FULLY HOMOMORPHIC ENCRYPTION

WITHOUT BOOTSTRAPPING

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SUMMARY: Gentry's bootstrapping technique is the most famous method of obtaining fully homomorphic encryption. In this paper I propose a new fully homomorphic encryption scheme on non-associative octonion ring over finite field without bootstrapping technique [1]. The security of the proposed fully homomorphic encryption scheme is based on computational difficulty to solve the multivariate algebraic equations of high degree while the almost all multivariate cryptosystems [2],[3],[4],[5],[6],[7] proposed until now are based on the quadratic equations avoiding the explosion of the coefficients. Because proposed fully homomorphic encryption scheme is based on multivariate algebraic equations with high degree or too many variables, it is against the Gröbner basis [8] attack, the differential attack, rank attack and so on.

The key size of this system and complexity for enciphering/deciphering become to be small enough to handle.

keywords: fully homomorphic encryption, multivariate algebraic equation, Gröbner basis, octonion

§1. Introduction

A cryptosystem which supports both addition and multiplication (thereby preserving the ring structure of the plaintexts) is known as fully homomorphic encryption (FHE) and is very powerful. Using such a scheme, any circuit can be homomorphically evaluated, effectively allowing the construction of programs which may be run on encryptions of their inputs to produce an encryption of their output. Since such a program never decrypts its input, it can be run by an untrusted party without revealing its inputs and internal state. The existence of an efficient and fully homomorphic cryptosystem would have great practical implications in the outsourcing of private computations, for instance, in the context of cloud computing.

With homomorphic encryption, a company could encrypt its entire database of e-mails and upload it to a cloud. Then it could use the cloud-stored data as desired—for example, to calculate the stochastic value of stored data. The results would be downloaded and decrypted without ever exposing the details of a single e-mail.

In 2009 Gentry, an IBM researcher, has created a homomorphic encryption scheme that makes it possible to encrypt the data in such a way that performing a mathematical operation on the encrypted information and then decrypting the result produces the same answer as performing an analogous operation on the unencrypted data[9],[10].

But in Gentry's scheme a task like finding a piece of text in an e-mail requires chaining together thousands of basic operations. His solution was to use a second layer of encryption, essentially to protect intermediate results when the system broke down and needed to be reset.

Some fully homomorphic encryption schemes were proposed until now [11],[12], [13],[14],[15].

In this paper I propose a fully homomorphic encryption scheme on non-associative octonion ring over finite field which is based on computational difficulty to solve the multivariate algebraic equations of high degree while the almost all multivariate cryptosystems [4],[5],[6],[7] proposed until now are based on the quadratic equations avoiding the explosion of the coefficients. Our scheme is against the Gröbner basis [8] attack, the differential attack, rank attack and so on.

Organization of this paper is as follows. In Sec.2 preliminaries for octonion operation are described. In Sec.3 we construct proposed fully homomorphic encryption scheme. In Sec.4 the procedure for proposed fully homomorphic encryption scheme is described. In Sec.5 re-encryption scheme is described. In Sec.6 we analyse proposed scheme to show that proposed scheme is immune from the Gröbner basis attacks by calculating the complexity to obtain the Gröbner basis for the multivariate algebraic equations**.** In Sec.7 we describe the size of the parameters and the complexity for enciphering and deciphering. In Sec.8 we describe conclusion. In Sec.9 we consider the composition of plaintext.

§2. Preliminaries for octonion operation

In this section we describe the operations on octonion ring and properties of octonion ring.

§2.1 Multiplication and addition on the octonion ring *O*

Let q be a fixed modulus to be as large prime as $O(2^{10})$.
Let O be the octonion [16]ring over a finite field $\boldsymbol{Fq}$.

$$O=\{(a_0,a_1,\ldots,a_7) \mid a_j \in \boldsymbol{Fq}\ (j=0,1,\ldots,7)\} \tag{1}$$

We define the multiplication and addition of $A,B \in O$ as follows.

$$A=(a_0,a_1,\ldots,a_7), \quad a_j \in \boldsymbol{Fq}\ (j=0,1,\ldots,7), \tag{2}$$

$$B=(b_0,b_1,\ldots,b_7),\ b_j \in \boldsymbol{Fq}\ (j=0,1,\ldots,7). \tag{3}$$

$$\begin{aligned} &AB \bmod q \\ &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 \bmod q, \\ &\quad a_0b_1 + a_1b_0 + a_2b_4 + a_3b_7 - a_4b_2 + a_5b_6 - a_6b_5 - a_7b_3 \bmod q, \\ &\quad a_0b_2 - a_1b_4 + a_2b_0 + a_3b_5 + a_4b_1 - a_5b_3 + a_6b_7 - a_7b_6 \bmod q, \\ &\quad a_0b_3 - a_1b_7 - a_2b_5 + a_3b_0 + a_4b_6 + a_5b_2 - a_6b_4 + a_7b_1 \bmod q, \\ &\quad a_0b_4 + a_1b_2 - a_2b_1 - a_3b_6 + a_4b_0 + a_5b_7 + a_6b_3 - a_7b_5 \bmod q, \\ &\quad a_0b_5 - a_1b_6 + a_2b_3 - a_3b_2 - a_4b_7 + a_5b_0 + a_6b_1 + a_7b_4 \bmod q, \\ &\quad a_0b_6 + a_1b_5 - a_2b_7 + a_3b_4 - a_4b_3 - a_5b_1 + a_6b_0 + a_7b_2 \bmod q, \\ &\quad a_0b_7 + a_1b_3 + a_2b_6 - a_3b_1 + a_4b_5 - a_5b_4 - a_6b_2 + a_7b_0 \bmod q) \end{aligned} \tag{4}$$

$$\begin{aligned} &A+B \bmod q \\ &=(a_0+b_0 \bmod q,\ a_1+b_1 \bmod q,\ a_2+b_2 \bmod q,\ a_3+b_3 \bmod q\ , \\ &\quad a_4+b_4 \bmod q,\ a_5+b_5 \bmod q,\ a_6+b_6 \bmod q,\ a_7+b_7 \bmod q\ ). \end{aligned} \tag{5}$$

Let

$$|A|^2 = a_0^2+a_1^2+\ldots+a_7^2 \bmod q. \tag{6}$$

If $|A|^2 \neq 0 \bmod q$, we can have A^{-1}, the inverse of A by using the algorithm **Octinv**(A) such that

$$A^{-1} = (a_0/|A|^2 \bmod q,\ -a_1/|A|^2 \bmod q,\ldots,\ -a_7/|A|^2 \bmod q) \leftarrow \textbf{Octinv}(A). \tag{7}$$

Here details of the algorithm **Octinv**(A) are omitted and can be looked up in the **Appendix A.**

§2.2 Order of the element in *O*

In this section we describe the order "J" of the element "A" in octonion ring, that is,

$$A^{J+1}=A \bmod q.$$

Theorem 1

Let $A:=(a_{10},a_{11},\ldots,a_{17})\in O$, $a_{1j}\in Fq \quad (j=0,1,\ldots,7)$.

Let $(a_{n0},a_{n1},\ldots,a_{n7}) := A^n \in O$, $a_{nj}\in Fq \quad (n=1,2,\ldots;j=0,1,\ldots,7)$.

a_{00}, a_{nj}'s$(n=1,2,\ldots;j=0,1,\ldots)$ and b_n's$(n=0,1,\ldots)$ satisfy the equations such that

$$N:= a_{11}^2+\ldots+a_{17}^2 \quad \bmod q$$

$$a_{00}:=1,\ b_0:=0,\ b_1:=1,$$

$$a_{n0}= a_{n-1,0}\,a_{10} - b_{n-1}N \bmod q\ ,(n=1,2,\ldots), \tag{8}$$

$$b_n = a_{n-1,0}+ b_{n-1}a_{10} \bmod q\ ,(n=1,2,\ldots), \tag{9}$$

$$a_{nj}= b_n a_{1j} \bmod q\ ,(n=1,2,\ldots;j=1,2,\ldots,7)\ . \tag{10}$$

(*Proof*:)

Here proof is omitted and can be looked up in the **Appendix B**.

Theorem 2

For an element $A=(a_{10},a_{11},\ldots,a_{17})\in O$,

$$A^{J+1}=A \bmod q,$$

where

$$J= \mathrm{LCM}\ \{q^2-1,\ q-1\}=q^2-1,$$

$$N:=a_{11}^2+ a_{12}^2+\ldots+a_{17}^2\neq 0 \bmod q.$$

(*Proof*:)

Here proof is omitted and can be looked up in the **Appendix C.**

§2.3. Property of multiplication over octonion ring *O*

A,B,C etc.$\in O$ satisfy the following formulae in general where A,B and C have the inverse A^{-1}, B^{-1} and C^{-1} mod q.

1) Non-commutative

$$AB \neq BA \mod q.$$

2) Non-associative

$$A(BC) \neq (AB)C \mod q.$$

3) Alternative

$$(AA)B=A(AB) \mod q, \quad (11)$$

$$A(BB)=(AB)B \mod q, \quad (12)$$

$$(AB)A=A(BA) \mod q. \quad (13)$$

4) Moufang's formulae [16],

$$C(A(CB))=((CA)C)B \mod q, \quad (14)$$

$$A(C(BC))=((AC)B)C \mod q, \quad (15)$$

$$(CA)(BC)=(C(AB))C \mod q, \quad (16)$$

$$(CA)(BC)=C((AB)C) \mod q. \quad (17)$$

5) For positive integers n,m, we have

$$(AB)B^n =((AB)B^{n-1})B=A(B(B^{n-1}B))=AB^{n+1} \mod q, \quad (18)$$

$$(AB^n)B =((AB)B^{n-1})B=A(B(B^{n-1}B))=AB^{n+1} \mod q, \quad (19)$$

$$B^n (BA) =B(B^{n-1}(BA))= ((BB^{n-1})B)A=B^{n+1}A \mod q, \quad (20)$$

$$B(B^n A)=B(B^{n-1}(BA))= ((BB^{n-1})B)A=B^{n+1}A \mod q. \quad (21)$$

From (12) and (19), we have

$$[(AB^n)B]B =[AB^{n+1}]B \mod q,$$

$$(AB^n)(BB) =[(AB^n)B]B =[AB^{n+1}]B= AB^{n+2} \mod q,$$

$$(AB^n)B^2= AB^{n+2} \mod q,$$

… …

$$(AB^n)B^m= AB^{n+m} \mod q.$$

In the same way we have

$$B^m(B^n A)= B^{n+m}A \mod q.$$

6) **Lemma 1**

$$A(B((AB)^n))=(AB)^{n+1} \mod q,$$

$$(((AB)^n)A)B =(AB)^{n+1} \mod q .$$

where n is a positive integer and B has the inverse B^{-1}.

(*Proof*:)

From (14) we have

$$B(A(B((AB)^n)=((BA)B)(AB)^n=(B(AB))(AB)^n=B(AB)^{n+1} \mod q.$$

Then

$$B^{-1}(B(A(B(AB)^n))= B^{-1}(B\,(AB)^{n+1}) \mod q,$$

$$A(B(AB)^n)= (AB)^{n+1} \mod q.$$

In the same way we have

$$(((AB)^n)A)B=(AB)^{n+1} \mod q.$$ q.e.d.

7) **Lemma 2**

$$A^{-1}(AB)= B \bmod q,$$

$$(BA)A^{-1}= B \bmod q .$$

(*Proof*:)

Here proof is omitted and can be looked up in the **Appendix D**.

8) **Lemma 3**

$$A(BA^{-1})= (AB)A^{-1} \bmod q.$$

(*Proof*:)

From (17) we substitute A^{-1} to C, we have

$$(A^{-1}A)(B\,A^{-1})= A^{-1}\,((AB)\,A^{-1}) \mod q,$$

$$(B\,A^{-1})= A^{-1}\,((AB)\,A^{-1}) \mod q.$$

We multiply A from left side ,

$$A(B\,A^{-1})= A(A^{-1}\,((AB)\,A^{-1}))= (AB)\,A^{-1} \mod q.$$ q.e.d.

