Revocable Hierarchical Identity-Based Encryption with Adaptive Security

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Abstract

Hierarchical identity-based encryption (HIBE) can be extended to revocable HIBE (RHIBE) if a private key of a user can be revoked when the private key is revealed or expired. Previously, many selectively secure RHIBE schemes were proposed, but it is still unsolved problem to construct an adaptively secure RHIBE scheme. In this work, we propose two RHIBE schemes in composite-order bilinear groups and prove their adaptive security under simple static assumptions. To prove the adaptive security, we use the dual system encryption framework, but it is not simple to use the dual system encryption framework in RHIBE since the security model of RHIBE is quite different with that of HIBE. We show that it is possible to solve the problem of the RHIBE security proof by carefully designing hybrid games.

Keywords: Hierarchical identity-based encryption, Key revocation, Adaptive security, Dual system encryption, Bilinear maps.

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1 Introduction

Hierarchical identity-based encryption (HIBE) is an important extension of identity-based encryption (IBE) [6] that uses the identity of a user as the public-key of the user. In HIBE, the identity of a user is represented as a hierarchical structure and a user with a private key can delegate his private key to next-level users. The concept of HIBE was introduced by Horwitz and Lynn [13] to reduce the burden of a private key generation in a trusted center and a secure HIBE scheme that supports arbitrary many levels are proposed by Gentry and Silverberg [11]. HIBE can be extended to broadcast encryption, forward-secure encryption, chosen-ciphertext secure encryption, and searchable encryption [1, 7, 8, 10] and it has many interesting applications like encryption systems for medical data and range query on encrypted data [2, 29].

To use an HIBE scheme in real applications, we should revoke the private key of a user if his private key is revealed or his credential is expired. Revocable HIBE (RHIBE) is an extension of HIBE that supports the revocation functionality by broadcasting an update key for non-revoked users per each time period. Previously, an efficient revocable IBE (RIBE) schemes were proposed by many researchers [3,17,21,23,26]. Seo and Emura [25] proposed the first RHIBE scheme by following the design strategy of Boldyreva et al. [3] that uses a binary tree and proved its selective security. After that, some efficient RHIBE schemes with improved parameters were proposed [19, 28], but these are also proven to be selectively secure.

The right security model of RHIBE is the adaptive model where an adversary can select a target in the challenge step. In RIBE, adaptively secure RIBE schemes were already proposed in [17, 21, 26]. However, all RHIBE schemes only provide the selective security where the challenge identity ID^* and the challenge time T^* should be submitted before receiving public parameters or the selective revocation list security where the challenge revocation set R^* should be additionally submitted [19, 25, 28]. Although an RHIBE scheme claimed to be adaptively secure was proposed in [27], the security proof that uses the dual system encryption technique has some flaws. The flaw is that the private key of $ID \in \text{Prefix}(ID^*)$ and the update key of $T = T^*$ cannot be directly converted from normal to semi-functional since the simple information theoretic argument doesn't work for this case. Therefore, the construction of an adaptively secure RHIBE scheme is an unsolved open problem.

1.1 Our Results

In this paper, we give an answer to this unsolved problem by proposing two RHIBE schemes in composite-order bilinear groups and proving their adaptive security under simple static assumptions.

We first propose an RHIBE-CS scheme by combining the HIBE and IBE schemes of Lewko and Waters [20] and the complete subtree (CS) scheme of Naor, Naor, and Lotspiech [22] in a modular way. For the construction of our RHIBE-CS scheme, we follow the modular design approach of Lee and Park [19] except that the underlying HIBE and IBE schemes are replaced by the schemes of Lewko and Waters. We then prove the adaptive security of our RHIBE-CS scheme by using the dual system encryption framework [20, 32]. However, the naive approach of dual system encryption does not work for RHIBE since an adversary can query a private key for ID that is a prefix of ID^* and an update key for T^* where ID^* is the challenge identity and T^* is the challenge time, and these private key and update key cannot be easily converted from normal to semi-functional. Thus, solving this problem of RHIBE when the dual system encryption was used is the core of the security proof. The main technical idea of solving this problem is described in the later part of this section.

Next, we propose an RHIBE-SD scheme by using the subset difference (SD) scheme instead of using the CS method to reduce the size of an update key. As mentioned before, we also follow the modular design approach of Lee and Park [19]. Our RHIBE-SD scheme has $O(r)$ number of group elements in an

Table 1: Comparison of revocable hierarchical identity-based encryption schemes

Scheme	PP Size	SK Size	UK Size	CT Size	Model	Assumption
SE (CS) [25]	$O(\ell)$	$O(\ell^2 \log N)$	$O(\ell r \log \frac{N}{r})$	$O(\ell)$	SE-IND	DBDH
SE (CS) [28]	$O(\ell)$	$O(\ell \log N)$	$O(\ell r \log \frac{N}{r})$	$O(1)$	SE-IND	q-Type
SE (SD) [28]	$O(\ell)$	$O(\ell \log^2 N)$	$O(\ell r)$	$O(1)$	SRL-IND	q-Type
LP (CS) [19]	$O(1)$	$O(\log N)$	$O(\ell + r \log \frac{N}{r})$	$O(\ell)$	SE-IND	q-Type
LP (SD) [19]	$O(1)$	$O(\log^2 N)$	$O(\ell + r)$	$O(\ell)$	SRL-IND	q-Type
Ours (CS)	$O(\ell)$	$O(\ell \log N)$	$O(\ell + r \log \frac{N}{r})$	$O(1)$	AD-IND	Static
Ours (SD)	$O(\ell)$	$O(\ell \log^2 N)$	$O(\ell + r)$	$O(1)$	AD-IND	Static

We let N be the number of maximum users in each level, r be the number of revoked users, and ℓ be the depth of a hierarchical identity. We count the number of group elements to measure the size of parameters. We use symbols SE-IND for selective IND-CPA, SRL-IND for selective revocation list IND-CPA, and AD-IND for adaptive IND-CPA.

update key and $O(\log^2 N_{max})$ number of group elements in a private key whereas our RHIBE-CS scheme has $O(r \log(N_{max}/r))$ number of group elements in an update key and $O(\log N_{max})$ number of group elements in a private key. The detailed comparison of RHIBE schemes is given in Table 1. To prove the adaptive security of our RHIBE-SD scheme, we carefully use the proof technique of Lee et al. [17] that was used to prove the adaptive security of their RIBE scheme.

1.2 Our Techniques

To prove the adaptive security of an HIBE scheme, the dual system encryption framework was introduced by Waters [32]. In the dual system encryption framework, ciphertexts and private keys can be normal or semi-functional in which a normal ciphertext can be decrypted by a normal or semi-functional private key whereas a semi-functional ciphertext cannot be decrypted by a semi-functional private key. To prove the adaptive security, a normal challenge ciphertext is changed to be semi-functional, and then each normal private key is changed to be semi-functional one by one through hybrid games. The main obstacle of this proof is to overcome the paradox of dual system encryption in which a simulator can check whether a private key is normal or semi-functional by decrypting a semi-functional ciphertext since a simulator can generate a ciphertext and a private key for any identity. Lewko and Waters [20] solved this problem by introducing the nominally semi-functional type of private keys where a semi-functional ciphertext can be decrypted by a nominally semi-functional private key. Note that an information theoretic argument should be given to argue that a nominally semi-functional key is indistinguishable from a semi-functional key.

For the security proof of an RHIBE scheme, one may simply use the dual system encryption technique that changes private keys and update keys from normal types to semi-functional types one by one through hybrid games. However, this simple strategy does not work since the adversary of RHIBE can query a private key for ID that is a prefix of ID^* and an update key for $T = T^*$ where ID^* and T^* are the challenge identity and time. That is, we cannot show the information theoretic argument for these private key and update key since ID is a prefix of ID^* and $T = T^*$. In HIBE, the restriction of an adversary that ID is not a prefix of ID^* is essentially used to show the information theoretic argument. Thus, it is not easy to prove the adaptive security of an RHIBE scheme by using the dual system encryption framework.

Our strategy to overcome this problem is that private keys and update keys of an RHIBE scheme are first divided into smaller component keys and then these component keys that are related to the same node in a binary tree are grouped together. Next, these component keys that belong to the same group are changed from normal types to semi-functional types one by one through hybrid games. Similar proof strategy was used in [16, 17, 24]. In particular we consider an RHIBE-CS scheme that use the CS method. A private key consists of many HIBE private keys that are related to a path in a binary tree and an update key also consists of many IBE private keys that are related to a cover set in a binary tree. By the grouping of HIBE private keys and IBE private keys with the same node, we can use the restriction of the RHIBE security model to show an information theoretic argument.

For example, if an adversary requests a private key for $ID \in \text{Prefix}(ID^*)$ and one HIBE private key of this private key is related to a node v^*, then all IBE private keys in update keys satisfy $T \neq T^*$ for this node v^* since this private key should be revoked on time T^* by the restriction of the security model. Thus, we first change IBE private key related to v^* from normal to semi-functional one by one by using $T \neq T^*$, and then we change HIBE private keys related to v^* from normal to semi-functional at once. Note that there is no paradox of dual system encryption when we change HIBE private keys from normal to semi-functional since IBE private keys are already semi-functional. Recall that an information theoretic argument is not needed if nominally semi-functional keys are not used. Similar argument also applies when the adversary requests an update key for $T = T^*$ and one IBE private key of this update key is related to a node v^* since we have $ID \notin \text{Prefix}(ID^*)$ for all HIBE private key for this node v^*.

To prove the adaptive security of our RHIBE-SD scheme, we also use the similar proof strategy that private keys and update keys are divided into smaller component keys and these component keys that belong to the same group are changed from normal to semi-functional. In our RHIBE-CS scheme, a group is simply defined by a node v_j in a binary tree. In our RHIBE-SD scheme, a group is defined as a set of subsets $S_{i,j}$ such that v_i is the same and the depth d_j of v_j is the same where $S_{i,j}$ is defined by two nodes v_i and v_j in a binary tree. To change HIBE private keys and IBE private keys in the same group from normal to semi-functional, we carefully design hybrid games since a group is very complex. Note that Lee et al. [17] also used this proof strategy to prove the adaptive security of their RIBE-SD scheme.

1.3 Related Work

An IBE scheme with key revocation was first proposed by Boneh and Franklin [6] in which each user should retrieve his private key from a trusted center for the identity $ID\|T$ per each time period T. Boldyreva, Goyal, and Kumar [3] proposed a scalable RIBE scheme by combining a fuzzy IBE scheme and the CS method in which an update key is broadcasted to non-revoked users per each time period. This design method that uses the CS method for key revocation was also used to build other adaptively secure RIBE schemes [21,26]. The SD method is an improvement on the CS method since the size of a broadcasting set can be reduced [22]. Lee et al. [17] proposed an RIBE scheme that uses the SD method to improve the size of an update key and proved its adaptive security under static assumptions. An RIBE scheme based on a binary tree cannot have short private keys and update keys. To overcome this problem, Park et al. [23] proposed an RIBE scheme with short private keys and update keys by using multilinear maps. In addition, RIBE schemes using lattice have been proposed [9, 14, 30].

As mentioned before, the first selectively secure RHIBE scheme was proposed by Seo and Emura [25] by combining the HIBE scheme of Boneh and Boyen [4] and the CS method. This RHIBE scheme is relatively inefficient since a user should retrieve all update keys generated by his ancestors to decrypt a ciphertext. To solve this problem of inefficiency, Seo and Emura [28] proposed another selectively secure RHIBE scheme with history-free updates that uses the CS (or SD) method where a user only needs to retrieve an update key generated by his parent. Recently, Lee and Park [19] proposed new RHIBE schemes with shorter private keys and update keys by combining a new HIBE scheme that has short intermediate private keys and the CS (or SD) method in a modular way. Following the announcement of our RHIBE scheme in this paper, an adaptively secure RHIBE scheme in prime order groups was proposed by Watanabe et al. [31].

An attribute-based encryption (ABE) scheme also can be extended to support the key revocation. A revocable ABE (RABE) scheme was also proposed by Boldyreva et al. [3] by combining a key-policy ABE scheme and the CS method and its selective revocation list security was claimed. To securely protect information stored in cloud storage, one may use an RABE scheme since it provides the access control on encrypted data as well as the key revocation. Sahai et al. [24] pointed out that RABE is not enough for cloud storage and then they proposed a revocable-storage ABE (RS-ABE) scheme that supports the key revocation and the ciphertext update. After that, Lee et al. showed that an RS-ABE scheme can be improved by using a self-updatable encryption (SUE) scheme [15, 16, 18].

2 Preliminaries

In this section, we introduce composite-order bilinear groups and complexity assumptions. Next, we define the syntax and the adaptive security model of RHIBE.

2.1 Notation

Let $\mathcal{I}$ be the identity space. A hierarchical identity ID with a depth k is defined as an identity vector $ID = (I_1, \dots, I_k) \in \mathcal{I}^k$. We let $ID|_j$ be a vector $(I_1, \dots, I_j)$ of size j derived from ID. If $ID = (I_1, \dots, I_k)$, then we have $ID = ID|_k$. We define $ID|_0 = \varepsilon$ (i.e. the empty string) for simplicity. The function $\text{Prefix}(ID|_k)$ returns a set of prefix vectors $\{ID|_j\}$ for all $1 \le j \le k$ where $ID|_k = (I_1, \dots, I_k) \in \mathcal{I}^k$ for some k. For two hierarchical identities $ID|_i$ and $ID|_j$ with $i < j$, $ID|_i$ is an ancestor of $ID|_j$ and $ID|_j$ is a descendant of $ID|_i$ if $ID|_i \in \text{Prefix}(ID|_j)$.

2.2 Binary Tree

A perfect binary tree $\mathcal{BT}$ is a tree data structure in which all internal nodes have two child nodes and all leaf nodes have the same depth. Let $N = 2^n$ be the number of leaf nodes in $\mathcal{BT}$. The number of all nodes in $\mathcal{BT}$ is $2N-1$ and we denote v_i as a node in $\mathcal{BT}$ for any $1 \le i \le 2N-1$. The depth d_i of a node v_i is the length of the path from a root node to the node. The root node of a tree has depth zero. The depth of $\mathcal{BT}$ is the length of the path from the root node to a leaf node. A level of $\mathcal{BT}$ is a set of all nodes at given depth.

Each node $v_i \in \mathcal{BT}$ has an identifier $L_i \in \{0,1\}^*$ which is a fixed and unique string. An identifier of each node is assigned as follows: Each edge in the tree is assigned with 0 or 1 depending on whether it is connected to the left or right child node. The identifier L_i of a node v_i is obtained by reading all labels of edges in a path from the root node to the node v_i. The root node has an empty identifier ε. For a node v_i, we define $\text{Label}(v_i)$ be the identifier L_i of v_i and $\text{Depth}(v_i)$ be the depth d_i of v_i.

A subtree $\mathcal{T}_i$ in $\mathcal{BT}$ is defined as a tree that is rooted at a node $v_i \in \mathcal{BT}$. A subset S_i is defined as a set of all leaf nodes in $\mathcal{T}_i$. For any two nodes $v_i, v_j \in \mathcal{BT}$ where v_j is a descendant of v_i, $\mathcal{T}_{i,j}$ is defined as a subtree $\mathcal{T}_i - \mathcal{T}_j$, that is, all nodes that are descendants of v_i but not v_j. A subset $S_{i,j}$ is defined as the set of leaf nodes in $\mathcal{T}_{i,j}$, that is, $S_{i,j} = S_i \setminus S_j$. For S_j and $S_{i,j}$, we define $\text{Label}(S_i) = \text{Label}(v_i)$ and $\text{Label}(S_{i,j}) = (\text{Label}(v_i), \text{Label}(v_j))$ respectively.

For a perfect binary tree $\mathcal{BT}$ and a subset R of leaf nodes, $\mathcal{ST}_R$ is defined as the Steiner Tree induced by the set R and the root node, that is, the minimal subtree of $\mathcal{BT}$ that connects all the leaf nodes in R and the root node.

2.3 Bilinear Groups of Composite Order

Let $N = p_1 p_2 p_3$ where p_1, p_2, and p_3 are distinct prime numbers. Let $\mathbb{G}$ and $\mathbb{G}_T$ be two multiplicative cyclic groups of same composite order N and g be a generator of $\mathbb{G}$. The bilinear map $e : \mathbb{G} \times \mathbb{G} \to \mathbb{G}_T$ has the following properties:

1. Bilinearity: $\forall u, v \in \mathbb{G}$ and $\forall a, b \in \mathbb{Z}_N$, $e(u^a, v^b) = e(u, v)^{ab}$.
2. Non-degeneracy: $\exists g$ such that $e(g, g)$ has order N, that is, $e(g, g)$ is a generator of $\mathbb{G}_T$.

We say that $\mathbb{G}$ is a bilinear group if the group operations in $\mathbb{G}$ and $\mathbb{G}_T$ as well as the bilinear map e are all efficiently computable. Furthermore, we assume that the description of $\mathbb{G}$ and $\mathbb{G}_T$ includes generators of $\mathbb{G}$ and $\mathbb{G}_T$ respectively. We use the notation $\mathbb{G}_{p_i}$ to denote the subgroups of order p_i of $\mathbb{G}$ respectively. Similarly, we use the notation $\mathbb{G}_{T,p_i}$ to denote the subgroups of order p_i of $\mathbb{G}_T$ respectively.

2.4 Complexity Assumptions

Assumption 1 (Subgroup Decision, SD)**.** Let $(N, \mathbb{G}, \mathbb{G}_T, e)$ be a description of the bilinear group of composite order $N = p_1 p_2 p_3$. Let g_1, g_2, g_3 be generators of subgroups $\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \mathbb{G}_{p_3}$ respectively. The SD assumption is that if the challenge tuple

$$D = ((N, \mathbb{G}, \mathbb{G}_T, e), g_1, g_3) \text{ and } Z$$

are given, no PPT algorithm $\mathcal{A}$ can distinguish $Z = Z_0 = X_1 \in \mathbb{G}_{p_1}$ from $Z = Z_1 = X_1 R_1 \in \mathbb{G}_{p_1 p_2}$ with more than a negligible advantage. The advantage of $\mathcal{A}$ is defined as $\mathrm{Adv}^{SD}_{\mathcal{A}}(\lambda) = \big|\Pr[\mathcal{A}(D, Z_0) = 0] - \Pr[\mathcal{A}(D, Z_1) = 0]\big|$ where the probability is taken over random choices of $X_1 \in \mathbb{G}_{p_1}$ and $R_1 \in \mathbb{G}_{p_2}$.

Assumption 2 (General Subgroup Decision, GSD)**.** Let $(N, \mathbb{G}, \mathbb{G}_T, e)$ be a description of the bilinear group of composite order $N = p_1 p_2 p_3$. Let g_1, g_2, g_3 be generators of subgroups $\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \mathbb{G}_{p_3}$ respectively. The GSD assumption is that if the challenge tuple

$$D = ((N, \mathbb{G}, \mathbb{G}_T, e), g_1, g_3, X_1 R_1, R_2 Y_1) \text{ and } Z$$

are given, no PPT algorithm $\mathcal{A}$ can distinguish $Z = Z_0 = X_2 Y_2 \in \mathbb{G}_{p_1 p_3}$ from $Z = Z_1 = X_2 R_3 Y_2 \in \mathbb{G}_{p_1 p_2 p_3}$ with more than a negligible advantage. The advantage of $\mathcal{B}$ is defined as $\mathrm{Adv}^{GSD}_{\mathcal{A}}(\lambda) = \big|\Pr[\mathcal{A}(D, T_0) = 0] - \Pr[\mathcal{A}(D, T_1) = 0]\big|$ where the probability is taken over random choices of $X_1, X_2 \in \mathbb{G}_{p_1}$, $R_1, R_2, R_3 \in \mathbb{G}_{p_2}$, and $Y_1, Y_2 \in \mathbb{G}_{p_3}$.

Assumption 3 (Composite Diffie-Hellman, ComDH)**.** Let $(N, \mathbb{G}, \mathbb{G}_T, e)$ be a description of the bilinear group of composite order $N = p_1 p_2 p_3$. Let g_1, g_2, g_3 be generators of subgroups $\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \mathbb{G}_{p_3}$ respectively. The ComDH assumption is that if the challenge tuple

$$D = ((N, \mathbb{G}, \mathbb{G}_T, e), g_1, g_2, g_3, g_1^a R_1, g_1^b R_2) \text{ and } Z$$

are given, no PPT algorithm $\mathcal{A}$ can distinguish $Z = Z_0 = e(g_1, g_1)^{ab}$ from $Z = Z_1 = e(g_1, g_1)^c$ with more than a negligible advantage. The advantage of $\mathcal{A}$ is defined as $\mathrm{Adv}^{ComDH}_{\mathcal{A}}(\lambda) = \big|\Pr[\mathcal{A}(D, Z_0) = 0] - \Pr[\mathcal{A}(D, Z_1) = 0]\big|$ where the probability is taken over random choices of $a, b, c \in \mathbb{Z}_N$, and $R_1, R_2 \in \mathbb{G}_{p_2}$.

2.5 Pseudo-Random Functions

A pseudo-random function (PRF) [12] is an efficiently computable function $F : \mathcal{K} \times \mathcal{X} \rightarrow \mathcal{Y}$ where $\mathcal{K}$ is the key space, $\mathcal{X}$ is the domain, and $\mathcal{Y}$ is the range. Let $F(k, \cdot)$ be an oracle for a uniformly chosen $k \in \mathcal{K}$ and $f(\cdot)$ be an oracle for a uniformly chosen function $f : \mathcal{X} \rightarrow \mathcal{Y}$. We say that a PRF is secure if for all efficient adversaries $\mathcal{A}$ the advantage $\mathrm{Adv}_{\mathcal{A}}^{PRF}(\lambda) = \big|\Pr[\mathcal{A}^{F(k,\cdot)} = 1] - \Pr[\mathcal{A}^{f(\cdot)} = 1]\big|$ is negligible.

2.6 Revocable HIBE

RHIBE is an extension of HIBE and it provides the revocation functionality in which each user can revoke child users if the private key of a child user is revealed. In RHIBE, each user additionally provides an update key UK per each time period and a child user can derive a (short-term) decryption key DK to decrypt a ciphertext by combining his (long-term) private key SK and the update key UK if he is not revoked in the update key. The syntax of RHIBE with history-free updates [28] is defined as follows:

Definition 2.1 (Revocable HIBE)**.** An RHIBE scheme with history-free updates for the identity space $\mathcal{I}$, the time space $\mathcal{V}$, and the message space $\mathcal{M}$, consists of seven algorithms **Setup**, **GenKey**, **UpdateKey**, **DeriveKey**, **Encrypt**, **Decrypt**, and **Revoke**, which are defined as follows:

Setup$(1^\lambda, L, N_{max})$: This algorithm takes as input a security parameter 1^λ and the maximum number N_{max} of users in each level. It outputs a master key MK, an (empty) revocation list RL_ε, a state ST_ε, and public parameters PP.

GenKey$(ID|_k, ST_{ID|_{k-1}}, PP)$: This algorithm takes as input a hierarchical identity $ID|_k = (I_1, \ldots, I_k) \in \mathcal{I}^k$, the state $ST_{ID|_{k-1}}$, and public parameters PP. It outputs a private key $SK_{ID|_k}$.

UpdateKey$(T, RL_{ID|_{k-1}}, DK_{ID|_{k-1},T}, ST_{ID|_{k-1}}, PP)$: This algorithm takes as input time $T \in \mathcal{V}$, a revocation list $RL_{ID|_{k-1}}$, a decryption key $DK_{ID|_{k-1},T}$, and public parameters PP. It outputs an update key $UK_{ID|_{k-1},T}$.

DeriveKey$(SK_{ID|_k}, UK_{ID|_{k-1},T}, PP)$: This algorithm takes as input a private key $SK_{ID|_k}$ for a hierarchical identity $ID|_k$, an update key $UK_{ID|_{k-1},T}$ for time T, and the public parameters PP. It outputs a decryption key $DK_{ID|_k,T}$.

Encrypt$(ID|_\ell, T, M, PP)$: This algorithm takes as input a hierarchical identity $ID|_\ell = (I_1, \ldots, I_\ell) \in \mathcal{I}^\ell$, time T, a message M, and the public parameters PP. It outputs a ciphertext $CT_{ID|_\ell,T}$.

Decrypt$(CT_{ID|_\ell,T}, DK_{ID'|_k,T'}, PP)$: This algorithm takes as input a ciphertext $CT_{ID|_\ell,T}$, a decryption key $DK_{ID'|_k,T'}$ and the public parameters PP. It outputs an encrypted message M.

