$\mathrm{Ext}_1\left(\mathrm{par}_1, \mathbf{u}_1, \{\mathbf{r}_i\}_{i=1}^{\tau}\right) = \mathrm{Ext}_1\left(\mathrm{par}_1, \mathbf{u}_2, \{\mathbf{r}_i\}_{i=1}^{\tau}\right).$$

Proof. Since $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{a}_j(\mathbf{g}_j)^k \bmod \mathbf{f}_j, j \in [\![m]\!]$ and $\mathbf{f}_j, j \in [\![m]\!]$ are co-prime, we have $\mathbf{u}_1 = \mathbf{d} + \mathbf{r}_1 \cdot \mathbf{f}, \mathbf{u}_2 = \mathbf{d} + \mathbf{r}_2 \cdot \mathbf{f}$.

Given $\{\mathbf{r}_i\}_{i=1}^{\tau}$, we have $\mathbf{p}_{zt} = \sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,i}$ and $\mathbf{h}_j^{'} = \mathbf{h}_j(\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{a}_i + \sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{s}_{i,j} \cdot \mathbf{f}_j)$. So,

$$\begin{aligned}
\left[\mathbf{p}_{zt} \cdot \mathbf{u}_1\right]_q &= ((\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot (\mathbf{d} + \mathbf{r}_1 \cdot \mathbf{f}) \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q) \bmod q \\
&= (\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{r}_1 \cdot \mathbf{f} / \mathbf{f}_j) + (\sum_{j=1}^{m} (\mathbf{h}_j^{'} \cdot \mathbf{d}) \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q \\
\left[\mathbf{p}_{zt} \cdot \mathbf{u}_2\right]_q &= ((\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot (\mathbf{d} + \mathbf{r}_2 \cdot \mathbf{f}) \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q) \bmod q \\
&= (\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{r}_2 \cdot \mathbf{f} / \mathbf{f}_j) + (\sum_{j=1}^{m} (\mathbf{h}_j^{'} \cdot \mathbf{d}) \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q
\end{aligned}.$$

By Lemma 3.6, $(\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{r}_1 \cdot \mathbf{f} / \mathbf{f}_j)$, $(\sum_{j=1}^{m} \mathbf{h}_j^{'} \cdot \mathbf{r}_2 \cdot \mathbf{f} / \mathbf{f}_j)$ are less than $q / 2^{\eta}$, respectively. Hence, the most significant bits of $\left[\mathbf{p}_{zt} \cdot \mathbf{u}_1\right]_q$, $\left[\mathbf{p}_{zt} \cdot \mathbf{u}_2\right]_q$ are decided by $(\sum_{j=1}^{m} (\mathbf{h}_j^{'} \cdot \mathbf{d}) \cdot (\mathbf{f}_j^{-1} \bmod q))$. That is, $\mathrm{Ext}_1\left(\mathrm{par}_1, \mathbf{u}_1, \{\mathbf{r}_i\}_{i=1}^{\tau}\right) = \mathrm{Ext}_1\left(\mathrm{par}_1, \mathbf{u}_2, \{\mathbf{r}_i\}_{i=1}^{\tau}\right)$.■

3.3 Security

The security of our constructions depends on new hardness assumptions, and seems to rely on hardness to solve shortest generator problems for principal ideal lattices. However, we currently do not know how to reduce the security of our construction to the shortest principal ideal generator problems.

Hardness assumptions for multilinear maps in [GGH13] are modeled as discrete logarithms and DDH assumptions in multilinear groups. Generating a level-k encoding of the product or distinguishing the product from random elements is unfeasible given the public parameters and $k+1$ level-1 encodings of random elements.

Garg, Gentry, and Halevi introduced the definition of GCDH/GDDH in [GGH13] to describe the hardness assumption of the GGH construction. Langlois, Stehlé, and Steinfeld extended the GCDH/GDDH to the ext-GCDH/ext-GDDH in [LSS14] to describe the security of the GGHLite scheme.

In the following, we adapt the definition of ext-GCDH/ext-GDDH in [LSS14] to our constructions. Consider the following process:

(1) $(\mathrm{par}_1) \leftarrow \mathrm{InstGen}_1(1^{\lambda})$.

(2) Choose an arbitrary positive integer k.

(3) For $t = 0$ to k:

Sample $\mathbf{w}_{t,i} \leftarrow D_{\mathbb{Z}^n,\sigma}, i \in [\![\tau]\!]$,

Generate level-1 encoding $\mathbf{u}_t = (\sum_{i=1}^{\tau} \mathbf{w}_{t,i} \cdot \mathbf{y}_i) \bmod \mathbf{q}$.

(4) Sample $\mathbf{r}_{0,i} \leftarrow D_{\mathbb{Z}^n,\sigma'}, i \in [\![\tau]\!]$.

(5) Compute $\mathbf{u} = \prod_{t=1}^{k} \mathbf{u}_t \bmod \mathbf{q}$.

(6) Set $\mathbf{v}_C = \mathbf{v}_D = \mathrm{Ext}_1\left(\mathrm{par}_1, \mathbf{u}, \{\mathbf{w}_{0,i}\}_{i=1}^{\tau}\right)$.

(7) Set $\mathbf{v}_R = \mathrm{Ext}_1\left(\mathrm{par}_1, \mathbf{u}, \{\mathbf{r}_{0,i}\}_{i=1}^{\tau}\right)$.

Definition 3.8 (ext-GCDH/ext-GDDH). The extraction k-graded CDH problem (ext-GCDH)

is, on input $\{\text{par}_1, \mathbf{u}_0, \cdots, \mathbf{u}_k\}$, to output the extraction encoding $\mathbf{v}_C$. The extraction k-graded DDH problem (ext-GDDH) distinguishes between $\mathbf{v}_D$ and $\mathbf{v}_R$, that is, between the distributions $D_{GDDH} = \{\text{par}_1, \mathbf{u}_0, \cdots, \mathbf{u}_k, \mathbf{v}_D\}$ and $D_{RAND} = \{\text{par}_1, \mathbf{u}_0, \cdots, \mathbf{u}_k, \mathbf{v}_R\}$.

As in [GGH13], our construction security depends on new assumptions that are unlikely to be reducible to more classical assumptions. We assume the ideal-GCDH/ ideal-GDDH is hard in our scheme.

3.4 Cryptanalysis

In this section, we describe some known attacks for our construction. In the following section, we will present a variant construction to thwart these attacks.

3.4.1 Average Attack

Since $\mathbf{q} = \mathbf{p}_0 \cdot \mathbf{f}$ is an encoding of zero in our construction, $[\mathbf{q} \cdot \mathbf{p}_{zt}]_q$ is not reduced modulo q. So, the following quantities are easily computed from public parameters through algebraic transformation.

$$\begin{aligned}\mathbf{u} &= [\mathbf{q} \cdot \mathbf{p}_{zt}]_q \\ &= (\mathbf{p}_0 \cdot \mathbf{f})(\sum_{j=1}^{m} \mathbf{h}'_j \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q \\ &= (\mathbf{p}_0 \cdot \sum_{j=1}^{m} \mathbf{h}'_j \cdot \mathbf{f} / \mathbf{f}_j) \bmod q \\ &= \mathbf{p}_0 \cdot \sum_{j=1}^{m} \mathbf{h}'_j \cdot \mathbf{f} / \mathbf{f}_j \\ &= \mathbf{p}_0 \cdot \mathbf{h}\end{aligned} \quad ,$$

where $\mathbf{p}_{zt} = \sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,i}$, $\mathbf{h}'_j = \mathbf{h}_j (\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{a}_i + \sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{s}_{i,j} \cdot \mathbf{f}_j)$, $\mathbf{h} = \sum_{j=1}^{m} \mathbf{h}'_j \cdot \mathbf{f} / \mathbf{f}_j$.

The above fourth equality holds because $\mathbf{f} / \mathbf{f}_j \in R, j \in [\![m]\!]$ and $\|\mathbf{p}_0\|, \|\mathbf{h}'_j\|, \|\mathbf{f} / \mathbf{f}_j\|$ all are small according to our parameter setting. That is, $\mathbf{u} = \mathbf{p}_0 \cdot \mathbf{h}$ is not reduced modulo q. So, one can compute a basis $\mathbf{P}_0$ of $\mathbf{p}_0$ from $\mathbf{u}, \mathbf{q}$ and a basis $\mathbf{F}$ of $\mathbf{f}$. However, the short generators for $\mathbf{p}_0, \mathbf{f}$ cannot currently be found using $\mathbf{P}_0$, $\mathbf{F}$ and $\mathbf{u}, \mathbf{q}$.

For the averaging attacks considered in [GGH13, LS14], the current countermeasure is to increase dimension of ideal lattice of our scheme. The security of our scheme is based on the difficulty of finding any short element of the secret element $\mathbf{f}$.

3.4.2 Lattice reduction attack

Given $\mathbf{u}_t = (\sum_{i=1}^{\tau} \mathbf{w}_{t,i} \cdot \mathbf{y}_i) \bmod \mathbf{q}$, to solve $\{\mathbf{w}_{t,i}\}_{i=1}^{\tau}$. If τ is set large enough, then this attack does not work for our construction.

4 Variant of basic ideal multilinear maps

From the cryptanalysis above, there exist easily computable bases in our scheme. These bases are related to secret ring elements and can threaten our scheme security. This is because $\mathbf{q}$ is an encoding of zero in our scheme above. The reason why we include the encoding of zero is

to obtain ideal multilinear maps.

4.1 Construction

Setting the parameters. Let λ be the security parameter, n the dimension of R. Concrete parameters are set as $\sigma = \sqrt{\lambda n}$, $\sigma' = \lambda n^{1.5}$, $q \geq 2^{\eta + O(\lambda)} n^{O(1)}$, $n = O(\lambda^2)$, $m = O(\lambda), l = O(n^2), \tau = O(n^2)$.

Instance generation: $(\mathrm{par}_2) \leftarrow \mathrm{InstGen}_2(1^\lambda)$.

(1) Choose a large enough prime q.

(2) Sample $\mathbf{f}_j \leftarrow D_{\mathbb{Z}^n,\sigma}$, $j \in [\![m]\!]$ and $\mathbf{p}_0 \leftarrow D_{\mathbb{Z}^n,q}$ such that all ideal lattices $\mathbf{f}_j$'s are coprime, $\mathbf{f}_j^{-1} \in \mathbb{k}$ and $\left\|\mathbf{f}_j^{-1}\right\| \leq l$.

(3) Set $\mathbf{f} = \prod_{j=1}^{m} \mathbf{f}_j$, $\mathbf{q} = \mathbf{p}_0 \cdot \mathbf{f}$ and $\beta = \|\mathbf{f}\|$.

(4) Sample $\mathbf{g} \leftarrow D_{\mathbb{Z}^n,\beta}, \mathbf{a}_i \leftarrow D_{\mathbb{Z}^n,\beta}, \mathbf{t}_i \leftarrow D_{\mathbb{Z}^n,\sigma'}, i \in [\![\tau]\!]$.

(5) Set $\mathbf{y}_i = (\mathbf{a}_i \mathbf{g} + \mathbf{t}_i \mathbf{f}) \bmod \mathbf{q}, i \in [\![\tau]\!]$.

(6) Sample $\mathbf{h}_j \leftarrow D_{\mathbb{Z}^n,\sigma}$, $\mathbf{s}_{0,i,j} \leftarrow D_{\mathbb{Z}^n,\sigma'}$, $j \in [\![m]\!], i \in [\![\tau]\!]$.

