A SAT-Based Algorithm for Finding Short Cycles in Shift Register Based Stream Ciphers

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Abstract. This paper addresses the problem of finding short cycles in the internal state space of shift register based stream ciphers. The absence of short cycles is a desirable property for stream ciphers because otherwise the keystream they generate may have small period. The existing Boolean Decision Diagram (BDD) based algorithms for finding cycles have limited capacity due to the excessive memory requirements of BDDs. The simulation-based algorithms can be applied to larger instances, however, they cannot guarantee the detection of all cycles of a given length. The same holds for general-purpose SAT-based model checkers. We present a SAT-based algorithm which can find all short cycles in real cryptographic systems with very large state spaces. The algorithm is evaluated by analyzing Trivium, Bivium, and Grain family of stream ciphers. We found that Trivium, Bivium, Grain-80 and Grain-128 contain short cycles whose existence, to our best knowledge, was previously unknown.

Keywords: Shift register, stream cipher, Trivium, Grain, cycle, SAT.

1 Introduction

SAT solvers are a powerful tool for finding solutions to Boolean satisfiability problem. Using backtracking and different heuristics, these solvers explore the exponential space of variable assignments looking for a satisfying assignment. In spite of NP-completeness of the SAT problem [1], today's SAT solvers are able to handle problem instances involving thousands of variables. This is sufficient for many practical SAT problems in formal verification, automatic test pattern generation, and logic synthesis [2].

In the past there were multiple attempts to use SAT solvers in cryptanalysis, including finding preimages for hash functions [3–5], generating collisions for hash functions [6], faking RSA signatures [7], recovering RSA private keys [8], finding (or disproving the existence of) weak keys in stream and block ciphers [9, 10], guess-and-determine attacks on stream ciphers [9], algebraic attacks [11–13], and side-channel attacks on software implementations of block ciphers [14]. So

called *logical cryptanalysis* has been introduced by Massacci and Marraro [15] as a general framework for encoding the low-level properties of cryptographic algorithms as SAT problems and then using efficient automated theorem-proving systems and SAT-solvers for reasoning about them. This approach has been followed by many, including [16–19]. However, it is generally quite difficult to make SAT solvers to produce meaningful results except for strongly reduced instances of cryptographic systems.

In this paper, we present a SAT-solver based algorithm which is able to produce results for important stream ciphers such as Trivium [20] and Grain [21] which are among the candidates that have continuously attracted attention since the end of the eStream project [22]. The presented algorithm can find all short cycles of a given length in the internal state space of a shift register-based stream cipher. Given that the internal state space of Trivium is of size 2^{288}, designing an algorithm which guarantees that *all* cycles of a given length are detected is far from trivial. To the best of our knowledge, none of the existing algorithms is able to do it. The existing Boolean Decision Diagram (BDD) based algorithms [23,24] have limited capacity due to the excessive memory requirements of BDDs. The simulation-based algorithms [25–28] can be applied to larger problem instances, however, they cannot guarantee the detection of all cycles of a given length, even if the length is short. The same holds for the SAT-based algorithms that do not limit cycle length [29] and general-purpose SAT-based model checkers, e.g. [30]. There are many algorithms for finding cycles of length one, including [31–33], but they cannot handle larger cycles.

The presented algorithm uses a SAT-solver for the identification of paths of a particular length k in the state transition graph of a shift register. The input fed into the SAT-solver is a propositional formula representing the unfolding of the transition relation of the shift register by k time steps. A satisfying assignment to this propositional formula corresponds to a valid path of length k in the state transition graph. Since each state of a shift register has a unique next state, once a path reaches a loop, it never leaves it. So, we can determine the presence of a cycle simply by checking whether the last state of the path occurs at least twice on this path. Furthermore, for shift registers with invertible state transition functions (which is the case for stream ciphers), we can take advantage of that fact that their state transition diagrams consist of pure cycles, without branches. In this case, if the last state of a path of length k is equal to its first state, we can conclude that we found a cycle of length k or a factor of k. We can than mark all cycle's states by adding them as constrains to the input formula of the SAT-solver and, in the following iterations, search only for paths in which the last state is not marked. We repeat the process iteratively until the SAT-solver cannot find any more satisfying assignments.

