

Random Linear Code Based Public Key Encryption Scheme RLCE

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Abstract

Lattice based encryption schemes and linear code based encryption schemes have received extensive attention in recent years since they have been considered as post-quantum candidate encryption schemes. Though LLL reduction algorithm has been one of the major cryptanalysis techniques for lattice based cryptographic systems, key recovery cryptanalysis techniques for linear code based cryptographic systems are generally scheme specific. In recent years, several important techniques such as Sidelnikov-Shestakov attack, filtration attacks, and algebraic attacks have been developed to crypt-analyze linear code based encryption schemes. Though most of these cryptanalysis techniques are relatively new, they prove to be very powerful and many systems have been broken using them. Thus it is important to design linear code based cryptographic systems that are immune against these attacks. This paper proposes linear code based encryption schemes RLCE and bRLCE which share many characteristics with random linear codes. Our analysis shows that the schemes RLCE and bRLCE are secure against existing attacks and we hope that the security of the RLCE/bRLCE schemes is equivalent to the hardness of decoding random linear codes. Example parameters for different security levels are recommended for the schemes RLCE and bRLCE. It is expected that the scheme RLCE with Reed-Solomon code has smaller public key sizes and is more efficient than Goppa code based McEliece encryption scheme.

Key words: Random linear codes; McEliece Encryption scheme; secure public key encryption scheme; linear code based encryption scheme

MSC 2010 Codes: 94B05; 94A60; 11T71; 68P25

1 Introduction

With rapid development for quantum computing techniques, our society is concerned with the security of current Public Key Infrastructures (PKI) which are fundamental for Internet services. The core components for current PKI infrastructures are based on public cryptographic techniques such as RSA and DSA. However, it has been shown that these public key cryptographic techniques could be broken by quantum computers. Thus it is urgent to develop public key cryptographic systems that are secure against quantum computing.

Since McEliece encryption scheme [21] was introduced more than thirty years ago, it has withstood many attacks and still remains unbroken for general cases. It has been considered as one of the candidates for post-quantum cryptography since it is immune to existing quantum computer algorithm attacks. The original McEliece cryptographic system is based on binary Goppa codes. Several variants have been introduced to replace Goppa codes in the McEliece encryption scheme. For instance, Niederreiter [23] proposed the use of generalized Reed-Solomon codes and later, Berger and Loidreau [3] proposed the use of sub-codes of generalized Reed-Solomon codes. Sidelnikov [28] proposed the use of Reed-Muller codes, Janwa and Moreno [14] proposed the use of algebraic geometry codes, Baldi et al [1] proposed the use of LDPC codes,

Misoczki et al [22] proposed the use of MDPC codes, and Löndahl and Johansson [17] proposed the use of convolutional codes. Most of them have been broken though MDPC/LDPC code based McEliece encryption scheme [1, 22] and the original binary Goppa code based McEliece encryption scheme are still considered secure.

Goppa code based McEliece encryption scheme is hard to attack since Goppa codes share many characteristics with random codes. Motivated by Faugere et al's [12] algebraic attacks against quasi-cyclic and dyadic structure based compact variants of McEliece encryption scheme, Faugere et al [11] designed an efficient algorithm to distinguish a random code from a high rate Goppa code. Márquez-Corbella and Pellikaan [18] simplified the distinguisher in [11] using Schur component-wise product of codes.

Sidelnikov and Shestakov [27] showed, for the generalized Reed-Solomon code based McEliece encryption scheme, one can efficiently recover the private parameters for the generalized Reed-Solomon code from the public key. Using component-wise product of codes and techniques from [27], Wieschebrink [33] showed that Berger and Loidreau's proposal [3] could be broken efficiently also. Couvreur et al [6] proposed a general distinguisher based filtration technique to recover keys for generalized Reed-Solomon code based McEliece scheme and Couvreur, Márquez-Corbella, and Pellikaan [7] used filtration attacks to break Janwa and Moreno's [14] algebraic geometry code based McEliece encryption scheme. The filtration attack was recently used by Couvreur et al [8] and Faugere et al [13] to attack Bernstein et al's [4] wild Goppa code based McEliece scheme.

General Goppa code based McEliece schemes are still immune from these attacks. However, based on the new development of cryptanalysis techniques against linear code based cryptographic systems in the recent years, it is important to systematically design random linear code based cryptographic systems defeating these attacks. Motivated by this observation, this paper presents a systematic approach of designing public key encryption schemes using any linear code. For example, we can even use Reed-Solomon codes to design McEliece encryption scheme while it is insecure to use Reed-Solomon codes in the original McEliece scheme. Since our design of linear code based encryption scheme embeds randomness in each column of the generator matrix, it is expected that, without the corresponding private key, these codes are as hard as random linear codes for decoding.

The most powerful message recovery attacks (not key recovery attacks) on McEliece cryptosystem is the information-set decoding attack which was introduced by Prange [25]. In an information-set decoding approach, one finds a set of coordinates of a received ciphertext which are error-free and that the restriction of the code's generator matrix to these positions is invertible. The original message can then be computed by multiplying the ciphertext with the inverse of the sub-matrix. Improvements of the information-set decoding attack have been proposed by Lee-Brickell [15], Leon [16], Stern [29], May-Meurer-Thomae [19], Becker-Joux-May-Meurer [2], and May-Ozerov [20]. Bernstein, Lange, and Peters [5] presented an exact complexity analysis on information-set decoding attack against McEliece cryptosystem. The attacks in [2, 5, 15, 16, 19, 20, 29] are against binary linear codes and are not applicable when the underlying field is $GF(p^m)$ for a prime p . Peters [24] presented an exact complexity analysis on information-set decoding attack against McEliece cryptosystem over $GF(p^m)$. These information-set decoding techniques (in particular, the exact complexity analysis in [5, 24]) are used to select example parameters for RLCE and bRLCE schemes in Section 5.