(7) Set $\mathbf{p}_{zt,i} = \left[\sum_{j=1}^{m} \mathbf{h}_j (\mathbf{a}_{i,j} + \mathbf{s}_{0,i,j} \cdot \mathbf{f}_j) \cdot (\mathbf{f}_j^{-1} \bmod q)\right]_q$, $i \in [\![\tau]\!]$, where $\mathbf{a}_{i,j} = \mathbf{a}_i \bmod \mathbf{f}_j$, $j \in [\![m]\!]$.

(8) Let $\mu = \lceil \log_\sigma q \rceil$ and $q_t = \sigma^{\mu - t}, t \in [\![\mu - 1]\!]$

(9) Set $\mathbf{q}_t = \mathbf{e}_t + \mathbf{p}_t \cdot \mathbf{f}$, $t \in [\![\mu - 1]\!]$ such that $\left\|\mathbf{q}_t^{-1}\right\| \leq n(\|\mathbf{q}_t\|)^{-1}$ and $\left\|\mathbf{q}_{t-1}\right\| \leq n\sigma \|\mathbf{q}_t\|$, where $\mathbf{p}_t \leftarrow D_{\mathbb{Z}^n,q_t}, \mathbf{e}_t \leftarrow D_{\mathbb{Z}^n,\beta}$.

(10) Set $\mathbf{p}_{zt,t,i} = (\sum_{j=1}^{m} \mathbf{h}_j (\mathbf{e}_{t,j} \mathbf{a}_{i,j} + \mathbf{s}_{t,i,j} \mathbf{f}_j)(\mathbf{f}_j^{-1} \bmod q)) \bmod q, t \in [\![\mu - 1]\!], i \in [\![\tau]\!]$, where $\mathbf{e}_{t,j} = \mathbf{e}_t \bmod \mathbf{f}_j$, $\mathbf{s}_{t,i,j} \leftarrow D_{\mathbb{Z}^n,\sigma}, t \in [\![\mu - 1]\!], i \in [\![\tau]\!], j \in [\![m]\!]$.

(11) Output the public parameters

$$\mathrm{par}_2 = \left\{q, \mathbf{q}, \left\{\mathbf{y}_i, \mathbf{p}_{zt,i}\right\}_{i \in [\![\tau]\!]}, \left\{\mathbf{q}_t, \left\{\mathbf{p}_{zt,t,i}\right\}_{i \in [\![\tau]\!]}\right\}_{t \in [\![\mu - 1]\!]}\right\}.$$

Generating a random level-k encoding: $\mathbf{u} \leftarrow \mathrm{Enc}_2(\mathrm{par}_2, k, \{\mathbf{w}_i\}_{i \in [\![\tau]\!]})$.

Choose $\mathbf{w}_i \leftarrow D_{\mathbb{Z}^n,\sigma'}$, $i \in [\![\tau]\!]$, generate a level-k encoding $\mathbf{u} = \sum_{i=1}^{\tau} \mathbf{w}_i \cdot (\mathbf{y}_i)^k \bmod \mathbf{q}$.

Adding encodings: $\mathbf{u} \leftarrow \mathrm{Add}_2(\mathrm{par}_2, k, \mathbf{u}_1, \cdots, \mathbf{u}_s)$.

Given s level-k encodings $\mathbf{u}_t, t \in [\![s]\!]$, their sum $\mathbf{u} = \sum_{t=1}^{s} \mathbf{u}_t \bmod \mathbf{q}$ is a level-k encoding.

Multiplying encodings: $\mathbf{u} \leftarrow \mathrm{Mul}_2(\mathrm{par}_2, 1, \mathbf{u}_1, \cdots, \mathbf{u}_k)$.

Given k level-1 encodings $\mathbf{u}_t, t \in [\![k]\!]$, their product $\mathbf{u} = \prod_{t=1}^{k} \mathbf{u}_t \bmod \mathbf{q}$ is a level-k encoding.

Zero-testing: $\mathrm{isZero}_2\left(\mathrm{par}_2, \mathbf{u}_0, \{\mathbf{r}_i\}_{i=1}^{\tau}\right)$.

Given a level-k encoding $\mathbf{u}_0$ and $\mathbf{r}_i \leftarrow D_{\mathbb{Z}^n,\sigma}, i \in [\![\tau]\!]$, determine whether $\mathbf{u}_0$ is an encoding of zero:

(1) For $t=1$ to $\mu-1$

Compute $\mathbf{u}_t = \mathbf{u}_{t-1} \bmod \mathbf{q}_t$ and $\mathbf{k}_t = (\mathbf{u}_{t-1} - \mathbf{u}_t)/\mathbf{q}_t$.

(2) Compute $\mathbf{v} = \left[\mathbf{p}_{zt} \cdot \mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{p}_{zt,t} \cdot \mathbf{k}_t\right]_q$, where $\mathbf{p}_{zt} = \left[\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,i}\right]_q$,

$\mathbf{p}_{zt,t} = \left[\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,t,i}\right]_q$.

(3) Checked whether $\|\mathbf{v}\|$ is short:

$$\mathrm{isZero}_2\left(\mathrm{par}_2, \mathbf{u}_0, \{\mathbf{r}_i\}_{i=1}^{\tau}\right) = \begin{cases} 1 & \text{if } \|\mathbf{v}\| < q/2^{\eta} \\ 0 & \text{otherwise} \end{cases}.$$

Extract: $sk \leftarrow \mathrm{Ext}_2\left(\mathrm{par}_2, \mathbf{u}_0, \{\mathbf{r}_i\}_{i=1}^{\tau}\right)$.

Given a level-k encoding $\mathbf{u}_0$ and $\mathbf{r}_i \leftarrow D_{\mathbb{Z}^n,\sigma}, i \in [\![\tau]\!]$, extract a bit string as follows:

(1) For $t=1$ to $\mu-1$

Compute $\mathbf{u}_t = \mathbf{u}_{t-1} \bmod \mathbf{q}_t$ and $\mathbf{k}_t = (\mathbf{u}_{t-1} - \mathbf{u}_t)/\mathbf{q}_t$.

(2) Compute $\mathbf{v} = \left[\mathbf{p}_{zt} \cdot \mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{p}_{zt,t} \cdot \mathbf{k}_t\right]_q$, where $\mathbf{p}_{zt} = \left[\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,i}\right]_q$,

$\mathbf{p}_{zt,t} = \left[\sum_{i=1}^{\tau} \mathbf{r}_i \mathbf{p}_{zt,t,i}\right]_q$.

(3) Extract the most significant bits $\mathrm{Ext}_2(\mathrm{par}_2, \mathbf{u}_0, \{\mathbf{r}_i\}_{i=1}^{\tau}) = \mathrm{Extract}(\mathrm{msbs}_{\eta}(\mathbf{v}))$.

Remark 4.1 (1) If one sets $m = O(\lambda^2)$ and $n=1$ for our variant, then the variant is extended to the ring $\mathbb{Z}$ of integers. In some sense this case is similar to the construction in [CLT15], however our construction is ideal multilinear maps, whereas their construction is approximate multilinear maps. Moreover, our variant needs to use the method of constructing new zero-testing parameter in [Gu15]. (2) One cannot set $m = O(\lambda^2)$ and $n=1$ for the ideal multilinear maps in Section 3. This is because the multiplier $\mathbf{q}_0$ is an integer and can be computed in this case. (3) One can set $m=1$ and $n = O(\lambda^2)$ for our variant to decrease time of computing inverse elements in generating instance algorithm.

4.2 Correctness

We first give Lemma 4.2 to show the correctness of our variant construction.

Lemma 4.2 If $\|\mathbf{p}^{-1}\| \le n(\|\mathbf{p}\|)^{-1}$ and $\|\mathbf{u}\|/\|\mathbf{p}\| \le \alpha$, then $\mathbf{u} \bmod \mathbf{p} = \mathbf{u} - \mathbf{k} \cdot \mathbf{p}$ such that $\|\mathbf{k}\| \le n^2(\alpha+1)$.

Proof. Since $\mathbf{u} \bmod \mathbf{p} = \mathbf{u} - \mathbf{k} \cdot \mathbf{p}$, then $\mathbf{k} = \mathbf{p}^{-1}(\mathbf{u} - \mathbf{u} \bmod \mathbf{p})$. So , we have

$$\begin{aligned}\|\mathbf{k}\| &= \|\mathbf{p}^{-1}(\mathbf{u}-\mathbf{u} \bmod \mathbf{p})\| \\ &\le n\|\mathbf{p}^{-1}\|\|(\mathbf{u}-\mathbf{u} \bmod \mathbf{p})\| \\ &\le n\|\mathbf{p}^{-1}\|(\|\mathbf{u}\|_\infty + \|\mathbf{u} \bmod \mathbf{p}\|) \\ &\le n\|\mathbf{p}^{-1}\|(\|\mathbf{u}\| + \|\mathbf{p}\|) \\ &\le n\|\mathbf{p}^{-1}\|(\alpha\|\mathbf{p}\| + \|\mathbf{p}\|) \\ &\le n^2(\alpha+1)\end{aligned}.$$

So, the proof is complete. ■

Similar as that in the construction of ideal multilinear maps, it is easy to prove that $\mathrm{InstGen}_2$, Enc_2, Add_2, Mul_2 are correct. Here we only require to prove that isZero_2 and Ext_2 are correct.

Lemma 4.3 For an arbitrary integer $k > 0$, the zero-testing algorithm $\mathrm{isZero}_2(\mathrm{par}_2, \mathbf{u}_0)$ correctly determines whether a level-k encoding $\mathbf{u}_0$ is an encoding of zero.

Proof. Given an arbitrary level-k encoding $\mathbf{u}_0$, we have $\mathbf{u}_0 = \mathbf{d} + \mathbf{r}\cdot\mathbf{f}$ and $\|\mathbf{u}_0\| < \|\mathbf{q}\|$ with $\|\mathbf{d}\| < \|\mathbf{f}\|$.

For $t \in [\![\mu - 1]\!]$, $\mathbf{u}_t = \mathbf{u}_{t-1} \bmod \mathbf{q}_t$, then $\|\mathbf{u}_t\| \le \|\mathbf{q}_t\|$. So, we have

$$\|\mathbf{u}_{t-1}\| / \|\mathbf{q}_t\| \le \|\mathbf{q}_{t-1}\| / \|\mathbf{q}_t\| \le n\sigma .$$

By Lemma 4.2 and $\|\mathbf{q}_t^{-1}\| \le n(\|\mathbf{q}_t\|)^{-1}$, we have $\|\mathbf{k}_t\| \le n^2(n\sigma+1)$.

For $t \in [\![\mu - 1]\!]$, $\mathbf{u}_t = \mathbf{u}_{t-1} \bmod \mathbf{q}_t$, $\mathbf{k}_t = (\mathbf{u}_{t-1} - \mathbf{u}_t)/\mathbf{q}_t$, then $\mathbf{u}_t = \mathbf{u}_{t-1} - \mathbf{k}_t \cdot \mathbf{q}_t$.

