Horizontal Side-Channel Attacks and Countermeasures on the ISW Masking Scheme*

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Abstract. A common countermeasure against side-channel attacks consists in using the masking scheme originally introduced by Ishai, Sahai and Wagner (ISW) at Crypto 2003, and further generalized by Rivain and Prouff at CHES 2010. The countermeasure is provably secure in the probing model, and it was showed by Duc, Dziembowski and Faust at Eurocrypt 2014 that the proof can be extended to the more realistic noisy leakage model. However the extension only applies if the leakage noise σ increases at least linearly with the masking order n, which is not necessarily possible in practice. In this paper we investigate the security of an implementation when the previous condition is not satisfied, for example when the masking order n increases for a constant noise σ. We exhibit two (template) horizontal side-channel attacks against the Rivain-Prouff's secure multiplication scheme and we analyze their efficiency thanks to several simulations and experiments.
Eventually, we describe a variant of Rivain-Prouff's multiplication that is still provably secure in the original ISW model, and also heuristically secure against our new attacks.

1 Introduction

Side-channel analysis is a class of cryptanalytic attacks that exploit the physical environment of a cryptosystem to recover some *leakage* about its secrets. To secure implementations against this threat, security developers usually apply techniques inspired from *secret sharing* [Bla79, Sha79] or *multi-party computation* [CCD88]. The idea is to randomly split a secret into several shares such that the adversary needs all of them to reconstruct the secret. For these schemes, the number of shares n in which the key-dependent data are split plays the role of a security parameter.

A common countermeasure against side-channel attacks consists in using the masking scheme originally introduced by Ishai, Sahai and Wagner (ISW) [ISW03]. The countermeasure achieves provable security in the so-called *probing security model* [ISW03], in which the adversary can recover a limited number of intermediate variables of the computation. This model has been argued to be practically relevant to address so-called *higher-order* side-channel attacks and it has been the basis of several efficient schemes to protect block ciphers [BFG15, CGP+12, Cor14, CRV15, GPQ11, GPS14, PR11, RP10].

More recently, it has been shown in [DDF14] that the probing security of an implementation actually implies its security in the more realistic *noisy leakage model* introduced in [PR13]. More precisely, if an implementation obtained by applying the compiler in [ISW03] is secure at order n in

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the probing model, then [DFS15, Theorem 3] shows that the success probability of distinguishing the correct key among $|\mathbb{K}|$ candidates is bounded above by $|\mathbb{K}| \cdot 2^{-n/9}$ if the leakage L_i on each intermediate variable X_i satisfies:

$$\mathrm{I}(X_i; L_i) \leqslant 2 \cdot (|\mathbb{K}| \cdot (28n + 16))^{-2} \ ,$$

where $\mathrm{I}(\cdot;\cdot)$ denotes the *mutual information* and where the index i ranges from 1 to the total number of intermediate variables.

In this paper we investigate what happens when the above condition is not satisfied. Since the above mutual information $I(X_i; L_i)$ can be approximated by $k/(8\sigma^2)$ in the Hamming weight model in $\mathbb{F}_{2^k}$, where σ is the noise in the measurement (see Appendix A), this amounts to investigating the security of Ishai-Sahai-Wagner's (ISW) implementations when the number of shares n satisfies:

$$n > c \cdot \sigma$$

As already observed in previous works [VGS14, CFG+10], the fact that the same share (or more generally several data depending on the same sensitive value) is manipulated several times may open the door to new attacks which are not taken into account in the probing model. Those attacks, sometimes called *horizontal* [CFG+10] or *(Template) algebraic* [ORSW12, VGS14] exploit the algebraic dependency between several intermediate results to discriminate key hypotheses.

In this paper, we exhibit two (horizontal) side channel attacks against the ISW multiplication algorithm. These attacks show that the use of this algorithm (and its extension proposed by Rivain and Prouff in [RP10]) may introduce a weakness with respect to horizontal side channel attacks if the sharing order n is such that $n > c \cdot \sigma^2$, where σ is the measurement noise. While the first attack is too costly (even for low noise contexts) to make it applicable in practice, the second attack, which essentially iterates the first one until achieving a satisfying likelihood, shows very good performances. For instance, when the leakages are simulated by noisy Hamming weights computed over $\mathbb{F}_{2^8}$ with $\sigma = 1$, it recovers all the shares of a 21-sharing. We also confirm the practicality of our attack with a real life experiment on a development platform embedding the `ATMega328` processor (see Section 7). Actually, in this context where the leakages are multivariate and not univariate as in our theoretical analyses and simulations, the attack appears to be more efficient than expected and recovers all the shares of a n-sharing when $n \geqslant 40$.

Eventually, we describe a variant of Rivain-Prouff's multiplication that is still provably secure in the original ISW model, and also heuristically secure against our new attacks. Our new countermeasure is similar to the countermeasure in [FRR+10], in that it can be divided in two steps: a "matrix" step in which starting from the input shares x_i and y_j, one obtains a matrix $x_i \cdot y_j$ with n^2 elements, and a "compression" step in which one uses some randomness to get back to a n-sharing c_i. Assuming a leak-free component, the countermeasure in [FRR+10] is proven secure in the noisy leakage model, in which the leakage function reveals all the bits of the internal state of the circuit, perturbed by independent binomial noise. Our countermeasure does not use any leak-free component, but is only heuristically secure in the noisy leakage model (see Section 8.2 for our security analysis).

2 Preliminaries

For two positive integers n and d, a (n, d)-*sharing* of a variable x defined over some finite field $\mathbb{F}_{2^k}$ is a random vector $(x_1, x_2, \ldots, x_n)$ over $\mathbb{F}_{2^k}$ such that $x = \sum_{i=1}^{n} x_i$ holds (*completeness equality*)

and any tuple of $d-1$ shares x_i is a uniform random vector over $(\mathbb{F}_{2^k})^{d-1}$. If $n = d$, the terminology simplifies to *n-sharing*. An algorithm with domain $(\mathbb{F}_{2^k})^n$ is said to be $(n-1)$th-order secure in the probing model if on input an n-sharing $(x_1, x_2, \ldots, x_n)$ of some variable x, it admits no tuple of $n-1$ or fewer intermediate variables that depends on x.

Calligraphic letters, like $\mathcal{X}$, are used to denote finite sets (*e.g.* $\mathbb{F}_{2^n}$). The corresponding large letter X denotes a random variable over $\mathcal{X}$, while the lower-case letter x a value over $\mathcal{X}$. The probability of an event ev is denoted by $\Pr[ev]$. The *probability distribution function* (pdf for short) of a *continuous* random variable X is denoted by $f_X(\cdot)$. It will sometimes be denoted by $\mathrm{p}_X(\cdot)$ if X is discrete. The pdf of the random variable $X|Y$ is denoted by $f_{X|Y}(\cdot)$. The expectation and the variance of a random variable X are respectively denoted by $\mathbb{E}[X]$ and $\mathrm{Var}[X]$. The covariance between two random variables X and Y is denoted by $\mathrm{Cov}[X, Y]$. The *Signal to Noise Ratio* (SNR) of a univariate noisy observation L of a random variable X defined as *the signal*, is defined as $\mathrm{SNR} \doteq \frac{\mathrm{Var}[\mathbb{E}[L|X]]}{\mathbb{E}[\mathrm{Var}[L|X]]}$ (where we recall that $\mathbb{E}[L \mid X]$ and $\mathrm{Var}[L \mid X]$ are both viewed as functions of the random variable X).

The Gaussian distribution of dimension t with t-size expectation vector $\boldsymbol{\mu}$ and $t \times t$ covariance matrix Σ is denoted by $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$. We recall that the corresponding probability density function (pdf) is defined for every $\boldsymbol{\ell} \in \mathbb{R}^t$ as:

$$f(\boldsymbol{\ell}) = \frac{1}{\sqrt{(2\pi)^t \det(\Sigma)}} \exp\left(-\frac{1}{2}(\boldsymbol{\ell} - \boldsymbol{\mu})' \cdot \Sigma^{-1} \cdot (\boldsymbol{\ell} - \boldsymbol{\mu})\right) , \tag{1}$$

where $(\cdot)'$ denotes the transposition operation and $\det(\cdot)$ denotes the matrix determinant. The corresponding cumulative distribution function (cdf) F is defined for every $(a_i, b_i)_{i \in [1..t]} \in ((\mathbb{R} \cup \{-\infty, +\infty\})^2)^t$ by

$$F(\boldsymbol{a}, \boldsymbol{b}) = \int_{a_t}^{b_t} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(\ell_1, \ell_2, \cdots, \ell_t)\ d\ell_1 d\ell_2 \cdots d\ell_t \ \doteq \int_{\boldsymbol{a}}^{\boldsymbol{b}} f(\boldsymbol{\ell})\ d\boldsymbol{\ell} , \tag{2}$$

with $\boldsymbol{\ell} \doteq (\ell_1, \ell_2, \cdots, \ell_t)$, $\boldsymbol{a} \doteq (a_1, a_2, \cdots, a_t)$ and $\boldsymbol{b} \doteq (b_1, b_2, \cdots, b_t)$.

If the dimension t equals 1, then the Gaussian distribution is said to be *univariate* and its covariance matrix is reduced to the variable variance denoted σ^2. Otherwise, it is said to be *multivariate*.