We can express $A(BA^{-1})$, $(AB)A^{-1}$ such that

$$ABA^{-1}.$$

9) From (13) and Lemma 2 we have

$$A^{-1}((A(BA^{-1}))A)= A^{-1}(A((BA^{-1})A))= (BA^{-1})A=B \bmod q,$$

$$(A^{-1}((AB)A^{-1}))A=((A^{-1}(AB))A^{-1})A= A^{-1}(AB)=B \bmod q\,.$$

10) **Lemma 4**

$$(BA^{-1})(AB)=B^2 \bmod q\,.$$

(*Proof*:)

From (17),

$$(BA^{-1})(AB)=B((A^{-1}A)B)=B^2 \mod q. \qquad \text{q.e.d.}$$

11) **Lemma 5**

$$(ABA^{-1})(ABA^{-1})= AB^2A^{-1} \bmod q.$$

(*Proof*:)

From (17),

$$(ABA^{-1})(ABA^{-1}) \mod q$$

$$=[A^{-1} (A^2(BA^{-1}))][(AB)A^{-1}]= A^{-1} \{[(A^2(BA^{-1}))(AB)]A^{-1}\} \mod q$$

$$= A^{-1} \{[(A(A(BA^{-1})))(AB)]A^{-1}\} \mod q$$

$$= A^{-1} \{[(A((AB)A^{-1}))(AB)]A^{-1}\} \mod q$$

$$= A^{-1} \{[(A(AB))A^{-1}))(AB)]A^{-1}\} \mod q.$$

We apply (15) to inside of [.],

$$= A^{-1} \{[(A((AB)(A^{-1}(AB)))]A^{-1}\} \mod q$$

$$= A^{-1} \{[(A((AB)B))]A^{-1}\} \mod q$$

$$= A^{-1} \{[A(A(BB))]A^{-1}\} \mod q$$

$$=\{ A^{-1} [A(A(BB))]\}A^{-1} \mod q$$

$$=(A(BB))A^{-1} \mod q$$

$$=AB^2A^{-1} \bmod q. \qquad \text{q.e.d.}$$

12) **Lemma 6**

$$(AB^mA^{-1})(\,AB^nA^{-1}\,)= AB^{m+n}A^{-1} \quad \bmod q.$$

(*Proof*:)

From (16),

$$[A^{-1}\,(A^2(B^mA^{-1}))][(AB^n)A^{-1}]= \{A^{-1}\,[(A^2(B^mA^{-1}))(AB^n)]\}A^{-1} \quad \bmod q$$

$$= A^{-1}\{\,[(A(A(B^mA^{-1}))(AB^n)]A^{-1}\} \quad \bmod q$$

$$= A^{-1}\{\,[(A((AB^m)A^{-1}))(AB^n)]A^{-1}\} \quad \bmod q$$

$$= A^{-1}\,\{\,[((A(AB^m))A^{-1}))(AB^n)]\,A^{-1}\} \quad \bmod q$$

$$= A^{-1}\,\{\,[((A^2B^m)A^{-1}))(AB^n)]\,A^{-1}\} \quad \bmod q.$$

We apply (15) to inside of { . },

$$= A^{-1}\,\{\,(A^2B^m)[A^{-1}((AB^n)A^{-1})]\} \quad \bmod q$$

$$= A^{-1}\,\{\,(A^2B^m)[A^{-1}(A(B^nA^{-1}))]\} \quad \bmod q$$

$$= A^{-1}\,\{\,(A^2B^m)(B^nA^{-1})\} \quad \bmod q$$

$$= A^{-1}\,\{\,(A^{-1}(A^3B^m))(B^nA^{-1})\} \quad \bmod q.$$

We apply (17) to inside of { . },

$$= A^{-1}\,\{\,A^{-1}([(A^3B^m)B^n]A^{-1})]\} \quad \bmod q$$

$$= A^{-1}\,\{\,A^{-1}((A^3B^{m+n})A^{-1})\} \quad \bmod q$$

$$= A^{-1}\,\{\,(A^{-1}(A^3B^{m+n}))A^{-1}\} \quad \bmod q$$

$$= A^{-1}\,\{\,(A^2B^{m+n})A^{-1}\} \quad \bmod q$$

$$=\{\,A^{-1}\,(A^2B^{m+n}))\,\}A^{-1} \quad \bmod q$$

$$=(AB^{m+n})A^{-1} \quad \bmod q$$

$$=AB^{m+n}A^{-1} \bmod q. \qquad \text{q.e.d}$$

13) $A \in O$ satisfies the following theorem.

Theorem 3

$$A^2=w\mathbf{1}+vA \bmod q,$$

where

$$^{\exists}w,v\in Fq,$$

$$\mathbf{1}=(1,0,0,0,0,0,0,0)\in O,$$

$$A=(a_0,a_1,\ldots,a_7)\in O.$$

(*Proof*:)

$$\begin{aligned}
&A^2 \bmod q\\
=(\;&a_0a_0-a_1a_1-a_2a_2-a_3a_3-a_4a_4-a_5a_5-a_6a_6-a_7a_7 \bmod q,\\
&a_0a_1+a_1a_0+a_2a_4+a_3a_7-a_4a_2+a_5a_6-a_6a_5-a_7a_3 \bmod q,\\
&a_0a_2-a_1a_4+a_2a_0+a_3a_5+a_4a_1-a_5a_3+a_6a_7-a_7a_6 \bmod q,\\
&a_0a_3-a_1a_7-a_2a_5+a_3a_0+a_4a_6+a_5a_2-a_6a_4+a_7a_1 \bmod q,\\
&a_0a_4+a_1a_2-a_2a_1-a_3a_6+a_4a_0+a_5a_7+a_6a_3-a_7a_5 \bmod q,\\
&a_0a_5-a_1a_6+a_2a_3-a_3a_2-a_4a_7+a_5a_0+a_6a_1+a_7a_4 \bmod q,\\
&a_0a_6+a_1a_5-a_2a_7+a_3a_4-a_4a_3-a_5a_1+a_6a_0+a_7a_2 \bmod q,\\
&a_0a_7+a_1a_3+a_2a_6-a_3a_1+a_4a_5-a_5a_4-a_6a_2+a_7a_0 \bmod q)
\end{aligned}$$

$$\begin{aligned}
=(&2a_0^2- L \bmod q,\ 2a_0a_1 \bmod q,\ 2a_0a_2 \bmod q,\ 2a_0a_3 \bmod q,\ 2a_0a_4 \bmod q,\ 2a_0a_5 \bmod q,\\
&2a_0a_6 \bmod q,\ 2a_0a_7 \bmod q)
\end{aligned}$$

where

$$\mathrm{L}= a_0^2+a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2 \bmod q.$$

Now we try to obtain $u, v\in Fq$ that satisfy $A^2=w\mathbf{1}+vA \bmod q$.

$$w\mathbf{1}+vA= w(1,0,0,0,0,0,0,0)+v(a_0,a_1,\ldots,a_7) \bmod q,$$

$$\begin{aligned}
A^2= (&2a_0^2- L \bmod q,\ 2a_0a_1 \bmod q,\ 2a_0a_2 \bmod q,\ 2a_0a_3 \bmod q,\ 2a_0a_4 \bmod q,\\
&2\ a_0a_5 \bmod q,2a_0a_6 \bmod q,\ 2a_0a_7 \bmod q).
\end{aligned}$$

Then we have

$$A^2=w\mathbf{1}+vA=-L\mathbf{1}+2\ a_0A \bmod q,$$

$$w= -L \bmod q,$$

$$v=2a_0 \bmod q. \qquad \text{q.e.d.}$$

14) **Theorem 4**

$$A^h=w_h\mathbf{1}+v_hA \bmod q$$

where h is an integer and $w_h,v_h \in Fq$.

(*Proof*:)

From Theorem 3

$$A^2=w_2\mathbf{1}+v_2A=-L\mathbf{1}+2a_0A \bmod q.$$

If we can express A^h such that

$$A^h=w_h\mathbf{1}+v_hA \bmod q \in O\ ,\ w_h,v_h \in Fq,$$

Then

$$A^{h+1}=(w_h\mathbf{1}+v_hA)A \bmod q$$

$$=w_hA+v_h(-L\mathbf{1}+2\ a_0A\ ) \bmod q$$

$$=-Lv_h\mathbf{1}+(\ w_h+2a_0v_h)A \bmod q.$$

We have

$$w_{h+1}= -L\ v_h \bmod q \in Fq,$$

$$v_{h+1}= w_h+2a_0v_h \bmod q \in Fq. \qquad \text{q.e.d.}$$

15) **Theorem 5**

$D \in O$ does not exist that satisfies the following equation.

$$B(AX)=DX \bmod q\ ,$$

where $B,A,D \in O$, and X is a variable.

(*Proof*:)

When $X=\mathbf{1}$, we have

$$BA=D \bmod q.$$

Then

$$B(AX)=(BA)X \bmod q.$$

We can select $C \in O$ that satisfies

$$B(AC) \neq (BA)C \bmod q. \tag{22}$$

We substitute $C \in O$ to X to obtain

$$B(AC) = (BA)C \bmod q. \tag{23}$$

(23) is contradictory to (22). q.e.d.

16) **Theorem 6**

$D \in O$ does not exist that satisfies the following equation.

$$C(B(AX)) = DX \bmod q \tag{24}$$

where $C,B,A,D \in O$, C has inverse $C^{-1} \bmod q$ and X is a variable.

B,A,C are non-associative, that is,

$$B(AC) \neq (BA)C \bmod q. \tag{25}$$

(*Proof*:)

If D exists, we have at X=**1**

$$C(BA) = D \bmod q.$$

Then

$$C(B(AX)) = (C(BA))X \bmod q.$$

We substitute C to X to obtain

$$C(B(AC)) = (C(BA))C \bmod q.$$

From (13)

$$C(B(AC)) = (C(BA))C = C((BA)C) \bmod q$$

Multiplying C^{-1} from left side ,

$$B(AC) = (BA)C \bmod q \tag{26}$$

(26) is contradictory to (25). q.e.d.

17) **Theorem 7**

D and $E \in O$ do not exist that satisfy the following equation.

$$C(B(AX)) = E\,(DX) \bmod q$$

where C,B,A,D and $E \in O$ have inverse and X is a variable.

A,B,C are non-associative, that is,

$$C(BA) \neq (CB)A \bmod q. \tag{27}$$

(*Proof*:)

If D and E exist, we have at X=**1**

$$C(BA) = ED \bmod q \tag{28}$$

We have at $X=(ED)^{-1}=D^{-1}E^{-1} \bmod q$.

$$C(B(A(D^{-1}E^{-1}))) = E\,(D(D^{-1}E^{-1})) \bmod q=\mathbf{1},$$

$$(C(B(A(D^{-1}E^{-1})))^{-1} \bmod q=\mathbf{1},$$

$$((ED)A^{-1})B^{-1})C^{-1} \bmod q=\mathbf{1},$$

$$ED = (CB)A \bmod q. \tag{29}$$

From (28) and (29) we have

$$C(BA) = (CB)A \bmod q. \tag{30}$$

(30) is contradictory to (27). q.e.d.

18) **Theorem 8**

$D \in O$ does not exist that satisfies the following equation.

$$A(B(A^{-1}X)) = DX \bmod q$$

where $B,A,D \in O$, A has inverse $A^{-1} \bmod q$ and X is a variable.

(*Proof*:)

If D exists, we have at X=**1**

$$A(BA^{-1}) = D \bmod q.$$

Then

$$A\,(B(A^{-1}X)) = (A(BA^{-1}))X \bmod q. \tag{31}$$

We can select $C \in O$ such that

$$(BA^{-1})(CA^2) \neq (BA^{-1})C)A^2 \bmod q. \qquad (32)$$

That is, (BA^{-1}), C and A^2 are non-associative.