So, we have

$$\mathbf{u}_0 = \mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{k}_t \cdot \mathbf{q}_t = \mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{k}_t \cdot (\mathbf{e}_t + \mathbf{p}_t \cdot \mathbf{f}).$$

Namely,

$$\mathbf{u}_0 \bmod \mathbf{f} = (\mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{k}_t \cdot (\mathbf{e}_t + \mathbf{p}_t \cdot \mathbf{f})) \bmod \mathbf{f} = (\mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{k}_t \cdot \mathbf{e}_t) \bmod \mathbf{f},$$

$$\mathbf{u}_0 \bmod \mathbf{f}_j = (\mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{k}_t \cdot \mathbf{e}_{t,j}) \bmod \mathbf{f}_j .$$

Moreover, we have

$$\begin{aligned}&\left\|\mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{k}_t \cdot \mathbf{e}_t\right\| \\ &\le \|\mathbf{u}_{\mu-1}\| + \sum_{t=1}^{\mu-1} \|\mathbf{k}_t \cdot \mathbf{e}_t\| \\ &\le \|\mathbf{u}_{\mu-1}\| + n\sum_{t=1}^{\mu-1} \|\mathbf{k}_t\|\|\mathbf{e}_t\| \\ &\le \|\mathbf{e}_{\mu-1} + \mathbf{p}_{\mu-1}\cdot\mathbf{f}\| + n\sum_{t=1}^{\mu-1} \|\mathbf{k}_t\|\|\mathbf{f}\| \\ &\le (\|\mathbf{p}_{\mu-1}\| + 1)\|\mathbf{f}\| + \mu n^3(n\sigma+1)\|\mathbf{f}\| \\ &\le 2\mu n^4 \sigma\beta\end{aligned}.$$

Similarly, we have

$$\begin{aligned}&\left\|\mathbf{u}_{\mu-1}+\sum_{t=1}^{\mu-1}\mathbf{k}_t\cdot\mathbf{e}_{t,j}\right\|\\&\le\left\|\mathbf{u}_{\mu-1}\right\|+\sum_{t=1}^{\mu-1}\left\|\mathbf{k}_t\cdot\mathbf{e}_{t,j}\right\|\\&\le(n\sigma+1)\left\|\mathbf{f}\right\|+n\mu\left\|\mathbf{k}_t\right\|\left\|\mathbf{e}_{t,j}\right\|\\&\le 2n\sigma\beta+n^3\sigma^2\mu\end{aligned}.$$

$$\begin{aligned}\mathbf{p}_{zt}&=\left[\sum_{i=1}^{\tau}\mathbf{r}_i\mathbf{p}_{zt,i}\right]_q\\&=\left[\sum_{i=1}^{\tau}\mathbf{r}_i\sum_{j=1}^{m}\mathbf{h}_j(\mathbf{a}_{i,j}+\mathbf{s}_{0,i,j}\cdot\mathbf{f}_j)\cdot(\mathbf{f}_j^{-1}\bmod q)\right]_q\\&=\left[\sum_{j=1}^{m}\mathbf{h}_j(\sum_{i=1}^{\tau}\mathbf{r}_i\mathbf{a}_{i,j}+\sum_{i=1}^{\tau}\mathbf{r}_i\mathbf{s}_{0,i,j}\cdot\mathbf{f}_j)\cdot(\mathbf{f}_j^{-1}\bmod q)\right]_q,\\&=\left[\sum_{j=1}^{m}\mathbf{h}_j(\mathbf{a}_j+\mathbf{s}_{0,j}\cdot\mathbf{f}_j)\cdot(\mathbf{f}_j^{-1}\bmod q)\right]_q\end{aligned}$$

$$\begin{aligned}\mathbf{p}_{zt,t}&=\left[\sum_{i=1}^{\tau}\mathbf{r}_i\mathbf{p}_{zt,t,i}\right]_q\\&=\left[\sum_{i=1}^{\tau}(\mathbf{r}_i\cdot\sum_{j=1}^{m}\mathbf{h}_j(\mathbf{e}_{t,j}\mathbf{a}_{i,j}+\mathbf{s}_{t,i,j}\mathbf{f}_j)(\mathbf{f}_j^{-1}\bmod q))\right]_q\\&=\left[\sum_{j=1}^{m}\mathbf{h}_j(\sum_{i=1}^{\tau}\mathbf{e}_{t,j}\mathbf{r}_i\mathbf{a}_{i,j}+\sum_{i=1}^{\tau}\mathbf{r}_i\mathbf{s}_{t,i,j}\mathbf{f}_j)(\mathbf{f}_j^{-1}\bmod q)\right]_q,\\&=\left[\sum_{j=1}^{m}\mathbf{h}_j(\mathbf{e}_{t,j}\mathbf{a}_j+\mathbf{s}_{t,j}\mathbf{f}_j)(\mathbf{f}_j^{-1}\bmod q)\right]_q\end{aligned}$$

where $\mathbf{a}_j=\sum_{i=1}^{\tau}\mathbf{r}_i\mathbf{a}_{i,j}$, $\mathbf{s}_{t,j}=\sum_{t=0}^{\mu-1}\mathbf{r}_i\mathbf{s}_{t,i,j}$.

Thus, we have

$$\begin{aligned}&\mathbf{v}\\&=\left[\mathbf{p}_{zt}\cdot\mathbf{u}_{\mu-1}+\sum_{t=1}^{\mu-1}\mathbf{p}_{zt,t}\cdot\mathbf{k}_t\right]_q\\&=\left[\sum_{j=1}^{m}\mathbf{h}_j(\mathbf{a}_j+\mathbf{s}_{0,j}\cdot\mathbf{f}_j)\cdot\left[\mathbf{f}_j^{-1}\right]_q\cdot\mathbf{u}_{\mu-1}+\sum_{t=1}^{\mu-1}\sum_{j=1}^{m}\mathbf{h}_j(\mathbf{e}_{t,j}\mathbf{a}_j+\mathbf{s}_{t,j}\mathbf{f}_j)\left[\mathbf{f}_j^{-1}\right]_q\cdot\mathbf{k}_t\right]_q,\\&=\left[\sum_{j=1}^{m}\mathbf{h}_j\mathbf{a}_j(\mathbf{u}_{\mu-1}+\sum_{t=1}^{\mu-1}\mathbf{k}_t\mathbf{e}_{t,j}+\mathbf{s}_j\mathbf{f}_j)\cdot\left[\mathbf{f}_j^{-1}\right]_q\right]_q\end{aligned}$$

where $\mathbf{s}_j=\sum_{t=0}^{\mu-1}\mathbf{k}_t\mathbf{s}_{t,j}$.

(1) If $\mathbf{u}_0$ is an encoding of zero, then $\mathbf{u}_0\bmod\mathbf{f}_j=0,\ j\in[\![m]\!]$, $\mathbf{u}\bmod\mathbf{f}=0$ and $\mathbf{d}=0$. So, $\mathbf{u}_0\bmod\mathbf{f}_j=(\mathbf{u}_{\mu-1}+\sum_{t=1}^{\mu-1}\mathbf{k}_t\cdot\mathbf{e}_{t,j})\bmod\mathbf{f}_j=0$, $\mathbf{u}_{\mu-1}+\sum_{t=1}^{\mu-1}\mathbf{k}_t\cdot\mathbf{e}_{t,j}=\mathbf{t}_j\mathbf{f}_j$.

Namely, $\mathbf{v}=\left[\sum_{j=1}^{m}\mathbf{h}_j\mathbf{a}_j(\mathbf{t}_j+\mathbf{s}_j)\right]_q=\sum_{j=1}^{m}\mathbf{h}_j\mathbf{a}_j(\mathbf{t}_j+\mathbf{s}_j)$.

Thus, we have

$$
\begin{aligned}
&\|\mathbf{v}\| \\
&= \left\|\sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j (\mathbf{t}_j + \mathbf{s}_j)\right\| \\
&\le m \left\|\mathbf{h}_j \mathbf{a}_j (\mathbf{t}_j + \mathbf{s}_j)\right\| \\
&\le mn^2 \|\mathbf{h}_j\| \|\mathbf{a}_j\| (\|\mathbf{t}_j\| + \|\mathbf{s}_j\|) \\
&\le mn^2 \cdot n\sigma \cdot \tau n^2 \sigma^2 \cdot 2(2n\sigma\beta + n^3\sigma^2\mu) \\
&\le n^{O(1)} \\
&\le q / 2^{\eta}
\end{aligned}
$$

(2) If $\mathbf{u}_0$ is not an encoding of zero, then $\mathbf{u}_0 \bmod \mathbf{f} \neq 0$. That is, $\exists j \in [\![m]\!], \mathbf{d} \bmod \mathbf{f}_j \neq 0$. So, $\mathbf{u}_0 \bmod \mathbf{f}_j = (\mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{k}_t \cdot \mathbf{e}_{t,j}) \bmod \mathbf{f}_j = \mathbf{d} \bmod \mathbf{f}_j$, $\mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{k}_t \cdot \mathbf{e}_{t,j} = \mathbf{t}_j \mathbf{f}_j + \mathbf{d}$.

Thus, we have

$$
\begin{aligned}
\|\mathbf{v}\| &= \left\|\left[\sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j (\mathbf{u}_{\mu-1} + \sum_{t=1}^{\mu-1} \mathbf{k}_t \mathbf{e}_{t,j} + \mathbf{s}_j \mathbf{f}_j) \cdot \left[\mathbf{f}_j^{-1}\right]_q\right]_q\right\| \\
&= \left\|\left[\sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j \left[\mathbf{f}_j^{-1}\right]_q \cdot (\mathbf{d} + \mathbf{t}_j \mathbf{f}_j + \mathbf{s}_j \mathbf{f}_j)\right]_q\right\| \\
&= \left\|\left[\sum_{j=1}^{m} \mathbf{h}_j \mathbf{d} \left[\mathbf{f}_j^{-1}\right]_q + \sum_{j=1}^{m} \mathbf{h}_j (\mathbf{t}_j + \mathbf{s}_j)\right]_q\right\| \\
&\ge \left\|\left[\sum_{j=1}^{m} \mathbf{h}_j \mathbf{d} \left[\mathbf{f}_j^{-1}\right]_q\right]_q\right\| - \left\|\left[\sum_{j=1}^{m} \mathbf{h}_j (\mathbf{t}_j + \mathbf{s}_j)\right]_q\right\| \\
&\ge q^{1-\varepsilon} - q / 2^{\eta} \\
&\ge q^{1-\varepsilon'}
\end{aligned}
$$

So, $\mathrm{isZero}_2(\mathrm{par}_2, \mathbf{u}_0)$ can correctly decide whether the encoding of $\mathbf{u}_0$ is zero. ■

Lemma 4.4 If two level-k encodings $\mathbf{u}_0^{(1)}, \mathbf{u}_0^{(2)}$ encode same level-0 element, then $\mathrm{Ext}_2\left(\mathrm{par}_2, \mathbf{u}_0^{(1)}, \{\mathbf{r}_i\}_{i=1}^{\tau}\right) = \mathrm{Ext}_2\left(\mathrm{par}_2, \mathbf{u}_0^{(2)}, \{\mathbf{r}_i\}_{i=1}^{\tau}\right)$.

Proof. Since $\mathbf{u}_0^{(1)} = \mathbf{u}_0^{(2)} = \mathbf{a}_j(\mathbf{g}_j)^k \bmod \mathbf{f}_j, \forall j \in [\![m]\!]$ and $\mathbf{f}_j, j \in [\![m]\!]$ are co-prime, we have $\mathbf{u}_0^{(1)} = \mathbf{d} + \mathbf{r}^{(1)} \cdot \mathbf{f}, \mathbf{u}_0^{(2)} = \mathbf{d} + \mathbf{r}^{(2)} \cdot \mathbf{f}$.