The *entropy* $\mathbb{H}(X)$ of a discrete r.v. X defined over $\mathbb{F}_{2^k}$ aims at measuring the amount of information provided by an observation of X. It is defined by $\mathbb{H}(X) = -\sum_{x \in \mathbb{F}_{2^k}} f_X(x) \log f_X(x)$. The *differential entropy* extends the notion of entropy to continuous, and possibly t-dimensional random variables; contrary to the entropy, the differential entropy can be negative. In the case of a real valued random variable $\boldsymbol{L}$, it is defined by:

$$\mathbb{H}(\boldsymbol{L}) = -\int_{\boldsymbol{\ell} \in \mathbb{R}^t} f_{\boldsymbol{L}}(\boldsymbol{\ell}) \log(f_{\boldsymbol{L}}(\boldsymbol{\ell})) d\boldsymbol{\ell} . \tag{3}$$

If $\boldsymbol{L}$ is a t-dimensional Gaussian r.v. with covariance matrix Σ (*i.e.* its pdf is defined by (1)), then its entropy satisfies the following equality:

$$\mathbb{H}(\boldsymbol{L}) = \frac{1}{2} \log((2\pi e)^t \det(\Sigma)) . \tag{4}$$

In the general case, there is no analytical expression for the differential entropy of a r.v. X whose pdf mixes more than one Gaussian pdf. However, upper and lower bounds can be derived [CP00].

3 Secure Multiplication Schemes

In this section, we recall the secure multiplication scheme over $\mathbb{F}_2$ introduced in [ISW03] and its extension to any field $\mathbb{F}_{2^k}$ proposed in [RP10].

Ishai-Sahai-Wagner's Scheme [ISW03]. Let $x^\star$ and $y^\star$ be binary values from $\mathbb{F}_2$ and let $(x_i)_{1\leq i\leq n}$ and $(y_i)_{1\leq i\leq n}$ be n-sharings of $x^\star$ and $y^\star$ respectively. To securely compute a sharing of $c = x^\star \cdot y^\star$ from $(x_i)_{1\leq i\leq n}$ and $(y_i)_{1\leq i\leq n}$, the ISW method works as follows:

1. For every $1 \leq i < j \leq n$, pick up a random bit $r_{i,j}$.
2. For every $1 \leq i < j \leq n$, compute $r_{j,i} = (r_{i,j} + x_i \cdot y_j) + x_j \cdot y_i$.
3. For every $1 \leq i \leq n$, compute $c_i = x_i \cdot y_i + \sum_{j\neq i} r_{i,j}$.

The above multiplication scheme achieves security at order $\lfloor n/2 \rfloor$ in the probing security model [ISW03].

The Rivain-Prouff Scheme. The ISW countermeasure was extended to $\mathbb{F}_{2^k}$ by Rivain and Prouff in [RP10]. As showed in [BBD+15], the SecMult algorithm below is secure in the ISW probing model against t probes for $n \geq t+1$ shares; the authors also show that with some additional mask refreshing, the Rivain-Prouff countermeasure for the full AES can be made secure with $n \geq t+1$ shares.

Algorithm 1 SecMult

Input: the n-sharings $(x_i)_{i\in[1..n]}$ and $(y_j)_{j\in[1..n]}$ of $x^\star$ and $y^\star$ respectively
Output: the n-sharing $(c_i)_{i\in[1..n]}$ of $x^\star \cdot y^\star$

1: **for** $i = 1$ **to** n **do**
2: **for** $j = i+1$ **to** n **do**
3: $r_{i,j} \leftarrow^{\$} \mathbb{F}_{2^k}$
4: $r_{j,i} \leftarrow (r_{i,j} + x_i \cdot y_j) + x_j \cdot y_i$
5: **end for**
6: **end for**
7: **for** $i = 1$ **to** n **do**
8: $c_i \leftarrow x_i \cdot y_i$
9: **for** $j = 1$ **to** n, $j \neq i$ **do** $c_i \leftarrow c_i + r_{i,j}$
10: **end for**
11: **return** $(c_1, c_1, \ldots, c_n)$

In Algorithm 1, one can check that each share x_i or y_j is manipulated n times, whereas each product $x_i y_j$ is manipulated a single time. This gives a total of $3n^2$ manipulations that can be observed through side channels.

4 Horizontal DPA Attack

4.1 Problem description.

Let $(x_i)_{i\in[1..n]}$ and $(y_i)_{i\in[1..n]}$ be respectively the n-sharings of $x^\star$ and $y^\star$ (namely, we have $x^\star = x_1 + \cdots + x_n$ and $y^\star = y_1 + \cdots + y_n$). We assume that an adversary gets, during the processing of

Algorithm 1, a single observation of each of the following random variables for $1 \leq i, j \leq n$:

$$L_i = \varphi(x_i) + B_i \tag{5}$$
$$L'_j = \varphi(y_j) + B'_j \tag{6}$$
$$L''_{ij} = \varphi(x_i \cdot y_j) + B''_{ij} \tag{7}$$

where φ is an unknown function which depends on the device architecture, where B_i, B'_j are Gaussian noise of standard deviation $\sigma/\sqrt{n}$, and B''_{ij} is Gaussian noise with standard deviation σ. Namely we assume that each x_i and y_j is processed n times, so by averaging the standard deviation is divided by a factor $\sqrt{n}$, which gives $\sigma/\sqrt{n}$ if we assume that the initial noise standard deviation is σ. The random variables associated to the ith share x_i and the jth share y_j are respectively denoted by X_i and Y_j. Our goal is to recover the secret variable $x^\star$ (and/or $y^\star$).

4.2 Complexity Lower Bound: Entropy Analysis of Noisy Hamming Weight Leakage

For simplicity, we first restrict ourselves to a leakage function φ equal to the Hamming weight of the variable being manipulated. In that case, the mutual information $\mathrm{I}(X; L)$ between the Hamming weight of a uniform random variable X defined over $\mathbb{F}_{2^k}$ and a noisy observation L of this Hamming weight can be approximated as:

$$\mathrm{I}(X; L) \simeq \frac{k}{8\sigma^2} \ , \tag{8}$$

if the noise being modeled by a Gaussian random variable has standard deviation σ. This approximation, whose derivation is given in Appendix A, is only true for large σ.

To recover a total of $2n$ shares (n shares of $x^\star$ and $y^\star$ respectively) from $3n^2$ Hamming weight leakages (namely each manipulation leaks according to (5)-(7) with $\varphi = \mathrm{HW}$), the total amount of information to be recovered is $2n \cdot k$ if we assume that the shares are i.i.d. with uniform distribution over $\mathbb{F}_{2^k}$. Therefore, since we have a total of $3n^2$ observations during the execution of Algorithm 1, we obtain from (8) that the noise standard deviation σ and the sharing order n must satisfy the following inequality for a side channel attack to be feasible:

$$3 \cdot n^2 \cdot \frac{k}{8\sigma^2} > 2n \cdot k \ . \tag{9}$$

We obtain an equality of the form $n > c \cdot \sigma^2$ for some constant c, as in a classical (vertical) side channel attack trying to recover $x^\star$ from n observations of intermediate variables depending on $x^\star$ [CJRR99]. This analogy between horizontal and vertical attacks has already been noticed in previous papers like [CFG+10] or [BJPW13]. Note that in principle the constant c is independent of the field degree k (which has also been observed in previous papers, see for instance [SVO+10]).

4.3 Attack With Perfect Hamming Weight Observations

We consider the particular case of perfect Hamming weight measurements (no noise). We show that even with perfect observations of the Hamming weight, depending on the finite-field representation, we are not always guaranteed to recover the secret variable $x^\star$.

More precisely, we consider the pdf of the random variable corresponding to perfect Hamming weight measurements:

$$\boldsymbol{H} \mid X_i \doteq (\mathrm{HW}(X_i), \mathrm{HW}(Y_j), \mathrm{HW}(X_i \cdot Y_j)) \mid X_i$$

defined for every $x_i \in \mathbb{F}_{2^k}$ and every triplet $(h_1, h_2, h_3) \in [0..k]^3$ by:

$$f_{\boldsymbol{H}|X_i}((h_1, h_2, h_3), x_i) \doteq \Pr[\mathrm{HW}(X_i) = h_1, \mathrm{HW}(Y_j) = h_2, \mathrm{HW}(X_i \cdot Y_j) = h_3 \mid X_i = x_i] \ . \quad (10)$$

If the probability distributions $f_{\boldsymbol{H}|X_i=x_i}$ are distinct for every $x_i \in \mathbb{F}_{2^k}$, then one can recover the secret variable $x^\star$ with overwhelming probability from enough measurements. However this property depends on the finite field $\mathbb{F}_{2^k}$ and its representation. For example in $\mathbb{F}_{2^4}$, if the representation $\mathbb{F}_{2^4} \simeq \mathbb{F}_2[t]/(t^4 + t + 1)$ is used then it may be checked that the finite field elements $x_i = t + 1$ and $x_i' = t^3 + t^2$ are associated to identical distributions f_{x_i} and $f_{x_i'}$; so we cannot distinguish between x_i and x_i' (the other field elements have distinct probability distributions). In $\mathbb{F}_{2^8}$, for the representation $\mathbb{F}_{2^8} \simeq \mathbb{F}_2[t]/(t^8+t^4+t^3+t+1)$ (as used in AES), all finite field elements have distinct probability distributions, but this is not always the case with other irreducible polynomials.

In summary, even with perfect observations of the Hamming weight, depending on the finite-field representation, we are not always guaranteed to recover the secret variable $x^\star$; however for the finite field representation used in AES the attack enables to recover the secret $x^\star$ for a large enough number of observations.