Substituing $X=CA$ in (31), we have

$$A\,(B(A^{-1}(CA)))=(A(BA^{-1}))(CA) \bmod q.$$

From Lemma 3

$$A\,(B((A^{-1}C)A)))=(A(BA^{-1}))(CA) \bmod q.$$

From (17)

$$A\,(B((A^{-1}C)A)))=A([(BA^{-1})C]A) \bmod q.$$

Multiply A^{-1} from left side we have

$$B((A^{-1}C)A))= ((BA^{-1})C)A \bmod q.$$

From Lemma 3

$$B(A^{-1}(CA))=((BA^{-1})C)A \bmod q.$$

Transforming CA to $((CA^2)A^{-1})$, we have

$$B(A^{-1}((CA^2)A^{-1}))=((BA^{-1})C)A \bmod q.$$

From (15) we have

$$((BA^{-1})(CA^2))A^{-1}=((BA^{-1})C)A \bmod q.$$

Multiply A from right side we have

$$((BA^{-1})(CA^2)=((BA^{-1})C)A^2 \bmod q. \qquad (33)$$

(33) is contradictory to (32). q.e.d.

§3. Concept of proposed fully homomorphic encryption scheme

Homomorphic encryption is a form of encryption which allows specific types of computations to be carried out on ciphertext and obtain an encrypted result which decrypted matches the result of operations performed on the plaintext. For instance, one person could add two encrypted numbers and then another person could decrypt the result, without either of them being able to find the value of the individual numbers.

§3.1 Definition of homomorphic encryption

A homomorphic encryption scheme **HE** := (**KeyGen; Enc; Dec; Eval**) is a quadruple of PPT (Probabilistic polynomial time) algorithms.

In this work, the message space M of the encryption schemes will be octonion ring, and the functions to be evaluated will be represented as arithmetic circuits over this ring, composed of addition and multiplication gates. The syntax of these algorithms is given as follows.

-Key-Generation. The algorithm **KeyGen**, on input the security parameter 1^{λ}, outputs (**sk**) ← **KeyGen**(1^{λ}) , where **sk** is a secret encryption/decryption key.

-Encryption. The algorithm **Enc**, on input system parameter [q], secret keys(**sk**) and a message $m \in U$, outputs a ciphertext $C \leftarrow$**Enc**(**sk**;m).

-Decryption. The algorithm **Dec**, on input system parameter [q], secret key(**sk**) and a ciphertext C, outputs a message $m^* \leftarrow$**Dec**(**sk**;C).

-Homomorphic-Evaluation. The algorithm **Eval**, on input system parameter q, an arithmetic circuit ckt, and a tuple of n ciphertexts ($C_1,\ldots, C_n$), outputs a ciphertext $C' \leftarrow$**Eval**(ckt; $C_1,\ldots, C_n$).

The security notion needed in this scheme is security against chosen plaintext attacks (**IND-CPA** security), defined as follows.

Definition 1 (**IND-CPA security**). A scheme HE is **IND-CPA** secure if for any PPT adversary A_d it holds that:

$$Adv^{CPA}_{HE}[\lambda] := |Pr[A_d(\mathbf{Enc}(\mathbf{sk};0)) = 1] - Pr[A_d(\mathbf{Enc}(\mathbf{sk};1)) = 1]| = negl(\lambda)$$

where (**sk**) ←**KeyGen**(1^{λ}).

§3.2 Definition of fully homomorphic encryption

A scheme HE is fully homomorphic if it is both compact and homomorphic with respect to a class of circuits. More formally:

Definition 2 (**Fully homomorphic encryption**). A homomorphic encryption scheme FHE :=(**KeyGen; Enc; Dec; Eval**) is fully homomorphic if it satisfies the following properties:

1. Homomorphism: Let $CR = \{CR_\lambda\}_{\lambda\in N}$ be the set of all polynomial sized arithmetic circuits. On input **sk** $\leftarrow$**KeyGen**(1^λ), $\forall$ckt $\in$ CR_λ, $\forall(m_1,\ldots,m_n) \in M^n$ where n = n(λ), $\forall(C_1,\ldots,C_n)$

where $C_i \leftarrow$ **Enc**(**sk**;m_i), it holds that:

$$\Pr[\mathbf{Dec}(\mathbf{sk};\mathbf{Eval}(\text{ckt}; C_1,\ldots,C_n)) \neq \text{ckt}(m_1,\ldots,m_n)] = \text{negl}(\lambda).$$

2. Compactness: There exists a polynomial $\mu = \mu(\lambda)$ such that the output length of **Eval** is at most μ bits long regardless of the input circuit ckt and the number of its inputs.

§3.3 Proposed fully homomorphic enciphering/deciphering functions

We propose a fully homomorphic encryption (FHE) scheme based on the enciphering/deciphering functions on octonion ring over ***Fq***.

I define the some parameters for describing FHE.

Let q be a prime more than 2.

Let $M=(m_0,m_1,\ldots,m_7) \in O$ be the plaintext to be encrypted.

Let $X=(x_0,\ldots,x_7) \in O[X]$ be a variable.

Let $E(M,X)$ and $D(M,X)$ be a enciphering/deciphering function of user A.

Let $C(X)=E(M,X)\in O[X]$ be the ciphertext.

$A_i \in O$ is selected randomly such that A_i^{-1} exists (i=1,...,k) which is the secret keys of user A.

$C(X)=E(M,X)$ is defined as follows.

$$C(X)=E(M,X):=A_1(\ldots(A_k(M(A_k^{-1}(\ldots(A_1^{-1}X)\ldots) \bmod q \in O[X], \quad (34)$$

$$=(e_{00}x_0+e_{01}x_1+\ldots+e_{07}x_7,$$

$$e_{10}x_0+e_{11}x_1+\ldots+e_{17}x_7,$$

$$\ldots \quad \ldots$$

$$e_{70}x_0+e_{71}x_1+\ldots+e_{77}x_7), \quad (35)$$

$$= \{e_{ij}\}(i,j=0,\ldots,7) \tag{36}$$

with $e_{ij} \in \boldsymbol{Fq}$ $(i,j=0,\ldots,7)$ which is published in cloud centre.

Let D be the deciphering function defined as follows.

$$G_1(X) := A_k^{-1}(\ldots(A_1^{-1}X)\ldots), \tag{37}$$

$$G_2(X) := A_1(\ldots(A_kX)\ldots), \tag{38}$$

$D :=$

$$G_1(C(G_2(\mathbf{1})) = A_k^{-1}(\ldots(A_1^{-1}(\ldots(A_1(\ldots(A_k(M(A_k^{-1}(\ldots(A_1^{-1}(A_1(\ldots(A_k\mathbf{1})\ldots) \tag{39}$$

$= M$

$= (m_0, m_1, \ldots, m_7)$.

§3.4 Elements on octonion ring assumption EOR($k,n;q$)

Here we describe the assumption on which the proposed scheme bases.

Elements on octonion ring assumption EOR($k,n;q$).

Let q be a prime more than 2. Let k and n be integer parameters. Let $\boldsymbol{A} := (A_1,\ldots,A_k) \in O^k$. Let $C_i(X) := A(X,M_i) = A_k(\ldots(A_1(M_i(A_1^{-1}(\ldots(A_k^{-1}X)\ldots) \bmod q \in O[X]$ where plaintexts $M_i := (m_{i0},\ldots,m_{i7}) \in O(i=1,\ldots,n)$, X is a variable.

In the **EOR**($k,n;q$) assumption, the adversary A_d is given $C_i(X)$ $(i=1,\ldots,n)$ randomly and his goal is to find a set of elements $\boldsymbol{A} = (A_1,\ldots,A_k) \in O^k$ with the order of the elements $A_1,\ldots, A_k$ and plaintexts $M_i(i=1,\ldots,n)$. For parameters $k = k(\lambda)$ and $n = n(\lambda)$ defined in terms of the security parameter λ and for any PPT adversary A_d we have

$$\Pr[A_k(\ldots(A_1(M_i(A_1^{-1}(\ldots(A_k^{-1}X)\ldots) \bmod q = C_i(X)\ (i=1,\ldots,n):$$

$$\boldsymbol{A} = (A_1,\ldots,A_k),\ M_i(i=1,\ldots,n) \leftarrow A_d(1^\lambda, C_i(X)\ (i=1,\ldots,n))] = \mathrm{negl}(\lambda).$$

To solve directly **EOR**($k,n;q$) assumption is known to be the problem for solving the multivariate algebraic equations of high degree which is known to be NP-hard.

§3.5 Syntax of proposed algorithms

The syntax of proposed scheme is given as follows.

-Key-Generation. The algorithm **KeyGen**, on input the security parameter 1^λ and system parameter q, outputs $\mathbf{sk} \leftarrow \mathbf{KeyGen}(1^\lambda)$, where **sk** is a secret encryption/dencryption key.

-Encryption. The algorithm **Enc**, on input system parameter q, and secret keys **sk** and a message $m \in M$, outputs a ciphertext $\mathbf{C}(X;\mathbf{sk},m) \leftarrow \mathbf{Enc}(\mathbf{sk};m)$.

-Decryption. The algorithm **Dec**, on input system parameter [q], secret keys **sk** and a ciphertext $\{c_{ij}\}$, outputs message **Dec**(**sk**;$\{c_{ij}\}$) where $\{c_{ij}\} \leftarrow$**Enc**(**sk**;m).

-Homomorphic-Evaluation. The algorithm **Eval**, on input system parameter q, an arithmetic circuit ckt, and a tuple of n ciphertexts ($\{c1_{ij}\},\ldots, \{cn_{ij}\}$), outputs an evaluated ciphertext $\{c'_{ij}\}\leftarrow$**Eval**(ckt; $\{c1_{ij}\},\ldots, \{cn_{ij}\}$).

Theorem 9

For any $M,N\in O$,

$$\text{if } E(M, X)= E(N, X) \bmod q \text{ , then } M= N \bmod q.$$

That is , if $M \neq N \bmod q$, then $E(M, X)\neq E(N, X) \bmod q$.

(*Proof*)

If $E(M, X)= E(N, X) \bmod q$, then

$$A_1(\ldots(A_k(M(A_k^{-1}(\ldots(A_1^{-1}X)\ldots)= (A_1(\ldots(A_k(N(A_k^{-1}(\ldots(A_1^{-1}X)\ldots) \bmod q.$$

We substitute $A_1(\ldots(A_kX)\ldots)$ to X in above expression, we obtain

$$A_1(..(A_k(M(A_k^{-1}(..(A_1^{-1}(A_1(..(A_kX)\ldots)=A_1(..(A_k(N(A_k^{-1}(..(A_1^{-1}(A_1(..(A_kX)\ldots) \bmod q,$$

$$A_1(\ldots(A_k(MX)\ldots)= A_1(\ldots(A_k(N X)\ldots) \bmod q,$$

$$MX =NX \bmod q.$$

We substitute **1** to X,

$$M=N \bmod q .$$ q.e.d

It is said that M and $E(M,X)$ corresponds one to one .

It is shown that the encrypting function $E(M, X)$ has the property of fully homomorphism.

§3.6 Addition/subtraction scheme on ciphertexts

Let $N=(n_0,n_1,\ldots,n_7)\in O$ be another plaintext to be encrypted.