For $t \in [\![\mu - 1]\!]$, $\mathbf{u}_t^{(1)} = \mathbf{u}_{t-1}^{(1)} \bmod \mathbf{q}_t$, $\mathbf{k}_t^{(1)} = (\mathbf{u}_{t-1}^{(1)} - \mathbf{u}_t^{(1)}) / \mathbf{q}_t$, we have $\mathbf{u}_t^{(1)} = \mathbf{u}_{t-1}^{(1)} - \mathbf{k}_t^{(1)} \cdot \mathbf{q}_t$.

So,

$$
\mathbf{u}_0^{(1)} = \mathbf{u}_{\mu-1}^{(1)} + \sum_{t=1}^{\mu-1} \mathbf{k}_t^{(1)} \cdot \mathbf{q}_t = \mathbf{u}_{\mu-1}^{(1)} + \sum_{t=1}^{\mu-1} \mathbf{k}_t^{(1)} \cdot (\mathbf{e}_t + \mathbf{p}_t \cdot \mathbf{f}).
$$

$$
\begin{aligned}
&\mathbf{u}_0^{(1)} \bmod \mathbf{f} \\
&= (\mathbf{u}_{\mu-1}^{(1)} + \sum_{t=1}^{\mu-1} \mathbf{k}_t^{(1)} \cdot (\mathbf{e}_t + \mathbf{p}_t \cdot \mathbf{f})) \bmod \mathbf{f} \\
&= \mathbf{u}_{\mu-1}^{(1)} + \sum_{t=1}^{\mu-1} \mathbf{k}_t^{(1)} \cdot \mathbf{e}_t) \bmod \mathbf{f} \\
&= \mathbf{d} \bmod \mathbf{f}
\end{aligned}
,
$$

$$
\mathbf{u}_0^{(1)} \bmod \mathbf{f}_j = (\mathbf{u}_{\mu-1}^{(1)} + \sum_{t=1}^{\mu-1} \mathbf{k}_t^{(1)} \cdot \mathbf{e}_{t,j}) \bmod \mathbf{f}_j = \mathbf{d} \bmod \mathbf{f}_j.
$$

Namely, we get $\mathbf{u}_{\mu-1}^{(1)} + \sum_{t=1}^{\mu-1} \mathbf{k}_t^{(1)} \cdot \mathbf{e}_{t,j} = \mathbf{d} + \mathbf{t}_j^{(1)} \mathbf{f}_j$.

So, we have

$$\begin{aligned}
\mathbf{v}^{(1)} &= \left[\mathbf{p}_{zt} \cdot \mathbf{u}_{\mu-1}^{(1)} + \sum_{t=1}^{\mu-1} \mathbf{p}_{zt,t} \cdot \mathbf{k}_t^{(1)} \right]_q \\
&= \left[\sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j (\mathbf{u}_{\mu-1}^{(1)} + \sum_{t=1}^{\mu-1} \mathbf{k}_t^{(1)} \mathbf{e}_{t,j} + \mathbf{s}_j^{(1)} \mathbf{f}_j) \cdot \left[\mathbf{f}_j^{-1} \right]_q \right]_q \\
&= \left[\sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j \left[\mathbf{f}_j^{-1} \right]_q \cdot (\mathbf{d} + \mathbf{t}_j^{(1)} \mathbf{f}_j + \mathbf{s}_j^{(1)} \mathbf{f}_j) \right]_q \\
&= \left[\sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j \left[\mathbf{f}_j^{-1} \right]_q \cdot (\mathbf{d} + (\mathbf{t}_j^{(1)} + \mathbf{s}_j^{(1)}) \mathbf{f}_j) \right]_q
\end{aligned}.$$

Similarly, we have

$$\begin{aligned}
\mathbf{v}^{(2)} &= \left[\mathbf{p}_{zt} \cdot \mathbf{u}_{\mu-1}^{(2)} + \sum_{t=1}^{\mu-1} \mathbf{p}_{zt,t} \cdot \mathbf{k}_t^{(2)} \right]_q \\
&= \left[\sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j \left[\mathbf{f}_j^{-1} \right]_q \cdot (\mathbf{d} + (\mathbf{t}_j^{(2)} + \mathbf{s}_j^{(2)}) \mathbf{f}_j) \right]_q
\end{aligned}.$$

By Lemma 4.3, we have

$$\left\| \sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j \cdot (\mathbf{t}_j^{(1)} + \mathbf{s}_j^{(1)}) \right\| \le q / 2^{\eta},$$

$$\left\| \sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j \cdot (\mathbf{t}_j^{(2)} + \mathbf{s}_j^{(2)}) \right\| \le q / 2^{\eta}.$$

Hence, the most significant bits of $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are decided by $\sum_{j=1}^{m} \mathbf{h}_j \mathbf{a}_j \mathbf{d} \cdot \left[\mathbf{f}_j^{-1} \right]_q$.

Namely, $\mathrm{Ext}_2\left(\mathrm{par}_2, \mathbf{u}_0^{(1)}, \{\mathbf{r}_i\}_{i=1}^{\tau}\right) = \mathrm{Ext}_2\left(\mathrm{par}_2, \mathbf{u}_0^{(2)}, \{\mathbf{r}_i\}_{i=1}^{\tau}\right)$. ■

4.3 Security

Variant security depends on new assumptions, and the extraction variant of GCDH/GDDH is called ext-GCDH/ext-GDDH. The ext-GCDH/ext-GDDH is introduced in [LSS14] to prove the security of the GGHLite scheme. We describe the security experiment of our variant below:

(1) $(\mathrm{par}_2) \leftarrow \mathrm{InstGen}_2(1^{\lambda})$.

(2) Choose an arbitrary positive integer k.

(3) For $t = 0$ to k:

Sample $\mathbf{w}_{t,i} \leftarrow D_{\mathbb{Z}^n,\sigma}, i \in [\![\tau]\!]$,

Generate level-1 encoding $\mathbf{u}_t = (\sum_{i=1}^{\tau} \mathbf{w}_{t,i} \cdot \mathbf{y}_i) \bmod \mathbf{q}$.

(4) Sample $\mathbf{r}_{0,i} \leftarrow D_{\mathbb{Z}^n,\sigma'}, i \in [\![\tau]\!]$.

(5) Compute $\mathbf{u} = \prod_{t=1}^{k} \mathbf{u}_t \bmod \mathbf{q}$.

(6) Set $\mathbf{v}_C = \mathbf{v}_D = \mathrm{Ext}_2\left(\mathrm{par}_2, \mathbf{u}, \{\mathbf{w}_{0,i}\}_{i=1}^{\tau}\right)$.

(7) Set $\mathbf{v}_R = \mathrm{Ext}_2\left(\mathrm{par}_2, \mathbf{u}, \{\mathbf{r}_{0,i}\}_{i=1}^{\tau}\right)$..

Definition 4.5 The extraction k-graded CDH problem (ext-GCDH) is, on input $\{\mathrm{par}_2, \mathbf{u}_0, \cdots, \mathbf{u}_k\}$, to output an extraction encoding $\mathbf{v}_C$. The extraction k-graded DDH problem (ext-GDDH) distinguishes between $\mathbf{v}_D$ and $\mathbf{v}_R$, that is, between the distributions

$D_{GDDH} = \{\text{par}_2, \mathbf{u}_0, \cdots, \mathbf{u}_k, \mathbf{v}_D\}$ and $D_{RAND} = \{\text{par}_2, \mathbf{u}_0, \cdots, \mathbf{u}_k, \mathbf{v}_R\}$.

4.4 Cryptanalysis

4.4.1 Easily computable quantities

Since $\mathbf{q} = \mathbf{p}_0 \cdot \mathbf{f}$, one can compute $\det(Rot(\mathbf{q}))$. If one can factor $\det(Rot(\mathbf{q}))$ to get $\det(Rot(\mathbf{q}_0))$ and $p_j = \det(Rot(\mathbf{f}_j))$, then one obtain principal ideal lattice generated by two elements (p_j, α_j), and reduce the principal ideal lattice to find $\mathbf{f}_j$. However, there currently exists efficient algorithm which computes small generator of principal ideal lattice.

Because $\mathbf{q}_i = \mathbf{e}_i + \mathbf{p}_i \cdot \mathbf{f}$, $\mathbf{p}_{zt,i} = \left[\sum_{j=1}^{m} \mathbf{h}_j (\mathbf{e}_{i,j} + \mathbf{r}_{i,j} \mathbf{f}_j)(\left[\mathbf{f}_j^{-1}\right]_q) \right]_q$, $i \in [\![\mu - 1]\!]$, one can compute cross-multiplying $\mathbf{z}_{i_1,i_2} = \left[\mathbf{q}_{i_1} \mathbf{p}_{zt,i_2} - \mathbf{q}_{i_2} \mathbf{p}_{zt,i_1} \right]_q$. It is easy to see that $\mathbf{z}_{i_1,i_2}$ is not reduced modulo q for some α such that $i_1, i_2 \geq \alpha \geq 1$. However, one cannot get common factor from $\mathbf{z}_{i_1,i_2}$.

5 Ideal multilinear maps using unimodular matrix

For the above ideal constructions, there are mainly two problems: (1) One can compute a basis of secret ideal lattice using the public parameters in Section 3; (2) One can only generate usable plaintext in level-1 encoding, because the plaintext of encoding is included in corresponding zero-testing parameter, and is not known.

In this section, we introduce unimodular matrices to solve the above problems. The reason using unimodular matrix is our construction work over the integers. We apply different unimodular matrices for level-0 encoding and level-1 encoding to thwart cross-multiplying of encodings.

5.1 Construction

Setting the parameters. Let λ be the security parameter, n the dimension of elements of R. Concrete parameters are set as $\sigma = \sqrt{\lambda n}$, $\sigma' = \lambda n^{1.5}$, $q \geq 2^{\eta + O(\lambda)} n^{O(1)}$, $n = O(\lambda^2)$, $m = O(\lambda)$, $l = O(n^2)$, $\tau = O(n^2)$.

Instance generation: $(\text{par}_3) \leftarrow \text{InstGen}_3(1^\lambda)$.

(1) Choose a large enough prime q.

(2) Sample $\mathbf{f}_j \leftarrow D_{\mathbb{Z}^n,\sigma}$, $j \in [\![m]\!]$ and $\mathbf{p}_0 \leftarrow D_{\mathbb{Z}^n,\sigma'}$ such that the ideal lattices $\mathbf{f}_j$'s are coprime, $\mathbf{f}_j^{-1} \in \mathbb{k}$ and $\left\| \mathbf{f}_j^{-1} \right\| \leq l$.

(3) Set $\mathbf{f} = \prod_{j=1}^{m} \mathbf{f}_j$, $\mathbf{q} = \mathbf{p}_0 \cdot \mathbf{f}$ and $\mu = \|\mathbf{f}\|$.

(4) Sample $\mathbf{g} \leftarrow D_{\mathbb{Z}^n,\mu}, \mathbf{a}_i \leftarrow D_{\mathbb{Z}^n,\mu}, \mathbf{e}_i, \mathbf{t}_i \leftarrow D_{\mathbb{Z}^n,\sigma'}, i \in [\![\tau]\!]$.

(5) Set $\mathbf{x}_i = (\mathbf{a}_i + \mathbf{e}_i \mathbf{f}) \bmod \mathbf{q}$, $\mathbf{y}_i = (\mathbf{a}_i \mathbf{g} + \mathbf{t}_i \mathbf{f}) \bmod \mathbf{q}, i \in [\![\tau]\!]$.