4.4 Maximum Likelihood Attack: Theoretical Attack with the full ISW State

For most field representations and leakage functions, the maximum likelihood approach used in the previous section (in particular in (10)), recovers the i-th share of $x^\star$ from an observation of L_i and an observation of (L_j', L_{ij}'') for every $j \in [1..n]$. It extends straightforwardly to noisy scenarios and we shall detail this extension in Section 5.1. However, the disadvantage of this approach is that it recovers each share separately, before rebuilding $x^\star$ and $y^\star$ from them. From a pure information theoretic point of view this is suboptimal since (1) the final purpose is not to recover all the shares perfectly but only the shared values and (2) only $3n$ observations are used to recover each share whereas the full tuple of $3n^2$ observations brings more information. Actually, the most efficient attack in terms of leakage exploitation consists in using the joint distribution of $(L_i, L_j', L_{ij}'')_{i,j\in[1..n]}$ to distinguish the correct hypothesis about $x^\star = x_1 + x_2 + \cdots + x_n$ and $y^\star = y_1 + y_2 + \cdots + y_n$.

As already observed in Section 3, during the processing of Algorithm 1, the adversary may get a tuple $(\ell_{ij})_{j\in[1..n]}$ (resp. $(\ell_{ij}')_{i\in[1..n]}$) of n observations for each L_i (resp. each L_j') and one observation ℓ_{ij}'' for each L_{ij}''. The full tuple of observations $(\ell_{ij}, \ell_{ij}', \ell_{ij}'')_{i,j}$ is denoted by $\boldsymbol{\ell}$, and we denote by $\boldsymbol{L}$ the corresponding random variable [1]. Then, to recover $(x^\star, y^\star)$ from $\boldsymbol{\ell}$, the *maximum likelihood approach* starts by estimating the pdfs $f_{\boldsymbol{L}|X^\star=x^\star,Y^\star=y^\star}$ for every possible $(x^\star, y^\star)$, and then estimates the following vector of distinguisher values for every hypothesis (x, y):

$$\mathrm{d}^\star_{\mathrm{ML}}(\boldsymbol{\ell}) \doteq \left(f_{\boldsymbol{L}|(X^\star,Y^\star)}(\boldsymbol{\ell}, (x, y))\right)_{(x,y)\in\mathbb{F}_{2^k}^2} \quad (11)$$

The pair (x, y) maximizing the above probability is eventually chosen.

At a first glance, the estimation of the pdfs $f_{\boldsymbol{L}|X^\star=x^\star,Y^\star=y^\star}$ seems to be challenging. However, it can be deduced from the estimations of the pdfs associated to the manipulations of the shares. Indeed, after denoting by $\mathrm{p}_{x,y}$ each probability value in the right-hand side of (11), and by using the law of total probability together with the fact that the noises are independent, we get:

$$2^{2kn} \cdot \mathrm{p}_{x,y} = \sum_{\substack{x_1,\cdots,x_n\in\mathbb{F}_{2^k} \\ x=x_1+\cdots+x_n}} \sum_{\substack{y_1,\cdots,y_n\in\mathbb{F}_{2^k} \\ y=y_1+\cdots+y_n}} \prod_{i,j=1}^{n} f_{L_i|X_i}(\ell_{ij}, x_i) \cdot f_{L_j'|Y_j}(\ell_{ij}', y_j) \cdot f_{L_{ij}''|X_iY_j}(\ell_{ij}'', x_i y_j) \ .$$

[1] In (5)-(7), it is assumed that the observations $(\ell_{ij})_{j\in[1..n]}$ and $(\ell_{ij}')_{i\in[1..n]}$ are averaged to build a single observation with noise divided by $\sqrt{n}$. This assumption is not done here in order to stay as general as possible.

Unfortunately, even if the equation above shows how to deduce the pdfs $f_{\boldsymbol{L}|(X^\star,Y^\star)}(\cdot,(x^\star,y^\star))$ from characterizations of the shares' manipulations, a direct processing of the probability has complexity $O(2^{2nk})$. By representing the sum over the x_i's as a sequence of convolution products, and thanks to Walsh transforms processing, the complexity can be easily reduced to $O(n2^{n(k+1)})$. The latter complexity stays however too high, even for small values of n and k, which led us to look at alternatives to this attack.

5 First Attack: Maximum Likelihood Attack on a Single Matrix Row

5.1 Attack Description

In this section, we explain how to recover each share x_i of $x^\star$ separately, by observing the processing of Algorithm 1. Applying this attack against all the shares leads to the full recovery of the sensitive value $x^\star$ with some success probability, which is essentially the product of the success probabilities of the attack on each share separately.

Given a share x_i, the attack consists in collecting the leakages on $(y_j, x_i \cdot y_j)$ for every $j \in [1..n]$. Therefore the attack is essentially a horizontal version of the classical (vertical) second-order side-channel attack, where each share x_i is multiplicatively masked over $\mathbb{F}_{2^k}$ by a random y_j for $j \in [1..n]$.

The most efficient attack to maximize the amount of information recovered on X_i from a tuple of observations $\boldsymbol{\ell} \doteq \ell_i, (\ell'_j, \ell''_{ij})_{j\in[1..n]} \hookleftarrow \boldsymbol{L} \doteq L_i, (L'_j, L''_{ij})_{j\in[1..n]}$ consists in applying a *maximum likelihood approach* [CJRR99,GHR15], which amounts to computing the following vector of distinguisher values:

$$\mathrm{d}_{\mathrm{ML}}(\boldsymbol{\ell}) \doteq \big(f_{\boldsymbol{L}|X_i}(\boldsymbol{\ell}, \hat{x}_i)\big)_{\hat{x}_i \in \mathbb{F}_{2^k}} \tag{12}$$

and in choosing the candidate $\hat{x}_i$ which maximizes the probability.

Let us respectively denote by $f(\cdot,\cdot)$, $f'(\cdot,\cdot)$ and $f''(\cdot,\cdot)$ the pdfs $f_{L_i|X_i}(\cdot,\cdot)$, $f_{L'_j|Y_j}(\cdot,\cdot)$ and $f_{L''_{ij}|X_iY_j}(\cdot,\cdot)$. Since the variables $L_i \mid X_i = \hat{x}_i$ and all the variables $(L'_j, L''_{ij} \mid X_i = \hat{x}_i)$ are mutually independent whatever $\hat{x}_i \in \mathbb{F}_{2^k}$, we have:

$$f_{\boldsymbol{L}|X_i}(\boldsymbol{\ell}, \hat{x}_i) = f(\ell_i, \hat{x}_i) \prod_{j=1}^{n} f_{(L'_j,L''_{ij})|X_i}((\ell'_j, \ell''_{ij}), \hat{x}_i) \ . \tag{13}$$

Applying the law of total probability, the pdf of $(L'_j, L''_{ij}) \mid X_i = \hat{x}_i$ can moreover be developed such that:

$$f_{(L'_j,L''_{ij})|X_i}((\ell'_j, \ell''_{ij}), \hat{x}_i) = \sum_{y\in\mathbb{F}_{2^k}} f_{(L'_j,L''_{ij})|(X_i,Y_j)}((\ell'_j, \ell''_{ij}), (\hat{x}_i, y)) \cdot \mathrm{p}_{Y_j}(y) \ , \tag{14}$$

that is

$$f_{(L'_j,L''_{ij})|X_i}((\ell'_j, \ell''_{ij}), \hat{x}_i) = 2^{-k} \sum_{y\in\mathbb{F}_{2^k}} f'(\ell'_j, y) \cdot f''(\ell''_{ij}, \hat{x}_i \cdot y) \ , \tag{15}$$

since the Y_j's are assumed to have uniform distribution and since the variables $L'_j \mid Y_j = y$ and $L''_{ij} \mid X_iY_j = \hat{x}_i \cdot y$ are independent.

Eventually, each score $f_{\boldsymbol{L}|X_i}(\boldsymbol{\ell}, \hat{x}_i)$ in (12) may be computed based on the following expression:

$$f_{\boldsymbol{L}|X_i}(\boldsymbol{\ell}, \hat{x}_i) = 2^{-nk} f(\ell_i, \hat{x}_i) \cdot \prod_{j=1}^{n} \Big(\sum_{y\in\mathbb{F}_{2^k}} f'(\ell'_j, y) \cdot f''(\ell''_{ij}, \hat{x}_i \cdot y) \Big) \ ,$$

where all the distributions are Gaussian ones (and hence can be easily evaluated). Hence, an approximation of the pdf in (12) can be deduced from the approximations (aka *templates*) of the distributions associated to the manipulations of the shares X_i, Y_j and X_iY_j.