Unless specified otherwise, we will use $q = 2^m$ or $q = p^m$ for a prime p and our discussion are based on the field $GF(q)$ through out this paper. Bold face letters such as **a**, **b**, **e**, **f**, **g** are used to denote row or column vectors over $GF(q)$. It should be clear from the context whether a specific bold face letter represents a row vector or a column vector.

2 Goppa codes and McEliece Public Key Encryption scheme

In this section, we briefly review Goppa codes and McEliece scheme. For given parameters $q, n \leq q$, and t , let $g(x)$ be a polynomial of degree t over $GF(q)$. Assume that $g(x)$ has no multiple zero roots and $\alpha_0, \dots, \alpha_{n-1} \in GF(q)$ be pairwise distinct which are not root of $g(x)$. The following subspace $C_{Goppa}(g)$ defines the code words of an $[n, k, d]$ binary Goppa code where $d \geq 2t + 1$. This binary Goppa code $C_{Goppa}(g)$ has dimension $k \geq n - tm$ and corrects t errors.

$$C_{Goppa}(g) = \left\{ c \in \{0, 1\}^n : \sum_{i=0}^{n-1} \frac{c_i}{x - \alpha_i} \equiv 0 \pmod{g(x)} \right\}.$$

Furthermore, if $g(x)$ is irreducible, then $C_{Goppa}(g)$ is called an irreducible Goppa code. The parity check matrix H for the Goppa codes looks as follows:

$$V_t(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0^1 & \alpha_1^1 & \cdots & \alpha_{n-1}^1 \\ \cdots & \cdots & \ddots & \cdots \\ \alpha_0^t & \alpha_1^t & \cdots & \alpha_{n-1}^t \end{pmatrix} \begin{pmatrix} \frac{1}{g(\alpha_0)} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \frac{1}{g(\alpha_{n-1})} \end{pmatrix} \quad (1)$$

where $\mathbf{x} = [\alpha_0, \dots, \alpha_{n-1}]$ and $\mathbf{y} = \left[\frac{1}{g(\alpha_0)}, \dots, \frac{1}{g(\alpha_{n-1})} \right]$.

The McEliece scheme [21] is described as follows. For the given parameters n and t , choose a binary Goppa code based on an irreducible polynomial $g(x)$ of degree t . Let G_s be the $k \times n$ generator matrix for the Goppa code. Select a random dense $k \times k$ nonsingular matrix S and a random $n \times n$ permutation matrix P . Note that the permutation matrix P is required only if the support $\alpha_0, \dots, \alpha_{n-1}$ is known to the public. Then the public key is $G = SG_sP$ which generates a linear code with the same rate and minimum distance as the code generated by G_s . The private key is G_s .

Encryption. For a k -bit message block \mathbf{m} , choose a random row vector \mathbf{e} of length n and weight t . Compute the cipher text $\mathbf{y} = \mathbf{m}G + \mathbf{e}$

Decryption. For a received ciphertext \mathbf{y} , first compute $\mathbf{y}' = \mathbf{y}P^{-1}$. Next use an error-correction algorithm to recover $\mathbf{m}' = \mathbf{m}S$ and compute the message \mathbf{m} as $\mathbf{m} = \mathbf{m}'S^{-1}$.

3 Random linear code based encryption schemes RLCE and bRLCE

We first present the design of Random Linear Code based Encryption schemes RLCE. The protocol proceeds as follows:

Key Setup. Let $n, k, d, t > 0$, and $r \geq 1$ be given parameters such that $n - k + 1 \geq d \geq 2t + 1$. Let G_s be a $k \times n$ generator matrix for an $[n, k, d]$ linear code such that there is an efficient decoding algorithm to correct at least t errors for this linear code given by G_s .

1. Let P be an $n \times n$ permutation matrix. Note that the permutation matrix P is required only if the support (e.g., $\alpha_0, \dots, \alpha_{n-1}$ for the Reed-Solomon code) for the underlying code is publicly known. Assume that $G_sP = [\mathbf{g}_0, \dots, \mathbf{g}_{n-1}]$
2. Let $C_0, C_1, \dots, C_{n-1} \in GF(q)^{k \times r}$ be $k \times r$ matrices drawn uniformly at random and let

$$G_1 = [\mathbf{g}_0, C_0, \mathbf{g}_1, C_1 \cdots, \mathbf{g}_{n-1}, C_{n-1}] \quad (2)$$

be the $k \times n(r + 1)$ matrix obtained by inserting the random matrices C_i into G_s .

3. Let $A_0, \dots, A_{n-1} \in GF(q)^{(r+1) \times (r+1)}$ be dense nonsingular $(r+1) \times (r+1)$ matrices chosen uniformly at random and let

$$A = \begin{pmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_{n-1} \end{pmatrix} \quad (3)$$

be an $n(r+1) \times n(r+1)$ nonsingular matrix.

4. Let S be a random dense $k \times k$ nonsingular matrix.
5. The public key is the $k \times n(r+1)$ matrix $G = SG_1A$ and the private key is (S, G_s, P, A) .

Encryption. For a row vector message $\mathbf{m} \in GF(q)^k$, choose a random row vector $\mathbf{e} = [e_0, \dots, e_{n(r+1)-1}] \in GF(q)^{n(r+1)}$ with the property

- For each $i \leq n-1$, the sub-vector $\mathbf{e}_i = [e_{i(r+1)}, \dots, e_{i(r+1)+r}]$ of \mathbf{e} is either a zero vector or all elements of \mathbf{e}_i are non-zero.
- There are at most t non-zero sub-vectors \mathbf{e}_i .

The cipher text is $\mathbf{y} = \mathbf{m}G + \mathbf{e}$.

