Tight on Budget?
Tight Bounds for r-Fold Approximate Differential Privacy

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September 5, 2018

Abstract

Many applications, such as anonymous communication systems, privacy-enhancing database queries, or privacy-enhancing machine-learning methods, require robust guarantees under thousands and sometimes millions of observations. The notion of r-fold approximate differential privacy (ADP) offers a well-established framework with a precise characterization of the degree of privacy after r observations of an attacker. However, existing bounds for r-fold ADP are loose and, if used for estimating the required degree of noise for an application, can lead to over-cautious choices for perturbation randomness and thus to suboptimal utility or overly high costs.

We present a numerical and widely applicable method for capturing the privacy loss of differentially private mechanisms under composition, which we call privacy buckets. With privacy buckets we compute provable upper and lower bounds for ADP for a given number of observations. We compare our bounds with state-of-the-art bounds for r-fold ADP, including Kairouz, Oh, and Viswanath's composition theorem (KOV), concentrated differential privacy and the moments accountant. While KOV proved optimal bounds for heterogeneous adaptive k-fold composition, we show that for concrete sequences of mechanisms tighter bounds can be derived by taking the mechanisms' structure into account. We compare previous bounds for the Laplace mechanism, the Gauss mechanism, for a timing leakage reduction mechanism, and for the stochastic gradient descent and we significantly improve over their results (except that we match the KOV bound for the Laplace mechanism, for which it seems tight). Our lower bounds almost meet our upper bounds, showing that no significantly tighter bounds are possible.

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1 Introduction

Approximate differential privacy (ADP [4]) has been designed to quantify, with two parameters (ε, δ), the privacy leakage of systems that require a careful trade-off between the system's usefulness and the system's privacy leakage. Since its introduction, ADP has been successfully used to quantify the privacy leakage of privacy-enhancing mechanisms in various applications, including query-response of sensitive databases, training deep neural networks [1] while protecting the training data, and even anonymous communication [26]. This privacy leakage, i.e., the (ε, δ) parameters, inevitably grows under continual observation; thus, privacy eventually deteriorates (see Apple's case [25]). In many application scenarios, continual attacker-observation is unavoidable, e.g., an attacker may have thousands if not hundreds of thousands of observation points.

Precisely computing the (ε, δ) parameters after r observations, called r-fold ADP bounds, is hard. However, imprecise bounds on (ε, δ) can lead to either a wrong perception of the privacy leakage (resulting, e.g., in unsatisfied customers) or to an over-cautious choice of system parameters (resulting in unnecessarily high costs). There is a rich body of work on approximating r-fold ADP-bounds [8, 14, 1, 20, 7, 2]. Early work did not take the shape of the mechanism's output distribution into account [8, 14] (*mechanism-oblivious* bounds). We show that the best mechanism-oblivious bounds are tight w.r.t. the Laplace mechanism but imprecise for many other mechanisms, e.g., the Gaussian mechanism, as these bounds inherently assume a worst-case behavior under composition. Recent work [1, 20, 7, 2], in contrast, introduced *mechanism-aware* bounds that take the shape of the output distribution of the mechanism into account and achieve significantly tighter bounds for some particular mechanism, such as the Gaussian mechanism. However, it was not clear how tight previous mechanism-aware bounds are and whether they can be further improved.

1.1 Contribution

We introduce a numerical method—*privacy buckets*—for computing upper and lower r-fold (ε, δ)-ADP bounds that take the mechanisms and their (fixed) noise parameters into account. To this end, we utilize a discretized version of the privacy loss random variable introduced by Dwork and Rothblum [7]. Our approach is sufficiently general to subsume the generic adaptive r-fold ADP bounds of prior work [14, 21]. We compare our upper bounds with state-of-the-art bounds and achieve significant improvements over all of them. Moreover, our lower bounds almost meet the upper bounds, showing that no significantly tighter bounds are possible.

Our evaluations illustrate that privacy buckets can give insights about the composition behavior of various mechanisms. We find that for the right choice of scale parameter and standard-deviation, the Laplace mechanism and the Gauss mechanism converge to the same privacy leakage, i.e., their (ε, δ) parameters coincide from a sufficiently high number of observations r onward.[1]

Our method is useful for deriving tight bounds for classical differential privacy mechanisms but can also be applied to any privacy analysis resulting in ADP. We exemplify this statement by computing bounds for the anonymous communication system Vuvuzela [26], the stochastic gradient descent mechanism for deep learning [1] and for timing-leakage histograms of a recently introduced browser extension for deniable communication [24].

2 Background and Related work

In this section, we review the notion of differential privacy, highlight an often implicit assumption in the analysis of differentially private mechanisms, generalize differential privacy to pairs of distributions, and position our work in the work from the literature.

Differential privacy Differential privacy (DP) [3] quantifies how close the distributions of outputs of a mechanism on two similar inputs are. We say that a mechanism M is ε-DP, if for any two closely related inputs D_1, D_2, $\forall S \subseteq \mathcal{U}$, $\Pr[M(D_1) \in S] \leq e^{\varepsilon} \cdot \Pr[M(D_2) \in S]$. To extend the applicability of DP, approximate differential privacy [4] (ADP) has been introduced, which allows for distributions to exceed a

[1] For the expert reader, this observation indicates that the result from Dwork and Rothblum [7], that subgaussian privacy loss variables compose (at most as badly) as a Gaussian privacy loss variable, can be generalized.

limiting factor ε, as long as this deviation can be limited to a value δ as follows: $\forall S \subseteq \mathcal{U},\ \Pr[M(D_1) \in S] \leq e^{\varepsilon} \cdot \Pr[M(D_2) \in S] + \delta$. In this work we focus on ADP.

2.1 Worst case distributions for ADP

Classically, differential privacy argues about the output of a probabilistic mechanism M that is run on similar inputs (e.g., neighboring databases). Since M is probabilistic, the application of M to any input D can be seen as a random variable with outputs from a distribution $M(D)$. Differential privacy requires the outputs of M on all pairs of neighboring databases w.r.t. an application specific sensitivity-metric, i.e., all pairs of distributions $M(D_1), M(D_2)$, where D_1 and D_2 are neighboring, to be closely related (quantified via the privacy parameters ε and δ).

Our approach operates on individual pairs of distributions. While this formalization is unconventional and at first glance might seem to restrict the applicability to particular queries, our approach leads to far more general results. We elaborate that considering pairs of distributions is very natural in the context of differentially private mechanisms, e.g., to analyze mechanisms in the presence of arbitrary adversarial queries. In the simplest case the proofs analyze mechanisms using pairs of inputs that are worst-case in the following sense: in terms of privacy these inputs are as bad as any other pair of inputs for a given sensitivity (see Definition 11). Applying such worst-case inputs (x_0, x_1) to a mechanism M leads to a pair of *worst-case distributions* $(M(x_0), M(x_1))$. Such worst-case inputs appear in many [6], but not all proofs for DP. In more complex cases, explicit worst-case distributions are used in the analysis [1].

To understand why worst-case distributions in general are an integral part of DP proofs, we now discuss, on an abstract level, how we typically prove that a mechanism satisfies DP.

Most differential privacy analyses implicitly use worst-case distributions For illustration, we consider a mechanism, where $M(D, q)$ (for a database D and a query q) can be divided into a precise response $f(D, q)$ to a query and an independent noise distribution N. In the simplest case, mechanisms are of the form $M(D, q) = f(D, q) + N$. Examples of this structure include the Laplace mechanism, the Gauss mechanism, as well as any other distribution of noise N added to some function $f(D, q)$, where N does not depend on $f(D, q)$. In all these cases, differential privacy guarantees can be calculated based solely on the distribution of the noise and on the sensitivity $\Delta f = \max_{D_1, D_2 \text{ neighboring}} |f(D_1, q) - f(D_2, q)|$.[2]

To show that M satisfies (approximate) differential privacy, the proof typically analyzes the two distributions N and $N + \Delta f$, implicitly assuming that $f(D_1, q) = 0$ and $f(D_2, q) = \Delta f$. The proof then argues that for any value $\Delta' < \Delta f$ the distributions N and $N + \Delta'$ also satisfy differential privacy. From this simplified analysis it can then be derived (implicitly) that for all other values of $f(D_1, q)$ and $f(D_2, q)$ s.t., $|f(D_1, q) - f(D_2, q)| \leq \Delta f$, $f(D_1, q) + N$ and $f(D_2, q) + N$ also satisfy differential privacy, which concludes the analysis. In any such analysis, there are worst-case distributions N and $N + \Delta f$, thus, our approach is compatible.

A prominent example of such an analysis is a recent work on a differentially private a mechanism for privacy-preserving stochastic gradient descent [1]. In this work, Abadi et al. first prove that a pair of a Gaussian distribution a Gaussian mixture distribution is worst case for their analysis, and they then estimate differential privacy for this pair of distributions.

Worst-case distributions for the Laplace mechanism As an example, let us consider counting queries q with sensitivity 1 to which Laplace noise is added: the mechanism M that gets a database D as input is defined as $M(D) := q(D) + \mathrm{LP}_{\lambda,0}$, where $\mathrm{LP}_{\lambda,\mu}$ is the Laplace distribution with scale parameter λ and mean μ (and $f(D, q) := q(D)$). In this example, it suffices to only consider $\mathrm{LP}_{\lambda,0}$ and $\mathrm{LP}_{\lambda,1}$, with means 0 and 1, instead of considering $M(D_0)$ and $M(D_1)$ for all possible combinations of neighboring databases D_0 and D_1: Let D_0 and D_1 be two neighboring databases where the true answers to a query q are $q(D_0) = x$ and $q(D_1) = x + 1$, respectively, for some value x.[3] We can map any output y drawn from $\mathrm{LP}_{\lambda,\mu}$ (for $\mu \in \{0, 1\}$)

[2] The notion of neighboring databases differs from application to application. That said, databases are typically called neighboring if they differ in at most one row.

[3] Since the differential privacy guarantee and analysis are symmetric, we can assume without loss of generality that $q(D_0) < q(D_1)$.

to $y+x$ to obtain the correct adversarial view for the respective scenario $M(D_i) = q(D_i) + \mathrm{LP}_{\lambda,0}$.

What if my differential privacy analysis doesn't implicitly use worst-case distributions? If the mechanism does not consist of and cannot be reduced to independent noise being added to a numerical value, the above simplified description does not immediately apply. However, Kairouz, Oh, and Viswanath [14], and Murtagh and Vadhan [22, Lemma 3.2 & Lemma 3.7] show that for any (ε, δ)-DP mechanism M there is a worst case mechanism $M_{\varepsilon,\delta}$ operating on a single bit such that ADP guarantees for $M_{\varepsilon,\delta}$ directly translate to ADP guarantees for M, even under r-fold adaptive composition.[4]

In more detail, they show [22, Lemma 3.2] that there is a probabilistic translation T that relates every differentially private mechanism M on two neighboring inputs D_0 and D_1 to a generic pair of distributions $M_{\varepsilon,\delta}(0)$ and $M_{\varepsilon,\delta}(1)$. They then leverage the post-processing property of differential privacy to show that analyzing $M_{\varepsilon,\delta}(0)$ and $M_{\varepsilon,\delta}(1)$ is sufficient, even under composition.

Thus, considering such worst-case distributions and their behavior under composition is sufficient for deriving bounds on the mechanism M. Moreover, even for different mechanisms and adaptively chosen neighboring inputs, one can compute sound bounds on differential privacy by only considering the distributions $M_{\varepsilon,\delta}(0)$ and $M_{\varepsilon,\delta}(1)$. This technique gives us a fall-back plan for computing bounds on the distributions of $M_{\varepsilon,\delta}(0)$ and $M_{\varepsilon,\delta}(1)$ if no more than ε and δ of the mechanism M is known. If more about the distribution of the output of M is known (ideally two exact worst-case output distributions) we can derive significantly tighter bounds.

2.2 Tight ADP on distributions

Approximate differential privacy is typically captured with two parameters ε and δ. In this work we show that considering not just one such pair of parameters, but a parameter space helps to derive tight adaptive r-fold ADP composition results. This parameter space of two distributions can be represented as a function $\delta(\varepsilon)$ such that $(\varepsilon, \delta(\varepsilon))$-ADP holds and $\delta(\varepsilon)$ is minimal (for $\varepsilon \geq 0$). To capture this minimality, we define (tight) differential privacy (generalized to pairs of distributions) and show how to precisely compute $\delta(\varepsilon)$.

Definition 1 ((Tight) ADP)**.** *Two distributions A and B over the universe $\mathcal{U}$ are (ε, δ)-ADP, if for every set $S \subseteq \mathcal{U}$,*

$$P_A(S) \leq e^{\varepsilon} P_B(S) + \delta(\varepsilon) \text{ and } P_B(S) \leq e^{\varepsilon} P_A(S) + \delta(\varepsilon),$$

where $P_A(x)$ denotes the probability of the event x in A and $P_B(x)$ denotes the probability of the event x in B. We call A and B tightly *$(\varepsilon, \delta(\varepsilon))$-ADP if they are $(\varepsilon, \delta(\varepsilon))$-ADP, and $\forall \delta' \leq \delta(\varepsilon)$ such that A and B are (ε, δ')-ADP we have $\delta(\varepsilon) = \delta'$.*

A note on utility and sensitivity. In the remainder of the paper, we consider pairs of distributions. In particular, analyzing the ADP-parameter of mechanisms amounts to analyzing the respective worst-case inputs or worst-case distributions for a given sensitivity. Hence, we can abstract away from the sensitivity of two inputs and we can abstract away from the utility functions of a task.

Computing a tight ADP bound. To see that and how we can compute $\delta(\varepsilon)$, first consider that any pair of distributions is $(\varepsilon, 1)$-ADP for arbitrary $\varepsilon \geq 0$. More precise bounds for distributions A and B can be captured by setting $\delta(\varepsilon)$ to the area between the probability distributions of B and a scaled-up version of A: we multiply every point of the curve of A with e^{ε} (which is not a probability distribution anymore, because it sums up to e^{ε} instead of to 1). Any area where B is larger than this scaled-up curve contains probability mass for events x outside of the multiplicative bound, i.e., for which we have $P_A(x) \leq e^{\varepsilon} P_B(x)$. The difference between those terms is precisely what we need to characterize. We refer to Figure 1 for the $(\varepsilon, \delta(\varepsilon))$-graph of possible (tight) ADP-bounds for two truncated Gaussian distributions (left side) and for a graphical depiction of this intuition (right side).

[4] For some mechanisms the privacy parameters (e.g., a Gaussian distribution's standard deviation σ) can be adaptively chosen for every run and can depend on previous adversarial observations. If privacy cannot be bounded per response or the number of relevant responses for computing privacyis unbounded, worst-case distributions might not exist [23].

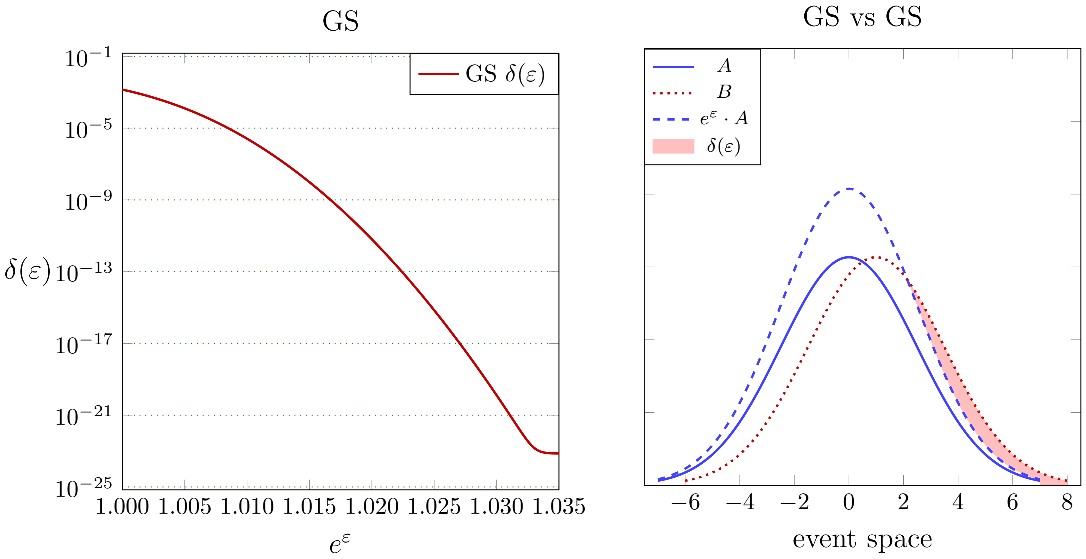


Figure 1: A graph depicting $\delta(\varepsilon)$ for the truncated Gauss mechanism (left) and a graphical depiction of how to compute $\delta(\varepsilon)$ for the truncated Gauss mechanism (right). Note that $e^{\varepsilon} \cdot A$ is not a probability distribution.

Lemma 1. *For every ε, two distributions A and B over a finite universe $\mathcal{U}$ are tightly (ε, δ)-ADP with*

$$\begin{aligned}\delta = \max \Bigg(& \sum_{x \in U} \max\left(P_A(x) - e^{\varepsilon} P_B(x), 0\right), \\ & \sum_{x \in U} \max\left(P_B(x) - e^{\varepsilon} P_A(x), 0\right) \Bigg)\end{aligned}$$

Proof. Let $\varepsilon \geq 0$ and let A and B be two distributions over the universe $\mathcal{U}$. We show the equivalence by first showing that (1) for every set S, the calculation describes an upper bound and then that (2) there exists a set S such that this bound is tight.