Revoke$(ID|_k, T, RL_{ID|_{k-1}}, ST_{ID|_{k-1}})$: This algorithm takes as input a hierarchical identity $ID|_k$, revocation time T, a revocation list $RL_{ID|_{k-1}}$, and a state $ST_{ID|_{k-1}}$. It updates the revocation list $RL_{ID|_{k-1}}$.

The correctness of RHIBE is defined as follows: For all MK and PP generated by **Setup**$(1^\lambda, L, N_{max})$, $SK_{ID|_k}$ generated by **GenKey**$(ID|_k, ST|_{k-1}, PP)$, $UK_{ID|_{k-1},T}$ generated by **UpdateKey**$(T, RL_{ID|_{k-1}}, DK_{ID|_{k-1},T}, ST_{ID|_{k-1}}, PP)$, and $CT_{ID|_\ell,T}$ generated by **Encrypt**$(ID|_\ell, T, M, PP)$, it is required that

- If $ID|_k \in \mathrm{Prefix}(ID|_\ell)$, $ID|_k \notin RL_{ID|_{k-1}}$, and $T = T'$, then **Decrypt**$(CT_{ID|_\ell,T}$, **DeriveKey**$(SK_{ID|_k}, UK_{ID|_{k-1},T}, PP), PP) = M$.

The adaptive security model of RHIBE can be defined by extending the adaptive security model of RIBE. We use the adaptive model of RHIBE by extending the selective model of Seo and Emura [25]. In the adaptive security model of RHIBE, an adversary can adaptively request a private key query for any ID and an update key query for time T. In the challenge step, the adversary selects the challenge identity ID^* and challenge time T^*, and two challenge messages M_0^*, M_1^* with some restrictions. After receiving the challenge ciphertext, the adversary guesses the encrypted message in the challenge ciphertext. The formal security definition of RHIBE is given as follows:

Definition 2.2 (Adaptive IND-CPA Security)**.** The adaptive IND-CPA security (AD-IND-CPA) of RHIBE is defined in terms of the following experiment between a challenger $\mathcal{C}$ and a PPT adversary $\mathcal{A}$:

1. **Setup**: $\mathcal{C}$ obtains a master key MK, a revocation list RL_ε, a state ST_ε, and public parameters PP by running **Setup**$(1^\lambda, L, N_{max})$. It keeps $MK, RL_\varepsilon, ST_\varepsilon$ to itself and gives PP to $\mathcal{A}$.

2. **Phase 1**: $\mathcal{A}$ adaptively requests a polynomial number of queries. These queries are processed as follows:

   - Create key: If it is a create key query for a hierarchical identity $ID|_k$, then $\mathcal{C}$ creates a private key $SK_{ID|_k}$ by running **GenKey**$(ID|_k, ST_{ID|_{k-1}}, PP)$ with the restriction that the private key $SK_{ID|_{k-1}}$ was already created.
   - Private key: If it is a private key query for a hierarchical identity $ID|_k$, then $\mathcal{C}$ reveals the private key $SK_{ID|_k}$ that was already created.
   - Update key: If it is an update key query for a hierarchical identity $ID|_{k-1}$ and time T, then $\mathcal{C}$ gives an update key $UK_{ID|_{k-1},T}$ by running **UpdateKey**$(T, RL_{ID|_{k-1}}, DK_{ID|_{k-1},T}, ST_{ID|_{k-1}}, PP)$ with the restrictions that $SK_{ID|_{k-1}}$ was already created and $ID|_{k-1}$ or one of its ancestor was not revoked on time T. Although we described this update key as a key query, we can assume that all update keys for created private keys are broadcasted to $\mathcal{A}$.
   - Decryption key: If it is a decryption key query for a hierarchical identity $ID|_k$ and time T, then $\mathcal{C}$ gives a decryption key $DK_{ID|_k,T}$ by running **DeriveKey**$(SK_{ID|_k}, UK_{ID|_{k-1},T}, PP)$ with the restriction that $SK_{ID|_{k-1}}$ was already created and $ID_{ID|_k}$ is not revoked in $UK_{ID|_{k-1},T}$.
   - Revocation: If it is a revocation query for a hierarchical identity $ID|_k$ and time T, then $\mathcal{C}$ updates a revocation list $RL_{ID|_{k-1}}$ by running **Revoke**$(ID|_k, T, RL_{ID|_{k-1}}, ST_{ID|_{k-1}})$ with the restriction: A revocation query for $ID|_k$ on time T cannot be requested if an update key query for $ID|_k$ on the time T was requested.

   Note that we assume that update key, decryption key, and revocation queries are requested in non-decreasing order of time.

3. **Challenge**: $\mathcal{A}$ submits a challenge hierarchical identity $ID^*|_\ell = (I_1^*, \ldots, I_\ell^*)$, challenge time T^*, and two challenge messages M_0^*, M_1^* with the following restrictions:

   - If a private key query for $ID|_k \in \text{Prefix}(ID^*|_\ell)$ was requested, then $ID|_k$ or one of its ancestors must be revoked at some time $T \leq T^*$.
   - A decryption key query for $ID|_k \in \text{Prefix}(ID^*|_\ell)$ on the challenge time T^* was not requested.

   $\mathcal{C}$ flips a random coin $\mu \in \{0,1\}$ and gives the challenge ciphertext $CT^*_{ID^*|_\ell,T^*}$ to $\mathcal{A}$ by running **Encrypt**$(ID^*|_\ell, T^*, M_\mu^*, PP)$.

4. **Phase 2**: $\mathcal{A}$ may continue to request a polynomial number of queries subject to the restrictions of the challenge step.

5. **Guess**: Finally, $\mathcal{A}$ outputs a guess $\mu' \in \{0,1\}$, and wins the game if $\mu = \mu'$.

The advantage of $\mathcal{A}$ is defined as $\mathrm{Adv}^{AD\text{-}IND\text{-}CPA}_{RHIBE,\mathcal{A}}(\lambda) = \left|\Pr[\mu = \mu'] - \frac{1}{2}\right|$ where the probability is taken over all the randomness of the experiment. An RHIBE scheme is AD-IND-CPA secure if for all probabilistic polynomial-time (PPT) adversary $\mathcal{A}$, the advantage of $\mathcal{A}$ in the above experiment is negligible in the security parameter λ.

Remark 1. The stronger security model of RHIBE is the insider security model that considers internal attackers. The insider security model was introduced by Seo and Emura [28], and this model allows the exposure of state information in addition to the private key in the private key query. The security model of this paper does not take into account the insider security since our RHIBE scheme does not provide the insider security.

3 Revocable HIBE with Complete Subtree

In this section, we propose an RHIBE scheme via the complete subtree method and prove its adaptive security under simple static assumptions.

3.1 Complete Subtree Method

The complete subtree (CS) method is a specific instance of the subset cover framework of Naor et al. [22]. We follow the definition the CS method in the work of Lee and Park [19].

CS.Setup(N_{max})**:** Let $N_{max} = 2^n$ for simplicity. It first sets a perfect binary tree $\mathcal{BT}$ of depth n. Each user is assigned to a different leaf node in $\mathcal{BT}$. The collection $\mathcal{S}$ is defined as $\{S_i\}$ where S_i is the set of all leaves in a subtree $\mathcal{T}_i$ with a subroot $v_i \in \mathcal{BT}$. It outputs the binary tree $\mathcal{BT}$.

CS.Assign$(\mathcal{BT}, v)$**:** Let v be a leaf node of $\mathcal{BT}$ that is assigned to a user ID. Let $(v_{k_0}, v_{k_1}, \ldots, v_{k_n})$ be the path from the root node $v_{k_0} = v_0$ to the leaf node $v_{k_n} = v$. It initializes a private set PV as an empty one. For all $j \in \{k_0, \ldots, k_n\}$, it adds S_j into PV. It outputs the private set $PV = \{S_j\}$.

CS.Cover$(\mathcal{BT}, R)$**:** It first computes the Steiner tree $\mathcal{ST}_R$. Let $\mathcal{T}_{k_1}, \ldots \mathcal{T}_{k_m}$ be all the subtrees of $\mathcal{BT}$ that hang off $\mathcal{ST}_R$, that is all subtrees whose roots $v_{k_1}, \ldots v_{k_m}$ are not in $\mathcal{ST}_R$ but adjacent to nodes of outdegree 1 in $\mathcal{ST}_R$. It initializes a cover set CV as an empty one. For all $i \in \{k_1, \ldots, k_m\}$, it adds S_i into CV. It outputs the cover set $CV = \{S_i\}$.

CS.Match(CV, PV)**:** It finds a common subset S_k with $S_k \in CV$ and $S_k \in PV$. If there exists a common subset, it outputs (S_k, S_k). Otherwise, it outputs $\perp$.

The correctness of the CS scheme requires that if $v \notin R$, then **CS.Match**$(CV, PV) = (S_k, S_k)$ for the same S_k where CV and PV are associated with R and v respectively.

Lemma 3.1 ([22])**.** *In the CS method, the size of a private set is $O(\log N_{max})$ and the size of a cover set is $O(r \log(N_{max}/r))$ where N_{max} is the maximum number of leaf nodes and r is the size of revoked users R.*

3.2 Construction

To build an RHIBE-CS scheme, we follow the design strategy of Lee and Park [19]. That is, we construct an RHIBE-CS scheme by combining HIBE and IBE schemes with special properties and the CS method. For our construction, we use the LW-HIBE scheme in composite-order bilinear groups as the underlying HIBE scheme for our RHIBE scheme. The LW-HIBE scheme has short ciphertexts similar to the BBG-HIBE scheme [5], but it is fully secure under static assumptions [20]. Lee and Park [19] also pointed out that the BBG-HIBE scheme also can be used to build a selectively secure RHIBE scheme. Our RHIBE-CS scheme is similar to that of Lee and Park [19] except that it uses composite-order bilinear groups and the underlying HIBE and IBE schemes are replaced by the HIBE and IBE schemes of Lewko and Waters [20]. However, we prove the adaptive security of our RHIBE scheme.

Let PRF be a pseudo-random function for $\mathcal{K} = \{0,1\}^\lambda$, $\mathcal{X} = \{0,1\}^*$, and $\mathcal{Y} = \mathbb{Z}_N$. Our RHIBE scheme for $\mathcal{I} = \mathbb{Z}_N$, $\mathcal{V} = \mathbb{Z}_N$, and $\mathcal{M} \in \mathbb{G}_T$ is described as follows:

RHIBE-CS.Setup$(1^\lambda, L, N_{max})$**:** Let λ be a security parameter, L be the maximum depth of a hierarchical identity, and N_{max} be the maximum number of users for each level.

1. It first generates bilinear groups $\mathbb{G}, \mathbb{G}_T$ of composite order $N = p_1 p_2 p_3$ where p_1, p_2, and p_3 are random primes. It selects random generators g_1, g_3 of $\mathbb{G}_{p_1}, \mathbb{G}_{p_3}$ respectively.
2. It selects a random exponent $\alpha \in \mathbb{Z}_N$ and chooses random elements $h, u_1, \ldots, u_L, v, w \in \mathbb{G}_{p_1}$. It outputs a master key $MK = \alpha$ and public parameters

$$PP = \Big((N, \mathbb{G}, \mathbb{G}_T, e),\ g = g_1, Y = g_3, h, u_1, \ldots, u_L,\ v, w,\ \Omega = e(g,g)^\alpha, N_{max} \Big).$$

We define $F(ID|_k) = (h \prod_{i=1}^k u_i^{I_i})$ for $ID|_k = (I_1, \ldots, I_k)$ and use the notation $SK_{ID|_0} = MK$.

RHIBE-CS.GenKey$(ID|_k, ST_{ID|_{k-1}}, PP)$**:** Let $ID|_k = (I_1, \ldots, I_k) \in \mathbb{Z}_N^k$ be a hierarchical identity with $k \geq 1$ and $ST_{ID|_{k-1}}$ be a state information.

1. If $ST_{ID|_{k-1}}$ is empty, then it obtains $\mathcal{BT}_{ID|_{k-1}}$ by running **CS.Setup**(N_{max}) and selects a random exponent $\beta_{ID|_{k-1}} \in \mathbb{Z}_N$ and a PRF key $z_{ID|_{k-1}} \in \mathcal{K}$. It sets $ST_{ID|_{k-1}} = (\mathcal{BT}_{ID|_{k-1}}, \beta_{ID|_{k-1}}, z_{ID|_{k-1}})$.
2. It assigns $ID|_k$ to a random leaf node $v \in \mathcal{BT}_{ID|_{k-1}}$ and obtains a private set $PV = \{S_j\}$ by running **CS.Assign**$(\mathcal{BT}_{ID|_{k-1}}, v)$.
   For each $S_j \in PV$, it proceeds as follows: It computes $\gamma_j = \mathbf{PRF}(z_{ID|_{k-1}}, L_j)$ where $L_j = \text{Label}(S_j)$. It selects random $r \in \mathbb{Z}_N$, $Y_0, Y_1, Y_{2,k+1}, \ldots, Y_{2,L} \in \mathbb{G}_{p_3}$ and creates an HIBE key

$$SK_{HIBE,S_j} = \Big(K_0 = g^{\gamma_j} F(ID|_k)^r Y_0,\ K_1 = g^{-r} Y_1,\ \big\{ K_{2,i} = u_i^r Y_{2,i} \big\}_{i=k+1}^L \Big).$$

3. Finally, it outputs a private key $SK_{ID|_k} = \big(PV, \{SK_{HIBE,S_j}\}_{S_j \in PV} \big)$ where the master key part of SK_{HIBE,S_j} is γ_j.

RHIBE-CS.UpdateKey$(T, RL_{ID|_{k-1}}, DK_{ID|_{k-1},T}, ST_{ID|_{k-1}}, PP)$**:** Let $DK_{ID|_{k-1},T} = (DK_{HIBE}, DK_{IBE})$ be a decryption key.

1. If $ST_{ID|_{k-1}}$ is empty, then it obtains $\mathcal{BT}_{ID|_{k-1}}$ by running **CS.Setup**(N_{max}) and selects a random exponent $\beta_{ID|_{k-1}} \in \mathbb{Z}_N$ and a PRF key $z_{ID|_{k-1}} \in \mathcal{K}$. It sets $ST_{ID|_{k-1}} = (\mathcal{BT}_{ID|_{k-1}}, \beta_{ID|_{k-1}}, z_{ID|_{k-1}})$.

2. It derives the set of revoked nodes R at time T from $RL_{ID|_{k-1}}$ and obtains a cover set $CV = \{S_i\}$ by running **CS.Cover**$(\mathcal{BT}_{ID|_{k-1}}, R)$.

   For each $S_i \in CV$, it proceeds as follows: It computes $\gamma_i = \mathbf{PRF}(z_{ID|_{k-1}}, L_i)$ where $L_i = \text{Label}(S_i)$. It selects random $r \in \mathbb{Z}_N$, $Y_0, Y_1 \in \mathbb{G}_{p_3}$ and creates an IBE key

$$SK_{IBE,S_i} = \Big(U_0 = g^{\beta_{ID|_{k-1}} - \gamma_i}(vw^T)^r Y_0,\ U_1 = g^{-r} Y_1\Big).$$

3. Let $DK_{HIBE} = (D_0, D_1, \{D_{2,i}\})$ and $DK_{IBE} = (V_0, V_1)$ where the master key parts are η and $\alpha - \eta$ respectively. It chooses a random exponent $\eta' \in \mathbb{Z}_N$ and creates temporal blinded HIBE and IBE keys

$$TBK_{HIBE} = \Big(A'_0 = D_0 \cdot g^{\eta'},\ A'_1 = D_1,\ \{A'_{2,i} = D_{2,i}\}_{i=k}^{L}\Big)$$
$$TBK_{IBE} = \Big(B'_0 = V_0 \cdot g^{-\beta_{ID|_{k-1}} - \eta'},\ B'_1 = V_1\Big).$$

4. Next, it chooses random $r', r'' \in \mathbb{Z}_N$, $Y'_0, Y'_1, Y'_{2,k}, \ldots, Y'_{2,L}, Y''_0, Y''_1 \in \mathbb{G}_{p_3}$ and randomizes blinded HIBE and IBE keys

$$BK_{HIBE} = \Big(A_0 = A'_0 \cdot F(ID|_{k-1})^{r'} Y'_0,\ A_1 = A'_1 \cdot g^{-r'} Y'_1,\ \{A_{2,i} = A'_{2,i} \cdot u_i^{r'} Y'_{2,i}\}_{i=k}^{L}\Big)$$
$$BK_{IBE} = \Big(B_0 = B'_0 \cdot (vw^T)^{r''} Y''_0,\ B_1 = B'_1 \cdot g^{-r''} Y''_1\Big).$$

5. Finally, it outputs an update key $UK_{ID|_{k-1},T} = \big(CV, \{SK_{IBE,S_i}\}_{S_i \in CV}, BK_{ID|_{k-1},T} = (BK_{HIBE}, BK_{IBE})\big)$ where the master key parts of SK_{IBE,S_i}, BK_{HIBE}, and BK_{IBE} are $\beta_{ID|_{k-1}} - \gamma_i$, η'', and $\alpha - \beta_{ID|_{k-1}} - \eta''$ for random $\eta'' = \eta + \eta'$ respectively.

RHIBE-CS.DeriveKey$(ID|_k, T, SK_{ID|_k}, UK_{ID|_{k-1},T}, PP)$**:** Let $ID|_k = (I_1, \ldots, I_k)$ with $k \geq 0$, $SK_{ID|_k} = (PV, \{SK_{HIBE,S_j}\}_{S_j \in PV})$, and $UK_{ID|_{k-1},T} = (CV, \{SK_{IBE,S_i}\}_{S_i \in CV}, BK_{ID|_{k-1},T})$ where $BK_{ID|_{k-1},T} = (BK_{HIBE}, BK_{IBE})$.

If $k = 0$, then $SK_{ID|_0} = MK = \alpha$ and $UK_{ID|_{-1},T}$ is empty. It proceeds as follows:

1. It first selects a random exponent $\eta \in \mathbb{Z}_N$. It chooses random $r, r' \in \mathbb{Z}_N$, $Y_0, Y_1, Y_{2,1}, \ldots, Y_{2,L}$, $Y'_0, Y'_1 \in \mathbb{G}_{p_3}$ and creates HIBE and IBE keys

$$DK_{HIBE} = \Big(D_0 = g^{\eta}(h)^r Y_0,\ D_1 = g^{-r} Y_1,\ \{D_{2,i} = u_i^r Y_{2,i}\}_{i=1}^{L}\Big),$$
$$DK_{IBE} = \Big(V_0 = g^{\alpha - \eta}(vw^T)^{r'} Y'_0,\ V_1 = g^{-r'} Y'_1\Big).$$

2. It outputs a decryption key $DK_{ID|_0,T} = \big(DK_{HIBE}, DK_{IBE}\big)$.

If $k \geq 1$, then it proceeds as follows:

1. It first obtains (S_i, S_i) by running **CS.Match**(CV, PV). If it fails, it outputs $\perp$. It then retrieves $SK_{HIBE,S_i} = (K_0, K_1, \{K_{2,i}\})$ from $SK_{ID|_k}$ and $SK_{IBE,S_i} = (U_0, U_1)$ from $UK_{ID|_{k-1},T}$ where the master key parts are γ_i and $\beta_{ID|_{k-1}} - \gamma_i$ respectively.

2. Let $BK_{HIBE} = (A_0, A_1, \{A_{2,i}\})$ and $BK_{IBE} = (B_0, B_1)$ where the master key parts are η and $\alpha - \beta_{ID|_{k-1}} - \eta$ respectively. It chooses a random exponent $\eta' \in \mathbb{Z}_N$ and creates temporal HIBE and IBE keys by combining the retrieved keys as

$$TDK_{HIBE} = \Big(D'_0 = A_0 A_{2,k}^{I_k} K_0 \cdot g^{\eta'},\ D'_1 = A_1 K_1,\ \big\{D'_{2,i} = A_{2,i} K_{2,i}\big\}_{i=k+1}^{L}\Big),$$
$$TDK_{IBE} = \Big(V'_0 = B_0 U_0 \cdot g^{-\eta'},\ V'_1 = B_1 U_1\Big).$$

3. Next, it chooses random $r, r' \in \mathbb{Z}_N$, $Y_0, Y_1, Y_{2,k+1}, \ldots, Y_{2,L}$, $Y'_0, Y'_1 \in \mathbb{G}_{p_3}$ and creates randomized HIBE and IBE keys

$$DK_{HIBE} = \Big(D_0 = D'_0 \cdot F(ID|_k)^r Y_0,\ D_1 = D'_1 \cdot g^{-r} Y_1,\ \big\{D_{2,i} = D'_{2,i} \cdot u_i^r Y_{2,i}\big\}_{i=k+1}^{L}\Big),$$
$$DK_{IBE} = \Big(V_0 = V'_0 \cdot (vw^T)^{r'} Y'_0,\ V_1 = V'_1 \cdot g^{-r'} Y'_1\Big).$$

4. Finally, it outputs a decryption key $DK_{ID|_k,T} = \big(DK_{HIBE}, DK_{IBE}\big)$ where the master key parts of DK_{HIBE} and DK_{IBE} are η'' and $\alpha - \eta''$ for random $\eta'' = \eta + \gamma_i + \eta'$ respectively.

RHIBE-CS.Encrypt$(ID|_\ell, T, M, PP)$**:** Let $ID|_\ell = (I_1, \ldots, I_\ell) \in \mathcal{I}^\ell$ be a hierarchical identity with $\ell \geq 1$. It first chooses a random exponent $t \in \mathbb{Z}_N$ and creates HIBE and IBE ciphertext headers

$$CH_{HIBE} = \Big(C_0 = g^t,\ C_1 = F(ID|_\ell)^t\Big),\ CH_{IBE} = \Big(E_0 = g^t,\ E_1 = (vw^T)^t\Big).$$

It outputs a ciphertext $CT_{ID|_k,T} = \big(CH_{HIBE}, CH_{IBE}, C = \Omega^t \cdot M\big)$.

RHIBE-CS.Decrypt$(CT_{ID|_\ell,T}, DK_{ID'|_k,T'}, PP)$**:** Let $CT_{ID|_\ell,T} = (CH_{HIBE}, CH_{IBE}, C)$ and $DK_{ID'|_k,T'} = (DK_{HIBE}, DK_{IBE})$ where $CH_{HIBE} = (C_0, C_1)$, $CH_{IBE} = (E_0, E_1)$, $DK_{HIBE} = (D_0, D_1, \{D_{2,i}\})$, and $DK_{IBE} = (V_0, V_1)$. From the ciphertext header and the decryption key, it computes two session keys as

$$EK_{HIBE} = e(C_0, D_0 \prod_{i=k+1}^{\ell} D_{2,i}^{I_i}) \cdot e(C_1, D_1),\ EK_{IBE} = e(E_0, V_0) \cdot e(E_1, V_1).$$

It outputs a decrypted message $M = C \cdot (EK_{HIBE} \cdot EK_{IBE})^{-1}$.

RHIBE-CS.Revoke$(ID|_k, T, RL_{ID|_{k-1}}, ST_{ID|_{k-1}})$**:** If $ID|_k$ is not assigned in $\mathcal{BT}_{ID|_{k-1}}$, then it outputs $\perp$. Otherwise, it updates $RL_{ID|_{k-1}}$ by adding $(ID|_k, T)$ to $RL_{ID|_{k-1}}$.

3.3 Correctness

Let $SK_{ID|_k} = (PV, \{SK_{HIBE,S_j}\})$ be a private key generated by the **GenKey** algorithm. The HIBE key SK_{HIBE,S_j} is similar to the private key of LW-HIBE except that the master key part is γ_j. Let $UK_{ID|_{k-1},T} = (CV, \{SK_{IBE,S_i}\}, BK_{ID|_{k-1},T} = (BK_{HIBE}, BK_{IBE}))$ be an update key generated by the **UpdateKey** algorithm. The master key part of SK_{IBE,S_i} is $\beta_{ID|_{k-1}} - \gamma_i$. The master key parts of BK_{HIBE} and BK_{IBE} are η'' and $\alpha - \beta_{ID|_{k-1}} - \eta''$ respectively since the master key parts of DK_{HIBE} and DK_{IBE} are η and $\alpha - \eta$, and exponents η' and $-\beta_{ID|_{k-1}} - \eta'$ are added to the temporal keys.