This paper contributes to the ongoing work by providing an algorithm which can find all short cycles in real cryptographic systems. We evaluated the presented algorithm by analyzing Trivium, Bivium, Grain family stream ciphers. We found that Trivium, Bivium, Grain-80 and Grain-128 contain short cycles

whose existence, to our best knowledge, was previously unknown (except for all-0 cycles).

The paper is organized as follows. Section 2 gives the background on shift registers, stream ciphers and SAT-solvers. Section 3 presents the new SAT-based algorithm. Section 4 summarizes the experimental results on four stream ciphers. Section 5 concludes the paper and discusses open problems.

2   Background

This section gives an introduction to shift registers, stream ciphers and SAT solvers and describes basic notions used in the paper.

2.1   Shift registers

As a model of a shift register, we use a deterministic finite state machine with a set of states $S = \{0,1\}^n$ and the state transition function $f : S \to S$. We are not concerned with inputs and outputs in this paper, so we assume that the input alphabet is empty and there is no output function.

The state transition function f defines the next state $s^+ = (x_1^+, x_2^+, \ldots, x_n^+)$ as $f(s)$, where $s = (x_1, x_2, \ldots, x_n)$ is a current state and $x_i \in \{0,1\}$ and $x_i^+ \in \{0,1\}$ are Boolean variables representing the bit number i of the current and next state, respectively , $\forall i \in \{1, 2, \ldots, n\}$. The bit number i of the output of f is computed by the Boolean function $f_i : S \to \{0,1\}$, $\forall i \in \{1, 2, \ldots, n\}$. In other words, the state transition function f defines the mapping

$$\begin{pmatrix} x_1 \\ \ldots \\ x_n \end{pmatrix} \to \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \ldots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}. \tag{1}$$

Note that such a general definition of shift registers includes the linear and non-linear feedback shift registers in the Fibonacci and the Galois configurations [34], shift registers with feedforward connections (e.g. used in Trivium), as well as Golomb's binary machines [35, p. 21].

Each state transition function f induces a directed graph called the *state transition graph* in which the nodes represent the states and the edges represent possible transitions between the states. The node x (the predecessor) in connected to node y (the successor) by an edge if $f(x) = y$. Nodes forming a loop are called a *cycle*. A node x satisfying $f(x) = x$ is regarded as a cycle of length 1. Every node has a unique successor, but some nodes may have no predecessors and some may have more than one predecessor. In general, a state transition graph consists of a number of cycles and a number of directed trees rooted in cycles. For example, Figure 1 shows the state transition graph for the state transition function

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} x_2 \\ x_3 \oplus x_1 x_2 \\ x_1 \oplus x_2 \end{pmatrix} \tag{2}$$
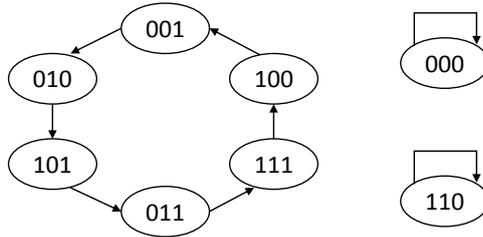
Fig. 1. The state transition graph of the function defined by (2).

where "⊕" is for the Boolean XOR (the Boolean AND in the product-terms is omitted).

If the state transition function f is invertible, so that for every y there is an x satisfying $f(x) = y$, then f is called a *permutation*, and the state transition diagram consists of pure cycles, without any trees rooted in them. Stream ciphers [20, 36] and hash functions [37, 38] typically use invertible state transition functions to prevent incremental reduction of the entropy of the state. The round function of a block cipher [39–41] has to be invertible in order to result in a unique decryption.