Let $C_1(X)= E(M, X)$ and $C_2(X)= E(N, X)$ be the ciphertexts.

$$C_1(X)\pm C_2(X) \bmod q =E(M,X) \pm E(N,X) \bmod q$$

$$=A_1(\ldots(A_k(M\,(A_k^{-1}\,(\ldots(A_1\,^{-1}X)\ldots)+A_1(\ldots(A_k(N(A_k^{-1}\,(\ldots(A_1\,^{-1}X)\ldots)\bmod q$$

$$=A_1(\ldots(A_k([M\pm N]\,(A_k^{-1}\,(\ldots(A_1\,^{-1}X)\ldots)\bmod q$$

$$E(M,X)\pm E\,(N,X)=E(M\pm N,X)\bmod q. \tag{40}$$

§3.7 Multiplication scheme on ciphertexts

Let $C_1(X)=E(M,X)$ and $C_2(X)=E\,(N,X)$ be the ciphertexts.

$$C_1(C_2(X))\bmod q=E(M,E\,(N,X))\bmod q$$

$$=A_1(\ldots(A_k(M\,(A_k^{-1}(A_1^{-1}(A_1\,(\ldots(A_k(N(A_k^{-1}(\ldots(A_1^{-1}X)\ldots)\bmod q$$

$$=A_1(\ldots(A_k(M(N\,(A_k^{-1}\,(\ldots(A_1^{-1}X)\ldots)\bmod q \tag{41}$$

§3.8 Inverse of $E(M,X)$

We define $E(M,X)^{(-1)}$, the inverse of $E(M,X)$ such that

$$E(M,E(M,X)^{(-1)})=E(M,E(M,X))^{(-1)}=X. \tag{42}$$

Theorem 10

In case that $M\in O$ has the inverse , that is $|M|^2\neq 0 \bmod q$,
if $E(M,E(M',X))=E(M',E(M,X))=X$,

$$M'=M^{-1}\bmod q,$$

That is,

$$E(M,X)^{(-1)}=E(M^{-1},X).$$

(*Proof*:)

$$E(M,E(M',X))$$

$$=A_1(\ldots(A_k(M\,(A_k^{-1}\,(\ldots(A_1\,^{-1}(A_1(\ldots(A_k(M'(A_k^{-1}\,(\ldots(A_1\,^{-1}X)\ldots)\bmod q=X,$$

$$A_1(\ldots(A_k(M(M'\,(A_k^{-1}\,(\ldots(A_1\,^{-1}X)\ldots)\bmod q=X.$$

We substitute $(A_1(\ldots(A_k\mathbf{1})\ldots)$ to X in above expression, we obtain

$$A_1(\ldots(A_k(MM')\ldots)=(A_1(\ldots(A_k\mathbf{1})\ldots)\bmod q.$$

$$MM'=\mathbf{1}\bmod q,$$

Then

$$M'=M^{-1} \bmod q,$$

That is

$$E(M, X)^{(-1)}=E(M^{-1}, X).$$

As the same manner, we obtain the same result from the equation

$$E(M', E(M, X))=X. \qquad \text{q.e.d.}$$

We define the element expression of $E(M, X)$ and $E(M', X)$ which is the inverse of $E(M, X)$ as follows.

$$E(M, X)=A_1(\ldots(A_k(M\,(A_k^{-1}\,(\ldots(A_1^{-1}X)\ldots) \bmod q \in O[X],$$

$$=(\, e_{00}x_0+e_{01}x_1+\ \ldots+e_{07}x_7,$$

$$e_{10}x_0+e_{11}x_1+\ \ldots+e_{17}\,x_7,$$

$$\ldots. \qquad \ldots.$$

$$e_{70}x_0+e_{71}x_1+\ \ldots+e_{77}\,x_7) \bmod q, \tag{43}$$

$$=\{e_{ij}\}(i,j=0,\ldots,7) \tag{44}$$

with $e_{ij}\in \boldsymbol{Fq}\ (i,j=0,\ldots,7)$.

$$E(M', X)=A_1(\ldots(A_k(M'\,(A_k^{-1}(\ldots(A_1^{-1}X)\ldots) \bmod q \in O[X],$$

$$=(\, e'_{00}x_0+e'_{01}x_1+\ \ldots+e'_{07}x_7,$$

$$e'_{10}x_0+e'_{11}x_1+\ \ldots+e'_{17}\,x_7,$$

$$\ldots. \qquad \ldots.$$

$$e'_{70}x_0+e'_{71}x_1+\ \ldots+e'_{77}\,x_7) \bmod q, \tag{45}$$

$$=\{e'_{ij}\}(i,j=0,\ldots,7) \tag{46}$$

with $e'_{ij}\in \boldsymbol{Fq}\ (i,j=0,\ldots,7)$.

Then we obtain

$$E(M, E(\mathrm{M}', X)) =$$

$$\{e_{00}(e'_{00}x_0+e'_{01}x_1+\ldots+e'_{07}x_7)+e_{01}(e'_{10}x_0+e'_{11}x_1+\ldots+e'_{17}\,x_7)+\ \ldots\ +e_{07}(e'_{70}x_0+e'_{71}x_1+\ \ldots+e'_{77}\,x_7),$$

$$e_{10}(e'_{00}x_0+e'_{01}x_1+\ldots+e'_{07}x_7)+e_{11}(e'_{10}x_0+e'_{11}x_1+\ldots+e'_{17}\,x_7)+\ \ldots\ +e_{17}(e'_{70}x_0+e'_{71}x_1+\ \ldots+e'_{77}\,x_7),$$

$$\ldots. \qquad \ldots.$$

$e_{70}(e'_{00}x_0+e'_{01}x_1+\ldots+e'_{07}x_7)+e_{71}(e'_{10}x_0+e'_{11}x_1+\ldots+e'_{17}x_7)+\ldots+e_{77}(e'_{70}x_0+e'_{71}x_1+\ldots+e'_{77}x_7)\}$

mod q.

From

$$E(M, E(M', X)) = X = (x_0,\ldots,x_7) \bmod q,$$

we obtain

$$\left.\begin{aligned} e_{00}e'_{00}+e_{01}e'_{10}+\ldots+e_{07}e'_{70}&=1 \bmod q\\ e_{10}e'_{00}+e_{11}e'_{10}+\ldots+e_{17}e'_{70}&=0 \bmod q\\ \cdots\quad&\cdots\\ e_{70}e'_{00}+e_{71}e'_{10}+\ldots+e_{77}e'_{70}&=0 \bmod q \end{aligned}\right\}$$

$$\left.\begin{aligned} e_{00}e'_{01}+e_{01}e'_{11}+\ldots+e_{07}e'_{71}&=0 \bmod q\\ e_{10}e'_{01}+e_{11}e'_{11}+\ldots+e_{17}e'_{71}&=1 \bmod q\\ \cdots\quad&\cdots\\ e_{70}e'_{01}+e_{71}e'_{11}+\ldots+e_{77}e'_{71}&=0 \bmod q \end{aligned}\right\}$$

$$\cdots\quad\cdots$$

$$\cdots\quad\cdots$$

$$\left.\begin{aligned} e_{00}e'_{07}+e_{01}e'_{17}+\ldots+e_{07}e'_{77}&=0 \bmod q\\ e_{10}e'_{07}+e_{11}e'_{17}+\ldots+e_{17}e'_{77}&=0 \bmod q\\ \cdots\quad&\cdots\\ e_{70}e'_{07}+e_{71}e'_{17}+\ldots+e_{77}e'_{77}&=1 \bmod q \end{aligned}\right\}$$

We solve the above 8 simultaneous equations so that we obtain the value of $e'_{ij} \in \boldsymbol{Fq}(i,j=0,\ldots,7)$.
That is,

$$E(M, X)^{(-1)} = E(M^{-1}, X) \bmod q$$

$$= (\, e'_{00}x_0+e'_{01}x_1+\ \ldots+e'_{07}x_7,$$

$$e'_{10}x_0+e'_{11}x_1+ \ldots+e'_{17}x_7,$$

$$\ldots \qquad \ldots$$

$$e'_{70}x_0+e'_{71}x_1+ \ldots+e'_{77}x_7) \bmod q, \tag{47}$$

$$=\{e'_{ij}\}(i,j=0,\ldots,7) \tag{48}$$

with $e'_{ij} \in \boldsymbol{Fq}$ $(i,j=0,\ldots,7)$.

§3.9 Division scheme on ciphertexts

Let $C_1(X)= E(M, X)$ and $C_2(X)= E(N, X)$ be the ciphertexts.

We try to make the ciphertext of M/N where N has the inverse.

First we calculate the inverse of $E(N,X)$ by using method described above to obtain $E(N,X)^{(-1)}$,

$$E(N, X)^{(-1)}= E(N^{-1}, X).$$

$$E(M, E(N, X)^{(-1)})= E(M, E(N^{-1}, X))$$

$$=A_1(\ldots(A_k(M(N^{-1}(A_k^{-1}(\ldots(A_1{}^{-1}X)\ldots) \bmod q \in O[X] \tag{49}$$

§3.10 Property of proposed fully homomorphic encryption

(**IND-CPA security**). Proposed fully homomorphic encryption is **IND-CPA** secure.

As adversary A_d does not know **sk**, A_d is not able to calculate M from the value of $E(M,X)$.

For any PPT adversary A_d it holds that:

$$\mathrm{Adv}^{\mathrm{CPA}}_{\mathrm{HE}}[\lambda] := |\Pr[A_d(E(M_0,X)) = 1] - \Pr[A_d((E(M_1,X))= 1]| = \mathrm{negl}(\lambda)$$

where **sk** $\leftarrow$**KeyGen**(1^λ).

(**Fully homomorphic encryption**). Proposed fully homomorphic encryption =(**KeyGen; Enc; Dec; Eval**) is fully homomorphic because it satisfies the following properties:

1. Homomorphism: Let $CR = \{CR_\lambda\}_{\lambda\in N}$ be the set of all polynomial sized arithmetic circuits. On input **sk** $\leftarrow$**KeyGen**(1^λ), $\forall$ckt $\in$ CR_λ, $\forall(m_1,\ldots,m_n) \in M^n$ where $n = n(\lambda)$, $\forall(\{c1_{ij}\},\ldots,\{cn_{ij}\})$ where $\{c_{rij}\} \leftarrow(E(M_r,X))$, $(r =1,\ldots,n)$,
we have D(**sk**;**Eval**(ckt; $\{c1_{ij}\},\ldots,\{cn_{ij}\}$)) = ckt$(m_1,\ldots,m_n)$.

Then it holds that:

$$\Pr[D(\mathbf{sk}; \mathbf{Eval}(\text{ckt}; \{c1_{ij}\},\ldots,\{cn_{ij}\})) \neq \text{ckt}(m_1,\ldots,m_n)] = \text{negl}(\lambda).$$

2. Compactness: As the output length of **Eval** is at most $k\log_2 q = k\lambda$ where k is a positive integer, there exists a polynomial $\mu = \mu(\lambda)$ such that the output length of **Eval** is at most μ bits long regardless of the input circuit ckt and the number of its inputs.

§4. Proposed fully homomorphic data-processing procedure

User A is the data holder generating the ciphertexts corresponding to the data which he holds. He wants that the information of the data which he holds is not revealed.

User B is the company which processes the enciphered data in cloud data centre without revealing and knowing the information of the data.

User A wants to obtain the result for processing the data, for example, stochastic value.

1)User A selects $A_i \in O$ randomly such that A_i^{-1} exists (i=1,...,k) which is the secret keys of user A.

Let M_i(i=0,…,n-1) be the plaintexts.

$C(X)=E(M_i, X)$ is defined as follows.

$$C(X)=E(M_i, X)=A_1(\ldots(A_k(M_i\,(A_k^{-1}\,(\ldots(A_1\,^{-1}X)\ldots)\ldots) \bmod q \in O[X]\,, \qquad (50a)$$

$$=(\, ei_{00}x_0+ei_{01}x_1+\ \ldots+ei_{07}x_7,$$

$$ei_{10}x_0+ei_{11}x_1+\ \ldots+ei_{17}\,x_7,$$

$$\ldots. \qquad \ldots.$$

$$ei_{70}x_0+ei_{71}x_1+\ \ldots+ei_{77}\,x_7), \qquad (50b)$$

$$= \{ei_{jk}\}(j,k=0,\ldots,7) \qquad (50c)$$

with $ei_{jk} \in Fq$ (i=0,…,n-1;j,k=0,…,7).

2) User A encrypts plaintexts $M_i \in O$ (i=0,...,n-1) to $C_i(X) \in O[X]$ (i=0,...,n-1) such that

$C_i(X)=E(M_i, X)$ (i=0,1,…,n-1)

where E(.) is the enciphering function to be secret.

3) User A sends [ei_{jk} (i=0,…,n-1;j,k=0,…,7)] to cloud data centre through the insecure line.

4) Later, user A sends his request for processing the data of user A to the user B.

For example 1, user A requests the value of the ciphertext corresponding to

$(M_1+M_2+M_3)/(M_2M_3) \in O$.