(6) Set $\mathbf{p}_{zt} = (\sum_{j=1}^{m} \mathbf{h}_j \cdot (\mathbf{f}_j^{-1} \bmod q)) \bmod q$, where $\mathbf{h}_j \leftarrow D_{\mathbb{Z}^n,\sigma}$, $j \in [\![m]\!]$.

(7) Sample randomly two unimodular matrices $\mathbf{S}, \mathbf{T}$ such that $\|\mathbf{S}\| = \|\mathbf{T}\| \leq n^{O(1)}$.

(8) Set $\mathbf{X}_i = \mathbf{S}Rot(\mathbf{x}_i)\mathbf{S}^{-1}, \mathbf{Y}_i = \mathbf{T}^{-1}Rot(\mathbf{y}_i)\mathbf{T}, i \in [\![\tau]\!]$ and $\mathbf{P}_{zt} = [\mathbf{S}Rot(\mathbf{p}_{zt})\mathbf{T}]_q$.

(9) Set $\mathbf{Q}_{x,t} = \mathbf{S}Rot(\mathbf{b}_{x,t}\mathbf{q})\mathbf{S}^{-1}$, $\mathbf{Q}_{y,t} = \mathbf{T}^{-1}Rot(\mathbf{b}_{y,t}\mathbf{q})\mathbf{T}$, where $\mathbf{b}_{x,t}, \mathbf{b}_{y,t} \leftarrow D_{\mathbb{Z}^n,\sigma}$, $t \in [\![n]\!]$. Let the columns of matrices $\mathbf{Q}_x, \mathbf{Q}_y \in \mathbb{Z}^{n^2 \times n}$ be $\mathbf{Q}_{x,t}, \mathbf{Q}_{y,t}$, respectively, where $\mathbf{Q}_{x,t}, \mathbf{Q}_{y,t}$ are arrange as n^2-dimension column vectors.

(10) Output the public parameters $\text{par}_3 = \left\{q, \mathbf{Q}_x, \mathbf{Q}_y, \{\mathbf{X}_i, \mathbf{Y}_i\}_{i \in [\![\tau]\!]}, \mathbf{P}_{zt}\right\}$.

Generating level-k random encodings: $\mathbf{U} \leftarrow \text{Enc}_3(\text{par}_3, k, \{w_i\}_{i \in [\![\tau]\!]})$.

Choose $w_i \leftarrow D_{\mathbb{Z},\sigma'}$, $i \in [\![\tau]\!]$, generate a level-0 encoding $\mathbf{D} = \sum_{i=1}^{\tau} w_i \cdot (\mathbf{X}_i)^k \bmod \mathbf{Q}_x$ and a level-k encoding $\mathbf{U} = \sum_{i=1}^{\tau} w_i \cdot (\mathbf{Y}_i)^k \bmod \mathbf{Q}_y$.

Adding encodings: $\mathbf{U} \leftarrow \text{Add}_3(\text{par}_3, k, \mathbf{U}_1, \cdots, \mathbf{U}_s)$.

Given s level-k encodings $\mathbf{U}_t$, $t \in [\![s]\!]$, their sum $\mathbf{U} = \sum_{t=1}^{s} \mathbf{U}_t \bmod \mathbf{Q}_y$ is a level-k encoding.

Multiplying encodings: $\mathbf{U} \leftarrow \text{Mul}_3(\text{par}_3, 1, \mathbf{U}_1, \cdots, \mathbf{U}_k)$.

Given k level-1 encodings $\mathbf{U}_t, t \in [\![k]\!]$, their product $\mathbf{U} = \prod_{t=1}^{k} \mathbf{U}_t \bmod \mathbf{Q}_y$ is a level-k encoding.

Zero Testing: $\text{isZero}_3(\text{par}_3, \mathbf{D}, \mathbf{U})$.

Given a level-k encoding $\mathbf{U}$ and a level-0 encoding $\mathbf{D}$, determine whether $\mathbf{U}$ is an encoding of zero. $\mathbf{V} = [\mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}]_q$ is computed in $\mathbb{Z}_q$ and checked whether $\|\mathbf{V}\|$ is short:

$$\text{isZero}_3(\text{par}_3, \mathbf{D}, \mathbf{U}) = \begin{cases} 1 & \text{if } \left\|[\mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}]_q\right\| < q/2^{\eta} \\ 0 & \text{otherwise} \end{cases}.$$

Extract: $sk \leftarrow \text{Ext}_3(\text{par}_3, \mathbf{U})$.

Given a level-k encoding $\mathbf{U}$ and a level-0 encoding $\mathbf{D}$, $\text{Ext}_3(\text{par}_3, \mathbf{D}, \mathbf{U}) = \text{Extract}_s(\text{msbs}_{\eta}([\mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}]_q))$.

Remark 5.1 (1) Unimodular matrices can be obtained by generating two lists of upper/lower triangular unimodular matrices $\mathbf{S}_i, \mathbf{T}_i \in \{0, 1, -1\}^{n \times n}$, $i \in [\![\beta]\!]$, and set $\mathbf{S} = \prod_{i=1}^{\beta} \mathbf{S}_i$, $\mathbf{T} = \prod_{i=1}^{\beta} \mathbf{T}_i$. (2) Since the dimension of R is n, the rank of the matrices generated by column vectors $\mathbf{Q}_{x,t}, \mathbf{Q}_{y,t}$ is n with high probability. So, one only require to provide n encodings $\mathbf{Q}_{x,t}, \mathbf{Q}_{y,t}$ of zero to reduce encodings into the parallelization of $\mathbf{Q}_x, \mathbf{Q}_y$. (3) When our construction is applied to multipartite Diffie-Hellman key exchange, $\mathbf{P}_{zt}, \mathbf{X}_i$ in the public parameters can be replaced by $\mathbf{P}_{zt,i} = [\mathbf{X}_i \mathbf{P}_{zt}]_q$, and in this case $\mathbf{S}$ does not require unimodular matrix.

5.2 Correctness

Lemma 5.2 $\text{InstGen}_3(1^{\lambda})$ is a probabilistic polynomial time algorithm.

Proof. One can efficiently generate a prime q. By [GGH13], one can sample $\mathbf{f}_j$ such that $\mathbf{f}_j^{-1} \in \mathbb{k}$ and $\left\|\mathbf{f}_j^{-1}\right\| \le l$ with high probability, and compute $\mathbf{f}$ and $\mathbf{q}$.

One can sample $\mathbf{g}, \mathbf{a}_i, \mathbf{e}_i, \mathbf{t}_i$ and compute $\mathbf{x}_i, \mathbf{y}_i, \mathbf{p}_{zt}$ can be generated in polynomial time.

Moreover, one can efficiently sample $\mathbf{S}, \mathbf{T}, \mathbf{b}_{x,t}, \mathbf{b}_{y,t}$ and compute $\mathbf{Q}_{x,t}, \mathbf{Q}_{y,t}$. It is easy to prove that the rank of $\mathbf{Q}_x, \mathbf{Q}_y$ is n with high probability. ■

Lemma 5.3 $\mathbf{U} \leftarrow \mathrm{Enc}_3(\mathrm{par}_3, k, \{w_i\}_{i\in[\![\tau]\!]})$ is a level-k encoding.

Proof. Since $\mathbf{D} = \sum_{i=1}^{\tau} w_i \cdot (\mathbf{X}_i)^k \bmod \mathbf{Q}_x$, we have

$$\begin{aligned}
\mathbf{D} &= \textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{X}_i)^k \bmod \mathbf{Q}_x \\
&= \textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{X}_i)^k - \sum_{t=1}^{n} \alpha_{x,t} \cdot \mathbf{Q}_{x,t} \\
&= \mathbf{S}(\textstyle\sum_{i=1}^{\tau} w_i \cdot Rot(\mathbf{x}_i)^k - \sum_{t=1}^{n} \alpha_{x,t} \cdot Rot(\mathbf{b}_{x,t}\mathbf{q}))\mathbf{S}^{-1} \\
&= \mathbf{S}Rot(\mathbf{d})\mathbf{S}^{-1}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{d} \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{x}_i)^k - \sum_{t=1}^{n} \alpha_{x,t} \cdot (\mathbf{b}_{x,t}\mathbf{q})) \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{x}_i)^k) \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot ((\mathbf{a}_i + \mathbf{e}_i\mathbf{f}) \bmod \mathbf{q})^k) \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{a}_i \bmod \mathbf{f}_j)^k) \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{a}_{i,j})^k) \bmod \mathbf{f}_j
\end{aligned}$$

Since $\mathbf{U} = \sum_{i=1}^{\tau} w_i \cdot (\mathbf{Y}_i)^k \bmod \mathbf{Q}_y$, we have

$$\begin{aligned}
\mathbf{U} &= \textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{Y}_i)^k \bmod \mathbf{Q}_y \\
&= \textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{Y}_i)^k - \sum_{t=1}^{n} \alpha_{y,t} \cdot \mathbf{Q}_{y,t} \\
&= \mathbf{T}^{-1}(\textstyle\sum_{i=1}^{\tau} w_i \cdot Rot(\mathbf{y}_i)^k - \sum_{t=1}^{n} \alpha_{y,t} \cdot Rot(\mathbf{b}_{y,t}\mathbf{q}))\mathbf{T} \\
&= \mathbf{T}^{-1}Rot(\mathbf{u})\mathbf{T}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{u} \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{y}_i)^k - \sum_{t=1}^{n} \alpha_{y,t} \cdot (\mathbf{b}_{y,t}\mathbf{q})) \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{y}_i)^k) \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot ((\mathbf{a}_i\mathbf{g} + \mathbf{t}_i\mathbf{f}) \bmod \mathbf{q})^k) \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{a}_i \bmod \mathbf{f}_j)^k (\mathbf{g} \bmod \mathbf{f}_j)^k) \bmod \mathbf{f}_j \\
&= (\textstyle\sum_{i=1}^{\tau} w_i \cdot (\mathbf{a}_{i,j})^k \cdot (\mathbf{g}_j)^k \bmod \mathbf{f}_j \\
&= ((\mathbf{d} \bmod \mathbf{f}_j) \cdot (\mathbf{g}_j)^k \bmod \mathbf{f}_j
\end{aligned}$$

So, $\mathbf{U}$ is a level-k encoding of $\mathbf{D}$. ■

Lemma 5.4 $\mathbf{U} \leftarrow \mathrm{Add}_3(\mathrm{par}_3, k, \mathbf{U}_1, \cdots, \mathbf{U}_s)$ is a level-k encoding.

Proof. Using rule of modulo operation, it is easy to prove that the sum $\mathbf{U}$ of level-k encodings $\mathbf{U}_t, t \in [\![s]\!]$ is a level-k encoding. ■

Lemma 5.5 $\mathbf{U} \leftarrow \mathrm{Mul}_3(\mathrm{par}_3, 1, \mathbf{U}_1, \cdots, \mathbf{U}_k)$ is a level-k encoding.