We show that a decryption key $DK_{ID|_k,T}$ generated by the **DeriveKey** algorithm is correctly derived from $SK_{ID|_k}$ and $UK_{ID|_{k-1},T}$. If $(ID|_k, T') \notin RL_{ID|_{k-1}}$ for all $T' \leq T$, then the master key parts of SK_{HIBE,S_i}

and SK_{IBE,S_i} are associated with γ_i and $\beta_{ID|_{k-1}} - \gamma_i$ since these keys are related to the same tree node by the correctness of the CS scheme. The master key parts of BK_{HIBE} and BK_{IBE} are associated with η and $\alpha - \beta_{ID|_{k-1}} - \eta$. Thus, the master key part of TDK_{HIBE} and TDK_{IBE} are associated with $\eta + \gamma_i + \eta'$ and $(\alpha - \beta_{ID|_{k-1}} - \eta) + (\beta_{ID|_{k-1}} - \gamma_i) - \eta' = \alpha - \eta - \gamma_i - \eta'$ respectively. Since the master key parts of DK_{HIBE} and DK_{IBE} are same with that of TDK_{HIBE} and TDK_{IBE}, two master key parts of DK_{HIBE} and DK_{IBE} are η'' and $\alpha - \eta''$ for some random η'' respectively.

Next, we show that an original message M outputted by the **Decrypt** algorithm is correctly derived from $CT_{ID|_\ell,T} = (CH_{HIBE}, CH_{IBE})$ and $DK_{ID'|_k,T'} = (DK_{HIBE}, DK_{IBE})$. If $ID'|_k \in \text{Prefix}(ID|_\ell)$, then a partial session key EK_{HIBE} is correctly derived by the following equation

$$e(C_0, D_0 \prod_{i=k+1}^{\ell} D_{2,i}^{I_i}) \cdot e(C_1, D_1) = e(g^t, g^{\eta''}) \cdot e(g^t, F(ID|_\ell)^r) \cdot e(F(ID|_\ell)^t, g^{-r}) = e(g,g)^{t\eta''}$$

If $T' = T$, then another partial session key EK_{IBE} is correctly derived by the following equation

$$e(E_0, V_0) \cdot e(E_1, V_1) = e(g^t, g^{\alpha - \eta''}(vw^T)^r Y_0) \cdot e((vw^T)^t, g^{-r} Y_1) = e(g,g)^{t(\alpha - \eta'')}$$

By multiplying two partial session keys, we have $e(g,g)^{t\alpha}$.

3.4 Security Analysis

To prove the adaptive security of our RHIBE-CS scheme, we use the dual system encryption proof technique of Lewko and Waters [20]. As mentioned before, we simply cannot change normal private keys and normal update keys into semi-functional keys one by one through hybrid games. Instead, we divide private keys and update keys into small component keys and these small component keys are grouped together if they are related to the same node in a binary tree. The security proof is described as follows.

Theorem 3.2. *The above RHIBE-CS scheme is AD-IND-CPA secure if the SD, GSD, and ComDH assumptions hold.*

Proof. We first define the semi-functional type of HIBE private keys, HIBE ciphertext, IBE private keys, and IBE ciphertexts. For the semi-functional type, we let g_2 denote a fixed generator of the subgroup $\mathbb{G}_{p_2}$.

HIBE.SK-SF1. Let $SK'_{HIBE} = (K'_0, K'_1, \{K'_{2,i}\}_{i=k+1}^L)$ be a normal HIBE key for $ID|_k$. It chooses random exponents $a_0, b_0, \{z_i\}_{i=k+1}^L \in \mathbb{Z}_N$ and outputs a semi-functional-type1 HIBE key $SK_{HIBE} = \big(K_0 = K'_0 g_2^{a_0}, K_1 = K'_1 g_2^{-b_0}, \{K_{2,i} = K'_{2,i} g_2^{b_0 z_i}\}_{i=k+1}^L\big)$.

HIBE.SK-SF2. Let $SK'_{HIBE} = (K'_0, K'_1, \{K'_{2,i}\}_{i=k+1}^L)$ be a normal HIBE key. It chooses a random exponent $a_0 \in \mathbb{Z}_N$ and outputs a semi-functional-type2 HIBE key $SK_{HIBE} = \big(K_0 = K'_0 g_2^{a_0}, K_1 = K'_1, \{K_{2,i} = K'_{2,i}\}_{i=k+1}^L\big)$.

HIBE.SK-SF. Let $SK'_{HIBE} = (K'_0, K'_1, \{K'_{2,i}\}_{i=k+1}^L)$ be a normal HIBE key. Let $\delta_0 \in \mathbb{Z}_N$ be a fixed random exponent that will be defined in RHIBE. It outputs a semi-functional HIBE key $SK_{HIBE} = \big(K_0 = K'_0 g_2^{\delta_0}, K_1 = K'_1, \{K_{2,i} = K'_{2,i}\}_{i=k+1}^L\big)$.

HIBE.CH-SF. Let $CH'_{HIBE} = (C'_0, C'_1)$ be a normal ciphertext. It chooses random exponents $c, d_0 \in \mathbb{Z}_N$ and outputs a semi-functional ciphertext $CH_{HIBE} = \big(C_0 = C'_0 g_2^c, C_1 = C'_1 g_2^{cd_0}\big)$.

Note that if a semi-functional-type1 HIBE key are used to decrypt a semi-functional HIBE ciphertext, then an additional random element $e(g_2,g_2)^{c(a_0+\sum_{i=k+1}^{\ell} b_0 z_i I_i - b_0 d_0)}$ is left. If $a_0+\sum_{i=k+1}^{\ell} b_0 z_i I_i = b_0 d_0$, then this HIBE key is *nominally* semi-functional-type1.

IBE.SK-SF1. Let $SK'_{IBE} = (U'_0, U'_1)$ be a normal IBE key for T. It chooses random exponents $a_1, b_1 \in \mathbb{Z}_N$ and outputs a semi-functional-type1 IBE key $SK_{IBE} = \big(U_0 = U'_0 g_2^{a_1}, U_1 = U'_1 g_2^{-b_1}\big)$.

IBE.SK-SF2. Let $SK'_{IBE} = (U'_0, U'_1)$ be a normal IBE key. It chooses a random exponent $a_1 \in \mathbb{Z}_N$ and outputs a semi-functional-type2 IBE key $SK_{IBE} = \big(U_0 = U'_0 g_2^{a_1}, U_1 = U'_1\big)$.

IBE.SK-SF. Let $SK'_{IBE} = (U'_0, U'_1)$ be a normal IBE key. Let $\delta_1 \in \mathbb{Z}_N$ be a fixed random exponent that will be defined in RHIBE. It outputs a semi-functional IBE key $SK_{IBE} = \big(U_0 = U'_0 g_2^{\delta_1}, U_1 = U'_1\big)$.

IBE.CH-SF. Let $CH'_{IBE} = (E'_0, E'_1)$ be a normal ciphertext. It chooses random exponents $c, d_1 \in \mathbb{Z}_N$ and outputs a semi-functional ciphertext $CH_{IBE} = \big(E_0 = E'_0 g_2^{c}, E_1 = E'_1 g_2^{c d_1}\big)$.

Note that if a semi-functional-type1 IBE key is used to decrypt a semi-functional IBE ciphertext, then an additional random element $e(g_2,g_2)^{c(a_1-b_1 d_1)}$ is left. If $a_1 = b_1 d_1$, then this IBE key is *nominally* semi-functional type-1.

We now define the semi-functional types of private keys, update keys, decryption keys, and ciphertexts in RHIBE by using the semi-functional HIBE and IBE types.

RHIBE-CS.SK-SF. To generate a semi-functional private key, it proceeds as follows.

1. It first creates a normal private key $SK'_{ID|_k} = (PV, \{SK'_{HIBE,S_j}\}_{S_j \in PV})$ by using MK where each SK'_{HIBE,S_j} is a normal HIBE key.
2. For each $S_j \in PV$, it fixes a random exponent $\delta_{j,0} \in \mathbb{Z}_N$ once for $S_j \in \mathcal{BT}_{ID|_{k-1}}$ and converts SK'_{HIBE,S_j} to a semi-functional HIBE key SK_{HIBE,S_j} with the exponent $\delta_{j,0}$.
3. It outputs a semi-functional private key $SK_{ID|_k} = \big(PV, \{SK_{HIBE,S_j}\}_{S_j \in PV}\big)$.

RHIBE-CS.UK-SF. To generate a semi-functional update key, it proceeds as follows.

1. It first creates a normal update key $UK'_{ID|_{k-1},T} = (CV, \{SK'_{IBE,S_i}\}_{S_i \in CV}, BK'_{ID|_{k-1},T} = (BK'_{HIBE}, BK'_{IBE}))$ by using MK where BK'_{HIBE} is a normal HIBE key, BK'_{IBE} and SK'_{IBE,S_i} are normal IBE keys.
2. For each $S_i \in CV$, it fixes a random exponent $\delta_{i,1} \in \mathbb{Z}_N$ once for $S_i \in \mathcal{BT}_{ID|_{k-1}}$ and converts SK'_{IBE,S_i} to a semi-functional IBE key SK_{IBE,S_j} with the exponent $\delta_{i,1}$.
3. It chooses a random exponent $a_0 \in \mathbb{Z}_N$ and fixes a random exponent $a_{ID|_{k-1}} \in \mathbb{Z}_N$ for $\mathcal{BT}_{ID|_{k-1}}$. It converts BK'_{HIBE} to a semi-functional HIBE key BK_{HIBE} with the exponent a_0. It also converts $BK'_{IBE,T}$ to a semi-functional IBE key BK_{IBE} with the exponent $a_{ID|_{k-1}} - a_0$. It sets a semi-functional $BK_{ID|_{k-1},T} = (BK_{HIBE}, BK_{IBE})$.
4. It outputs a semi-functional update key $UK_{ID|_{k-1},T} = \big(CV, \{SK_{IBE,S_i}\}_{S_i \in CV}, BK_{ID|_{k-1},T}\big)$.

RHIBE-CS.DK-SF. To generate a semi-functional decryption key, it proceeds as follows.

1. It first creates a normal decryption key $DK'_{ID|_k,T} = (DK'_{HIBE}, DK'_{IBE})$ by using MK where DK'_{HIBE} is a normal HIBE key, DK'_{IBE} is a normal IBE key.

Table 2: Hybrid games from $\mathbf{G}_0$ to $\mathbf{G}_6$

Game	$CT =$ (CH_{HIBE}, CH_{IBE}, C)	$SK =$ $(\{SK_{HIBE}\})$	$UK =$ $(\{SK_{IBE}\}, BK_{HIBE}, BK_{IBE})$	$DK =$ (DK_{HIBE}, DK_{IBE})
$\mathbf{G}_0$	(N, N, N)	({N})	({N}, N, N)	(N, N)
$\mathbf{G}_1$	(N, N, N)	({N})	({N}, N, N)	(N, N)
$\mathbf{G}_2$	(SF, SF, N)	({N})	({N}, N, N)	(N, N)
$\mathbf{G}_3$	(SF, SF, N)	({SF})	({SF}, N, N)	(N, N)
$\mathbf{G}_4$	(SF, SF, N)	({SF})	({SF}, SF, SF)	(N, N)
$\mathbf{G}_5$	(SF, SF, N)	({SF})	({SF}, SF, SF)	(SF2, SF2)
$\mathbf{G}_6$	(SF, SF, R)	({SF})	({SF}, SF, SF)	(SF2, SF2)

We use symbols N for normal, SF2 for semi-functional-type2, SF for semi-functional, and R for random.

2. It chooses random exponents $a_0, a_1 \in \mathbb{Z}_N$. It converts DK'_{HIBE} to a semi-functional-type2 HIBE key DK_{HIBE} with the exponent a_0. It also converts DK'_{IBE} to a semi-functional-type2 IBE key DK_{IBE} with the exponent a_1.
3. It outputs a semi-functional decryption key $DK_{ID|_k,T} = (DK_{HIBE}, DK_{IBE})$.

RHIBE-CS.CT-SF. To generate a semi-functional ciphertext, it proceeds as follows.

1. It first creates a normal ciphertext $CT'_{ID|_\ell,T} = (CH'_{HIBE}, CH'_{IBE}, C')$ where CT'_{HIBE} is a normal HIBE ciphertext, CT'_{IBE} is a normal IBE ciphertext.
2. It chooses random exponents $c, d_0, d_1 \in \mathbb{Z}_N$. It converts $CH'_{HIBE,ID|_\ell}$ to a semi-functional CH_{HIBE} with the exponents c, d_0. It also converts CH'_{IBE} to a semi-functional $CH_{IBE,T}$ with the exponents c, d_1.
3. It outputs a semi-functional ciphertext $CT_{ID|_\ell,T} = (CH_{HIBE}, CH_{IBE}, C')$.

The security proof consists of a sequence of hybrid games $\mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_6$. The first game will be the original security game and the last one will be a game in which an adversary has no advantage. The structure of games is given in Table 2. We define the games as follows:

Game $\mathbf{G}_0$. This game is the original security game. In this game, all private keys, update keys, decryption keys and the challenge ciphertext are normal.

Game $\mathbf{G}_1$. In the game $\mathbf{G}_1$, the PRFs that are used in the generation of private keys and update keys are changed to be truly random functions.

Game $\mathbf{G}_2$. In this game, the challenge ciphertext is changed to be semi-functional. All other keys are still normal.

Game $\mathbf{G}_3$. Next, we define a new game $\mathbf{G}_3$. In this game, all private keys and all update keys are changed to be semi-functional. The process of changing from the game $\mathbf{G}_2$ to the game $\mathbf{G}_3$ is the essential part of this security proof and it consists of very complex steps. The detailed strategy of arguing that an adversary cannot distinguish the change of these games is given in Lemma 3.5.

Game $\mathbf{G}_4$. This game $\mathbf{G}_4$ is similar to the game $\mathbf{G}_3$ except that the remaining blinded keys in update keys are changed to be semi-functional.

Game $\mathbf{G}_5$. In this game $\mathbf{G}_5$, the remaining decryption keys are changed to be semi-functional. That is, all private keys, update keys, decryption keys, and the challenge ciphertext are now semi-functional.

Game $\mathbf{G}_6$. In the final game $\mathbf{G}_6$, the session key in the semi-functional challenge ciphertext is changed to be random. In this game, the adversary cannot distinguish the challenge messages since the session key is random.

Let $\mathrm{Adv}_{\mathcal{A}}^{G_j}$ be the advantage of $\mathcal{A}$ in the game $\mathbf{G}_j$. We have that $\mathrm{Adv}_{RHIBE,\mathcal{A}}^{AD\text{-}IND\text{-}CPA}(\lambda) = \mathrm{Adv}_{\mathcal{A}}^{G_0}$, and $\mathrm{Adv}_{\mathcal{A}}^{G_5} = 0$. From the following Lemmas 3.3, 3.4, 3.5, 3.7, 3.8, and 3.11, we obtain the equation

$$\begin{aligned}\mathrm{Adv}_{RHIBE,\mathcal{A}}^{AD\text{-}IND\text{-}CPA}(\lambda) &\leq \sum_{j=1}^{6} \left|\mathrm{Adv}_{\mathcal{A}}^{G_{j-1}} - \mathrm{Adv}_{\mathcal{A}}^{G_j}\right| \\ &\leq O(q_{sk}+q_{uk})\mathrm{Adv}_{\mathcal{B}}^{PRF}(\lambda) + \mathrm{Adv}_{\mathcal{B}}^{SD}(\lambda) + \\ &\quad O(q_{sk}\log N_{max} + q_{uk} r_{max} \log N_{max} + q_{dk})\mathrm{Adv}_{\mathcal{B}}^{GSD}(\lambda) + \mathrm{Adv}_{\mathcal{B}}^{ComDH}(\lambda)\end{aligned}$$

where q_{sk}, q_{uk}, and q_{dk} are the number of private key, update key, and decryption key queries respectively. This completes the proof. □

Lemma 3.3. *If the PRF is secure, then no probabilistic polynomial-time (PPT) adversary can distinguish $\mathbf{G}_0$ from $\mathbf{G}_1$ with a non-negligible advantage.*

This proof of Lemma 3.3 is relatively straightforward from the security of PRF. That is, we can use additional hybrid games that change a PRF to a truly random function. Note that there are at most $O(q_{sk} + q_{uk})$ number of binary trees in the security proof where q_{sk} is the number of private key queries and q_{uk} is the number of update key queries. We omit the proof of this lemma.

Lemma 3.4. *If the SD assumption holds, then no PPT adversary can distinguish $\mathbf{G}_1$ from $\mathbf{G}_2$ with a non-negligible advantage.*

Proof. Suppose there exists an adversary $\mathcal{A}$ that distinguishes $\mathbf{G}_1$ from $\mathbf{G}_2$ with a non-negligible advantage. A simulator $\mathcal{B}$ that solves the SD assumption using $\mathcal{A}$ is given: a challenge tuple $D = ((N,\mathbb{G},\mathbb{G}_T,e),g_1,g_3)$ and Z where $Z = Z_0 = X_1 \in \mathbb{G}_{p_1}$ or $Z = Z_1 = X_1R_1 \in \mathbb{G}_{p_1p_2}$. Then $\mathcal{B}$ that interacts with $\mathcal{A}$ is described as follows:

Setup: $\mathcal{B}$ first chooses random exponents $h', u'_1, \ldots, u'_L, v', w', \alpha \in \mathbb{Z}_N$. It sets $MK = \alpha$ and publishes $PP = \big((N,\mathbb{G},\mathbb{G}_T,e), g = g_1, Y = g_3, h = g^{h'}, u_1 = g^{u'_1}, \ldots, u_L = g^{u'_L}, v = g^{v'}, w = g^{w'}, \Omega = e(g,g)^{\alpha}\big)$.
Phase 1: $\mathcal{B}$ creates normal keys by running normal algorithms except that each γ_j is randomly chosen in $\mathbb{Z}_N$ instead of calculating it by running PRF.
Challenge: For challenge $ID^*|_\ell$ and T^*, $\mathcal{B}$ builds $CH_{HIBE} = \big(C_0 = Z, C_1 = (Z)^{h'+\sum_{i=1}^{\ell} u'_i I^*_i}\big)$ and $CH_{IBE} = \big(E_0 = Z, E_1 = (Z)^{v'+w'T^*}\big)$. Next, it flips a random coin $\mu \in \{0,1\}$ and creates a challenge ciphertext $CT^*_{ID^*|_\ell,T^*} = \big(CH_{HIBE}, CH_{IBE}, C = e(Z,g)^{\alpha} \cdot M^*_\mu\big)$.
Phase 2: Same as Phase 1.
Guess: $\mathcal{A}$ outputs a guess μ'. If $\mu = \mu'$, then $\mathcal{B}$ outputs 1. Otherwise, it outputs 0.

If $Z = Z_0 = X_1$, then the simulation is the same as $\mathbf{G}_1$. If $Z = Z_1 = X_1R_1$, then it is the same as $\mathbf{G}_2$ since the challenge ciphertext is semi-functional by implicitly setting $d_0 \equiv h' + \sum_{i=1}^{\ell} u'_i I^*_i \bmod p_2$ and $d_1 \equiv$

$v' + w'T^* \bmod p_2$. Note that d_0 and d_1 are random since $h', u'_1, \dots, u'_L, v', w'$ modulo p_2 are not correlated with their values modulo p_1 by the Chinese Remainder Theorem (CRT). This completes our proof. □

Lemma 3.5. *If the GSD assumption holds, then no PPT adversary can distinguish $\mathbf{G}_2$ from $\mathbf{G}_3$ with a non-negligible advantage.*

Proof. For the proof of this lemma, we cannot use simple hybrid games that change a normal private key (or normal update key) to a semi-functional private key (or semi-functional update key) one by one since the adversary of RHIBE can query a private key for $ID|_k \in \text{Prefix}(ID^*|_\ell)$ and an update key for T^*. Note that these normal keys cannot directly converted to semi-functional keys since an information theoretic argument cannot be used.

To solve this problem, we first divide each private key and update key into small HIBE keys and IBE keys. Recall that a private key $SK_{ID|_k}$ consists of many HIBE keys and an update key $UK_{ID|_{k-1},T}$ consists of many IBE keys and a blinded key where each HIBE key (or an IBE key) is associated with a node v_j (or a subset S_j) in $\mathcal{BT}_{ID|_{k-1}}$. Next, HIBE keys and IBE keys that are related to the same node v_j in $\mathcal{BT}_{ID|_{k-1}}$ are grouped together. To uniquely identify a node $v_j \in \mathcal{BT}_{ID|_{k-1}}$, we define a node identifier NID of this node as a string $ID|_{k-1} \| L_j$ where $L_j = \text{Label}(S_j)$. To prove this lemma, we change normal HIBE keys (or normal IBE keys) that are related to the same node identifier NID into semi-functional HIBE keys (or semi-functional IBE keys) by defining additional hybrid games. This additional hybrid games are performed for all node identifiers that are used in the key queries of the adversary.

For additional hybrid games that change HIBE keys (or IBE keys) that are related to the same node identifier $NID = ID|_{k-1} \| L_j$ from normal keys to semi-functional keys, we need to define an index pair (i_n, i_c) for an HIBE key (or an IBE key) that is related to the node $v_j \in \mathcal{BT}_{ID|_{k-1}}$ where i_n is a node index and i_c is a counter index. Suppose that an HIBE key (or an IBE key) is related to a node NID. The node index i_n for the HIBE key (or the IBE key) is assigned as follows: If the node $v_j \in \mathcal{BT}_{ID|_{k-1}}$ with a node identifier NID appears first time in key queries, then we set i_n as the number of distinct node identifiers in previous key queries plus one. If the node identifier NID already appeared before in key queries, then we set i_n as the value i'_n of previous HIBE key (or IBE key) with the same node identifier. The counter index i_c of an HIBE key is assigned as follows: If the node identifier NID appears first time in HIBE keys, then we set i_c as one. If the node identifier NID appeared before in HIBE keys, then we set i_c as the number of HIBE keys with the same node identifier that appeared before plus one. Similarly, we assigns the counter index i_c of an IBE key.

For the security proof, we define a sequence of additional hybrid games $\mathbf{G}_{2,1}, \dots, \mathbf{G}_{2,h}, \dots, \mathbf{G}_{2,q_n}$ where $\mathbf{G}_2 = \mathbf{G}_{2,0}$, $\mathbf{G}_3 = \mathbf{G}_{2,q_n}$, and q_n is the number of all node identifiers that are used in HIBE keys of private keys and IBE keys of update keys. The structure of hybrid games is given in Table 3. In the game $\mathbf{G}_{2,h}$ for $1 \le h \le q_n$, the challenge ciphertext is semi-functional, HIBE keys and IBE keys with a node index $i_n \le h$ are semi-functional, the remaining HIBE keys and IBE keys with a node index $i_n > h$ are normal, and all blinded keys in update keys are still normal.

Let $\text{Adv}_{\mathcal{A}}^{G_j}$ be the advantage of $\mathcal{A}$ in the game $\mathbf{G}_j$. From the following Lemma 3.6, we have the following equation

$$\text{Adv}_{\mathcal{A}}^{G_2} - \text{Adv}_{\mathcal{A}}^{G_3} \le \sum_{h=1}^{q_n} \left| \text{Adv}_{\mathcal{A}}^{G_{2,h-1}} - \text{Adv}_{\mathcal{A}}^{G_{2,h}} \right| \le O(q_{sk} \log N_{max} + q_{uk} r_{max} \log N_{max}) \text{Adv}_{\mathcal{B}}^{GSD}(\lambda)$$

where q_{sk} and q_{uk} are the number of private key and update key queries respectively. This completes the proof. □

Table 3: Hybrid games from $\mathbf{G}_2$ to $\mathbf{G}_3$

Game	$SK_{HIBE} \in SK$ or $SK_{IBE} \in UK$ with an index (i_n, i_c)						
	$i_n = 1$	$i_n = 2$	$\dots$	$i_n = h-1$	$i_n = h$	$\dots$	$i_n = q_n$
$\mathbf{G}_{2,0}$	N	N	$\dots$	N	N	$\dots$	N
$\mathbf{G}_{2,1}$	SF	N					
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\mathbf{G}_{2,h-1}$	SF	SF	$\dots$	SF	N	$\dots$	N
$\mathbf{G}_{2,h}$	SF	SF		SF	SF		N
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\mathbf{G}_{2,q_n}$	SF	SF	$\dots$	SF	SF	$\dots$	SF

We use symbols N for normal and SF for semi-functional.

Lemma 3.6. *If the GSD assumption holds, then no PPT adversary can distinguish $\mathbf{G}_{2,h-1}$ from $\mathbf{G}_{2,h}$ with a non-negligible advantage.*

Proof. We first divide the adversaries into two types based on the behavior of adversaries on the node index h: Type-h-I and Type-h-II. Let $ID^*|_\ell$ be the challenge hierarchical identity and T^* be the challenge time. For the node index h, an adversary can query HIBE keys for all ID or it can query HIBE key at least one ID. The adversary types are formally defined as follows:

Type-h-I. An adversary is Type-h-I if all HIBE keys with the node index h satisfy $ID|_k \notin \text{Prefix}(ID^*|_\ell)$ and at least one IBE key with the node index h satisfies $T = T^*$.