(1) We show that $\forall S \subseteq \mathcal{U}$,

$$\begin{aligned} & P_A(x \in S : x) - e^{\varepsilon} P_B(x \in S : x) \\ \leq & \sum_{x \in U} \max\left(P_A(x) - e^{\varepsilon} P_B(x), 0\right) \end{aligned}$$

The inverse direction then follows analogously.

$$\begin{aligned} & P_A(x \in S : x) - e^{\varepsilon} P_B(x \in S : x) \\ = & \sum_{x \in S} P_A(x) - e^{\varepsilon} P_B(x) \\ \leq & \sum_{x \in S} \max\left(P_A(x) - e^{\varepsilon} P_B(x), 0\right) \\ \leq & \sum_{x \in U} \max\left(P_A(x) - e^{\varepsilon} P_B(x), 0\right) \end{aligned}$$

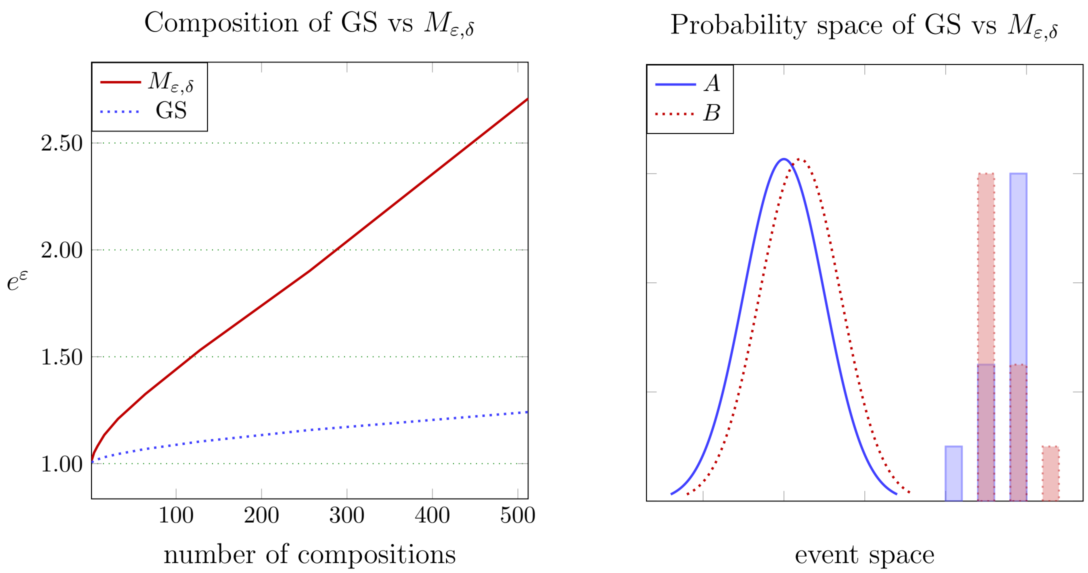


Figure 2: Comparison between the truncated Gauss mechanisms (blue) and a randomized response mechanism $M_{\varepsilon,\delta}$ (red). We show how e^ε evolves in both cases for a fixed $\delta = 10^{-4}$ (left side). Note that both graphs start at the same point, but quickly diverge. For ease of understanding we depict the probability distributions of interest for both mechanisms (right); here, randomized response directly consists of only 4 possible events. For both mechanisms, we portray the two distributions A and B.

(2) Let $S := \{x \in \mathcal{U} \text{ s.t. } \Pr[x \in A] \geq e^\varepsilon \Pr[x \in B]\}$. Then,

$$\begin{aligned}
&P_A(x \in S : x) - e^\varepsilon P_B(x \in S : x)\\
=&\sum_{x \in S} P_A(x) - e^\varepsilon P_B(x)\\
=&\sum_{x \in U} \max\left(P_A(x) - e^\varepsilon P_B(x), 0\right).
\end{aligned}$$

Analogously, for $S := \{x \in \mathcal{U} \text{ s.t. } \Pr[x \in B] \geq e^\varepsilon \Pr[x \in A]\}$,

$$\begin{aligned}
&P_B(x \in S : x) - e^\varepsilon P_A(x \in S : x)\\
=&\sum_{x \in S} P_B(x) - e^\varepsilon P_A(x)\\
=&\sum_{x \in U} \max\left(P_B(x) - e^\varepsilon P_A(x), 0\right).
\end{aligned}$$

Thus, for every pair of distributions A and B and for every $\varepsilon \geq 0$ the distributions are tightly (ε, δ)-differentially private, where δ is calculated as described. □

If only one pair $\varepsilon, \delta(\varepsilon)$ is considered, composition can only be based on very limited information about the distributions. In this case, for all we know, the distributions could actually have the shape of the randomized response distributions $M_{\varepsilon,\delta}(0)$ and $M_{\varepsilon,\delta}(1)$.[5] However, by considering more information we can derive much better composition bounds. Figure 2 shows ADP guarantees of two Gaussian distributions in contrast to $M_{\varepsilon,\delta}(0/1)$ under $r = 512$-fold composition (left side) and graphically depict those distributions (right side).

With this tight ADP-bound, we can formulate our main result.

Main result (informal version). *Let $M_1, \ldots, M_r$ be mechanisms with worst-case distributions. Let $\prod_{i=1}^r M_i$ be their sequential composition. For every $\varepsilon \geq 0$ our numerical method privacy buckets derives*

[5] Kairouz, Oh, and Viswanath [14] proved that if a mechanism satisfies (ε, δ)-ADP, it cannot have more leakage than $M_{\varepsilon,\delta}$.

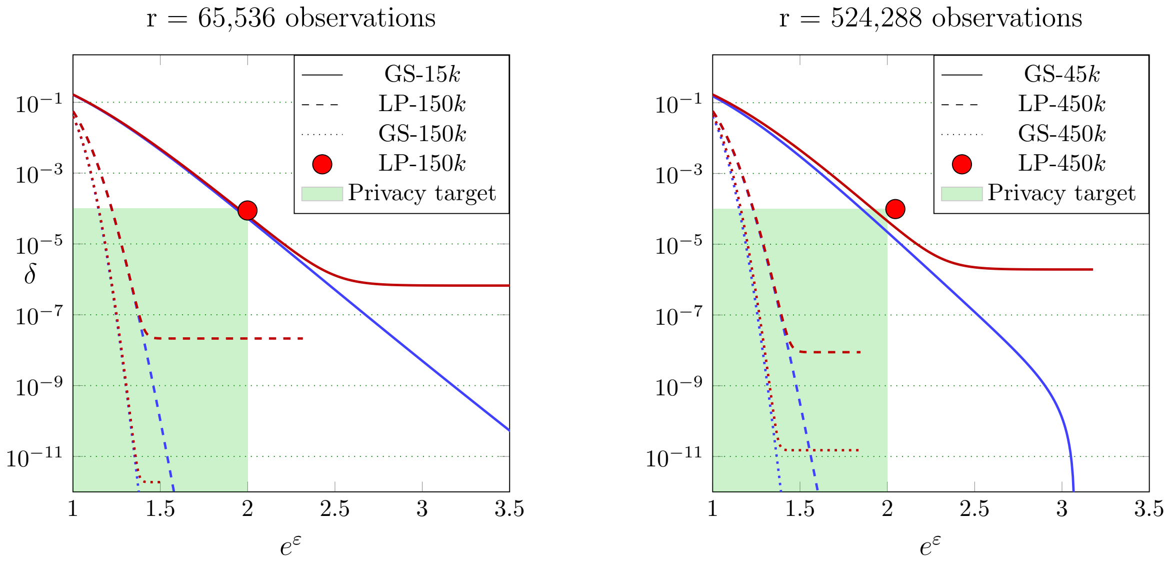


Figure 3: Vuvuzela analysis: upper (red) and lower (blue) bounds on δ (log-scale) for different e^{ε}. Originally recommended mechanisms with 150k (left) and 450k (right) messages overhead per round, analyzed with previous bounds [8] (red dot) and privacy buckets (dotted lines), vs our recommended mechanism with 15k (left) and 45k (right) overhead, analyzed using privacy buckets (solid line).

upper and lower bounds $\delta^{\mathrm{up}}(\varepsilon)$ and $\delta^{\mathrm{low}}(\varepsilon)$ for tight ADP $\delta(\varepsilon)$ as in Definition 1 for $\prod_{i=1}^{r} M_i$:

$$\delta^{\mathrm{low}}(\varepsilon) \leq \delta(\varepsilon) \leq \delta^{\mathrm{up}}(\varepsilon).$$

2.3 Practical relevance of tight privacy bounds

To further highlight the importance of tight privacy bounds for actual mechanisms and protocols, we briefly discuss as a case study the Vuvuzela [26] protocol, which is an anonymous communication system tailored towards messengers. The Vuvuzela paper argues that the only leakage of their strong anonymity mechanisms is the patterns of communication between entities. To limit this leakage, they apply noise to the patterns by sending a random number of dummy messages, where the number of messages follows a truncated Laplace distribution. Vuvuzela has two relevant protocol parts that can be analyzed separately: the dialing protocol (to establish contact) and the communication protocol (to transfer messages).

To quantify the improvements of a tight analysis, Figure 3 plots the $\delta(\varepsilon)$ for various values of ε for a number r of $65,536$ and $524,288$ observations for the conversation protocol, one of the two relevant parts of their system. The original paper [26] proposed to increase privacy with dummy messages that are distributed according to a Laplace distribution. We propose to use a Gaussian distribution with a smaller mean, significantly reducing the noise-overhead. The original paper proposed noise overheads with 150k and 450k noise messages per round and privacy guarantees. We show that with Gaussian noise and a tighter analysis, we can achieve better privacy bounds with only 15k and 45k noise messages per round than the respective previously proposed configurations, reducing the overhead by a factor of 10 while achieving better privacy bounds.

In detail, for $r = 524,288$ the Laplace noise 450k yields a privacy bound δ that is almost 4 orders of magnitude lower, and the corresponding Gaussian noise (with the same variance and mean) a more than 6 orders of magnitude decreased bound compared to their original guarantees. Furthermore, even Gaussian noise with only a mean of 15k meets the privacy requirements of $e^{\varepsilon} \leq 2$ and $\delta \leq 10^{-4}$ for $r = 65,536$ observations. We also analyze the dialing protocol, where similar improvements are possible: 5 times lower Gaussian noise suffices for matching their best guarantees and using Laplace noise incurs a privacy bound improvement of 3 orders of magnitude, whereas comparable Gaussian noise leads to an improvement of 4 orders of magnitude.[6]

2.4 Composition of differential privacy

One of the main advantages of differential privacy is the fact that guarantees are still sound under composition, albeit with increasing values for ε and δ.

Definition 2 (k-fold DP of a mechanism)**.** *A randomized algorithm M with domain $\mathcal{D}$ and range $\mathcal{U}$ is k-fold (ε, δ)-differentially private for sensitivity s if for all $S \subseteq \mathcal{U}^k$ and for all $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \mathcal{D}^k$ such that $\forall 1 \le i \le k.\ ||x_i - y_i||_1 \le s$:*

$$\begin{aligned}&\Pr[(M(x_1), \dots, M(x_k)) \in S]\\ \le & e^{\varepsilon} \Pr[(M(y_1), \dots, M(y_k)) \in S] + \delta\end{aligned}$$

Note that when we describe differential privacy in terms of distributions over the worst-case inputs, the composition of differential privacy is equivalent to considering differential privacy for product distributions. If x_0, x_1 are the worst-case inputs for a mechanism M, resulting in the distributions $M(x_0)$ and $M(x_1)$, then the k-fold composition is described in Definition 1 on the distributions $A = M(x_0)^k$ and $B = M(x_1)^k$. Similarly, a composition of two different mechanisms M and M' with worst-case inputs (in the sense of Section 2.1) x_0, x_1 and x'_0, x'_1 respectively, boils down to Definition 1 on the distributions $A = M(x_0) \times M'(x'_0)$ and $B = M(x_1) \times M'(x'_1)$.

The main composition results we compare our work with are: naive composition, slightly less naive composition and two composition result with improved bounds [8, 14]. We recall these results here.

Lemma 2 (Naïve Composition)**.** *Let (A_1, B_1) and (A_2, B_2) be two pairs of distributions, such that A_1 and B_1 are $(\varepsilon_1, \delta_1)$-differentially private and A_2 and B_2 are $(\varepsilon_2, \delta_2)$-differentially private. Then $A_1 \times A_2$ and $B_1 \times B_2$ are $(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2)$-differentially private.*

Lemma 3 (Adaptive Composition)**.** *Let (A_1, B_1) and (A_2, B_2) be two pairs of distributions, such that A_1 and B_1 are $(\varepsilon_1, \delta_1)$-differentially private and A_2 and B_2 are $(\varepsilon_2, \delta_2)$-differentially private. Then $A_1 \times A_2$ and $B_1 \times B_2$ are $(\varepsilon_1 + \varepsilon_2, \delta_1 + (1 - \delta_1) \cdot \delta_2)$-differentially private.*

Lemma 4 (Boosting and Differential Privacy (Advanced Composition) [8])**.** *Let $(A_1, B_1), \dots, (A_k, B_k)$ be pairs of distributions, such that A_i and B_i are (ε, δ)-differentially private for all $i \in \{1, \dots, k\}$. Then $A_1 \times \dots \times A_k$ and $B_1 \times \dots \times B_k$ are $(\hat{\varepsilon}_{\hat{\delta}}, \hat{\delta})$-differentially private, where $\hat{\delta} = k \cdot \delta$ and $\hat{\varepsilon}_{\hat{\delta}} = O\left(k\varepsilon^2 + \varepsilon\sqrt{k \log\left(e + (\varepsilon\sqrt{k}/\hat{\delta})\right)}\right)$*

Lemma 5 (Kairouz et al.'s Composition [14])**.** *For any $\varepsilon \ge 0$ and $\delta \in [0, 1]$, the class of (ε, δ)-differentially private mechanisms satisfies*

(ε', δ')-differential privacy

under k-fold composition, for all $i \in \{0, \dots, \lfloor k/2 \rfloor\}$ where $\varepsilon' = (k - 2i)\varepsilon$ and $\delta' = 1 - (1 - \delta)^k (1 - \delta_i)$

$$\delta_i = \frac{\sum_{\ell=0}^{i-1} \binom{k}{\ell} \left(e^{(k-\ell)\varepsilon} - e^{(k-2i+\ell)\varepsilon}\right)}{(1 + e^{\varepsilon})^k}$$

These composition results allow for deriving differential privacy guarantees under composition in a black-box manner, i.e., only depending on ε and δ. Consequently, these results are oblivious to how the underlying distributions actually compose and present, in a way, worst-case results under composition. Thus, we cannot expect that they come close to the tight differential privacy guarantee of the composed distributions. In the remainder of this paper we introduce, prove sound and discuss our main idea: approximating the distributions A_1, A_2, B_1, B_2 in a way that allows for a sound calculation of a differential-privacy guarantee that takes into account features of the distribution even under manifold composition. Moreover, we use the same technique to derive a lower bound for the guarantee, to bound the (unknown) tight differential privacy guarantee from both directions.

[6] For a more detailed description of our Vuvuzela analysis, we refer the interested reader to Section 7.

2.5 Related work

Mechanism-oblivious bounds for adaptive composition Early composition bounds for r-fold ADP [8, 14, 21] only provide mechanism-oblivious bounds, i.e., these bounds are oblivious to the actual mechanisms. These results only rely on the initial values $(\varepsilon_0, \delta_0)$. Our work, in contrast, is mechanism-aware in the sense that it takes the shape of the distributions (/mechanisms) into account. Our results yield mechanism-aware δ-tight bounds for r-fold composition and thereby lead to significantly tighter bounds.

Mechanism-aware bounds for adaptive composition Recent work [7, 2, 1, 20] partially takes the shape of the mechanism into account by computing the Rényi divergence of the corresponding worst-case distributions, i.e., the moments of the distribution of ratios, to achieve tighter privacy bounds. Similarly, Abadi et al. [1] use the moments accountant based on Rényi divergence to find tighter bounds. These approaches, in special cases, find better composition results than the best mechanism-oblivious composition theorem. As shown in our comparisons, however, our bounds are even tighter than prior mechanism-aware bounds and—in contrast to all previous work—also include lower bounds and thereby a means to estimating their precision. Additionally, our work provides tight bounds for very low epsilon, even epsilon = 0, i.e., the total variation (also called statistical distance), which is used to formalize statistical indistinguishability.

Adaptively chosen privacy parameters As in previous work [8, 1, 14] our technique satisfies adaptive composition [8] in the following sense: sequences of mechanisms are composed where each query can be adaptively chosen by the attacker and depend on previously observed responses, but the noise distributions of each mechanism have to be independent of these previously observed responses to the attacker. This kind of adaptive composition results does not hold for some mechanisms that achieve ADP under continual observation that use carefully correlated noise and/or only use noise when necessary [5, 9, 12, 13]. Nevertheless, the proofs of these adaptive mechanisms can still benefit from our results as they often over-approximate a subset of these correlated distributions with independent distributions, e.g., in order to apply Azuma's Inequality [12] (which is stated for independent distributions).

