Mastrovito Form of Non-recursive Karatsuba Multiplier for All Trinomials

Yin Li, Xingpo Ma, Yu Zhang and Chuanda Qi

Abstract—We present a new type of bit-parallel non-recursive Karatsuba multiplier over $GF(2^m)$ generated by an arbitrary irreducible trinomial. This design effectively exploits Mastrovito approach and shifted polynomial basis (SPB) to reduce the time complexity and Karatsuba algorithm to reduce its space complexity. We show that this type of multiplier is only one T_X slower than the fastest bit-parallel multiplier for all trinomials, where T_X is the delay of one 2-input XOR gate. Meanwhile, its space complexity is roughly 3/4 of those multipliers. To the best of our knowledge, it is the first time that our scheme has reached such a time delay bound. This result outperforms previously proposed non-recursive Karatsuba multipliers.

Index Terms—Mastrovito multiplier, Karatsuba, shifted polynomial basis, trinomials.

1 Introduction

Efficient hardware implementation of multiplication over $GF(2^m)$ is one of the main topics studied in recent years, as it is frequently required in many areas such as coding theory and public key cryptography [1], [2]. A number of algorithms for efficient $GF(2^m)$ multiplication have been proposed, one of the most attractive approaches is the Karatsuba algorithm (KA) [3]. The KA was originally applied in the digit number multiplication and can be directly shifted to polynomial multiplication [4]. It is a divide-and-conquer algorithm which works by recursively breaking down a problem into two or more sub-problems. For polynomial multiplication, the key idea of the KA saves coefficient multiplications at the cost of extra coefficient additions. When using polynomial basis (PB) representation, the $GF(2^m)$ multiplication consists of a polynomial multiplication and a modular reduction. Let $f(x)$ be the irreducible polynomial that defines $GF(2^m)$, $A(x), B(x)$ be two arbitrary elements of $GF(2^m)$. Then the field multiplication is defined as $C(x) = A(x) \cdot B(x) \bmod f(x)$. Since the KA can optimize polynomial multiplication $A(x) \cdot B(x)$, it can be easily adopted to design efficient $GF(2^m)$ multipliers, either for sub-quadratic complexity multipliers (apply KA recursively and cost $O(m^\delta)$ circuit gates with $1 < \delta < 2$) [5], [6] or hybrid multipliers [10], [11].

Generally, the hybrid multipliers [10], [11] firstly perform a few iterations of KA to reduce the whole space complexities, and then perform school-book multiplication algorithm over the smaller input operands to achieve relatively higher speed. Therefore, this kind of multipliers provided a trade-off between the space and time complexities. Specially, there is another type of hybrid multiplier [14] which only apply KA once in the polynomial multiplication. We name this type of multiplier as non-recursive Karatsuba multiplier. Compared with the bit-parallel multipliers without using divide-and-conquer algorithms [16], [17], [24], [25], the space complexity of this multiplier is roughly reduced by $1/4$, with time complexity increased by a small number of T_X, where T_X is the delay of one 2-input XOR gate. For example, the classic scheme [14] is at least 2 T_X slower than the fastest bit-parallel multiplier [16], [17]. During recent years, several schemes [12], [13], [18], [23] have been proposed to accelerate this multiplier, some of which have their time complexities close to the fastest ones. However, these schemes work by using either specific irreducible polynomials (e.g, all-one polynomial (AOP), equally-spaced trinomial (EST)) which are quite rare or a modified KA that increased the space complexity.

Mastrovito [7] provided a novel way that transforms the $GF(2^m)$ multiplication into a matrix-vector multiplication. A product matrix $\mathbf{M}$ is introduced to combine the polynomial multiplication and modular reduction together. Thus, the field multiplication is carried out by $\mathbf{c} = \mathbf{M} \cdot \mathbf{b}$, where $\mathbf{c}$, $\mathbf{b}$ are the coefficient vectors of $C(x)$ and $B(x)$ and $\mathbf{M}$ is constructed from $A(x)$ and $f(x)$ presented previously. Empirically, the Mastrovito multiplier is generally faster than other type of multipliers, but the organization of $\mathbf{M}$ is always its implementation bottleneck. In [16], Fan and Hasan proposed a new Mastrovito multiplier based on shifted polynomial basis (SPB) which can simplify $\mathbf{M}$. This type of multiplier is considered as the fastest bit-parallel multiplier so far. More explicitly, this architecture contains m^2 AND and m^2-1 XOR gates with time delay of $T_A+(1+\lceil \log_2 m \rceil)T_X$ (for good field it is equal to $T_A + \lceil \log_2 m \rceil T_X$), where T_A is the delay of one 2-input AND gate.

In this paper, we apply the idea of Mastrovito to KA and describe a new architecture for non-recursive Karatsuba multiplier for all irreducible trinomials. Field multiplication is firstly partitioned into two parts according to KA, then each of which is formulize by a Mas-

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trovito matrix-vector multiplication. Explicit formulae of related product matrices using SPB are studied. Our scheme fully takes advantage of Mastrovito algorithm and SPB and achieves faster implementation speed. It is argued that this new multiplier is only one T_X slower than the fastest multipliers for trinomials. Meanwhile, by applying KA and exploiting the overlapped entries of different Mastrovito matrices, the space complexity of the proposed multiplier is still roughly 3/4 of those multipliers. To the best of our knowledge, it is the first time our scheme has achieved such a time delay bound compared with previously proposed bit-parallel non-recursive Karatsuba multipliers for trinomials.

The remainder of this paper is organized as follows: In section 2, we first briefly introduce some basic concepts and recall the Karatsuba and Mastrovito algorithms. Then, based on combination of these two algorithms, a new type of bit-parallel multiplier architecture is proposed in the following section. Section 4 presents the comparison between the proposed multiplier and some others. The last section summarizes the results and draws some conclusions.

2 Notation and Preliminary

In this section, we briefly review some notations and algorithms used throughout this paper.

Consider a binary extension field generated with an irreducible trinomial $GF(2^m) \cong \mathbb{F}_2[x]/(f(x))$ where $f(x) = x^m + x^k + 1$. Let x be a root of $f(x)$, and then the set $M = \{x^{m-1}, \cdots, x, 1\}$ constitutes a polynomial basis (PB). The shifted polynomial basis (SPB) proposed by Fan and Dai [15] was derived from polynomial basis. It can be obtained by multiplying the set M by a certain exponentiation of x:

Definition 1 *[15] Let v be an integer and the ordered set $M = \{x^{m-1}, \cdots, x, 1\}$ be a polynomial basis of $GF(2^m)$ over $\mathbb{F}_2$. The ordered set $x^{-v}M := \{x^{i-v} | 0 \le i \le m-1\}$ is called the shifted polynomial basis(SPB) with respect to M.*

It is easy to check that the field multiplication using SPB is nearly the same as that using PB except a certain parameter:

$$C(x)x^{-v} = A(x)x^{-v} \cdot B(x)x^{-v} \bmod f(x).$$

The advantage of SPB over PB is that it can simplify the modular reduction if v is properly chosen. For trinomial $x^m + x^k + 1$, it has been proved that the optimal value of v here is k or $k-1$ [15]. In this study, we choose that v equals k and use this denotation thereafter.

The Karatsuba algorithm (KA) optimized the polynomial multiplication $D(x) = A(x) \cdot B(x)$ by partitioning each polynomial into two halves.

$$\begin{aligned} AB =& (A_H x^n + A_L) \cdot (B_H x^n + B_L) \\ =& A_H B_H x^{2n} + [(A_H + A_L)(B_H + B_L) \\ & + A_H B_H + A_L B_L] x^n + A_L B_L, \end{aligned} \quad (1)$$

where $n = m/2$, A_H, A_L and B_H, B_L are two halves of $A(x)$ and $B(x)$, respectively. When m is odd, the formula is almost the same as (1). We note that the addition and subtraction are the same in $GF(2^m)$. The above expression saves one partial multiplication at the cost of three extra partial additions. For VLSI implementation of expression (1), it leads to more XOR gate delay than the ordinary polynomial multiplication.

Polynomial multiplication $D(x) = A(x) \cdot B(x)$ can be implemented as a matrix-vector multiplication

$$\mathbf{d} = \mathbf{A} \cdot \mathbf{b},$$

where $\mathbf{b} = [b_0, b_1, \cdots, b_{m-1}]^T$ and $\mathbf{d} = [d_0, d_1, \cdots, d_{m-1}]^T$ are the coefficient vectors of $B(x)$ and $D(x)$, respectively. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-2} & a_{m-3} & a_{m-4} & \cdots & a_0 & 0 \\ a_{m-1} & a_{m-2} & a_{m-3} & \cdots & a_1 & a_0 \\ 0 & a_{m-1} & a_{m-2} & \cdots & a_2 & a_1 \\ 0 & 0 & a_{m-1} & \cdots & a_3 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{m-1} & a_{m-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{m-1} \end{bmatrix}. \quad (2)$$

Then, we can perform the modular reduction $C(x) = D(x) \bmod f(x)$. The Mastrovito algorithm [7] provides a way to combine the polynomial multiplication and modular reduction into a matrix-vector multiplication, i.e.,

$$\mathbf{c} = \mathbf{M} \cdot \mathbf{b}.$$

The matrix $\mathbf{M}$ is called *product matrix* which is constructed from $\mathbf{A}$ and $f(x)$, and $\mathbf{c} = [c_0, c_1, \cdots, c_{m-1}]^T$ is the coefficient vector of $C(x)$. The main problem of Mastrovito algorithm is that its implementation relies on the organization of $\mathbf{M}$. Several algorithms are given to construct $\mathbf{M}$ efficiently or make it simpler, some of them are presented in the literature [15], [16], [24], [25], [26], [27].

In the following section, we will use Mastrovito approach to speed up KA and describe a new architecture for bit-parallel non-recursive Karatsuba multiplier. We first introduce some notations pertaining to matrices and vectors which are already presented in [26]: $\mathbf{Z}(i,:), \mathbf{Z}(:,j)$ and $\mathbf{Z}(i,j)$ represent the ith row vector, jth column vector, and the entry with position (i,j) in matrix $\mathbf{Z}$, respectively. $\mathbf{Z}[\uparrow i], \mathbf{Z}[\downarrow i]$ represent up and down shift of matrix $\mathbf{Z}$ by i rows and feeding the vacancies with zero.