Type-h-II. An adversary is Type-h-II if all IBE keys with the node index h satisfy $T \neq T^*$. In this case, at least one HIBE key with h satisfies $ID|_k \in \text{Prefix}(ID^*|_\ell)$, or all HIBE keys with h satisfy $ID|_k \notin \text{Prefix}(ID^*|_\ell)$.

These two types cover all possible strategies of adversaries related to the node index h since the remaining case of at least one HIBE key satisfies $ID|_k \in \text{Prefix}(ID^*|_\ell)$ and at least one IBE key satisfies $T = T^*$ does not occur by the restriction of the security model. Note that Type-h-1 and Type-h-2 adversaries only check the conditions of the HIBE keys and IBE keys related to the node index h, and do not check the conditions of other node index. Thus, if the node index h is given, then there are only two possible types of adversaries.

We next show that this lemma holds for two types of the adversary. To guess the type of the adversary, we can simply toss a coin since there are only two types for the node index h. If an adversary is Type-h-I, then all HIBE keys with the node index h are changed to be semi-functional through hybrid games by using the restriction $ID|_k \notin \text{Prefix}(ID^*|_\ell)$. After that, the remaining IBE keys with h are change to be semi-functional. Note that there is no paradox of the dual system encryption when the remaining IBE keys are changed since HIBE keys are already semi-functional. If an adversary is Type-h-II, then all IBE keys with h are changed to be semi-functional by using the restriction $T \neq T^*$ and then the remaining HIBE keys are changed to be semi-functional.

For the Type-h-I adversary $\mathcal{A}_I$, we define hybrid games $\mathbf{H}_{1,1}, \mathbf{H}_{1,2}, \dots, \mathbf{H}_{q_c,1}, \mathbf{H}_{q_c,2}, \mathbf{H}'_{q_c,1}, \mathbf{H}'_{q_c,2}, \dots, \mathbf{H}'_{1,1}, \mathbf{H}'_{1,2}, \mathbf{H}''$ where $\mathbf{G}_{2,h-1} = \mathbf{H}_{0,2}$, $\mathbf{H}_{q_c,2} = \mathbf{H}'_{q_c+1,2}$, $\mathbf{H}'' = \mathbf{G}_{2,h}$, and q_c is the maximum number of HIBE keys

Table 4: Hybrid games from $\mathbf{G}_{2,h-1}$ to $\mathbf{G}_{2,h}$ for Type-h-I

Game	SK_{HIBE} with an index (h,i_c)						SK_{IBE} with
	$i_c=1$	$i_c=2$	$\dots$	$i_c=h_c$	$\dots$	$i_c=q_c$	an index (h,i_c)
$\mathbf{H}_{0,2}$	N						
$\mathbf{H}_{1,1}$	SF1	N	$\dots$	N	$\dots$	N	N
$\mathbf{H}_{1,2}$	SF2						
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
$\mathbf{H}_{h_c-1,2}$				N			
$\mathbf{H}_{h_c,1}$	SF2	SF2	$\dots$	SF1	$\dots$	N	N
$\mathbf{H}_{h_c,2}$				SF2			
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
$\mathbf{H}_{q_c-1,2}$						N	
$\mathbf{H}_{q_c,1}$						SF1	
$\mathbf{H}_{q_c,2}$	SF2	SF2	$\dots$	SF2	$\dots$	SF2	N
$\mathbf{H}'_{q_c,1}$						SF1$'$	
$\mathbf{H}'_{q_c,2}$						SF	
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
$\mathbf{H}'_{2,2}$	SF2						N
$\mathbf{H}'_{1,1}$	SF1$'$	SF	$\dots$	SF	$\dots$	SF	N
$\mathbf{H}'_{1,2}$	SF						N
$\mathbf{H}''$	SF						SF

We use symbols N for normal, SF1 for semi-functional-type1, SF2 for semi-functional-type2, SF1$'$ for semi-functional-type1 with an additional δ, and SF for semi-functional.

for the node index h. The structure of hybrid games is given in Table 4. These games are formally defined as follows:

Game $\mathbf{H}_{h_c,1}$. This game $\mathbf{H}_{h_c,1}$ for $1 \le h_c \le q_c$ is almost the same as $\mathbf{G}_{2,h-1}$ except the generation of HIBE keys and IBE keys with the node index h. An IBE key with an index pair (h,i_c) is generated as normal. An HIBE key with an index pair (h,i_c) is generated as follows:

- $i_c < h_c$: It generates a normal SK'_{HIBE,S_j} and converts this key to a semi-functional-type2 SK_{HIBE,S_j} by selecting a new random exponent $a_0 \in \mathbb{Z}_N$.
- $i_c = h_c$: It generates a normal SK'_{HIBE,S_j} for S_j and converts the key to a semi-functional-type1 SK_{HIBE,S_j} by selecting new random exponents $a_0, b_0, \{z_i\} \in \mathbb{Z}_N$.
- $i_c > h_c$: It simply creates a normal HIBE key.

Recall that if $a_0 + \sum_{i=k+1}^{\ell} b_0 z_i I_i = b_0 d_0$, then this HIBE key is *nominally* semi-functional-type1 where d_0 is the exponent of the challenge HIBE ciphertext.

Game $\mathbf{H}_{h_c,2}$. This game $\mathbf{H}_{h_c,2}$ is almost the same as $\mathbf{H}_{h_c,1}$ except that the HIBE key for the index pair $(h, i_c = h_c)$ is generated with $b_0 = 0$. That is, this HIBE key is generated as semi-functional-type2. In the game $\mathbf{H}_{q_c,2}$, all HIBE keys with the node index h are semi-functional-type2, but all IBE keys with the node index h are still normal.

Game $\mathbf{H}'_{h_c,1}$. This game $\mathbf{H}'_{h_c,1}$ is almost the same as $\mathbf{H}_{h_c,1}$ except the generation of an HIBE key with an index pair $(h, i_c \geq h_c)$. This HIBE key is generated as follows:

- $i_c = h_c$: It first generates a semi-functional-type1 $SK'_{HIBE,S_j} = (K'_0, K'_1, \{K'_{2,i}\})$ as the same as $\mathbf{H}_{h_c,1}$ with random exponents $a_0, b_0, \{z_i\}$. Let $\delta_{j,0} \in \mathbb{Z}_N$ be a random exponent which is fixed for the subset S_j. It creates a semi-functional-type1 HIBE key $SK_{HIBE,S_j} = \big(K_0 = K'_0 g_2^{\delta_{j,0}}, K_1 = K''_1, \{K_{2,i} = K''_{2,i}\}\big)$.
- $i_c > h_c$: It generates a semi-functional HIBE key with a fixed exponent $\delta_{j,0}$ which is chosen for the subset S_j.

Game $\mathbf{H}'_{h_c,2}$. This game $\mathbf{H}'_{h_c,2}$ is almost the same as $\mathbf{H}'_{h_c,1}$ except that the HIBE key with the index pair $(h, i_c = h_c)$ is generated with $b_0 = 0$. In the game $\mathbf{H}'_{1,2}$, all HIBE keys with the node index h are semi-functional where a fixed $\delta_{j,0}$ is used for a subset S_j, but all IBE keys with the node index h are still normal.

Game $\mathbf{H}''$. This game $\mathbf{H}''$ is the same as $\mathbf{G}_{2,h}$. Compared to the game $\mathbf{H}'_{1,2}$, all normal IBE keys with the node index h are changed to be semi-functional by using a fixed $\delta_{i,1}$ for a subset S_i.

Let $\mathrm{Adv}_{\mathcal{A}_I}^{H_i}$ be the advantage of $\mathcal{A}_I$ in a game $\mathbf{H}_i$. From the following Lemmas 3.12, 3.13, 3.14, 3.15, and 3.16, we obtain the following equation

$$
\begin{aligned}
\mathrm{Adv}_{\mathcal{A}_I}^{H_{0,2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H''} \leq & \sum_{h_c=1}^{q_c} \left|\mathrm{Adv}_{\mathcal{A}_I}^{H_{h_c-1,2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H_{h_c,1}}\right| + \sum_{h_c=1}^{q_c} \left|\mathrm{Adv}_{\mathcal{A}_I}^{H_{h_c,1}} - \mathrm{Adv}_{\mathcal{A}_I}^{H_{h_c,2}}\right| + \\
& \sum_{h_c=1}^{q_c} \left|\mathrm{Adv}_{\mathcal{A}_I}^{H'_{h_c+1,2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H'_{h_c,1}}\right| + \sum_{h_c=1}^{q_c} \left|\mathrm{Adv}_{\mathcal{A}_I}^{H'_{h_c,1}} - \mathrm{Adv}_{\mathcal{A}_I}^{H'_{h_c,2}}\right| + \left|\mathrm{Adv}_{\mathcal{A}_I}^{H'_{1,2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H''}\right| \\
\leq & \, O(q_c)\mathrm{Adv}_{\mathcal{B}}^{GSD}(\lambda).
\end{aligned}
$$

For the Type-h-II adversary $\mathcal{A}_{II}$, we define hybrid games $\mathbf{I}_{1,1}, \mathbf{I}_{1,2}, \ldots, \mathbf{I}_{q_c,1}, \mathbf{I}_{q_c,2}, \mathbf{I}'_{q_c,1}, \mathbf{I}'_{q_c,2}, \ldots, \mathbf{I}'_{1,1}, \mathbf{I}'_{1,2}, \mathbf{I}''$ where $\mathbf{G}_{2,h-1} = \mathbf{I}_{0,2}$, $\mathbf{I}'' = \mathbf{G}_{2,h}$, and q_c is the maximum number of IBE keys for the node index h. These games are formally defined as follows:

Game $\mathbf{I}_{h_c,1}$. This game $\mathbf{I}_{h_c,1}$ for $1 \leq h_c \leq q_c$ is almost the same as $\mathbf{G}_{1,h-1}$ except the generation of HIBE keys and IBE keys with the node index h. An HIBE key with an index pair (h, i_c) is generated as normal. An IBE key with an index pair (h, i_c) is generated as follows:

- $i_c < h_c$: It generates a normal SK'_{IBE,S_i} and converts this key to a semi-functional-type2 SK_{IBE,S_i} by selecting a new random exponent $a_1 \in \mathbb{Z}_N$.
- $i_c = h_c$: It generates a normal SK'_{IBE,S_i} and converts this key to a semi-functional-type1 SK_{IBE,S_i} by selecting new random exponents $a_1, b_1 \in \mathbb{Z}_N$.
- $i_c > h_c$: It simply creates a normal IBE key.

Recall that if $a_1 = b_1 d_1$, then this IBE key is *nominally* semi-functional-type1 where d_1 is the exponent of the challenge IBE ciphertext.

Game $\mathbf{I}_{h_c,2}$. This game $\mathbf{I}_{h_c,2}$ is almost the same as $\mathbf{I}_{h_c,1}$ except that the IBE key for the index pair $(h, i_c = h_c)$ is generated with $b_1 = 0$. That is, this IBE key is generated as semi-functional-type2. In the game $\mathbf{I}_{q_c,2}$, all IBE keys with the node index h are semi-functional-type2, but all HIBE keys with the node index h are still normal.

Game $\mathbf{I}'_{h_c,1}$. This game $\mathbf{I}'_{h_c,1}$ is almost the same as $\mathbf{I}_{h_c,1}$ except the generation of an IBE key with an index pair $(h, i_c \geq h_c)$. This IBE key is generated as follows:

- $i_c = h_c$: It first generates a semi-functional-type1 $SK'_{IBE,S_i} = (U'_0, U'_1)$ as the same as $\mathbf{I}_{h_c,1}$ with random exponents $a_1, b_1 \in \mathbb{Z}_N$. Let $\delta_{i,1} \in \mathbb{Z}_N$ be a random exponent fixed for the subset S_i. It creates a semi-functional-type1 IBE key $SK_{IBE,S_i} = \big(U_0 = U'_0 g_2^{\delta_{i,1}}, U_1 = U'_1\big)$.
- $i_c > h_c$: It generates a semi-functional IBE key with a fixed exponent $\delta_{i,1}$ which is chosen for the subset S_i.

Game $\mathbf{I}'_{h_c,2}$. This game $\mathbf{I}'_{h_c,2}$ is almost the same as $\mathbf{I}'_{h_c,1}$ except that the IBE key with the index pair $(h, i_c = h_c)$ is generated with $b_1 = 0$. This modification is similar to the game $\mathbf{I}'_{h_c,1}$. In the game $\mathbf{I}'_{1,2}$, all IBE keys with the node index h are semi-functional where a fixed $\delta_{i,1}$ is used for a subset S_i, but all HIBE keys with the node index h are still normal.

Game $\mathbf{I}''$. This game $\mathbf{I}''$ is the same as $\mathbf{G}_{2,h}$. Compared to the game $\mathbf{I}'_{1,2}$, all normal HIBE keys with the node index h are changed to be semi-functional by using a fixed $\delta_{j,0}$ for a subset S_j.

Let $\mathrm{Adv}^{I_i}_{\mathcal{A}_{II}}$ be the advantage of $\mathcal{A}_{II}$ in a game $\mathbf{I}_i$. From the following Lemmas 3.17, 3.18, 3.19, 3.20, and 3.21, we obtain the following equation

$$\begin{aligned}
\mathrm{Adv}^{I_{0,2}}_{\mathcal{A}_{II}} - \mathrm{Adv}^{I''}_{\mathcal{A}_{II}} \leq\ & \sum_{h_c=1}^{q_c} \left|\mathrm{Adv}^{I_{h_c-1,2}}_{\mathcal{A}_{II}} - \mathrm{Adv}^{I_{h_c,1}}_{\mathcal{A}_{II}}\right| + \sum_{h_c=1}^{q_c} \left|\mathrm{Adv}^{I_{h_c,1}}_{\mathcal{A}_{II}} - \mathrm{Adv}^{I_{h_c,2}}_{\mathcal{A}_{II}}\right| + \\
& \sum_{h_c=1}^{q_c} \left|\mathrm{Adv}^{I'_{h_c+1,2}}_{\mathcal{A}_{II}} - \mathrm{Adv}^{I'_{h_c,1}}_{\mathcal{A}_{II}}\right| + \sum_{h_c=1}^{q_c} \left|\mathrm{Adv}^{I'_{h_c,1}}_{\mathcal{A}_{II}} - \mathrm{Adv}^{I'_{h_c,2}}_{\mathcal{A}_{II}}\right| + \left|\mathrm{Adv}^{I'_{1,2}}_{\mathcal{A}_{II}} - \mathrm{Adv}^{I''}_{\mathcal{A}_{II}}\right| \\
\leq\ & O(q_c)\mathrm{Adv}^{GSD}_{\mathcal{B}}(\lambda).
\end{aligned}$$

This completes our proof. □

Lemma 3.7. *If the GSD assumption holds, then no PPT adversary can distinguish $\mathbf{G}_3$ from $\mathbf{G}_4$ with a non-negligible advantage.*

Proof. Suppose there exists an adversary $\mathcal{A}$ that distinguishes $\mathbf{G}_3$ from $\mathbf{G}_4$ with a non-negligible advantage. A simulator $\mathcal{B}$ that solves the GSD assumption using $\mathcal{A}$ is given: a challenge tuple $D = ((N, \mathbb{G}, \mathbb{G}_T, e), g_1, g_3, X_1R_1, R_2Y_1)$ and Z where $Z = Z_0 = X_2Y_2$ or $Z = Z_1 = X_2R_3Y_2$. Then $\mathcal{B}$ that interacts with $\mathcal{A}$ is described as follows:

Setup: $\mathcal{B}$ first chooses random exponents $h', u'_1, \ldots, u'_L, v', w', \alpha \in \mathbb{Z}_N$. It sets $MK = \alpha$ and publishes $PP = \big((N, \mathbb{G}, \mathbb{G}_T, e), g = g_1, Y = g_3, h = g^{h'}, u_1 = g^{u'_1}, \ldots, u_L = g^{u'_L}, v = g^{v'}, w = g^{w'}, \Omega = e(g,g)^{\alpha}\big)$.

Phase 1: For each query, $\mathcal{B}$ proceeds as follows: If this is a private key query, then it creates a semi-functional one by using R_2Y_1. That is, for each $S_j \in PV$, it builds a normal HIBE key and converts it to a semi-functional HIBE key by raising a fixed random exponent $\delta_{j,0} \in \mathbb{Z}_N$ to R_2Y_1.
If this is an update key query for $ID|_{k-1}$ and T, then it creates each component as follows:

- It first fixes a random exponent $\beta_{ID|_{k-1}} \in \mathbb{Z}_N$ for $\mathcal{BT}_{ID|_{k-1}}$.
- For each $S_i \in CV$, it builds a normal IBE key by using $\beta_{ID|_{k-1}}$ and converts it to a semi-functional IBE key by raising a fixed random exponent $\delta_{i,1}$ to $R_2 Y_1$.
- Next, it chooses random $\eta', r', r'' \in \mathbb{Z}_N$, $Y_0', Y_1', \{Y_{2,i}'\}_{i=k+1}^{L}, Y_0'', Y_1'' \in \mathbb{G}_{p_3}$ and builds

$$BK_{HIBE} = \big(A_0 = (Z)^{\eta'} F(ID|_{k-1})^{r'} Y_0', A_1 = g^{-r'} Y_1', \{A_{2,i} = u_i^{r'} Y_{2,i}'\}\big),$$
$$BK_{IBE} = \big(B_0 = g^{\alpha} (Z)^{-\beta_{ID|_{k-1}} - \eta'} (vw^T)^{r''} Y_0'', B_1 = g^{-r''} Y_1''\big).$$

  It creates a semi-functional blinded key $BK_{ID|_{k-1},T} = (BK_{HIBE}, BK_{IBE})$.

If this is a decryption key query, then it creates a normal one by using MK.
Challenge: For challenge $ID^*|_\ell$ and T^*, $\mathcal{B}$ builds $CH_{HIBE} = \big(C_0 = X_1 R_1,\ C_1 = (X_1 R_1)^{h' + \sum_{i=1}^{\ell} u_i' I_i^*}\big)$ and $CH_{IBE} = (E_0 = X_1 R_1,\ E_1 = (X_1 R_1)^{v' + w' T^*})$. Next, it flips a random coin $\mu \in \{0,1\}$ and creates a semi-functional $CT_{ID^*|_\ell, T^*} = (CH_{HIBE}, CH_{IBE}, C = e(X_1 R_1, g)^{\alpha} \cdot M_\mu^*)$.
Phase 2: Same as Phase 1.
Guess: $\mathcal{A}$ outputs a guess μ'. If $\mu = \mu'$, then $\mathcal{B}$ outputs 1. Otherwise, it outputs 0.

If $Z = Z_0 = X_2 Y_2$, then the simulation is the same as $\mathbf{G}_3$ since all blinded keys are normal. If $Z = Z_1 = X_2 R_3 Y_2$, then the simulation is the same as $\mathbf{G}_4$ since all blinded keys are semi-functional by implicitly setting $a_0 \equiv c\eta' \bmod p_2$ and $a_{ID|_{k-1}} - a_0 \equiv -c\beta_{ID|_{k-1}} - c\eta' \bmod p_2$ where $c = \log_{g_2}(R_3)$. In this case, the random η' is fresh one for each blinded key and two random values $\eta' \bmod p_2$ and $\beta_{ID|_{k-1}} \bmod p_2$ are independent of their values in modulo p_1 by CRT. Note that there is no paradox of dual system encryption since HIBE keys in a private key and IBE keys in an update key are already semi-functional. □

Lemma 3.8. *If the GSD assumption holds, then no PPT adversary can distinguish $\mathbf{G}_4$ from $\mathbf{G}_5$ with a non-negligible advantage.*

Proof. For the security proof, we define a sequence of hybrid games $\mathbf{G}_{4,1}, \mathbf{G}_{4,2}, \ldots, \mathbf{G}_{4,q_{dk}}$ where $\mathbf{G}_4 = \mathbf{G}_{4,0}$ and q_{dk} is the number of decryption key queries. The structure of hybrid games is given in Table 5. In the game $\mathbf{G}_{4,h_d}$ for $1 \leq h_d \leq q_{dk}$, all private keys, all update keys, and the challenge ciphertext are generated as semi-functional, but decryption keys are generated as follows: The first h_d decryption keys are generated as semi-functional and the remaining decryption keys are generated as normal.

To show that an adversary cannot distinguish $\mathbf{G}_{4,h_d-1}$ from $\mathbf{G}_{4,h_d}$, we additionally define games $\mathbf{J}_{h_d,1}, \mathbf{J}_{h_d,2}$ where $\mathbf{J}_{h_d,2} = \mathbf{G}_{4,h_d}$. These games are defined as follows:

Game $\mathbf{J}_{h_d,1}$. This game is almost similar to the game $\mathbf{G}_{4,h_d-1}$ except the generation of h_dth decryption key. Let $DK'_{HIBE} = (D_0', D_1', \{D_{2,i}'\})$ be a normal HIBE key and $DK'_{IBE} = (V_0', V_1')$ be a normal IBE key. The h_dth decryption key consists of a semi-functional-type1 HIBE key $DK_{HIBE} = (D_0 = D_0' g_2^{a_0}, D_1 = D_1' g_2^{-b_0}, \{D_{2,i} = D_{2,i}' g_2^{b_0 z_i}\})$ and a semi-functional-type1 IBE key $DK_{IBE} = (V_0 = V_0' g_2^{a_1}, V_1 = V_1' g_2^{-b_1})$ where $a_0, b_0, \{z_i\}, a_1, b_1$ are random exponents in $\mathbb{Z}_N$.

Game $\mathbf{J}_{h_d,2}$. In this game, the HIBE key and IBE key in the h_dth decryption key is changed to be semi-functional-type2. Recall that a decryption key is semi-functional if HIBE key and IBE key are semi-functional-type2. It is obvious that $\mathbf{J}_{h_d,2} = \mathbf{G}_{4,h_d}$.

Table 5: Hybrid games from $\mathbf{G}_4$ to $\mathbf{G}_5$

Game	$DK = (DK_{HIBE}, DK_{IBE})$ with an index i_d						
	$i_d = 1$	$i_d = 2$	…	$i_d = h_d - 1$	$i_d = h_d$	…	$i_d = q_{dk}$
$\mathbf{J}_{0,2}$	(N, N)	(N, N)					
$\mathbf{J}_{1,1}$	(SF1, SF1)	(N, N)	…	(N, N)	(N, N)	…	(N, N)
$\mathbf{J}_{1,2}$	(SF2, SF2)	(N, N)					
⋮	⋮	⋮		⋮	⋮		⋮
$\mathbf{J}_{h_d-1,1}$				(SF1, SF1)	(N, N)		
$\mathbf{J}_{h_d-1,2}$	(SF2, SF2)	(SF2, SF2)	…	(SF2, SF2)	(N, N)	…	(N, N)
$\mathbf{J}_{h_d,1}$				(SF2, SF2)	(SF1, SF1)		
$\mathbf{J}_{h_d,2}$					(SF2, SF2)		
⋮	⋮	⋮		⋮	⋮		⋮
$\mathbf{J}_{q_{dk},1}$	(SF2, SF2)	(SF2, SF2)	…	(SF2, SF2)	(SF2, SF2)	…	(SF1, SF1)
$\mathbf{J}_{q_{dk},2}$							(SF2, SF2)

We use symbols N for normal, SF1 for semi-functional-type1, and SF2 for semi-functional-type2.

Note that if a semi-functional-type1 decryption key is used to decrypt a semi-functional challenge ciphertext, then a random element $e(g_2, g_2)^{c(a_0+\sum_{i=k+1}^{\ell} b_0 z_i I_i + a_1 - b_0 d_0 - b_1 d_1)}$ is left where d_0, d_1 are random exponents in semi-functional CH_{HIBE}, CH_{IBE} respectively. If $a_0 + \sum_{i=k+1}^{\ell} b_0 z_i I_i + a_1 \equiv b_0 d_0 + b_1 d_1 \bmod p_2$, then this decryption key is *nominally* semi-functional-type1.