Proof. Since $\mathbf{U}_t$, $t \in [\![k]\!]$ are level-1 encodings, we have $\mathbf{U}_t = \mathbf{T}^{-1} Rot(\mathbf{u}_t)\mathbf{T}$ and $\mathbf{u}_t \bmod \mathbf{f}_j = \mathbf{u}_{t,j}\mathbf{g}_j$. So, we get

$$\mathbf{U} = \prod_{t=1}^{k} \mathbf{U}_t \bmod \mathbf{Q}_y = \mathbf{T}^{-1} Rot(\mathbf{u})\mathbf{T},$$

$$\begin{aligned}
&\mathbf{u} \bmod \mathbf{f}_j \\
&= (\prod_{t=1}^{k} \mathbf{u}_t \bmod \mathbf{q}) \bmod \mathbf{f}_j \\
&= (\prod_{t=1}^{k} (\mathbf{u}_t \bmod \mathbf{f}_j)) \bmod \mathbf{f}_j . \\
&= (\prod_{t=1}^{k} \mathbf{u}_{t,j}\mathbf{g}_j) \bmod \mathbf{f}_j \\
&= (\prod_{t=1}^{k} \mathbf{u}_{t,j})(\mathbf{g}_j)^k \bmod \mathbf{f}_j
\end{aligned}$$

So, $\mathbf{U}$ is a level-k encoding. ■

Lemma 5.6 For an arbitrary integer $k > 0$, the zero-testing algorithm $\mathrm{isZero}_3(\mathrm{par}_3, \mathbf{D}, \mathbf{U})$ correctly determines whether a level-k encoding $\mathbf{U}$ is an encoding of zero.

Proof. Given a level-0 encoding $\mathbf{D}$ and an arbitrary level-k encoding $\mathbf{U}$, we have $\mathbf{D} = \mathbf{S}Rot(\mathbf{d})\mathbf{S}^{-1}$, $\mathbf{U} = \mathbf{T}^{-1}Rot(\mathbf{u})\mathbf{T}$, $\mathbf{u} = \mathbf{a} + \mathbf{r} \cdot \mathbf{f}$ and $\|\mathbf{d}\| \le \|\mathbf{S}\|\|\mathbf{D}\|\|\mathbf{S}^{-1}\|$, $\|\mathbf{a}\| < \|\mathbf{f}\|$, $\|\mathbf{u}\| \le \|\mathbf{T}\|\|\mathbf{U}\|\|\mathbf{T}^{-1}\|$.

(1) If $\mathbf{U}$ is an encoding of zero, then $\mathbf{u} \bmod \mathbf{f}_j = 0, j \in [\![m]\!]$. Since $\mathbf{f}_j, j \in [\![m]\!]$ are co-prime, $\mathbf{u} \bmod \mathbf{f} = 0$. That is, $\mathbf{d} \bmod \mathbf{f} = 0$ and $\mathbf{d} = 0$ according to $\|\mathbf{d}\| < \|\mathbf{f}\|$. So, we have

$$\begin{aligned}
\|\mathbf{V}\| &= \left\| [\mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}]_q \right\| \\
&= \left\| \left[\mathbf{S}Rot(\mathbf{d})\mathbf{S}^{-1} \cdot \mathbf{S}Rot(\mathbf{p}_{zt})\mathbf{T} \cdot \mathbf{T}^{-1}Rot(\mathbf{u})\mathbf{T} \right]_q \right\| \\
&= \left\| [\mathbf{S}Rot(\mathbf{d} \cdot \mathbf{p}_{zt} \cdot \mathbf{u})\mathbf{T}]_q \right\| \\
&= \left\| \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{d} \cdot \mathbf{h}_j \cdot [\mathbf{f}_j^{-1}]_q \cdot \mathbf{r} \cdot \mathbf{f})\mathbf{T} \right]_q \right\| \\
&= \left\| \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{d}\mathbf{h}_j\mathbf{r} \cdot \mathbf{f} / \mathbf{f}_j)\mathbf{T} \right]_q \right\| \qquad . \\
&\le n^2 \|\mathbf{S}\| \left\| Rot(\sum_{j=1}^{m} \mathbf{d}\mathbf{h}_j\mathbf{r} \cdot \mathbf{f} / \mathbf{f}_j) \right\| \|\mathbf{T}\| \\
&\le n^2 \|\mathbf{S}\| \left\| \sum_{j=1}^{m} \mathbf{d}\mathbf{h}_j\mathbf{r} \cdot \mathbf{f} / \mathbf{f}_j \right\| \|\mathbf{T}\| \\
&\le mn^5 \|\mathbf{S}\| \|\mathbf{d}\| \|\mathbf{h}_j\| \cdot \|\mathbf{r}\| \cdot \|\mathbf{f} / \mathbf{f}_j\| \|\mathbf{T}\| \\
&\le q / 2^{\eta}
\end{aligned}$$

(2) If $\mathbf{U}$ is not an encoding of zero, then $\mathbf{u} \bmod \mathbf{f} \neq 0$. That is,

$\exists j \in [\![m]\!], \mathbf{a} \bmod \mathbf{f}_j \neq 0$. So, we have

$$\begin{aligned}
\|\mathbf{V}\| &= \left\| \left[\mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}\right]_q \right\| \\
&= \left\| \left[\mathbf{S}Rot(\mathbf{d} \cdot \mathbf{p}_{zt} \cdot \mathbf{u})\mathbf{T}\right]_q \right\| \\
&= \left\| \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \cdot \left[\mathbf{f}_j^{-1}\right]_q \cdot (\mathbf{a} + \mathbf{rf})\mathbf{T} \right]_q \right\| \\
&= \left\| \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{a} \cdot \left[\mathbf{f}_j^{-1}\right]_q \mathbf{T} + \mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{r} \cdot \mathbf{f} / \mathbf{f}_j)\mathbf{T} \right]_q \right\| \\
&\geq \left\| \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{a} \cdot \left[\mathbf{f}_j^{-1}\right]_q \mathbf{T} \right]_q \right\| - n^2 \|\mathbf{S}\| \left\| Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{r} \cdot \mathbf{f} / \mathbf{f}_j) \right\| \|\mathbf{T}\| \\
&\geq q^{1-\varepsilon} - q / 2^{\eta} \\
&\geq q^{1-\varepsilon'}
\end{aligned}$$

So, $\mathrm{isZero}_3(\mathrm{par}_3, \mathbf{D}, \mathbf{U})$ correctly decides the encoding of $\mathbf{U}$. ■

Lemma 5.7 Given a level-0 encoding $\mathbf{D} = \mathbf{S}Rot(\mathbf{d})\mathbf{S}^{-1}$ and two level-k encodings $\mathbf{U}_t = \mathbf{T}^{-1}Rot(\mathbf{u}_t)\mathbf{T}$, $t \in [\![2]\!]$. If $\mathbf{U}_t$, $t \in [\![2]\!]$ encode same level-0 element, namely $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{a}_j(\mathbf{g}_j)^k \bmod \mathbf{f}_j, \forall j \in [\![m]\!]$, then

$$\mathrm{Ext}_3(\mathrm{par}_3, \mathbf{D}, \mathbf{U}_1) = \mathrm{Ext}_3(\mathrm{par}_3, \mathbf{D}, \mathbf{U}_2), \text{ where.}$$

Proof. Since $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{a}_j(\mathbf{g}_j)^k \bmod \mathbf{f}_j, \forall j \in [\![m]\!]$ and $\mathbf{f}_j, j \in [\![m]\!]$ are co-prime, we get $\mathbf{u}_1 = \mathbf{a} + \mathbf{r}_1 \cdot \mathbf{f}, \mathbf{u}_2 = \mathbf{a} + \mathbf{r}_2 \cdot \mathbf{f}$. So, we have

$$\begin{aligned}
\left[\mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}_1\right]_q &= \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \cdot \left[\mathbf{f}_j^{-1}\right]_q \cdot (\mathbf{a} + \mathbf{r}_1\mathbf{f}))\mathbf{T} \right]_q \\
&= \left(\left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{a} \cdot \left[\mathbf{f}_j^{-1}\right]_q)\mathbf{T} \right]_q + \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{r}_1 \cdot \mathbf{f} / \mathbf{f}_j)\mathbf{T} \right]_q \right) \bmod q \\
\left[\mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}_2\right]_q &= \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \cdot \left[\mathbf{f}_j^{-1}\right]_q \cdot (\mathbf{a} + \mathbf{r}_2\mathbf{f}))\mathbf{T} \right]_q \\
&= \left(\left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{a} \cdot \left[\mathbf{f}_j^{-1}\right]_q)\mathbf{T} \right]_q + \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{r}_2 \cdot \mathbf{f} / \mathbf{f}_j)\mathbf{T} \right]_q \right) \bmod q
\end{aligned}$$

By Lemma 5.6, we have that $\left\| \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{r}_t \cdot \mathbf{f} / \mathbf{f}_j)\mathbf{T} \right]_q \right\| \leq q / 2^{\eta}, t \in [\![2]\!]$. Hence, the most significant bits of $\left[\mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}_1\right]_q, \left[\mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}_2\right]_q$ are all decided by the first term $\left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{dh}_j \mathbf{a} \cdot \left[\mathbf{f}_j^{-1}\right]_q)\mathbf{T} \right]_q$. ■

5.3 Security

The security of our constructions depends on new hardness assumptions, and seems to rely on hardness to solve shortest generator problems for principal ideal lattices. However, at present we do not know how to reduce the security of our construction to the shortest principal ideal generator problems.

In the following, we adapt the definition of ext-GCDH/ext-GDDH in [LSS14] to our constructions. Consider the following process:

(1) $(\mathrm{par}_3) \leftarrow \mathrm{InstGen}_3(1^{\lambda})$.

(2) Choose an arbitrary positive integer k.

(3) For $t=0$ to k:

Sample $w_{t,i} \leftarrow D_{\mathbb{Z},\sigma'}, i \in [\![\tau]\!]$

Generate level-1 encoding of $\mathbf{D}_t = (\sum_{i=1}^{\tau} w_{t,i} \cdot \mathbf{X}_i) \bmod \mathbf{Q}_x$:

$\mathbf{U}_t = (\sum_{i=1}^{\tau} w_{t,i} \cdot \mathbf{Y}_i) \bmod \mathbf{Q}_y$.

(4) Sample $r_{0,i} \leftarrow D_{\mathbb{Z},\sigma'}, i \in [\![\tau]\!]$ and generate $\mathbf{R}_0 = (\sum_{i=1}^{\tau} r_{0,i} \cdot \mathbf{X}_i) \bmod \mathbf{Q}_x$.

(5) Compute $\mathbf{U} = \prod_{t=1}^{k} \mathbf{U}_t \bmod \mathbf{Q}_y$.

(6) Set $\mathbf{V}_C = \mathbf{V}_D = \mathrm{Ext}_3(\mathrm{par}_3, \mathbf{D}_0, \mathbf{U})$.

(7) Set $\mathbf{V}_R = \mathrm{Ext}_3(\mathrm{par}_3, \mathbf{R}_0, \mathbf{U})$.

Definition 3.8 (ideal-ext-GCDH/ideal-ext-GDDH). The extraction k-graded CDH problem (ideal-ext-GCDH) is, on input $\{\mathrm{par}_3, \mathbf{U}_0, \cdots, \mathbf{U}_k\}$, to output an extraction encoding $\mathbf{V}_C \in \mathbb{Z}^{n \times n}$. The extraction k-graded DDH problem (ideal-ext-GDDH) distinguishes between $\mathbf{V}_D$ and $\mathbf{V}_R$, that is, between the distributions $D_{GDDH} = \{\mathrm{par}_3, \mathbf{U}_0, \cdots, \mathbf{U}_k, \mathbf{V}_D\}$ and $D_{RAND} = \{\mathrm{par}_3, \mathbf{U}_0, \cdots, \mathbf{U}_k, \mathbf{V}_R\}$.