Probabilistic differential privacy (PDP) vs ADP It might appear preferable to only use δ such that it is only the probability of distinguishing events, in order to guarantee pure ε-DP with probability $(1-\delta)$ (which is also called PDP [17, 11]). However, if delta would only contain distinguishing events, both ε and δ would grow linearly in the number of compositions. Thus, better ε-bounds can only be achieved by allowing some of the probability mass of the non-distinguishing events to be hidden within the δ parameter. While using PDP with distinguishing events has an intuitive interpretation, it is not closed under post-processing [18]. Hence, this work concentrates on ADP.

Optimal mechanisms for a given utility function Recent work [10, 15] made progress on finding optimal mechanisms for DP for a large class of utility functions. These results concentrate on single observations and do not characterize how these mechanisms behave under r-fold composition.

Dependencies The work of Liu, Chakraborty and Mittal [16] discusses the importance of correctly measuring the sensitivity of databases for differential privacy. They show that in real-world examples entries can be correlated and thus cannot be independently exchanged as in DP's basic definition. Their approach, however, finally results in the same techniques as in previous work being used to achieve the same goal: noise applied to database queries results in differential privacy, although the sensitivity is calculated in a more complex manner. Our results can directly be applied in such a setting as well: given the (final) distributions that potentially consider dependent entries we calculate differential privacy guarantees for these distributions.

3 Privacy buckets of distributions

3.1 Informal description of privacy buckets

Generic bounds for differential privacy under continual observation [8, 14] are stated independently of the shape of the underlying distributions, simply based on the ADP guarantees before the composition. This obliviousness is both strength and weakness: the exact shape of the distribution does not need to be characterized to apply these results, but they cannot devise tight bounds that are derivable from the shape of the distributions. We now introduce an alternative approach: we approximate the distributions with an explicit focus on their most important features for ADP, the privacy loss of atomic events.

Recall from Lemma 1 that for distributions A and B over the universe $\mathcal{U}$ we can calculate a value $\delta(\varepsilon)$ for every value $\varepsilon \geq 0$ so that A and B are tightly $(\varepsilon, \delta(\varepsilon))$-ADP:

$$\begin{aligned}\delta(\varepsilon) = \max\Bigg(&\sum_{x\in\mathcal{U}} \max\left(P_A(x) - e^{\varepsilon} P_B(x), 0\right),\\ &\sum_{x\in\mathcal{U}} \max\left(P_B(x) - e^{\varepsilon} P_A(x), 0\right)\Bigg),\end{aligned}$$

For simplicity we consider $\delta(\varepsilon) = \sum_{x\in\mathcal{U}} \max\left(P_A(x) - e^{\varepsilon} P_B(x), 0\right)$ for now. Consequently, the contribution of each atomic event $x \in \mathcal{U}$ to $\delta(\varepsilon)$ is $\delta(x, \varepsilon) = \max(P_A(x) - e^{\varepsilon} P_B(x), 0)$ and their sum is $\sum_x \delta(x, \varepsilon) = \delta(\varepsilon)$. This is of course not surprising. Let us observe that if $P_B(x) = 0$, we have $\delta(x, \varepsilon) = P_A(x)$. We can combine all atomic events x with $P_B(x) = 0$ into one non-atomic event x_∞ of all such events.

For events x with $P_B(x) > 0$, let $\mathcal{L}^{(x)}_{(A||B)} = \ln \frac{P_A(x)}{P_B(x)}$ be the logarithmic privacy loss of x [7]. For ease of use, we define the privacy loss without the logarithm as $e\mathcal{L}^{(x)}_{(A||B)} = e^{\mathcal{L}^{(x)}_{(A||B)}} = \frac{P_A(x)}{P_B(x)}$, i.e., as the ratio of two probabilities. Based on the privacy loss, we can calculate the contribution $\delta(x, \varepsilon)$ of an atomic event x as

$$\begin{aligned}\delta(x, \varepsilon) &= \max\left(P_A(x) - e^{\varepsilon} P_B(x), 0\right)\\ &= \max\left(P_A(x) - e^{\varepsilon} \frac{P_A(x)}{e\mathcal{L}^{(x)}_{(A||B)}}, 0\right)\\ &= P_A(x) \cdot \max\left(\left(1 - \frac{e^{\varepsilon}}{e\mathcal{L}^{(x)}_{(A||B)}}\right), 0\right).\end{aligned}$$

Combining the contributions of several events For any two disjoint events x, y with the same privacy loss $p = e\mathcal{L}^{(x)}_{(A||B)} = e\mathcal{L}^{(y)}_{(A||B)}$, their contribution can be combined without loss of information to

$$\delta(x \cup y, \varepsilon) = \delta(x, \varepsilon) + \delta(y, \varepsilon) = (P_A(x) + P_A(y)) \cdot \max\left(\left(1 - \frac{e^{\varepsilon}}{p}\right), 0\right),$$

requiring us to only remember the privacy loss p and the sum of their probabilities $P_A(x) + P_A(y)$. In other words, we can combine all atomic events with the same ratio without losses. If we allow for a slight imprecision, we can soundly combine disjoint events x and y with approximately the same privacy loss by summing the probabilities $P_A(x) + P_A(y)$ and choosing $p = \max(e\mathcal{L}^{(x)}_{(A||B)}, e\mathcal{L}^{(y)}_{(A||B)})$ and yield $\delta(\varepsilon)(x \cup y) \geq \delta(x, \varepsilon) + \delta(y, \varepsilon)$.

Constructing privacy buckets from atomic events To render our approach feasible, we fix a finite set of privacy loss values $\{f^i | i \in \{-n, \ldots, n\}\}$ based on a factor f that parametrizes the coarseness of the values and a limit $n \in \mathbb{N}$ that limits the number of values we consider. We then collect all atomic events x with a similar privacy loss into one combined event, which we call a *bucket* as follows.

Given a factor $f > 1$, the bucket $\mathcal{B}(i)$ summarizes all atomic events where $f^{i-1} < e\mathcal{L}^{(x)}_{(A||B)} \leq f^i$ (illustrated in Figure 4). The value of $\mathcal{B}(i)$ is the sum over the probabilities $P_A(x)$ of all those atomic events

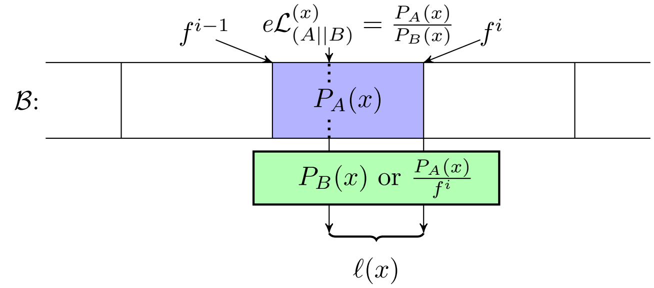

f^{i-1}
$e\mathcal{L}^{(x)}_{(A||B)} = \frac{P_A(x)}{P_B(x)}$
f^i
$\mathcal{B}$:
$P_A(x)$
$P_B(x)$ or $\frac{P_A(x)}{f^i}$
$\ell(x)$

Figure 4: Placement of an element x into a bucket when $f^{i-1} < e\mathcal{L}^{(x)}_{(A||B)} \leq f^i$. Buckets store f^i and $P_A(x)$ (accumulated over all elements in the bucket). We approximate $P_B(x)$ with $\frac{P_A(x)}{f^i}$, accepting an error of $\ell(x) = P_B(x) - \frac{P_A(x)}{f^i}$.

Buckets for given parameters f and n.

Bucket factor:	f^{-n}	f^{-n+1}	$\cdots$	f^{-2}	f^{-1}	f^0	f^1	f^2	$\cdots$	f^{n-1}	f^n	$> f^n$
Index:	$-n$	$-n+1$	$\cdots$	-2	-1	0	1	2	$\cdots$	$n-1$	n	∞

Figure 5: Depiction of the buckets (separately) constructed for both $\mathcal{B}_A$ and $\mathcal{B}_B$. For $\mathcal{B}_A$ each bucket $\mathcal{B}_A(i)$ with $i \in \{-n+1, \ldots, n\}$ contains all elements $x \in \mathcal{U}$ with $f^{i-1}P_B(x) \leq P_A(x) \leq f^i P_B(x)$, the bucket $\mathcal{B}_A(-n)$ contains all elements with $P_A(x) \leq f^{-n}P_B(x)$ and the bucket $\mathcal{B}_A(\infty)$ contains all elements with $P_A(x) > f^n P_B(x)$.

(according to distribution A). Here, $e\mathcal{L}^{(x)}_{(A||B)} \leq f^i$ guarantees soundness, whereas $f^{i-1} < e\mathcal{L}^{(x)}_{(A||B)}$ limits the imprecision of our approximation: For each $P_B(x)$ we introduce an error of $\ell(x) = P_B(x) - \frac{P_A(x)}{f^i} \leq P_A(x) \cdot \left(\frac{1}{f^{i-1}} - \frac{1}{f^i}\right)$. We define "corner buckets" that collect all atomic events with privacy loss outside of $[f^{-n}, f^n]$, where $\mathcal{B}(-n)$ contains all atomic events with a very small privacy loss and $\mathcal{B}(\infty)$ contains all atomic events with a very large (or even infinite) privacy loss. Using these $n+2$ buckets, we can now compute an upper bound for $\delta(\varepsilon)$ ADP as $\sum_{i\in\{-n,\ldots,n\}} \mathcal{B}(i) \cdot \max\left(\left(1 - \frac{e^\varepsilon}{f^i}\right), 0\right) + \mathcal{B}(\infty)$.

This buckets representation does not only allow us to directly derive a bound on $\delta(\varepsilon)$. It also is particularly well suited for calculating ADP after composing several pairs of distributions. Note that we include "privacy loss" values that are smaller than 1. Those values by definition cannot influence $\delta(\varepsilon)$ directly, but they are crucial for computing tight bounds under composition.

Composition Consider a pair of distributions, say (A_1, B_1) and (A_2, B_2) over universes $\mathcal{U}_1$,$\mathcal{U}_2$, where A_1, A_2 are independent and B_1, B_2 are independent. We first create $\mathcal{B}_1$ from (A_1, B_1) and $\mathcal{B}_2$ from (A_2, B_2). For each event $(x, y) \in \mathcal{U}_1 \times \mathcal{U}_2$ where x was placed in $\mathcal{B}_1(i)$ and y was placed in $\mathcal{B}_2(j)$ we can now immediately derive an upper bound for the privacy loss: $\frac{P_{A_1}(x)}{P_{B_1}(x)} \cdot \frac{P_{A_2}(y)}{P_{B_2}(y)} \leq f^{i+j}$ (c.f. Figure 8).

Since A_1 and A_2 are independent and B_1 and B_2 are independent, we can generate a new set of buckets $\mathcal{B}'(i) = \sum_{j,k,j+k=i} \mathcal{B}_1(j) \cdot \mathcal{B}_2(k)$ and for these buckets $\mathcal{B}'$ we can directly compute $\delta'(\varepsilon)$ s.t. for every choice of $\varepsilon \geq 0$, $(A_1 \times A_2, B_1 \times B_2)$ satisfy $(\varepsilon, \delta'(\varepsilon))$-ADP.

Squaring Very fine-grained privacy buckets have a small approximation error, but require more buckets to avoid a significant mass of the privacy loss to fall into the ∞-bucket. Moreover, when composing privacy buckets, the bucket list naturally broadens, i.e., the buckets that are farther away from the middle bucket (with factor f^0) gain higher values. When creating privacy buckets for a given number n, this effect leads to a trade-off between the granularity (i.e., the choice of the bucket factor f) and the expected number of compositions: the smaller the value of f, the more precise the privacy buckets model of the features of the

$P_A(x)$	the probability that x happens in A.
ε, δ	parameters for ADP.
$\mathcal{U}$	universe of all atomic events.
f	factor (close to 1) with $f > 1$.
∞	symbol for any ratio $> f^n$.
n	index of the last bucket before ∞.
N	bucket indexes $\{-n, \ldots, n\}$.
N_∞	bucket indexes with ∞, $N \cup \{\infty\}$.
T	composition tree
🍃(A, B, f, n)	leaf node of A/B privacy buckets without error correction with indexes N_∞ and ratios $\{\leq f^{-n}, \ldots, \leq f^n, > f^n\}$.
$T_1 \times T_2$	node for composition of T_1 and T_2
$\blacktriangledown T$	node for squaring of T
A_T	$(A_1, \ldots, A_k)$ for a composition tree T with leafs 🍃$(A_i, B_i, f, n)_{i=1}^k$
B_T	$(B_1, \ldots, B_k)$ for a composition tree T with leafs 🍃$(A_i, B_i, f, n)_{i=1}^k$
$\mathcal{B}_T(i)$ for $i \in N_\infty$	privacy bucket of tree T with index i.
$\mathcal{B}_T^{⚛}(x)$ for $x \in \mathcal{U}$	impact of the atomic event x in tree T.
$\ell_T(i), \ell_T^{⚛}(x)$	"real" error correction term for index i or atomic event x.
$\tilde{\ell}_T(i), \tilde{\ell}_T^{⚛}(x)$	bound on the maximum error, "virtual" error correction term.
$\iota_T(x)$	index of x w.r.t. composition tree T.
j_ε	smallest integer j such that $f^j \geq e^\varepsilon$.
S_i	the set of atomic events that contribute to $\mathcal{B}(i)$.

Figure 6: Notation for our privacy buckets.

distributions, but the fewer compositions before a significant amount of events reaches the corner buckets $\big(\mathcal{B}(-n)$ and $\mathcal{B}(\infty)\big)$, which again reduces the precision. To counter this effect, we introduce an additional operation which we call *squaring*: we square the factor f, thus halving the precision of the privacy buckets, and merge the privacy buckets into these new, more coarse-grained privacy buckets. Squaring allows us to start with much more fine-grained privacy buckets and reduce the granularity as we compose, which can significantly improve the overall precision of the approach. We choose to square f instead of increasing it to an arbitrary f' to ease the computation of the new privacy buckets: we simply combine buckets $2i - 1$ and $2i$ with factors f^{2i-1} and f^{2i} into the new bucket i with factor $(f^2)^i = f^{2i}$. We refer to Figure 9 for a graphical depiction of squaring.Figure 10 describes as an algorithm how we suggest buckets to be created for practical purposes.

3.2 A formal description of privacy buckets

We now formalize privacy buckets, our approximation of the pair of distributions based on the privacy loss of all atomic events, which is sufficient for calculating (ε, δ)-ADP, the *privacy buckets*, and that comes with an efficient way for computing r-fold (ε, δ)-ADP from a sequence of privacy buckets.

The infinity symbol ∞ In this paper we will write ∞ to describe the corner case accumulated in the largest bucket $\mathcal{B}(\infty)$ of our bucket lists. We consider ∞ to be a distinct symbol and in an abuse of notation, we use the following mathematical rules to interact with it: $\infty > i$ for all $i \in \mathbb{Z}$. $\quad \infty + i = \infty$ for all $i \in \mathbb{Z}$.

The composition of privacy buckets is commutative but not associative. For example, consider three

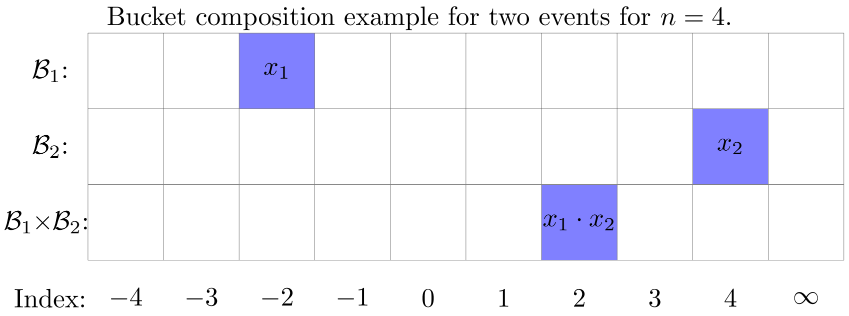

Bucket composition example for two events for $n = 4$.
$\mathcal{B}_1$:
x_1
$\mathcal{B}_2$:
x_2
$\mathcal{B}_1 \times \mathcal{B}_2$:
$x_1 \cdot x_2$
Index: -4 -3 -2 -1 0 1 2 3 4 ∞

Figure 7: Depiction of how individual events x_1 with index -2 and x_2 with index 4 compose into their new bucket with index $-2 + 4 = 2$.