5) User B receives the request from user A and downloads q,$C_1(X)$,$C_2(X)$,$C_3(X)$ from

cloud data centre.He calculates the ciphertext $C^*(X)$ corresponding to $(M_1+M_2+M_3)/(M_2M_3)$ as follows.

$$C_1(X)=E(M_1,X)=A_1(\ldots(A_k(M_1\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q, \quad (51)$$

$$C_2(X)=E(M_2,X)=A_1(\ldots(A_k(M_2(A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q, \quad (52)$$

$$C_3(X)=E(M_3,X)=A_1(\ldots(A_k(M_3(A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q, \quad (53)$$

$$K_1(X):=C_1(X)+C_2(X)+C_3(X)=A_1(\ldots(A_k([M_1+M_2+M_3](A_k^{-1}(\ldots(A_1^{-1}X)\ldots)\bmod q, \quad (54)$$

$$K_2(X):=C_2(C_3(X))=E(M_2,E(M_3,X))=A_1(\ldots(A_k(M_2(M_3(A_k^{-1}(\ldots(A_1^{-1}X)\ldots)\bmod q, \quad (55)$$

$$K_3(X):=K_2(X)^{(-1)}\bmod q, \quad (56)$$

where $K_3(K_2(X))=K_2(K_3(X))=X \bmod q$.

$K_3(X)$ is obtained by using the method described in §3.8 Inverse of $E(M,X)$ and applying to $K_2(X)^{(-1)}$.

Then $K_3(X)$ is given such that

$$K_3(X)=A_1(\ldots(A_k(M_3^{-1}\ (M_2^{-1}\ (A_k^{-1}\ (\ldots(A_1^{-1}\mathrm{X})\ldots)\bmod q, \quad (57)$$

because

$$A_1(\ldots(A_k(M_2(M_3\ (A_k^{-1}\ (\ldots(A_1^{-1}\ (A_1(\ldots(A_k(M_3^{-1}\ (M_2^{-1}\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)$$

$$=A_1(\ldots(A_k(M_2(M_3\ (M_3^{-1}\ (M_2^{-1}\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)$$

$$=A_1(\ldots(A_k(M_2(M_3\ (M_3^{-1}\ (M_2^{-1}\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)=X \bmod q. \quad (58)$$

$$C^*(X)=K_1(K_3(X))\bmod q. \quad (59a)$$

$$=(c_{00}x_0+\ldots c_{07}x_7,$$

$$\ldots \quad \ldots$$

$$c_{70}x_0+\ldots+c_{77}x_7) \quad (59b)$$

$$=\{c_{ij}\}(i,j=0,\ldots,7) \quad (59c)$$

with $c_{ij}\in \boldsymbol{Fq}$ $(i,j=0,\ldots,7)$.

6) User B sends $C^*(X)$ as the ciphertext corresponding to the plaintext $(M_1+M_2+M_3)/(M_2M_3)$ to user A.

7) User A receives $C^*(X)$ from user B.

8) User A deciphers $C^*(X)$ to obtain the plaintext $(M_1+M_2+M_3)/(M_2M_3)$ by using the secret keys of user A , $A_i(i=0,\ldots,k)$ such that

$$A_k^{-1}\ (\ldots(A_1^{-1}\ (\ C^*[\ A_1(\ldots(A_k\mathbf{1})\ldots)]\ldots)$$

$= A_k^{-1}(\ldots(A_1^{-1}\{K_1(K_3[A_1(\ldots(A_k\mathbf{1})\ldots)])\})\ldots)$

$=A_k^{-1}(\ldots(A_1^{-1}(A_1(\ldots(A_k\{[M_1+M_2+M_3](A_k^{-1}(\ldots(A_1^{-1}(A_1(\ldots(A_k(M_3^{-1}(M_2^{-1}(A_k^{-1}(\ldots(A_1^{-1}$

$(A_1(\ldots(A_k\mathbf{1})\ldots)\}\ldots)$

$=[M_1 + M_2+ M_3](M_3^{-1}(M_2^{-1}(A_k^{-1}(\ldots(A_1^{-1}(A_1(\ldots(A_k\mathbf{1})\ldots)$

$=[M_1 + M_2+ M_3](M_3^{-1}(M_2^{-1}))$

$=[M_1 + M_2+ M_3](M_2M_3)^{-1}$

$=[M_1+M_2+M_3]/(M_2M_3) \bmod q.$ (60)

9) Next user A sends his request for processing the data of user A to the user B. For example 2, user A requests the value of the ciphertext corresponding to $(M_1+M_2+M_3)/(M_1+M_2M_3) \in O$.

10) User B receives the request from user A and he calculates the ciphertext $C^*(X)$ corresponding to $(M_1+M_2+M_3)/(M_1+M_2M_3)$ as follows.

$C_1(X)=E(M_1,X)=A_1(\ldots(A_k(M_1(A_k^{-1}(\ldots(A_1^{-1}X)\ldots) \bmod q,$ (61)

$C_2(X)=E(M_2,X)=A_1(\ldots(A_k(M_2(A_k^{-1}(\ldots(A_1^{-1}X)\ldots) \bmod q,$ (62)

$C_3(X)=E(M_3,X)=A_1(\ldots(A_k(M_3(A_k^{-1}(\ldots(A_1^{-1}X)\ldots) \bmod q,$ (63)

$K_1(X)=C_1(X)+C_2(X)+C_3(X)=A_1(\ldots(A_k([M_1+M_2+M_3](A_k^{-1}(\ldots(A_1^{-1}X)\ldots) \bmod q,$ (64)

$K_2(X)=C_2(C_3(X))=E(M_2,E(M_3,X))=A_1(\ldots(A_k(M_2(M_3(A_k^{-1}(\ldots(A_1^{-1}X)\ldots) \bmod q,$ (65)

$K_3(X)= C_1(X)+K_2(X)$

$=A_1(\ldots(A_k(M_1(A_k^{-1}(\ldots(A_1^{-1}X)\ldots)+ A_1(\ldots(A_k(M_2(M_3(A_k^{-1}(\ldots(A_1^{-1}X)\ldots) \bmod q,$

$=A_1(\ldots(A_k[M_1(A_k^{-1}(\ldots(A_1^{-1}X)\ldots)+M_2(M_3(A_k^{-1}(\ldots(A_1^{-1}X)\ldots)])\ldots) \bmod q,$ (66)

Let

$C_1^*(X):= K_1(X)$ (67)

and

$C_2^*(X):=K_3(X).$ (68)

11) User B sends $C_1^*(X)= K_1(X)$ and $C_2^*(X)=K_3(X)$ as the ciphertext to user A.

12) User A receives $C_1^*(X)= K_1(X)$ and $C_2^*(X)=K_3(X)$ from user B.

13) User A deciphers $C_1^*(X)= K_1(X)$ and $C_2^*(X)=K_3(X)$ to obtain the plaintext $(M_1+M_2+M_3)/(M_1+M_2M_3)$ by using the secret keys of user A as follows.

$R_1:=A_k^{-1}(\ldots(A_1^{-1}(C_1*(A_1(\ldots(A_k\mathbf{1})\ldots)$

$=A_k^{-1}(\ldots(A_1^{-1}(K_1(A_1(\ldots(A_k\mathbf{1})\ldots)\ldots)$

$=M_1+M_2+M_3 \bmod q,$ (69)

$R_2:=A_k^{-1}(\ldots(A_1^{-1}(C_2*(A_1(\ldots(A_k\mathbf{1})\ldots)$

$=A_k^{-1}(\ldots(A_1^{-1}(A_1(\ldots(A_k[M_1(A_k^{-1}(\ldots(A_1^{-1}(A_1(\ldots(A_k\mathbf{1})\ldots)+$

$M_2(M_3(A_k^{-1}(\ldots(A_1^{-1}(A_1(\ldots(A_k\mathbf{1})\ldots)])\ldots),$

$=M_1+M_2(M_3\,\mathbf{1}) \bmod q,$

$=M_1+M_2M_3 \bmod q.$ (70)

$R_1(R_2)^{-1}=(M_1+M_2+M_3)(M_1+M_2M_3)^{-1} \bmod q$

$=(M_1+M_2+M_3)/(M_1+M_2M_3) \bmod q.$ (71)

Here we notice that from theorem 11, $D\in O$ does not exist such that $C_2*(X)=A_k^{-1}(\ldots(A_1^{-1}(A_1(\ldots(A_k(D(A_k^{-1}(\ldots(A_1^{-1}(A_1(\ldots(A_kX)\ldots)$.

Theorem 11

Let $M_1,M_2,M_3\in O$ be non-associative each other.
$D\in O$ does not exist such that

$$M_1X+M_2(M_3X)=DX.$$

(*Proof*:)

When $X=\mathbf{1}$, we obtain

$$M_1+M_2M_3=D.$$

Then

$$M_1X+M_2(M_3X)=(M_1+M_2M_3)X \qquad ①$$

Let E be non-associative to M_2,M_3, that is,

$$M_2(M_3E)\neq(M_2M_3)E. \qquad ②$$

From ①,when $X=E$,

$$M_1E+M_2(M_3E)=(M_1+M_2M_3)E$$

$$M_2(M_3E)=(M_2M_3)E \qquad ③$$

It is contradictory.

q.e.d.

§5. Proposed re-encryption scheme

Now I describe the re-encryption schemes on octonion ring over Fq by using the proposed enciphering/deciphering functions.

User A selects $A_i \in O$ randomly such that A_i^{-1} exists (i=1,...,k) which is the secret keys of user A.

User A selects $B_i \in O$ randomly such that B_i^{-1} exists (i=1,...,k) which is the secret keys of user A.

Let M_i be a plaintext (i=0,…,n-1).

Ciphertext $E(M_i,X;A)$ *is* defined as follows.

$$E(M_i,X;A)=A_1(\dots(A_k(M_i(A_k^{-1}(\dots(A_1^{-1}X)\dots) \bmod q \in O[X], \quad (72a)$$

$$= (\, ei_{00}x_0+ei_{01}x_1+ \dots+ei_{07}x_7,$$

$$ei_{10}x_0+ei_{11}x_1+ \dots+ei_{17}x_7,$$

$$\dots. \qquad \dots.$$

$$ei_{70}x_0+ei_{71}x_1+ \dots+ei_{77}x_7) \quad \bmod q, \quad (72b)$$

$$=\{ei_{jk}\}(\, i=0,\dots,n\text{-}1;j,k=0,\dots,7) \quad (72c)$$

with $ei_{jk} \in \boldsymbol{Fq}$ (i=0,…,n-1;j,k=0,…,7) which is the secret keys of user A.

We show how to re-encipher $E(M_i,X;A)$ *to* $E(M_i,X;B)$ by using $B_i \in O$ as follows.

$$C_{A\text{-}B}=B_1(\dots(B_k(A_k^{-1}(\dots(A_1^{-1}(E[M_i,A_1(\dots(A_k(B_k^{-1}(\dots(B_1^{-1}X)\dots);A])\dots) \bmod q \quad (73a)$$

$$= B_1(\dots(B_k(M_i(B_k^{-1}(\dots(B_1^{-1}X)\dots) \bmod q$$

$$= E(M_i, X;B) \quad (73b)$$

User A sends the re-encryption keys $K_{RE}1$, $K_{RE}2$ to cloud data centre.

$$K_{RE}1(X):= B_1(\dots(B_k(A_k^{-1}(\dots(A_1^{-1}(X)\dots) \bmod q, \quad (74)$$

$$K_{RE}2(X):= A_1(\dots(A_k(B_k^{-1}(\dots(B_1^{-1}X)\dots) \bmod q, \quad (75)$$

where we notice that

$$K_{RE}1(K_{RE}2(X))$$

$$= B_1(\dots(B_k(A_k^{-1}(\dots(A_1^{-1}(A_1(\dots(A_k(B_k^{-1}(\dots(B_1^{-1}X)\dots)\dots)=X \quad \bmod q. \quad (76)$$

Cloud data centre replaces $E(M_i,X;A)$ into $E(M_i,X;B)$ by using the re-encryption

keys $K_{RE}1(X)$, $K_{RE}2(X)$ as follows.

$$K_{RE}1(E(M_i, K_{RE}2(X);A))= E(M_i, X;B)\ (i=0,\ldots,n\text{-}1). \quad (77)$$

Cloud data centre sends $E(M_i,X;B)$ to user B who wants information M_i (i=0,…,n-1).