Let $\mathrm{Adv}_{\mathcal{A}}^{J_{h_d,i}}$ be the advantage of $\mathcal{A}$ in the game $\mathbf{J}_{h_d,i}$. We have that $\mathrm{Adv}_{\mathcal{A}}^{G_4} = \mathrm{Adv}_{\mathcal{A}}^{J_{0,2}}$ and $\mathrm{Adv}_{\mathcal{A}}^{G_5} = \mathrm{Adv}_{\mathcal{A}}^{J_{q_{dk},2}}$. From the following Lemmas 3.9 and 3.10, we obtain the following equation

$$\mathrm{Adv}_{\mathcal{A}}^{G_4} - \mathrm{Adv}_{\mathcal{A}}^{G_5} \leq \sum_{h_d=1}^{q_{dk}} \left(\left| \mathrm{Adv}_{\mathcal{A}}^{J_{h_d-1,2}} - \mathrm{Adv}_{\mathcal{A}}^{J_{h_d,1}} \right| + \left| \mathrm{Adv}_{\mathcal{A}}^{J_{h_d,1}} - \mathrm{Adv}_{\mathcal{A}}^{J_{h_d,2}} \right| \right) \leq O(q_{dk}) \mathrm{Adv}_{\mathcal{B}}^{GSD}(\lambda).$$

This completes our proof. □

Lemma 3.9. *If the GSD assumption holds, then no PPT adversary can distinguish $\mathbf{J}_{h_d-1,2}$ from $\mathbf{J}_{h_d,1}$ with a non-negligible advantage.*

Proof. Suppose there exists an adversary $\mathcal{A}$ that distinguishes $\mathbf{J}_{h_d-1,2}$ from $\mathbf{J}_{h_d,1}$ with a non-negligible advantage. A simulator $\mathcal{B}$ that solves the GSD assumption using $\mathcal{A}$ is given: a challenge tuple $D = ((N, \mathbb{G}, \mathbb{G}_T, e), g_1, g_3, X_1R_1, R_2Y_1)$ and Z where $Z = Z_0 = X_2Y_2$ or $Z = Z_1 = X_2R_3Y_2$. Then $\mathcal{B}$ that interacts with $\mathcal{A}$ is described as follows:

Setup: $\mathcal{B}$ first chooses random exponents $h', u'_1, \ldots, u'_L, v', w', \alpha \in \mathbb{Z}_N$. It sets $MK = \alpha$ and publishes $PP = \big((N, \mathbb{G}, \mathbb{G}_T, e), g = g_1, Y = g_3, h = g^{h'}, u_1 = g^{u'_1}, \ldots, u_L = g^{u'_L}, v = g^{v'}, w = g^{w'}, \Omega = e(g,g)^{\alpha}\big)$.

Phase 1: For each query, $\mathcal{B}$ proceeds as follows: If this is a private key (or update key) query, then it creates a semi-functional one by using MK and R_2Y_1.

If this is an h_dth decryption key query for $ID|_k$ and T, then it handles this query as follows:

- $j < h_d$: It creates a semi-functional decryption key by using MK and $R_2 Y_1$.
- $j = h_d$: It chooses random $\eta', r', r'' \in \mathbb{Z}_N$, $Y_0', Y_1', \{Y_{2,i}'\}_{i=k+1}^{L}, Y_0'', Y_1'' \in \mathbb{G}_{p_3}$ and builds

$$DK_{HIBE} = \big(D_0 = (Z)^{\eta' + (h' + \sum_{i=1}^{k} u_i' I_i) r'} Y_0', D_1 = (Z)^{-r'} Y_1', \{D_{2,i} = (Z)^{u_i' r'} Y_{2,i}'\}_{i=k+1}^{L}\big),$$
$$DK_{IBE} = \big(V_0 = g^{\alpha} (Z)^{-\eta' + (v' + w'T) r''} Y_0'', V_1 = (Z)^{-r''} Y_1''\big).$$

  It creates a decryption key $DK_{ID|_k, T} = (DK_{HIBE}, DK_{IBE})$.

- $j > h_d$: It creates a normal decryption key by using MK.

Challenge: For challenge $ID^*|_\ell$ and T^*, $\mathcal{B}$ builds $CH_{HIBE} = \big(C_0 = X_1 R_1, C_1 = (X_1 R_1)^{h' + \sum_{i=1}^{\ell} u_i' I_i^*}\big)$ and $CH_{IBE} = (E_0 = X_1 R_1, E_1 = (X_1 R_1)^{v' + w'T^*})$. Next, it flips a random coin $\mu \in \{0,1\}$ and creates a semi-functional $CT_{ID^*|_\ell, T^*} = (CH_{HIBE}, CH_{IBE}, C = e(X_1 R_1, g)^{\alpha} \cdot M_\mu^*)$.
Phase 2: Same as Phase 1.
Guess: $\mathcal{A}$ outputs a guess μ'. If $\mu = \mu'$, then $\mathcal{B}$ outputs 1. Otherwise, it outputs 0.

If $Z = Z_0 = X_2 Y_2$, then the simulation is the same as $\mathbf{J}_{h_d - 1, 2}$ since the h_dth decryption key is normal. If $Z = Z_1 = X_2 R_3 Y_2$, then the simulation is almost the same as $\mathbf{J}_{h_d, 1}$ except that the h_dth decryption key is nominally semi-functional-type1 by implicitly setting

$$\begin{aligned}
&a_0 \equiv c\eta' + cr'(h' + \textstyle\sum_{i=1}^{k} u_i' I_i) \bmod p_2,\ b_0 \equiv cr' \bmod p_2,\ z_i \equiv u_i' \bmod p_2,\\
&a_1 \equiv -c\eta' + cr''(v' + w'T) \bmod p_2,\ b_1 \equiv cr'' \bmod p_2,\\
&d_0 \equiv (h' + \textstyle\sum_{i=1}^{\ell} u_i' I_i^*) \bmod p_2,\\
&d_1 \equiv (v' + w'T^*) \bmod p_2
\end{aligned}$$

where $c = \log_{g_2}(R_3)$. If $ID|_k \in \text{Prefix}(ID^*|_\ell)$ and $T = T^*$, then the h_dth key is nominally semi-functional-type1 since the following equation holds

$$\begin{aligned}
&a_0 + \textstyle\sum_{i=k+1}^{\ell} b_0 z_i I_i + a_1\\
&\equiv \big(c\eta' + cr'(h' + \textstyle\sum_{i=1}^{k} u_i' I_i)\big) + \textstyle\sum_{i=k+1}^{\ell} cr' u_i' I_i + \big(-c\eta' + cr''(v' + w'T)\big)\\
&\equiv c\eta' + cr'(h' + \textstyle\sum_{i=1}^{\ell} u_i' I_i) - c\eta' + cr''(v' + w'T)\\
&\equiv (cr')(h' + \textstyle\sum_{i=1}^{\ell} u_i' I_i) + (cr'')(v' + w'T)\\
&\equiv b_0 d_0 + b_1 d_1 \bmod p_2.
\end{aligned}$$

Note that we solve the paradox of dual system encryption by introducing the nominally semi-functional decryption key.

To finish the proof, we should argue that the adversary cannot distinguish a nominally semi-functional decryption key from a semi-functional one. For this argument, we can easily show an information theoretic argument by using the restriction of a decryption key query in the security model and CRT. We omit the details of this argument since it is similar to that of Lemma 3.12. □

Lemma 3.10. *If the GSD assumption holds, then no PPT adversary can distinguish $\mathbf{J}_{h_d,1}$ from $\mathbf{J}_{h_d,2}$ with a non-negligible advantage.*

Proof. The proof of this lemma is almost the same as that of Lemma 3.9, except the generation of the h_dth decryption key. The h_dth decryption key for $ID|_k$ and T is generated as follows:

- $j = h_d$: It chooses random $\eta', r', r'', a'_0, a'_1 \in \mathbb{Z}_N$, $Y'_0, Y'_1, \{Y'_{2,i}\}_{i=k+1}^L, Y''_0, Y''_1 \in \mathbb{G}_{p_3}$ and builds

$$DK_{HIBE} = \big(D_0 = (Z)^{\eta' + (h' + \sum_{i=1}^k u'_i I_i) r'} (R_2 Y_1)^{a'_0} Y'_0, D_1 = (Z)^{-r'} Y'_1, \{D_{2,i} = (Z)^{u'_i r'} Y'_{2,i}\}_{i=k+1}^L\big),$$
$$DK_{IBE} = \big(V_0 = g^\alpha (Z)^{-\eta' + (v' + w'T) r''} (R_2 Y_1)^{a'_1} Y''_0, V_1 = (Z)^{-r''} Y''_1\big).$$

It creates $DK_{ID|_k, T} = (DK_{HIBE}, DK_{IBE})$.

Note that this h_dth decryption key is no longer correlated with the challenge ciphertext since D_0 and V_0 are randomized by using a'_0 and a'_1 respectively.

If $Z = Z_1 = X_2 R_3 Y_2$, then the simulation is the same as $\mathbf{J}_{h_d,1}$ by implicitly setting

$$a_0 \equiv c\eta' + cr'(h' + \textstyle\sum_{i=1}^k u'_i I_i) + c_2 a'_0 \bmod p_2,\ b_0 \equiv cr' \bmod p_2,\ z_i \equiv u'_i \bmod p_2,$$
$$a_1 \equiv -c\eta' + cr''(v' + w'T) + c_2 a'_1 \bmod p_2,\ b_1 \equiv cr'' \bmod p_2$$

where $c = \log_{g_2}(R_3)$ and $c_2 = \log_{g_2}(R_2)$. If $Z = Z_0 = X_2 Y_2$, then the simulation is the same as $\mathbf{J}_{h_d,2}$ by implicitly setting $a_0 \equiv c_2 a'_0 \bmod p_2$ and $a_1 \equiv c_2 a'_1 \bmod p_2$ where $c_2 = \log_{g_2}(R_2)$. □

Lemma 3.11. *If the ComDH assumption holds, then no PPT adversary can distinguish $\mathbf{G}_5$ from $\mathbf{G}_6$ with a non-negligible advantage.*

Proof. Suppose there exists an adversary $\mathcal{A}$ that distinguish $\mathbf{G}_5$ from $\mathbf{G}_6$ with a non-negligible advantage. A simulator $\mathcal{B}$ that solves the ComDH assumption using $\mathcal{A}$ is given: a challenge tuple $D = ((N, \mathbb{G}, \mathbb{G}_T, e), g_1, g_2, g_3, g_1^a R_1, g_1^b R_2)$ and Z where $Z = Z_0 = e(g_1, g_1)^{ab}$ or $Z = Z_1 = e(g_1, g_1)^c$. Then $\mathcal{B}$ that interacts with $\mathcal{A}$ is described as follows:

Setup: $\mathcal{B}$ first chooses random exponents $h', u'_1, \ldots, u'_L, v', w' \in \mathbb{Z}_N$. It implicitly sets $\alpha = a$ from $g_1^a R_1$ and publishes $PP = \big((N, \mathbb{G}, \mathbb{G}_T, e), g = g_1, Y = g_3, h = g^{h'}, u_1 = g^{u'_1}, \ldots, u_L = g^{u'_L}, v = g^{v'}, w = g^{w'}, \Omega = e(g, g_1^a R_1)\big)$.

Phase 1: For each query, $\mathcal{B}$ creates a semi-functional key by using $g_1^a R_1$ and g_2. Note that it cannot create a normal update key (and a normal decryption key) since g_1^a is not given.

Challenge: For challenge $ID^*|_\ell$ and T^*, $\mathcal{B}$ builds $CH_{HIBE} = \big(C_0 = g_1^b R_2, C_1 = (g_1^b R_2)^{h' + \sum_{i=1}^\ell u'_i I_i^*}\big)$ and $CH_{IBE} = (E_0 = g_1^b R_2, E_1 = (g_1^b R_2)^{v' + w'T^*})$. Next, it flips a random coin $\mu \in \{0,1\}$ and creates a challenger ciphertext $CT_{ID^*|_\ell, T^*} = (CH_{HIBE}, CH_{IBE}, C = Z \cdot M_\mu^*)$.

Phase 2: Same as Phase 1.

Guess: $\mathcal{A}$ outputs a guess μ'. If $\mu = \mu'$, then $\mathcal{B}$ outputs 1. Otherwise, it outputs 0.

If $Z = Z_0$, then the simulation is the same as $\mathbf{G}_5$. If $Z = Z_1$, then the simulation is the same as $\mathbf{G}_6$ since C is random. □

3.5 Type-h-I Adversary

Lemma 3.12. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish $\mathbf{H}_{h_c-1,2}$ from $\mathbf{H}_{h_c,1}$ with a non-negligible advantage.*

Proof. Suppose there exists an adversary $\mathcal{A}_I$ that distinguishes $\mathbf{H}_{h_c-1,2}$ from $\mathbf{H}_{h_c,1}$ with a non-negligible advantage. A simulator $\mathcal{B}$ that solves the GSD assumption using $\mathcal{A}_I$ is given: a challenge tuple $D = ((N, \mathbb{G}, \mathbb{G}_T, e), g_1, g_3, X_1 R_1, R_2 Y_1)$ and Z where $Z = Z_0 = X_2 Y_2$ or $Z = Z_1 = X_2 R_3 Y_2$. Then $\mathcal{B}$ that interacts with $\mathcal{A}_I$ is described as follows:

Setup: $\mathcal{B}$ first chooses random exponents $h', u'_1, \ldots, u'_L, v', w', \alpha \in \mathbb{Z}_N$. It sets $MK = \alpha$ and publishes $PP = \big((N, \mathbb{G}, \mathbb{G}_T, e), g = g_1, Y = g_3, h = g^{h'}, u_1 = g^{u'_1}, \ldots, u_L = g^{u'_L}, v = g^{v'}, w = g^{w'}, \Omega = e(g,g)^{\alpha}\big)$.

Phase 1: For each query, $\mathcal{B}$ proceeds as follows: If this is a decryption key query, then it creates a normal key.

If this is an HIBE key in a private key or an IBE key in an update key query with indexes (i_n, i_c), then $\mathcal{B}$ handles this key as follows:

- **Case** $i_n < h$: It builds a normal key by using MK and converts this key to a semi-functional one with fixed random exponents $\delta_{j,0}, \delta_{j,1} \in \mathbb{Z}_N$ for the subset S_j by using $R_2 Y_1$.
- **Case** $i_n = h$: If this is an IBE key, then it creates a normal IBE key by using MK. If this is an HIBE key, then it proceeds as follows:
  - $i_c < h_c$: It builds a normal HIBE key and converts this key to a semi-functional-type2 key by raising a random exponent $a_0 \in \mathbb{Z}_N$ to $R_2 Y_1$.
  - $i_c = h_c$: It chooses random elements $Y'_0, Y'_1, \{Y'_{2,i}\} \in \mathbb{G}_{p_3}$ and creates an HIBE key

    $$SK_{HIBE,S_j} = \big(K_0 = g^{\gamma_j} (Z)^{h' + \sum_{i=1}^{k} u'_i I_i} Y'_0,\ K_1 = (Z)^{-1} Y'_1,\ \{K_{2,i} = (Z)^{u'_i} Y'_{2,i}\}\big).$$

  - $i_c > h_c$: It creates a normal HIBE key by using MK.
- **Case** $i_n > h$: It creates a normal HIBE key or a normal IBE key.

Challenge: For challenge $ID^*|_\ell$ and T^*, $\mathcal{B}$ builds $CH_{HIBE} = \big(C_0 = X_1 R_1, C_1 = (X_1 R_1)^{h' + \sum_{i=1}^{\ell} u'_i I^*_i}\big)$ and $CH_{IBE} = \big(E_0 = X_1 R_1, E_1 = (X_1 R_1)^{v' + w' T^*}\big)$. Next, it flips a random coin $\mu \in \{0,1\}$ and creates a semi-functional $CT_{ID^*|_\ell, T^*} = (CH_{HIBE}, CH_{IBE}, C = e(X_1 R_1, g)^{\alpha} \cdot M^*_\mu)$.

Phase 2: Same as Phase 1.

Guess: $\mathcal{A}$ outputs a guess μ'. If $\mu = \mu'$, then $\mathcal{B}$ outputs 1. Otherwise, it outputs 0.

If $Z = Z_0 = X_2 Y_2$, then the simulation is the same as $\mathbf{H}_{h_c - 1, 2}$ since the HIBE key with $(i_n = h) \wedge (i_c = h_c)$ is normal. If $Z = Z_1 = X_2 R_3 Y_2$, then the simulation is almost the same as $\mathbf{H}_{h_c, 1}$ except that the HIBE key with $(i_n = h) \wedge (i_c = h_c)$ is nominally semi-functional-type1 by implicitly setting

$$a_0 \equiv c(h' + \textstyle\sum_{i=1}^{k} u'_i I_i) \bmod p_2,\ b_0 \equiv c \bmod p_2,\ z_i \equiv u'_i \bmod p_2,$$
$$d_0 \equiv (h' + \textstyle\sum_{i=1}^{\ell} u'_i I^*_i) \bmod p_2$$

where $c = \log_{g_2}(R_3)$. Note that the paradox of dual system encryption is solved by introducing the nominally semi-functional-type1 key. That is, the simulator cannot check whether the HIBE key is normal or nominally semi-functional-type1 since the exponents $a_0, b_0, \{z_i\}$ of the HIBE key are correlated to the exponent d_0 of the challenge HIBE ciphertext.

Next, we should argue that the Type-h-I adversary cannot distinguish a nominally semi-functional-type1 HIBE key from a semi-functional-type1 HIBE key. For this argument, we show an information theoretic argument by using the fact that $ID|_k \notin \text{Prefix}(ID^*|_\ell)$ for all HIBE keys with the node index h. Suppose there exists an unbounded Type-h-I adversary. If the HIBE keys is with $(i_n = h) \wedge (i_c = h_c)$, then the adversary can gather the exponents a_0, b_0 from the HIBE key and d_0 from the challenge HIBE ciphertext. We easily show that $h' + \sum_{i=1}^{k} u'_i I_i \bmod p_2$ looks random to the adversary since $h' + u'_j I_j$ is a pair-wise independent function, $\exists j$ such that $I_j \neq I^*_j$ if $ID|_k \notin \text{Prefix}(ID^*|_\ell)$, and $h' \bmod p_2$ and $u'_j \bmod p_2$ are information theoretically hidden to the adversary by the CRT. This completes our proof. $\square$

Lemma 3.13. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish* $\boldsymbol{H}_{h_c,1}$ *from* $\boldsymbol{H}_{h_c,2}$ *with a non-negligible advantage.*

Proof. The proof of this lemma is almost the same as that of Lemma 3.12. The only difference is the generation of an HIBE key with indexes $(i_n = h, i_c = h_c)$. This HIBE key is generated as follows:

- $i_c = h_c$: It chooses random $a'_0 \in \mathbb{Z}_N$, $Y'_0, Y'_1, \{Y'_{2,i}\} \in \mathbb{G}_{p_3}$ and creates an HIBE key

$$SK_{HIBE,S_j} = \big(K_0 = g^{\gamma_j}(Z)^{h'+\sum_{i=1}^{k} u'_i I_i} Y'_0 (R_2 Y_1)^{a'_0},\ K_1 = (Z)^{-1} Y'_1,\ \{K_{2,i} = (Z)^{u'_i} Y'_{2,i}\}\big).$$

Note that the exponent of this HIBE key is no longer correlated with the exponent of the challenge HIBE ciphertext since K_0 is randomized by a'_0.

Let $c = \log_{g_2}(R_3)$ and $c_2 = \log_{g_2}(R_2)$. If $Z = Z_1 = X_2 R_3 Y_2$, then the simulation is the same as $\mathbf{H}_{h_c,1}$ since the HIBE key is semi-functional-type1 by implicitly setting $a_0 \equiv c(h' + \sum_{i=1}^{k} u'_i I_i) + c_2 a'_0 \bmod p_2$, $b_0 \equiv c \bmod p_2$, and $z_i \equiv u'_i \bmod p_2$. If $Z = Z_0 = X_2 Y_2$, then the simulation is the same as $\mathbf{H}_{h_c,2}$ since the HIBE key is semi-functional-type2 by implicitly setting $a_0 \equiv c_2 a'_0 \bmod p_2$. □

Lemma 3.14. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish* $\boldsymbol{H}'_{h_c+1,2}$ *from* $\boldsymbol{H}'_{h_c,1}$ *with a non-negligible advantage.*

Proof. The proof of this lemma is almost the same as that of Lemma 3.13. The only difference is that the element K_0 of the HIBE key with the indexes $(i_n = h, i_c = h_c)$ that is generated in Lemma 3.13 is additionally multiplied by $(R_2 Y_1)^{\delta'_{j,0}}$ where a fixed exponent $\delta'_{j,0}$ is related with the node v_j as follows:

$$SK_{HIBE,S_j} = \big(K_0 = g^{\gamma_j}(Z)^{h'+\sum_{i=1}^{k} u'_i I_i} Y'_0 (R_2 Y_1)^{a'_0} (R_2 Y_1)^{\delta'_{j,0}},\ K_1 = (Z)^{-1} Y'_1,\ \{K_{2,i} = (Z)^{u'_i} Y'_{2,i}\}\big).$$

Let $c = \log_{g_2}(R_3)$ and $c_2 = \log_{g_2}(R_2)$. If $Z = Z_0 = X_2 Y_2$, then the simulation is the same as $\mathbf{H}'_{h_c+1,2}$ since the HIBE key is semi-functional-type2 by implicitly setting $a_0 \equiv c_2 a'_0 + c_2 \delta'_{j,0} \bmod p_2$. If $Z = Z_1 = X_2 R_3 Y_2$, then the simulation is the same as $\mathbf{H}'_{h_c,1}$ since the HIBE key is semi-functional-type1 by implicitly setting $a_0 \equiv c(h' + \sum_{i=1}^{k} u'_i I_i) + c_2 a'_0 + c_2 \delta'_{j,0} \bmod p_2$, $b_0 \equiv c \bmod p_2$, and $z_i \equiv u'_i \bmod p_2$. □

Lemma 3.15. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish* $\boldsymbol{H}'_{h_c,1}$ *from* $\boldsymbol{H}'_{h_c,2}$ *with a non-negligible advantage.*

Proof. The proof of this lemma is almost the same as that of Lemma 3.12. The only difference is that each element K_0 of HIBE keys with the indexes $(i_n = h, i_c = h_c)$ that is generated in Lemma 3.12 is additionally multiplied by $(R_2 Y_1)^{\delta'_{j,0}}$ where a fixed exponent $\delta'_{j,0}$ is related with the node v_j as follows:

$$SK_{HIBE,S_j} = \big(K_0 = g^{\gamma_j}(Z)^{h'+\sum_{i=1}^{k} u'_i I_i} Y'_0 (R_2 Y_1)^{\delta'_{j,0}},\ K_1 = (Z)^{-1} Y'_1,\ \{K_{2,i} = (Z)^{u'_i} Y'_{2,i}\}\big).$$

Let $c = \log_{g_2}(R_3)$ and $c_2 = \log_{g_2}(R_2)$. If $Z = Z_1 = X_2 R_3 Y_2$, then the simulation is the same as $\mathbf{H}'_{h_c,1}$ since the HIBE key is semi-functional-type1 by implicitly setting $a_0 \equiv c(h' + \sum_{i=1}^{k} u'_i I_i) + c_2 \delta'_{j,0} \bmod p_2$, $b_0 \equiv c \bmod p_2$, and $z_i \equiv u'_i \bmod p_2$. Recall that $h' + \sum_{i=1}^{k} u'_i I_i$ looks random to an adversary by the analysis in Lemma 3.12. If $Z = Z_0 = X_2 Y_2$, then the simulation is the same as $\mathbf{H}'_{h_c,2}$ since the HIBE key is semi-functional by implicitly setting $\delta_{j,0} \equiv c_2 \delta'_{j,0} \bmod p_2$. □

Lemma 3.16. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish* $\boldsymbol{H}'_{0,2}$ *from* $\boldsymbol{H}''$ *with a non-negligible advantage.*

Proof. The proof of this lemma is the important part of the security proof since it changes the IBE key for T^* from a normal type to a semi-functional type. It should be noted that this changes from normal to semi-functional cannot be handled by introducing a nominally semi-functional type since an information theoretic argument for T^* cannot be used. To solve this problem, we directly change normal keys with the index h to semi-functional keys without introducing nominally semi-functional keys.

Many part of this proof is similar to that of Lemma 3.12 except that the generation of HIBE keys and IBE keys with the node index h. These keys with the node index $i_n = h$ are generated as follows:

- **Case** $i_n = h$: Let $\delta'_{j,0}$ be a fixed exponent in $\mathbb{Z}_N$ for the subset S_j in this node index h.