Conditions for invertibility are well-known [35] for Fibonacci NLFSRs whose state transition function is of type

$$\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \rightarrow \begin{pmatrix} x_2 \\ x_3 \\ \dots \\ f(x_1, \dots, x_n) \end{pmatrix}.$$

An n-bit Fibonacci NLFSR is invertible if and only if the Boolean function updating its nth bit has the form:

$$f(x_1, \dots, x_n) = x_1 \oplus g(x_2, \dots, x_n).$$

where $g : \{0, 1\}^{n-1} \rightarrow \{0, 1\}$ does not depend on x_1.

For Galois NLFSRs, a sufficient condition for invertibility has been derived in [42]. An n-bit Galois NLFSR is invertible if its state transition function is of type

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{pmatrix} \rightarrow \begin{pmatrix} x_2 \oplus g_1 \\ x_3 \oplus g_2(x_2) \\ x_4 \oplus g_3(x_2, x_3) \\ \dots \\ x_1 \oplus g_n(x_2, x_3 \dots, x_n) \end{pmatrix}.$$

2.2 Stream ciphers

A stream cipher is a symmetric cryptographic primitive that allows two parties which share a secret key to communicate confidential information [43]. Stream

ciphers generate a stream of pseudo-random bits, called *the keystream*, given a secret key and a public random initialization vector (IV). The keystream is combined with the plaintext message before the message is send, typically by bitwise XORing. The received ciphertext is again XORed with the keystream to recover the original plaintext. As a result, the message cannot be read while in transit or storage by unauthorized parties who do not posses the secret key.

The stream ciphers considered in this paper are based on one or more shift registers with linear or non-linear state transition functions as well as an output function that maps the internal register state to keystream bits. Stream ciphers have two working phases: (1) an initialization phase and (2) a keystream generation phase. During the initialization, key and IV are usually mixed by shifting the shift registers while updating them with a combination of the state transition function and the output function. During the keystream generation, the shift registers are shifted and their state transition functions are evaluated, while the keystream is generated from the internal state using the output function.

2.3 SAT solvers

SAT solvers are software tools that employ efficient heuristics to decide whether a set of constraints has a solution. Constraints are typically represented by a propositional formula in the Conjunctive Normal Form (CNF). In a CNF, each variable symbol in the expression, x or $\overline{x}$, is called a *literal*. A *clause* is a disjunction (Boolean OR) of literals. CNF is a conjunction (Boolean AND) of clauses.

SAT solvers are widely used in Electronic Design Automation (EDA) for formal verification, automatic test pattern generation, and logic synthesis [2], though they are also popular in a growing number of other domains.

Modern SAT solvers that are typically based on the Davis-Putnam-Logemann-Loveland (DPLL) algorithm [44] which is a refinement of the earlier Davis-Putnam algorithm [45]. Most efficient DPLL-based SAT solvers include GRASP [46], where a powerful conflict analysis procedure was introduced, and Chaff [47], where a particularly efficient implementation of Boolean constraint propagation and a novel low overhead decision strategy were introduced. Solvers that employ these techniques are called *conflict-driven* SAT solvers. In this paper we use MiniSat [48], a conflict-driven SAT solver which is designed to be easily adaptable to different domains. Other reasons for choosing MiniSat is its competitive performance, code availability, and flexibility in adding arbitrary Boolean constraints

MiniSat employs a backtracking-based, depth-first search algorithm to find a satisfying assignment of variables for a set of clauses. The algorithm branches on a variable by picking it heuristically, guessing its value to be TRUE or FALSE and examining whether the values of other variables depend on this guess. The affected variables are then assigned and the search continues until no more assignments can be made. During this propagation phase, a conflict may be detected, i.e. a clause that cannot be satisfied may be found (because all its literals became false). If this happens, a so called *conflict clause* (or *learned clause*) which describes the wrong guesses that lead to the conflict is constructed and

added to the SAT problem. Assumptions are then canceled by backtracking until the conflict clause becomes a unit, at which point it is propagated and the search process continues. The conflict clauses drive the backtracking, guiding the algorithm in choosing the best next guess, and speed up future conflicts by caching the reason for the conflict [48]. Eventually, either a satisfiable assignment is found, or the search tree is exhausted implying that no satisfying assignment exists.