Decryption. For a received cipher text $\mathbf{y} = [y_0, \dots, y_{n(r+1)-1}]$, compute

$$\mathbf{y}A^{-1} = [y'_0, \dots, y'_{n(r+1)-1}] = \mathbf{m}SG_1 + \mathbf{e}A^{-1}$$

where

$$A^{-1} = \begin{pmatrix} A_0^{-1} & & & \\ & A_1^{-1} & & \\ & & \ddots & \\ & & & A_{n-1}^{-1} \end{pmatrix} \quad (4)$$

Let $\mathbf{y}' = [y'_0, y'_{r+1}, \dots, y'_{(n-1)(r+1)}]$ be the row vector of length n selected from the length $n(r+1)$ row vector $\mathbf{y}A^{-1}$. Then $\mathbf{y}' = \mathbf{m}SG_sP + \mathbf{e}'$ for some error vector $\mathbf{e}' \in GF(q)^n$. For $i \leq n-1$, $\mathbf{e}'[i]$ is the first element of $\mathbf{e}_iA_i^{-1}$ where $\mathbf{e}_i = [e_{i(r+1)}, \dots, e_{i(r+1)+r}]$ is a sub-vector of \mathbf{e} . Thus $\mathbf{e}'[i] \neq 0$ only if \mathbf{e}_i is non-zero. Since there are at most t non-zero sub-vectors \mathbf{e}_i , the Hamming weight of $\mathbf{e}' \in GF(q)^n$ is at most t . Using the efficient decoding algorithm, one can compute $\mathbf{m}' = \mathbf{m}S$. Finally, \mathbf{m} is computed by $\mathbf{m} = \mathbf{m}'S^{-1}$.

In the above RLCE encryption scheme, each sub-vector \mathbf{e}_i of the error vector \mathbf{e} is either a zero vector or all elements of \mathbf{e}_i are non-zero. If the error vector \mathbf{e} is chosen from $GF(2)^{n(r+1)}$ then, for each sub vector $\mathbf{y}_i = [y_{i(r+1)}, \dots, y_{i(r+1)+r}]$ of the ciphertext \mathbf{y} , the following vector

$$\mathbf{y}_i + y_{i(r+1)} = [y_{i(r+1)} + y_{i(r+1)}, \dots, y_{i(r+1)+r} + y_{i(r+1)}]$$

is error-free no matter whether \mathbf{e}_i is a zero or non-zero sub-vector. If one prefers to work over the binary field or to select binary error vectors, we recommend the following revised RLCE scheme bRLCE.

Key setup. Let $n, k, d, t > 0$, and $r \geq 1$ be given parameters such that $n - k + 1 \geq d \geq 2t + 1$. Let $G_s = [\mathbf{g}_0, \dots, \mathbf{g}_{n-1}]$ be a $k \times n$ generator matrix for an $[n, k, d]$ linear code such that there is an efficient decoding algorithm to correct at least t errors for this linear code given by G_s .

1. Let $G_1 \in GF(q)^{k \times n(r+1)}$, $A \in GF(q)^{n(r+1) \times n(r+1)}$, and $S \in GF(q)^{k \times k}$ be matrices constructed in the same way as in the scheme RLCE.

2. Let P be an $n(r+1) \times n(r+1)$ permutation matrix.

3. The public key is the $k \times n(r+1)$ matrix $G = SG_1AP$ and the private key is (S, G_s, P, A) .

Encryption. For a row vector message $\mathbf{m} \in GF(q)^k$, choose a random row vector $\mathbf{e} = [e_0, \dots, e_{n(r+1)-1}] \in GF(q)^{n(r+1)}$ such that the Hamming weight of \mathbf{e} is at most t . The cipher text is $\mathbf{y} = \mathbf{m}G + \mathbf{e}$.

Decryption. For a received cipher text $\mathbf{y} = [y_0, \dots, y_{n(r+1)-1}]$, compute

$$\mathbf{y}P^{-1}A^{-1} = [y'_0, \dots, y'_{n(r+1)-1}] = \mathbf{m}SG_1 + \mathbf{e}P^{-1}A^{-1}$$

Let $\mathbf{y}' = [y'_0, y'_{r+1}, \dots, y'_{(n-1)(r+1)}]$ be the row vector of length n selected from the length $n(r+1)$ row vector $\mathbf{y}P^{-1}A^{-1}$. Then $\mathbf{y}' = \mathbf{m}SG_s + \mathbf{e}'$ for some error vector $\mathbf{e}' \in GF(q)^n$. Let $\mathbf{e}'' = \mathbf{e}P^{-1}A^{-1} = [e''_0, \dots, e''_{n(r+1)-1}]$. For $i \leq n-1$, $\mathbf{e}'[i]$ is the first element of $\mathbf{e}''_i A_i^{-1}$ where $\mathbf{e}''_i = [e''_{i(r+1)}, \dots, e''_{i(r+1)+r}]$ is a sub-vector of \mathbf{e}'' . Thus $\mathbf{e}'[i] \neq 0$ only if \mathbf{e}''_i is non-zero. Since there are at most t non-zero sub-vectors \mathbf{e}''_i , the Hamming weight of $\mathbf{e}' \in GF(q)^n$ is at most t . Thus \mathbf{m} could be decrypted from \mathbf{y}' efficiently.

4 Robustness of RLCE codes against existing attacks

4.1 Randomness of generator matrix columns

We first use the following theorem to show that any single column of the underlying generator matrix G_s could be completely randomized in a RLCE public key G .