  If this is an HIBE key, then it selects random $r' \in \mathbb{Z}_N, Y'_0, Y'_1, \{Y'_{2,i}\} \in \mathbb{G}_{p_3}$ and creates an HIBE key

$$SK_{HIBE,S_j} = \big(K_0 = Z \cdot F(ID|_k)^{r'} Y'_0 \cdot (R_2 Y_1)^{\delta'_{j,0}},\ K_1 = g^{-r'} Y'_1,\ \{K_{2,i} = u_i^{r'} Y'_{2,i}\}\big).$$

  If this is an IBE key, then it selects random $r'' \in \mathbb{Z}_N, Y''_0, Y''_1 \in \mathbb{G}_{p_3}$ and creates an IBE key

$$SK_{IBE,S_i} = \big(U_0 = g^{\beta_{ID|_{k-1}}} (Z)^{-1} (vw^T)^{-r''} Y''_0,\ U_1 = g^{-r''} Y''_1\big).$$

If $Z = Z_0 = X_2 Y_2$, then the simulation is the same as $\mathbf{H}'_{1,2}$ since all HIBE keys with h are semi-functional and all IBE keys with h are normal. If $Z = Z_1 = X_2 R_3 Y_2$, then the simulation is the same as $\mathbf{H}''$ since it implicitly sets $\delta_{j,0} = c + c_2 \delta'_{j,0} \bmod p_2$ and $\delta_{j,1} = -c \bmod p_2$ where $c = \log_{g_2}(R_3)$ and $c_2 = \log_{g_2}(R_2)$.

We now show that the paradox of dual system encryption does not occur. To check whether an IBE key with h is normal or semi-functional, the simulator may try to decrypt a semi-functional ciphertext by deriving a decryption key from these keys with h. However, the simulator always derive a semi-functional decryption key from those keys since the HIBE key with h is already semi-functional. Thus, the simulator cannot check whether the IBE key with h is normal or semi-functional since the decryption always fails. This completes our proof. □

3.6 Type-h-II Adversary

Lemma 3.17. *If the GSD assumption holds, then no PPT Type-h-II adversary can distinguish* $\boldsymbol{I}_{h_c-1,2}$ *from* $\boldsymbol{I}_{h_c,1}$ *with a non-negligible advantage.*

Lemma 3.18. *If the GSD assumption holds, then no PPT Type-h-II adversary can distinguish* $\boldsymbol{I}_{h_c,1}$ *from* $\boldsymbol{I}_{h_c,2}$ *with a non-negligible advantage.*

Lemma 3.19. *If the GSD assumption holds, then no PPT Type-h-II adversary can distinguish* $\boldsymbol{I}'_{h_c+1,2}$ *from* $\boldsymbol{I}'_{h_c,1}$ *with a non-negligible advantage.*

Lemma 3.20. *If the GSD assumption holds, then no PPT Type-h-II adversary can distinguish* $\boldsymbol{I}'_{h_c,1}$ *from* $\boldsymbol{I}'_{h_c,2}$ *with a non-negligible advantage.*

Lemma 3.21. *If the GSD assumption holds, then no PPT Type-h-II adversary can distinguish* $\boldsymbol{I}'_{1,2}$ *from* $\boldsymbol{I}''$ *with a non-negligible advantage.*

The proofs of Lemmas 3.17, 3.18, 3.19, 3.20, and 3.21 are almost the same as those of Lemmas 3.12, 3.13, 3.14, 3.15, and 3.16 respectively except that IBE keys are first changed to semi-functional by using the restriction $T \neq T^*$ and the master key part of an IBE key is set with the exponent $\beta_{ID|_{k-1}} - \gamma_i$. Note that the IBE scheme is a specific case of the HIBE scheme. We omit the detailed proofs of these lemmas.

4 Revocable HIBE with Subset Difference

In this section, we propose an RHIBE-SD scheme by combining HIBE, IBE, and SD schemes and prove its adaptive security under simple static assumptions.

4.1 Subset Difference Method

The subset difference (SD) method is also a specific instance of the subset cover framework of Naor et al. [22]. We also follow the SD definition of Lee and Park [19].

SD.Setup(N_{max})**:** Let $N_{max} = 2^n$ for simplicity. It first sets a perfect binary tree $\mathcal{BT}$ of depth n. Each user is assigned to a different leaf node in $\mathcal{BT}$. The collection $\mathcal{S}$ of SD is the set of all subsets $\{S_{i,j}\}$ where $v_i, v_j \in \mathcal{BT}$ and v_j is a descendant of v_i. It outputs the binary tree $\mathcal{BT}$.

SD.Assign$(\mathcal{BT}, ID)$**:** Let v be the leaf node of $\mathcal{BT}$ that is assigned to the user ID. Let $(v_{k_0}, v_{k_1}, \ldots, v_{k_n})$ be the path from the root node v_{k_0} to the leaf node $v_{k_n} = v$. For all $i, j \in \{k_0, \ldots, k_n\}$ such that v_j is a descendant of v_i, it adds the subset $S_{i,j}$ defined by two nodes v_i and v_j in the path into PV. It outputs the private set $PV = \{S_{i,j}\}$.

SD.Cover$(\mathcal{BT}, R)$**:** It first sets a subtree $\mathcal{T}$ as the Steiner Tree $\mathcal{ST}_R$ that is the minimum subtree of $\mathcal{BT}$ that connects all the leaf nodes in R and the root node, and then it builds a cover set CV iteratively by removing nodes from $\mathcal{T}$ until $\mathcal{T}$ consists of just a single node as follows:

1. It finds two leaf nodes v_i and v_j in $\mathcal{T}$ such that the least-common-ancestor v of v_i and v_j does not contain any other leaf nodes of $\mathcal{T}$ in its subtree. Let v_l and v_k be the two child nodes of v such that v_i is a descendant of v_l and v_j is a descendant of v_k. If there is only one leaf node left, it makes $v_i = v_j$ to the leaf node, v to be the root of $\mathcal{T}$ and $v_l = v_k = v$.
2. If $v_l \neq v_i$, then it adds the subset $S_{l,i}$ to CV; likewise, if $v_k \neq v_j$, it adds the subset $S_{k,j}$ to CV.
3. It removes from $\mathcal{T}$ all the descendants of v and makes v a leaf node.

It outputs the cover set $CV = \{S_{i,j}\}$.

SD.Match(CV, PV)**:** It finds two subsets $S_{i,j} \in CV$ and $S_{i',j'} \in PV$ such that $(v_i = v_{i'}) \wedge (d_j = d_{j'}) \wedge (v_j \neq v_{j'})$ where d_j is the depth of v_j. If it found two subsets, then it outputs $(S_{i,j}, S_{i',j'})$. Otherwise, it outputs $\perp$.

Lemma 4.1 ([22])**.** *In the SD method, the size of a private set if $O(\log^2 N_{max})$ and the size of a cover set is $O(r)$ where N_{max} is the maximum number of leaf nodes and r is the size of revoked users R.*

4.2 Construction

Our RHIBE-SD scheme is also very similar to that of Lee and Park [19] except that the underlying HIBE and IBE schemes are replaced by the LW-HIBE and LW-IBE schemes. We define $\text{GMLabel}(S_{i,j}) = (GL = \text{Label}(v_i)\|\text{Depth}(v_j), L_j = \text{Label}(v_j))$ where $S_{i,j} = (v_i, v_j)$. Let $\Delta_{i,I}$ be a Lagrange coefficient which is defined as $\Delta_{i,I}(x) = \prod_{j \in I, j \neq i} \frac{x-j}{i-j}$ for an index $i \in \mathbb{Z}_N$ and a set of indexes I in $\mathbb{Z}_N$.

Let PRF be a pseudo-random function for $\mathcal{K} = \{0,1\}^\lambda$, $\mathcal{X} = \{0,1\}^*$, and $\mathcal{Y} = \mathbb{Z}_N$. Our RHIBE scheme for $\mathcal{I} = \mathbb{Z}_N$, $\mathcal{V} = \mathbb{Z}_N$, and $\mathcal{M} \in \mathbb{G}_T$ is described as follows:

RHIBE-SD.Setup$(1^\lambda, L, N_{max})$**:** Let λ be a security parameter, L be the maximum depth of a hierarchical identity, and N_{max} be the maximum number of users for each level.

1. It first generates bilinear groups $\mathbb{G}, \mathbb{G}_T$ of composite order $N = p_1 p_2 p_3$ where p_1, p_2, and p_3 are random primes. It selects random generators g_1, g_3 of $\mathbb{G}_{p_1}, \mathbb{G}_{p_3}$ respectively.
2. It selects a random exponent $\alpha \in \mathbb{Z}_N$ and chooses random elements $h, u_1, \ldots, u_L, v, w \in \mathbb{G}_{p_1}$. It outputs a master key $MK = \alpha$ and public parameters

$$PP = \Big((N, \mathbb{G}, \mathbb{G}_T, e),\ g = g_1, Y = g_3, h, u_1, \ldots, u_L,\ v, w,\ \Omega = e(g,g)^\alpha, N_{max} \Big).$$

We define $F(ID|_k) = (h \prod_{i=1}^{k} u_i^{I_i})$ for $ID|_k = (I_1, \ldots, I_k)$ and use the notation $SK_{ID|_0} = MK$.

RHIBE-SD.GenKey$(ID|_k, ST_{ID|_{k-1}}, PP)$**:** Let $ID|_k = (I_1, \ldots, I_k) \in \mathcal{I}^k$ be a hierarchical identity with $k \geq 1$, and $ST_{ID|_{k-1}}$ be a state information.

1. If $ST_{ID|_{k-1}}$ is empty, then it obtains $\mathcal{BT}_{ID|_{k-1}}$ by running **SD.Setup**(N_{max}) and selects a random exponent $\beta_{ID|_{k-1}} \in \mathbb{Z}_N$ and a PRF key $z_{ID|_{k-1}} \in \mathcal{K}$. It sets $ST_{ID|_{k-1}} = (\mathcal{BT}_{ID|_{k-1}}, \beta_{ID|_{k-1}}, z_{ID|_{k-1}})$.
2. It assigns $ID|_k$ to a random leaf node $v \in \mathcal{BT}_{ID|_{k-1}}$ and obtains a private set $PV = \{S_{i,j}\}$ by running **SD.Assign**$(\mathcal{BT}_{ID|_{k-1}}, ID|_k)$.
   For each $S_{i,j} \in PV$, it sets $(GL, L_j) = \text{GMLabel}(S_{i,j})$ and proceeds as follows: It defines $f_{GL}(x) = a_{GL}x + \beta_{ID|_{k-1}}$ by computing $a_{GL} = \mathbf{PRF}(z_{ID|_{k-1}}, GL)$. It selects random $r \in \mathbb{Z}_N$, $Y_0, Y_1, Y_{2,k+1}, \ldots, Y_{2,L} \in \mathbb{G}_{p_3}$ and creates an HIBE key

$$SK_{HIBE,S_{i,j}} = \Big(K_0 = g^{f_{GL}(L_j)} F(ID|_k)^r Y_0,\ K_0 = g^{-r} Y_1,\ \big\{ K_0 = u_i^r Y_{2,i} \big\}_{i=k+1}^{L} \Big).$$

3. Finally, it outputs a private key $SK_{ID|_k} = \big(PV, \{SK_{HIBE,S_{i,j}}\}_{S_{i,j} \in PV} \big)$. Note that the master key part of $SK_{HIBE,S_{i,j}}$ is $f_{GL}(L_j) = a_{GL} L_j + \beta_{ID|_{k-1}}$.

RHIBE-SD.UpdateKey$(T, RL_{ID|_{k-1}}, DK_{ID|_{k-1},T}, ST_{ID|_{k-1}}, PP)$**:** Let $DK_{ID|_{k-1},T} = (DK_{HIBE}, DK_{IBE})$ be a decryption key.

1. If $ST_{ID|_{k-1}}$ is empty, then it obtains $\mathcal{BT}_{ID|_{k-1}}$ by running **SD.Setup**(N_{max}) and selects a random exponent $\beta_{ID|_{k-1}} \in \mathbb{Z}_N$ and a PRF key $z_{ID|_{k-1}} \in \mathcal{K}$. It sets $ST_{ID|_{k-1}} = (\mathcal{BT}_{ID|_{k-1}}, \beta_{ID|_{k-1}}, z_{ID|_{k-1}})$.
2. It derives the set R of revoked identities at time T from $RL_{ID|_{k-1}}$ and obtains a cover set $CV = \{S_{i,j}\}$ by running **SD.Cover**$(\mathcal{BT}_{ID|_{k-1}}, R)$.
   For each $S_{i,j} \in CV$, it sets $(GL, L_j) = \text{GMLabel}(S_{i,j})$ and proceeds as follows: It defines $f_{GL}(x) = a_{GL}x + \beta_{ID|_{k-1}}$ by computing $a_{GL} = \mathbf{PRF}(z_{ID|_{k-1}}, GL)$. It selects random $r \in \mathbb{Z}_N$, $Y_0, Y_1 \in \mathbb{G}_{p_3}$ and creates an IBE key

$$SK_{IBE,S_{i,j}} = \Big(U_0 = g^{f_{GL}(L_j)} (vw^T)^r Y_0,\ U_1 = g^{-r} Y_1 \Big).$$

3. Let $DK_{HIBE} = (D_0, D_1, \{D_{2,i}\})$ and $DK_{IBE} = (V_0, V_1)$ where the master key parts are η and $\alpha - \beta_{ID|_{k-1}} - \eta$ respectively. It chooses random $\eta' \in \mathbb{Z}_N$ and creates temporal blinded HIBE and IBE keys

$$TBK_{HIBE} = \Big(A'_0 = D_0 \cdot g^{\eta'},\ A'_1 = D_1,\ \big\{ A'_{2,i} = D_{2,i} \big\}_{i=k}^{L} \Big)$$
$$TBK_{IBE} = \Big(B'_0 = V_0 \cdot g^{-\beta_{ID|_{k-1}} - \eta'},\ B'_1 = V_1 \Big).$$

4. Next, it chooses random $r', r'' \in \mathbb{Z}_N$, $Y'_0, Y'_1, Y'_{2,k}, \dots, Y'_{2,L}$, $Y''_0, Y''_1 \in \mathbb{G}_{p_3}$ and randomizes blinded HIBE and IBE keys

$$BK_{HIBE} = \Big(A_0 = A'_0 \cdot F(ID|_{k-1})^{r'} Y'_0,\ A_1 = A'_1 \cdot g^{-r'} Y'_1,\ \big\{ A_{2,i} = A'_{2,i} \cdot u_i^{r'} Y'_{2,i} \big\}_{i=k}^{L} \Big)$$
$$BK_{IBE} = \Big(B_0 = B'_0 \cdot (vw^T)^{r''} Y''_0,\ B_1 = B'_1 \cdot g^{-r''} Y''_1 \Big).$$

5. Finally, it outputs an update key $UK_{ID|_{k-1},T} = \big(CV, \{SK_{IBE,S_{i,j}}\}_{S_{i,j} \in CV}, BK_{ID|_{k-1},T} = (BK_{HIBE}, BK_{IBE}) \big)$ where the master key parts of $SK_{IBE,S_{i,j}}$, BK_{HIBE}, and BK_{IBE} are $f_{GL}(L_j) = a_{GL} L_j + \beta_{ID|_{k-1}}$, η', and $\alpha - \beta_{ID|_{k-1}} - \eta'$ for some random η' respectively.

RHIBE-SD.DeriveKey$(ID|_k, T, SK_{ID|_k}, UK_{ID|_{k-1},T}, PP)$**:** Let $ID|_k = (I_1, \dots, I_k)$ with $k \geq 0$, $SK_{ID|_k} = (PV, \{SK_{HIBE,S_{i,j}}\}_{S_{i,j} \in PV})$, and $UK_{ID|_{k-1},T} = (CV, \{SK_{IBE,S_{i,j}}\}_{S_{i,j} \in CV}, BK_{ID|_{k-1},T})$ where $BK_{ID|_{k-1},T} = (BK_{HIBE}, BK_{IBE})$.

If $k = 0$, then $SK_{ID|_0} = MK$ and UK is empty. It proceeds as follows:

1. It first selects a random exponent $\eta \in \mathbb{Z}_N$. It chooses random $r, r' \in \mathbb{Z}_N$, $Y_0, Y_1, Y_{2,1}, \dots, Y_{2,L}$, $Y'_0, Y'_1 \in \mathbb{G}_{p_3}$ and creates HIBE and IBE keys

$$DK_{HIBE} = \Big(D_0 = g^{\eta} (h)^r Y_0,\ D_1 = g^{-r} Y_1,\ \big\{ D_{2,i} = u_i^r Y_{2,i} \big\}_{i=1}^{L} \Big),$$
$$DK_{IBE} = \Big(V_0 = g^{\alpha - \eta} (vw^T)^{r'} Y'_0,\ V_1 = g^{-r'} Y'_1 \Big).$$

2. It outputs a decryption key $DK_{ID|_0,T} = (DK_{HIBE}, DK_{IBE})$.

If $k \geq 1$, then it proceeds as follows:

1. It first obtains $(S_{i,j}, S_{i',j'})$ by running **SD.Match**(CV, PV). If it fails, it outputs $\perp$. It then retrieves $SK_{HIBE,S_{i',j'}} = (K_0, K_1, \{K_{2,i}\})$ from $SK_{ID|_k}$ and $SK_{IBE,S_{i,j}} = (U_0, U_1)$ from $UK_{ID|_{k-1},T}$ where the master key parts are $f_{GL}(L_{j'}) = a_{GL} L_{j'} + \beta_{ID|_{k-1}}$ and $f_{GL}(L_j) = a_{GL} L_j + \beta_{ID|_{k-1}}$ respectively.
2. Let $BK_{HIBE} = (A_0, A_1, \{A_{2,i}\})$ and $BK_{IBE} = (B_0, B_1)$ where the master key parts are η and $\alpha - \beta_{ID|_{k-1}} - \eta$ respectively. Next, it calculates two Lagrange coefficients $\Delta_{L_{j'},I}(0) = \frac{-L_j}{L_{j'} - L_j} \bmod N$ and $\Delta_{L_j,I}(0) = \frac{-L_{j'}}{L_j - L_{j'}} \bmod N$ for the set $I = \{L_j, L_{j'}\}$ and creates temporal HIBE and IBE keys by selecting a random exponent $\eta' \in \mathbb{Z}_N$ as

$$TDK_{HIBE} = \Big(D'_0 = A_0 A_{2,k}^{I_k} (K_0)^{\Delta_{L_{j'},I}(0)} \cdot g^{\eta'},\ D'_1 = A_1 (K_1)^{\Delta_{L_{j'},I}(0)},\ \big\{ D'_{2,i} = A_{2,i} (K_{2,i})^{\Delta_{L_{j'},I}(0)} \big\}_{i=k+1}^{L} \Big),$$
$$TDK_{IBE} = \Big(V'_0 = B_0 (U_0)^{\Delta_{L_j,I}(0)} \cdot g^{-\eta'},\ V'_1 = B_1 (U_1)^{\Delta_{L_j,I}(0)} \Big).$$

3. After that, it chooses random $r, r' \in \mathbb{Z}_N$, $Y_0, Y_1, Y_{2,k+1}, \dots, Y_{2,L}$, $Y'_0, Y'_1 \in \mathbb{G}_{p_3}$ and creates randomized HIBE and IBE keys

$$DK_{HIBE} = \Big(D_0 = D'_0 \cdot F(ID|_k)^r Y_0,\ D_1 = D'_1 \cdot g^{-r} Y_1,\ \big\{ D_{2,i} = D'_{2,i} \cdot u_i^r Y_{2,i} \big\}_{i=k+1}^{L} \Big),$$
$$DK_{IBE} = \Big(V_0 = V'_0 \cdot (vw^T)^{r'} Y'_0,\ V_1 = V'_1 \cdot g^{-r'} Y'_1 \Big).$$

4. Finally, it outputs a decryption key $DK_{ID|_k,T} = \big(DK_{HIBE}, DK_{IBE}\big)$. Note that the master key parts of DK_{HIBE} and DK_{IBE} are η'' and $\alpha - \eta''$ for some random $\eta'' = \eta + f_{GL}(L_{j'})\Delta_{L_{j'},I}(0) + \eta'$ respectively since the following equation holds

$$\begin{aligned}
&\alpha - \beta_{ID|_{k-1}} - \eta + f_{GL}(L_j)\Delta_{L_j,I}(0) - \eta' \\
&= \alpha - \beta_{ID|_{k-1}} + f_{GL}(L_j)\Delta_{L_j,I}(0) - (\eta'' - f_{GL}(L_{j'})\Delta_{L_{j'},I}(0)) \\
&= \alpha - \beta_{ID|_{k-1}} + \beta_{ID|_{k-1}} - \eta'' = \alpha - \eta''.
\end{aligned}$$

RHIBE-SD.Encrypt$(ID|_\ell, T, M, PP)$**:** It is the same as the algorithm in Section 3.2.

RHIBE-SD.Decrypt$(CT_{ID|_\ell,T}, DK_{ID'|_k,T'}, PP)$**:** It is the same as the algorithm in Section 3.2.

RHIBE-SD.Revoke$(ID|_k, T, RL_{ID|_{k-1}}, ST_{ID|_{k-1}})$**:** It is the same as the algorithm in Section 3.2.

4.3 Correctness

To show the correctness of the above RHIBE-SD scheme, we only show that a decryption key is correctly derived from a private key and an update key since other parts are almost the same as those of the RHIBE-CS scheme.

Let $SK_{ID|_k} = (PV, \{SK_{HIBE,S_{i,j}}\})$ be a private key generated by the **GenKey** algorithm. The master key part of $SK_{HIBE,S_{i,j}}$ is $f_{GL}(L_j) = a_{GL}L_j + \beta_{ID|_{k-1}}$. Let $UK_{ID|_{k-1},T} = (CV, \{SK_{IBE,S_{i,j}}\}, BK_{ID|_{k-1},T} = (BK_{HIBE}, BK_{IBE}))$ be an update key generated by the **UpdateKey** algorithm. The master key part of $SK_{IBE,S_{i,j}}$ is $f_{GL}(L_j) = a_{GL}L_j + \beta_{ID|_{k-1}}$. The master key parts of BK_{HIBE} and BK_{IBE} are η'' and $\alpha - \beta_{ID|_{k-1}} - \eta''$ respectively since the master key parts of DK_{HIBE} and DK_{IBE} are η and $\alpha - \eta$, and exponents η' and $-\beta_{ID|_{k-1}} - \eta'$ are added to the temporal keys.

We show that a decryption key $DK_{ID|_k,T}$ generated by the **DeriveKey** algorithm is correctly derived from $SK_{ID|_k}$ and $UK_{ID|_{k-1},T}$. If $(ID|_k, T') \notin RL_{ID|_{k-1}}$ for all $T' \leq T$, then the master key parts of $SK_{HIBE,S_{i',j'}}$ and $SK_{IBE,S_{i,j}}$ are associated with $f_{GL}(L_{j'}) = a_{GL}L_{j'} + \beta_{ID|_{k-1}}$ and $f_{GL}(L_j) = a_{GL}L_j + \beta_{ID|_{k-1}}$ where $L_{j'} \neq L_j$ by the correctness of the SD scheme. If two Lagrange coefficients are multiplied, then we have the following equation

$$\begin{aligned}
&f_{GL}(L_{j'})\Delta_{L_{j'},I}(0) + f_{GL}(L_j)\Delta_{L_j,I}(0) \\
&= (a_{GL}L_{j'} + \beta_{ID|_{k-1}})\frac{-L_j}{L_{j'} - L_j} + (a_{GL}L_j + \beta_{ID|_{k-1}})\frac{-L_{j'}}{L_j - L_{j'}} \\
&= (-a_{GL}L_{j'}L_j - \beta_{ID|_{k-1}}L_j + a_{GL}L_jL_{j'} + \beta_{ID|_{k-1}}L_{j'})\frac{1}{L_{j'} - L_j} \\
&= \beta_{ID|_{k-1}}\frac{L_{j'} - L_j}{L_{j'} - L_j} = \beta_{ID|_{k-1}}.
\end{aligned}$$

The master key parts of BK_{HIBE} and BK_{IBE} are associated with η and $\alpha - \beta_{ID|_{k-1}} - \eta$. Thus, the master key parts of TDK_{HIBE} and TDK_{IBE} are associated with $\eta + f_{GL}(L_{j'})\Delta_{L_{j'},I}(0) + \eta'$ and $\alpha - \beta_{ID|_{k-1}} - \eta + f_{GL}(L_j)\Delta_{L_j,I}(0) - \eta'$ respectively. If we implicitly sets $\eta'' = \eta + f_{GL}(L_{j'})\Delta_{L_{j'},I}(0) + \eta'$, then we have the following equation

$$
\begin{aligned}
&\alpha - \beta_{ID|_{k-1}} - \eta + f_{GL}(L_j)\Delta_{L_j,I}(0) - \eta' \\
&= \alpha - \beta_{ID|_{k-1}} + f_{GL}(L_j)\Delta_{L_j,I}(0) - (\eta'' - f_{GL}(L_{j'})\Delta_{L_{j'},I}(0)) \\
&= \alpha - \beta_{ID|_{k-1}} + (f_{GL}(L_j)\Delta_{L_j,I}(0) + f_{GL}(L_{j'})\Delta_{L_{j'},I}(0)) - \eta'' \\
&= \alpha - \beta_{ID|_{k-1}} + \beta_{ID|_{k-1}} - \eta'' = \alpha - \eta''.
\end{aligned}
$$

Since the master key parts of DK_{HIBE} and DK_{IBE} are same with that of TDK_{HIBE} and TDK_{IBE}, two master key parts of DK_{HIBE} and DK_{IBE} are η'' and $\alpha - \eta''$ for some random η'' respectively.