In practice, one often makes use of the equivalent (averaged) log-likelihood distinguisher $\mathrm{d}'_{\mathrm{ML}}(\cdot)$ which, in our case, may be defined as:

$$\mathrm{d}'_{\mathrm{ML}}(\boldsymbol{\ell}) = \frac{1}{n}\log \mathrm{d}_{\mathrm{ML}}(\boldsymbol{\ell}) + k\log 2 \tag{16}$$

$$\simeq \left(\frac{1}{n}\Big(\log f(\ell_i,\hat{x}_i) + \sum_{j=1}^{n}\log\{\sum_{y\in\mathbb{F}_{2^k}} f'(\ell'_j,y)\cdot f''(\ell''_{ij},\hat{x}_i\cdot y)\}\Big)\right)_{\hat{x}_i\in\mathbb{F}_{2^k}} . \tag{17}$$

Remark 1. The same approach described in this section can be applied to iteratively recover each share y_j of y. The attack description can be straightforwardly deduced by exchanging the roles of X_i and Y_j (and the indices i and j). For instance, (14) becomes:

$$f_{(L_i,L''_{ij})|Y_j}((\ell_i,\ell''_{ij}),\hat{y}_j) = \sum_{x\in\mathbb{F}_{2^k}} f_{(L_i,L''_{ij})|(X_i,Y_j)}((\ell_i,\ell''_{ij}),(x,\hat{y}_j))\cdot \mathrm{p}_{X_i}(x) \ . \tag{18}$$

5.2 Complexity Analysis

As mentioned previously, given a share x_i, the attack consists in collecting the leakages on $(y_j, x_i\cdot y_j)$ for every $j\in[1..n]$. Therefore the attack is essentially an horizontal version of the classical (vertical) second-order side-channel attack. In principle the number n of leakage samples needed to recover x_i with good probability (aka the attack complexity) should consequently be $n=\mathcal{O}(\sigma^4)$ [CJRR99, GHR15, SVO+10]. This holds when multiplying two leakages both with noise having σ as standard deviation. However here the leakage on y_j has a noise with a standard deviation $\sigma/\sqrt{n}$ instead of σ (thanks to the averaging step). Therefore the noise of the product becomes $\sigma^2/\sqrt{n}$ (instead of σ^2), which gives after averaging with n measurements a standard deviation of σ^2/n, and therefore an attack complexity satisfying $n=\mathcal{O}(\sigma^2)$, as in a classical first-order side-channel attack.

5.3 Numerical Experiments

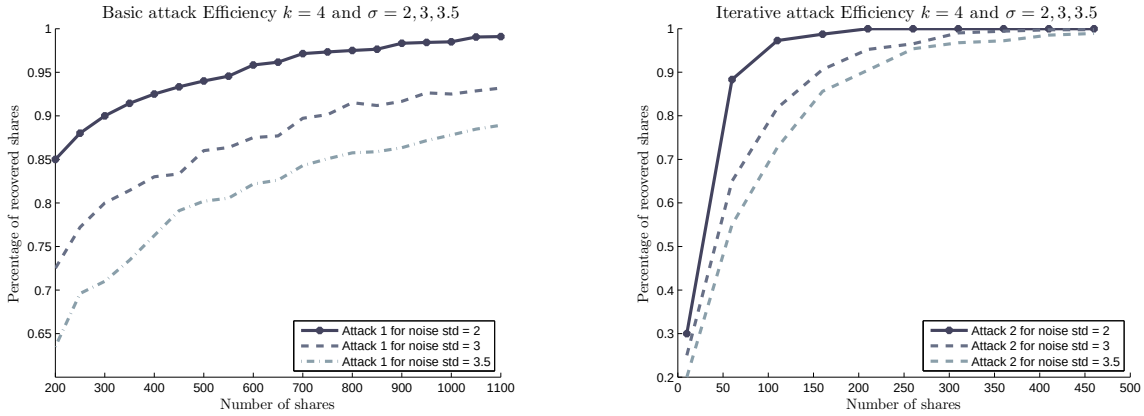
The attack presented in Sect. 5.1 has been implemented against each share x_i of a value x, with the leakages being simulated according to (5)-(7) with $\varphi=\mathrm{HW}$. For the noise standard deviation σ and the sharing order n, different values have been tested to enlighten the relation between these two parameters. We stated that an attack succeeds iff the totality of the n shares x_i have been recovered, which leads to the full recovery of $x^\star$. We recall that, since the shares x_i are manipulated n times, measurements for the leakages L_i and L'_j have noise standard deviations $\sigma/\sqrt{n}$ instead of σ. For efficiency reasons, we have chosen to work in the finite field $\mathbb{F}_{2^4}$ (namely $k=4$ in previous analyses).

For various noise standard deviations σ with $\mathrm{SNR}=k(2\sigma)^{-2}$ (*i.e.* $\mathrm{SNR}=\sigma^{-2}$ for $k=4$), Table 1 gives the average minimum number n of shares required for the attack to succeed with probability strictly greater than 0.5 (the averaging being computed over 300 attack iterations). The attack complexity $n=\mathcal{O}(\sigma^2)$ argued in Sect. 5.2 is confirmed by the trend of these numerical experiments. Undeniably, this efficiency is quickly too poor for practical applications where n is

small (clearly lower than 10) and the SNR is high (smaller than 1). However, it must be noticed that the attack quickly recovers 90% of the shares even for $\sigma = 2, 3$ (see Figure 1a), and the shares which are not recovered (because they are not given the maximum likelihood) have a good ranking. Consequently, combining this attack with one of the Key Enumeration (KEA) techniques recently developed (see *e.g.* [GGP$^+$15, MOOS15]) should significantly increase the attack efficiency.

σ (SNR)	0 ($+\infty$)	0.2 (25)	0.4 (6.25)	0.6 (2.77)	0.8 (1.56)	1 (1)
n	12	14	30	73	160	284

Table 1: First attack: number of shares n as a function of the noise σ to succeed with probability > 0.5



(a) Basic Attack

(b) Iterative Attack

Fig. 1: Percentage of recovered shares with respect to n for $\sigma = 2, 3, 3.5$ and $k = 4$

6 Second Attack: Iterative Attack

6.1 Attack Description

From the discussions in Sect. 4.4, and in view of the poor efficiency of the previous attack, we investigated another strategy which targets all the shares simultaneously. Essentially, the core idea of our second attack described below is to apply several attacks recursively on the x_i's and y_j's, and to refine step by step the likelihood of each candidate for the tuple of shares. Namely, we start by applying the attack described in Section 5.1 in order to compute, for every i, a likelihood probability for each hypothesis $X_i = x$ (x ranging over $\mathbb{F}_{2^k}$); then we apply the same attack in order to compute, for every j, a likelihood probability for each hypothesis $Y_j = y$ (y ranging over $\mathbb{F}_{2^k}$) with the single difference that the probability $\mathrm{p}_{X_i}(x)$ in (18) is replaced by the likelihood probability which was just

computed[2]. Then, one reiterates the attack to refine the likelihood probabilities $(\mathrm{p}_{X_i}(x))_{x\in\mathbb{F}_{2^k}}$, by evaluating (14) with the uniform distribution $\mathrm{p}_{Y_j}(y)$ being replaced by the likelihood probability $\text{new-p}_{Y_j}(y)$ which has been previously computed. The scheme is afterwards repeated until the maximum taken by the pdfs of each share X_i and Y_j is greater than some threshold β. In order to have better results, we perform the whole attack a second time, by starting with the computation of the likelihood probability for each hypothesis $Y_j = y$ instead of starting by $X_i = x$.

We give the formal description of the attack processing in Algorithm 2 (in order to have the complete attack, one should perform the while loop a second time, by rather starting with the computation of $\text{new-p}_{Y_j}(y)$ instead of $\text{new-p}_{X_i}(x)$).

Algorithm 2 Iterative Maximum Likelihood Attack

Input: a threshold β, an observation ℓ_i of each L_i, an observation ℓ'_j of each L'_j and one observation ℓ''_{ij} of each L''_{ij} (the random variables being defined as in (5)-(7))

Output: a n-tuple of pdfs $(\mathrm{p}_{X_i})_i$ (resp. $(\mathrm{p}_{Y_i})_i$) such that, for every $i \in [1..n]$, at least one $\hat{x}_i$ (resp. $\hat{y}_j$) satisfies $\mathrm{p}_{X_i}(\hat{x}_i) \geqslant \beta$ (resp. $\mathrm{p}_{Y_i}(\hat{y}_j) \geqslant \beta$)

1: **for** $i = 1$ **to** n **do**
2: **for** $x \in \mathbb{F}_{2^k}$ **do** # Initialize the likelihood of each candidate for X_i
3: $\mathrm{p}_{X_i}(x) = f_{L_i|X_i}(\ell_i, x)$
4: **end for**
5: **for** $y \in \mathbb{F}_{2^k}$ **do** # Initialize the likelihood of each candidate for Y_i
6: $\mathrm{p}_{Y_i}(y) = f_{\boldsymbol{L}'_i|Y_i}(\ell'_i, y_i)$
7: $\text{new-p}_{Y_i}(y) = \mathrm{p}_{Y_i}(y)$
8: **end for**
9: **end for**

10: **while** end $\neq n$ **do**
11: end $\leftarrow 0$
12: **for** $i = 1$ **to** n **do**
13: **for** $x \in \mathbb{F}_{2^k}$ **do** # Compute/Update the likelihood of each candidate for X_i
14: $\text{new-p}_{X_i}(x) = 2^{-(2n+1)k} \frac{\mathrm{p}_{X_i}(x)}{\sum_{x'\in\mathbb{F}_{2^k}} \mathrm{p}_{X_i}(x')} \prod_{j=1}^{n} \sum_{y\in\mathbb{F}_{2^k}} \frac{\text{new-p}_{Y_j}(y)}{\sum_{y'\in\mathbb{F}_{2^k}} \text{new-p}_{Y_j}(y')} \cdot f_{L''_{ij}|X_iY_j}(\ell''_{ij}, x \cdot y)$
15: **end for**
16: **end for**
17: **for** $i = 1$ **to** n **do**
18: **for** $y \in \mathbb{F}_{2^k}$ **do** # Compute/Update the likelihood of each candidate for Y_i
19: $\text{new-p}_{Y_i}(y) = 2^{-(2n+1)k} \frac{\mathrm{p}_{Y_i}(y)}{\sum_{y'\in\mathbb{F}_{2^k}} \mathrm{p}_{Y_i}(y')} \prod_{j=1}^{n} \sum_{x\in\mathbb{F}_{2^k}} \frac{\text{new-p}_{X_j}(x)}{\sum_{x'\in\mathbb{F}_{2^k}} \text{new-p}_{X_j}(x')} \cdot f_{L''_{ij}|X_iY_j}(\ell''_{ij}, x \cdot y)$
20: **end for**
21: **end for**
22: **for** $i = 1$ **to** n **do**
23: **if** $\max_x(\text{new-p}_{X_i}(x)) \geqslant \beta$ and $\max_y(\text{new-p}_{Y_i}(y)) \geqslant \beta$ **then**
24: end $++$
25: **end if**
26: **end for**
27: **end while**

[2] Actually to get the probability of $X_i \mid \boldsymbol{L}$ instead of $\boldsymbol{L} \mid X_i$, Bayes' Formula is applied which explains the division by the sum of probabilities in the lines 14 and 19 in Algorithm 2.