Theorem 4.1 *Let $\mathbf{g}_0 \in GF(q)^k$ be a non-zero column vector of the generator matrix G_s . For any randomly chosen full rank $k \times (r+1)$ matrix $R_0 \in GF(q)^{k \times (r+1)}$, there exists a $k \times k$ nonsingular matrix S , a $(r+1) \times (r+1)$ matrix A_0 , and a $k \times r$ matrix $C_0 \in GF(q)^{k \times r}$ such that*

$$R_0 = S[\mathbf{g}_0, C_0]A_0 \quad (5)$$

Proof. By the fundamental properties of matrix equivalence, for two $m \times n$ matrices A, B of the same rank, there exist invertible $m \times m$ matrix P and $n \times n$ invertible matrix Q such that $A = PBQ$. The theorem could be proved using this property and the details are omitted here. \square

Let $R = [R_0, \dots, R_{n-1}] \in GF(q)^{k \times n(r+1)}$ be a fixed random linear code generator matrix. Theorem 4.1 shows that for any generator matrix G_s (e.g., a Reed-Solomon code generator matrix), we can choose matrices S and A_0 so that the first $r+1$ columns of the RLCE scheme public key G (constructed from G_s) are identical to R_0 . However, we cannot use Theorem 4.1 to continue the process of choosing A_1, \dots, A_{n-1} to obtain $G = R$ since S is fixed after A_0 is chosen. Indeed, this observed property is essential for the security of RLCE scheme. If one could continue the process of Theorem 4.1 and obtain matrices $S, A_0, A_1, \dots, A_{n-1}$ such that $G = R$, then the adversary may use the same process to recover an equivalent private key. In the following, we present a general theorem regarding this observation.

Theorem 4.2 *Let $R = [R_0, \dots, R_{n-1}] \in GF(q)^{k \times n(r+1)}$ and $G_s = [\mathbf{g}_0, \dots, \mathbf{g}_{n-1}] \in GF(q)^{k \times n}$ be two fixed linear code generator matrices of rank k . If $r > k-1$, then one can find $A_0, \dots, A_{n-1} \in GF(q)^{r \times r}$ and $C_0, \dots, C_{n-1} \in GF(q)^{k \times r}$ such that $R = [\mathbf{g}_0, C_0, \dots, \mathbf{g}_{n-1}, C_{n-1}]A$ where A is in the format of (3).*

Proof. Without loss of generality, we may assume that $r = k-1$. For each $0 \leq i \leq n-1$, choose a random matrix $C_i \in GF(q)^{k \times r}$ such that $G_i = [\mathbf{g}_i, C_i]$ is an $k \times k$ invertible matrix. Let $A = G_i^{-1}R_i$. Then the theorem is proved. \square

Theorem 4.2 shows that in the RLCE/bRLCE schemes, we must have $r \leq k - 1$. Otherwise, for a given public key $G \in GF(q)^{k \times n(r+1)}$, the adversary can choose a Reed-Solomon code generator matrix $G_s \in GF(q)^{k \times n}$ and compute $A_0, \dots, A_{n-1} \in GF(q)^{r \times r}$ and $C_0, \dots, C_{n-1} \in GF(q)^{k \times r}$ such that $G = [\mathbf{g}_0, C_0, \dots, \mathbf{g}_{n-1}, C_{n-1}]A$. In other words, the adversary can use the decryption algorithm corresponding to the generator matrix G_s to break the RLCE scheme

Theorem 4.2 also implies an efficient decryption algorithm for random $[n, k]$ linear codes with sufficiently small t of errors. Specifically, for an $[n, k]$ linear code with generator matrix $R \in GF(q)^{k \times n}$, if $t \leq \frac{n-k^2}{2k}$, then one can divide R into $m = 2t + k$ blocks $R = [R_0, \dots, R_{m-1}]$. Theorem 4.2 can then be applied to construct an equivalent $[m, k]$ Reed-Solomon code with generator matrix $G_s \in GF(q)^{k \times m}$. Thus it is sufficient to decrypt the equivalent Reed-Solomon code instead of the original random linear code. For McEliece based encryption scheme, Bernstein, Lange, and Peters [5] recommends the use of 0.75 ($= k/n$) as the code rate. Thus Theorem 4.2 has no threat on these schemes.

For $t \leq \frac{n-k^2}{2k}$, the adversary is guaranteed to succeed in breaking the system. Since multiple errors might be located within the same block R_i with certain probability, for a given t that is slightly larger than $\frac{n-k^2}{2k}$, the adversary still has a good chance to break the system using the above approach. It is recommended that t is significantly larger than $\frac{n-k^2}{2k}$. For RLCE/bRLCE schemes, this means that r should be significantly smaller than k . This is normally true since k is very larger for secure RLCE/bRLCE schemes.

In following sections, we list heuristic and experimental evidences that G shares the properties of random linear codes. Thus the security of the RLCE scheme is believed to be equivalent to decoding a random linear code which is **NP-hard**.

4.2 Niederreiter's scheme and Sidelnikov-Shestakov's attack

Sidelnikov and Shestakov's cryptanalysis technique [27] was used to analyze Niederreiter's scheme which is based on generalized Reed-Solomon codes. Let $\alpha = (\alpha_0, \dots, \alpha_{n-1})$ be n distinct elements of $GF(q)$ and let $v = (v_0, \dots, v_{n-1})$ be nonzero (not necessarily distinct) elements of $GF(q)$. The generalized Reed-Solomon (GRS) code of dimension k , denoted by $GRS_k(\alpha, v)$, is defined by the following subspace.

$$GRS_k(\alpha, v) = \{(v_0 f(\alpha_0), \dots, v_{n-1} f(\alpha_{n-1})) : f(x) \in GF(q)[x]_k\}$$

where $GF(q)[x]_k$ is the set of polynomials in $GF(q)[x]$ of degree less than k . $GF(q)[x]_k$ is a vector space of dimension k over $GF(q)$. For each code word $c = (v_0 f(\alpha_0), \dots, v_{n-1} f(\alpha_{n-1}))$, $f(x) = f_0 + f_1 x + \dots + f_{k-1} x^{k-1}$ is called the associate polynomial of the code word c that encodes the message (f_0, \dots, f_{k-1}) . $GRS_k(\alpha, v)$ is an $[n, k, d]$ MDS code where $d = n - k + 1$.