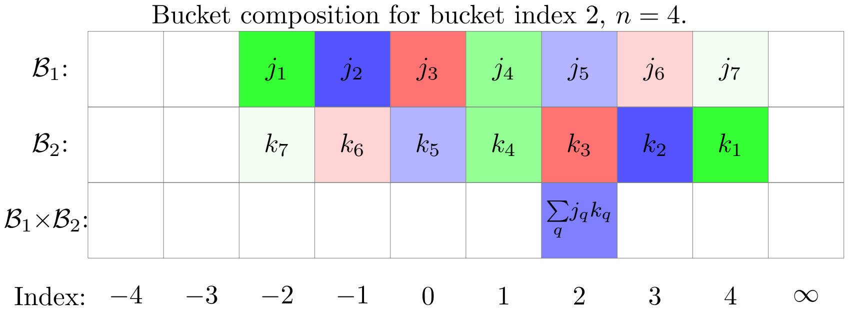

Bucket composition for bucket index 2, $n = 4$.
$\mathcal{B}_1$:
j_1 j_2 j_3 j_4 j_5 j_6 j_7
$\mathcal{B}_2$:
k_7 k_6 k_5 k_4 k_3 k_2 k_1
$\mathcal{B}_1 \times \mathcal{B}_2$:
$\sum_q j_q k_q$
Index: -4 -3 -2 -1 0 1 2 3 4 ∞

Figure 8: Depiction of the bucket composition for the (new) bucket with index $i = 2$. We calculate the value of bucket i by summing over the product of all $\mathcal{B}_1(j_t) \cdot \mathcal{B}_2(k_t)$ for $t \in \{1, \ldots, 7\}$. Graphically, buckets with the same color are combined. Note that none of the buckets ∞, -3 and -4 are used for the composition, as for all $j \in \{-4, \ldots, 4\}$, $\infty + j \neq 2$, $-3 + j \neq 2$ and $-4 + j \neq 2$.

events x, y, z in buckets with factors $f^{-0.6n}, f^{-0.6n}, f^{0.5n}$ respectively. If we compose x with y and the result (which is in f^{-n}) with z, the event will land in $f^{-0.5n}$. If we compose y with z and then the result (which is in $f^{-0.1n}$) with x they will land in $f^{-0.7n}$. Moreover, when and how often the squaring is performed influences the resulting privacy buckets. Hence, we need to keep track of the order in which we applied composition and squaring. To this end, we define *composition trees.*

Definition 3 (Composition trees)**.** *For two tuples of distributions $(A_1, \ldots, A_{\mathcal{W}})$ and $(B_1, \ldots, B_{\mathcal{W}})$ of the same size $\mathcal{W}$, a* composition tree *is a tree with three kinds of nodes: leaves ($T = \leaf(A_i, B_i, f, n)$) that are labeled with a pair of distributions A_i and B_i a factor $f > 1$ and a $n \in \mathbb{N}$; squaring nodes ($T = \blacktriangledown T_1$) with exactly one child node; and composition nodes ($T = T_1 \times T_2$) with exactly two child nodes.*

The bucket factor *f_T for a composition tree T is $f_{\leaf(A,B,f,n)} := f$, $f_{\blacktriangledown T_1} := (f_{T_1})^2$, and $f_{T_1 \times T_2} := f_{T_1}$ if $f_{T_1} = f_{T_2}$ and undefined otherwise. The* last bucket index *n_T of a composition tree T is always constant: $n_{\leaf(A,B,f,n)} := n$, $n_{\blacktriangledown T} = n_T$, and $n_{T_1 \times T_2} := n_{T_1}$ if $n_{T_1} = n_{T_2}$ and undefined otherwise.*

For the distributions over which each composition tree is defined, we write $A_{\leaf(A,B,f,n)} = A$, $B_{\leaf(A,B,f,n)} = B$, $A_{T_1 \times T_2} = A_{T_1} \times A_{T_2}$, and $A_{\blacktriangledown T_1} = A_{T_1}$ and analogously $B_{T_1 \times T_2} = B_{T_1} \times B_{T_2}$, and $B_{\blacktriangledown T_1} = B_{T_1}$. We write $\mathcal{U}_T$ for their support as $\mathcal{U}_T = [A_T] \cup [B_T]$.

We call a composition tree T valid *if for the product distributions $A_T = \Pi_{k=1}^{\mathcal{W}} A_k$ all A_j, A_s are pairwise independent ($j, s \in \{1, \ldots, \mathcal{W}\}$) and analogously for $B_T = \Pi_{k=1}^{\mathcal{W}} B_k$ all B_j, B_s are pairwise independent ($j, s \in \{1, \ldots, \mathcal{W}\}$), f_T and n_T are defined, $f_T > 1$ and n_T is an even natural number (i.e., there is a $q \in \mathbb{N}$ such that $n = 2q$). We sometimes write f instead of f_T, n instead of n_T, A instead of A_T, and B instead of B_T if T is clear from the context.*

We now define the privacy buckets associated with a valid composition tree T, starting with the base case of leaf nodes.

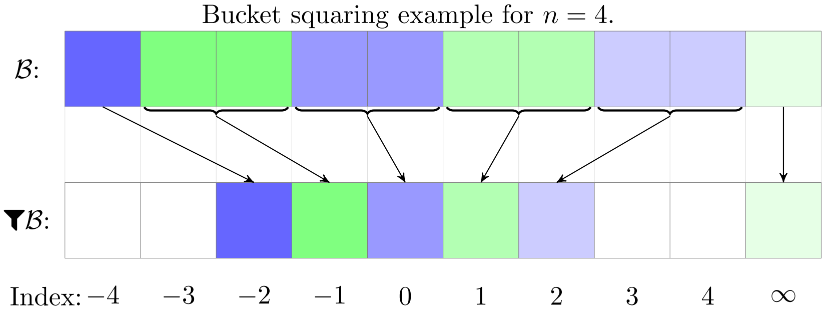

Bucket squaring example for $n = 4$.
$\mathcal{B}$:
$\blacktriangledown\mathcal{B}$:
Index: -4
-3
-2
-1
0
1
2
3
4
∞

Figure 9: Depiction of the bucket squaring. Events from each bucket $\mathcal{B}(i)$ are moved into bucket $\mathcal{B}(\lceil i/2 \rceil)$, with the exception of $\mathcal{B}(\infty)$, which remains unchanged.

```
**BucketDelta**$(A, B, f, n, t, \varepsilon)$:
$T$ = Construct privacy buckets with $(A, B, f, n)$
**for** $i$ **from** 0 **to** $t$ **do**
  $T' = T \times T$ (Compose $T$ with itself, without squaring)
  **if** $\mathcal{B}_{T'}(\infty) > 2.2 \cdot \mathcal{B}_T(\infty)$ **then**
    $T = \blacktriangledown T$ (Apply squaring to $T$, changing $f$ to $f^2$)
    $T = T \times T$ (Compose $T$ with itself, now after squaring)
**return** $\delta_T(\varepsilon) = \sum_{i \in \{-n,\ldots,n\}} \mathcal{B}_T(i) \cdot \max\left(\left(1 - \frac{e^\varepsilon}{f^i}\right), 0\right) + \mathcal{B}_T(\infty)$
```

Figure 10: Depiction of how we create buckets – for simplicity without error correction terms and for the common special case where we compose the same distributions ($A_1 = A_2 = \ldots = A_r$ and $B_1 = B_2 = \ldots = B_r$). We use repeated squaring to compute r-fold DP for $r = 2^t$ compositions. Here T tracks the compositions and squarings we have already performed.

Computing delta and evaluating a composition tree

Definition 4 (Privacy buckets of a composition tree)**.** *Let T be a valid composition tree with $f := f_T$, $\mathcal{U} := \mathcal{U}_T$ and $n := n_T$. For $N_\infty := \{-n, -n+1, \ldots, n\} \cup \{\infty\}$, we define the A_T/B_T* privacy buckets *$\mathcal{B}_T : N_\infty \to [0, 1]$ recursively as follows.*

If $T = \varnothing(A, B, f, n)$, we define for $i \in N_\infty$,

$$\mathcal{B}_{\varnothing(A,B,f,n)}(i) = \textstyle\sum_{x \in S_i} P_A(x),$$

where the sets S_i are defined as follows:

$$S_\infty = \{x \in \mathcal{U}. P_A(x) > f^n P_B(x)\}$$
$$\forall i \in \{-n+1, \ldots, n\}\; S_i = \left\{x \in \mathcal{U}.\; f^{i-1} P_B(x) < P_A(x) \leq f^i P_B(x)\right\}$$
$$S_{-n} = \left\{x \in \mathcal{U}. P_A(x) \leq f^{-n} P_B(x)\right\}.$$

If $T = T_1 \times T_2$, we define

$$\mathcal{B}_{T_1 \times T_2}(i) := \begin{cases} \sum_{j+k=i} \mathcal{B}_{T_1}(j) \cdot \mathcal{B}_{T_2}(k) & i \in \mathbb{N} \setminus \{-n\} \\ \sum_{j+k \leq -n} \mathcal{B}_{T_1}(j) \cdot \mathcal{B}_{T_2}(k) & i = -n \\ \sum_{j+k > n} \mathcal{B}_{T_1}(j) \cdot \mathcal{B}_{T_2}(k) & i = \infty \end{cases}$$

If $T = \blacktriangledown T_1$,

$$\mathcal{B}_{\blacktriangledown T_1}(i) := \begin{cases} \mathcal{B}_{T_1}(2i-1) + \mathcal{B}_{T_1}(2i) & i \in [-n/2+1, n/2] \\ \mathcal{B}_{T_1}(\infty) & i = \infty \\ 0 & \textit{otherwise} \end{cases}$$

Note that since the sets S_i for $i \in \{-n, \ldots, n\} \cup \{\infty\}$ describe a partitioning of $\mathcal{U}$, we have $\sum_{i \in \{-n,\ldots,n\} \cup \{\infty\}} \mathcal{B}(i) = 1$.

We next define ADP directly on a privacy bucket list. For all atomic events x in $S_i \neq S_\infty$, we know that $P_A(x) \leq f^i P_B(x)$. We perform a slight over-approximation by treating this inequality as an equality and then use $P_A(x) - P_A(x)/f^i$ as in Lemma 1. For $x \in S_\infty$, we add $P_A(x)$ to δ, counting them as total privacy-breakdowns.

Definition 5 (Delta). *Let T be a valid composition tree labeled with $f := f_T$ and $n := n_T$, then*

$$\delta_T(\varepsilon) = \mathcal{B}_T(\infty) + \textstyle\sum_{i \in \{-n,\ldots,n\}} \max(0, \mathcal{B}_T(i) \cdot (1 - \frac{e^\varepsilon}{f^i}))$$

We say that the privacy buckets with composition tree T are (ε, δ)-ADP, if $\delta_T(\varepsilon) \leq \delta$.

3.3 Buckets per atomic event

For discussing our results and their soundness, we compare the differential privacy guarantees of privacy buckets with the real differential privacy guarantees (calculating which might not be feasible). To this end and for talking about individual atomic events, we assign an index to each such event. The index specifies the (one) bucket the respective event influences. For privacy buckets that have been created from distributions (and not composed), this index is simply the bucket the event was assigned to. After composition, the index depends on how the indexes of the respective buckets interacted: in the most simple case, if x_1 and x_2 are events with indexes i and j, then the event (x_1, x_2) will have the index $i + j$. However, the corner cases can modify the index, as the index can only be in the set $\{-n, \ldots, n, \infty\}$.

Definition 6 (Index of an event according to a composition tree). *For a valid composition tree T with $A_T = \Pi_{k=1}^{\mathcal{W}} A_k$ and $B_T = \Pi_{k=1}^{\mathcal{W}} B_k$, and $\mathcal{U}_T = \Pi_{k=1}^{\mathcal{W}} \mathcal{U}_i$, $f := f_T$, and $n := n_T$, we define the set of indexes for atomic events $x = (x_1, \ldots, x_{\mathcal{W}}) \in \Pi_{k=1}^{\mathcal{W}} \mathcal{U}_k$ as follows.*

First, we define for $T = \text{🍃}(A_k, B_k, f, n)$ and consequently for atomic elements $x_k \in \mathcal{U}_k$ with $k \in \{1, \ldots, \mathcal{W}\}$, the index of x_k as

$$\iota_T(x_k) := \begin{cases} l & \textit{if } l \in \{-n+1, \ldots, n\} \wedge \\ & f^{l-1} P_{B_k}(x_k) < P_{A_k}(x_k) \leq f^l P_{B_k}(x_k) \\ \infty & \textit{if } P_{A_k}(x_k) > f^n P_{B_k}(x_k) \\ -n & \textit{otherwise} \end{cases}$$

For a pair of composition trees T_1, T_2 and for $T = T_1 \times T_2$ we define the index of $x = (x_1, x_2) \in A_{T_1} \times A_{T_2}$ as

$$\iota_T(x) = \iota_{T_1 \times T_2}(x_1, x_2) := \begin{cases} -n & \textit{if } \iota_{T_1}(x_1) + \iota_{T_2}(x_2) < -n \\ \infty & \textit{if } \iota_{T_1}(x_1) + \iota_{T_2}(x_2) > n \\ \iota_{T_1}(x_1) + \iota_{T_2}(x_2) & \textit{otherwise,} \end{cases}$$

where we assume that $\forall y, z \in \mathbb{Z},\ y + \infty = \infty > z$.

For $T = \blacktriangledown T_1$ we define the index of $x \in A_T$ as

$$\iota_T(x) = \iota_{\blacktriangledown T_1}(x) := \begin{cases} \lceil \iota_{T_1}(x)/2 \rceil & \textit{if } \iota_{T_1}(x) \neq \infty \\ \infty & \textit{otherwise,} \end{cases}$$

Recall that composition is not necessarily associative, i.e., there are composition trees T_1, T_2, T_3 and x_1, x_2, x_3 such that

$$\iota_{(T_1 \times T_2) \times T_3}(x_1, x_2, x_3) \neq \iota_{T_1 \times (T_2 \times T_3)}(x_1, x_2, x_3).$$

Soundness of differential privacy guarantees for privacy buckets We can now show the soundness of the bounds on ADP we calculate using privacy buckets. We will show that if privacy buckets are (ε, δ)-ADP, then the distributions from which they were created (either directly or via composition) are also (ε, δ)-ADP. Simply put, the guarantees we calculate are sound.

We begin by showing a helpful lemma that directly follows our main strategy: all atomic events x that are assigned an index $\iota_{T(x)} = i \neq \infty$ (according to a composition tree T) satisfy $P_A(x) \leq f^i P_B(x)$.

Lemma 6. *Let T be a valid composition tree For all $x \in \mathcal{U}_T$ with $\iota_T(x) \neq \infty$ and for $f = f_T$, we have $P_{A_T}(x) \leq f^{\iota_T(x)} P_{B_T}(x)$, i.e., $f^{\iota_T(x)} \leq e\mathcal{L}^{(x)}_{(A_T||B_T)}$.*

Proof. We show the lemma by a structural induction over the composition tree T. Let $x \in \mathcal{U}_T$ with $\iota_T(x) \neq \infty$.

For leafs (i.e., $T = \text{🍃}(A, B, f, n)$), if $\iota_{\text{🍃}(A,B,f,n)}(x) = -n$, it follows that

$$\begin{aligned} &P_A(x) \leq f^{-n} P_B(x) \\ \Leftrightarrow\ &P_A(x) \leq f^{\iota_{\text{🍃}(A,B,f,n)}(x)} P_B(x). \end{aligned} \tag{1}$$

Thus, for all $\iota_{\text{🍃}(A,B,f,n)}(x) \neq \infty$ we get from Definition 6 and Equation (1) that

$$P_A(x) \leq f^{\iota_{\text{🍃}(A,B,f,n)}(x)} P_B(x).$$

For composition nodes (i.e., $T = T_1 \times T_2$), where T_1 and T_2 are valid composition trees, let $x = (x_1, x_2)$ with $x_1 \in \mathcal{U}_1$ and $x_2 \in \mathcal{U}_2$, $f := f_{T_1} = f_{T_2}$. We know from Definition 6 that $\iota_{T_1 \times T_2} \neq \infty \Rightarrow \iota_{T_1} \neq \infty \wedge \iota_{T_2} \neq \infty$. Moreover,

$$\begin{aligned} P_{A_{T_1 \times T_2}}(x) &= P_{A_{T_1}}(x_1) \cdot P_{A_{T_2}}(x_2) \\ &\overset{\text{IH}}{\leq} \left(f^{\iota_{T_1}(x_1)} P_{B_{T_1}}(x_1)\right) \cdot \left(f^{\iota_{T_2}(x_2)} P_{B_{T_2}}(x_2)\right) \\ &= f^{\iota_{T_1}(x_1) + \iota_{T_2}(x_2)} \left(P_{B_{T_1}}(x_1) P_{B_{T_1}}(x_2)\right) \\ &\leq f^{\iota_{T_1 \times T_2}(x)} P_{B_{T_1 \times T_2}}(x) \end{aligned}$$

Note that $f^{\iota_{T_1}(x_1) + \iota_{T_2}(x_2)} \leq f^{\iota_{T_1 \times T_2}(x)}$ holds by definition.

For squaring nodes (i.e., $T = \blacktriangledown T_1$) with $f := f_{T_1}$ (and consequently $f_{\blacktriangledown T_1} = f^2$), we know that $\iota_T(x) \neq \infty \Leftrightarrow \iota_{T_1}(x) \neq \infty$. By definition, we have

$$\begin{aligned} P_{A_{\blacktriangledown T_1}}(x) = P_{A_{T_1}}(x) &\overset{\text{IH}}{\leq} f^{\iota_{T_1}(x)} P_{B_{T_1}}(x) = f^{2\iota_{T_1}(x)/2} P_{B_{T_1}}(x) \\ &\leq (f^2)^{\lceil \iota_{T_1}(x)/2 \rceil} P_{B_{T_1}}(x) = (f_{\blacktriangledown T_1})^{\iota_{\blacktriangledown T_1}(x)} P_{B_{\blacktriangledown T_1}}(x) \end{aligned}$$

□

Lemma 7 (Bucket values are sums over atomic events)**.** *Let T be a valid composition tree. Then, for all $i \in \{-n_T, \ldots, n_T, \infty\}$,*

$$\mathcal{B}_T(i) = \sum\nolimits_{x \in \mathcal{U}_T \ s.t. \ \iota_T(x) = i} P_{A_T}(x).$$

Proof. We show the lemma via structural induction over T. Let $N := \{-n_T, \ldots, n_T\}$.