6.2 Numerical Experiments

The iterative attack described in Algorithm 2 has been tested against leakages simulations defined exactly as in Section 5.3. As previously we stated that an attack succeeds if the totality of the n shares x_i have been recovered, which leads to the full recovery of $x^\star$. For various noise standard deviations σ with $\mathrm{SNR} = k(2\sigma)^{-2}$, Table 2 gives the average minimum number of shares n required for the attack to succeed with probability strictly greater than 0.5 (the averaging being computed over 300 attack iterations). The first row corresponds to $k = 4$, and the second row to $k = 8$ (the corresponding SNRs are $\mathrm{SNR}_4 = \sigma^{-2}$ and $\mathrm{SNR}_8 = (\sqrt{2}\sigma^2)^{-1}$). Numerical experiments yield greatly improved results in comparison to those obtained by running the basic attack. Namely, in $\mathbb{F}_{2^4}$, for a noise $\sigma = 0$, the number of shares required is 2, while 12 shares were needed for the basic attack, and the improvement is even more confirmed with a growing σ: for a noise $\sigma = 1$, the number of shares required is 25, while 284 shares were needed for the basic attack. It can also be observed that the results for shares in $\mathbb{F}_{2^4}$ and $\mathbb{F}_{2^8}$ are relatively close, the number of shares being most likely slightly smaller for shares in $\mathbb{F}_{2^4}$ than in $\mathbb{F}_{2^8}$. This observation is in-line with the lower bound in (9), where the cardinality 2^k of the finite field plays no role. Once again, it may be observed that the attack quickly recovers 90% of the shares even for $\sigma \in \{2, 3, 3.5\}$ (see Figure 1b), and the shares which are not recovered (because they are not given the maximum likelihood) have a good ranking. Consequently, combining this attack with KEA should still increase the attack efficiency.

σ $(\mathrm{SNR}_4, \mathrm{SNR}_8)$	0 $(+\infty, +\infty)$	0.2 $(25, 17.67)$	0.4 $(6.25, 4.41)$	0.6 $(2.77, 1.96)$	0.8 $(1.56, 1.10)$	1 $(1, 0.7071)$
n (for $\mathbb{F}_{2^4}$)	2	2	3	6	13	25
n (for $\mathbb{F}_{2^8}$)	5	6	8	11	16	21

Table 2: Iterative attack: number of shares n as a function of the noise σ to succeed with probability > 0.5 in $\mathbb{F}_{2^4}$ (first row) and in $\mathbb{F}_{2^8}$ (second row).

7 Practical Results

7.1 Setup

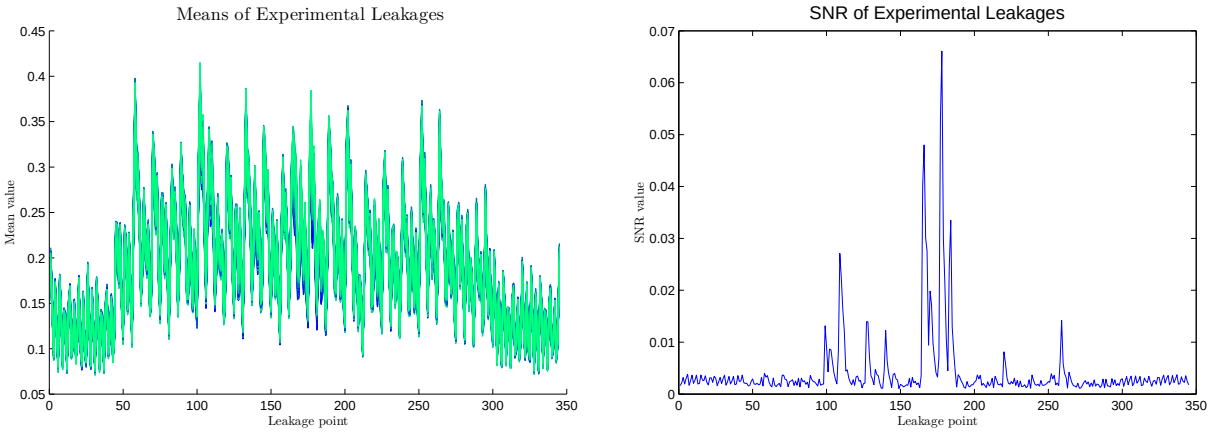
In order to provide real life experiments of the attack described in this work we have mounted it against a development platform embedding the `ATMega328` processor. The presence of decoupling capacitors between the ground pins of the processor and the reference ground did not allowed us to correctly measure the power consumption of the processor. We therefore de-soldered the ground pins of the processor and connected them to a 20 Ohm resistor whose other end was connected to the reference ground. After this preparation, we used a passive probe connected to an oscilloscope in order to register the current absorbed by the processor by measuring the difference of potential at the ends of the resistor. Thanks to a probe bandwidth of 500MHz, we pre-filtered all the frequencies higher than 200MHz. We moreover used a sampling rate of 100MHz on the oscilloscope as the best compromise between accuracy and measurement trace size.

7.2 Leakage Characterization

For our implementation of Algorithm 1, the `mov` instruction is used to manipulate the shares x_i, y_j and the multiplication results $x_i \cdot y_j$. We therefore chose to target it in our attack experiments.

An advantage of targeting a single instruction that manipulates all the values is that we obtain homogeneous leakages for all the manipulated data. Furthermore, the `mov` instruction may be found in many different architectures and we therefore think that our attack can be reproduced quite easily.

For the leakage characterization phase, we measured $200,000$ leakages of the `mov` instruction parametrized with randomly generated values. Each measure was composed of 340 points, which is essentially the size of a relevant sample of instantaneous measures for the `mov` instruction in our setup. This campaign allowed us to characterize the leakage related to the processing `mov y,x` for `x` ranging between 0 and 255 and for `y` being constant (our implementation uses the same destination register for all the shares); more precisely, each `x` was associated to a mean vector $\mu_{\mathtt{x}} \in \mathbb{R}^{340}$ and a covariance matrix $\Sigma_{\mathtt{x}} \in \mathbb{R}^{340} \times \mathbb{R}^{340}$. The 256 means are plotted in Figure 2a. To reduce the dimension of our *templates*, we afterwards estimated the signal-to-noise ratio of the acquisitions at each point in order to identify the best points of interest for our attack. The results are plotted in Figure 2b.



(a) Mean vectors of all 256 classes.

(b) SNR on each leakage point.

Fig. 2: Results of the leakage characterization phase.

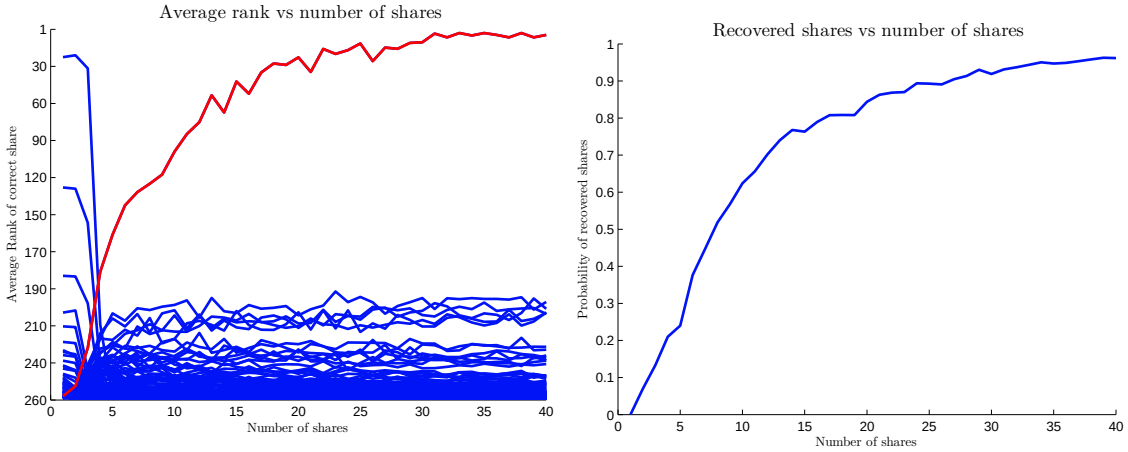
The peaks in 2b allowed us to choose a first set of points. In addition, a simple maximum likelihood attack on each of these points separately has also been mounted in order to find those who provided the most information. This step finally provided us with a set of 11 points to mount our attack. It may be observed that the best SNR obtained with our setup is around 0.07 which, for our simulations, corresponds to the case $k = 16$ and $\sigma = 3.75$, and $k = 256$ and $\sigma = 3.2$. Our experiments should therefore be compared to the simulations plotted in the gray dot-and-dashed curve in Fig. 1a (for the basic attack) and in Fig. 1b (for the iterative attack).