Besides, two extra types of operations are also introduced: $\mathbf{Z}[\circlearrowleft i]$ represents cyclic shift of $\mathbf{Z}$ by upper i rows. $\mathbf{Z}[\Uparrow i]$ and $\mathbf{Z}[\Downarrow i]$ represent appending i zero vectors to the top and bottom of $\mathbf{Z}$, respectively. For example,

$\mathbf{Z}[\circlearrowleft 2]$, $\mathbf{Z}[\Uparrow 1]$ and $\mathbf{Z}[\Downarrow 2]$ are given by

$$\begin{aligned}
&\mathbf{Z}[\circlearrowleft 2] = [\mathbf{Z}(3,:)^T, \cdots, \mathbf{Z}(m,:)^T, \mathbf{Z}(1,:)^T, \mathbf{Z}(2,:)^T]^T,\\
&\mathbf{Z}[\Uparrow 1] = [\mathbf{Z}(1,:)^T, \cdots, \mathbf{Z}(m,:)^T, \mathbf{0}]^T,\\
&\mathbf{Z}[\Downarrow 2] = [\mathbf{0}, \mathbf{0}, \mathbf{Z}(1,:)^T, \cdots, \mathbf{Z}(m,:)^T]^T.
\end{aligned}$$

3 Mastrovito form of Karatsuba Multiplier

In this section, we firstly introduce a matrix form of Karatsuba algorithm for $GF(2^m)$ multiplication using SPB representation. Then, we develop an efficient approach to calculate the product matrix, based on modified *sub-expression sharing* [7]. Accordingly, a fast bit-parallel Karatsuba multiplier architecture is proposed.

3.1 Matrix form of Karatsuba algorithm

Let $f(x) = x^m + x^k + 1$ be an irreducible trinomial generating the finite field $GF(2^m)$. Provided that $A, B \in GF(2^m)$ are two arbitrary elements in SPB representation, namely,

$$A = x^{-k}\sum_{i=0}^{m-1} a_i x^i,\ B = x^{-k}\sum_{i=0}^{m-1} b_i x^i.$$

The field multiplication consists of performing polynomial multiplication with parameter x^{-k} and then reducing the product modulo $f(x)$, i.e.,

$$\begin{aligned}
C =& A\cdot B \bmod f(x)\\
=& x^{-2k}\cdot\left(\sum_{i=0}^{m-1} a_i x^i\right)\cdot\left(\sum_{i=0}^{m-1} b_i x^i\right) \bmod f(x)\\
=& x^{-k}\sum_{i=0}^{m-1} c_i x^i.
\end{aligned}$$

We partition A and B into two halves and apply KA to the field multiplication. Two cases are considered:

Case 1, m **is even.** Let $m = 2n$ and

$$A = (A_2x^n + A_1)x^{-k},\ B = (B_2x^n + B_1)x^{-k},$$

where $A_1 = \sum_{i=0}^{n-1} a_i x^i, A_2 = \sum_{i=0}^{n-1} a_{i+n}x^i, B_1 = \sum_{i=0}^{n-1} b_i x^i, B_2 = \sum_{i=0}^{n-1} b_{i+n}x^i$. Then,

$$\begin{aligned}
C =& A\cdot B \bmod f(x)\\
=& (A_2x^n + A_1)x^{-k}\cdot(B_2x^n + B_1)x^{-k} \bmod f(x)\\
=& \left(A_2B_2x^{2n} + (A_2B_1 + A_1B_2)x^n + A_1B_1\right)x^{-2k} \bmod f(x)\\
=& \left[A_2B_2x^{2n} + A_1B_1 + (A_2B_2 + A_1B_1)x^n\right.\\
&\left.+(A_1 + A_2)(B_1 + B_2)x^n\right]x^{-2k} \bmod f(x)\\
=& \left[(A_2x^{2n} + A_2x^n)B_2 + (A_1x^n + A_1)B_1\right.\\
&\left.+(A_1 + A_2)(B_1 + B_2)x^n\right]x^{-2k} \bmod f(x)
\end{aligned} \tag{3}$$

Case 2, m **is odd.** Let $m = 2n+1$ and

$$A = (A_2x^n + A_1)x^{-k},\ B = (B_2x^{n+1} + B_1)x^{-k},$$

where $A_1 = \sum_{i=0}^{n-1} a_i x^i, A_2 = \sum_{i=0}^{n} a_{i+n}x^i, B_1 = \sum_{i=0}^{n} b_i x^i, B_2 = \sum_{i=0}^{n-1} b_{i+n+1}x^i$. Then,

$$\begin{aligned}
C =& A\cdot B \bmod f(x)\\
=& (A_2x^n + A_1)\cdot(B_2x^{n+1} + B_1)x^{-2k} \bmod f(x)\\
=& \left(A_2B_2x^{2n+1} + (A_2B_1 + A_1B_2x)x^n + A_1B_1\right)x^{-2k} \bmod f(x)\\
=& \left[A_2B_2x^{2n+1} + A_1B_1 + (A_2B_2x + A_1B_1)x^n\right.\\
&\left.+(A_1 + A_2)(B_1 + B_2x)x^n\right]x^{-2k} \bmod f(x)\\
=& \left[(A_2x^n + A_2)B_2x^{n+1} + (A_1x^n + A_1)B_1\right.\\
&\left.+(A_1 + A_2)(B_1 + B_2x)x^n\right]x^{-2k} \bmod f(x)
\end{aligned} \tag{4}$$

In order to compute (3) and (4), we use following notations:

$$S_1 = \begin{cases}(A_2x^n + A_2)B_2x^n + (A_1x^n + A_1)B_1, & (m \text{ is even}),\\ (A_2x^n + A_2)B_2x^{n+1} + (A_1x^n + A_1)B_1, & (m \text{ is odd}),\end{cases}$$

$$S_2 = \begin{cases}(A_1 + A_2)(B_1 + B_2), & (m \text{ is even}),\\ (A_1 + A_2)(B_1 + B_2x), & (m \text{ is odd}).\end{cases}$$

Therefore, the field multiplication is given by

$$C = S_1x^{-2k} + S_2x^{n-2k} \bmod f(x).$$

Let $U = \sum_{i=0}^{n-1} u_i x^i$ and $V = \sum_{i=0}^{n-1} v_i x^i$ be the results of $A_1 + A_2$ and $B_1 + B_2$, respectively[1]. According to previous description presented in Section 2, we know that polynomial multiplication $(A_2x^n + A_2)B_2x^i (i = n, n+1)$, $(A_1x^n + A_1)B_1$ and $S_2 = UV$ can be rewritten as matrix-vector multiplication form. In addition, multiplying by x^{-2k} or x^{n-2k} only affects the products degree and does not change their coefficients. Thus, we write $S_2x^{n-2k} = \mathbf{U'}\cdot\mathbf{v}$ where $\mathbf{U'}$ is slightly different from the multiplicative matrix with respect to S_2. For example, if m is even, the form of $\mathbf{U'}$ is as follows:

$$\mathbf{U'} = \begin{array}{c}
n-2k\\ n-2k+1\\ n-2k+2\\ \vdots\\ 2n-2k-2\\ 2n-2k-1\\ 2n-2k\\ 2n-2k+1\\ \vdots\\ 3n-2k-3\\ 3n-2k-2
\end{array}\begin{bmatrix}
u_0 & 0 & \cdots & 0 & 0\\
u_1 & u_0 & \cdots & 0 & 0\\
u_2 & u_1 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
u_{n-2} & u_{n-3} & \cdots & u_0 & 0\\
u_{n-1} & u_{n-2} & \cdots & u_1 & u_0\\
0 & u_{n-1} & \cdots & u_2 & u_1\\
0 & 0 & \cdots & u_3 & u_2\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & u_{n-1} & u_{n-2}\\
0 & 0 & \cdots & 0 & u_{n-1}
\end{bmatrix}.$$

Different from the matrix in (2), there are labels in the left side that indicate the exponent of indeterminate x for each line. Notice that S_1x^{-2k} consists of two sub-polynomial multiplications which correspond two matrix-vector multiplications. However, these multiplications can be implemented using only one matrix-vector multiplication. More explicitly, $S_1x^{-2k} = \mathbf{A'}\cdot\mathbf{b}$, where $\mathbf{A'}$ is constructed from $A_2x^n + A_2, A_1x^n + A_1$ and labeled by $\{-2k, -2k+1, \cdots, 2m-2-2k\}$ for the coefficients degree. The organization of $\mathbf{A'}$ varies according to the parity of m. Two cases are considered:

1. when m is odd, $U = A_1 + A_2 = \sum_{i=0}^{n} u_i x^i$ and $V = B_1 + B_2x = \sum_{i=0}^{n} v_i x^i$

Case 1: m **is even.**

$$S_1 x^{-2k} = \mathbf{A'} \cdot \mathbf{b} = \begin{bmatrix} \mathbf{A}_{L1}, & \mathbf{0}_{n\times n} \\ \mathbf{A}_{L1}+\mathbf{A}_{L2}, & \mathbf{A}_{H1} \\ \mathbf{A}_{L2}, & \mathbf{A}_{H1}+\mathbf{A}_{H2} \\ \mathbf{0}_{n\times n}, & \mathbf{A}_{H2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}. \quad (5)$$

where $\mathbf{b}_1, \mathbf{b}_2$ represent the coefficient vectors of B_1, B_2, $\mathbf{0}_{n\times n}$ is a $n \times n$ zero matrix, $\mathbf{A}_{L1}$ and $\mathbf{A}_{H1}$ are $n \times n$ lower-triangular Toeplitz matrices, $\mathbf{A}_{L2}$ and $\mathbf{A}_{H2}$ are $n \times n$ upper-triangular Toeplitz matrices[2].

$$\mathbf{A}_{L1} = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix},$$

$$\mathbf{A}_{L2} = \begin{bmatrix} 0 & a_{n-1} & \cdots & a_2 & a_1 \\ 0 & 0 & \cdots & a_3 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_{H1} = \begin{bmatrix} a_n & 0 & \cdots & 0 \\ a_{n+1} & a_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1} & a_{m-2} & \cdots & a_n \end{bmatrix},$$

$$\mathbf{A}_{H2} = \begin{bmatrix} 0 & a_{m-1} & \cdots & a_{n+2} & a_{n+1} \\ 0 & 0 & \cdots & a_{n+3} & a_{n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{m-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Case 2: m **is odd.**

$$S_1 x^{-2k} = \mathbf{A'} \cdot \mathbf{b} = \begin{bmatrix} \mathbf{A}_{L1}, & \mathbf{0}_{(n+1)\times n} \\ \mathbf{A}_{L1}+\mathbf{A}_{L2}, & \mathbf{A}_{H1} \\ \mathbf{A}_{L2}, & \mathbf{A}_{H1}+\mathbf{A}_{H2} \\ \mathbf{0}_{(n+1)\times(n+1)}, & \mathbf{A}_{H2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}. \quad (6)$$

where $\mathbf{0}_{(n+1)\times n}$ and $\mathbf{0}_{(n+1)\times(n+1)}$ represent a $(n+1) \times n$ zero matrix and a $(n+1)\times(n+1)$ zero matrix, respectively. $\mathbf{A}_{L1}$ is a $n \times (n+1)$ lower-triangular Toeplitz matrix, $\mathbf{A}_{H1}$ is a $n \times n$ lower-triangular Toeplitz matrix, $\mathbf{A}_{L2}$ is a $n \times (n+1)$ upper-triangular Toeplitz matrix and $\mathbf{A}_{H2}$ is a $n \times n$ upper-triangular Toeplitz matrix.