3 Description of the Algorithm

The pseudocode of the presented algorithm is given as Algorithm 1. The algorithm 1 finds cycles of length k and factors of k in a shift register with an invertible state transition function f.

3.1 Transition relation

The state transition function f of an n-bit shift register induces a transition relation $T \subseteq S \times S$ which can be described by the following propositional formula:

$$T(s, s^+) = \bigwedge_{i=1}^{n} (x_i^+ \ \leftrightarrow \ f_i(x_1, \ldots, x_n)) \tag{3}$$

where "$\wedge$" and "$\leftrightarrow$" are the Boolean logic operations AND and EQUALITY, respectively.

For example, the transition relation for the state transition function in the example (2) is given by:

$$T(s, s^+) = (x_1^+ \leftrightarrow x_2) \wedge (x_2^+ \leftrightarrow x_3 \oplus x_1 x_2) \wedge (x_3^+ \leftrightarrow x_1 \oplus x_2). \tag{4}$$

3.2 Unfolding of transition relation

At step 2 of the algorithm, the unfolding of the transition relation T by k time steps is constructed.

The *unfolding* of a transition relation by k time steps is typically performed by taking the AND of the transition relations from the current state to the next state, from the next state to the one after next state, etc., from the state at the time step $k - 1$ to the state at the time step k [30]. For instance, for the example defined by (2), the unfolding by two time steps is computed as $T(s, s^+) \wedge T(s^+, s^{++})$, where $T(s, s^+)$ is given by (4) and $T(s^+, s^{++})$ is given by

$$T(s^+, s^{++}) = (x_1^{++} \leftrightarrow x_2^+) \wedge (x_2^{++} \leftrightarrow x_3^+ \oplus x_1^+ x_2^+) \wedge (x_3^{++} \leftrightarrow x_1^+ \oplus x_2^+).$$

A possible satisfying assignment for the expression $T(s, s^+) \wedge T(s^+, s^{++})$ is $s = (101)$, $s^+ = (011)$, and $s^{++} = (111)$. This assignment corresponds to the path $101 \rightarrow 011 \rightarrow 111$ in the state transition graph in Figure 1.

Algorithm 1 An algorithm for finding cycles of length k and factors of k in the internal state space of an n-bit shift register with an invertible state transition function $f : \{0,1\}^n \to \{0,1\}^n$.

1: $T = \bigwedge_{i=1}^{n}(x_i^+ \leftrightarrow f_i(x_1,\ldots,x_n))$; /* transition relation induced by f */
2: $T_{-k\ldots0} = \bigwedge_{i=-k}^{-1} T(s_i,s_{i+1})$; /* unfolding of T from the time step $-k$ to 0 */
3: $F = T_{-k\ldots0}$; /* propositional formula representing $T_{0\ldots-k}$ */
4: $F = F \wedge (s_{-k} \leftrightarrow s_0)$; /* adding a constrain requiring the last state of a path */
5:                    /* of length k, s_0, to be equal to the first state, s_{-k} */
6: $C(s_0) = 0$; /* propositional formula representing the set states of all currently */
7:                /* found cycles expressed in terms of variables of the state s_0 */
8: Find the set M_k of all factors of k;
9: **while** $\mathrm{Sat}(F)$ **do**
10:    **for** each $m \in M_k$ **do**
11:        **if** $s_{-m} = s_0$ **then**
12:            **for** $(j = 0;\ j > -m;\ j--)$ **do**
13:                $C(s_0) = C(s_0) \vee (s_0 \leftrightarrow a_j)$;   /* marking a cycle of length m */
14:                /*a_j is the assignment of variables of the state s_j returned by $\mathrm{Sat}(F)$*/
15:            **end for**
16:            $F = F \wedge \neg C(s_0)$; /* adding a constrain describing marked cycles */
17:            **break**;
18:        **end if**
19:    **end for**
20: **end while**
21: **return** all cycles contained in $C(s_0)$

3.3   Reverse direction of unfolding

Let $T_{0\ldots k}$ denotes the unfolding of the transition relation T from the time step 0 to the time step k:

$$T_{0\ldots k} = \bigwedge_{i=0}^{k-1} T(s_i,s_{i+1}).$$

Note that we need to switch from the previous notation $T(s,s^+)$ to the notation $T(s_i,s_{i+1})$ in order to describe longer sequences of states.