7.3 Attacks

Thanks to the characterization described in the previous section, we performed the attacks of Sect. 5.1 and Sect. 6 against our implementation of Algorithm 1 parametrized with different orders n.

Maximum Likelihood Attack Experiments As a first experiment we tested the attack of Section 5.1 in order to evaluate the evolution of the rank of the correct hypothesis on a single share x_i when n varies. In 3a we plot the ranking of the correct hypothesis, in red, among all 256 byte values, for each n between 1 and 40. The results have been obtained by averaging the result among 100 repetitions of the same attack for each order. From 3a we can observe that the correct hypothesis is ranked among the first 50 values as soon as $n > 10$. We also observe that we need $n > 35$ for the correct hypothesis to be firmly ranked first by the basic attack.

During the same attack we have also evaluated the average number of shares of $x^\star$ correctly retrieved for each order n. This allowed us to obtain an estimation of the minimum order n required to successfully mount the attack. The result of such evaluation is depicted in 3b. Even if $n = 40$ appears to be a necessary condition to directly recover all the shares, the post-processing of our attack results with a KEA algorithm [GGP+15, MOOS15] should allow to recover them for smaller n ($n = 10$ seems to be achievable at a reasonable cost).



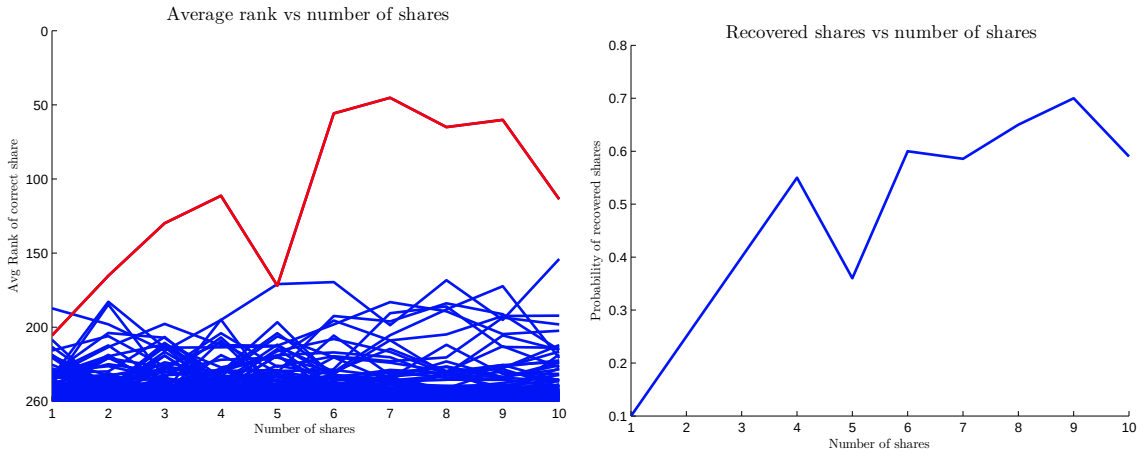
(a) Avg rank of the correct hypothesis vs number of shares.

(b) Avg rate of correctly retrieved shares.

Fig. 3: Results for the attack of Section 5.1 when n varies between 1 and 40.

Iterative Attack Experiments We have afterwards mounted the attack described in Section 6 in order to measure the success probability with respect to the masking order. As before we provide the average ranking of the correct hypothesis (in red) for different values of the order n. Due to time constraints only 10 repetitions of the attack have been averaged. The evolution of the rank of the correct hypothesis is depicted in 4a. As before, we also provide the average number of shares correctly retrieved for each order n in 4b.

Even if we did not got results as smooth as for previous experiments, we can observe an overall better detection rate for the iterative attack if we compare to the basic one (as actually expected). Furthermore, we have averaged the results of the above attack on 10 repetitions when $n = 20, 30, 40$. For such order we have obtained an average number of correctly retrieved shares of $0.88, 0.93$ and 0.97, respectively. We thus remark again that by using KEA post-processing on the results of our



(a) Avg rank of the correct hypothesis vs number of shares.

(b) Avg number of correctly retrieved shares.

Fig. 4: Results for the attack of Section 6 when n varies between 1 and 10.

attack the correct hypothesis should be recovered with a reasonable number of shares. In particular it seems reasonable to assume that $n = 10$ provides better results and lower costs with respect to the basic attack.

Experiments Conclusions. We have successfully proved the effectiveness of our attack on a real implementation using the `ATMega328` processor. We obtain better results on our experiments than those predicted by theory in 5.1 and 6 for similar SNR values. We have investigated such behavior and we conjecture that the better results are due to the use of a multi-variate attack on 11 points for the experimental attacks, where the theoretical results are computed for a mono-variate attack. The more points allows for an improvement of the global SNR which can explain the disparity between theory and practice success probabilities.

8 A Countermeasure against the previous Attacks

8.1 Description

In the following, we describe a countermeasure against the previous attack against the Rivain-Prouff algorithm. More precisely, we describe a variant of Algorithm 1, called RefSecMult, to compute an n-sharing of $c = x^\star \cdot y^\star$ from $(x_i)_{i\in[1..n]}$ and $(y_i)_{i\in[1..n]}$. Our new algorithm is still provably secure in the original ISW probing model, and heuristically secure against the horizontal side-channel attacks described the in previous sections.

As observed in [FRR+10], the ISW and Rivain-Prouff countermeasures can be divided in two steps: a "matrix" step in which starting from the input shares x_i and y_j, one obtains a matrix $x_i \cdot y_j$ with n^2 elements, and a "compression" step in which one uses some randomness to get back to a n-sharing c_i. Namely the matrix elements $(x_i \cdot y_j)_{1\leq i,j\leq n}$ form a n^2-sharing of $x^\star \cdot y^\star$:

$$x^\star \cdot y^\star = \left(\sum_{i=1}^{n} x_i\right) \cdot \left(\sum_{j=1}^{n} y_j\right) = \sum_{1\leq i,j\leq n} x_i \cdot y_j \tag{19}$$

Algorithm 3 RefSecMult

Input: n-sharings $(x_i)_{i\in[1..n]}$ and $(y_j)_{j\in[1..n]}$ of $x^\star$ and $y^\star$ respectively
Output: an n-sharing $(c_i)_{i\in[1..n]}$ of $x^\star \cdot y^\star$

$M_{ij} \leftarrow \mathsf{MatMult}((x_1,\ldots,x_n),(y_1,\ldots,y_n))$
for $i = 1$ **to** n **do**
  for $j = i+1$ **to** n **do**
    $r_{i,j} \leftarrow^{\$} \mathbb{F}_{2^k}$
    $r_{j,i} \leftarrow (r_{i,j} + M_{ij}) + M_{ji}$
  end for
end for
for $i = 1$ **to** n **do**
  $c_i \leftarrow M_{ii}$
  for $j = 1$ **to** n, $j \neq i$ **do** $c_i \leftarrow c_i + r_{i,j}$
end for
return $(c_1, c_1, \ldots, c_n)$

and the goal of the compression step is to securely go from such n^2-sharing of $x^\star \cdot y^\star$ to a n-sharing of $x^\star \cdot y^\star$.

Our new countermeasure (Algorithm 3) uses the same compression step as Rivain-Prouff, but with a different matrix step, called MatMult (Algorithm 4), so that the shares x_i and y_j are not used multiple times (as when computing the matrix elements $x_i \cdot y_j$ in Rivain-Prouff). Eventually the MatMult algorithm outputs a matrix $(M_{ij})_{1\leq i,j\leq n}$ which is still a n^2-sharing of $x^\star \cdot y^\star$, as in (19); therefore using the same compression step as Rivain-Prouff, Algorithm 3 outputs a n-sharing of $x^\star \cdot y^\star$, as required.