Similarly, we assume the ideal-ext-GCDH/ideal-ext-GDDH is hard in the definition 3.8.

5.4 Cryptanalysis

5.4.1 Easily computable quantities

Since $\mathbf{Q}_{x,t} = \mathbf{S}Rot(\mathbf{b}_{x,t}\mathbf{q})\mathbf{S}^{-1}, \mathbf{Q}_{y,t} = \mathbf{T}^{-1}Rot(\mathbf{b}_{y,t}\mathbf{q})\mathbf{T}$ we have

$$\det(\mathbf{Q}_{x,t}) = \det(\mathbf{S}Rot(\mathbf{b}_{x,t}\mathbf{q})\mathbf{S}^{-1}) = \det(Rot(\mathbf{b}_{x,t}\mathbf{q})) = \det(Rot(\mathbf{b}_{x,t}))\det(Rot(\mathbf{q})),$$

$$\det(\mathbf{Q}_{y,t}) = \det(\mathbf{T}^{-1}Rot(\mathbf{b}_{y,t}\mathbf{q})\mathbf{T}) = \det(Rot(\mathbf{b}_{y,t}\mathbf{q})) = \det(Rot(\mathbf{b}_{y,t}))\det(Rot(\mathbf{q})),$$

where $\mathbf{S}, \mathbf{T}$ are unimodular matrices.

So, one can obtain $\det(Rot(\mathbf{q}))$ using Euclid GCD algorithm.

Again since $\mathbf{Q}_{y,t} = \mathbf{T}^{-1}Rot(\mathbf{b}_{y,t}\mathbf{q})\mathbf{T}$ is encoding of zero, we have

$$\begin{aligned}
\mathbf{V}_t &= \left[\mathbf{P}_{zt}\mathbf{Q}_{y,t}\right]_q \\
&= \left[\mathbf{S}Rot(\mathbf{p}_{zt})\mathbf{T} \cdot \mathbf{T}^{-1}Rot(\mathbf{b}_{y,t}\mathbf{q})\mathbf{T}\right]_q \\
&= \left[\mathbf{S}Rot(\mathbf{p}_{zt}\mathbf{b}_{y,t}\mathbf{q})\mathbf{T}\right]_q \\
&= \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{h}_j\mathbf{b}_{y,t}\mathbf{p}_0 \cdot \mathbf{f} \cdot \left[\mathbf{f}_j^{-1}\right]_q)\mathbf{T}\right]_q \\
&= \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{h}_j\mathbf{b}_{y,t}\mathbf{p}_0 \cdot \mathbf{f}/\mathbf{f}_j)\mathbf{T}\right]_q
\end{aligned}$$

By our parameter setting, we get $\mathbf{V}_t = \left[\mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{h}_j\mathbf{b}_{y,t}\mathbf{p}_0 \cdot \mathbf{f}/\mathbf{f}_j)\mathbf{T}\right]_q$ is not reduced modulo q. Namely, $\mathbf{V}_t = \mathbf{S}Rot(\sum_{j=1}^{m} \mathbf{h}_j\mathbf{b}_{y,t}\mathbf{p}_0 \cdot \mathbf{f}/\mathbf{f}_j)\mathbf{T}$. Similarly, one can obtain $\det(Rot(\mathbf{q}_0))$.

Using $\mathbf{q} = \mathbf{p}_0\mathbf{f}$, we have $\det(Rot(\mathbf{f})) = \det(Rot(\mathbf{q}))/\det(Rot(\mathbf{p}_0))$. However, only

given $\det(Rot(\mathbf{f}))$, there currently exists no efficient algorithm which finds $\mathbf{f}$.
When our construction uses the method in Section 4, then one cannot compute $\det(Rot(\mathbf{q}_0))$, $\det(Rot(\mathbf{f}))$. However, if $\det(Rot(\mathbf{q}))$ can be factored, then $\det(Rot(\mathbf{f}))$ can also be revealed. Identically, $\mathbf{f}$ cannot be efficiently solved.

5.4.2 Lattice reduction attack

Since $\mathbf{Q}_{y,t} = \mathbf{T}^{-1} Rot(\mathbf{b}_{y,t}\mathbf{q})\mathbf{T}, \mathbf{Q}_{x,t} = \mathbf{S} Rot(\mathbf{b}_{x,t}\mathbf{q})\mathbf{S}^{-1}$ are all over the integers, one can attempt to use lattice reduction algorithm to find the secret elements in our construction. Using $\mathbf{Q}_{y,t}, \mathbf{Q}_{x,t}$, we generate lattices as follows:

$$L_y(t_1,t_2) = \begin{pmatrix} \mathbf{Q}_{y,t_1} \\ \mathbf{Q}_{y,t_2} \end{pmatrix}, \quad L_x(t_1,t_2) = \begin{pmatrix} \mathbf{Q}_{x,t_1} \\ \mathbf{Q}_{x,t_2} \end{pmatrix}.$$

Applying lattice reduction algorithm, we get $\mathbf{E}_y Rot(\mathbf{q})\mathbf{T}$ and $\mathbf{E}_x Rot(\mathbf{q})\mathbf{S}^{-1}$. According to current lattice reduction algorithm, in general $\mathbf{E}_y$ or $\mathbf{E}_x$ is not an identity matrix. If $\mathbf{E}_y = \mathbf{E}_x = \mathbf{I}$, then one can further get $Rot(\mathbf{q})$ by $Rot(\mathbf{q})\mathbf{T}$ and $Rot(\mathbf{q})\mathbf{S}^{-1}$, and finally solve $\mathbf{T}, \mathbf{S}$. After known $\mathbf{T}, \mathbf{S}$, how to break our construction is described in the following subsection.
Thus, one must set a large enough dimension in the construction to thwart the lattice reduction attack.

5.4.3 Attack known T, S

Given $\mathbf{T}$, $\mathbf{S}$, one computes $\mathbf{q}_{y,t} = \mathbf{b}_{y,t}\mathbf{q} = \mathbf{T}\mathbf{Q}_{y,t}\mathbf{T}^{-1}$, $\mathbf{q}_{x,t} = \mathbf{b}_{x,t}\mathbf{q} = \mathbf{S}^{-1}\mathbf{Q}_{x,t}\mathbf{S}$. Using $\mathbf{q}_{y,t}, \mathbf{q}_{x,t}$, one can solve a basis $\mathbf{Q} = \mathbf{E}_1 Rot(\mathbf{q})$ of $\mathbf{q}$.
By $\mathbf{X}_i = \mathbf{S} Rot(\mathbf{x}_i)\mathbf{S}^{-1}, \mathbf{Y}_i = \mathbf{T}^{-1} Rot(\mathbf{y}_i)\mathbf{T}$, we have

$$Rot(\mathbf{x}_i) = \mathbf{S}^{-1}\mathbf{X}_i\mathbf{S},$$
$$Rot(\mathbf{y}_i) = \mathbf{T}\mathbf{Y}_i\mathbf{T}^{-1}.$$

So, $\mathbf{x}_i = (\mathbf{a}_i + \mathbf{e}_i\mathbf{f}) \bmod \mathbf{q}$, $\mathbf{y}_i = (\mathbf{a}_i\mathbf{g} + \mathbf{t}_i\mathbf{f}) \bmod \mathbf{q}$. According to the relation of $\mathbf{x}_i, \mathbf{y}_i$, we have $\mathbf{z}_{i_1,i_2} = \mathbf{x}_{i_1}\mathbf{y}_{i_2} - \mathbf{x}_{i_2}\mathbf{y}_{i_1} = \mathbf{r}_{i_1,i_2}\mathbf{f}$. Thus, one can obtain a basis of $\mathbf{f}$ applying $\mathbf{z}_{i_1,i_2}$.

In the following attack, we remove matrices $\mathbf{T}, \mathbf{S}$ in the public parameters. For simplicity, we write $\mathbf{q}$ as $\mathbf{q}_{y,1}$.

Without loss of generality, assume $\mathbf{B} = \mathbf{E} \cdot Rot(\mathbf{f})$ is a basis of $\mathbf{f}$, where $\mathbf{E}$ is a unimodular matrix. We choose $\mathbf{x}_1 = (\mathbf{a}_1 + \mathbf{s}_1 \cdot \mathbf{f}) \bmod \mathbf{q}$ and $\mathbf{y}_1 = (\mathbf{a}_1 \cdot \mathbf{g} + \mathbf{t}_1 \cdot \mathbf{f}) \bmod \mathbf{q}$ and assume that $Rot(\mathbf{x}_1), \mathbf{B}$ are co-prime. Namely, $\det(Rot(\mathbf{x}_1)), \det(\mathbf{B})$ are co-prime. Otherwise, we can choose other pair of $\mathbf{x}_i, \mathbf{y}_i$.

Using LLL algorithm, one can compute $\mathbf{U}_1, \mathbf{U}_2$ such that $\mathbf{U}_1 \cdot Rot(\mathbf{x}_1) + \mathbf{U}_2 \cdot \mathbf{B} = \mathbf{U}_3$ and $\mathbf{U}_3$ is a unimodular matrix. So, $\mathbf{U}_3^{-1} \cdot \mathbf{U}_1 \cdot Rot(\mathbf{x}_1) + \mathbf{U}_3^{-1} \cdot \mathbf{U}_2 \cdot \mathbf{B} = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix.

By $\mathbf{x}_1 = (\mathbf{a}_1 + \mathbf{s}_1 \cdot \mathbf{f}) \bmod \mathbf{q} = \mathbf{a}_1 + \mathbf{s} \cdot \mathbf{f}$, $\mathbf{y}_1 = (\mathbf{a}_1 \cdot \mathbf{g} + \mathbf{t}_1 \cdot \mathbf{f}) \bmod \mathbf{q} = \mathbf{a}_1 \cdot \mathbf{g} + \mathbf{t} \cdot \mathbf{f}$, we have

$$
\begin{aligned}
&\mathbf{U}_3^{-1}\cdot\mathbf{U}_1\cdot Rot(\mathbf{x}_1)+\mathbf{U}_3^{-1}\cdot\mathbf{U}_2\cdot\mathbf{B}\\
&=\mathbf{U}_3^{-1}\mathbf{U}_1\cdot Rot(\mathbf{a}_1+\mathbf{s}\cdot\mathbf{f})+\mathbf{U}_3^{-1}\mathbf{U}_2\cdot\mathbf{E}\cdot Rot(\mathbf{f})\\
&=\mathbf{U}_3^{-1}\mathbf{U}_1\cdot Rot(\mathbf{a}_1)+\mathbf{U}_3^{-1}\mathbf{U}_1 Rot(\mathbf{s})\cdot Rot(\mathbf{f})+\mathbf{U}_3^{-1}\mathbf{U}_2\cdot\mathbf{E}\cdot Rot(\mathbf{f})\\
&=\mathbf{U}_3^{-1}\mathbf{U}_1\cdot Rot(\mathbf{a}_1)+(\mathbf{U}_3^{-1}\mathbf{U}_1 Rot(\mathbf{s})+\mathbf{U}_3^{-1}\mathbf{U}_2\mathbf{E})\cdot Rot(\mathbf{f})\\
&=\mathbf{U}_4\cdot Rot(\mathbf{a}_1)+\mathbf{U}_5\cdot Rot(\mathbf{f})\\
&=\mathbf{I}
\end{aligned},
$$

$$
\begin{aligned}
\mathbf{B}_1&=\mathbf{U}_4\cdot Rot(\mathbf{y}_1)\\
&=\mathbf{U}_4\cdot Rot(\mathbf{a}_1\cdot\mathbf{g}+\mathbf{t}\cdot\mathbf{f})\\
&=\mathbf{U}_4\cdot Rot(\mathbf{a}_1)\cdot Rot(\mathbf{g})+\mathbf{U}_4 Rot(\mathbf{t})\cdot Rot(\mathbf{f})\\
&=(\mathbf{I}-\mathbf{U}_5\cdot Rot(\mathbf{f}))\cdot Rot(\mathbf{g})+\mathbf{U}_4 Rot(\mathbf{t})\cdot Rot(\mathbf{f})\\
&=Rot(\mathbf{g})-(\mathbf{U}_5 Rot(\mathbf{g})+\mathbf{U}_4 Rot(\mathbf{t}))\cdot Rot(\mathbf{f})\\
&=Rot(\mathbf{g})-\mathbf{U}_6\cdot Rot(\mathbf{f})
\end{aligned}.
$$