User A sends the $K_{DE}1(X)$, $K_{DE}2(X)$ to user B such that

$$K_{DE}1(X):= B_1(\ldots(B_k(X)\ldots) \mod q, \quad (78)$$

$$K_{DE}2(X):= B_k^{-1}(\ldots(B_1^{-1}X)\ldots) \mod q, \quad (79)$$

where we notice that

$$K_{DE}1(K_{DE}2(X))=X \mod q. \quad (80)$$

User B deciphers $E(M_i,X;B)$ to obtain M_i as follows.

$$K_{DE}2(\ E(M_i, K_{DE}1(X);B))=M_i \mod q\ (i=0,\ldots,n\text{-}1). \quad (81)$$

§6. Analysis of proposed scheme

Here we analyze the proposed fully homomorphism encryption scheme.

§6.1 Computing plaintext M and A_i (i=1,…,k) from coefficients of ciphertext $E(M,X;A)$ to be published

Ciphertext $E(M,X;A)$ *is* published by cloud data centre as follows.

$$E(M,X;A)=A_1(\ldots(A_k(M(A_k^{-1}\ (\ldots(A_1\ ^{-1}X)\ldots) \mod q \in O[X]\ , \quad (82a)$$

$$=(\ e_{00}x_0+e_{01}x_1+\ \ldots+e_{07}x_7,$$

$$e_{10}x_0+e_{11}x_1+\ \ldots+e_{17}\ x_7,$$

$$\ldots\ldots\qquad\ldots\ldots$$

$$e_{70}x_0+e_{71}x_1+\ \ldots+e_{77}\ x_7) \mod q, \quad (82b)$$

$$=\{e_{jr}\}(j,r=0,\ldots,7) \quad (82c)$$

with $e_{jr}\in \boldsymbol{Fq}$ (j,r=0,…,7) which is published,
where $A_i \in O$ to be selected randomly such that A_i^{-1} exists (i=1,…,k) are the secret keys of user A.

We try to find plaintext M from coefficients of $E(M,X;A)$, $e_{jr}\in \boldsymbol{Fq}$ (j,r =0,…,7).

In case that n=7, the number of unknown variables (M, A_i (i=1,…,7)) is 64, the

number of equations is 65 such that

$$| E(M,\mathbf{1};A)|^2=e_{00}^2+e_{10}^2+\ldots+e_{70}^2 =|M|^2= m_0^2+m_1^2+\ldots+m_7^2 \bmod q, \quad (83)$$

$$\left.\begin{aligned} &F_{00}(M,A_1,A_2,\ldots,A_7)=e_{00} \bmod q,\\ &F_{01}(M,A_1,A_2,\ldots,A_7)=e_{01} \bmod q,\\ &\quad\cdots\qquad\cdots\\ &F_{07}(M,A_1,A_2,\ldots,A_7)=e_{07} \bmod q,\\ &\quad\cdots\qquad\cdots\\ &\quad\cdots\qquad\cdots\\ &F_{77}(M,A_1,A_2,\ldots,A_7)=e_{77} \bmod q, \end{aligned}\right\} \quad (84)$$

where $F_{00},\ldots,F_{77}$ are the 15^{th} algebraic multivariate equations.

Then the complexity G required for solving above simultaneous equations by using Gröbner basis is given [8]such as

$$G=({}_{64+dreg}C_{dreg})^w =({}_{512}C_{64})^w=O(2^{655})>> O(2^{80}), \quad (85)$$

where w=2.39, and

$$d_{reg} = 448(=64*(15-1)/2+(2-1)/2 - 0\sqrt{}(64*(225-1)/6+(4-1)/6)). \quad (86)$$

Next we try to find plaintexts M_1 and M_2 from two ciphertexts such as

$$E(M_1,X;A)=A_1(\ldots(A_k(M_1(A_k^{-1} (\ldots(A_1{}^{-1}X)\ldots) \bmod q\in O[X] , \quad (87)$$

$$E(M_2,X;A)=A_1(\ldots(A_k(M_2(A_k^{-1} (\ldots(A_1{}^{-1}X)\ldots) \bmod q\in O[X]. \quad (88)$$

We obtain 130 equations as follows.

$$| E(M_i,\mathbf{1};A)|^2=e_{i00}^2+e_{i10}^2+\ldots+e_{i70}^2 =|M_i|^2= m_{i0}^2+m_{i1}^2+\ldots+m_{i7}^2 \bmod q, \quad (89)$$

$$\left.\begin{aligned} F_{00}(M_i,A_1,A_2,\ldots,A_7)&=e_{i00} \bmod q,\\ F_{01}(M_i,A_1,A_2,\ldots,A_7)&=e_{i01} \bmod q,\\ \cdots \quad & \quad \cdots\\ F_{07}(M_i,A_1,A_2,\ldots,A_7)&=e_{i07} \bmod q,\\ \cdots \quad & \quad \cdots\\ \cdots \quad & \quad \cdots\\ F_{77}(M_i,A_1,A_2,\ldots,A_7)&=e_{i77} \bmod q, \end{aligned}\right\} \quad (90)$$

$$(i=1,2),$$

where $F_{00},\ldots, F_{77}$ are the 15th algebraic multivariate equations.

The number of unknown variables (M_1, M_2, A_i (i=1,…,7)) is 72, and the number of equations is 130.

Then the complexity G required for solving above simultaneous equations by using Gröbner basis is given such that

$$G=({}_{72+dreg}C_{dreg})^w, \quad (91)$$

where we are not able to compute the d_{reg} accurately.

So we adopt 15 as d_{reg}.

$$G>G'=({}_{72+dreg}C_{dreg})^w=({}_{87}C_{15})^w=O(2^{130})>>O(2^{80}). \quad (92)$$

The complexity G required for solving above simultaneous equations by using Gröbner basis is enough large for secure.

§6.2 Computing plaintext M_i and d_{ijk} (i,j,k=0,…,7)

We try to computing plaintext M_i and d_{ijk} (i,j,k=0,...,7) from coefficients of ciphertext $E(M_i,X;A)$ to be published.

At first let $E(Y,X,A) \in O[X,Y]$ be the enciphering function such as

$$E(Y,X;A):=A_1(\ldots(A_k(Y\,(A_k^{-1}\,(\ldots(A_1\,^{-1}X)\ldots) \bmod q \in O[X,Y]\,, \quad (93a)$$

$$=(d_{000}x_0y_0+d_{001}x_0y_1+\ \ldots+d_{077}x_7y_7,$$

$$d_{100}x_0y_0+d_{101}x_0y_1+\ \ldots+d_{177}x_7y_7,$$

$$\ldots\ldots \quad \ldots\ldots$$

$$d_{700}x_0y_0+d_{701}x_0y_1+\ \ldots+d_{777}x_7y_7) \bmod q, \quad (93b)$$

$$=\{d_{ijk}\}(i,j,k=0,\ldots,7) \quad (93c)$$

with $d_{ijk} \in \boldsymbol{Fq}$ $(i,j,k = 0,\ldots,7)$.

Next we substitute M_i to Y, where

$$M_i=(m_{i0},m_{i1},\ldots,m_{i7}) \in O. \tag{94}$$

We have

$$E(M_i,X;A):=A_1(\ldots(A_k(M_i\,(A_k^{-1}\,(\ldots(A_1\,^{-1}X)\ldots) \bmod q \in \quad O[X]\,, \tag{95a}$$

$$=(d_{000}x_0m_0+d_{001}x_0m_1+\ \ldots+d_{077}x_7m_7,$$

$$d_{100}x_0m_0+d_{101}x_0m_1+\ \ldots+d_{177}x_7m_7,$$

$$\ldots. \qquad\qquad \ldots.$$

$$d_{700}x_0m_0+d_{701}x_0m_1+\ \ldots+d_{777}x_7m_7) \bmod q, \tag{95b}$$

$$=\{d_{ijk}\}(i,j,k=0,\ldots,7) \tag{95c}$$

with $d_{ijk} \in \boldsymbol{Fq}$ $(i,j,k = 0,\ldots,7)$.

Then we obtain 64 equations from (43) and (95b) as follows.

$$\left.\begin{array}{l} d_{000}m_{i0}+d_{001}m_{i1}+\ \ldots+d_{007}m_{i7}=e_{00} \\ d_{010}m_{i0}+d_{011}m_{i1}+\ \ldots+d_{017}m_{i7}=e_{01} \\ \qquad\ldots. \qquad\qquad \ldots. \\ d_{070}m_{i0}+d_{071}m_{i1}+\ \ldots+d_{077}m_{i7}=e_{07} \end{array}\right\} \tag{96a}$$

$$\left.\begin{array}{l} d_{100}m_{i0}+d_{101}m_{i1}+\ \ldots+d_{107}m_{i7}=e_{10} \\ d_{110}m_{i0}+d_{111}m_{i1}+\ \ldots+d_{117}m_{i7}=e_{11} \\ \quad\ldots. \qquad\qquad \ldots. \\ d_{170}m_{i0}+d_{171}m_{i1}+\ \ldots+d_{177}m_{i7}=e_{17} \end{array}\right\} \tag{96b}$$

$$\begin{array}{l} \ldots. \qquad\qquad \ldots. \\ \ldots. \qquad\qquad \ldots. \end{array}$$

$$\left.\begin{array}{l} d_{700}m_{i0}+d_{701}m_{i1}+\ \ldots+d_{707}m_{i7}=e_{70} \\ d_{710}m_{i0}+d_{711}m_{i1}+\ \ldots+d_{717}m_{i7}=e_{71} \\ \qquad\ldots. \qquad\qquad \ldots. \\ d_{770}m_{i0}+d_{771}m_{i1}+\ \ldots+d_{777}m_{i7}=e_{77} \end{array}\right\} \tag{96c}$$

For $M_1,\ldots, M_8$ we obtain the same equations, the number of which is 512.

We also obtain the 8 equations such as

$$| E(M_i,\mathbf{1};A)|^2= |M_i|^2= m_{i0}^2+m_{i1}^2+\ldots+m_{i7}^2 \bmod q,(i=0,\ldots,7). \quad (97)$$

The number of unknown variables M_i and d_{ijk} $(i,j,k=0,...,7)$ is 576(=512+64).

The number of equations is 520(=512+8).

Then the complexity G required for solving above simultaneous quadratic algebraic equations by using Gröbner basis is given such as

$$G\approx G'=({}_{520+dreg}C_{dreg})^w =({}_{763}C_{243})^w=O(2^{1634})>> 2^{80}, \quad (98)$$

where G' is the complexity required for solving 520 simultaneous quadratic algebraic equations with 520 variables by using Gröbner basis,

where w=2.39,

and

$$d_{reg} = 243(=520*(2-1)/2 - \sqrt{}(520*(4-1)/6) \quad (99)$$

It is thought to be difficult computationally to solve the above simultaneous algebraic equations by using Gröbner basis.

§7. The size of the modulus q and the complexity for enciphering/ deciphering

We consider the size of the system parameter q. We select the size of q such that $O(q^8)$,the order of the plaintext is larger than $O(2^{80})$. Then we need to select modulus $q=O(2^{10})$.

In case of k=7, $q=O(2^{10})$, the size of $e_{ij}\in \boldsymbol{Fq}$ (i,j=0,…,7) which are the coefficients of elements in $E(M,X;A)=A_1(\ldots(A_k(M(A_k^{-1}(\ldots(A_1^{-1}X)\ldots)$ mod $q \in O[X]$ is $(64)(\log_2 q)$bits =640bits, and the size of system parameters $[q]$ is less than 20bits.

In case of k=7 , $q=O(2^{10})$, the complexity to obtain $E(M,X;A)$ is

$(15*512+14*36)(\log_2 q)^2= O(2^{20})$ bit-operations

and the complexity required for deciphering is

$(576)(\log_2 q)^2 = O(2^{16})$ bit-operations.

The complexity to obtain $E(M,X;A)$ by using the element expression of $E(M,X;A)$ is

$(512)(\log_2 q)^2 = O(2^{16})$ bit-operations.

On the other hand the complexity of the enciphering and deciphering in RSA scheme is

$O(2(\log n)^3) = O(2^{34})$ bit-operations

where the size of modulus n is 2048bits.

Then our scheme requires small memory space and complexity to encipher and decipher so that we are able to implement our scheme to the mobile device.

§8. Conclusion

We proposed the new fully homomorphism encryption scheme based on the octonion ring over finite field that requires small memory space and complexity to encipher and decipher. It was shown that our scheme is immune from the Gröbner basis attacks by calculating the complexity to obtain the Gröbner basis for the multivariate algebraic equations**.**

The proposed scheme does not require a "bootstrapping" process so that the complexity to encipher and decipher is not large.