Niederreiter's scheme [23] replaces the binary Goppa codes in McEliece scheme using GRS codes. The first attack on Niederreiter scheme is presented by Sidelnikov and Shestakov [27]. Wieschebrink [32] revised Niederreiter's scheme by inserting random column vectors into random positions of G_s before obtaining the public key G . Couvreur et al [6] showed that Wieschebrink's revised scheme is insecure under the product code attacks.

Berger and Loidreau [3] recommend the use of sub codes of Niederreiter's scheme to avoid Sidelnikov and Shestakov's attack. Specifically, in Berger and Loidreau's scheme, one uses a random $(k-l) \times k$ matrix S' of rank $k-l$ instead of the $k \times k$ matrix S to compute the public key $G = S'G_s$.

For smaller values of l , Wieschebrink [33] shows that a private key (α, v) for Berger and Loidreau scheme [3] could be recovered using Sidelnikov-Shestakov algorithm. For larger values of l , Wieschebrink used Schur product code to recover the secret values for Berger-Loidreau scheme. Let $G = SG_s$ be the $(k-l) \times n$ public generator matrix for Berger-Loidreau scheme, $\mathbf{r}_0, \dots, \mathbf{r}_{k-l-1}$ be the rows of G , and f_0, \dots, f_{k-l-1} be the associated polynomials to those rows. For two row vectors $\mathbf{a}, \mathbf{b} \in GF(q)^n$, the component wise product

$\mathbf{a} * \mathbf{b} \in GF(q)^n$ is defined as

$$\mathbf{a} * \mathbf{b} = (a_0 b_0, \dots, a_{n-1} b_{n-1}) \quad (6)$$

By the definition in (6), it is straightforward to observe that

$$\mathbf{r}_i * \mathbf{r}_j = (v_0^2 f_i(\alpha_0) f_j(\alpha_0), \dots, v_{n-1}^2 f_i(\alpha_{n-1}) f_j(\alpha_{n-1})). \quad (7)$$

For $2k - 1 \leq n - 2$, if the code generated by $\mathbf{r}_i * \mathbf{r}_j$ equals $GRS_{2k-1}(\alpha, \mathbf{v}')$ for $\mathbf{v}' = (v_0^2, \dots, v_{n-1}^2)$, then the Sidelnikov-Shestakov algorithm could be used to recover the values α and v . For $2k - 1 \leq n - 2$, if the code generated by $\mathbf{r}_i * \mathbf{r}_j$ does not equal $GRS_{2k-1}(n, \mathbf{v}')$, then the attack fails. Wieschebrink claimed that the probability that the attack fails is very small. For the case of $2k - 1 > n - 2$, Wieschebrink applied Sidelnikov-Shestakov algorithm on the component wise product code of a shortened code of the original $GRS_k(\alpha, \mathbf{v})$.

The crucial step in Sidelnikov and Shestakov attack is to use the echelon form $E(G) = [I|G']$ of the public key to get minimum weight codewords that are co-related to each other supports. In the encryption scheme RLCE, each column of the public key G contains mixed randomness. Thus the echelon form $E(G) = [I|G']$ obtained from the public key G could not be used to build any useful equation system. In other words, it is expected that Sidelnikov and Shestakov attack does not work against the RLCE scheme.

4.3 Filtration attacks

Using distinguisher techniques [11], Couvreur et al. [6] designed a filtration technique to attack GRS code based McEliece scheme. The filtration technique was further developed by Couvreur et al [8] to attack wild Goppa code based McEliece scheme. In the following, we briefly review the filtration attack in [8]. For two codes C_1 and C_2 of length n , the star product code $C_1 * C_2$ is the vector space spanned by $\mathbf{a} * \mathbf{b}$ for all pairs $(\mathbf{a}, \mathbf{b}) \in C_1 \times C_2$ where $\mathbf{a} * \mathbf{b}$ is defined in (6). For $C_1 = C_2$, $C_1 * C_1$ is called the square code of C_1 . It is showed in [8] that

$$\dim C_1 \times C_2 \leq \left\{ n, \dim C_1 \dim C_2 - \binom{\dim(C_1 \cap C_2)}{2} \right\}. \quad (8)$$

Furthermore, the equality in (8) is attained for most randomly selected codes C_1 and C_2 of a given length and dimension. Note that for $C = C_1 = C_2$ and $\dim C = k$, the equation (8) becomes $\dim C^{*2} \leq \min \left\{ n, \binom{k+1}{2} \right\}$.

Couvreur et al [8] showed that the square code of an alternant code of extension degree 2 may have an unusually low dimension when its actual rate is larger than its designed rate. Specifically, Couvreur et al created a family of nested codes (called a filtration) defined as follows, for any $a \in \{0, \dots, n - 1\}$:

$$C^a(0) \supseteq C^a(1) \supseteq \dots \supseteq C^a(q + 1). \quad (9)$$

Roughly speaking, $C^a(j)$ consists in the codewords of C which correspond to polynomials which have a zero of order j at position a . The first two elements of this filtration are just punctured and shortened versions of C and the rest of them can be computed from C by computing star products and solving linear systems. The support values $\alpha_0, \dots, \alpha_{n-1}$ (the private key) for the Goppa code could be recovered using this nested family of codes efficiently.

The crucial part of the filtration technique is the efficient algorithm to compute the nested family of codes in (9). For our RLCE scheme, the public generator matrix G contains random columns. Thus linear equations constructed in Couvreur et al [8] could not be solved and the nested family (9) could not be computed correctly. Furthermore, the important characteristics for a code C to be vulnerable is that one can find a related code C_1 of dimension k such that the dimension of the square code of C_1 has a dimension significantly less than $\min \left\{ n, \binom{k+1}{2} \right\}$.