If $T = \text{🍃}(A, B, f, n)$: Let $i \in N \cup \{\infty\}$. By Definitions 4 and 6 with S_i as in Definition 4,

$$\mathcal{B}_{\text{🍃}(A,B,f,n)}(i) = \sum\nolimits_{x \in S_i} P_A(x) = \sum\nolimits_{x, \iota(x) = i} P_A(x).$$

Otherwise, assume the lemma holds for composition trees T_1 and T_2. If $T = T_1 \times T_2$, we have for $i \in N \setminus \{-n_T\}$,

$$\begin{aligned} \mathcal{B}_{T_1 \times T_2}(i) &= \sum_{j,k \in N . j+k=i} \mathcal{B}_{T_1}(j) \cdot \mathcal{B}_{T_2}(k) \\ &\overset{\text{IH}}{=} \sum_{j,k \in N \ s.t. \ j+k=i} \left(\sum_{x_1 \in \mathcal{U}_{T_1} s.t. \ \iota_{T_1}(x_1) = j} \mathcal{B}_{T_1}(x_1) \right) \cdot \left(\sum_{x_2 \in \mathcal{U}_{T_2} s.t. \ \iota_{T_2}(x_2) = k} \mathcal{B}_{T_2}(x_2) \right) \\ &= \sum_{x = (x_1, x_2) \in \mathcal{U}_{T_1} \times \mathcal{U}_{T_2} \ s.t. \ \iota_{T_1}(x_1) + \iota_{T_2}(x_2) = i} \mathcal{B}_{T_1}(x_1) \cdot \mathcal{B}_{T_2}(x_2) \end{aligned}$$

We know from Definition 6 that $\iota_T(x) = \iota_{T_1}(x_1) + \iota_{T_2}(x_2)$, since $\iota_T(x) \in \{-n+1, \dots, n\}$.

$$
\begin{aligned}
&= \sum_{x=(x_1,x_2)\in \mathcal{U}_{T_1}\times \mathcal{U}_{T_2} \ s.t.\ \iota_T(x)=i} \mathcal{B}_{T_1}(x_1)\cdot \mathcal{B}_{T_2}(x_2)\\
&\overset{\text{Definition 6}}{=} \sum_{x=(x_1,x_2)\in \mathcal{U} \ s.t.\ \iota_T(x)=i} \mathcal{B}_{T_1\times T_2}(x).
\end{aligned}
$$

For $i \in \{-n_T, \infty\}$ the proof follows analogously, where for $-n_T$ we have $j + k \leq -n_T$ and we know from Definition 6 that $\iota_T(x) = -n_T$ is equivalent to $\iota_{T_1}(x_1) + \iota_{T_2}(x_2) \leq -n_T$. For ∞ we have $j + k > n$ and we know from Definition 6 that $\iota_T(x) = \infty$ is equivalent to $\iota_{T_1}(x_1) + \iota_{T_2}(x_2) \geq n_T$.

If $T = \blacktriangledown T_1$, we have for $i \in \{-n_T, \dots, -n_T/2 - 1, n_T/2 + 1, \dots, n_T\}$, $\mathcal{B}_{\blacktriangledown T_1}(i) = 0 = \sum_{x\in\emptyset} P_{A_1}(x) = \sum_{x\in\mathcal{U}_1, \iota_{\blacktriangledown T_1}=i} P_{A_{T_1}}(x)$.

For $i = \infty$, we have $\mathcal{B}_{\blacktriangledown T_1}(\infty) = \mathcal{B}_{T_1}(\infty)$, so the statement follows from the IH. For $i \in \{-n_T/2 + 1, \dots, n_T/2\}$ we have

$$
\begin{aligned}
&\mathcal{B}_{\blacktriangledown T_1}(i) = \mathcal{B}_{T_1}(2i) + \mathcal{B}_{T_1}(2i-1)\\
&\overset{\text{IH}}{=} \textstyle\sum_{x\in\mathcal{U}_{T_1}, \iota_{T_1}(x)=2i} P_{A_{T_1}}(x) + \sum_{x\in\mathcal{U}_{T_1}, \iota_{T_1}(x)=2i-1} P_{A_{T_1}}(x)\\
&= \textstyle\sum_{x\in\mathcal{U}_{\blacktriangledown T_1} \iota_T(x)=i} P_{A_{\blacktriangledown T_1}}(x).
\end{aligned}
$$

The statement for $\mathcal{B}_{\blacktriangledown T_1}(-n_T/2)$ follows analogously. □

We now state the first theorem of our paper: the ADP bounds calculated based on privacy buckets are sound.

Theorem 1 (Buckets are sound)**.** *Let X and Y be two distributions and let $T_{X||Y}$ and $T_{Y||X}$ be valid composition trees with $A_{T_{X||Y}} = B_{T_{Y||X}} = X$ and $B_{T_{X||Y}} = A_{T_{Y||X}} = Y$.*

Then for every $\varepsilon \geq 0$ and for any $\delta \geq \max\left(\delta_{T_{X||Y}}(\varepsilon), \delta_{T_{Y||X}}(\varepsilon)\right)$, X and Y are (ε, δ)-ADP.

The theorem follows quite trivially from the proof of Lemma 13 in the subsequent chapter. We still present a self-contained proof as it could be helpful in understanding the soundness of our privacy buckets.

Proof. We show that $\delta_{T_{X||Y}}(\varepsilon) \leq \delta$ implies $\delta \geq \sum_{x\in\mathcal{U}_{T_{X||Y}}} \max(P_X(x) - e^{\varepsilon} P_Y(x), 0)$ (one direction in Lemma 1); the proof for $T_{Y||X}$ and the other direction follows analogously. Let $n = n_{T_{X||Y}}$, $N = \{-n, \dots, n\}$, $\mathcal{U} = \mathcal{U}_{T_{X||Y}}$ and $f = f_{T_{X||Y}}$. By definition,

$$
\delta_{T_{X||Y}}(\varepsilon) = \textstyle\sum_{i\in N} \left(\max\left(0, \mathcal{B}_{T_{X||Y}}(i)\cdot(1 - e^{\varepsilon}/f^i)\right)\right) + \mathcal{B}_{T_{X||Y}}(\infty).
$$

We ignore $\mathcal{B}_{T_{X||Y}}(\infty)$ for now and apply Lemma 7 and get

$$
\begin{aligned}
&\textstyle\sum_{i\in N} \left(\max\left(0, \sum_{x\in\mathcal{U}.\iota_{T_{X||Y}}(x)=i} P_X(x)\cdot(1 - \frac{e^{\varepsilon}}{f^i})\right)\right)\\
&= \textstyle\sum_{i\in N. f^i > e^{\varepsilon}} \left(\sum_{x\in\mathcal{U}.\iota_{T_{X||Y}}(x)=i} P_X(x)\cdot(1 - \frac{e^{\varepsilon}}{f^i})\right)
\end{aligned}
$$

Using Lemma 6 we get

$$
\textstyle\sum_{x\in\mathcal{U}.\iota_T(x)\in N \wedge f^{\iota_{T_{X||Y}}(x)} > e^{\varepsilon}} \max\left(0, P_X(x) - e^{\varepsilon} P_Y(x)\right).
$$

With $\mathcal{B}_{T_{X||Y}}(\infty)$ (where we also apply Lemma 7) we yield

$$
\begin{aligned}
&\textstyle\sum_{x\in\mathcal{U}.\iota_{T_{X||Y}}(x)\in N. f^{\iota_{T_{X||Y}}(x)} > e^{\varepsilon}} \max\left(0, P_X(x) - e^{\varepsilon} P_Y(x)\right)\\
&+ \textstyle\sum_{x\in\mathcal{U}.\iota_{T_{X||Y}}(x)=\infty} P_X(x)\\
&\geq \textstyle\sum_{x\in\mathcal{U}} \max\left(0, P_X(x) - e^{\varepsilon} P_Y(x)\right).
\end{aligned}
$$

We repeat the calculation analogously for $T_{Y||X}$ and then we use Lemma 1 to see that X and Y are indeed (ε, δ)-ADP. □

4 Reducing and bounding the error

We have already presented a sound way of approximating a distribution pair by creating privacy buckets. Our calculations from the previous section lead to sound and, in many cases, better results than generic composition theorems from the literature. In this section we explore the precision of our results: we define error (correction) terms that help us to both find a lower bound on the differential privacy guarantee for the considered distributions even under manifold composition, and to find a tighter guarantee for differential privacy.

We distinguish between two types of error correction (EC) terms: the *real EC term* ℓ that captures the value we use to tighten our result in a sound way and the *virtual EC term* $\tilde{\ell}$ that captures the maximal influence an EC term can have. The virtual EC term accurately captures the difference between the probability an event x appears to have in the alternative distribution (using the bucket factor) $\frac{P_A(x)}{f^\iota}$ and the probability that it actually has in the alternative $P_B(x)$. In some cases, however, we misplace an event such that it ends up in a bucket with an index that is too large: events x that should not be considered for the overall guarantee, i.e., that have $P_A(x) - e^\varepsilon P_B(x)) < 0$ can appear in a bucket with index i s.t. $e^\varepsilon < f^i$. Thus, correctly calculating the EC term while possibly misplacing events can lead to wrong results.

There are two reasons for why events can be misplaced: First, when composing privacy buckets, events can be misplaced by one bucket. We take care of this by not including the EC terms of a certain number of buckets, depending on the number of compositions. Second, when events are put into the smallest bucket (with index $-n$), they can be arbitrarily "misplaced", particularly after a composition. To counter this effect, we introduce the real EC term, in which we do not include the error of the smallest bucket (with index $-n$).

4.1 Buckets with error correction terms

Our strategy is as follows. Assume two distributions A and B: Whenever we add an event x to a bucket $\mathcal{B}(i)$, we store the difference between the probability that the event occurs in A, adjusted by the bucket factor, and the probability that the same event occurs in B: $P_B(x) - \frac{P_A(x)}{f^i}$. Recall that the main purpose of the buckets is to keep track of the ratio between those two probabilities. We sum up all these *error correction terms* (or EC terms) per individual bucket $\mathcal{B}(i)$ and yield EC terms $\ell(i)$. We refer to Figure 4 (in Section 3.1) for a graphical intuition of our error correction. As an example consider one bucket $\mathcal{B}(i)$, containing events $x \in S_i$ for a set S_i:

$$\begin{aligned}\ell(i) - \frac{\mathcal{B}(i)}{f^i} &= \sum_{x \in S_i} \left(P_B(x) - \frac{P_A(x)}{f^i} \right) \\ &\quad - \frac{\sum_{x \in S_i} P_A(x)}{f^i} \\ &= \sum_{x \in S_i} P_B(x).\end{aligned}$$

Thus, only considering one additional value per bucket, we can precisely remember the probability that the events occurred in B and we can then use this probability to calculate a more precise differential privacy guarantee. We omit the EC terms for the bucket $\mathcal{B}(\infty)$, as there is no bucket factor attached to it (so there is no value the error correction term could correct).

Although the error correction precisely captures the error per event x, we need to be careful which events we consider for calculating δ. Consider the bucket $\mathcal{B}(j)$ with $f^{j-1} < e^\varepsilon \leq f^j$. If we were precise in our calculations, we would only consider *some* of the events from the bucket, namely the ones with $P_A(x) \leq e^\varepsilon P_B(x)$, but since we combined them all into one bucket, we cannot distinguish the individual events anymore. To retain a sound guarantee, we don't consider the EC term of this bucket when calculating δ. Under composition this error slightly increases, as events can be misplaced by more than one bucket when we compose the buckets and thus decrease $P_B(x)$'s approximation. Consequently, every composition increases the number of buckets for which we don't consider an EC term. Whenever events land in the bucket with index $-n$, an arbitrary misplacement can occur and our aforementioned strategy does not suffice. Thus, we distinguish between the *virtual EC term* $\tilde{\ell}$, which applies to index $-n$ and the *real EC term* ℓ, where we always set $\ell(-n) = 0$. For our sound upper bound on δ we will use the real EC term ℓ for all buckets that are unaffected by potential misplacements, and we will use the slightly too large $\tilde{\ell}$ and ignore potential misplacements to derive a lower bound on δ.

For the composition, we want to calculate the error correction (EC) term for the combined events: given events x_1 and x_2 with (individual) error terms $P_{B_1}(x_1) - \frac{P_{A_1}(x)}{f^{\iota_1}}$ and $P_{B_2}(x_2) - \frac{P_{A_2}(x_2)}{f^{\iota_2}}$ we want (in the typical case, ignoring corner cases) to have an EC term for the pair of the form $P_{B_1 \times B_2}((x_1, x_2)) - \frac{P_{A_1 \times A_2}((x_1,x_2))}{f^{\iota_1+\iota_2}}$. However, the buckets cannot keep track of the value for $P_{B_1 \times B_2}((x_1, x_2))$– recall that this is precisely why we have introduced the error terms. Fortunately, we can calculate the EC terms from the previous terms ℓ_{T_1}, ℓ_{T_2}, the bucket values $\mathcal{B}_{T_1}, \mathcal{B}_{T_2}$, and the bucket factor f as

$$\ell_{T_1 \times T_2}(i) := \textstyle\sum_{j+k=i} \frac{\mathcal{B}_{T_1}(j)}{f^j} \ell_{T_2}(k) + \frac{\mathcal{B}_{T_2}(k)}{f^k} \ell_{T_1}(j) + \ell_{T_1}(j)\ell_{T_2}(k).$$

Similarly, for the squaring, we quantify how the error terms change when we modify the buckets. Although each new bucket is composed of two previous buckets, the bucket factor actually only changes for one half of the values: the evenly indexed buckets $\mathcal{B}_T(2i)$ with factor f^{2i} are now moved into buckets $\mathcal{B}_{\blacktriangledown T}(i)$ with the same factor $(f^2)^i$ and thus their EC terms are still correct. The other half of buckets $\mathcal{B}_T(2i-1)$ with factor f^{2i-1} are moved into the same buckets $\mathcal{B}_{\blacktriangledown T}(i)$ with factor $(f^2)^i$ and thus the EC terms need to be modified to capture this change in the bucket factor, based on the previous EC terms ℓ_T and bucket values $\mathcal{B}_T$:

$$\ell_{\blacktriangledown T}(i) := \ell_T(2i-1) + \mathcal{B}_T(2i-1)\left(\frac{1}{f^{2i-1}} - \frac{1}{f^{2i}}\right) + \ell_T(2i).$$

Definition 7 (Privacy buckets with error correction terms)**.** *Let T be a valid composition tree with $n := n_T$ and let $N = \{-n, \dots, n\}$. We define A_T/B_T* privacy buckets with EC terms *as as follows. $\mathcal{B}_T$, f_T, and n_T are defined exactly as in Definition 4, whereas $\tilde{\ell}_T$, ℓ_T, and u_T are defined as follows*

$$\begin{aligned}
\tilde{\ell}_{🍃(A,B,f,n)}(i) &:= \begin{cases} \sum_{x \in \mathcal{U}, \iota(x)=i} P_B(x) - \frac{P_A(x)}{f^i} & \textit{if } i \in N \\ 0 & \textit{if } i = \infty \end{cases} \\
\ell_{🍃(A,B,f,n)}(i) &:= \begin{cases} \tilde{\ell}_{🍃(A,B,f,n)}(i) & \textit{if } i \in N \setminus \{-n\} \\ 0 & \textit{if } i \in \{-n, \infty\} \end{cases} \\
u_{🍃(A,B,f,n)} &:= 1
\end{aligned}$$

For composition we require that $f_{T_1} = f_{T_2}$ and we write $f = f_{T_1}$. To ease readability we define $V(j,k,x,y) = \frac{\mathcal{B}_{T_1}(j)}{f^j} y(k) + \frac{\mathcal{B}_{T_2}(k)}{f^k} x(j) + x(j)y(k)$ and based on V we define the EC terms as

$$\begin{aligned}
\tilde{\ell}_{T_1 \times T_2}(i) &:= \begin{cases} \sum_{j+k=i} V(j,k,\tilde{\ell}_{T_1},\tilde{\ell}_{T_2}) & \textit{if } i \in N \setminus \{-n\} \\ \sum_{j+k \leq -n} V(j,k,\tilde{\ell}_{T_1},\tilde{\ell}_{T_2}) & \textit{if } i = -n \\ 0 & \textit{if } i = \infty \end{cases} \\
\ell_{T_1 \times T_2}(i) &:= \begin{cases} \sum_{j+k=i} V(j,k,\ell_{T_1},\ell_{T_2}) & \textit{if } i \in N \setminus \{-n\} \\ 0 & \textit{if } i \in \{-n, \infty\} \end{cases} \\
u_{T_1 \times T_2}(i) &:= u_{T_1} + u_{T_2}
\end{aligned}$$

To ease the readability we define a function $W(i,x) := x(2i-1) + \mathcal{B}_1(2i-1)\left(\frac{1}{f_{T_1}^{2i-1}} - \frac{1}{f_{T_1}^{2i}}\right) + x(2i)$. We define the EC terms as

$$\begin{aligned}
\tilde{\ell}_{\blacktriangledown T_1}(i) &:= \begin{cases} W(i,\tilde{\ell}_{T_1}) & \textit{if } i \in [-n/2+1, n/2] \\ \tilde{\ell}_{T_1}(-n) & i = -n/2 \\ 0 & \textit{otherwise} \end{cases} \\
\ell_{\blacktriangledown T_1}(i) &:= \begin{cases} W(i,\ell_{T_1}) & \textit{if } i \in [-n/2+1, n/2] \\ 0 & \textit{otherwise} \end{cases} \\
u_{\blacktriangledown T_1}(i) &:= \lceil u_{T_1}/2 \rceil + 1
\end{aligned}$$

4.2 Buckets and error correction terms per element

Before we can show the first helpful lemmas for the soundness of our error correction (EC) terms, we introduce the impact that each individual event x has on the bucket terms that are influenced by x. We first simply define these terms per element separately and then continue by showing that each bucket value (and EC term) is simply the sum over the respective terms of all elements contributing to this bucket. This marks a significant step in the correctness (and tightness) of our results: Although we only consider a few values (one bucket value and one EC value per bucket) we still capture all individual events. The only exception to this precision then comes from misplaced events, which we will analyze subsequently. To distinguish terms per element from our previous (accumulated) terms, we mark terms considering only individual (atomic) events with a special symbol ⚛.