§9. Considering composition of plaintext

At end of this chapter we consider the composition of the plaintext.
In section 3 and 4 we adopt $M=(m_0,\ldots,m_7)\in O$ as the plaintext where $m_i \in Fq$ $(i=0,\ldots,7)$.

The scheme described in section 3 and 4 has the homomorphic property, that is,

$$M=(m_0,\ldots,m_7),\ N=(n_0,\ldots,n_7)$$

$$E(M,X)+E(N,X)=E(M+N,X) \bmod q,$$

$$E(M,E(N,X))=E(MN,X) \bmod q.$$

We notice that

$$M+N \bmod q = (m_0,\ldots,m_7) + (n_0,\ldots,n_7) \bmod q$$

$$= (m_0+n_0 \bmod q,\ldots, m_7+n_7 \bmod q).$$

$$MN \bmod q$$

$$= (m_0n_0-m_1n_1- m_2n_2- m_3n_3-m_4n_4- m_5n_5-m_6n_6-m_7n_7 \bmod q,$$

$$m_0n_1+m_1n_0+m_2n_4+m_3n_7-m_4n_2+m_5n_6-m_6n_5-m_7n_3 \bmod q,$$

$$m_0n_2-m_1n_4+m_2n_0+m_3n_5+m_4n_1-m_5n_3+m_6n_7-m_7n_6 \bmod q,$$

$$m_0n_3-m_1n_7-m_2n_5+m_3n_0+m_4n_6+m_5n_2-m_6n_4+m_7n_1 \bmod q,$$

$$m_0n_4+m_1n_2 - m_2n_1 - m_3n_6+m_4n_0+m_5n_7+m_6n_3 - m_7n_5 \bmod q,$$

$$m_0n_5 - m_1n_6+m_2n_3 - m_3n_2 - m_4n_7+m_5n_0+m_6n_1+m_7n_4 \bmod q,$$

$$m_0n_6+m_1n_5 - m_2n_7+m_3n_4 - m_4n_3 - m_5n_1+m_6n_0+m_7n_2 \bmod q,$$

$$m_0n_7+m_1n_3+m_2n_6 - m_3n_1+m_4n_5 - m_5n_4 - m_6n_2+m_7n_0 \bmod q).$$

From the viewpoint of m_i and n_i, it is not clear that this scheme has multiplicative homomorphism.

For practical use, we can also adopt the following method to have the multiplicative homomorphism clearly.

We select the element $B=(b_0,\ldots,b_7)\in O$ such that,

$$L_B:=|B|^2= b_0^2+b_1^2+\ldots+b_7^2 \bmod q=0,$$

and

$$b_0\neq 0 \bmod q.$$

We define the partial set of O, S_B by

$$S_B:=\{B=(b_0,\ldots,b_7)\in O \mid |B|^2= b_0^2+b_1^2+\ldots+b_7^2 \bmod q=0,\ b_0\neq 0 \bmod q\ \}.$$

Let u be the plaintext to belong to the set of the plaintext $P=\{u \mid u\in \boldsymbol{Fq}\}$.

Let $v\in \boldsymbol{Fq}$ be the random number.

We define the medium text M by

$$M:=u\mathbf{1}+vB\in O,$$

and

$$|M|^2=(u+vb_0)^2+v^2(b_1^2+\ldots+b_7^2)\neq 0 \bmod q.$$

Then

$$u^2+2vb_0u+v^2(b_0^2+b_1^2+\ldots+b_7^2)\neq 0 \bmod q,$$

$$(u+2vb_0)\,u\neq 0 \bmod q,$$

We have

$$u\neq 0 \bmod q$$

and

$$u+2vb_0\neq 0 \bmod q.$$

Theorem 12

$$M^q=(u\mathbf{1}+vB)^q \bmod q = M = u\mathbf{1}+vB\in O,$$

(*Proof*:)

From Theorem 3

$$B^2= -L\mathbf{1}+2\,b_0B=2\,b_0B \bmod q,$$

$$M^2=(u\mathbf{1}+vB)\,(u\mathbf{1}+vB) \bmod q,$$

$$=u^2\mathbf{1}+(uv+vu)B+v^2B^2 \bmod q,$$

$$= u^2\mathbf{1}+(2uv+2\,b_0v^2)B \bmod q \bmod q,$$

$$M^3=(u^2\mathbf{1}+(2uv+2\,b_0v^2)B)\,(u\mathbf{1}+vB) \bmod q,$$

$$= u^3\mathbf{1}+(3u^2v+2\,b_0uv^2+2\,b_0\,(2uv+2\,b_0v^2)v)B \bmod q,$$

$$= u^3\mathbf{1}+(3u^2v+3(2\,b_0)uv^2+(2\,b_0)^2v^3)B \bmod q,$$

$$M^4=(u^3\mathbf{1}+(3u^2v+3(2\,b_0)uv^2+(2\,b_0)^2v^3)B)\,(u\mathbf{1}+vB) \bmod q,$$

$$= u^4\mathbf{1}+(4u^3v+4(2\,b_0)u^2v^2+4(2\,b_0)^2uv^3+(2\,b_0)^3v^4)B \bmod q,$$

…. …. ….

$$M^q= u^q\mathbf{1}+(qu^{q-1}v+q(2\,b_0)u^{q-2}v^2+\ldots+q(2\,b_0)^{q-2}uv^{q-1}+(2\,b_0)^{q-1}v^q)B \bmod q$$

$$= u\mathbf{1}+(2\,b_0)^{q-1}v^qB \bmod q.$$

Because $(2\ b_0)^{q-1}=1 \bmod q$ and $v^{\,q}=v \bmod q$,

we have

$$M^{\,q} = u\mathbf{1}+vB=M \bmod q. \qquad \text{q.e.d.}$$

Lemma 7

$$M_1M_2= u_1u_2\mathbf{1}+(u_1v_2+v_1u_2 +2v_1v_2\ b_0)B \bmod q$$

where

$$M_1=(u_1\mathbf{1}+v_1B),$$

$$M_2=(u_2\mathbf{1}+v_2B).$$

(*proof*)

$$M_1M_2=(u_1\mathbf{1}+v_1B)\ (\mathrm{u}m_2\mathbf{1}+v_2B)$$

$$=u_1u_2\mathbf{1}+(u_1v_2+v_1u_2)B+v_1v_2B^2.$$

From $B^2= 2\ b_0B \bmod q$,

$$=u_1u_2\mathbf{1}+(u_1v_2+v_1u_2)B+v_1v_2 2\ b_0\mathrm{B}$$

$$=u_1u_2\mathbf{1}+(u_1v_2+v_1u_2 +2v_1v_2\ b_0)B \qquad \text{q.e.d.}$$

Here we consider the multiplicative operation on the ciphertexts.
From section 3 “Multiplication scheme on ciphertexts” we have

$$C_1(C_2(X)) \bmod q =E(M_1\ ,\ E\ (M_2,X)) \bmod q$$

where

$$C_1(X)= E(M_1,X) =A_1(\ldots(A_k(M_1(A_k^{-1}\ (\ldots(A_1\ ^{-1}\mathrm{X})\ldots) \bmod q$$

and

$$C_2(X)= E\ (M_2,X) =A_1(\ldots(A_k(M_2(A_k^{-1}\ (\ldots(A_1\ ^{-1}\mathrm{X})\ldots) \bmod q$$

are the ciphertexts.

Then we have

$$C_1(C_2(X)) \bmod q =E(M_1\ ,\ E\ (M_2,X)) \bmod q$$

$$=A_1(\ldots(A_k(M_1(A_k^{-1}\ (A_1^{-1}(A_1\ (\ldots(A_k(M_2(A_k^{-1}\ (\ldots(A_1\ ^{-1}X)\ldots) \bmod q$$

$$= A_1(\ldots(A_k(M_1(M_2\ (A_k^{-1}\ (\ldots(A_1\ ^{-1}X)\ldots) \bmod q.$$

Substituting $(u_1\mathbf{1}+v_1B)$, $(u_2\mathbf{1}+v_2B)$ to M_1,M_2 we have

$$= A_1(\ldots(A_k((u_1\mathbf{1}+v_1B)\ ((u_2\mathbf{1}+v_2B)\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q.$$

$$= A_1(\ldots(A_k((u_1\mathbf{1})((u_2\mathbf{1}+v_2B)\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q.$$

$$+ A_1(\ldots(A_k((v_1B)\ ((u_2\mathbf{1}+v_2B)\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q.$$

$$= A_1(\ldots(A_k((u_1\mathbf{1})((u_2\mathbf{1})\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q.$$

$$+ A_1(\ldots(A_k((u_1\mathbf{1})((v_2B)\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q$$

$$+ A_1(\ldots(A_k((v_1B)\ ((m_2\mathbf{1})\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q$$

$$+ A_1(\ldots(A_k((v_1B)\ ((v_2B)\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q.$$

$$= A_1(\ldots(A_k((u_1u_2\mathbf{1})\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q.$$

$$+ A_1(\ldots(A_k((u_1v_2B)\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q.$$

$$+ A_1(\ldots(A_k((v_1u_2B)\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q$$

$$+ A_1(\ldots(A_k((v_1v_2B\,B)\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q.$$

$$= A_1(\ldots(A_k((u_1u_2\mathbf{1}+ u_1v_2B + v_1u_2B + v_1v_2B\,B)\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q$$

$$= A_1(\ldots(A_k([u_1u_2\mathbf{1}+ (u_1v_2 + v_1u_2 +2b_0\ v_1v_2)B]\ (A_k^{-1}\ (\ldots(A_1^{-1}X)\ldots)\bmod q.$$

Then we have

$$C_1(C_2(X))\bmod q = E(M_1\ ,E\ (M_2,X))\bmod q$$

$$= E((u_1\mathbf{1}+v_1B)\ ,E\ ((u_2\mathbf{1}+v_2B),X))\bmod q$$

$$= E((u_1u_2\mathbf{1}+ (u_1v_2 + v_1u_2 +2b_0\ v_1v_2)B)\ ,\ X))\bmod q.$$

It is shown that in this method we have the multiplicative homomorphism on the plaintext u clearly.

§10.Acknowledgments

This paper is the revised chapter 4 of my work “Fully Homomorphic Encryption without bootstrapping” published in March, 2015 which was published by LAP LAMBERT Academic Publishing, Saarbrücken/Germany [1].

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Appendix A:

Octinv(*A*) ----------

S ← $a_0^2+a_1^2+\ldots+a_7^2$ mod *q*.
% S^{-1} mod *q*
q[1] ← q div S ;% integer part of q/S
r[1] ← q mod S ;% residue
k ←1
q[0] ← q
r[0] ← S
while r[k] ≠ 0
begin
k← k + 1
q[k] ← r[k−2] div r[k−1]
r[k] ← r[k−2] mod [rk−1]
end
Q [k−1] ← (-1)∗q[k−1]
L[k−1] ← 1
i ← k−1
while i > 1
begin
Q[i−1] ← (-1)∗Q[i] ∗q[i−1] + L[i]
L[i−1] ← Q[i]
i← i−1
end

invS ← Q[1] mod *q*
invA[0] ← a_0∗invS mod *q*
For *i*=1,…,7,
invA[*i*] ← (-1)∗a_i∗invS mod *q*
Return A^{-1}= (invA[0], invA[1],…, invA[7])

Appendix B:

Theorem 1

Let $A=(a_{10},a_{11},\ldots,a_{17})\in O$, $a_{1j}\in \boldsymbol{Fq}$ $(j=0,1,\ldots,7)$.

Let $A^n=(a_{n0},a_{n1},\ldots,a_{n7})\in O$, $a_{nj}\in \boldsymbol{Fq}$ $(n=1,\ldots,7;j=0,1,\ldots,7)$.

a_{00}, a_{nj} 's$(n=1,2,\ldots;j=0,1,\ldots)$and b_n's $(n=0,1,\ldots)$ satisfy the equations such that

$N= a_{11}^2+\ldots+a_{17}^2 \bmod q$

$a_{00}=1,\ b_0=0,\ b_1=1,$

$a_{n0}= a_{n-1,0}\,a_{10} - b_{n-1}N \bmod q\ ,(n=1,2,\ldots)$ (8)

$b_n= a_{n-1,0}+ b_{n-1}a_{10} \bmod q\ ,(n=1,2,\ldots)$ (9)

$a_{nj}= b_n a_{1j} \bmod q\ ,(n=1,2,\ldots;j=1,2,\ldots,7)$. (10)

(*Proof*:)

We use mathematical induction method.