To get experimental evidence that RLCE codes share similarity with random linear codes with respect to the above mentioned filtration attacks, we carried out several experiments using Shoup's NTL library [26]. The source code for our experiments is available at [31]. In the experiments, we used Reed-Solomon codes over $GF(2^{10})$. The RLCE parameters are chosen as the 80-bit security parameter $n = 560$, $k = 380$, $t = 90$, and $r = 1$ (see Section 5 for details). For each given 380×560 generator matrix G_s of Reed-Solomon code, we selected another random 380×560 matrix $C \in GF(2^{10})^{380 \times 560}$ and selected 2×2 matrices A_0, \dots, A_{559} . Each column \mathbf{c}_i in C is inserted in G_s after the column \mathbf{g}_i . The extended generator matrix is multiplied by $A = \text{diag}[A_0, \dots, A_{559}]$ from the right hand side to obtain the public key matrix $G \in GF(2^{10})^{380 \times 1120}$. For each $i = 0, \dots, 1119$, the matrix G_i is used to compute the product code, where G_i is obtained from G by deleting the i th column from G . In our experiments, all of these product codes have dimension 1119. We repeated the above experiments 100 times for 100 distinct Reed-Solomon generator matrices and the results remained the same. Since $\min\{1119, \binom{381}{2}\} = 1119$, the experimental results meet our expectation that RLCE behaves like a random linear code. We did the same experiments for the dual code of the above code. That is, for a 180×560 generator matrix G_s of the dual code, the same procedure has been taken. In this time, after deleting one column from the resulting public key matrix, the product code always had dimension 1119 which is the expected dimension for a random linear code. In an early draft of this paper, we used Maple 2015 to carry out the experiments. In that experiments, we did not check the invertible property of the randomly generated 2×2 matrices A_0, \dots, A_{559} . Thus the previously reported experimental results are not accurate. The experimental evidence confirms our expectation that RLCE scheme behaves like a random linear code.

4.4 Algebraic attacks

Faugere, Otmani, Perret, and Tillich [12] developed an algebraic attack against quasi-cyclic and dyadic structure based compact variants of McEliece encryption scheme. In a high level, the algebraic attack from [12] tries to find $\mathbf{x}^*, \mathbf{y}^* \in GF(q)^n$ such that $V_t(\mathbf{x}^*, \mathbf{y}^*)$ is the parity check matrix for the underlying alternant codes of the compact variants of McEliece encryption scheme. $V_t(\mathbf{x}^*, \mathbf{y}^*)$ can then be used to break the McEliece scheme. Note that this $V_t(\mathbf{x}^*, \mathbf{y}^*)$ is generally different from the original parity check matrix $V_t(\mathbf{x}, \mathbf{y})$ in (1). The parity check matrix $V_t(\mathbf{x}^*, \mathbf{y}^*)$ was obtained by solving an equation system constructed from

$$V_t(\mathbf{x}^*, \mathbf{y}^*)G^T = 0, \quad (10)$$

where G is the public key. The authors of [12] employed the special properties of quasi-cyclic and dyadic structures (which provide additional linear equations) to rewrite the equation system obtained from (10) and then calculate $V_t(\mathbf{x}^*, \mathbf{y}^*)$ efficiently.

Faugere, Gauthier-Umaña, Otmani, Perret, and Tillich [11] used the algebraic attack in [12] to design an efficient Goppa code distinguisher to distinguish a random matrix from the matrix of a Goppa code whose rate is close to 1. For instance, [11] showed that the binary Goppa code obtained with $m = 13$ and $r = 19$ corresponding to a 90-bit security key is distinguishable.

It is challenging to mount the above mentioned algebraic attacks on RLCE encryption schemes. Assume that the RLCE scheme is based on Reed-Solomon code. Let G be the public key and (S, G_s, A, P) be the private key. The parity check matrix for a Reed-Solomon code is in the format of

$$V_t(\alpha) = \begin{pmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{t+1} & \alpha^{2(t+1)} & \dots & \alpha^{(t+1)(n-1)} \end{pmatrix}. \quad (11)$$

The algebraic attack in [11, 12] requires one to obtain a parity check matrix $V_t(\alpha^*)$ for the underlying Reed-Solomon code from the public key G , where α^* may be different from α . Assume that $V_t(\alpha^*)(P^T)^{-1} = [\mathbf{v}_0, \dots, \mathbf{v}_{n-1}] \in GF(q)^{(t+1) \times n}$ where $V_t(\alpha^*)$ is a parity check matrix for the underlying Reed-Solomon code. Let $V'_t(\alpha^*) \in GF(q)^{(t+1) \times 2n}$ be a $(t+1) \times 2n$ matrix obtained from $V_t(\alpha^*)$ by inserting r column vectors $\mathbf{0}$ after each column of $V_t(\alpha^*)(P^T)^{-1}$. That is,

$$V'_t(\alpha^*) = [\mathbf{v}_0, \mathbf{0}, \mathbf{v}_1, \mathbf{0}, \dots, \mathbf{v}_{n-1}, \mathbf{0}].$$

Then we have

$$\begin{aligned} V'_t(\alpha^*)G_1^T &= V'_t(\alpha^*)[\mathbf{g}_0, C_0, \dots, \mathbf{g}_{n-1}, C_{n-1}]^T \\ &= V_t(\alpha^*)(P^T)^{-1}[\mathbf{g}_0, \dots, \mathbf{g}_{n-1}]^T \\ &= V_t(\alpha^*)(P^T)^{-1}P^T G_s^T \\ &= \mathbf{0}. \end{aligned} \tag{12}$$