It is common to unfold the transition relation T in the forward direction, as shown above, however, we have chosen to unfold T in the *reverse* order, from the time step $-k$ to the time step 0 (step 2):

$$T_{-k\ldots0} = \bigwedge_{i=-k}^{-1} T(s_i,s_{i+1}). \tag{5}$$

In this way, the states from the previous time steps rather than the next ones are added to the unfolding. As a result, the last state of the unfolded transition relation is always s_0, independently of the depth of unfolding. For example, the unfolding by two time steps is computed as $T_{-2,-1,0} = T(s_{-2},s_{-1}) \wedge T(s_{-1},s_0)$. In this case, the path $101 \to 011 \to 111$ in the state transition graph in Figure 1

corresponds to the satisfying assignment $s_{-2} = (101)$, $s_{-1} = (011)$, and $s_0 = (111)$. This way of unfolding reduces the number of extra clauses added to F at each iteration of SAT-solver. A similar strategy was used in the algorithm [29].

3.4 Finding satisfying assignments

Once the transition relation is unfolded, the propositional formula F representing the unfolding $T_{-k\ldots0}$ is constructed using the standard procedure [30] (step 3). Then a constrain $s_{-k} \leftrightarrow s_0$ which means that the last state of a path of length k, s_0, should to be equal to the first state of the path, s_{-k}, is added to F (step 4) and the SAT-solver is called to find a satisfying assignment for F (step 9).

The function $\mathrm{Sat}(F)$ takes the expression F and returns TRUE if there exists an assignment of variables for the states $s_{-k}, \ldots, s_1, s_0$ which makes F true. Such a satisfying assignment corresponds to a path of length k in the state transition graph induced by the state transition function f. A path has length k if it makes k transitions between the states. Since we added to F the constrain $s_{-k} \leftrightarrow s_0$, we also know that in the path returned by the SAT-solver the last state is equal to the first. This means we found a cycle which can either be of length k, or a factor of k.

To find the length of the cycle, for each of the factors of k, m, we check if the last state of the path, s_0, is equal to s_{-m} (step 11). In the pseudocode, the set M_k containing all factors of k is computed at step 8, before the first run of the SAT-solver. If a shorter than k cycle if found, the **for**-loop breaks.

For instance, a possible satisfying assignment for the unfolding by two steps $T_{-2,-1,0} = T(s_{-2}, s_{-1}) \wedge T(s_{-1}, s_0)$ in the example (2) is $s_{-2} = (110)$, $s_{-1} = (110)$, and $s_0 = (110)$. This assignment corresponds to the path $110 \rightarrow 110 \rightarrow 110$ in the state transition graph in Figure 1. By checking the set of factors $M_2 = \{1, 2\}$ of 2, we can determine that this path corresponds to a cycle of length 1 and stop checking other factors.

3.5 Identifying and marking cycles

If Algorithm 1 finds that $s_0 = s_{-m}$ for some $m \in M_k$, the corresponding cycle of length m is marked by adding constraints to the input formula of the SAT-solver, F. In the following iterations, the algorithm only searches for paths in which the last state is not marked.

Clearly, if the last state of a path of length k does not belong to a cycle of length at most k, then no states of this path belongs to such a cycle either. In our case, the last state of a path is always s_0. Therefore, it is sufficient to express the constraints in terms of variables of the state s_0 only. By adding these constraints to F, we restrict F in such a way that no satisfying assignment for F returned by the SAT solver contains a state of a previously found cycle.