[step 1]

When $n=1$, (8) holds because

$a_{10}= a_{00}\,a_{10} - b_0N=a_{10} \bmod q.$

(9) holds because

$b_1= a_{00}+ b_0a_{10} =a_{00} =1 \bmod q.$

(10) holds because

$a_{1j}= b_1a_{1j} = a_{1j} \bmod q\ ,(j=1,2,\ldots,7)$

[step 2]

When $n=k$,

If it holds that

$a_{k0}= a_{k-1,0}\,a_{10} - b_{k-1}N \bmod q\ ,(k=2,3,4,\ldots)$,

$b_k= a_{k-1,0}+ b_{k-1}a_{10} \bmod q,$

$a_{kj}= b_k a_{1j} \bmod q\ ,(j=1,2,\ldots,7),$

from (9)

$b_{k-1}= a_{k-2,0}+ b_{k-2}a_{10} \bmod q\ ,(k=2,3,4,\ldots),$

then

$A^{k+1}=A^kA=(a_{k0}, b_ka_{11},\ldots, b_ka_{17})(a_{10},a_{11},\ldots,a_{17})$

$=(a_{k0}\,a_{10} - b_kN,\ a_{k0}\,a_{11}+ b_ka_{11}\,a_{10},\ldots,\ a_{k0}\,a_{17}+ b_ka_{17}\,a_{10})$

$=(a_{k0}\,a_{10} - b_kN,\ (a_{k0} + b_ka_{10})a_{11},\ldots,\ (a_{k0} + b_ka_{10})a_{17})$

$=(a_{k+1,0},\ b_{k+1,0}\,a_{11},\ldots,\ b_{k+1,0}\,a_{17}),$

as was required. q.e.d.

Appendix C:

Theorem 2

For an element $A=(a_{10},a_{11},\ldots,a_{17})\in O$,

$A^{J+1}=A \bmod q$,

where

$J:= LCM\ \{q^2-1,q-1\}=q^2-1$,

$N:=a_{11}^2+ a_{12}^2+\ldots+a_{17}^2\neq 0 \bmod q$.

(*Proof*:)

From (8) and (9) it comes that

$a_{n0}= a_{n-1,0}\, a_{10} - b_{n-1}N \bmod q$,

$b_n= a_{n-1,0}+ b_{n-1}a_{10} \bmod q$,

$a_{n0}\, a_{10} + b_n\, N= (a_{n-1,0}\, a_{10} - b_{n-1}N)\, a_{10} +(a_{n-1,0}+ b_{n-1}a_{10})N= a_{n-1,0}\, a_{10}^2 + a_{n-1,0}\, N \bmod q$,

$b_n\, N= a_{n-1,0}\, a_{10}^2 + a_{n-1,0}\, N- a_{n0}\, a_{10} \bmod q$,

$b_{n-1}\, N= a_{n-2,0}\, a_{10}^2 + a_{n-2,0}\, N- a_{n-1,0}\, a_{10} \bmod q$,

$a_{n0}= 2\, a_{10}a_{n-1,0} - (\, a_{10}^2 +N)\, a_{n-2,0} \bmod q$, $(n=1,2,\ldots)$.

1) In case that $-N\neq 0 \bmod q$ is quadratic non-residue of prime q,

Because $-N\neq 0 \bmod q$ is quadratic non-residue of prime q,

$(-N)^{(q-1)/2}=-1 \bmod q$.

$a_{n0} - 2\, a_{10}\, a_{n-1,0} +(\, a_{10}^2 +N)\, a_{n-2,0}=0 \bmod q$,

$a_{n0}=(\beta^n(a_{10}-\alpha) + (\beta- a_{10})\alpha^n)/(\, \beta- \alpha)$ *over* $Fq[\alpha]$

$b_n=(\beta^n-\alpha^n)/(\, \beta- \alpha)$ *over* $Fq[\alpha]$

where α,β are roots of algebraic quadratic equation such that

$t^2-2a_{10}t+a_{10}^2+N=0$.

$\alpha = a_{10} + \sqrt{-N}$ *over* $Fq[\alpha]$,

$\beta = a_{10} - \sqrt{-N}$ *over* $Fq[\alpha]$.

We can calculate β^{q^2} as follows.

$\beta^{q^2} = (a_{10} - \sqrt{-N})^{q^2}$ *over* $Fq[\alpha]$

$= (a_{10}{}^{q} - \sqrt{-N}(-N)^{(q-1)/2})^{q}$ *over* $Fq[\alpha]$

$= (a_{10} \quad - \sqrt{-N}(-N)^{(q-1)/2})^{q}$ *over* $Fq[\alpha]$

$= (a_{10}{}^{q} - \sqrt{-N}(-N)^{(q-1)/2}(-N)^{(q-1)/2})$ *over* $Fq[\alpha]$

$= a_{10} \quad - \sqrt{-N}(-1)(-1) \quad$ *over* $\boldsymbol{Fq}[\alpha]$

$= a_{10} - \sqrt{-N}$ *over* $\boldsymbol{Fq}[\alpha]$

$= \beta$ *over* $Fq[\alpha]$.

In the same manner we obtain

$\alpha^{q^2} = \alpha$ *over* $\boldsymbol{Fq}[\alpha]$.

$a_{q^2,0} = (\beta^{q^2}(a_{10} - \alpha) + (\beta - a_{10})\alpha^{q^2})/(\beta - \alpha)$

$=(\beta(a_{10}-\alpha) + (\beta- a_{10})\alpha)/(\beta- \alpha)=a_{10} \mod q.$

$b_{q^2} = (\beta^{q^2} - \alpha^{q^2})/(\beta - \alpha) = 1 \mod q.$

Then we obtain

$A^{q2}=(a_{q2,0}, b_{q2}a_{11}, \ldots, b_{q2}a_{17})$

$=(a_{10}, a_{11}, \ldots, a_{17})=A \mod q$

2) In case that $-N \neq 0 \mod q$ is quadratic residue of prime q

$a_{n0}=(\beta^n(a_{10}-\alpha) + (\beta- a_{10})\alpha^n)/(\beta- \alpha) \mod q,$

$b_{n0}=(\beta^n-\alpha^n)/(\beta- \alpha) \mod q ,$

As $\alpha,\beta \in \boldsymbol{Fq}$, from Fermat's little Theorem

$\beta^q=\beta \mod q,$

$\alpha^q=\alpha \mod q.$

Then we have

$a_{q0}=(\beta^q(a_{10}-\alpha) + (\beta- a_{10})\alpha^q)/(\beta- \alpha) \mod q$

$=(\beta(a_{10}-\alpha) + (\beta- a_{10})\alpha)/(\beta- \alpha) \mod q$

$=a_{10} \mod q$

$b_q=(\beta^q-\alpha^q)/(\beta- \alpha)=1 \mod q.$

Then we have

$a^q=(a_{q0},b_q a_{11},\ldots,b_q a_{17})$

$=(a_{10}, a_{11},\ldots,a_{17})=a \bmod q.$

We therefore arrive at the equation such as

$A^{J+1}=A \bmod q$ for arbitrary element $A\in O$,

where

$J=$ LCM $\{ q^2-1, q-1\}=q^2-1,$

as was required. q.e.d.

We notice that

in case that $-N=0 \bmod q$

$a_{00}=1, b_0=0, b_1=1,$

From (8)

$a_{n0}= a_{n-1,0}\, a_{10} \bmod q ,(n=1,2,\ldots),$

then we have

$a_{n0}= a_{10}{}^n \bmod q ,(n=1,2,\ldots).$

$a_{q0}= a_{10}{}^q = a_{10} \bmod q.$

From (9),

$b_n= a_{n-1,0}+ b_{n-1}a_{10} \bmod q ,(n=1,2,\ldots)$

$= a_{10}{}^{n-1}+ b_{n-1}a_{10} \bmod q$

$= 2a_{10}{}^{n-1}+ b_{n-2}a_{10}{}^2 \bmod q$

… …

$= (n-1)a_{10}{}^{n-1}+ b_1 a_{10}{}^{n-1} \bmod q$

$= na_{10}{}^{n-1} \bmod q.$

Then we have

$a_{nj}= na_{10}{}^{n-1}a_{1j} \bmod q ,(n=1,2,\ldots;j=1,2,\ldots,7) .$

$a_{qj}= qa_{10}{}^{q-1}a_{1j} \bmod q =0,(j=1,2,\ldots,7) .$

Appendix D:

Lemma 2

$$A^{-1}(AB)= B$$
$$(BA)A^{-1}= B$$

(*Proof*:)

$A^{-1}= (a_0/|A|^2 \bmod q, -a_1/|A|^2 \bmod q,\ldots, -a_7/|A|^2 \bmod q)$.

$AB \bmod q$

$= (a_0b_0-a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7 \bmod q,$
$a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3 \bmod q,$
$a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6 \bmod q,$
$a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1 \bmod q,$
$a_0b_4+a_1b_2-a_2b_1-a_3b_6+a_4b_0+a_5b_7+a_6b_3-a_7b_5 \bmod q,$
$a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4 \bmod q,$
$a_0b_6+a_1b_5-a_2b_7+a_3b_4-a_4b_3-a_5b_1+a_6b_0+a_7b_2 \bmod q,$
$a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0 \bmod q).$

$[A^{-1}(AB)]_0$
$=\{ a_0(a_0b_0-a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7)$
$+a_1(a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3)$
$+ a_2(a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6)$
$+a_3(a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1)$
$+a_4(a_0b_4+a_1b_2-a_2b_1-a_3b_6+a_4b_0+a_5b_7+a_6b_3-a_7b_5)$
$+ a_5(a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4)$
$+a_6(a_0b_6+a_1b_5-a_2b_7+a_3b_4-a_4b_3-a_5b_1+a_6b_0+a_7b_2)$
$+a_7(a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0)\} /|A|^2 \bmod q$

$=\{(a_0^2+a_1^2+\ldots+a_7^2)\, b_0\} /|A|^2 = b_0 \bmod q$

where $[M]_n$ denotes the n-th element of $M \in O$.

$[A^{-1}(AB)]_1$
$=\{ a_0(a_0b_1+a_1b_0+a_2b_4+a_3b_7-a_4b_2+a_5b_6-a_6b_5-a_7b_3)$
$-a_1(a_0b_0-a_1b_1- a_2b_2- a_3b_3-a_4b_4- a_5b_5-a_6b_6-a_7b_7)$
$-a_2(a_0b_4+a_1b_2-a_2b_1-a_3b_6+a_4b_0+a_5b_7+a_6b_3-a_7b_5)$

$-a_3(a_0b_7+a_1b_3+a_2b_6-a_3b_1+a_4b_5-a_5b_4-a_6b_2+a_7b_0)$

$+a_4(a_0b_2-a_1b_4+a_2b_0+a_3b_5+a_4b_1-a_5b_3+a_6b_7-a_7b_6)$

$-a_5(a_0b_6+a_1b_5-a_2b_7+a_3b_4-a_4b_3-a_5b_1+a_6b_0+a_7b_2)$

$+a_6(a_0b_5-a_1b_6+a_2b_3-a_3b_2-a_4b_7+a_5b_0+a_6b_1+a_7b_4)$

$+a_7(a_0b_3-a_1b_7-a_2b_5+a_3b_0+a_4b_6+a_5b_2-a_6b_4+a_7b_1)\}/|A|^2 \bmod q$

$=\{(a_0^2+a_1^2+\ldots+a_7^2)\,b_1\}/|A|^2=b_1 \bmod q.$

Similarly we have

$[A^{-1}(AB)]_i=b_i \bmod q$ $(i=2,3,\ldots,7)$.

Then

$$A^{-1}(AB)= B \bmod q. \qquad \text{q.e.d.}$$