We cannot build an equation system for the unknown $V'_t(\alpha^*)$ from the public key $G = SG_1A$ directly since the identity (12) only shows the relationship between $V'_t(\alpha^*)$ and G_1 . In other words, in order to build an equation system for $V'_t(\alpha^*)$, one also needs to use unknown variables for the non-singular matrix A and the permutation matrix P . That is, we have

$$V'_t(\alpha^*)(A^{-1})^T G^T = V'_t(\alpha^*)(GA^{-1})^T = V'_t(\alpha^*)G_1^T S^T = \mathbf{0}. \tag{13}$$

with an unknown α^* , a permutation matrix P to obtain $V'_t(\alpha^*)$ from $V_t(\alpha^*)$, and an unknown matrix $A = \text{diag}[A_0, \dots, A_{n-1}]$ which consists of n dense nonsingular $(r+1) \times (r+1)$ matrices $A_i \in GF(q)^{(r+1) \times (r+1)}$ as defined in (3). In order to find a solution α^* , one first needs to take a potential permutation matrix P^{-1} to obtain $V_t(\alpha^*)(P^T)^{-1}$. Then, using the identity $V'_t(\alpha^*)(A^{-1})^T G^T = \mathbf{0}$, one can build an equation system of $k(t+1)$ equations in $n(r+1)^2 + 1$ unknowns. In case that $k(t+1) \geq n(r+1)^2 + 1$, one may use improved variants of Buchberger's Gröbner basis algorithms such as F_4 or F_5 (see, e.g., Faugere [10, 9]) to find a solution α^* . Since the number of permutation matrices P is huge, it is expected that the algebraic attack does not work against the encryption scheme RLCE.

5 Practical considerations

In order to reduce the public key size, it is preferred to use a smaller r for the RLCE and bRLCE encryption schemes. Indeed, the experimental results show that $r = 1$ is sufficient for RLCE/bRLCE to behave like a random linear code. As mentioned in the introduction section, the most powerful message recovery attack (not private key recovery attack) on McEliece encryption schemes is the information-set decoding attack. For RLCE/bRLCE encryption schemes, the information-set decoding attack is based on the number of columns in the public key G instead of the number of columns in the private key G_s . For the same error weight t , the probability to find error-free coordinates in $(r+1)n$ coordinates are different from the probability to find error-free coordinates in n coordinates. Thus for the same security level, the larger the value r , the larger error weight t is needed for the encryption scheme bRLCE. On the other hand, for the encryption scheme RLCE, the information-set decoding attack is based on the number of columns in the private key G_s since $(r+1)t$ errors are inserted to the ciphertext (instead of t errors for the bRLCE scheme).

Taking into account of the cost of recovering McEliece encryption scheme plaintext messages from ciphertexts using the information-set decoding methods, we generated a recommended list of parameters for RLCE/bRLCE schemes in Table 1 using the PARI/GP script by Peters's [24]. For the recommended parameters, the default underlying linear code is taken as the Reed-Solomon code over $GF(q)$ and the value of r is taken as 1. For the purpose of comparison, we also list the recommended parameters from [5]

for binary Goppa code based McEliece encryption scheme. The authors in [5, 24] proposed the use of semantic secure message coding approach so that one can store the public key as a systematic generator matrix. For binary Goppa code based McEliece encryption scheme, the systematic generator matrix public key is $k(n - k)$ bits. For RLCE encryption scheme over $GF(q)$, the systematic generator matrix public key is $k(n(r + 1) - k) \log q$ bits. It is observed that bRLCE scheme has the largest public key size and RLCE scheme has the smallest public key size. Specifically, for the same security level, the public key size for the bRLCE scheme is approximately five times larger than the public key size for binary Goppa code based McEliece encryption scheme and the public key size for the RLCE scheme is approximately 50% smaller than the public key size for binary Goppa code based McEliece encryption scheme. For example, for the security level of 80 bits, the RLCE-MDS scheme has a public key of size 29KB, the binary Goppa code based McEliece encryption scheme has a public key of size $56.2 \approx 2 \times 29$ KB, and the bRLCE-MDS scheme has a public key of size $267 \approx 5 \times 56.2$ KB.

Table 1: Parameters for RLCE and bRLCE: n, k, t, q , key size ($r = 1$ for all parameters), where “360, 200, 80, 2^8 , 101KB” under column “bRLCE-MDS code” represents $n = 360, k = 200, t = 80$, and $q = 2^8$.

Security	bRLCE-MDS code	RLCE-MDS code	binary Goppa code [5]
60	360, 200, 80, 2^8 , 101KB	200, 80, 50, 2^7 , 22KB	1024, 524, 50, 19.8KB
80	560, 380, 90, 2^8 , 267KB	220, 100, 60, 2^7 , 29KB	1632, 1269, 34, 56.2KB
128	1020, 660, 180, 2^9 , 0.98MB	360, 200, 80, 2^8 , 101KB	2960, 2288, 57, 187.7KB
192	1560, 954, 203, 2^{10} , 2.46MB	600, 400, 100, 2^8 , 312KB	4624, 3468, 97, 489.4KB
192		554, 300, 127, 2^8 , 236KB	
256	2184, 1260, 412, 2^{10} , 4.88MB	770, 400, 185, 2^9 , 500KB	6624, 5129, 117, 0.9MB

6 Conclusions

In this paper, we presented techniques for designing general random linear code based public encryption schemes using any linear code. Heuristics and experiments encourages us to think that the proposed schemes are immune against existing attacks on linear code based encryption schemes such as Sidelnikov-Shestakov attack, filtration attacks, and algebraic attacks. In addition to being a post-quantum cryptographic technique, our scheme RLCE has recently been used by Wang and Desmedt [30] to design fully homomorphic encryption schemes.

References

- [1] M. Baldi, M. Bodrato, and F. Chiaraluce. A new analysis of the mceliece cryptosystem based on qc-ldpc codes. In *Security and Cryptography for Networks*, pages 246–262. Springer, 2008.
- [2] A. Becker, A. Joux, A. May, and A. Meurer. Decoding random binary linear codes in $2^{n/20}$: How $1 + 1 = 0$ improves information set decoding. In *EUROCRYPT 2012*, pages 520–536. Springer, 2012.
- [3] T.P Berger and P. Loidreau. How to mask the structure of codes for a cryptographic use. *Designs, Codes and Cryptography*, 35(1):63–79, 2005.