$$\mathbf{A}_{L1} = \begin{bmatrix} a_0 & 0 & \cdots & 0 & 0 \\ a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 & 0 \end{bmatrix},$$

$$\mathbf{A}_{L2} = \begin{bmatrix} 0 & a_{n-1} & \cdots & a_1 & a_0 \\ 0 & 0 & \cdots & a_2 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1} \end{bmatrix},$$

$$\mathbf{A}_{H1} = \begin{bmatrix} a_n & 0 & \cdots & 0 \\ a_{n+1} & a_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-2} & a_{m-3} & \cdots & a_n \end{bmatrix},$$

$$\mathbf{A}_{H2} = \begin{bmatrix} a_{m-1} & \cdots & a_{n+2} & a_{n+1} \\ 0 & \cdots & a_{n+3} & a_{n+2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{m-1} \end{bmatrix}.$$

Similarly, the organization of $\mathbf{U'}$ is easier than that of $\mathbf{A'}$, i.e., $\mathbf{U'} = [\mathbf{U}_1^T, \mathbf{U}_2^T]^T$, where

$$\mathbf{U}_1 = \begin{bmatrix} u_0 & 0 & \cdots & 0 \\ u_1 & u_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{t-1} & u_{t-2} & \cdots & u_0 \end{bmatrix},$$

and

$$\mathbf{U}_2 = \begin{bmatrix} 0 & u_{t-1} & \cdots & u_2 & u_1 \\ 0 & 0 & \cdots & u_3 & u_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & u_{t-1} \end{bmatrix}.$$

Here, if m is even, $t = n = \frac{m}{2}$, else if m is odd, $t = n+1 = \frac{m+1}{2}$.

Example 3.1. Consider the field multiplication using SPB representation over $GF(2^5)$ with the underlying irreducible trinomial $x^5 + x^2 + 1$. The parameter $k = 2$ and SPB is defined as $\{x^{-2}, x^{-1}, 1, x, x^2\}$. Assume that $A = \sum_{i=0}^{4} a_i x^{i-2}$ and $B = \sum_{i=0}^{4} b_i x^{i-2}$ are two elements in $GF(2^5)$, we partition A, B as $A = A_2 + A_1 x^{-2}, B = B_2 + B_1 x^{-2}$, where

$$A_1 = a_1 x + a_0,\ A_2 = a_4 x^2 + a_3 x + a_2,$$
$$B_1 = b_2 x^2 + b_1 x + b_0,\ B_2 = b_4 x + b_3.$$

According to equation (4), then

$$\begin{aligned} A \cdot B &= [(A_2 x^2 + A_2) B_2 x^3 + (A_1 x^2 + A_1) B_1 \\ &\quad + (A_2 + A_1)(B_2 x + B_1) x^2] x^{-4} \\ &= S_1 x^{-4} + S_2 x^{-2}. \end{aligned}$$

Therefore, the matrices $\mathbf{A'}$ and $\mathbf{U'}$ are given by

$$\mathbf{A'} = \begin{array}{c} -4 \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \left[\begin{array}{ccc:cc} a_0 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 & 0 \\ a_0 & a_1 & a_0 & 0 & 0 \\ a_1 & a_0 & a_1 & a_2 & 0 \\ 0 & a_1 & a_0 & a_3 & a_2 \\ 0 & 0 & a_1 & a_2{+}a_4 & a_3 \\ 0 & 0 & 0 & a_3 & a_2{+}a_4 \\ 0 & 0 & 0 & a_4 & a_3 \\ 0 & 0 & 0 & 0 & a_4 \end{array}\right], \quad (7)$$

2. Please note that the matrix in the right side actually contains $4n = 2m$ rows, but the last row is $\mathbf{0}$, which does not affect the result.

and

$$\mathbf{U}' = \begin{matrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{matrix} \begin{bmatrix} u_0 & 0 & 0 \\ u_1 & u_0 & 0 \\ u_2 & u_1 & u_0 \\ 0 & u_2 & u_1 \\ 0 & 0 & u_2 \end{bmatrix}, \tag{8}$$

where $u_2 = a_4, u_1 = a_3 + a_1, u_0 = a_0 + a_2$. We have $S_1 x^{-4} = \mathbf{A}' \cdot \mathbf{b}, S_2 x^{-2} = \mathbf{U}' \cdot \mathbf{v}$.

3.2 Reduction process

It is easy to check that the products $S_1 x^{-2k}, S_2 x^{n-2k}$ contain the terms of degrees out of the range $[-k, m-k-1]$. To compute the field multiplication, we have to perform the reduction operation for these two expressions. According to Mastrovito scheme, the reduction can be regarded as the construction of product matrices from $\mathbf{A}'$ and $\mathbf{U}'$ using the equation $x^m = x^k + 1$. Denoted by $\mathbf{M}_A$ and $\mathbf{M}_U$ the product matrices of $S_1 x^{-2k} \bmod f(x)$ and $S_2 x^{n-2k} \bmod f(x)$, respectively. In the following, we investigate the construction details for these two matrices.

3.2.1 Construction of $\boldsymbol{M}_A$

Note that the matrix $\mathbf{A}'$ contains $2m-1$ rows, each of which corresponds to the polynomial degree from $-2k$ to $2m-2-2k$, we can reduce the rows labeled by $\{-2k, -2k+1, \cdots, -k-1\}$ (the first k rows), and by $\{m-k, m-k+1, \cdots, 2m-2-k\}$ (the last $m-k-1$ rows) of the matrix $\mathbf{A}'$ to obtain $\mathbf{M}_A$. According to the form of $f(x) = x^m + x^k + 1$ and the SPB definition, we mainly utilize following equations:

$$\begin{cases} x^i = x^{m+i} + x^{i+k}, \text{ for } i = -2k, \cdots, -k-1; \\ x^i = x^{i-m+k} + x^{i-m}, \text{ for } i = m-k, m-k+1, \\ \qquad\qquad \cdots, 2m-2k-2. \end{cases} \tag{9}$$

Corresponding to (9), the computation of $\mathbf{M}_A$ consists of adding the rows of $\mathbf{A}'$ labeled by i to those rows labeled by $m+i, i+k$ (or $i-m+k, i-m$). More explicitly, we eliminate the row $-2k, -2k+1, \cdots, -k-1$ by adding them to the row $-k, \cdots, -1$ and $m-2k, \cdots, m-k-1$, and eliminate the row $m-k, \cdots, 2m-2k-2$ by adding them to the row $0, \cdots, m-k-2$ and $-k, \cdots, m-2k-2$. These operations can be implemented by matrix additions. We will utilize following three $m \times m$ matrices:

$$\begin{aligned} \mathbf{A}_U &= [\mathbf{A}'(1,:)^T, \cdots, \mathbf{A}'(k,:)^T, \underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{m-k}]^T, \\ \mathbf{A}_M &= [\mathbf{A}'(k+1,:)^T, \cdots, \mathbf{A}'(k+m,:)^T]^T, \\ \mathbf{A}_L &= [\underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{k}, \mathbf{A}'(k+m+1,:)^T, \cdots, \mathbf{A}'(2m-1,:)^T, \mathbf{0}]^{\mathbf{T}}. \end{aligned}$$

Obviously, the nonzero parts of $\mathbf{A}_U$, $\mathbf{A}_L$ are the rows of $\mathbf{A}'$ whose labels are out of the range $[-k, m-k-1]$. Then, the product matrix M_A is obtained as follows:

$$\begin{aligned} \mathbf{M}_A = &\mathbf{A}_M + \mathbf{A}_U + \mathbf{A}_U[\downarrow (m-k)] \\ &+ \mathbf{A}_L + \mathbf{A}_L[\uparrow k]. \end{aligned} \tag{10}$$

Since the structure of $\mathbf{M}_A$ highly influences the efficiency of our multiplier, we make two important observations about the construction of $\mathbf{M}_A$.

Observation 3.1. If m is even, $\mathbf{A}_M + \mathbf{A}_U[\downarrow (m-k)] + \mathbf{A}_L[\uparrow k]$ is

$$\begin{bmatrix} \mathbf{A}_{L1} + \mathbf{A}_{L2}, & \mathbf{A}_{H1} + \mathbf{A}_{H2} \\ \mathbf{A}_{L1} + \mathbf{A}_{L2}, & \mathbf{A}_{H1} + \mathbf{A}_{H2} \end{bmatrix} [\circlearrowleft k]. \tag{11}$$

If m is odd, this expression can be rewritten as

$$\begin{bmatrix} \begin{pmatrix} \mathbf{A}'_{L1} + \mathbf{A}'_{L2} \\ \mathbf{A}_{L1} + \mathbf{A}_{L2} \end{pmatrix} [\circlearrowleft 1], & \begin{matrix} \mathbf{A}_{H1} + \mathbf{A}_{H2} \\ \mathbf{A}'_{H1} + \mathbf{A}'_{H2} \end{matrix} \end{bmatrix} [\circlearrowleft k]. \tag{12}$$

where $\mathbf{A}'_{L1} = \mathbf{A}_{L1}[\Downarrow 1]$, $\mathbf{A}'_{H1} = \mathbf{A}_{H1}[\Downarrow 1]$ and $\mathbf{A}'_{L2} = \mathbf{A}_{L2}[\Uparrow 1]$, $\mathbf{A}'_{H2} = \mathbf{A}_{H2}[\Uparrow 1]$.