The constraints are added to F as $F = F \wedge \neg C(s_0)$ (step 13), where $C(s_0)$ is the propositional formula describing the characteristic function of the set of states of all currently found cycles expressed in terms of variables of the state s_0

and "¬" is the Boolean logic operation NOT. Before the first run of the SAT-solver, at step 6, $C(s_0)$ is initialized to 0. If the algorithm finds a cycle of length m (step 11), each of the m states of this cycle, s_j, $j \in \{0, 1, \ldots, m-1\}$, is added to $C(s_0)$ as $C(s_0) = C(s_0) \vee (s_0 \leftrightarrow a_j)$, where a_j is an n-bit vector corresponding to the assignment of variables of the state s_j returned by the SAT-solver.

In the presented approach, the number of extra clauses added to F at each iteration is equal to the total number of states in the cycles found at this iteration, i.e. at most k. So, a formula of size $O(nk)$ is added to F, where n is the number of variables. Contrary, in a general case, a formula of size $O(np \log p))$, where p is the depth of unfolding, has to be added to to F to mark cycles [49].

If a satisfying assignment does not exist, then there is no cycle of length k or a factor of k among the unmarked states of the state transition graph. This implies that all cycles of length k and a factor of k are already found and marked. The algorithm terminates and returns the set of all found cycles (step 21).

4 Analysis of Trivium, Bivium and Grain

We applied the presented algorithm to Trivium, Bivium, Grain-80 and Grain-128 stream ciphers. This section shows the results and compares them to the results of the SAT-based algorithm from [29].

It might be worth mentioning that a random permutation over a set of p elements is expected to have an average of $\log p$ cycles [50]. The number of r-cycles is expected to decrease as $1/r$, for any fixed integer $1 \le r \le p$ [50].

4.1 Analysis of Trivium

We used the presented algorithm to search for short cycles in Trivium stream cipher [20]. Trivium has a 288-bit internal state in which only 3 out of 288 bits are updated non-trivially. The rest of bits shift the content of the previous bit.

Let $(x_1, x_2, \ldots, x_{288})$ be the variables representing the bits of a current state of Trivium. At each clock cycle, the bit number i of the next state, denoted by x_i^+, is computed as:

$$x_1^+ = x_{288} \oplus x_{287}x_{286} \oplus x_{243} \oplus x_{69}$$
$$x_{94}^+ = x_{93} \oplus x_{92}x_{91} \oplus x_{171} \oplus x_{66}$$
$$x_{178}^+ = x_{177} \oplus x_{176}x_{175} \oplus x_{264} \oplus x_{162}$$

and $x_i^+ = x_{i-1}$ for all other x_i, $i \in \{2, 3, \ldots, 93, 95, \ldots, 177, 179, \ldots, 288\}$.

We searched for cycles of length k, for $k = 1, 2, 3, \ldots$ until the timeout of 12 hours was reached at $k = 31$. The results are shown in Table 1. The 2nd row in the table shows the number N_k of cycles of length k found by the Algorithm 1. The 3rd row shows CPU time, t. The experiments were run on a PC with Intel Core i7-4600U CPU at 2.1 GHz with 7.7 GB RAM running under Ubuntu 14.04 LTS.

k	1	2	3	4	5	6	7	8	9	10	11	12
N_k	1	0	21	0	0	0	0	0	0	1	1	2
t	0.004s	0.003s	0.011s	0.004s	0.014s	0.030s	0.040s	0.091s	0.099s	0.359s	0.261s	0.384s

k	13	14	15	16	17	18	19	20	21	22
N_k	0	0	1	0	0	0	0	0	0	0
t	0.034s	0.748s	0.565s	7.275s	10.604s	53.161s	1m2.358s	3m18.442s	3m5.661s	48.548s

k	23	24	25	26	27	28	29	30
N_k	0	0	0	0	0	0	0	0
t	0.285s	15m10.014s	14m43.513s	0.125s	8m46.997s	151m44.853s	0.239s	511m19.397s

Table 1. The number N_k of cycles of length k found in Trivium; t is the runtime.