Definition 8 (Privacy buckets with EC terms per element)**.** *Let T be a valid composition tree with $n := n_T$ an $f := f_T$ and $N = \{-n, \dots, n\}$.*

For $T = 🍃(A, B, f, n)$ with $\mathcal{U}_T =: \mathcal{U}$, we define for all $x \in \mathcal{U}$

$$\begin{aligned}
\mathcal{B}^{⚛}_{🍃(A,B,f,n)}(x) &:= P_A(x)\\
\widetilde{\ell}^{⚛}_{🍃(A,B,f,n)}(x) &:= \begin{cases} P_B(x) - \frac{P_A(x)}{f^{\iota_{🍃(A,B,f,n)}(x)}} & \iota_{🍃(A,B,f,n)}(x) \in N,\\ 0 & \iota_{🍃(A,B,f,n)}(x) = \infty,\end{cases}\\
\ell^{⚛}_{🍃(A,B,f,n)}(x) &:= \begin{cases} \widetilde{\ell}^{⚛}_{🍃(A,B,f,n)}(x) & \iota_{🍃(A,B,f,n)}(x) \in N \setminus \{-n\},\\ 0 & \iota_{🍃(A,B,f,n)}(x) \in \{-n, \infty\}.\end{cases}
\end{aligned}$$

For $T = T_1 \times T_2$with $\mathcal{U}_i = \mathcal{U}_{T_i}$ (for $i \in \{1,2\}$), we define for all $x = (x_1, x_2)$ with $x_1 \in \mathcal{U}_1$ and $x_2 \in \mathcal{U}_2$

$$\mathcal{B}^{⚛}_{T_1 \times T_2}(x) := \mathcal{B}_{T_1}(x_1) \cdot \mathcal{B}_{T_2}(x_2)$$

and we define the EC terms as

$$\begin{aligned}
&\text{if } \iota_{T_1 \times T_2}(x) \in \{-n, \dots, n\}\\
\widetilde{\ell}^{⚛}_{T_1 \times T_2}(x) &:= \left(\frac{\mathcal{B}^{⚛}_{T_1}(x_1)}{f^{\iota_{T_1}(x_1)}} + \widetilde{\ell}^{⚛}_{T_1}(x_1)\right)\widetilde{\ell}^{⚛}_{T_2}(x_2) + \widetilde{\ell}^{⚛}_{T_1}(x_2)\left(\frac{\mathcal{B}^{⚛}_{T_2}(x_2)}{f^{\iota_{T_2}(x_2)}} + \widetilde{\ell}^{⚛}_{T_2}(x_2)\right) - \widetilde{\ell}^{⚛}_{T_1}(x_1)\widetilde{\ell}^{⚛}_{T_2}(x_2)\\
&\text{if } \iota_{T_1 \times T_2}(x) \in \{\infty\}\\
&\widetilde{\ell}^{⚛}_{T_1 \times T_2}(x) := 0\\
&\text{if } \iota_{T_1 \times T_2}(x) \in \{-n+1, \dots, n, \infty\}\\
\ell^{⚛}_{T_1 \times T_2}(x) &:= \left(\frac{\mathcal{B}^{⚛}_{T_1}(x_1)}{f^{\iota_{T_1}(x_1)}} + \ell^{⚛}_{T_1}(x_1)\right)\ell^{⚛}_{T_2}(x_2) + \ell^{⚛}_{T_1}(x_2)\left(\frac{\mathcal{B}^{⚛}_{T_2}(x_2)}{f^{\iota_{T_2}(x_2)}} + \ell^{⚛}_{T_2}(x_2)\right) - \ell^{⚛}_{T_1}(x_1)\ell^{⚛}_{T_2}(x_2)\\
&\text{if } \iota_{T_1 \times T_2}(x) \in \{-n, \infty\}\\
&\ell^{⚛}_{T_1 \times T_2}(x) := 0.
\end{aligned}$$

For a squaring node ($T = \blacktriangledown T_1$), we keep the bucket value as $\mathcal{B}^{⚛}_{\blacktriangledown T_1}(x) := \mathcal{B}^{⚛}_{T_1}(x_1)$ and we define the EC *terms as follows for $f = f_{T_1}$:*

$$\begin{aligned}
&\text{if } \iota_{T_1}(x) \in \{-n, \dots, n\}\\
\tilde{\ell}^{\circledast}_{\blacktriangledown T_1}(x) &:= \tilde{\ell}^{\circledast}_{T_1}(x) + \mathcal{B}^{\circledast}_{T_1}(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2\cdot\lceil \iota_{T_1}(x)/2 \rceil}} \right)\\
&\text{if } \iota_{T_1}(x) \in \{\infty\}\\
&\tilde{\ell}^{\circledast}_{\blacktriangledown T_1}(x) := 0\\
\\
&\text{if } \iota_{T_1}(x) \in \{-n+1, \dots, n\}\\
\ell^{\circledast}_{\blacktriangledown T_1}(x) &:= \ell^{\circledast}_{T_1}(x) + \mathcal{B}^{\circledast}_{T_1}(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2\cdot\lceil \iota_{T_1}(x)/2 \rceil}} \right)\\
&\text{if } \iota_{T_1}(x) \in \{-n, \infty\}\\
&\ell^{\circledast}_{\blacktriangledown T_1}(x) := 0.
\end{aligned}$$

We now show our first important lemma for the soundness of our buckets and EC terms: the terms we defined just previously indeed characterize the impact of each individual event on the overall bucket values and EC terms. These terms indeed are just the sum of the respective values per element for all elements of an index that equals the bucket index.

Lemma 8 (All values are sums over atomic events)**.** *Let T be a valid composition tree, labeled with $n \in \mathbb{N}$. Then, the following statements hold for all $i \in \{-n, \dots, n, \infty\}$ and $x \in \mathcal{U}_T$:*

- $\mathcal{B}_T(i) = \sum_{x \in \mathcal{U} \text{ s.t. } \iota_T(x)=i} \mathcal{B}^{\circledast}_T(x)$
- $\tilde{\ell}_T(i) = \sum_{x \in \mathcal{U} \text{ s.t. } \iota_T(x)=i} \tilde{\ell}^{\circledast}_T(x)$
- $\ell_T(i) = \sum_{x \in \mathcal{U} \text{ s.t. } \iota_T(x)=i} \ell^{\circledast}_T(x)$

Proof. We show the lemma via structural induction over T.

If $T = \varnothing(A, B, f, n)$: Let $i \in \{-n, \dots, n, \infty\}$ and $x \in \mathcal{U}_T$.

- By definition, $\mathcal{B}^{\circledast}_T(x) = P_A(x)$ (c.f., Definition 8). Thus, $\mathcal{B}^{\circledast}_T(i) = \sum_{x \text{ s.t. } \iota(x)=i} P_A(x) = \sum_{x \text{ s.t. } \iota(x)=i} \mathcal{B}^{\circledast}_T(x)$.
- If $i \in \{-n, \dots, n\}$, then $\tilde{\ell}_T(i) = \sum_{x \text{ s.t. } \iota(x)=i} P_B(x) - \frac{P_A(x)}{f^i} = \sum_{x \text{ s.t. } \iota(x)=i} \tilde{\ell}^{\circledast}_T(x)$. Otherwise $\tilde{\ell}_T(i) = 0 = \sum_{x \text{ s.t. } \iota(x)=i} 0 = \sum_{x \text{ s.t. } \iota(x)=i} \tilde{\ell}^{\circledast}_T(x)$.
- If $i \in \{-n+1, \dots, n\}$, then $\ell_T(i) = \sum_{x \text{ s.t. } \iota(x)=i} P_B(x) - \frac{P_A(x)}{f^i} = \sum_{x \text{ s.t. } \iota(x)=i} \ell^{\circledast}_T(x)$. Otherwise $\ell_T(i) = 0 = \sum_{x \text{ s.t. } \iota(x)=i} 0 = \sum_{x \text{ s.t. } \iota(x)=i} \ell^{\circledast}_T(x)$.

If $T = T_1 \times T_2$: We assume the lemma holds for the composition trees T_1 and T_2.
For $i \in \{-n+1, \dots, n\}$ and $x \in \mathcal{U}_T$ and $f := f_T$

$$\begin{aligned}
\mathcal{B}_{T_1 \times T_2}(i) &= \sum_{j,k \in \{-n,\dots,n\} \text{ s.t.} j+k=i} \mathcal{B}_{T_1}(j) \cdot \mathcal{B}_{T_2}(k)\\
&\overset{IV}{=} \sum_{j,k \in \{-n,\dots,n\} \text{ s.t. } j+k=i} \left(\sum_{x_1 \in \mathcal{U}_1 \text{s.t. } \iota_{T_1}(x_1)=j} \mathcal{B}^{\circledast}_{T_1}(x_1) \right) \cdot \left(\sum_{x_2 \in \mathcal{U}_2 \text{s.t. } \iota_{T_2}(x_2)=k} \mathcal{B}^{\circledast}_{T_2}(x_2) \right)\\
&= \sum_{x=(x_1,x_2) \in \mathcal{U}_1 \times \mathcal{U}_2 \text{ s.t. } \iota_{T_1}(x_1)+\iota_{T_2}(x_2)=i} \mathcal{B}^{\circledast}_{T_1}(x_1) \cdot \mathcal{B}^{\circledast}_{T_2}(x_2)
\end{aligned}$$

We know from Definition 6 that $\iota_T(x) = \iota_{T_1}(x_1) + \iota_{T_2}(x_2)$, since $\iota_T(x) \in \{-n+1, \dots, n\}$.

$$\begin{aligned}
&= \sum_{x=(x_1,x_2)\in\mathcal{U}_1\times\mathcal{U}_2 \ s.t. \ \iota_T(x)=i} \mathcal{B}^{\text{⚛}}_{T_1}(x_1) \cdot \mathcal{B}^{\text{⚛}}_{T_2}(x_2) \\
&= \sum_{x=(x_1,x_2)\in\mathcal{U} \ s.t. \ \iota_T(x)=i} \mathcal{B}^{\text{⚛}}_{T_1\times T_2}(x).
\end{aligned}$$

For $i \in \{-n, \infty\}$ the proof follows analogously, where for $-n$ we have $j + k \leq -n$ and we know from Definition 6 that $\iota_T(x) = -n$ is equivalent to $\iota_{T_1}(x_1) + \iota_{T_2}(x_2) \leq -n$ and for ∞ we have $j + k > n$ and we know from Definition 6 that $\iota_T(x) = \infty$ is equivalent to $\iota_{T_1}(x_1) + \iota_{T_2}(x_2) \geq n$.

For the virtual error, we distinguish the following cases:

- $\iota_T(x) \in \{-n+1, \dots, n\}$. Then,

$$\begin{aligned}
&\tilde{\ell}_{T_1\times T_2}(i) \\
&= \sum_{(k,l)\in\{-n,\dots,n\}^2, k+l=i} \left(\frac{\mathcal{B}_{T_1}(k)}{f^k} + \tilde{\ell}_{T_1}(k)\right)\tilde{\ell}_{T_2}(l) + \tilde{\ell}_{T_1}(k)\left(\frac{\mathcal{B}_{T_2}(l)}{f^l} + \tilde{\ell}_{T_2}(l)\right) - \tilde{\ell}_{T_1}(k)\tilde{\ell}_{T_2}(l) \\
&= \sum_{(k,l)\in\{-n,\dots,n\}^2, k+l=i} \frac{\mathcal{B}_{T_1}(k)}{f^k}\tilde{\ell}_{T_2}(l) + \tilde{\ell}_{T_1}(k)\frac{\mathcal{B}_{T_2}(l)}{f^l} + \tilde{\ell}_{T_1}(k)\tilde{\ell}_{T_2}(l) \\
&= \sum_{(k,l)\in\{-n,\dots,n\}^2, k+l=i} \left(\frac{\sum_{x_1\in\mathcal{U}_1 s.t. \ \iota_{T_1}(x_1)=k} \mathcal{B}^{\text{⚛}}_{T_1}(x_1)}{f^k}\left(\sum_{x_2\in\mathcal{U}_2 s.t. \ \iota_{T_2}(x_2)=l} \tilde{\ell}^{\text{⚛}}_{T_2}(x_2)\right)\right. \\
&\quad + \left(\sum_{x_1\in\mathcal{U}_1 s.t. \ \iota_{T_1}(x_1)=k} \tilde{\ell}^{\text{⚛}}_{T_1}(x_1)\right)\frac{\sum_{x_2\in\mathcal{U}_2 s.t. \ \iota_{T_2}(x_2)=l} \mathcal{B}^{\text{⚛}}_{T_2}(x_2)}{f^l} \\
&\quad \left. + \left(\sum_{x_1\in\mathcal{U}_1 s.t. \ \iota_{T_1}(x_1)=k} \tilde{\ell}^{\text{⚛}}_{T_1}(x_1)\right)\left(\sum_{x_2\in\mathcal{U}_2 s.t. \ \iota_{T_2}(x_2)=l} \tilde{\ell}^{\text{⚛}}_{T_2}(x_2)\right)\right) \\
&= \sum_{(k,l)\in\{-n,\dots,n\}^2, k+l=i} \ \sum_{x_1\in\mathcal{U}_1 s.t. \ \iota_{T_1}(x_1)=k} \ \sum_{x_2\in\mathcal{U}_2 s.t. \ \iota_{T_2}(x_2)=l} \left(\frac{\mathcal{B}^{\text{⚛}}_{T_1}(x_1)}{f^k}\tilde{\ell}^{\text{⚛}}_{T_2}(x_2)\right. \\
&\quad \left. + \tilde{\ell}^{\text{⚛}}_{T_1}(x_1)\frac{\mathcal{B}^{\text{⚛}}_{T_2}(x_2)}{f^l} + \tilde{\ell}^{\text{⚛}}_{T_1}(x_1)\tilde{\ell}^{\text{⚛}}_{T_2}(x_2)\right) \\
&= \sum_{(x_1,x_2)\in\mathcal{U}_1\times\mathcal{U}2 \ s.t. \ \iota_{T_1}(x_1)+\iota_{T_2}(x_2)=i} \left(\frac{\mathcal{B}^{\text{⚛}}_{T_1}(x_1)}{f^{\iota_{T_1}(x_1)}}\tilde{\ell}^{\text{⚛}}_{T_2}(x_2) + \tilde{\ell}^{\text{⚛}}_{T_1}(x_1)\frac{\mathcal{B}^{\text{⚛}}_{T_2}(x_2)}{f^{\iota_{T_2}(x_2)}} + \tilde{\ell}^{\text{⚛}}_{T_1}(x_1)\tilde{\ell}^{\text{⚛}}_{T_2}(x_2)\right)
\end{aligned}$$

  We know from Definition 6 that $\iota_T(x) = \iota_{T_1}(x_1) + \iota_{T_2}(x_2)$, since $\iota_T(x) \in \{-n+1, \dots, n\}$.

$$= \sum_{x\in\mathcal{U} \ s.t. \ \iota_T(x)=i} \tilde{\ell}^{\text{⚛}}_{T_1\times T_2}(x)$$

- $\iota_T(x) = -n$. The proof of the case from above follows analogously with $k + l \leq -n$, since we know from Definition 6 that $\iota_T(x) = -n$ is equivalent to $\iota_{T_1}(x_1) + \iota_{T_2}(x_2) \leq -n$.