Proof According to the definitions of $\mathbf{A}_U, \mathbf{A}_M$ and $\mathbf{A}_L$, we have

$$\begin{aligned} &\mathbf{A}_U[\downarrow (m-k)] = [\underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{m-k}, \mathbf{A}'(1,:)^T, \cdots, \mathbf{A}'(k,:)^T]^T, \\ &\mathbf{A}_M = [\mathbf{A}'(k+1,:)^T, \cdots, \mathbf{A}'(k+m,:)^T]^T, \\ &\mathbf{A}_L[\uparrow k] = [\mathbf{A}'(k+m+1,:)^T, \cdots, \mathbf{A}'(2m-1,:)^T, \underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{k+1}]^T. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{A}_U[\downarrow (m-k)] + \mathbf{A}_L[\uparrow k] = &[\mathbf{A}'(k+m+1,:)^T, \cdots, \mathbf{A}'(2m-1,:)^T, \\ &\mathbf{0}, \mathbf{A}'(1,:)^T, \cdots, \mathbf{A}'(k,:)^T]^T. \end{aligned}$$

Thus, $\mathbf{A}_M + \mathbf{A}_U[\downarrow (m-k)] + \mathbf{A}_L[\uparrow k]$ is given by

$$\begin{aligned} &[\mathbf{A}'(k+m+1,:)^T + \mathbf{A}'(k+1,:)^T, \cdots, \mathbf{A}'(2m-1,:)^T + \mathbf{A}'(m-1,:)^T, \\ &\mathbf{A}'(m,:)^T, \mathbf{A}'(1,:)^T + \mathbf{A}'(m+1,:)^T \cdots, \mathbf{A}'(k,:)^T + \mathbf{A}'(k+m,:)^T]. \end{aligned}$$

In addition, above matrix can also be rewritten as $\mathbf{E}[\circlearrowleft k]$, where

$$\begin{aligned} &\mathbf{E} = [\mathbf{A}'(1,:)^T + \mathbf{A}'(m+1,:)^T \cdots, \mathbf{A}'(k,:)^T + \mathbf{A}'(k+m,:)^T, \\ &\mathbf{A}'(k+1,:)^T + \mathbf{A}'(k+m+1,:)^T, \cdots, \mathbf{A}'(m-1,:)^T + \mathbf{A}'(2m-1,:)^T, \\ &\quad \mathbf{A}'(m,:)^T]. \end{aligned}$$

It is clear that $\mathbf{E}$ is equal to the upper m rows of $\mathbf{A}'$ plus its lower $m-1$ rows. According to the form of $\mathbf{A}'$ presented in (5) and (6), we can obtain that the explicit formula of $\mathbf{E}$. If m is even,

$$\mathbf{E} = \begin{bmatrix} \mathbf{A}_{L1} + \mathbf{A}_{L2}, & \mathbf{A}_{H1} + \mathbf{A}_{H2} \\ \mathbf{A}_{L1} + \mathbf{A}_{L2}, & \mathbf{A}_{H1} + \mathbf{A}_{H2} \end{bmatrix}.$$

If m is odd, the organization of $\mathbf{E}$ is a little complicated:

$$\mathbf{E} = \begin{bmatrix} \begin{matrix} (\mathbf{A}'_{L1} + \mathbf{A}'_{L2})(2 \sim n+1,:), \\ \mathbf{A}_{L1} + \mathbf{A}_{L2} \\ (\mathbf{A}'_{L1} + \mathbf{A}'_{L2})(1,:) \end{matrix}, & \begin{matrix} \mathbf{A}_{H1} + \mathbf{A}_{H2} \\ \mathbf{A}'_{H1} + \mathbf{A}'_{H2} \end{matrix} \end{bmatrix},$$

where $(\mathbf{A}'_{L1} + \mathbf{A}'_{L2})(2 \sim n+1,:)$ represents the lower n row vectors of $\mathbf{A}'_{L1} + \mathbf{A}'_{L2}$. The above expression can also be rewritten as:

$$\mathbf{E} = \begin{bmatrix} \begin{pmatrix} \mathbf{A}'_{L1} + \mathbf{A}'_{L2} \\ \mathbf{A}_{L1} + \mathbf{A}_{L2} \end{pmatrix} [\circlearrowleft 1], & \begin{matrix} \mathbf{A}_{H1} + \mathbf{A}_{H2} \\ \mathbf{A}'_{H1} + \mathbf{A}'_{H2} \end{matrix} \end{bmatrix}.$$

We then obtain the formulae (11) and (12) immediately.□

Observation 3.2. Matrix-vector multiplication $\mathbf{M}[\circlearrowleft i] \cdot \mathbf{b}$ costs the same logic gates as $\mathbf{M} \cdot \mathbf{b}$, where $\mathbf{M}$ is a $m \times m$ matrix and $\mathbf{b}$ is a $1 \times m$ vector.

Proof Note that $\mathbf{M}[\circlearrowleft\ i]\cdot\mathbf{b} = (\mathbf{M}\cdot\mathbf{b})[\circlearrowleft\ i]$. So the computation of $(\mathbf{M}\cdot\mathbf{b})[\circlearrowleft\ i]$ can be obtained by cyclic shift of $\mathbf{M}\cdot\mathbf{b}$. Then, the conclusion is direct. □

Furthermore, it is also clear that

$$\begin{aligned}\mathbf{A}_U + \mathbf{A}_L =&[\mathbf{A'}(1,:)^T,\cdots,\mathbf{A'}(k,:)^T,\mathbf{A'}(k+m+1,:)^T,\\ &\cdots,\mathbf{A'}(2m-1,:)^T,\mathbf{0}]\end{aligned} \tag{13}$$

Denoted by $\mathbf{M}_{A,1}$ the matrix presented either in (11) or (12), and by $\mathbf{M}_{A,2}$ the matrix presented in (13). Combined with Observation 3.1, (10) can be simplified as:

$$\mathbf{M}_A = \mathbf{M}_{A,1} + \mathbf{M}_{A,2}.$$

Therefore, we can utilize the same strategy presented in [21], where $S_1x^{-2k} \bmod f(x)$ can be implemented as:

$$\begin{aligned}S_1x^{-2k} \bmod f(x) &= \textstyle\sum_{i=0}^{m-1} s_i x^i\\ &= \mathbf{M}_A\cdot\mathbf{b}\\ &= \mathbf{M}_{A,1}\cdot\mathbf{b} + \mathbf{M}_{A,2}\cdot\mathbf{b}\\ &= [\mathbf{M}_{A,1},\mathbf{M}_{A,2}]\cdot[\mathbf{b},\mathbf{b}]^T.\end{aligned} \tag{14}$$

We implement the above expression by following steps:

- Perform row-vector products

$$\begin{aligned}&[\mathbf{M}_{A,1}(i,1)\cdot b_0,\cdots,\mathbf{M}_{A,1}(i,m)\cdot b_{m-1},\\ &\quad\mathbf{M}_{A,2}(i,1)\cdot b_0,\cdots,\mathbf{M}_{A,2}(i,m)\cdot b_{m-1}],\end{aligned} \tag{15}$$

  $i = 1,2,\cdots,m$ in parallel.
- Sum up all the $2m$ entries of each row using binary XOR tree, i.e.,

$$s_{i-1} = \sum_{j=1}^{m}\mathbf{M}_{A,1}(i,j)\cdot b_{j-1} + \sum_{j=1}^{m}\mathbf{M}_{A,2}(i,j)\cdot b_{j-1}, \tag{16}$$

  $i = 1,2,\cdots,m$.

Special case $m = 2k$. When m is even and $m = 2k$ ($k = n$), we can obtain the simplest form of $\mathbf{M}_{A,1}$ and $\mathbf{M}_{A,2}$, where

$$\mathbf{M}_{A,1} = \begin{bmatrix}\mathbf{A}_{L1}+\mathbf{A}_{L2}, & \mathbf{A}_{H1}+\mathbf{A}_{H2}\\ \mathbf{A}_{L1}+\mathbf{A}_{L2}, & \mathbf{A}_{H1}+\mathbf{A}_{H2}\end{bmatrix},$$

and

$$\mathbf{M}_{A,2} = \begin{bmatrix}\mathbf{A}_{L1}, & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n}, & \mathbf{A}_{H2}\end{bmatrix}.$$

By swapping and combining some overlapped entries, expression (14) now can be rewritten as:

$$\begin{aligned}\mathbf{M}_{A,1}\cdot\mathbf{b} + \mathbf{M}_{A,2}\cdot\mathbf{b} &= \begin{bmatrix}\mathbf{A}_{L2}, & \mathbf{A}_{H1}\\ \mathbf{A}_{L2}, & \mathbf{A}_{H1}\end{bmatrix}\cdot\mathbf{b}\\ &+\begin{bmatrix}\mathbf{0}_{n\times n}, & \mathbf{A}_{H2}\\ \mathbf{A}_{L1}, & \mathbf{0}_{n\times n}\end{bmatrix}\cdot\mathbf{b}.\end{aligned} \tag{17}$$

In this case, we just compute two submatrix-vector multiplications and add them up to obtain $S_1x^{-2k} \bmod f(x)$.

3.2.2 Construction of $\boldsymbol{M}_U$

Analogous with the construction procedure of $\mathbf{M}_A$, the reduction of $\mathbf{U'}$ can also be obtained by shifting of $\mathbf{U'}$ and adding specific rows of $\mathbf{U'}$ to itself. The form of $\mathbf{M}_U$ varies according to the values of m and k. There are six cases need to be considered:

1) m even, $m > 2k+2$;
2) m even, $m = 2k$ or $m = 2k+2$;
3) m even, $m < 2k$;
4) m odd, $m > 2k+1$;
5) m odd, $m = 2k+1$;
6) m odd, $m < 2k+1$.

Case 1 and 4: Note that the following equation holds with respect to these two cases:

$$\begin{aligned}&n-2k > -k+1, && m \text{ is even and } m > 2k+2,\\ &n-2k > -k, && m \text{ is odd and } m > 2k+1.\end{aligned}$$

We only need to reduce the rows which are labeled with $x^{m-k},\cdots,x^{m+n-2k-2}$ ($x^{m+n-2k-1}$ for odd m) of $\mathbf{U}'$. The equation $x^i = x^{i-m+k} + x^{i-m}, i = m-k,\cdots,m+n-2k-2$ is used for reduction. Then, the product matrix $\mathbf{M}_U$ is partitioned as $\mathbf{M}_U = \mathbf{M}_{U,1} + \mathbf{M}_{U,2}$, where

$$\begin{aligned}&\mathbf{M}_{U,1} =\\ &\quad[\mathbf{U}_2(k+1,:)^T,\cdots,\mathbf{U}_2(n-1,:)^T,\mathbf{0},\mathbf{U}_1(1,:)^T,\cdots,\\ &\qquad\mathbf{U}_1(n,:)^T,\mathbf{U}_2(1,:)^T,\cdots,\mathbf{U}_2(k,:)^T]^T,(m \text{ even}),\\ &\quad[\mathbf{U}_2(k+1,:)^T,\cdots,\mathbf{U}_2(n,:)^T,\mathbf{U}_1(1,:)^T,\cdots,\\ &\qquad\mathbf{U}_1(n+1,:)^T,\mathbf{U}_2(1,:)^T,\cdots,\mathbf{U}_2(k,:)^T]^T,(m \text{ odd}),\end{aligned}$$

and

$$\begin{aligned}&\mathbf{M}_{U,2} =\\ &[\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{k},\mathbf{U}_2(k+1,:)^T,\cdots,\mathbf{U}_2(n-1,:)^T,\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{n+1}]^T,(m \text{ even}),\\ &[\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{k},\mathbf{U}_2(k+1,:)^T,\cdots,\mathbf{U}_2(n,:)^T,\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{n+1}]^T,(m \text{ odd}).\end{aligned}$$

Case 2 and Case 5: In these two case, one can check that $S_2x^{n-2k} \bmod f(x) = S_2x^{n-2k}$, thus requiring no reduction for the matrix $\mathbf{U'}$ and the product matrix $\mathbf{M}_U$ is equal to $\mathbf{U'}$ itself.