In total, we found 27 cycles: one cycle of length 1 (all-0), 21 cycles of length 3, one cycle of length 10, one cycle of length 11, 2 cycles of length 12, and one cycle of length 15. Note, however, that these cycles do not directly affect security of Trivium because none of them contains a state required for Trivium's initialization. Trivium is initialized by loading the following state into its state register:

$$(x_1, x_2, \ldots, x_{93}) = (K_1, \ldots, K_{80}, 0, \ldots, 0)$$
$$(x_{94}, x_{95}, \ldots, x_{177}) = (IV_1, \ldots, IV_{80}, 0, \ldots, 0)$$
$$(x_{178}, x_{179}, \ldots, x_{288}) = (0, \ldots, 0, 1, 1, 1)$$

where $(K_1, \ldots, K_{80})$ is an 80-bit key and $(IV_1, \ldots, IV_{80})$ is an 80-bit initialization value (IV), and clocking the cipher 4×288 times.

4.2 Analysis of Bivium

We also applied the presented algorithm to Bivium stream cipher [51], a simplified version of Trivium.

Bivium [51] has a 177-bit internal state in which only 2 out of 177 bits are updated non-trivially. Using the same notation as in the previous section, Bivium can be specified as:

$$x_1^+ = x_{177} \oplus x_{176}x_{175} \oplus x_{69} \oplus x_{162}$$
$$x_{94}^+ = x_{93} \oplus x_{92}x_{91} \oplus x_{171} \oplus x_{66}$$

and $x_i^+ = x_{i-1}$ for all other x_i, $i \in \{2, 3, \ldots, 93, 95, \ldots, 177\}$.

Due to a smaller search space, the timeout of 12 hours was reached at $k = 44$. In Bivium, we found only one cycle of length 1 (all-0) and 5 cycles of length 3. As in Trivium's case, these cycles do not directly affect security of Bivium because none of them contains a state required for Bivium's initialization. Bivium is initialized by loading the following state into the state register:

$$(x_1, x_2, \ldots, x_{93}) = (K_1, \ldots, K_{80}, 0, \ldots, 0)$$
$$(x_{94}, x_{95}, \ldots, x_{177}) = (IV_1, \ldots, IV_{80}, 0, \ldots, 0)$$

where $(K_1, \ldots, K_{80})$ is an 80-bit key and $(IV_1, \ldots, IV_{80})$ is an 80-bit IV, and clocking the cipher 4 times.

4.3 Analysis of Grain-80

The stream cipher Grain-80 [36] is constructed from an 80-bit NLFSR and an 80-bit LFSR. The LFSR is known to have the maximum period of $2^{80} - 1$ since it uses a primitive generator polynomial of degree 80. The period of the NLFSR is unknown, however, the input stage of the NLFSR is fed from the output stage of the LFSR to guarantee the lower bound of $2^{80} - 1$ on the overall period of the two combined shift registers.

We investigated the state space of the 80-bit NLFSR of Grain-80 for the case when it is disconnected from the LFSR. In this case, its feedback function is specified as

$$
\begin{aligned}
x_{80}^+ = {} & x_{64} \oplus x_{61} \oplus x_{53} \oplus x_{46} \oplus x_{38} \oplus x_{34} \oplus x_{29} \oplus x_{22} \oplus x_{16} \oplus x_{10} \oplus x_1 \oplus x_{64}x_{61} \\
& \oplus x_{38}x_{34} \oplus x_{16}x_{10} \oplus x_{61}x_{53}x_{46} \oplus x_{34}x_{29}x_{22} \oplus x_{64}x_{46}x_{29}x_{10} \oplus x_{61}x_{53}x_{38}x_{34} \\
& \oplus x_{64}x_{61}x_{22}x_{16} \oplus x_{64}x_{61}x_{53}x_{46}x_{38} \oplus x_{34}x_{29}x_{22}x_{16}x_{10} \oplus x_{53}x_{46}x_{38}x_{34}x_{29}x_{22}
\end{aligned}
$$

and $x_i^+ = x_{i+1}$ for all other x_i, $i \in \{1, 2, \ldots, 79\}$.