Algorithm 4 MatMult

Input: the n-sharings $(x_i)_{i\in[1..n]}$ and $(y_j)_{j\in[1..n]}$ of $x^\star$ and $y^\star$ respectively
Output: the n^2-sharing $(M_{ij})_{i\in[1..n],j\in[1..n]}$ of $x^\star \cdot y^\star$

if $n = 1$ **then**
  $\boldsymbol{M} \leftarrow [x_1 \cdot y_1]$
else
  $\boldsymbol{X}^{(1)} \leftarrow (x_1,\ldots,x_{n/2})$, $\boldsymbol{X}^{(2)} \leftarrow (x_{n/2+1},\ldots,x_n)$
  $\boldsymbol{Y}^{(1)} \leftarrow (y_1,\ldots,y_{n/2})$, $\boldsymbol{Y}^{(2)} \leftarrow (y_{n/2+1},\ldots,y_n)$
  $\boldsymbol{M}^{(1,1)} \leftarrow \mathsf{MatMult}(\boldsymbol{X}^{(1)},\boldsymbol{Y}^{(1)})$
  $\boldsymbol{X}^{(1)} \leftarrow \mathsf{RefreshMasks}(\boldsymbol{X}^{(1)})$, $\boldsymbol{Y}^{(1)} \leftarrow \mathsf{RefreshMasks}(\boldsymbol{Y}^{(1)})$
  $\boldsymbol{M}^{(1,2)} \leftarrow \mathsf{MatMult}(\boldsymbol{X}^{(1)},\boldsymbol{Y}^{(2)})$
  $\boldsymbol{M}^{(2,1)} \leftarrow \mathsf{MatMult}(\boldsymbol{X}^{(2)},\boldsymbol{Y}^{(1)})$
  $\boldsymbol{X}^{(2)} \leftarrow \mathsf{RefreshMasks}(\boldsymbol{X}^{(2)})$, $\boldsymbol{Y}^{(2)} \leftarrow \mathsf{RefreshMasks}(\boldsymbol{Y}^{(2)})$
  $\boldsymbol{M}^{(2,2)} \leftarrow \mathsf{MatMult}(\boldsymbol{X}^{(2)},\boldsymbol{Y}^{(2)})$
  $\boldsymbol{M} \leftarrow \begin{bmatrix} \boldsymbol{M}^{(1,1)} & \boldsymbol{M}^{(1,2)} \\ \boldsymbol{M}^{(2,1)} & \boldsymbol{M}^{(2,2)} \end{bmatrix}$
end if
return $\boldsymbol{M}$

As illustrated in Fig. 5, the MatMult algorithm is recursive and computes the $n \times n$ matrix in four sub-matrix blocs. This is done by splitting the input shares x_i and y_j in two parts, namely $\boldsymbol{X}^{(1)} = (x_1,\ldots,x_{n/2})$ and $\boldsymbol{X}^{(2)} = (x_{n/2+1},\ldots,x_n)$, and similarly $\boldsymbol{Y}^{(1)} = (y_1,\ldots,y_{n/2})$ and $\boldsymbol{Y}^{(2)} = (y_{n/2+1},\ldots,y_n)$, and recursively processing the four sub-matrix blocs corresponding to $\boldsymbol{X}^{(u)} \times \boldsymbol{Y}^{(v)}$

for $1 \leq u, v \leq 2$. To prevent the same share x_i from being used twice, each input block $\boldsymbol{X}^{(u)}$ and $\boldsymbol{Y}^{(v)}$ is refreshed before being used a second time, using a mask refreshing algorithm. An example of such mask refreshing, hereafter called RefreshMasks, can for instance be found in [DDF14]; see Algorithm 5. Since the mask refreshing does not modify the xor of the input $n/2$-vectors $\boldsymbol{X}^{(u)}$ and $\boldsymbol{Y}^{(v)}$, each sub-matrix block $\boldsymbol{M}^{(u,v)}$ is still a $n^2/4$-sharing of $(\oplus \boldsymbol{X}^{(u)}) \cdot (\oplus \boldsymbol{X}^{(v)})$, and therefore the output matrix $\boldsymbol{M}$ is still a n^2-sharing of $x^\star \cdot y^\star$, as required. Note that without the RefreshMasks, we would have $M_{ij} = x_i \cdot y_j$ as in Rivain-Prouff.

Algorithm 5 RefreshMasks

Input: $a_1, \ldots, a_n$
Output: $c_1, \ldots, c_n$ such that $\sum_{i=1}^n c_i = \sum_{i=1}^n a_i$
1: **For** $i = 1$ **to** n **do** $c_i \leftarrow a_i$
2: **for** $i = 1$ to n do **do**
3: **for** $j = i + 1$ to n do **do**
4: $r \leftarrow \{0,1\}^k$
5: $c_i \leftarrow c_i + r$
6: $c_j \leftarrow c_j + r$
7: **end for**
8: **end for**
9: **return** $c_1, \ldots, c_n$

Since the RefreshMask algorithm has complexity $\mathcal{O}(n^2)$, it is easy to see that the complexity of our RefSecMult algorithm is $\mathcal{O}(n^2 \log n)$ (instead of $\mathcal{O}(n^2)$ for the original Rivain-Prouff countermeasure in Alg. 1). Therefore for a circuit of size $|C|$ the complexity is $\mathcal{O}(|C| \cdot n^2 \log n)$, instead of $\mathcal{O}(|C| \cdot n^2)$ for Rivain-Prouff. The following lemma shows the soundness of our RefSecMult countermeasure.

Lemma 1 (Soundness of RefSecMult). *The RefSecMult algorithm, on input n-sharings $(x_i)_{i\in[1..n]}$ and $(y_j)_{j\in[1..n]}$ of $x^\star$ and $y^\star$ respectively, outputs an n-sharing $(c_i)_{i\in[1..n]}$ of $x^\star \cdot y^\star$.*

Proof. We prove recursively that the MatMult algorithm, taking as input n-sharings $(x_i)_{i\in[1..n]}$ and $(y_j)_{j\in[1..n]}$ of $x^\star$ and $y^\star$ respectively, outputs an n^2-sharing M_{ij} of $x^\star \cdot y^\star$. The lemma for RefSecMult will follow, since as in Rivain-Prouff the lines 2 to 12 of Alg. 3 transform a n^2-sharing M_{ij} of $x^\star \cdot y^\star$ into a n-sharing of $x^\star \cdot y^\star$.

The property clearly holds for $n = 1$. Assuming that it holds for $n/2$, since the RefreshMasks does not change the xor of the input $n/2$-vectors $\boldsymbol{X}^{(u)}$ and $\boldsymbol{Y}^{(v)}$, each sub-matrix block $\boldsymbol{M}^{(u,v)}$ is still an $n^2/4$-sharing of $(\oplus \boldsymbol{X}^{(u)}) \cdot (\oplus \boldsymbol{X}^{(v)})$, and therefore the output matrix $\boldsymbol{M}$ is still an n^2-sharing of $x^\star \cdot y^\star$, as required. This proves the lemma. □

Remark 2. The description of our countermeasure requires that n is a power of two, but it is easy to modify the countermeasure to handle any value of n. Namely in Algorithm 4, for odd n it suffices to split the inputs x_i and y_j in two parts of size $(n-1)/2$ and $(n+1)/2$ respectively, instead of $n/2$.

8.2 Security Analysis.

Proven security in the ISW probing model. We prove that our RefSecMult algorithm achieves at least the same level of security of Rivain-Prouff, namely it is secure in the ISW probing model

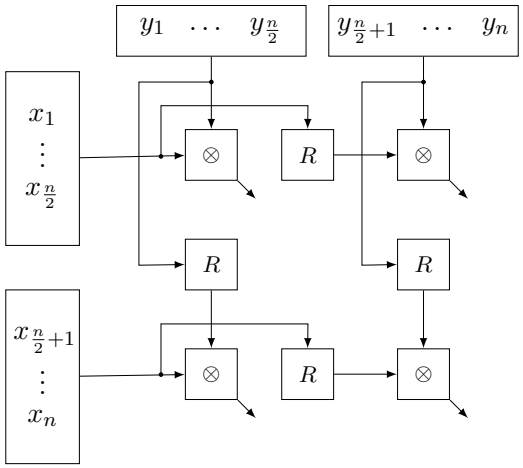


Fig. 5: The recursive MatMult algorithm, where R represents the RefreshMasks Algorithm, and $\otimes$ represents a recursive call to the MatMult algorithm.

against t probes for $n \geq t+1$ shares. For this we use the refined security model against probing attacks recently introduced in [BBD+15], called t-SNI security. This stronger definition of t-SNI security enables to prove that a gadget can be used in a full construction with $n \geq t+1$ shares, instead of $n \geq 2t+1$ for the weaker definition of t-NI security (corresponding to the original ISW security proof). The authors of [BBD+15] show that the ISW (and Rivain-Prouff) multiplication gadget does satisfy this stronger t-SNI security definition. They also show that with some additional mask refreshing satisfying the t-SNI property (such as RefreshMasks), the Rivain-Prouff countermeasure for the full AES can be made secure with $n \geq t+1$ shares.

The following lemma shows that our RefSecMult countermeasure achieves the t-SNI property; we provide the proof in Appendix B. The proof is essentially the same as in [BBD+15] for the Rivain-Prouff countermeasure; namely the compression step is the same, and for the matrix step, in the simulation we can assume that all the randoms in RefreshMasks are given to the adversary. The t-SNI security implies that our RefSecMult algorithm is also composable, with $n \geq t+1$ shares.

Lemma 2 (t-SNI of RefSecMult). *Let $(x_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n}$ be the input shares of the SecMult operation, and let $(c_i)_{1 \leq i < n}$ be the output shares. For any set of t_1 intermediate variables and any subset $|\mathcal{O}| \leq t_2$ of output shares such that $t_1 + t_2 < n$, there exists two subsets I and J of indices with $|I| \leq t_1$ and $|J| \leq t_1$, such that those t_1 intermediate variables as well as the output shares $c_{|\mathcal{O}}$ can be perfectly simulated from $x_{|I}$ and $y_{|J}$.*

Heuristic security against horizontal-DPA attacks. We stress that the previous lemma only proves the security of our countermeasure against t probes for $n \geq t+1$, so it does not prove that our countermeasure is secure against the horizontal-DPA attacks described in the previous sections, since such attacks use information about n^2 intermediate variables instead of only $n-1$.