Case 3 and Case 6: Since $m < 2k$ (or $m < 2k+1$), we have $n < k$ for both even and odd m. The following formula holds:

$$\begin{aligned}&m+n-2k-2 < m-k-2, && m \text{ is even and } m < 2k,\\ &m+n-2k-1 < m-k-1, && m \text{ is odd and } m < 2k+1.\end{aligned}$$

Therefore, we only need to reduce the rows which are labeled with $x^{n-2k},\cdots,x^{-k-1}$ of $\mathbf{U}'$. The equation $x^i = x^{m+i} + x^{k+i}, i = n-2k,\cdots,-k-1$ is used for reduction. Analogous to Case 1 and 4, the product matrix $\mathbf{M}_U =$

$\mathbf{M}_{U,1} + \mathbf{M}_{U,2}$, where

$$\begin{aligned}\mathbf{M}_{U,1} = \\ &[\mathbf{U}_1(k-n+1,:)^T, \cdots, \mathbf{U}_1(n,:)^T, \mathbf{U}_2(1,:)^T, \cdots, \\ &\mathbf{U}_2(n-1,:)^T, \mathbf{0}, \mathbf{U}_1(1,:)^T, \cdots, \mathbf{U}_1(k-n,:)^T]^T, (m \text{ even}), \\ &[\mathbf{U}_1(k-n+1,:)^T, \cdots, \mathbf{U}_1(n+1,:)^T, \mathbf{U}_2(1,:)^T, \cdots, \\ &\mathbf{U}_2(n+1,:)^T, \mathbf{U}_1(1,:)^T, \cdots, \mathbf{U}_1(k-n,:)^T]^T, (m \text{ odd}),\end{aligned}$$

and

$$\mathbf{M}_{U,2} = [\underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{n}, \mathbf{U}_1(1,:)^T, \cdots, \mathbf{U}_1(k-n,:)^T, \underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{m-k}]^T.$$

3.3 Complexity Analysis

3.3.1 Complexity analysis for $S_1 x^{-2k} \bmod f(x)$

Applying Observation 3.1 and 3.2, we have the following theorem.

Theorem 1 *The m row-vector products of (15) only requires $\frac{m^2}{2}$ AND gates for even m and $\frac{m^2-1}{2}$ AND gates for odd m.*

Proof According to (10), (11), (12), (13) and the organization of $\mathbf{A'}$, it is clear that all the non-zero entries of $\mathbf{M}_{A,1}$ and $\mathbf{M}_{A,2}$ are included in four sub-matrices $\mathbf{A}_{L1}, \mathbf{A}_{L2}$, $\mathbf{A}_{H1}, \mathbf{A}_{H2}$. Applying observation 3.1 and 3.2, we know the partial row-vector products are sufficient to obtain all the row-vector products for $\mathbf{M}_A \cdot \mathbf{b}$. More explicitly, if m is even, the row-vector products of $[\mathbf{A}_{L1}+\mathbf{A}_{L2}, \mathbf{A}_{H1}+\mathbf{A}_{H2}] \cdot \mathbf{b}$ is sufficient. It requires only $n^2 + n^2 = \frac{m^2}{2}$ AND gates to compute these products. If m is odd, the row-vector products of $[\mathbf{A}_{L1} + \mathbf{A}_{L2}] \cdot \mathbf{b}_1$ and $[\mathbf{A'}_{H1} + \mathbf{A'}_{H2}] \cdot \mathbf{b}_2$ are sufficient, which requires $(n+1) \cdot n + (n+1) \cdot n = \frac{m^2-1}{2}$ AND gates. □

Next, we investigate the number of XOR gates needed in (16). Notice that $\mathbf{M}_{A,1}$ and $\mathbf{M}_{A,2}$ share some common entries. If these common entries multiply the same vector entries during the computation of row-vector products presented in (15), the results stay the same. When adding up all the products of each row, some XOR gates can be saved by sharing the common items. This technique is so called *sub-expression sharing* [9]. However, when we use binary XOR tree to sum up the products of each row, the common items between two binary trees should be utilized carefully. For example, two expressions $c_1 = z_0 + z_1 + z_2 + z_3$ and $c_2 = z_0 + z_2 + z_3 + z_4$ share three common items, but only one XOR will be saved when using binary tree. The computation details are as follows:

$$c_1 = \underbrace{[\underbrace{[z_0 + z_2]}_{\text{I}} + \underbrace{[z_1 + z_3]}_{\text{I}}]}_{\text{II}}, \quad c_2 = \underbrace{[\underbrace{[z_0 + z_2]}_{\text{I}} + \underbrace{[z_3 + z_4]}_{\text{I}}]}_{\text{II}}.$$

The sub-expressions labeled by the same Roman number are calculated simultaneously. It is clear that the common item z_3 cannot save one more XOR gate unless the binary tree is not used.

In [8], [20], the authors have shown that if two binary XOR trees share k common items, only $k - W(k)$ XOR gates can be saved, where $W(k)$ is the Hamming weight of the binary representation of k. Obviously, one can check that c_1, c_2 of above expression can save only $3 - W(3) = 1$ XOR gate. Thus, we use the similar trick to save XOR gates in the accumulating processes in (16). Firstly, we consider n intermediate values $P_0, P_1, \cdots, P_{n-1}$ ($n+1$ values for odd m), where

$$[P_0, \cdots, P_{n-1}]^T = [\mathbf{A}_{L1} + \mathbf{A}_{L2}, \mathbf{A}_{H1} + \mathbf{A}_{H2}] \cdot \mathbf{b},$$

if m is even, or

$$[P_0, \cdots, P_n]^T = \begin{bmatrix} \mathbf{A}_{L1}+\mathbf{A}_{L2} \\ (\mathbf{A'}_{L1}+\mathbf{A'}_{L2})(1,:) \end{bmatrix}, \mathbf{A'}_{H1}+\mathbf{A'}_{H2} \cdot \mathbf{b},$$

if m is odd. Please note that the matrices appeared in above formulae contain all the entries of $\mathbf{M}_{A,1}$ and $\mathbf{M}_{A,2}$. Then, we compute the accumulations of the bit-wise products related to these n (or $n+1$) intermediate values using binary XOR tree in parallel. At the same time, the accumulations presented in (16) are also calculated by sharing common items with $P_0, P_1, \cdots, P_{n-1}$ ($P_0, P_1, \cdots, P_n$ for odd m).

For example, note that $s_1 = \mathbf{M}_{A,1}(2,:) \cdot \mathbf{b} + \mathbf{M}_{A,2}(2,:) \cdot \mathbf{b}$. If m is even and $0 < k < n$, $\mathbf{M}_{A,1}(2,:) \cdot \mathbf{b}$ has m items overlapped with P_{k+1} and $\mathbf{M}_{A,2}(2,:) \cdot \mathbf{b}$ has 2 terms overlapped with P_1. According to previous assertion, the computation of s_1 totally can reuse $m - W(m)$ and $2 - W(2)$ XOR gates from P_k and P_1, respectively. Also notice that s_1 only has $m+2$ nonzero entries, so we only need $m + 1 - (m - W(m)) - (2 - W(2)) = W(m) + W(2) - 1 = W(m)$ XOR gates to implement s_1. Table 1 indicates the explicit number of XOR gates which can be saved using the binary tree based *sub-expression sharing* trick, if m is even and $0 < k < n$. For simplicity purpose, we divide each coefficient s_i into two parts, i.e., $\mathbf{M}_{A,1}(i+1,:) \cdot \mathbf{b}$, $\mathbf{M}_{A,2}(i+1,:) \cdot \mathbf{b}$, and indicate their overlapped values and the number of saved XOR gates, independently. One can find more details in the appendix.

Moreover, in this case, the number of XOR gates needed by (16) without optimization is $\frac{3m^2-m}{2} - km + k^2 + k$. Also notice that the computation of the intermediate values $P_0, P_1, \cdots P_{n-1}$ requires $(m-1)n = \frac{m^2-m}{2}$ XOR gates. Adding up these two formulae and subtracting the number of saved XOR gates presented in Table 1, we can obtain the explicit number of XOR gates required by $S_1 x^{-2k} \bmod f(x)$:

$$\text{\#XOR: } \frac{m^2 - 3m}{2} + \sum_{i=1}^{k} W(i) + \sum_{i=1}^{m-k-1} W(i) + mW(m).$$

The circuit delay of $S_1 x^{-2k} \bmod f(x)$ equals the depth of the biggest XOR tree. According to (11), (13) and (14), it is easy to check that

$$\text{Delay: } T_A + \lceil \log_2(2m - k - 1) \rceil T_X.$$

Note that the number of #AND gates for $S_1 x^{-2k} \bmod f(x)$ is already given by theorem 1. The space and time complexity of other cases are summarized in Table 2.