Using LLL algorithm, one can compute $\mathbf{U}_7,\mathbf{U}_8$ such that $\mathbf{U}_7\cdot\mathbf{B}_1+\mathbf{U}_8\cdot\mathbf{B}=\mathbf{U}_9$ and $\mathbf{U}_9$ is a unimodular matrix. So, we have

$$\mathbf{U}_9^{-1}\mathbf{U}_7\cdot\mathbf{B}_1+\mathbf{U}_9^{-1}\mathbf{U}_8\cdot\mathbf{B}=\mathbf{I}$$

So,

$$
\begin{aligned}
&\mathbf{U}_9^{-1}\mathbf{U}_7\cdot\mathbf{B}_1+\mathbf{U}_9^{-1}\mathbf{U}_8\cdot\mathbf{B}\\
&=\mathbf{U}_9^{-1}\mathbf{U}_7\cdot(Rot(\mathbf{g})-\mathbf{U}_6\cdot Rot(\mathbf{f}))+\mathbf{U}_9^{-1}\mathbf{U}_8\cdot\mathbf{E}\cdot Rot(\mathbf{f})\\
&=\mathbf{U}_9^{-1}\mathbf{U}_7\cdot Rot(\mathbf{g})-(\mathbf{U}_9^{-1}\mathbf{U}_7\cdot\mathbf{U}_6+\mathbf{U}_9^{-1}\mathbf{U}_8\cdot\mathbf{E})\cdot Rot(\mathbf{f})\\
&=\mathbf{U}_{10}\cdot Rot(\mathbf{g})-\mathbf{U}_{11}Rot(\mathbf{f})\\
&=\mathbf{I}
\end{aligned}.
$$

Given a level-1 encoding $\mathbf{u}=\sum_{i=1}^{\tau}w_i\cdot\mathbf{y}_i \bmod \mathbf{q}$, we have

$$
\begin{aligned}
\mathbf{X}&=\mathbf{U}_{10}Rot(\mathbf{u})\\
&=\mathbf{U}_{10}\cdot Rot(\textstyle\sum_{i=1}^{\tau}w_i\cdot\mathbf{y}_i)\\
&=\mathbf{U}_{10}\cdot Rot(\textstyle\sum_{i=1}^{\tau}w_i\cdot(\mathbf{a}_i\cdot\mathbf{g}+\mathbf{t}_i\cdot\mathbf{f}))\\
&=\mathbf{U}_{10}\cdot\textstyle\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{a}_i\cdot\mathbf{g})+\mathbf{U}_{10}\cdot\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{t}_i\cdot\mathbf{f})\\
&=\mathbf{U}_{10}\cdot Rot(\mathbf{g})\textstyle\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{a}_i)+\mathbf{U}_{10}\cdot\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{t}_i)\cdot Rot(\mathbf{f})\\
&=(\mathbf{I}+\mathbf{U}_{11}Rot(\mathbf{f}))\textstyle\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{a}_i)+\mathbf{U}_{10}\cdot\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{t}_i)\cdot Rot(\mathbf{f})\\
&=\textstyle\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{a}_i)+(\mathbf{U}_{11}\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{a}_i)+\mathbf{U}_{10}\cdot\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{t}_i))\cdot Rot(\mathbf{f})\\
&=\textstyle\sum_{i=1}^{\tau}Rot(w_i\cdot\mathbf{a}_i)+\mathbf{U}_{12}\cdot Rot(\mathbf{f})
\end{aligned}.
$$

So, we can obtain $\mathbf{a}=\sum_{i=1}^{\tau}w_i\cdot\mathbf{a}_i+\mathbf{r}\cdot\mathbf{f}$ by arranging the first row of $\mathbf{X}$. Namely, we can get the level-0 encoding $\mathbf{a}$ corresponding to $\mathbf{u}$.

6 Asymmetric ideal multilinear maps

Asymmetric ideal multilinear maps with different group are required in some applications.

Similar to [GGH13], we briefly describe asymmetric variant as follow.

In the asymmetric construction, we sample different $\mathbf{g}_{j,t} \leftarrow D_{\mathbb{Z}^n,\sigma}, t = 1,...,\beta$. The element of the form $(\mathbf{a}_{j,t} \cdot \mathbf{g}_{j,t}) \bmod \mathbf{f}_j$ is a level-1 encoding relative to the t-th generator $\mathbf{g}_{j,t}$. We denote by vectors the different levels of encoding. For a level-0 encoding $\mathbf{a}$, the encoding $\mathbf{c}$ with an index vector $\mathbf{w} = (w_1,...,w_\beta) \in \mathbb{N}^\beta$ is satisfied to $\mathbf{c} \bmod \mathbf{f}_j = (\mathbf{a} \prod_{t=1}^{\beta} (\mathbf{g}_{j,t})^{w_t}) \bmod \mathbf{f}_j$. So, we give the public parameters $\mathbf{x}_{i,t} = (\mathbf{a}_{i,t} + \mathbf{s}_{i,t} \cdot \mathbf{f}) \bmod \mathbf{q}$ and $\mathbf{y}_{i,t} = (\mathbf{a}_{i,t}\mathbf{g}_t + \mathbf{t}_{i,t} \cdot \mathbf{f}) \bmod \mathbf{q}$, $i \in [\![\tau]\!]$, $t \in [\![\beta]\!]$, where $\mathbf{a}_{i,t} = \mathbf{a}_{i,j,t} \bmod \mathbf{f}_j$, $\mathbf{g}_{j,t} = \mathbf{g}_t \bmod \mathbf{f}_j$. Finally, we choose unimodular matrices $\mathbf{T}, \mathbf{S}$ and generate the public parameters $\text{par}_4 = \left\{ q, \mathbf{Q}_x, \mathbf{Q}_y, \left\{ \mathbf{X}_{i,t}, \mathbf{Y}_{i,t} \right\}_{i \in [\![\tau]\!], t \in [\![\beta]\!]}, \mathbf{P}_{zt} \right\}$. For the variant in Section 4, we can similarly obtain its asymmetric variant.

7 Commutative variant

In our ideal construction using unimodular matrix, the dimension n must be large enough to guarantee security and $\tau > n^2 + \lambda$ is the lowest requirement to avoid algebraic equation attack. As a result, the public parameter size of our construction is too large to be practical. To decrease the public parameter size, we use polynomial ring instead of the ring of integers.

We use $R_1 = \mathbb{Z}[y] / < y^{n_1} + 1 >$ and $R_{1,q} = \mathbb{Z}_q[y] / < y^{n_1} + 1 >$ instead of $\mathbb{Z}$ and $\mathbb{Z}_q$ for our ideal construction using unimodular matrices. It is easy to verify that our ideal construction is still correct under this case.

8 One-round Multipartite Diffie-Hellman Key Exchange Protocol

In this section, we present one round multipartite Diffie-Hellman key exchange protocols using our ideal constructions.

8.1 Basic construction

We describe the construction of one-round multipartite Diffie-Hellman key exchange protocol using our ideal multilinear maps in Section 3 and 4. As in [GGH13], protocol security relies on the hardness assumption of the ext-GDDH.

Setup(1^λ). For $\forall t \in [\![2]\!]$, output $(\text{par}_t) \leftarrow \text{InstGen}_t(1^\lambda)$ as the public parameter.

Publish(par_t, j). Let N be the number of participants, $j \in [\![N]\!]$. Each party j samples random elements $\mathbf{w}_{j,i} \leftarrow D_{\mathbb{Z}^n,\sigma'}, i \in [\![\tau]\!]$ which is used as secret key, computes and publishes level-1 encoding $\mathbf{u}_j \leftarrow \text{Enc}_t\left(\text{par}_t, 1, \left\{ \mathbf{w}_{j,i} \right\}_{i \in [\![\tau]\!]}\right)$ as a public key.

KeyGen$(\text{par}_t, j, \left\{ \mathbf{w}_{j,i} \right\}_{i \in [\![\tau]\!]}, \left\{ \mathbf{u}_j \right\}_{j \neq i})$. Each party j computes $\mathbf{c}_j = \prod_{k \neq j} \mathbf{u}_k$ and extracts the common secret key $sk = \text{Ext}_t(\text{par}_t, \mathbf{c}_j, \left\{ \mathbf{w}_{j,i} \right\}_{i \in [\![\tau]\!]})$.

Theorem 8.1 Suppose that ext-GDDH is hard, then our construction above is one-round multipartite Diffie-Hellman key exchange protocol.

Proof. The proof is similar as Theorem 2 in [GGH13]. ■

8.2 Construction using unimodular matrix

Because one only require to generate level-1 encoding in one-round multipartite Diffie-Hellman key exchange protocol, one can directly give $\mathbf{P}_{zt,i} = \left[\mathbf{X}_i \mathbf{P}_{zt}\right]_q$ to remove $\mathbf{X}_i$ and $\mathbf{P}_{zt}$, and $\mathbf{S}$ must not be a unimodular matrix. So, we generate the public parameters is $\text{par}_3 = \left\{q, \mathbf{Q}_y, \left\{\mathbf{P}_{zt,i}, \mathbf{Y}_i\right\}_{i\in[\![\tau]\!]}\right\}$.

$\mathbf{Setup}(1^\lambda)$. Output $(\text{par}_3) \leftarrow \text{InstGen}_3(1^\lambda)$ as the public parameter.

$\mathbf{Publish}(\text{par}_3, j)$. Let N be the number of participants, $j \in [\![N]\!]$. Each party j samples random elements $w_{j,i} \leftarrow D_{\mathbb{Z},\sigma'}, i \in [\![\tau]\!]$ which is used as secret key, computes and publishes level-1 encoding $\mathbf{U}_j \leftarrow \text{Enc}_3\left(\text{par}_3, 1, \left\{w_{j,i}\right\}_{i\in[\![\tau]\!]}\right)$ as a public key.

$\mathbf{KeyGen}(\text{par}_3, j, \left\{\mathbf{w}_{j,i}\right\}_{i\in[\![\tau]\!]}, \left\{\mathbf{U}_j\right\}_{j\neq i})$. Each party j computes $\mathbf{C}_j = \prod_{k\neq j} \mathbf{U}_k$ and extracts the common secret key $sk = \text{Ext}_3(\text{par}_3, \mathbf{C}_j, \left\{w_{j,i}\right\}_{i\in[\![\tau]\!]})$.

Theorem 8.2 Suppose that ideal-ext-GDDH is hard, then our construction is one-round multipartite Diffie-Hellman key exchange protocol.

Proof. The proof is similar as Theorem 2 in [GGH13]. ■

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