- $\iota_T(x) = \infty$.

$$\begin{aligned}
&\tilde{\ell}_{T_1\times T_2}(i) \\
&= 0 = \sum_{x\in\mathcal{U} \ s.t. \ \iota_T(x)=i} 0 \\
&= \sum_{x\in\mathcal{U} \ s.t. \ \iota_T(x)=i} \tilde{\ell}^{\text{⚛}}_{T_1\times T_2}(x).
\end{aligned}$$

If $T = \blacktriangledown T_1$:

We assume the lemma holds for a composition tree T_1, we have for $i \in \{-n, \dots, -n/2-1, n/2+1, \dots, n\}$ and $x \in \mathcal{U}_T$

$$\mathcal{B}_{\blacktriangledown T_1}(i) = 0 = \sum_{x \in \emptyset} \mathcal{B}^{\text{⚛}}_{\blacktriangledown T_1}(x) = \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{\blacktriangledown T} = i} \mathcal{B}^{\text{⚛}}_{\blacktriangledown T_1}(x)$$

For $i = \infty$, we have

$$\mathcal{B}_{\blacktriangledown T_1}(\infty) = \mathcal{B}_{T_1}(\infty) \overset{\text{IH}}{=} \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x) = \infty} \mathcal{B}^{\text{⚛}}_{T_1}(x) = \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{\blacktriangledown T_1}(x) = \infty} \mathcal{B}^{\text{⚛}}_{\blacktriangledown T_1}(x).$$

For $i \in \{-n/2+1, \dots, n/2\}$ we have

$$\begin{aligned}
\mathcal{B}_{\blacktriangledown T_1}(i) &= \mathcal{B}_{T_1}(2i) + \mathcal{B}_{T_1}(2i-1) \\
&\overset{\text{IH}}{=} \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x) = 2i} \mathcal{B}^{\text{⚛}}_{T_1}(x) + \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x) = 2i-1} \mathcal{B}^{\text{⚛}}_{T_1}(x) \\
&= \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x) = 2i} \mathcal{B}^{\text{⚛}}_{\blacktriangledown T_1}(x) + \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x) = 2i-1} \mathcal{B}^{\text{⚛}}_{\blacktriangledown T_1}(x) \\
&= \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{\blacktriangledown T}(x) = i} \mathcal{B}^{\text{⚛}}_{\blacktriangledown T_1}(x).
\end{aligned}$$

For $i = -n/2$ we have

$$\begin{aligned}
\mathcal{B}_{\blacktriangledown T_1}(-n/2) &= \mathcal{B}_1(-n) \\
&\overset{\text{IH}}{=} \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x) = -n} \mathcal{B}_{T_1}(x) \\
&= \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x) = -n} \mathcal{B}^{\text{⚛}}_{\blacktriangledown T_1}(x) \\
&= \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{\blacktriangledown T_1}(x) = -n/2} \mathcal{B}^{\text{⚛}}_{\blacktriangledown T_1}(x).
\end{aligned}$$

We hence go forward to show the lemma for the EC terms.

For the EC terms and for $i \in \{-n, \dots, -n/2-1, n/2+1, \dots, n\}$

$$\tilde{\ell}_{\blacktriangledown T_1}(i) = 0 = \sum_{x \in \emptyset} \tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x) = \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{\blacktriangledown T_1} = i} \tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x)$$

For $i = \infty$, we have

$$\begin{aligned}
\tilde{\ell}_{\blacktriangledown T_1}(\infty) = 0 &= \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{\blacktriangledown T_1}(x) = \infty} 0 \\
&= \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x) = \infty} \tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x) = \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{\blacktriangledown T_1}(x) = \infty} \tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x).
\end{aligned}$$

For $i \in \{-n/2+1, \ldots, n/2\}$, let $f := f_{T_1}$. Then we have

$$\begin{aligned}
\tilde{\ell}_{\blacktriangledown T_1}(i) &= \tilde{\ell}_{T_1}(2i-1) + \mathcal{B}_{T_1}(2i-1)\left(\frac{1}{f^{2i-1}} - \frac{1}{f^{2i}}\right) + \tilde{\ell}_{T_1}(2i) \\
&\overset{\text{IH}}{=} \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x)=2i-1} \tilde{\ell}^{\text{⚛}}_{T_1}(x) + \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x)=2i-1} \mathcal{B}^{\text{⚛}}_{T_1}(x)\left(\frac{1}{f^{2i-1}} - \frac{1}{f^{2i}}\right) \\
&\quad + \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x)=2i} \tilde{\ell}^{\text{⚛}}_{T_1}(x) \\
&= \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x)=2i-1} \tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x) - \mathcal{B}^{\text{⚛}}_{T_1}(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2\cdot\lceil \iota_{T_1}(x)/2\rceil}}\right) \\
&\quad + \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x)=2i-1} \mathcal{B}^{\text{⚛}}_{T_1}(x)\left(\frac{1}{f^{2i-1}} - \frac{1}{f^{2i}}\right) \\
&\quad + \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x)=2i} \tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x) - \mathcal{B}^{\text{⚛}}_{T_1}(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2\cdot\lceil \iota_{T_1}(x)/2\rceil}}\right) \\
&= \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x)=2i-1} \tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x) - \mathcal{B}^{\text{⚛}}_{T_1}(x) \cdot \left(\frac{1}{f^{2i-1}} - \frac{1}{f^{2i}}\right) + \mathcal{B}^{\text{⚛}}_{T_1}(x)\left(\frac{1}{f^{2i-1}} - \frac{1}{f^{2i}}\right) \\
&\quad + \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{T_1}(x)=2i} \tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x) \\
&= \sum_{x \in \mathcal{U} \text{ s.t. } \iota_{\blacktriangledown T_1}(x)=i} \tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x)
\end{aligned}$$

The proof for $\tilde{\ell}_{\blacktriangledown T_1}(i)$ in case $i = -n/2$ and the $\ell_{\blacktriangledown T_1}(i)$ follow analogously to the proof for $\tilde{\ell}_{\blacktriangledown T_1}(i)$ with the exception that the case $-n/2$ is analogous to the case ∞ instead to the cases $i \in \{-n+1, \ldots, n\}$ for $\ell_{\blacktriangledown T_1}(i)$. □

With Lemma 8 we now have a powerful tool for proving a set of properties for our EC terms that will ultimately allow us to show the soundness of our results: We can relate every bucket value and every EC term to the underlying events and can thus analyze our properties per event.

4.3 Helpful properties of error correction terms

In this rather technical subsection we present and show a set of helpful properties of our EC terms that we require for our proof of soundness (and for our lower bound). We show that all error terms are positive (which means that not considering one of them can only increase the δ of our result), we show that our real EC term is always smaller than the virtual EC term, which we use for proving the soundness of the approximation (Lemma 13). Finally, we show that for every event x, the virtual EC term after an arbitrary amount of composition and squaring following the composition tree T still precisely captures $P_B(x) - \frac{P_A(x)}{f^{\iota_T}}$.

Lemma 9 (Positive real and virtual error correction terms)**.** *Let T be a valid composition tree with $n := n_T$. Then for all $i \in \{-n, \ldots, n, \infty\}$, both the real and virtual EC terms are positive, i.e., $\ell_T(i) \geq 0$ and $\tilde{\ell}_T(i) \geq 0$.*

Proof. We show the lemma via structural induction over T. For leaf nodes $T = \text{🍃}(A, B, f, n)$, the real EC term of an initial bucketing is calculated as the sum of EC terms for each $x \in \mathcal{U}$, which are either $P_B(x) - \frac{P_A(x)}{f^{\iota_T(x)}}$ or 0. By definition we know that $P_A(x) \leq f^{\iota_T(x)} P_B(x)$, so all these values are positive. For composition $T_1 \times T_2$ with $f_{T_1} = f_{T_2} =: f$ we have either 0 or $V(j,k,x,y) = \frac{\mathcal{B}_{T_1}(j)}{f^j} y(k) + \frac{\mathcal{B}_{T_2}(k)}{f^k} x(j) + x(j)y(k)$, which is the sum and product of positive terms (the latter we know from the induction invariant). Analogously we notice that for squaring $\blacktriangledown T_1$ with $f_{T_1} =: f$ we have either 0 or $\ell_{T_1}(2i-1) + \mathcal{B}_{T_1}(2i-1)\left(\frac{1}{f^{2i-1}} - \frac{1}{f^{2i}}\right) + \ell_{T_1}(2i)$, which again consists purely of positive terms (again via induction invariant).

More precisely, we distinguish the following cases:

For $T = \varnothing(A, B, f, n)$, the real EC term of an initial bucketing is calculated as the sum of EC terms for each $x \in \mathcal{U}$, $\ell^{\circledast}_T(x) = P_B(x) - \frac{P_A(x)}{f^{\iota_T(x)}}$ if $\iota_T(x) \notin \{-n, \infty\}$ and 0 otherwise. For $\iota_T(x) \in \{-n, \dots, n\}$ by definition we have $P_A(x) \le f^{\iota_T(x)} P_B(x)$. Thus, for all $i \in \{-n, \dots, n, \infty\}$ are positive, i.e., $\ell_T(i) \ge 0$.

For $T = T_1 \times T_2$, $\mathcal{B}_T$ with $f_{T_1} = f_{T_2} =: f$, by induction hypothesis, ℓ_{T_1} and ℓ_{T_2} are positive. We calculate the composed EC terms as either 0 (if $i \in \{-n, \infty\}$) or as

$$\ell_{T_1 \times T_2}(i) = \ell_{T_1 \times T_2}(i) = \sum_{j,k \ s.t.\ j+k=i} \left(\left(\frac{\mathcal{B}_{A_1}(j)}{f^j} \right) \ell_{T_2}(k) + \left(\frac{\mathcal{B}_{T_2}(k)}{f^k} \right) \ell_{T_1}(j) + \ell_{T_1}(j)\ell_{T_2}(k) \right),$$

which is positive as well since all the EC terms and all bucket terms are positive.

For $T = \blacktriangledown T_1$, We calculate, with $f := f_{T_1}$, the EC terms as either 0 (if $i \in \{-n, \dots, -n/2-1, n/2+1, \dots, n, \infty\}$) or as

$$\ell_{\blacktriangledown T_1}(i) = \blacktriangledown \ell_{T_1}(i) = \ell_{T_1}(2i-1) + \mathcal{B}_{T_1}(2i-1)\left(\frac{1}{f^{2i-1}} - \frac{1}{f^{2i}}\right) + \ell_{T_1}(2i),$$

which is positive as well since all the EC terms and all bucket terms are positive.[7] Analogously, we can show that the virtual EC terms $\tilde{\ell}$ are positive as well. □

We now show that the real EC term is smaller than the virtual EC term.

Lemma 10 (The real error ℓ is smaller than the virtual error $\tilde{\ell}$)**.** *Let T be a valid composition tree labeled $n \in \mathbb{N}$ with $\mathcal{U} := \mathcal{U}_T$. Then, the real error is always smaller than the virtual error: $\ell^{\circledast}_T(x) \le \tilde{\ell}^{\circledast}_T(x)$.*

Proof. We show the lemma via structural induction over T.

For $T = \varnothing(A, B, f, n)$: We know that $\tilde{\ell}^{\circledast}_{\varnothing(A,B,f,n)}(x) \ge 0$. By definition, since $u_{\varnothing(A,B,f,n)} = 1$, either $\ell^{\circledast}_{\varnothing(A,B,f,n)}(x) = 0$ or $\ell^{\circledast}_{\varnothing(A,B,f,n)}(x) = \tilde{\ell}^{\circledast}_{\varnothing(A,B,f,n)}(x)$ holds. Thus, $\ell^{\circledast}_{\varnothing(A,B,f,n)}(x) \le \tilde{\ell}^{\circledast}_{\varnothing(A,B,f,n)}(x)$.

For $T = T_1 \times T_2$: Let $\mathcal{U}_1$ be the universe of T_1 and $\mathcal{U}_2$ be the universe of T_2. By induction hypothesis, $\ell^{\circledast}_{T_1} \le \tilde{\ell}^{\circledast}_{T_1}$ and $\ell^{\circledast}_{T_2} \le \tilde{\ell}^{\circledast}_{T_2}$. Let $f = f_{T_1} = f_{T_2}$. For $\iota_T(x) = -n$, $\ell^{\circledast}_{T_1 \times T_2}(x) = 0$. By Lemma 9 we know that $0 \le \tilde{\ell}^{\circledast}_{T_1 \times T_2}(x)$, hence $\ell^{\circledast}_{T_1 \times T_2}(x) = 0 \le \tilde{\ell}^{\circledast}_{T_1 \times T_2}(x)$. For $\iota_{T_1 \times T_2}(x) \ne -n$, with $x_1 \in \mathcal{U}_1$ and $x_2 \in \mathcal{U}_2$ we have

$$\begin{aligned}
\ell^{\circledast}_{T_1 \times T_2}(x) &= \left(\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} + \ell^{\circledast}_{T_1}(x_1) \right) \ell^{\circledast}_{T_2}(x_2) + \left(\frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} + \ell^{\circledast}_{T_2}(x_2) \right) \ell^{\circledast}_{T_1}(x_1) - \ell^{\circledast}_{T_1}(x_1)\ell^{\circledast}_{T_2}(x_2) \\
&= \left(\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} \right) \underbrace{\ell^{\circledast}_{T_2}(x_2)}_{\overset{\text{IH}}{\le} \tilde{\ell}^{\circledast}_{T_2}(x_2)} + \left(\frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} \right) \underbrace{\ell^{\circledast}_{T_1}(x_1)}_{\overset{\text{IH}}{\le} \tilde{\ell}^{\circledast}_{T_1}(x_1)} + \underbrace{\ell^{\circledast}_{T_1}(x_1)}_{\overset{\text{IH}}{\le} \tilde{\ell}^{\circledast}_{T_1}(x_1)} \underbrace{\ell^{\circledast}_{T_2}(x_2)}_{\overset{\text{IH}}{\le} \tilde{\ell}^{\circledast}_{T_2}(x_2)} \\
&\overset{\text{IH}}{\le} \left(\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} \right) \tilde{\ell}^{\circledast}_{T_2}(x_2) + \left(\frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} \right) \tilde{\ell}^{\circledast}_{T_1}(x_1) + \tilde{\ell}^{\circledast}_{T_1}(x_1)\tilde{\ell}^{\circledast}_{T_2}(x_2) \\
&= \tilde{\ell}^{\circledast}_{T_1} \times \tilde{\ell}^{\circledast}_{T_2}(x) = \tilde{\ell}^{\circledast}_{T_1 \times T_2}(x)
\end{aligned}$$

[7] Note that in the case $-n/2$ there is only one term instead of two. This term, however, is still positive.

For $T = \blacktriangledown T_1$: This case directly holds by induction hypothesis, as the squaring operation is analogously defined for the real and the virtual error.

□

We now show our main lemma for the lower bound on δ: the virtual EC term is precise for any event with an index other than ∞. We can directly use this lemma to get a lower bound for δ if we ignore the bucket with index ∞. Note that although the virtual error is precise on a per-event basis, events can still be misplaced and thus negatively contribute to δ if we use the virtual EC term. For our upper bound on δ we circumvent this problem by over-approximating misplaced events (using the real EC term) and by not using EC terms in some buckets with a bucket factor f^i close to e^ε.

Lemma 11 (Characterizing the virtual error after compositions and rescaling)**.** *Let T be a valid composition tree with $A := A_T$, $B := B_T$, $\mathcal{U} := \mathcal{U}_T$, $n := n_T$, and $f := f_T$. Then, for $x \in \mathcal{U}$ with $\iota_T(x) \neq \infty$ we have*

$$\widetilde{\ell}^{\circledast}_{T}(x) = P_B(x) - \frac{P_A(x)}{f^{\iota_T(x)}}$$

Proof. We show the lemma via structural induction over T. For $T = \varnothing(A, B, f, n)$, the statement follows by construction:

$$\widetilde{\ell}^{\circledast}_{T}(x), = P_B(x) - \frac{P_A(x)}{f^{\iota_T(x)}}$$

and $f_{\varnothing(A,B,f,n)} = f$.