TABLE 1
The overlapped values and saved #XOR, if m even, $0 < k < n$

First part	Overlapped	Saved #XOR	Second part	Overlapped	Saved #XOR
$\mathbf{M}_{A,1}(1,:)\cdot\mathbf{b}$	P_k	$m-W(m)$	$\mathbf{M}_{A,2}(1,:)\cdot\mathbf{b}$	P_0	$1-W(1)$
$\mathbf{M}_{A,1}(2,:)\cdot\mathbf{b}$	P_{k+1}	$m-W(m)$	$\mathbf{M}_{A,2}(2,:)\cdot\mathbf{b}$	P_1	$2-W(2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(n-k,:)\cdot\mathbf{b}$	P_{n-1}	$m-W(m)$	$\mathbf{M}_{A,2}(k,:)\cdot\mathbf{b}$	P_{k-1}	$k-W(k)$
$\mathbf{M}_{A,1}(n-k+1,:)\cdot\mathbf{b}$	P_0	$m-W(m)$	$\mathbf{M}_{A,2}(k+1,:)\cdot\mathbf{b}$	P_{k+1}	$m-k-1-W(m-k-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m-k,:)\cdot\mathbf{b}$	P_{n-1}	$m-W(m)$	$\mathbf{M}_{A,2}(n-1,:)\cdot\mathbf{b}$	P_{n-1}	$n+1-W(n+1)$
$\mathbf{M}_{A,1}(m-k+1,:)\cdot\mathbf{b}$	P_0	$m-W(m)$	$\mathbf{M}_{A,2}(n,:)\cdot\mathbf{b}$	P_0	$n-W(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m,:)\cdot\mathbf{b}$	P_{k-1}	$m-W(m)$	$\mathbf{M}_{A,2}(m-1,:)\cdot\mathbf{b}$	P_{n-1}	$1-W(1)$

TABLE 2
The space and time complexity of $S_1x^{-2k} \bmod f(x)$

case	#AND	#XOR	Delay
m even, $m<2k$	$\frac{m^2}{2}$	$\frac{m^2-3m}{2}+\sum_{i=1}^{k}W(i)+\sum_{i=1}^{m-k-1}W(i)+mW(m)$	$T_A+(\lceil\log_2(m+k)\rceil)T_X$
m even, $m=2k$	$\frac{m^2}{2}$	$\frac{m^2-m}{2}$	$T_A+(1+\lceil\log_2(\frac{m}{2})\rceil)T_X$
m odd, $m\geq 2k+1$ $n=\frac{m-1}{2}$	$\frac{m^2-1}{2}$	$\frac{m^2-2m-1}{2}+(n+1)W(m)+nW(n+1)+(m-k-1)W(n)$ $\sum_{i=1}^{k}W(i)+\sum_{i=1}^{n}W(i)+\sum_{i=1}^{n-k}W(i)$	$T_A+\lceil\log_2(2m-k-1)\rceil)T_X$
m odd, $m<2k+1$ $n=\frac{m-1}{2}$	$\frac{m^2-1}{2}$	$\frac{m^2-2m-1}{2}+(n+1)W(m)+nW(n+1)+kW(n)$ $\sum_{i=1}^{m-k-1}W(i)+\sum_{i=1}^{n}W(i)+\sum_{i=1}^{k-n}W(i)$	$T_A+(\lceil\log_2(m+k)\rceil)T_X$

Special case $m=2k$. According to (17), the first submatrix has its upper $\frac{m}{2}$ rows equal to its lower $\frac{m}{2}$ rows. Thus, we only need to compute $[\mathbf{A}_{L2},\mathbf{A}_{H1}]\cdot\mathbf{b}$. In addition, both $\mathbf{A}_{L2}$ and $\mathbf{A}_{H1}$ are triangular matrices, one can easily check that each row of $[\mathbf{A}_{L2},\mathbf{A}_{H1}]$ consists of at most $\frac{m}{2}$ nonzero entries. Hence, the corresponding submatrix-vector multiplication cost $\frac{m^2}{4}$ AND gates and $\frac{m^2-2m}{4}$ XOR gates with delay of $T_A+\lceil\log_2(\frac{m}{2})\rceil T_X$. The computation of the second submatrix-vector multiplication is similar. It totally requires $\frac{m^2}{4}$ AND gates and $\frac{m^2}{4}-m+1$ XOR gates with delay of $T_A+\lceil\log_2(\frac{m}{2})\rceil T_X$. Finally, $m-1$ XOR gates are needed to add these two results up, which lead to one more T_X delay.

Example 3.2. Consider the reduction of $\mathbf{A}'$ presented in Example 3.1. The construction of $\mathbf{M}_A$ is based on the following equation:

$$\begin{cases} x^i = x^{5+i}+x^{i+2}, \text{ for } i=-4,-3;\\ x^i = x^{i-3}+x^{i-5}, \text{ for } i=3,4. \end{cases}$$

Then $\mathbf{M}_A=\mathbf{M}_{A,1}+\mathbf{M}_{A,2}$ where

$$\mathbf{M}_{A,1}=\begin{array}{c} -2\\ -1\\ 0\\ 1\\ 2 \end{array}\left[\begin{array}{ccc:cc} a_0 & a_1 & a_0 & a_4 & a_3\\ a_1 & a_0 & a_1 & a_2 & a_4\\ 0 & a_1 & a_0 & a_3 & a_2\\ a_0 & 0 & a_1 & a_2+a_4 & a_3\\ a_1 & a_0 & 0 & a_3 & a_2+a_4 \end{array}\right],$$

and

$$\mathbf{M}_{A,2}=\begin{array}{c} -2\\ -1\\ 0\\ 1\\ 2 \end{array}\left[\begin{array}{ccc:cc} a_0 & 0 & 0 & 0 & 0\\ a_1 & a_0 & 0 & 0 & 0\\ 0 & 0 & 0 & a_4 & a_3\\ 0 & 0 & 0 & 0 & a_4\\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $S_1x^{-4} \bmod x^5+x^2+1 = [\mathbf{M}_{A,1},\mathbf{M}_{A,2}]\cdot[\mathbf{b},\mathbf{b}]^T$. Firstly, it is easy to see that $\frac{5^2-1}{2}=12$ AND gates are needed for the bitwise multiplication. We then evaluate the number of required XOR gates. According to the description of Subsection 3.3, we use three intermediate values:

$$\begin{aligned} P_0 =& [a_1b_1+a_0b_2]+[a_4b_3+a_3b_4]+a_0b_0,\\ P_1 =& [a_1b_2+a_2b_3]+a_4b_4+[a_1b_0+a_0b_1],\\ P_2 =& [a_1b_1+a_0b_2]+[a_3b_3+a_2b_4]. \end{aligned}$$

To sum up the row entries with respect to $[\mathbf{M}_{A,1}\cdot\mathbf{b},\mathbf{M}_{A,2}\cdot\mathbf{b}]$, we have the binary tree of each row as follows:

Row 1: $\underbrace{\underline{[a_1b_1+a_0b_2]}+\underline{[a_4b_3+a_3b_4]}+[a_0b_0}_{P_0}\underbrace{+a_0b_0]}_{P_0}$,

Row 2: $\underbrace{\underline{[a_1b_2+a_2b_3]}+a_4b_4+\underline{[a_1b_0+a_0b_1]}}_{P_1}\underbrace{+\underline{[a_1b_0+a_0b_1]}}_{P_1}$,

Row 3: $\underbrace{\underline{[a_1b_1+a_0b_2]}+\underline{[a_3b_3+a_2b_4]}}_{P_2}\underbrace{+\underline{[a_4b_3+a_3b_4]}}_{P_0}$,

$$\text{Row 4: } \underbrace{[a_1b_2 + a_2b_3]}_{P_1} + \underbrace{[a_4b_4}_{P_1} + \underbrace{a_0b_0] + [a_4b_3 + a_3b_4]}_{P_0},$$
$$\text{Row 5: } \underbrace{[a_3b_3 + a_2b_4]}_{P_2} + \underbrace{a_4b_4 + [a_1b_0 + a_0b_1]}_{P_1}.$$

Additions in the square brackets of the above expressions are firstly computed in parallel, and the underlined ones can be saved as we compute them in P_0, P_1, P_2. One can check that Row 4 and Row 5 require $5-(5-W(3)-W(2))-(1-W(1)) = 5-2 = 3$ and $4-(3-W(3))-(2-W(2)) = 2$ XOR gates, respectively. Similarly, the binary trees of Row 1-3 cost 5 XOR gates in all. Plus 11 XOR gates needed in computation of P_0, P_1, P_2, we totally need 21 [3] XOR gates for $S_1x^{-4} \bmod x^5+x^2+1$ with delay $T_A + (\lceil \log_2 7 \rceil T_X) = T_A + 3T_X$.

3.3.2 Complexity analysis for $S_2x^{n-2k} \bmod f(x)$

Apparently, we compute $S_2x^{n-2k} \bmod f(x)$ as follows:

$$\begin{aligned} S_2x^{n-2k} \bmod f(x) &= \mathbf{M}_U \cdot \mathbf{v} \\ &= \mathbf{M}_{U,1} \cdot \mathbf{v} + \mathbf{M}_{U,2} \cdot \mathbf{v} \\ &= [\mathbf{M}_{U,1}, \mathbf{M}_{U,2}] \cdot [\mathbf{v}, \mathbf{v}]^T \end{aligned} \quad (18)$$

The computation of $S_2x^{n-2k} \bmod f(x)$ consists of the precomputation of U, V and matrix-vector multiplication presented as above. Firstly, $2n$ XOR gates are needed for precomputation of U, V which cost one T_X in parallel. Then, note that there are some common items between $\mathbf{M}_{U,1}$ and $\mathbf{M}_{U,2}$. So the matrix-vector multiplication $[\mathbf{M}_{U,1}, \mathbf{M}_{U,2}] \cdot [\mathbf{v}, \mathbf{v}]^T$ can follow the same line of the computing strategy presented in subsection 3.3.1. One T_A is needed for bitwise parallel multiplication and the required number of T_X is equal to the depth of the biggest binary trees related to non-zero entries of each row of $[\mathbf{M}_{U,1} \cdot \mathbf{v}, \mathbf{M}_{U,2} \cdot \mathbf{v}]$, which varies according to m and k. For simplicity, we only evaluate the upper bound of the number of T_X for some cases. The space and time complexities of (18) are summarized in Table 3.

Example 3.3. Consider the reduction of $\mathbf{U}'$ in Example 3.1. Since $5 = 2 \cdot 2 + 1$, there is no reduction needed here. We have $\mathbf{M}_U = \mathbf{U}'$. We note that the computation of U, V requires 4 XOR. Plus 4 XOR gates used in $\mathbf{M}_U \cdot \mathbf{v}$, it totally costs 8 XOR gates for $S_2x^{-2} \bmod x^5+x^2+1$. In addition, the delay of $\mathbf{M}_U \cdot \mathbf{v}$ is $T_A + \lceil \log_2 3 \rceil T_X = T_A + 2T_X$. Plus one T_X for computation U, V, the delay of $S_2x^{-2} \bmod x^5+x^2+1$ is $T_A + 3T_X$ which is equal to that of $S_1x^{-4} \bmod x^5+x^2+1$.

4 Complexity and Comparison

4.1 Theoretic complexity

Based on the delay of the two expressions presented in Table 2 and 3, we immediately have the following proposition.

3. Apparently, more XOR gates can be saved in this example, but we prefer to use the general formulae presented in Table 2.