4.4 Security Analysis

We also use the dual system encryption proof technique of Lewko and Waters [20] to prove the adaptive security of our RHIBE-SD scheme. The overall strategy of this security proof is somewhat similar to that of our RHIBE-CS scheme, but we use a different grouping method of small component keys because of the difference between the CS method and the SD method. The details of the security proof are given as follows.

Theorem 4.2. *The above RHIBE-SD scheme is AD-IND-CPA secure if the SD, GSD, and ComDH assumptions hold.*

Proof. We first define the semi-functional types of private keys, update keys, decryption keys, and ciphertexts in RHIBE by using the semi-functional types of HIBE and IBE in Theorem 3.2. For the semi-functional type, we let g_2 denote a fixed generator of the subgroup $\mathbb{G}_{p_2}$.

RHIBE-SD.SK-SF. To generate a semi-functional private key, it proceeds as follows.

1. It first creates a normal private key $SK'_{ID|_k} = (PV, \{SK'_{HIBE,S_{i,j}}\}_{S_{i,j} \in PV})$ by using MK where each $SK'_{HIBE,S_{i,j}}$ is a normal HIBE key.
2. For each $S_{i,j} \in PV$, it chooses a random exponent $\delta_{i,j} \in \mathbb{Z}_N$ once for $S_{i,j} \in \mathcal{BT}_{ID|_{k-1}}$ and converts $SK'_{HIBE,S_{i,j}}$ to a semi-functional $SK_{HIBE,S_{i,j}}$ with the exponent $\delta_{i,j}$.
3. It outputs a semi-functional private key $SK_{ID|_k} = \big(PV, \{SK_{HIBE,S_{i,j}}\}_{S_{i,j} \in PV}\big)$.

RHIBE-SD.UK-SF. To generate a semi-functional update key, it proceeds as follows.

1. It first creates a normal update key $UK'_{ID|_{k-1},T} = (CV, \{SK'_{IBE,S_{i,j}}\}_{S_{i,j} \in CV}, BK'_{ID|_{k-1},T} = (BK'_{HIBE}, BK'_{IBE}))$ by using MK where BK'_{HIBE} is a normal HIBE key, $SK'_{IBE,S_{i,j}}$ and BK'_{IBE} are normal IBE keys.
2. For each $S_{i,j} \in CV$, it chooses a random exponent $\delta_{i,j} \in \mathbb{Z}_N$ once for $S_{i,j}$ and converts a normal $SK'_{IBE,S_{i,j}}$ to a semi-functional $SK_{IBE,S_{i,j}}$ with the exponent $\delta_{i,j}$.
3. It chooses a random exponent $a_0 \in \mathbb{Z}_N$ and fixes a random exponent $a_{ID|_{k-1}} \in \mathbb{Z}_N$ for $\mathcal{BT}_{ID|_{k-1}}$. It converts BK'_{HIBE} to a semi-functional HIBE key BK_{HIBE} with the exponent a_0. It also converts BK'_{IBE} to a semi-functional IBE key BK_{IBE} with the exponent $a_{ID|_{k-1}} - a_0$. It sets a semi-functional $BK_{ID|_{k-1},T} = (BK_{HIBE}, BK_{IBE})$.
4. It outputs a semi-functional update key $UK_{ID|_{k-1},T} = \big(CV, \{SK_{IBE,S_{i,j}}\}_{S_{i,j} \in CV}, BK_{ID|_{k-1},T}\big)$.

RHIBE-SD.DK-SF. To generate a semi-functional decryption key, it proceeds as follows.

1. It first creates a normal decryption key $DK'_{ID|_k,T} = (DK'_{HIBE}, DK'_{IBE})$ by using MK where DK'_{HIBE} is a normal HIBE key and DK'_{IBE} is normal IBE key.
2. It chooses random exponents $a_0, a_1 \in \mathbb{Z}_N$. It converts DK'_{HIBE} to a semi-functional-type2 HIBE key DK_{HIBE} with the exponent a_0. It also converts DK'_{IBE} to a semi-functional-type2 IBE key DK_{IBE} with the exponent a_1.
3. It outputs a semi-functional decryption key $DK_{ID|_k,T} = \big(DK_{HIBE}, DK_{IBE}\big)$.

RHIBE-SD.CT-SF. To generate a semi-functional ciphertext, it proceeds as follows.

1. It first creates a normal ciphertext $CT'_{ID|_\ell,T} = (CH'_{HIBE}, CH'_{IBE}, C')$ where CH'_{HIBE} is a normal HIBE ciphertext and CH'_{IBE} is a normal IBE ciphertext.
2. It chooses random exponents $c, d_0, d_1 \in \mathbb{Z}_N$. It converts CH'_{HIBE} to a semi-functional CH_{HIBE} with the exponents c, d_0. It also converts CH'_{IBE} to a semi-functional CH_{IBE} with the exponents c, d_1.
3. It outputs a semi-functional ciphertext $CT_{ID|_\ell,T} = \big(CH_{HIBE}, CH_{IBE}, C'\big)$.

The security proof consists of the sequence of hybrid games $\mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_6$ defined in Theorem 3.2. The first game $\mathbf{G}_0$ is the original security game and the last one $\mathbf{G}_6$ is a game such that the adversary has no advantage. We omit the definition of these games since they are given in Theorem 3.2.

Let $\text{Adv}_{\mathcal{A}}^{G_j}$ be the advantage of $\mathcal{A}$ in the game $\mathbf{G}_j$. We have that $\text{Adv}_{RHIBE,\mathcal{A}}^{AD\text{-}IND\text{-}CPA}(\lambda) = \text{Adv}_{\mathcal{A}}^{G_0}$ and $\text{Adv}_{\mathcal{A}}^{G_6} = 0$. From the following Lemmas 4.3, 4.4, 4.5, 4.7, 4.8 and 4.9, we obtain the following equation

$$\begin{aligned}\text{Adv}_{RHIBE,\mathcal{A}}^{AD\text{-}IND\text{-}CPA}(\lambda) &\leq \sum_{j=1}^{6} \left|\text{Adv}_{\mathcal{A}}^{G_{j-1}} - \text{Adv}_{\mathcal{A}}^{G_j}\right| \\ &\leq O(q_{sk} + q_{uk})\text{Adv}_{\mathcal{B}}^{PRF}(\lambda) + \text{Adv}_{\mathcal{B}}^{SD}(\lambda) + \\ &\quad O((q_{sk}\log N_{max} + q_{uk} r_{max})(q_{sk} + q_{uk}) + q_{dk})\text{Adv}_{\mathcal{B}}^{GSD}(\lambda) + \text{Adv}_{\mathcal{B}}^{ComDH}(\lambda)\end{aligned}$$

where q_{sk}, q_{uk}, and q_{dk} are the number of private key, update key, and decryption key queries respectively. This completes our proof. □

Lemma 4.3. *If the PRF is secure, then no PPT adversary can distinguish $\mathbf{G}_0$ from $\mathbf{G}_1$ with a non-negligible advantage.*

Lemma 4.4. *If the SD assumption holds, then no PPT adversary can distinguish $\mathbf{G}_1$ from $\mathbf{G}_2$ with a non-negligible advantage.*

The proofs of Lemmas 4.3 and 4.4 are the same as those of Lemmas 3.3 and 3.4.

Lemma 4.5. *If the GSD assumption holds, then no PPT adversary can distinguish $\mathbf{G}_2$ from $\mathbf{G}_3$ with a non-negligible advantage.*

Proof. For the proof of this lemma, we cannot use simple hybrid games that change a normal private key (or normal update key) to a semi-functional private key (or semi-functional update key) one by one since the adversary of RHIBE can query a private key for $ID|_k \in \text{Prefix}(ID^*|_\ell)$ and an update key for T^*. Note that these normal keys cannot directly converted to semi-functional keys since an information theoretic argument cannot be used.

To solve this problem, we first divide each private key and update key into small HIBE keys and IBE keys. Recall that a private key $SK_{ID|_k}$ consists of many HIBE keys and an update key $UK_{ID|_{k-1},T}$ consists of many IBE keys and a blinded key where each HIBE key (or an IBE key) is associated with a subset $S_{i,j}$ in $\mathcal{BT}_{ID|_{k-1}}$. Next, HIBE keys and IBE keys that are related to the same group of a subset $S_{i,j}$ in $\mathcal{BT}_{ID|_{k-1}}$ are grouped together. To uniquely identify the group of a subset $S_{i,j} \in \mathcal{BT}_{ID|_{k-1}}$, we define a group identifier GID of this subset as a string $ID|_{k-1}\|L_i\|d_j$ where $(L_i, L_j) = \text{Label}(S_{i,j})$ and $d_j = \text{Depth}(S_j)$. To prove this lemma, we change normal HIBE keys and normal IBE keys that are related to the same group identifier into semi-functional keys by defining additional hybrid games. This additional hybrid games are performed for all group identifiers that are used in the key queries of the adversary.

For additional hybrid games that change HIBE keys (or IBE keys) that are related to the same group identifier $GID = ID|_{k-1}\|L_i\|d_j$ from normal keys to semi-functional keys, we need to state additional information of a subset $S_{i,j}$ in $\mathcal{BT}_{ID|_{k-1}}$. Note that an HIBE key for $S_{i,j}$ and an IBE key for $S_{i',j'}$ share the same polynomial $f(x)$ if $(L_i = L_{i'}) \wedge (d_j = d_{j'})$ since they belong to the same group. Thus we associate an HIBE key (or an IBE key) with an index pair (i_g, i_m, i_c) to state additional information where i_g is a group index, i_m is a member index, and i_c is a counter index.

Suppose that an HIBE key (or an IBE key) is related with a subset $S_{i,j}$, Then this key has a group identifier $GID = ID|_{k-1}\|L_i\|d_j$ and a member label L_j. The group index i_g for HIBE keys (or IBE keys) is assigned as follows: If the group identifier GID appears first time in queries, then we set i_g as the number of distinct group identifiers in previous queries plus one. If the group identifier GID already appeared before in queries, then we set i_g as the value i'_g of previous HIBE key (or IBE key) with the same group identifier GID. The member index i_m for the group index i_g is assigned as follows: If the member label L_j for this group identifier GID appears first time in queries, then we set i_m as the number of distinct members for this group identifier GID in previous queries plus one. If the member label L_j for this group identifier already appeared before in queries, then we set i_m as the value i'_m of previous one. The counter index i_c is assigned as follows: If the group identifier and member label (GID, L_j) appears first time in queries, then we set i_c as one. If the group identifier and member label (GID, L_j) appeared before in queries, then we set i_c as the number of queries with the group identifier and member label (GID, L_j) that appeared before plus one.

For the security proof, we additionally define a sequence of games $\mathbf{G}_{2,1}, \ldots, \mathbf{G}_{2,h}, \ldots, \mathbf{G}_{2,q_g}$ where $\mathbf{G}_2 = \mathbf{G}_{2,0}, \mathbf{G}_3 = \mathbf{G}_{2,q_g}$, and q_g is the maximum number of group identifiers that are used in private keys and update keys. In the game $\mathbf{G}_{2,h}$ for $1 \le h \le q_g$, the challenge ciphertext is semi-functional, HIBE keys and IBE keys with a group identifier $i_g \le h$ are semi-functional, the remaining HIBE keys and IBE keys with a group index $i_g > h$ are normal, and all blinded keys in update keys are still normal.

Let $\text{Adv}_{\mathcal{A}}^{G_j}$ be the advantage of $\mathcal{A}$ in the game $\mathbf{G}_j$. From the following Lemma 4.6, we have the following equation

$$\begin{aligned}\text{Adv}_{\mathcal{A}}^{G_2} - \text{Adv}_{\mathcal{A}}^{G_3} &\le \sum_{h=1}^{q_g} \left|\text{Adv}_{\mathcal{A}}^{G_{2,h-1}} - \text{Adv}_{\mathcal{A}}^{G_{2,h}}\right| \\ &\le O((q_{sk} \log N_{max} + q_{uk} r_{max})(q_{sk} + q_{uk}))\text{Adv}_{\mathcal{B}}^{GSD}(\lambda).\end{aligned}$$

This completes the proof. $\square$

Lemma 4.6. *If the GSD assumption holds, then no PPT adversary can distinguish $\mathbf{G}_{2,h-1}$ from $\mathbf{G}_{2,h}$ with a non-negligible advantage.*

Table 6: Hybrid games from $\mathbf{G}_{2,h-1}$ to $\mathbf{G}_{2,h}$ for Type-h-I

Game	SK_{HIBE}, SK_{IBE} with $h, (i_m \neq h_m^*, i_c)$						$SK_{HIBE}, SK_{IBE} with$
	$(1,1)$	$(1,2)$	$\dots$	(h_m,h_c)	$\dots$	(q_m,q_c)	$h,(i_m = h_m^*, i_c)$
$\mathbf{H}_{(0,q_c),2}$	N						
$\mathbf{H}_{(1,1),1}$	SF1	N	$\dots$	N	$\dots$	N	N
$\mathbf{H}_{(1,1),2}$	SF2						
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
$\mathbf{H}_{(h_m,h_c-1),2}$				N			
$\mathbf{H}_{(h_m,h_c),1}$	SF2	SF2	$\dots$	SF1	$\dots$	N	N
$\mathbf{H}_{(h_m,h_c),2}$				SF2			
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
$\mathbf{H}_{(q_m,q_c-1),2}$						N	
$\mathbf{H}_{(q_m,q_c),1}$						SF1	
$\mathbf{H}_{(q_m,q_c),2}$	SF2	SF2	$\dots$	SF2	$\dots$	SF2	N
$\mathbf{H}'_{(q_m,q_c),1}$						SF1$'$	
$\mathbf{H}'_{(q_m,q_c),2}$						SF	
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
$\mathbf{H}'_{(1,2),2}$	SF2						N
$\mathbf{H}'_{(1,1),1}$	SF1$'$						N
$\mathbf{H}'_{(1,1),2}$	SF	SF	$\dots$	SF	$\dots$	SF	N
$\mathbf{H}''_1$	SF						SF2
$\mathbf{H}''_2$	SF						SF

We use symbols N for normal, SF1 for semi-functional-type1, SF2 for semi-functional-type2, SF1$'$ for semi-functional-type1 with an additional δ, and SF for semi-functional.

Proof. We first divide the adversaries into two types based on the behavior of adversaries on the group index h: Type-h-I and Type-h-II. Let $ID^*|_\ell$ be the challenge hierarchical identity and T^* be the challenge time respectively. The adversary types are formally defined as follows:

Type-h-I. An adversary is Type-h-I if at least one HIBE key with the group index h satisfies $ID|_k \in \text{Prefix}(ID^*|_\ell)$ or at least one IBE key with the group index h satisfies $T = T^*$. More specifically, this adversary can be divided as follows:

- Type-h-I-A. All HIBE keys with the group index h satisfy $ID|_k \notin \text{Prefix}(ID^*|_\ell)$ and at least one IBE key with the group index h satisfies $T = T^*$.
- Type-h-I-B. At least one HIBE key with the group index h satisfies $ID|_k \in \text{Prefix}(ID^*|_\ell)$ and all IBE keys with the group index h satisfy $T \neq T^*$.

- Type-h-I-C. At least one HIBE key with the group index h satisfies $ID|_k \in \text{Prefix}(ID^*|_\ell)$ and at least one IBE key with the group index h satisfies $T = T^*$.

Type-h-II. An adversary is Type-h-II if all HIBE keys with the group index h satisfy $ID|_k \notin \text{Prefix}(ID^*|_\ell)$ and all IBE keys with the group index h satisfy $T \neq T^*$.

Note that these two types of adversaries cover all possible strategies related to the group index h.

Let's assume that the group index h for this game is defined in $\mathcal{BT}_{ID|_{k-1}}$. Let CV^* be the cover set of an update key for the challenge time T^* and the revoked set R^* at time T^*, and PV^* be the private set of an private key for an hierarchical identity $ID^*|_k \in \text{Prefix}(ID^*|_\ell)$. Let h_m^* be a member index of the group index h such that the HIBE key for $ID^*|_k$ or the IBE key for T^* belong to the member index h_m^*. Note that we can randomly guess h_m^* since it is polynomially bounded in Lemma 4.10. If the adversary is Type-h-I-A, then there is only one member index h_m^* since CV^* is a partition. If the adversary is In Type-h-I-B, then there is only one member index h_m^* since PV^* is related with a path. If the adversary is Type-h-I-C, the member index h_m^* of CV^* with the group index h should be the same as that of PV^* with the same group index h in the SD method since $ID^*|_k \in R^*$ by the restriction of the security model. If the adversary is Type-h-II, then there is no member index h_m^* since the adversary does not request a key query for $ID^*|_k$ or T^*. We next show that this lemma holds for two types of the adversary. To guess the type of the adversary, we simply toss a coin since there are only two types for the group index h.

For the Type-h-I adversary $\mathcal{A}_I$, we define hybrid games $\mathbf{H}_{(1,1),1}, \mathbf{H}_{(1,1),2}, \ldots, \mathbf{H}_{(q_m,q_c),1}, \mathbf{H}_{(q_m,q_c),2}, \mathbf{H}'_{(q_m,q_c),1}, \mathbf{H}'_{(q_m,q_c),2}, \ldots, \mathbf{H}'_{(1,1),1}, \mathbf{H}'_{(1,1),2}, \mathbf{H}''_1, \mathbf{H}''_2$ where $\mathbf{G}_{2,h-1} = \mathbf{H}_{(0,q_c),2}$, $\mathbf{H}''_2 = \mathbf{G}_{2,h}$, q_m is the maximum number of distinct member subsets of the group index h, and q_c is the maximum number of queries for one member subset. The structure of hybrid games is given in Table 6. These games are formally defined as follows:

Game $\mathbf{H}_{(h_m,h_c),1}$. This game $\mathbf{H}_{(h_m,h_c),1}$ for $1 \leq h_m \leq q_m$ and $1 \leq h_c \leq q_c$ is almost the same as $\mathbf{G}_{2,h-1}$ except the generation of HIBE keys and IBE keys with the group index h. These HIBE keys and IBE keys with indexes $(i_g = h, i_m, i_c)$ are generated as follows:

- **Case** $i_g < h$: The HIBE (or IBE) keys are generated as semi-functional.
- **Case** $i_g = h$: The HIBE (or IBE) keys are generated as follows:
  - $(i_m \neq h_m^* \wedge i_m < h_m)$ or $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c < h_c)$:
    If this is an HIBE key, then it generates a normal $SK'_{HIBE,S_{i,j}}$ and converts this key to a semi-functional-type2 $SK_{HIBE,S_{i,j}}$ by selecting a new random exponent $a_0 \in \mathbb{Z}_N$.
    If this is an IBE key, then it generates a normal $SK'_{IBE,S_{i,j}}$ and converts this key to a semi-functional-type2 $SK_{IBE,S_{i,j}}$ by selecting a new random exponent $a_1 \in \mathbb{Z}_N$.
  - $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c = h_c)$:
    If this is an HIBE key, then it generates a normal $SK'_{HIBE,S_{i,j}}$ and converts this key to a semi-functional-type1 $SK_{HIBE,S_{i,j}}$ by selecting new random exponents $a_0, b_0, \{z_i\} \in \mathbb{Z}_N$.
    If this is an IBE key, then it generates a normal $SK'_{IBE,S_{i,j}}$ and converts this key to a semi-functional-type1 $SK_{IBE,S_{i,j}}$ by selecting new random exponents $a_1, b_1 \in \mathbb{Z}_N$.
  - $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c > h_c)$ or $(i_m \neq h_m^* \wedge i_m > h_m)$:
    It simply creates a normal key.
  - $(i_m = h_m^*)$: It simply creates a normal key.
- **Case** $i_g > h$: The HIBE (or IBE) keys are generated as normal.

Recall that if $a_0 + \sum_{i=k+1}^{\ell} b_0 z_i I_i = b_0 d_0$, then this HIBE key is *nominally* semi-functional-type1. Similarly, if $a_1 = b_1 d_1$, then this IBE key is *nominally* semi-functional type-1.

Game $\mathbf{H}_{(h_m,h_c),2}$. This game $\mathbf{H}_{(h_m,h_c),2}$ is almost the same as $\mathbf{H}_{(h_m,h_c),1}$ except that the HIBE key (or the IBE key) with the indexes $(i_g = h, i_m, i_c)$ such that $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c = h_c)$ is generated with $b_0 = b_1 = 0$. In the game $\mathbf{H}_{(q_m,q_c),2}$, all HIBE keys and IBE keys with the group index h are semi-functional-type2 except that HIBE keys and IBE keys with the member index h_m^* are normal.

Game $\mathbf{H}'_{(h_m,h_c),1}$. This game $\mathbf{H}'_{(h_m,h_c),1}$ is almost the same as $\mathbf{H}_{(h_m,h_c),1}$ except the generation of an HIBE (or IBE) key with the indexes $(i_g = h, i_m, i_c)$ such that $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c \geq h_c)$ or $(i_m \neq h_m \wedge i_m > h_m)$. These HIBE (or IBE) keys are generated as follows:

- $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c = h_c)$: Let $\delta_{i,j}$ be a random exponent in $\mathbb{Z}_N$ that is fixed for this member subset $S_{i,j}$.
  If this is an HIBE key, then it generates a semi-functional-type1 $SK'_{HIBE,S_{i,j}} = (K'_0, K'_1, \{K'_{2,i}\})$ as the same as $\mathbf{H}_{(h_m,h_c),1}$ and creates a semi-functional-type1 $SK_{HIBE,S_{i,j}} = \big(K_0 = K'_0 g_2^{\delta_{i,j}}, K_1 = K'_1, \{K_{2,i} = K'_{2,i}\}\big)$ with a fixed δ.
  If this is an IBE private key, then it generates a semi-functional-type1 $SK'_{IBE,S_{i,j}} = (K'_0, K'_1)$ as the same as $\mathbf{H}_{(h_m,h_c),1}$ and creates a semi-functional-type1 $SK_{IBE,S_{i,j}} = \big(K_0 = K''_0 g_2^{\delta_{i,j}}, K_1 = K''_1\big)$ with a fixed δ.
- $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c > h_c)$ or $(i_m \neq h_m^* \wedge i_m > h_m)$:
  It creates a semi-functional HIBE (or IBE) key by using the fixed exponent $\delta_{i,j}$ for this member subset $S_{i,j}$.

Game $\mathbf{H}'_{(h_m,h_c),2}$. This game $\mathbf{H}'_{(h_m,h_c),2}$ is almost the same as $\mathbf{H}'_{(h_m,h_c),1}$ except that the HIBE key or IBE key with the indexes $(i_g = h, i_m, i_c)$ such that $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c = h_c)$ is generated with $b_0 = b_1 = 0$. In the game $\mathbf{H}'_{(1,1),2}$, all HIBE keys and all IBE keys with the group index h except the keys with the member index h_m^* are semi-functional where a fixed $\delta_{i,j}$ is used for each member.

Game $\mathbf{H}''_1$. This game $\mathbf{H}''_1$ is very similar to the game $\mathbf{H}'_{(1,1),2}$ except that the remaining HIBE keys and IBE keys with the member index h_m^* are changed to be semi-functional-type2 by using a random exponent.

Game $\mathbf{H}''_2$. This game $\mathbf{H}''_2$ is the same as $\mathbf{G}_{2,h}$. Compared to the game $\mathbf{H}''_1$, the HIBE keys and IBE keys with the member index h_m^* are changed to be semi-functional by using a fixed $\delta_{i,j}$ for this member subset $S_{i,j}$.