- [4] D. Bernstein, T. Lange, and C. Peters. Wild McEliece. In *Selected Areas in Cryptography*, pages 143–158. Springer, 2011.
- [5] D.J. Bernstein, T. Lange, and C. Peters. Attacking and defending the McEliece cryptosystem. In *Post-Quantum Cryptography*, pages 31–46. Springer, 2008.
- [6] A. Couvreur, P. Gaborit, V. Gauthier-Umaña, A. Otmani, and J.-P. Tillich. Distinguisher-based attacks on public-key cryptosystems using Reed–Solomon codes. *Designs, Codes and Cryptography*, pages 1–26, 2013.
- [7] A. Couvreur, I. Márquez-Corbella, and R. Pellikaan. A polynomial time attack against algebraic geometry code based public key cryptosystems. In *Proc. ISIT*, pages 1446–1450. IEEE, 2014.
- [8] A. Couvreur, A. Otmañi, and J.-P. Tillich. Polynomial time attack on wild McEliece over quadratic extensions. In *Advances in Cryptology–EUROCRYPT 2014*, pages 17–39. Springer, 2014.
- [9] J.-C. Faugere. A new efficient algorithm for computing Gröbner bases without reduction to 0 (F5). In *Proc. ISSAC*, pages 75–83.
- [10] J.-C. Faugere. A new efficient algorithm for computing Gröbner bases (F4). *J. Pure and Applied Algebra*, 139(1):61–88, 1999.
- [11] J.-C. Faugere, V. Gauthier-Umaña, A. Otmani, L. Perret, and J.-P. Tillich. A distinguisher for high-rate mceliece cryptosystems. *Information Theory, IEEE Transactions on*, 59(10):6830–6844, 2013.
- [12] J.-C. Faugere, A. Otmani, L. Perret, and J.-P. Tillich. Algebraic cryptanalysis of McEliece variants with compact keys. In *Eurocrypt 2010*, pages 279–298. Springer, 2010.
- [13] J.-C. Faugere, L. Perret, and F. De Portzamparc. Algebraic attack against variants of mceliece with goppa polynomial of a special form. In *Advances in Cryptology–ASIACRYPT 2014*, pages 21–41. Springer, 2014.
- [14] H. Janwa and O. Moreno. Mceliece public key cryptosystems using algebraic-geometric codes. *Designs, Codes and Cryptography*, 8(3):293–307, 1996.
- [15] P. J. Lee and E. F. Brickell. An observation on the security of McEliece’s public-key cryptosystem. In *EUROCRYPT’88*, pages 275–280. Springer, 1988.
- [16] J. Leon. A probabilistic algorithm for computing minimum weights of large error-correcting codes. *IEEE Trans. Information Theory*, 34(5):1354–1359, 1988.
- [17] C. Löndahl and T. Johansson. A new version of mceliece pkc based on convolutional codes. In *Information and Communications Security*, pages 461–470. Springer, 2012.
- [18] I. Márquez-Corbella and R. Pellikaan. Error-correcting pairs for a public-key cryptosystem. *arXiv preprint arXiv:1205.3647*, 2012.
- [19] A. May, A. Meurer, and E. Thomae. Decoding random linear codes in $\tilde{O}(2^{0.054n})$. In *ASIACRYPT 2011*, pages 107–124. Springer, 2011.
- [20] A. May and I. Ozerov. On computing nearest neighbors with applications to decoding of binary linear codes. In *EUROCRYPT 2015*, pages 203–228. Springer, 2015.

- [21] R.J. McEliece. A public-key cryptosystem based on algebraic coding theory. *DSN progress report*, 42(44):114–116, 1978.
- [22] R. Misoczki, J.-P. Tillich, N. Sendrier, and P. Barreto. MDPC-McEliece: New McEliece variants from moderate density parity-check codes. In *Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on*, pages 2069–2073. IEEE, 2013.
- [23] H. Niederreiter. Knapsack-type cryptosystems and algebraic coding theory. *Problems of Control and Information Theory*, 15(2):159–166, 1986.
- [24] C. Peters. Information-set decoding for linear codes over F_q . In *Post-Quantum Cryptography*, pages 81–94. Springer, 2010.
- [25] Eugene Prange. The use of information sets in decoding cyclic codes. *IRE Trans. Information Theory*, 8(5):5–9, 1962.
- [26] Victor Shoup. NTL: A library for doing number theory, 2001.
- [27] V. M Sidelnikov and S. Shestakov. On insecurity of cryptosystems based on generalized Reed-Solomon codes. *Discrete Mathematics and Applications*, 2(4):439–444, 1992.
- [28] V.M. Sidelnikov. A public-key cryptosystem based on binary reed-muller codes. *Discrete Mathematics and Applications*, 4(3):191–208, 1994.
- [29] J. Stern. A method for finding codewords of small weight. In *Coding theory and applications*, pages 106–113. Springer, 1989.
- [30] Y. Wang and Y. Desmedt. Towards fully homomorphic encryption schemes from codes. In *Submitted*, pages 1–1. Springer Press, 2015.
- [31] Yongge Wang. RLCE implementation <http://webpages.uncc.edu/yonwang/rlce>, 2015.
- [32] C. Wieschebrink. Two NP-complete problems in coding theory with an application in code based cryptography. In *Proc. IEEE ISIT*, pages 1733–1737. IEEE Press, 2006.
- [33] C. Wieschebrink. Cryptanalysis of the Niederreiter public key scheme based on GRS subcodes. In *Post-Quantum Cryptography*, pages 61–72. Springer, 2010.