Proposition 1 *Let T_{s_1} and T_{s_2} denote the delay of S_1x^{-2k} and S_2x^{n-2k} modulo $f(x)$, respectively. Then T_{s_2} is no greater than T_{s_1}.*

Proof According to circuit delay expressions in Table 2 and 3, it is clear that both T_{s_1} and T_{s_2} contain 1 T_A. We only need to compare the coefficients of T_X in these tables. Still consider the six cases indicated in subsection 3.2.2. We have

$$\begin{aligned} &\text{Case 1: } 1 + \lceil \log_2(m-k-1) \rceil \le \lceil \log_2(2m-k-1) \rceil, \\ &\text{Case 2: } 1 + \lceil \log_2(\frac{m}{2}) \rceil \le \lceil \log_2 m \rceil, \\ &\text{Case 3: } 1 + \lceil \log_2(k) \rceil \le \lceil \log_2(m+k) \rceil, \\ &\text{Case 4: } 1 + \lceil \log_2(m-k) \rceil \le \lceil \log_2(2m-k) \rceil, \\ &\text{Case 5: } 1 + \lceil \log_2(\frac{m-1}{2}) \rceil \le \lceil \log_2(m+k) \rceil, \\ &\text{Case 6: } 1 + \lceil \log_2(k+1) \rceil \le \lceil \log_2(m+k) \rceil. \end{aligned}$$

One can directly check that T_{s_2} is smaller or at most equal to T_{s_1}, which conclude the proposition. □

Proposition 1 ensure that $S_1x^{-2k} \bmod f(x)$ and $S_2x^{n-2k} \bmod f(x)$ can be implemented simultaneously, and the time delay is T_{s_1}. Finally, we add up these values to obtain the ultimate result which requires m XOR gates and one T_X in parallel. As a result, we obtain the total space complexity of the proposed multiplier by summing up all these related expressions.

If m is even:

$$\begin{aligned} &\#\text{AND} : \tfrac{3m^2}{4}, \\ &\#\text{XOR} : \tfrac{3m^2}{4} - \tfrac{m}{2} + O(m \log_2 m)^*, \\ &\text{Delay} : \begin{cases} T_A + (1 + \lceil \log_2(2m-k-1) \rceil) T_X, (m > 2k), \\ T_A + (1 + \lceil \log_2(m+k) \rceil) T_X, (m < 2k). \end{cases} \end{aligned} \quad (19)$$

If m is odd:

$$\begin{aligned} &\#\text{AND} : \tfrac{3m^2+2m-1}{4}, \\ &\#\text{XOR} : \tfrac{3m^2}{4} + \tfrac{m}{2} + O(m \log_2 m)^*, \\ &\text{Delay} : \begin{cases} T_A + (1 + \lceil \log_2(2m-k-1) \rceil) T_X, (m \ge 2k+1), \\ T_A + (1 + \lceil \log_2(m+k) \rceil) T_X, (m < 2k+1). \end{cases} \end{aligned} \quad (20)$$

We note that the formulae for the counts of XOR gates in Table 2 and 3 contain the sum of hamming weights related to certain integers, denoted by σ. The expression $\sum_{i=1}^{\sigma} W(i)$ can be roughly written as $\frac{\sigma}{2} \log_2 \sigma$ [8], [20]. Notice that $\sigma \le m$, so we use the expression $O(m \log_2 m)^*$ instead to make the complexity formulae simpler. For the special case, i.e, m is even and $m = 2k$, the explicit formulae with respect to the space and time complexities are given in Table 5.

4.2 Comparison

As the most important contribution of this study, the time delay of our multiplier is summarized in Table 5. Compared with the fastest bit-parallel multiplier [16], [17], our proposal only requires one more T_X. In addition, it is especially attractive if the corresponding circuit

TABLE 3
The space and time complexity of $S_2 x^{n-2k} \bmod f(x)$

Case	#AND	#XOR	Delay
m even, $m > 2k+2$	$\frac{m^2}{4}$	$\frac{m^2}{4} + \sum_{i=1}^{\frac{m}{2}-k-1} W(i) + 1$	$< T_A + (1 + \lceil \log_2(m-k-1) \rceil) T_X$
m even, $m = 2k, 2k+2$	$\frac{m^2}{4}$	$\frac{m^2}{4} + 1$	$T_A + (1 + \lceil \log_2(\frac{m}{2}) \rceil) T_X$
m even, $m < 2k$	$\frac{m^2}{4}$	$\frac{m^2}{4} + \sum_{i=1}^{k-\frac{m}{2}} W(i) + 1$	$< T_A + (1 + \lceil \log_2 k \rceil) T_X$
m odd, $m > 2k+1$	$\frac{m^2+2m+1}{4}$	$\frac{m^2+2m-3}{4} + \sum_{i=1}^{\frac{m-1}{2}-k} W(i)$	$< T_A + (1 + \lceil \log_2(m-k) \rceil) T_X$
m odd, $m = 2k+1$	$\frac{m^2+2m+1}{4}$	$\frac{m^2+2m-3}{4}$	$T_A + (1 + \lceil \log_2(\frac{m-1}{2}) \rceil) T_X$
m odd, $m < 2k+1$	$\frac{m^2+2m+1}{4}$	$\frac{m^2+2m-3}{4} + \sum_{i=1}^{k-\frac{m-1}{2}} W(i)$	$< T_A + (1 + \lceil \log_2(k+1) \rceil) T_X$

TABLE 4
Time delay for SPB multiplier using $f(x) = x^m + x^k + 1$

$m < 2k$	$T_A + (1 + \lceil \log_2(m+k) \rceil)\, T_X$
$m = 2k$	$T_A + (1 + \lceil \log_2 m \rceil)\, T_X$
$2k \leq m-1$	$T_A + (1 + \lceil \log_2(2m-k-1) \rceil)\, T_X$

delay is $T_A + (1 + \lceil \log_2 m \rceil) T_X$, this happens frequently if m, k satisfy:

$$\lceil \log_2(m+k) \rceil = \lceil \log_2 m \rceil,\ m < 2k,$$
$$\lceil \log_2(2m-k-1) \rceil = \lceil \log_2 m \rceil,\ 2k \leq m-1.$$

In fact, for the range $100 \leq m \leq 1023$ with cryptographic interests, there exist 1405 irreducible trinomials and 457 trinomials satisfying the above condition.

In Table 5, we give a comparison of several different bit-parallel multipliers for irreducible trinomials. All these multipliers are using PB representations except particular description. It is clear that our scheme is faster than other Karatsuba-based multiplier and still has roughly 25% logic gates gain. In [13], [18], [23], the authors investigated the speedup of Karatsuba multiplier independently. None of them had given such a precise time bound for all the trinomials. This is the first time for us to show that Karatsuba-based multiplier can always be only $1T_X$ slower than the fastest bit-parallel multipliers so far.

5 Conclusion

In this paper, we have constructed a matrix-vector form of Karatsuba algorithm and proposed a novel bit-parallel $GF(2^m)$ multiplier based on this approach. Mastrovito scheme and shifted polynomial basis are combined together to reduce the gate delay. A binary tree based *subexpression sharing* approach is utilized to exploit common items sharing efficiently. As a result, it is argued that our proposal matches the fastest non-recursive Karatsuba multipliers and is only one T_X slower than the fastest bit-parallel multipliers where no divide-and-conquer algorithm is applied.

Furthermore, we note that, based on the similarity between SPB and Montgomery multiplier, the proposed scheme can be easily moved to design bit-parallel $GF(2^m)$ Montgomery multiplier. Finally, the space and time trade-off enables our scheme to apply in acceleration of scalar multiplication under some area constraint platforms. We next work on Mastrovito-Karatsuba multiplier for pentanomials.

Appendix
The overlapped values for the computation of $s_i, i = 0, \cdots, m-1$

Table 6-10 give the overlapped values and number of saved XOR gates for $S_1 x^{-2k} \bmod f(x)$ of other cases.

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TABLE 5
Comparison of Some Bit-Parallel Multipliers for Irreducible Trinomials

Multiplier	# AND	# XOR	Time delay
$x^m + x^k + 1, 1 < k \leq \frac{m-1}{2}$			
Montgomery [29], school-book [28]	m^2	$m^2 - 1$	$T_A + (2 + \lceil \log_2 m \rceil) T_X$
Mastrovito [24] [25] [26]	m^2	$m^2 - 1$	$T_A + (2 + \lceil \log_2 m \rceil) T_X$
Mastrovito [27]	m^2	$m^2 - 1$	$T_A + (\lceil \log_2 (2m + 2k - 3) \rceil) T_X$
SPB Mastrovito [16]	m^2	$m^2 - 1$	$T_A + \lceil \log_2 (2m - k - 1) \rceil T_X$
Montgomery [17]	m^2	$m^2 - 1$	$T_A + \lceil \log_2 (2m - k - 1) \rceil T_X$
Karatsuba [14]	$\frac{3m^2+2m-1}{4}$	$\frac{3m^2}{4} + 4m + k - \frac{23}{4}$ (m odd)	$T_A + (3 + \lceil \log_2 (m-1) \rceil) T_X$
	$\frac{3m^2}{4}$	$\frac{3m^2}{4} + \frac{5m}{2} + k - 4$ (m even)	
Modified Karatsuba [18]	$\frac{m^2}{2} + (m-k)^2$	$\frac{m^2}{2} + (m-k)^2 + 2k$	$T_A + (2 + \lceil \log_2 (m-1) \rceil) T_X$
Modified Karatsuba [13]	$m^2 - k^2$	$m^2 + k - k^2 - 1 (1 < k < \frac{m}{3})$	$\leq T_A + (2 + \lceil \log_2 m \rceil) T_X$
		$m^2 + 4k - k^2 - m - 1 (\frac{m}{3} \leq k < \frac{m-1}{2})$	
		$m^2 + 2k - k^2 (k = \frac{m-1}{2})$	
Montgomery squaring [20]	$\frac{3m^2+2m-1}{4}$	$\frac{3m^2}{4} + O(m \log_2 m)$ (m odd)	$\leq T_A + (3 + \lceil \log_2 m \rceil) T_X$
	$\frac{3m^2}{4}$	$\frac{3m^2}{4} + O(m \log_2 m)$ (m even)	$T_A + (2 + \lceil \log_2 m \rceil) T_X$
Chinese Remainder Theorem [31]	Δ	$\Delta + 3k - m$ (Type-A)	$T_A + \lceil \log_2 (\Theta) \rceil T_X$
	Δ	$\Delta + 2k - m + kW(k)$ (Type-B)	$T_A + \lceil \log_2 (3m - 3k - 1) \rceil T_X$
SPB Mastrovito-Karatsuba	$\frac{3m^2+2m-1}{4}$	$\frac{3m^2}{4} + \frac{m}{2} + O(m \log_2 m)$ (m odd)	$T_A + (1 + \lceil \log_2 (2m - k - 1) \rceil) T_X$
	$\frac{3m^2}{4}$	$\frac{3m^2}{4} - \frac{m}{2} + O(m \log_2 m)$ (m even)	
where $\Delta = m^2 + \frac{(m-k)(m-1-3k)}{2}$ ($\frac{m-1}{3} \leq k < \frac{m}{2}$, $2^{v-1} < k \leq 2^v$), $\Theta = \max(3m-3k-1, 2m-2k+2^v)$			
$x^m + x^k + 1, m = 2k$			
Mastrovito [24] [26]	m^2	$m^2 - \frac{m}{2}$	$T_A + (1 + \lceil \log_2 (m-1) \rceil) T_X$
Montgomery [29], school-book [28]	m^2	$m^2 - \frac{m}{2}$	$T_A + (1 + \lceil \log_2 (m-1) \rceil) T_X$
Mastrovito [27]	m^2	$m^2 - \frac{m}{2}$	$T_A + (\lceil \log_2 (\frac{3m}{2}) \rceil) T_X$
SPB Mastrovito [16]	m^2	$m^2 - \frac{m}{2}$	$T_A + \lceil \log_2 (\frac{3m}{2}) \rceil T_X$
SPB Karatsuba [23]	$\frac{3m^2}{4}$	$\frac{3m^2}{4} + m + 1$	$T_A + (1 + \lceil \log_2 (m-1) \rceil) T_X$
Modified Karatsuba [13]	$\frac{3m^2}{4}$	$\frac{3m^2}{4} + m - 1$	$T_A + (1 + \lceil \log_2 (m+2) \rceil) T_X$
Chinese Remainder Theorem [31]	$\frac{7m^2-2m}{8}$	$\frac{7m^2+2m}{8}$ (Type-A)	$T_A + \lceil \log_2 (2m) \rceil T_X$
	$\frac{7m^2-2m}{8}$	$\frac{7m^2-2m}{8} + kW(k)$ (Type-B)	$T_A + \lceil \log_2 (\frac{3m}{2}) \rceil T_X$
SPB Mastrovito-Karatsuba	$\frac{3m^2}{4}$	$\frac{3m^2}{4} + \frac{m}{2} + 1$	$T_A + (1 + \lceil \log_2 m \rceil) T_X$
$x^m + x^k + 1, m < 2k$			
Mastrovito [24] [26]	m^2	$m^2 - 1$	$T_A + (\lceil \log_2 (2^r m) \rceil) T_X$
Mastrovito [27]	m^2	$m^2 - 1$	$T_A + (\lceil \log_2 (\Theta) \rceil) T_X$
SPB Mastrovito [16]	m^2	$m^2 - 1$	$T_A + \lceil \log_2 (m + k) \rceil T_X$
Montgomery [17]	m^2	$m^2 - 1$	$T_A + \lceil \log_2 (m + k) \rceil T_X$
SPB Mastrovito-Karatsuba	$\frac{3m^2+2m-1}{4}$	$\frac{3m^2}{4} + \frac{m}{2} + O(m \log_2 m)$ (m odd)	$T_A + (1 + \lceil \log_2 (m + k) \rceil) T_X$
	$\frac{3m^2}{4}$	$\frac{3m^2}{4} - \frac{m}{2} + O(m \log_2 m)$ (m even)	
where $r = \lceil \frac{m-1}{m-k} \rceil$, $\Theta = 2^r \big(m - 1 - (r-2)(m-k)\big) - 2(m-k) + 1$			