Let $\mathrm{Adv}_{\mathcal{A}_I}^{H_i}$ be the advantage of $\mathcal{A}_I$ in a game $\mathbf{H}_i$. From the following Lemmas 4.10, 4.11, 4.12, 4.13, 4.14, and 4.15, we obtain the following equation

$$\begin{aligned}
&\mathrm{Adv}_{\mathcal{A}_I}^{H_{(1,0),2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H_2''} \\
&\leq \sum_{h_m=1}^{q_m}\sum_{h_c=1}^{q_c} \left|\mathrm{Adv}_{\mathcal{A}_I}^{H_{(h_m,h_c-1),2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H_{(h_m,h_c),1}}\right| + \sum_{h_m=1}^{q_m}\sum_{h_c=1}^{q_c} \left|\mathrm{Adv}_{\mathcal{A}_I}^{H_{(h_m,h_c),1}} - \mathrm{Adv}_{\mathcal{A}_I}^{H_{(h_m,h_c),2}}\right| + \\
&\quad \sum_{h_m=1}^{q_m}\sum_{h_c=1}^{q_c} \left|\mathrm{Adv}_{\mathcal{A}_I}^{H'_{(h_m,h_c+1),2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H'_{(h_m,h_c),1}}\right| + \sum_{h_m=1}^{q_m}\sum_{h_c=1}^{q_c} \left|\mathrm{Adv}_{\mathcal{A}_I}^{H'_{(h_m,h_c),1}} - \mathrm{Adv}_{\mathcal{A}_I}^{H'_{(h_m,h_c),2}}\right| + \\
&\quad \left|\mathrm{Adv}_{\mathcal{A}_I}^{H'_{(1,1),2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H_1''}\right| + \left|\mathrm{Adv}_{\mathcal{A}_I}^{H_1''} - \mathrm{Adv}_{\mathcal{A}_I}^{H_2''}\right| \\
&\leq \sum_{h_m=1}^{q_m}\sum_{h_c=1}^{q_c} O(q_{sk}+q_{uk})\mathrm{Adv}_{\mathcal{B}}^{GSD}(\lambda).
\end{aligned}$$

For the Type-h-II adversary $\mathcal{A}_{II}$, we define hybrid games $\mathbf{I}_{(1,1),1}, \mathbf{I}_{(1,1),2}, \dots, \mathbf{I}_{(q_m,q_c),1}, \mathbf{I}_{(q_m,q_c),2}, \mathbf{I}'_{(q_m,q_c),1}, \mathbf{I}'_{(q_m,q_c),2}, \dots, \mathbf{I}'_{(1,1),1}, \mathbf{I}'_{(1,1),2}$ where $\mathbf{G}_{2,h-1} = \mathbf{I}_{(0,q_c),2}$ and $\mathbf{I}'_{(1,1),2} = \mathbf{G}_{2,h}$. The games are formally defined as follows:

Game $\mathbf{I}_{(h_m,h_c),1}$. This game $\mathbf{I}_{(h_m,h_c),1}$ is almost the same as $\mathbf{H}_{(h_m,h_c),1}$ except that there is no case $i_m = h_m^*$ since the adversary is Type-h-II.

Game $\mathbf{I}_{(h_m,h_c),2}$. This game $\mathbf{I}_{(h_m,h_c),2}$ is almost the same as $\mathbf{H}_{(h_m,h_c),2}$ except that there is no case $i_m = h_m^*$ since the adversary is Type-h-II.

Game $\mathbf{I}'_{(h_m,h_c),1}$. This game $\mathbf{I}'_{(h_m,h_c),1}$ is almost the same as $\mathbf{H}'_{(h_m,h_c),1}$ except that there is no case $i_m = h_m^*$ since the adversary is Type-h-II.

Game $\mathbf{I}'_{(h_m,h_c),2}$. This game $\mathbf{I}'_{(h_m,h_c),2}$ is almost the same as $\mathbf{H}'_{(h_m,h_c),2}$ except that there is no case $i_m = h_m^*$ since the adversary is Type-h-II. In the game $\mathbf{I}'_{(1,1),2}$, all HIBE keys and all IBE keys with the group index h are semi-functional where a fixed $\delta_{i,j}$ is used for each member.

Let $\mathrm{Adv}_{\mathcal{A}_{II}}^{I_i}$ be the advantage of $\mathcal{A}_{II}$ in a game $\mathbf{I}_i$. From the following Lemmas 4.16, 4.17, 4.18, and 4.19, we can obtain the equation

$$\mathrm{Adv}_{\mathcal{A}_{II}}^{I_{(0,q_c),2}} - \mathrm{Adv}_{\mathcal{A}_{II}}^{I'_{(1,1),2}} \leq \sum_{h_m=1}^{q_m}\sum_{h_c=1}^{q_c} \mathrm{Adv}_{\mathcal{B}}^{GSD}(\lambda)$$

.

Let E_I, E_{II} be the event such that an adversary behave like the Type-h-I, Type-h-II adversary respectively. From the above three inequalities for three types, we have the following inequality

$$\begin{aligned}
\mathrm{Adv}_{\mathcal{A}}^{G_{2,h-1}} - \mathrm{Adv}_{\mathcal{A}}^{G_{2,h}} &\leq \Pr[E_I]\left|\mathrm{Adv}_{\mathcal{A}_I}^{H_{(0,q_c),2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H_2''}\right| + \Pr[E_{II}]\left|\mathrm{Adv}_{\mathcal{A}_{II}}^{I_{(0,q_c),2}} - \mathrm{Adv}_{\mathcal{A}_{II}}^{I'_{(1,1),2}}\right| \\
&\leq \left|\mathrm{Adv}_{\mathcal{A}_I}^{H_{(0,q_c),2}} - \mathrm{Adv}_{\mathcal{A}_I}^{H_2''}\right| + \left|\mathrm{Adv}_{\mathcal{A}_{II}}^{I_{(0,q_c),2}} - \mathrm{Adv}_{\mathcal{A}_{II}}^{I'_{(1,1),2}}\right|.
\end{aligned}$$

This completes our proof. □

Lemma 4.7. *If the GSD assumption holds, then no PPT adversary can distinguish $\boldsymbol{G}_3$ from $\boldsymbol{G}_4$ with a non-negligible advantage.*

Lemma 4.8. *If the GSD assumption holds, then no PPT adversary can distinguish $\boldsymbol{G}_4$ from $\boldsymbol{G}_5$ with a non-negligible advantage.*

Lemma 4.9. *If the ComDH assumption holds, then no PPT adversary can distinguish $\boldsymbol{G}_5$ from $\boldsymbol{G}_6$ with a non-negligible advantage.*

The proofs of Lemmas 4.7, 4.8 and 4.9 are the same as those of Lemmas 3.7, 3.8 and 3.11.

4.4.1 Type-h-I Adversary

Lemma 4.10. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish $\boldsymbol{H}_{(h_m,h_c-1),2}$ from $\boldsymbol{H}_{(h_m,h_c),1}$ with a non-negligible advantage.*

Proof. Suppose there exists an adversary $\mathcal{A}_I$ that distinguishes $\mathbf{H}_{(h_m,h_c-1),2}$ from $\mathbf{H}_{(h_m,h_c),1}$ with a non-negligible advantage. A simulator $\mathcal{B}$ that solves the GSD assumption using $\mathcal{A}_I$ is given: a challenge tuple $D = ((N,\mathbb{G},\mathbb{G}_T,e), g_1, g_3, X_1R_1, R_2Y_1)$ and Z where $Z = Z_0 = X_2Y_2$ or $Z = Z_1 = X_2R_3Y_2$. Then $\mathcal{B}$ that interacts with $\mathcal{A}_I$ is described as follows:

Setup: $\mathcal{B}$ first chooses random exponents $h', u'_1, \ldots, u'_L, v', w', \alpha \in \mathbb{Z}_N$. It sets $MK = \alpha$ and publishes $PP = \big((N,\mathbb{G},\mathbb{G}_T,e), g = g_1, Y = g_3, h = g^{h'}, u_1 = g^{u'_1}, \ldots, u_L = g^{u'_L}, v = g^{v'}, w = g^{w'}, \Omega = e(g,g)^{\alpha}\big)$.

Phase 1: Let h^*_m be a member index of the group index h such that the HIBE key for $ID^*|_k$ or the IBE key for T^* belong to the member index h^*_m such that $1 \le h^*_m \le q_m$ where q_m is the maximum number of members in the group index h. As mentioned before, there is only one index h^*_m in the Type-h-I adversary. By randomly selecting an index, $\mathcal{B}$ can correctly guess h^*_m with the probability of $1/q_m$. Note that $q_m \le q_{sk} + q_{uk}$ since the private set of a private key is related with a path and the cover set of an update key is a partition where q_{sk} is the number of private key queries and q_{uk} is the number of update key queries of the adversary.

For each query, $\mathcal{B}$ proceeds as follows: If this is a decryption key query, then it creates a normal one since it knows MK. If this is an HIBE key or an IBE key with indexes (i_g, i_m, i_c), then it handles this query as follows:

- **Case** $i_g < h$: It builds a normal key by using MK and converts the key to a semi-functional one with a fixed random exponent $\delta'_{i,j}$ for the subset $S_{i,j}$ by using R_2Y_1.
- **Case** $i_g = h$: It generates the key as follows:
  - $(i_m \ne h^*_m \wedge i_m < h_m)$ or $(i_m \ne h^*_m \wedge i_m = h_m) \wedge (i_c < h_c)$:
    If this is an HIBE key, then it builds a normal key and converts the key to a semi-functional-type2 $SK_{HIBE,S_{i,j}}$ with a new random exponent $a'_0 \in \mathbb{Z}_N$ by using R_2Y_1.
    If this is an IBE key, then it builds a normal key and converts the key to a semi-functional-type2 $SK_{IBE,S_{i,j}}$ with a new random exponent $a'_1 \in \mathbb{Z}_N$ by using R_2Y_1.
  - $(i_m \ne h^*_m \wedge i_m = h_m) \wedge (i_c = h_c)$:
    If this is an HIBE key, then it chooses random $Y'_0, Y'_1, \{Y'_{2,i}\} \in \mathbb{G}_{p_3}$ and creates an HIBE key

$$SK_{HIBE,S_{i,j}} = \big(K_0 = g^{f_{GL}(L_j)}(Z)^{h'+\sum_{i=1}^{k} u'_i I_i} Y'_0,\ K_1 = (Z)^{-1} Y'_1,\ \{K_{2,i} = (Z)^{u'_i} Y'_{2,i}\}\big).$$

    If this is an IBE key, then it chooses random $Y'_0, Y'_1 \in \mathbb{G}_{p_3}$ and creates an IBE key

$$SK_{IBE,S_{i,j}} = \big(K_0 = g^{f_{GL}(L_j)}(Z)^{v'+w'T} Y'_0,\ K_1 = (Z)^{-1} Y'_1\big).$$

  - $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c > h_c)$ or $(i_m \neq h_m^* \wedge i_m > h_m)$: It creates a normal key by using MK.
  - $(i_m = h_m^*)$: It creates a normal key by using MK.

- **Case** $i_g > h$: It creates a normal key by using MK.

Challenge: For challenge $ID^*|_\ell$ and T^*, $\mathcal{B}$ builds $CH_{HIBE} = \big(C_0 = X_1R_1, C_1 = (X_1R_1)^{h'+\sum_{i=1}^{\ell} u_i' I_i^*}\big)$ and $CH_{IBE} = (C_0 = X_1R_1, C_1 = (X_1R_1)^{v'+w'T^*})$. Next, it flips a random coin $\mu \in \{0,1\}$ and creates the semi-functional challenger ciphertext $CT_{ID^*|_\ell, T^*} = (CH_{HIBE}, CH_{IBE}, C = e(X_1R_1, g)^\alpha \cdot M_\mu^*)$.
Phase 2: Same as Phase 1.
Guess: $\mathcal{A}$ outputs a guess μ'. If $\mu = \mu'$, then $\mathcal{B}$ outputs 1. Otherwise, it outputs 0.

If $Z = Z_0 = X_2Y_2$, then the simulation is the same as $\mathbf{H}_{(h_m, h_c-1),2}$ since the HIBE key (or the IBE key) with $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c = h_c)$ and the semi-functional challenge ciphertext are correctly distributed. If $Z = Z_1 = X_2R_3Y_2$, then the simulation is almost the same as $\mathbf{H}_{(h_m,h_c),1}$ except that the HIBE key (or the IBE key) with $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c = h_c)$ is generated as nominally semi-functional-type1 by implicitly setting $a_0 \equiv \log_{g_2}(R_3)(h' + \sum_{i=1}^{k} u_i' I_i) \mod p_2$ (or $a_1 \equiv \log_{g_2}(R_3)(v' + w'T) \mod p_2$), $b_0 \equiv b_1 \equiv \log_{g_2}(R_3) \mod p_2$, and $z_i \equiv u_i' \mod p_2$. Note that we solve the paradox of dual system encryption by introducing the nominally semi-functional-type1 key.

Next, we should argue that the Type-h-I adversary cannot distinguish a nominally semi-functional-type1 key from a semi-functional-type1 key. For this argument, we show an information theoretic argument by using the fact that $ID|_k \notin \text{Prefix}(ID^*|_\ell)$ for all HIBE keys with indexes $(i_g = h, i_m, i_c)$ such that $i_m \neq h_m^*$, and $T \neq T^*$ for all IBE keys with indexes $(i_g = h, i_m, i_c)$ such that $i_m \neq h_m^*$. The analysis of this information theoretic argument is the same as that in Lemma 3.12. This completes our proof. □

Lemma 4.11. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish* $\mathbf{H}_{(h_m,h_c),1}$ *from* $\mathbf{H}_{(h_m,h_c),2}$ *with a non-negligible advantage.*

Proof. The proof of this lemma is almost the same as that of Lemma 4.10 except the generation of the key with indexes $i_g = h$ and $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c = h_c)$. This key with the group index h is generated as follows:

- $(i_m \neq h_m^* \wedge i_m = h_m) \wedge (i_c = h_c)$:
  If this is an HIBE key, then it chooses random $a_0' \in \mathbb{Z}_N$, $Y_0', Y_1', \{Y_{2,i}'\} \in \mathbb{G}_{p_3}$ and creates an HIBE key
  $$SK_{HIBE,S_{i,j}} = \big(K_0 = g^{f_{GL}(L_j)}(Z)^{h'+\sum_{i=1}^{k} u_i' I_i} Y_0' (R_2Y_1)^{a_0'},\ K_1 = (Z)^{-1} Y_1',\ \{K_{2,i} = (Z)^{u_i'} Y_{2,i}'\}\big).$$
  If this is an IBE key, then it chooses random $a_1' \in \mathbb{Z}_N$, $Y_0', Y_1' \in \mathbb{G}_{p_3}$ and creates an IBE key
  $$SK_{IBE,S_{i,j}} = \big(K_0 = g^{f_{GL}(L_j)}(Z)^{v'+w'T} Y_0' (R_2Y_1)^{a_1'},\ K_1 = (Z)^{-1} Y_1'\big).$$

Note that this HIBE key (or IBE key) is no longer correlated with the challenge ciphertext since K_0 is randomized by $(R_2Y_1)^{a_0'}$ (or $(R_2Y_1)^{a_1'}$). □

Lemma 4.12. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish* $\mathbf{H}'_{(h_m,h_c-1),2}$ *from* $\mathbf{H}'_{(h_m,h_c),1}$ *with a non-negligible advantage.*

Lemma 4.13. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish* $\mathbf{H}'_{(h_m,h_c),1}$ *from* $\mathbf{H}'_{(h_m,h_c),2}$ *with a non-negligible advantage.*

The proofs of Lemmas 4.12 and 4.13 are almost the same as those of Lemmas 4.10 and 4.11 respectively. The only difference is that K_0 of an HIBE key and K_0 of an IBE key with indexes $(i_g = h, i_m, i_c)$ such that $i_m \neq h_m^*$ that are generated in Lemmas 4.10 and 4.11 respectively are additionally multiplied by $(R_2Y_1)^{\delta'_{i,j}}$ where $\delta'_{i,j}$ is a fixed exponent that is related with the member subset $S_{i,j}$. This modification is possible since R_2Y_1 is given in the assumption. We omit the detailed proofs of these lemmas.

Lemma 4.14. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish $\boldsymbol{H}'_{(1,0),2}$ from $\boldsymbol{H}''_1$ with a non-negligible advantage.*

Proof. The proof of this lemma is the important part of the security proof since it changes the HIBE key for $ID^*|_k \in \text{Prefix}(ID^*|_\ell)$ and the IBE key for T^* from a normal type to a semi-functional type. It should be noted that this changes from normal to semi-functional cannot be handled by introducing a nominally semi-functional type since an information theoretic argument for $ID^*|_k$ and T^* cannot be used. Recall that h_m^* be the member index that is related to $ID^*|_k$ and T^*. To solve this problem, we directly change normal keys for h_m^* to semi-functional keys without introducing nominally semi-functional keys, and then we argue that the paradox of dual system encryption can be solved by the property of the Lagrange interpolation method.

Many parts of this proof is similar to that of Lemma 4.10 except the generation of HIBE keys and IBE keys with the group index $i_g = h$. These keys with the group index $i_g = h$ are generated as follows:

- **Case** $i_g = h$: Let $\delta'_{i,j}$ be a fixed exponent in $\mathbb{Z}_N$ for each member $S_{i,j}$ in this group index h.
  - $(i_m \neq h_m^*)$: If this is an HIBE key, then it selects random $r_1 \in \mathbb{Z}_N$, $Y'_0, Y'_1, \{Y'_{2,i}\} \in \mathbb{G}_{p_3}$ and creates an HIBE key
  $$SK_{HIBE,S_{i,j}} = \big(K_0 = (Z)^{L_j} g^{\beta_{ID|_{k-1}}} F(ID|_k)^{r_1} Y'_0 \cdot (R_2Y_1)^{\delta'_{i,j}},\ K_1 = g^{-r_1}Y'_1,\ \{K_{2,i} = u_i^{r_1} Y'_{2,i}\}\big).$$
  If this is an IBE key, then it selects random $r_2 \in \mathbb{Z}_N$, $Y'_0, Y'_1 \in \mathbb{G}_{p_3}$ and creates an IBE key
  $$SK_{IBE,S_{i,j}} = \big(U_0 = (Z)^{L_j} g^{\beta_{ID|_{k-1}}} (vw^T)^{r_2} Y'_0 \cdot (R_2Y_1)^{\delta'_{i,j}},\ U_1 = g^{-r_2}Y'_1\big).$$
  - $(i_m = h_m^*)$: If this is an HIBE key, then it selects random $r_1 \in \mathbb{Z}_N$, $Y'_0, Y'_1, \{Y'_{2,i}\} \in \mathbb{G}_{p_3}$ and creates an HIBE key
  $$SK_{HIBE,S_{i,j}} = \big(K_0 = (Z)^{L_j} g^{\beta_{ID|_{k-1}}} F(ID|_k)^{r_1} Y'_0,\ K_1 = g^{-r_1}Y'_1,\ \{K_{2,i} = u_i^{r_1} Y'_{2,i}\}\big).$$
  If this is an IBE key, then it selects random $r_2 \in \mathbb{Z}_N$, $Y'_0, Y'_1 \in \mathbb{G}_{p_3}$ and creates an IBE key
  $$SK_{IBE,S_{i,j}} = \big(U_0 = (Z)^{L_j} g^{\beta_{ID|_{k-1}}} (vw^T)^{r_2} Y'_0,\ U_1 = g^{-r_2}Y'_1\big).$$

If $Z = Z_0 = X_2Y_2$, then the simulation is the same as $\mathbf{H}'_{(1,0),2}$ since all HIBE keys and IBE keys with the group index h implicitly uses a random polynomial $f_{GL}(x) \equiv \log_g(X_2) \cdot x + \beta_{ID|_{k-1}} \mod p_1$ and it implicitly sets $\delta_{i,j} \equiv \log_{g_2}(R_2)\delta'_{i,j} \mod p_2$ for each member index $i_m \neq h_m^*$. If $Z = Z_1 = X_2R_3Y_2$, then the simulation is the same as $\mathbf{H}''_1$ since it implicitly sets $\delta_{i,j} = \log_{g_2}(R_3)L_j \mod p_2$ for the member index h_m^*. As mentioned before, the HIBE key for $ID^*|_k$ and the IBE key for T^* should belong to the same member index h_m^* by the restriction $ID^*|_k \in R^*$ of the security model.

We now show that the paradox of dual system encryption can be solved. To check whether an HIBE key for h_m^* and an IBE key for h_m^* are normal or semi-functional, the simulator may try to decrypt a semi-functional ciphertext by deriving a decryption key from these keys for h_m^*. However, the simulator cannot derive a decryption key from those keys since the Lagrange interpolation method does not work for the same h_m^* since only one point of $f_{GL}(x)$ is revealed. Recall that the Lagrange interpolation method requires two points of $f_{GL}(x)$ to derive $f_{GL}(0)$. Thus, the simulator cannot check whether these two keys for the same h_m^* are normal or semi-functional. This completes our proof. $\square$

Lemma 4.15. *If the GSD assumption holds, then no PPT Type-h-I adversary can distinguish $\mathbf{H}_1''$ from $\mathbf{H}_2''$ with a non-negligible advantage.*

Proof. The proof is similar to that of Lemma 4.14 except the generation of HIBE keys and IBE keys with the group index $i_g = h$ and the member index $i_m = h_m^*$. These keys with the indexes $i_g = h$ and $i_m = h_m^*$ are generated as follows:

- **Case** $i_g = h$: Let $\delta'_{i,j}$ be a fixed exponent in $\mathbb{Z}_N$ for each member $S_{i,j}$ in this group index h.
  - ($i_m = h_m^*$): If this is an HIBE key, then it selects random $r_1 \in \mathbb{Z}_N$, $Y_0', Y_1', \{Y_{2,i}'\} \in \mathbb{G}_{p_3}$ and creates an HIBE key

    $$SK_{HIBE,S_{i,j}} = \big(K_0 = (Z)^{L_j} g^{\beta_{ID|_{k-1}}} F(ID|_k)^{r_1} Y_0' \cdot (R_2 Y_1)^{\delta'_{i,j}},\ K_1 = g^{-r_1} Y_1',\ \{K_{2,i} = u_i^{r_1} Y_{2,i}'\}\big).$$

    If this is an IBE key, then it selects random $r_2 \in \mathbb{Z}_N$, $Y_0', Y_1' \in \mathbb{G}_{p_3}$ and creates an IBE key

    $$SK_{IBE,S_{i,j}} = \big(U_0 = (Z)^{L_j} g^{\beta_{ID|_{k-1}}} (vw^T)^{r_2} Y_0' \cdot (R_2 Y_1)^{\delta'_{i,j}},\ U_1 = g^{-r_2} Y_1'\big).$$

If $Z = Z_1 = X_2 R_3 Y_2$, then the simulation is the same as $\mathbf{H}_1''$ since it implicitly sets $\delta_{i,j} = \log_{g_2}(R_3) L_j + \log_{g_2}(R_2)\delta'_{i,j} \mod p_2$ for the member index h_m^*. If $Z = Z_0 = X_2 Y_2$, then the simulation is the same as $\mathbf{H}_2''$ since all HIBE keys and IBE keys with the group index h implicitly uses a random polynomial $f_{GL}(x) \equiv \log_g(X_2) \cdot x + \beta_{ID|_{k-1}} \mod p_1$ and it implicitly sets $\delta_{i,j} \equiv \log_{g_2}(R_2)\delta'_{i,j} \mod p_2$ for all member indexes. This completes our proof. $\square$

4.4.2 Type-*h*-II Adversary

Lemma 4.16. *If the GSD assumption holds, then no PPT Type-h-II adversary can distinguish $\mathbf{I}_{(h_m,h_c-1),2}$ from $\mathbf{I}_{(h_m,h_c),1}$ with a non-negligible advantage.*

Lemma 4.17. *If the GSD assumption holds, then no PPT Type-h-II adversary can distinguish $\mathbf{I}_{(h_m,h_c),1}$ from $\mathbf{I}_{(h_m,h_c),2}$ with a non-negligible advantage.*

Lemma 4.18. *If the GSD assumption holds, then no PPT Type-h-II adversary can distinguish $\mathbf{I}'_{(h_m,h_c+1),2}$ from $\mathbf{I}'_{(h_m,h_c),1}$ with a non-negligible advantage.*

Lemma 4.19. *If the GSD assumption holds, then no PPT Type-h-II adversary can distinguish $\mathbf{I}'_{(h_m,h_c),1}$ from $\mathbf{I}'_{(h_m,h_c),2}$ with a non-negligible advantage.*

The proofs of Lemmas 4.16, 4.17, 4.18, and 4.19 are almost the same as those of Lemmas 4.10, 4.11, 4.12, and 4.13 respectively except that there is no case $i_m = h_m^*$ since the Type-*h*-II adversary does not request an HIBE key for $ID^*|_k \in \text{Prefix}(ID^*|_\ell)$ and an IBE key for T^*. We omit the detailed proofs of these lemmas.

5 Conclusion

In this work, we proposed two RHIBE schemes by combining LW-HIBE and LW-IBE schemes in composite-order bilinear groups, and the CS (or SD) scheme in a modular way, and then we proved the adaptive security of our RHIBE schemes by using the dual system encryption technique. As mentioned before, we carefully re-designed hybrid games to use the dual system encryption technique since a naive approach of dual system encryption does not work. Our RHIBE schemes are the first RHIBE schemes that achieve the adaptive security.

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