The timeout of 12 hours was reached at $k = 46$. We found one cycle of length 1 (all-0), one cycle of length 3 and one cycle of length 12. Note that in the full version Grain-80, when the NLFSR is combined with the LFSR, the cycles of length 3 and 12 will translate into the cycles of length $3 \times (2^{80} - 1)$ and $12 \times (2^{80} - 1)$.

Grain-80 is initialized by loading an 80-bit key into the NLFSR, loading a 64-bit IV into the first 64-bits of the LFSR, filling the rest of LFSR with 1s, and clocking the cipher 160 times without producing keystream. Since no restrictions are imposed on the 80-bit key loaded into the NLFSR, the key may potentially be one of the states of the short cycles.

4.4 Analysis of Grain-128

Similarly to Grain-80, the stream cipher Grain-128 is constructed from an 128-bit NLFSR and a 128-bit LFSR which is feeding the input stage of the NLFSR [21].

We investigated the state space of the 128-bit NLFSR of Grain-128 for the case when it is separated from the LFSR. In this case, its feedback function is specified as

$$
\begin{aligned}
x_{128}^+ = {} & x_1 \oplus x_{27} \oplus x_{57} \oplus x_{92} \oplus x_{97} \oplus x_4x_{68} \oplus x_{12}x_{14} \oplus x_{18}x_{19} \oplus x_{28}x_{60} \\
& \oplus x_{41}x_{49} \oplus x_{62}x_{66} \oplus x_{69}x_{85}
\end{aligned}
$$

and $x_i^+ = x_{i+1}$ for all other x_i, $i \in \{1, 2, \ldots, 127\}$.

The timeout of 12 hours was reached at $k = 32$. We found one cycle of length 1 (all-0), one cycle of length 7 one cycle of length 8. Note that in the full version

Grain-128, when the NLFSR is combined with the LFSR, the cycles of length 7 and 8 will translate into the cycles of length $7 \times (2^{128} - 1)$ and $8 \times (2^{128} - 1)$.

Grain-128 is initialized by loading an 128-bit key into the NLFSR, loading a 96-bit IV into the first 96-bits of the LFSR, filling the rest of LFSR with ones, and clocking the cipher 256 times without producing keystream. Since no restrictions are imposed on the 128-bit key loaded into the NLFSR, the key may potentially be one of the states of the short cycles.

4.5   Comparison with the SAT-based algorithm from [29]

The algorithm presented in [29] is a SAT-based algorithm which does not limit the cycle length. It was developed for the analysis of so called $(n, 2)$ *random Boolean networks* (or *Kauffman networks*) which are used in physics to model spin glasses [52] and in biology to study cell differentiation [53], immune response [54], and evolution [55]. The $(n, 2)$ random Boolean networks use non-invertible n-variable state transition functions in which each Boolean function depends on at most two variables assigned at random. They are known to have a small number of short cycles [56]. While quite successful for the $(n, 2)$ random Boolean network case, the algorithm [29] does not seem suitable for the invertible mappings in which *all* states belong to some cycle. This makes the search for all cycles unfeasible for large n.

We have applied the algorithm [29] to Trivium, Bivium, Grain-80 and Grain-128. It all four cases, the algorithm [29] was able to find all-0 cycles, but no other short cycles. The process was killed due to the memory blow-up.

5   Conclusion

In this paper, we presented a SAT-based algorithm for finding short cycles and applied it to Trivium, Bivium, Grain-80 and Grain-128 stream ciphers. We found that all four ciphers contain short cycles whose existence, to our best knowledge, was previously unknown (except for all-0 cycles).

Note that the presented algorithm is applicable not only to shift-register based systems, but also to any invertible mapping of type $\{0, 1\}^n \rightarrow \{0, 1\}^n$ including Feistel networks [57], T-functions [58], and univariate permutation polynomials modulo 2^n [59].

Our next steps are to investigate if the discovered cycles open new possibilities for cryptanalysis and to apply the presented algorithm to other cryptographic systems. It is particularly interesting to investigate systems which use only a secret key and IV for their initialization, without fixing the remaining state bits (if any) to constants.

6   Acknowledgements

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