TABLE 6
The overlapped values and saved #XOR, if m even, $n < k \leq m-1$

First part	Overlapped	Saved #XOR	Second part	Overlapped	Saved #XOR
$\mathbf{M}_{A,1}(1,:)\cdot\mathbf{b}$	P_{k-n}	$m-W(m)$	$\mathbf{M}_{A,2}(1:)\cdot\mathbf{b}$	P_0	$1-W(1)$
$\mathbf{M}_{A,1}(2,:)\cdot\mathbf{b}$	P_{k-n+1}	$m-W(m)$	$\mathbf{M}_{A,2}(2,:)\cdot\mathbf{b}$	P_1	$2-W(2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m-k,:)\cdot\mathbf{b}$	P_{n-1}	$m-W(m)$	$\mathbf{M}_{A,2}(n,:)\cdot\mathbf{b}$	P_{n-1}	$n-W(n)$
$\mathbf{M}_{A,1}(m-k+1,:)\cdot\mathbf{b}$	P_0	$m-W(m)$	$\mathbf{M}_{A,2}(n+1,:)\cdot\mathbf{b}$	P_0	$n+1-W(n+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m+n-k,:)\cdot\mathbf{b}$	P_{n-1}	$m-W(m)$	$\mathbf{M}_{A,2}(k,:)\cdot\mathbf{b}$	P_{k-n-1}	$k-W(k)$
$\mathbf{M}_{A,1}(m+n-k+1,:)\cdot\mathbf{b}$	P_0	$m-W(m)$	$\mathbf{M}_{A,2}(k+1,:)\cdot\mathbf{b}$	P_{k-n+1}	$m-k-1-W(m-k-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m,:)\cdot\mathbf{b}$	P_{k-n-1}	$m-W(m)$	$\mathbf{M}_{A,2}(m-1,:)\cdot\mathbf{b}$	P_{n-1}	$1-W(1)$

TABLE 7
The overlapped values and saved #XOR, if m odd, $0 < k \leq n-1$

first part	Overlapped	Saved #XOR	second part	Overlapped	Saved #XOR
$\mathbf{M}_{A,1}(1,:)\cdot\mathbf{b}$	P_k, P_{k+1}	$m-W(n+1)-W(n)$	$\mathbf{M}_{A,2}(1,:)\cdot\mathbf{b}$	P_0	$1-W(1)$
$\mathbf{M}_{A,1}(2,:)\cdot\mathbf{b}$	P_{k+1}, P_{k+2}	$m-W(n+1)-W(n)$	$\mathbf{M}_{A,2}(2,:)\cdot\mathbf{b}$	P_1	$2-W(2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(n-k,:)\cdot\mathbf{b}$	P_{n-1}, P_n	$m-W(n+1)-W(n)$	$\mathbf{M}_{A,2}(k,:)\cdot\mathbf{b}$	P_{k-1}	$k-W(k)$
$\mathbf{M}_{A,1}(n-k+1,:)\cdot\mathbf{b}$	P_0	$m-W(m)$	$\mathbf{M}_{A,2}(k+1,:)\cdot\mathbf{b}$	P_k, P_{k+1}	$2n-k-W(n-k)-W(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m-k,:)\cdot\mathbf{b}$	P_{n-1}	$m-W(m)$	$\mathbf{M}_{A,2}(n,:)\cdot\mathbf{b}$	P_{n-1}, P_n	$n+1-W(1)-W(n)$
$\mathbf{M}_{A,1}(m-k+1,:)\cdot\mathbf{b}$	P_0, P_1	$m-W(n+1)-W(n)$	$\mathbf{M}_{A,2}(n+1,)\cdot\mathbf{b}$	P_0	$n-W(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m,:)\cdot\mathbf{b}$	P_{k-1}, P_k	$m-W(n+1)-W(n)$	$\mathbf{M}_{A,2}(m-1,:)\cdot\mathbf{b}$	P_n	$1-W(1)$

TABLE 8
The overlapped values and saved #XOR, if m odd, $n < k \leq m-1$

first part	Overlapped	Saved #XOR	second part	Overlapped	Saved #XOR
$\mathbf{M}_{A,1}(1,:)\cdot\mathbf{b}$	P_{k-n}	$m-W(m)$	$\mathbf{M}_{A,2}(1,:)\cdot\mathbf{b}$	P_0	$1-W(1)$
$\mathbf{M}_{A,1}(2,:)\cdot\mathbf{b}$	P_{k-n+1}	$m-W(m)$	$\mathbf{M}_{A,2}(2,:)\cdot\mathbf{b}$	P_1	$1-W(2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m-k,:)\cdot\mathbf{b}$	P_n	$m-W(m)$	$\mathbf{M}_{A,2}(n,:)\cdot\mathbf{b}$	P_{n-1}	$n-W(n)$
$\mathbf{M}_{A,1}(m-k+1,:)\cdot\mathbf{b}$	P_0, P_1	$m-W(n+1)-W(n)$	$\mathbf{M}_{A,2}(n+1,:)\cdot\mathbf{b}$	P_0^*	$n+1-W(n)-W(1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m+n-k,:)\cdot\mathbf{b}$	P_{n-1}, P_n	$m-W(n+1)-W(n)$	$\mathbf{M}_{A,2}(k,:)\cdot\mathbf{b}$	P_{k-n-1}^*	$k-W(n)-W(k-n)$
$\mathbf{M}_{A,1}(m+n-k+1,:)\cdot\mathbf{b}$	P_0	$m-W(m)$	$\mathbf{M}_{A,2}(k+1,:)\cdot\mathbf{b}$	P_{k-n+1}	$m-k-1-W(m-k-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m,:)\cdot\mathbf{b}$	P_{k-n-1}	$m-W(m)$	$\mathbf{M}_{A,2}(m-1,:)\cdot\mathbf{b}$	P_n	$W(1)$
* represent that one intermediate value contains two separate parts of items overlapped with $\mathbf{M}_{A,2}(i,:)\cdot\mathbf{b}$, $i = n+1,\cdots,k$					

TABLE 9
The overlapped values and saved #XOR, if m odd, $k = n$

first part	Overlapped	Saved #XOR	second part	Overlapped	Saved #XOR
$\mathbf{M}_{A,1}(1,:)\cdot\mathbf{b}$	P_0	$m-W(m)$	$\mathbf{M}_{A,2}(1,:)\cdot\mathbf{b}$	P_0	$1-W(1)$
$\mathbf{M}_{A,1}(2,:)\cdot\mathbf{b}$	P_1	$m-W(m)$	$\mathbf{M}_{A,2}(2,:)\cdot\mathbf{b}$	P_1	$2-W(2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(n+1,:)\cdot\mathbf{b}$	P_n	$m-W(m)$	$\mathbf{M}_{A,2}(n,:)\cdot\mathbf{b}$	P_{n-1}	$n-W(n)$
$\mathbf{M}_{A,1}(n+2,:)\cdot\mathbf{b}$	P_0, P_1	$m-W(n+1)+W(n)$	$\mathbf{M}_{A,2}(n+1,:)\cdot\mathbf{b}$	P_0	$n-W(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{M}_{A,1}(m-1,:)\cdot\mathbf{b}$	P_{n-2}, P_{n-1}	$m-W(n+1)+W(n)$	$\mathbf{M}_{A,2}(m-2,:)\cdot\mathbf{b}$	P_{n-2}	$2-W(2)$
$\mathbf{M}_{A,1}(m,:)\cdot\mathbf{b}$	P_{n-1}, P_n	$m-W(n+1)+W(n)$	$\mathbf{M}_{A,2}(m-1,:)\cdot\mathbf{b}$	P_{n-1}	$1-W(1)$

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