As illustrated in Fig. 5, the main difference between the new RefSecMult algorithm and the original SecMult algorithm (Alg. 1) is that we keep refreshing the x_i shares and the y_j shares blockwise between the processing of the finite field multiplications $x_i \cdot y_j$. Therefore, as opposed to what happens in SecMult, we never have the same x_i being multiplied by all y_j's for $1 \leq j \leq n$.

Therefore an attacker cannot accumulate information about a specific share x_i, which heuristically prevents the attacks described in this paper.

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A Mutual Information Approximation

In this section, we develop the mutual information between the Hamming weight of uniform random variable X defined over $\mathbb{F}_{2^k}$ and a noisy observation L of this Hamming weight, the noise being modeled by a Gaussian random variable with 0 mean and standard deviation σ. By definition, we have $L = \mathrm{HW}(X) + B$, with $B \sim \mathcal{N}(0, \sigma^2)$. By definition of the entropy, we have:

$$\mathbb{H}(X|L) = \mathbb{H}(X, L) - \mathbb{H}(L) \ . \tag{20}$$

Given X, the distribution of L is Gaussian with standard deviation σ. The differential entropy corresponding to a Gaussian distribution with standard deviation σ is $\ln(\sigma\sqrt{2\pi e})$, whereas the entropy of a uniform random variable over $\mathbb{F}_{2^k}$ is k. We therefore get:

$$\mathbb{H}(X, L) = \mathbb{H}(X) + \mathbb{H}(B) = k + \ln(\sigma\sqrt{2\pi e}) \tag{21}$$

The distribution of L is the sum of two distributions. We can model the distribution of $\mathrm{HW}(X)$ as Gaussian with standard deviation $\sigma_k = \sqrt{k}/2$; this is true for large k. The sum of two independent and normally distributed random variables has a normal distribution. Moreover, its mean and variance are simply obtained by summing those of the added variables. In our context, this gives:

$$\mathbb{H}(L) \simeq \ln\left(\sqrt{(\sigma_k^2 + \sigma^2)2\pi e}\right) \simeq \ln\left(\sqrt{(k/4 + \sigma^2)2\pi e}\right) \tag{22}$$

Combining (20)-(22) leads to:

$$\begin{aligned}\mathbb{H}(X \mid L) &\simeq k + \ln(\sigma\sqrt{2\pi e}) - \ln\left(\sqrt{(k/4 + \sigma^2)2\pi e}\right) \simeq k + \frac{1}{2}\ln\left(\frac{\sigma^2}{k/4 + \sigma^2}\right) \\ &\simeq k + \frac{1}{2}\ln\left(\frac{1}{1 + k/(4\sigma^2)}\right) \simeq k - \frac{k}{8\sigma^2} \ ,\end{aligned}$$

after approximating $\ln(1 - U)$ by $-U$, and $k/(k + 4\sigma^2)$ by $k/4\sigma^2$ (which is true when $\sigma^2 \gg k$). Eventually, the amount of information $\mathrm{I}(X; L) \doteq \mathbb{H}(X) - \mathbb{H}(X \mid L)$ given by the noisy Hamming weight leakage can be approximated for $\sigma^2 \gg k$ by:

$$\mathrm{I}(X; L) \simeq \frac{k}{8\sigma^2}$$

B Proof of Lemma 2

Our proof is essentially the same as in [BBD+15]. We construct two sets I and J corresponding to the input shares of $x^\star$ and $y^\star$ respectively. We denote by M_{ij} the result of the subroutine $\mathsf{MatMult}((x_1, \ldots, x_n), (y_1, \ldots, y_n))$. From the definition of MatMult and RefreshMasks, it is easy to see that each M_{ij} can be perfectly simulated from x_i and y_j; more generally any internal variable within MatMult can be perfectly simulated from x_i and/or y_j for some i and j; for this it suffices to simulate the randoms in RefreshMasks exactly as they are generated in RefreshMasks.

We divide the internal probes in 4 groups. The four groups are processed separately and sequentially, that is we start with all probes in Group 1, and finish with all probes in Group 4.

- Group 1: If M_{ii} is probed, add i to I and J.

- Group 2: If $r_{i,j}$ or $c_{i,j}$ is probed (for any $i \neq j$), add i to I and J.

Note that after the processing of Group 1 and 2 probes, we have $I = J$; we denote by U the common value of I and J after the processing of Group 1 and 2 probes.

- Group 3: If $M_{ij} \oplus r_{i,j}$ is probed: if $i \in U$ or $j \in U$, add $\{i, j\}$ to both I and J.
- Group 4: If M_{ij} is probed (for any $i \neq j$), then add i to I and j to J. If some probe in $\mathsf{MatMult}$ requires the knowledge of x_i and/or y_j, add i to I and/or j to J.

We have $|I| \leq t_1$ and $|J| \leq t_1$, since for every probe we add at most one index in I and J. The simulation of probed variables in groups 1 and 4 is straightforward. Note that for $i < j$, the variable r_{ij} is used in all partial sums c_{ik} for $k \geq j$; moreover r_{ij} is used in $r_{ij} \oplus M_{ij}$, which is used in r_{ji}, which is used in all partial sums c_{jk} for $k \geq i$. Therefore if $i \notin U$, then r_{ij} is not probed and does not enter in the computation of any probed c_{ik}; symmetrically if $j \notin U$, then r_{ji} is not probed and does not enter in the computation of any probed c_{jk}.

For any pair $i < j$, we can now distinguish 4 cases:

- Case 1: $\{i, j\} \in U$. In that case, we can perfectly simulate all variables r_{ij}, M_{ij}, $M_{ij} \oplus r_{ij}$, M_{ji} and r_{ji}. In particular, we let $r_{ij} \leftarrow \mathbb{F}_{2^k}$, as in the real circuit.
- Case 2: $i \in U$ and $j \notin U$. In that case we simulate $r_{ij} \leftarrow \mathbb{F}_{2^k}$, as in the real circuit. If $M_{ij} \oplus r_{i,j}$ is probed (Group 3), we can also simulate it since $i \in U$ and $j \in J$ by definition of the processing of Group 3 variables.
- Case 3: $i \notin U$ and $j \in U$. In that case r_{ij} has not been probed, nor any variable c_{ik}, since otherwise $i \in U$. Therefore r_{ij} is not used in the computation of any probed variable (except r_{ji}, and possibly $M_{ij} \oplus r_{i,j}$). Therefore we can simulate $r_{ji} \leftarrow \mathbb{F}_{2^k}$; moreover if $M_{ij} \oplus r_{ij}$ is probed, we can perfectly simulate it using $M_{ij} \oplus r_{ij} = M_{ji} \oplus r_{ji}$, since $j \in U$ and $i \in J$ by definition of the processing of Group 3 variables.
- Case 4: $i \notin U$ and $j \notin U$. If $M_{ij} \oplus r_{i,j}$ is probed, since r_{ij} is not probed and does not enter into the computation of any other probed variable, we can perfectly simulate such probe with a random value.

From cases 1, 2 and 3, we obtain that for any $i \neq j$, we can perfectly simulate any variable r_{ij} such that $i \in U$. This implies that we can also perfectly simulate all partial sums c_{ik} when $i \in U$, including the output variables c_i. Finally, all probed variables are perfectly simulated.

We now consider the simulation of the output variables c_i. We must show how to simulate c_i for all $i \in \mathcal{O}$, where $\mathcal{O}$ is an arbitrary subset of $[1, n]$ such that $t_1 + |\mathcal{O}| < n$. For $i \in U$, such variables are already perfectly simulated, as explained above. We now consider the output variables c_i with $i \notin U$. We construct a subset of indices V as follows: for any probed Group 3 variable $M_{ij} \oplus r_{ij}$ such that $i \notin U$ and $j \notin U$ (this corresponds to Case 4), we put j in V if $i \in \mathcal{O}$, otherwise we put i in V. Since we have only considered Group 3 probes, we must have $|U| + |V| \leq t_1$, which implies $|U| + |V| + |\mathcal{O}| < n$. Therefore there exists an index $j^\star \in [1, n]$ such that $j^\star \notin U \cup V \cup \mathcal{O}$. For any $i \in \mathcal{O}$, we can write:

$$c_i = M_{ii} \oplus \bigoplus_{j \neq i} r_{ij} = r_{i,j^\star} \oplus \left(M_{ii} \oplus \bigoplus_{j \neq i, j^\star} r_{ij} \right)$$

We claim that neither $r_{i,j^\star}$ nor $r_{j^\star,i}$ do enter into the computation of any probed variable or other $c_{i'}$ for $i' \in \mathcal{O}$. Namely $i \notin U$ so neither $r_{i,j^\star}$ nor any partial sum c_{ik} was probed; similarly $j^\star \notin U$ so neither $r_{j^\star,i}$ nor any partial sum $c_{j^\star,k}$ was probed, and the output $c_{j^\star}$ does not have to be simulated since by definition $j^\star \notin \mathcal{O}$. Finally if $i < j^\star$ then $M_{i,j^\star} \oplus r_{i,j^\star}$ was not probed since otherwise $j^\star \in V$ (since $i \in \mathcal{O}$); similarly if $j^\star < i$ then $M_{j^\star,i} \oplus r_{j^\star,i}$ was not probed since otherwise we would have $j^\star \in V$ since $j^\star \notin \mathcal{O}$. Therefore, since neither $r_{i,j^\star}$ nor $r_{j^\star,i}$ are used elsewhere, we can perfectly simulate c_i by generating a random value. This proves the Lemma.