For $T = T_1 \times T_2$ with $A_i := A_{T_i}$, $B_i := B_{T_i}$, $\mathcal{U}_i := \mathcal{U}_{T_i}$, $f := f_T$, and set $A := A_1 \times A_2$ and $B := B_1 \times B_2$. By induction hypothesis, the statement holds for $\widetilde{\ell}^{\circledast}_{T_1}$ and $\widetilde{\ell}^{\circledast}_{T_2}$. By definition of the EC term composition we get for all $x_1 \in \mathcal{U}_1$ and $x_2 \in \mathcal{U}_2$

$$\begin{aligned}
\widetilde{\ell}^{\circledast}_{T_1 \times T_2}(x) = &\left(\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} + \widetilde{\ell}^{\circledast}_{T_1}(x_1)\right)\widetilde{\ell}^{\circledast}_{T_2}(x_2) + \left(\frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} + \widetilde{\ell}^{\circledast}_{T_2}(x_2)\right)\widetilde{\ell}^{\circledast}_{T_1}(x_1) - \widetilde{\ell}^{\circledast}_{T_1}(x_1)\widetilde{\ell}^{\circledast}_{T_2}(x_2) \\
= &\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} \cdot \widetilde{\ell}^{\circledast}_{T_2}(x_2) + \frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} \cdot \widetilde{\ell}^{\circledast}_{T_1}(x_1) + \widetilde{\ell}^{\circledast}_{T_1}(x_1)\widetilde{\ell}^{\circledast}_{T_2}(x_2) \\
\overset{\text{IH}}{=} &\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} \cdot \left(P_{B_2}(x_2) - \frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}}\right) \\
&+ \frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} \cdot \left(P_{B_1}(x_1) - \frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}}\right) \\
&+ \left(P_{B_1}(x_1) - \frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}}\right) \cdot \left(P_{B_2}(x_2) - \frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}}\right) \\
= &\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} \cdot P_{B_2}(x_2) - \frac{P_A(x)}{f^{\iota_T(x)}} \\
&+ \frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} \cdot P_{B_1}(x_1) - \frac{P_A(x)}{f^{\iota_T(x)}} \\
&+ P_B(x) - \frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} \cdot P_{B_2}(x_2) - \frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} \cdot P_{B_1}(x_1) + \frac{P_A(x)}{f^{\iota_T(x)}} \\
= &P_B(x) - \frac{P_A(x)}{f^{\iota_T(x)}}
\end{aligned}$$

For $T = \blacktriangledown T_1$, where T_1 is a composition tree over the distributions A/B over the universe $\mathcal{U}$, we know that for all $x \in \mathcal{U}$, $\iota_{\blacktriangledown T_1}(x) \in \{-n/2, \ldots, n/2\} \cup \{\infty\}$. Since the index ∞ is excluded in our lemma, we focus on the remaining values for the index. Note that the bucket factor in this case changes from $f_{T_1} =: f$ (of the child node) to $f_{\blacktriangledown T_1} = (f_{T_1})^2 = f^2$ (of the squaring node). By induction hypothesis, we have

$$\widetilde{\ell}^{\circledast}_{T_1}(x) = P_B(x) - \frac{P_{A_1}(x)}{f^{\iota_T(x)}}$$

Consequently and since $\iota_{\blacktriangledown T_1}(x) \in \{-n/2, \dots, n/2\}$ and $\mathcal{B}^{\text{⚛}}_{T_1} = P_A(x)$, we get,

$$\begin{aligned}
\tilde{\ell}^{\text{⚛}}_{\blacktriangledown T_1}(x) &= \tilde{\ell}^{\text{⚛}}_{T_1}(x) + \mathcal{B}^{\text{⚛}}_{T_1}(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2 \cdot \lceil \iota_{T_1}(x)/2 \rceil}} \right) \\
&\overset{\text{IH}}{=} P_B(x) - \frac{P_A(x)}{f^{\iota_{T_1}(x)}} + P_A(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2 \cdot \lceil \iota_{T_1}(x)/2 \rceil}} \right) \\
&= P_B(x) - \frac{P_A(x)}{f^{\iota_{T_1}(x)}} + P_A(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2\iota_T(x)}} \right) \\
&= P_B(x) - \frac{P_A(x)}{(f^2)^{\iota_T(x)}}.
\end{aligned}$$

□

4.4 The approximated delta with error correction

Finally, we define how to calculate a sound upper bound on δ based on privacy buckets with EC terms. We note that when using the real EC term, events cannot harm the soundness by being misplaced as a result of parts of the event having been placed in the smallest bucket (with index $-n$). However, every composition can misplace events into the next larger bucket. This slight misplacement poses a problem for a small number of buckets with a bucket factor f^i just slightly larger than e^ε, as they can now contain events that should have been placed in a lower bucket (with factor $f^{i^*} < e^\varepsilon$) and that now actually have a negative contribution to δ: $P_A(x) - e^\varepsilon P_B(x) < 0$. All composition trees for privacy buckets carry a value $u = 1$ at each leaf that increases by 1 for every composition and that is halved by squaring. If j_ε is the index of the bucket with the smallest bucket factor larger than e^ε, we don't consider the the EC term for buckets with index $i < j_\varepsilon + u$ and instead fall back to Definition 5 for those buckets. For the remaining buckets with $i \geq j_\varepsilon + u$, which typically is the vast majority of buckets, we make use of the real EC term to reduce the error.

Definition 9 (Approximated delta with error correction)**.** *Let T be a valid composition tree with $A := A_T$, $\mathcal{U} := \mathcal{U}_T$, $n := n_T$, and $f := f_T$.*

We define $\delta_T(\varepsilon)$ with $j_\varepsilon \in \mathbb{N}$ such that $f^{j_\varepsilon - 1} < e^\varepsilon \leq f^{j_\varepsilon}$ as

$$\begin{aligned}
\delta_T(\varepsilon) := &\sum_{i \in \{j_\varepsilon, \dots, j_\varepsilon + u_T - 1\}} \mathcal{B}_T(i) - \frac{e^\varepsilon \mathcal{B}_T(i)}{f^i} \\
&+ \sum_{i \in \{j_\varepsilon + u_T, \dots, n\}} \left(\mathcal{B}_T(i) - e^\varepsilon \left(\frac{\mathcal{B}_T(i)}{f^i} + \ell_T(i) \right) \right) + \mathcal{B}_T(\infty)
\end{aligned}$$

Moreover, for all individual events $x \in \mathcal{U}$ we define

$$\delta^{\text{⚛}}_T(x, \varepsilon) := \begin{cases} P_A(x) \cdot \left(1 - \frac{e^\varepsilon}{f^{\iota_T(x)}}\right) & \text{1. } \textit{if } j_\varepsilon \leq \iota_T(x) \leq j_\varepsilon + u_T - 1 \\ P_A(x) - e^\varepsilon \left(\frac{P_A(x)}{f^{\iota_T(x)}} + \ell^{\text{⚛}}_T(x) \right) & \text{2. } \textit{if } j_\varepsilon + u_T \leq \iota_T(x) \leq n \\ P_A(x) & \text{3. } \textit{if } \iota_T(x) = \infty \\ 0 & \text{4. } \textit{otherwise} \end{cases}$$

Let for a composition tree T, $\varepsilon \geq 0$ and j_ε s.t., $f_T^{j_\varepsilon - 1} < e^\varepsilon \leq f_T^{j_\varepsilon}$, $\delta_T^{\text{low}} := \sum_{i \in \{j_\varepsilon, \dots, n_T\}} \max\Big(0, \mathcal{B}_T(i) - e^\varepsilon \left(\frac{\mathcal{B}_T(i)}{(f_T)^i} + \tilde{\ell}_T(i) \right) \Big)$

Note that if $j > n$, we only consider elements in the bucket B_∞.

Next we show that the real EC terms are bounded by the value of u_T: For every event x the real EC term $\ell^{\text{⚛}}_T(x)$ can never exceed a fraction of $\frac{1}{f^{\iota_T(x) - u}} - \frac{1}{f^{\iota_T(x)}}$ of the probability of the event. Intuitively, this means that the value of the real EC term can never be larger than what a *misplacement by u buckets* would result in.

Lemma 12 (An upper bound for ℓ)**.** *Let T be a valid composition tree with $A := A_T$, $\mathcal{U} := \mathcal{U}_T$, $f := f_T$, and $u := u_T$. Let $\varepsilon \geq 0$ and with $j_\varepsilon \in \mathbb{N}$ such that $f^{j_\varepsilon - 1} < e^\varepsilon \leq f^{j_\varepsilon}$.*

If $j_\varepsilon + u \leq \iota_T(x) \neq \infty$ $(x \in \mathcal{U})$, then the EC term never makes a negative contribution to the approximated delta with EC: $\ell^{\text{⚛}}_T(x) \leq P_A(x) \left(\frac{1}{f^{\iota_T(x)-u}} - \frac{1}{f^{\iota_T(x)}} \right)$.

Proof. We show the lemma via structural induction over T.

Let $T = \text{🍃}(A, B, f, n)$**.** If $\iota_{\text{🍃}(A,B,f,n)}(x) = -n$ then

$$\ell^{\text{⚛}}_{\text{🍃}(A,B,f,n)}(x) = 0 \leq P_A(x) \cdot \left(\frac{1}{f^{-n-1}} - \frac{1}{f^{-n}} \right).$$

Otherwise, if $\iota_{\text{🍃}(A,B,f,n)}(x) > -n$, we know that by definition of $\iota_{\text{🍃}(A,B,f,n)}(x)$ we have $f^{\iota_{\text{🍃}(A,B,f,n)}(x)-1} P_B(x) \leq P_A(x)$

$$\begin{aligned}
\ell^{\text{⚛}}_{\text{🍃}(A,B,f,n)}(x) =& P_B(x) - \frac{P_A(x)}{f^{\iota_{\text{🍃}(A,B,f,n)}(x)}} \\
\leq& \frac{P_A(x)}{f^{\iota_{\text{🍃}(A,B,f,n)}(x)-1}} - \frac{P_A(x)}{f^{\iota_{\text{🍃}(A,B,f,n)}(x)}}.
\end{aligned}$$

Let $T = T_1 \times T_2$**.** If $\iota_{T_1 \times T_2}(x) = -n$, then $\ell^{\text{⚛}}_{T_1 \times T_2}(x) = 0 \leq \frac{P_A(x)}{f^{\iota_{T_1 \times T_2}(x) - u_{T_1 \times T_2}}} - \frac{P_A(x)}{f^{\iota_{T_1 \times T_2}(x)}}$, since $u_{T_1 \times T_2} \geq 0$. Otherwise, by induction hypothesis, the statement holds for ℓ_{T_1} ℓ_{T_2}. For $x_1 \in \mathcal{U}_1$ and $x_2 \in \mathcal{U}_2$ we know that $\iota_T(x) = \iota_{T_1}(x_1) + \iota_{T_2}(x_2)$, since $\iota_T \neq \infty$. Moreover, we know that $P_A(x) = P_{A_1}(x_1) \cdot P_{A_2}(x_2)$ and $u := u_{T_1 \times T_2} = u_{T_1} + u_{T_2}$ and we get

$$\begin{aligned}
\ell^{\text{⚛}}_{T_1 \times T_2}(x) &= \left(\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} + \ell^{\text{⚛}}_{T_1}(x_1) \right) \ell^{\text{⚛}}_{T_2}(x_2) + \left(\frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} + \ell^{\text{⚛}}_{T_2}(x_2) \right) \ell^{\text{⚛}}_{T_1}(x_1) - \ell^{\text{⚛}}_{T_1}(x_1) \ell^{\text{⚛}}_{T_2}(x_2) \\
&= \left(\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}} \right) \ell^{\text{⚛}}_{T_2}(x_2) + \left(\frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}} \right) \ell^{\text{⚛}}_{T_1}(x_1) + \ell^{\text{⚛}}_{T_1}(x_1) \ell^{\text{⚛}}_{T_2}(x_2)
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{IH}}{\leq} \left(\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}}\right)\left(\frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)-(u-u_{T_1})}} - \frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}}\right)\\
&\quad + \left(\frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}}\right)\left(\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)-u_{T_1}}} - \frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}}\right)\\
&\quad + \left(\frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)-u_{T_1}}} - \frac{P_{A_1}(x_1)}{f^{\iota_{T_1}(x_1)}}\right)\left(\frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)-(u-u_{T_1})}} - \frac{P_{A_2}(x_2)}{f^{\iota_{T_2}(x_2)}}\right)\\
&= \frac{P_A(x)}{f^{\iota_T(x)-(u-u_{T_1})}} - \frac{P_A(x)}{f^{\iota_T(x)}}\\
&\quad + \frac{P_A(x)}{f^{\iota_{T_2}(x_2)+\iota_{T_1}(x_1)-u_{T_1}}} - \frac{P_A(x)}{f^{\iota_{T_2}(x_2)+\iota_{T_1}(x_1)}}\\
&\quad + \frac{P_A(x)}{f^{\iota_T(x)-u_{T_1}-(u-u_{T_1})}} - \frac{P_A(x)}{f^{\iota_T(x)-(u-u_{T_1})}} - \frac{P_A(x)}{f^{\iota_T(x)-u_{T_1}}} + \frac{P_A(x)}{f^{\iota_T(x)}}\\
&= \frac{P_A(x)}{f^{\iota_T(x)-u_{T_2}}} - \frac{P_A(x)}{f^{\iota_T(x)}}\\
&\quad + \frac{P_A(x)}{f^{\iota_T(x)-u_{T_1}}} - \frac{P_A(x)}{f^{\iota_T(x)}}\\
&\quad + \frac{P_A(x)}{f^{\iota_T(x)-u}} - \frac{P_A(x)}{f^{\iota_T(x)-u_{T_2}}} - \frac{P_A(x)}{f^{\iota_T(x)-u_{T_1}}} + \frac{P_A(x)}{f^{\iota_T(x)}}\\
&= \frac{P_A(x)}{f^{\iota_T(x)-u}} - \frac{P_A(x)}{f^{\iota_T(x)}}
\end{aligned}
$$

Let $T = \blacktriangledown T_1$. For $f = f_{T_1}$ we have $f_{\blacktriangledown T_1} = f^2$ and $u = u_{\blacktriangledown T_1} = \lceil u_{T_1}/2 \rceil + 1$. We know that $\ell^{⚛}_{\blacktriangledown T_1}(x) = \ell^{⚛}_{T_1}(x) + \mathcal{B}^{⚛}_{T_1}(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2\cdot\lceil \iota_{T_1}(x)/2\rceil}}\right)$. Since we excluded $\iota_T(x) = \infty = \iota_{T_1}$ and $j_\varepsilon + u \leq \iota_T(x)$, we know that $\iota_T(x) \in \{0, \ldots, n/2\}$.

Thus,

$$
\begin{aligned}
\ell^{⚛}_{\blacktriangledown T_1}(x) &= \ell^{⚛}_{T_1}(x) + \mathcal{B}^{⚛}_{T_1}(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2\cdot\lceil \iota_{T_1}(x)/2\rceil}}\right)\\
&\overset{\text{IH}}{\leq} \frac{P_A(x)}{f^{\iota_{T_1}(x)-u_1}} - \frac{P_A(x)}{f^{\iota_{T_1}(x)}} + \mathcal{B}^{⚛}_{T_1}(x) \cdot \left(\frac{1}{f^{\iota_{T_1}(x)}} - \frac{1}{f^{2\cdot\lceil \iota_{T_1}(x)/2\rceil}}\right)\\
&= \frac{P_A(x)}{f^{\iota_{T_1}(x)-u_1}} - \frac{P_A(x)}{f^{\iota_{T_1}(x)}} + \frac{P_A(x)}{f^{\iota_{T_1}(x)}} - \frac{P_A(x)}{f^{2\cdot\lceil \iota_{T_1}(x)/2\rceil}}\\
&= \frac{P_A(x)}{(f^2)^{\frac{\iota_{T_1}(x)-u_1}{2}}} - \frac{P_A(x)}{(f^2)^{\iota_T(x)}}\\
&\leq \frac{P_A(x)}{(f^2)^{\lceil \iota_{T_1}(x)/2\rceil-(\lceil u_1/2\rceil+1)}} - \frac{P_A(x)}{(f^2)^{\iota_T(x)}}\\
&= \frac{P_A(x)}{(f^2)^{\iota_T(x)-u}} - \frac{P_A(x)}{(f^2)^{\iota_T(x)}}
\end{aligned}
$$

□

From Lemma 12 we can deduct that no event in a bucket with index $i \geq j_\varepsilon + u$ can have a negative impact on δ. Since moreover for each event we consider an impact that is at least as large as the actual impact of the event (as in the precise calculation of δ from Lemma 1) we can show the soundness of our result:

Lemma 13 (Soundness of the approximated delta with error correction)**.** *Let T be a valid composition tree with $A := A_T$, $B := B_T$, and $\mathcal{U} := \mathcal{U}_T$. Then, for all $\varepsilon \geq 0$, the following statement holds:*

$$\delta_T(\varepsilon) \geq \sum_{x \in \mathcal{U}} \max\left(0, P_A(x) - e^{\varepsilon} P_B(x)\right)$$

Proof. Let $f = f_T$, $j_\varepsilon \in \mathbb{N}$, s.t. $f^{j_\varepsilon - 1} < e^{\varepsilon} \leq f^{j_\varepsilon}$.

We first show that $\delta_T(\varepsilon) = \sum_{x \in \mathcal{U}} \delta^{\text{⚛}}_T(x, \varepsilon)$. Let $N^- = \{j_\varepsilon, \ldots, j_\varepsilon + u - 1\}$ and $N^+ = \{j_\varepsilon + u, \ldots, n\}$.

$$\begin{aligned}
\delta_T(\varepsilon) = & \sum_{i \in N^-} \mathcal{B}_T(i) \cdot \left(1 - \frac{e^{\varepsilon}}{f^i}\right) \\
& + \sum_{i \in N^+} \left(\mathcal{B}_T(i) - e^{\varepsilon}\left(\frac{\mathcal{B}_T(i)}{f^i} + \ell_T(i)\right)\right) + \mathcal{B}_T(\infty) \\
= & \sum_{i \in N^-} \left(\sum_{x \in \mathcal{U} \ s.t. \ \iota_T(x) = i} \mathcal{B}^{\text{⚛}}_T(x)\right) \cdot \left(1 - \frac{e^{\varepsilon}}{f^i}\right) \\
& + \sum_{i \in N^+} \left(\left(\sum_{x \in \mathcal{U} \ s.t. \ \iota_T(x) = i} \mathcal{B}^{\text{⚛}}_T(x)\right)\right. \\
& \left. - e^{\varepsilon}\left(\frac{\left(\sum_{x \in \mathcal{U} \ s.t. \ \iota_T(x) = i} \mathcal{B}^{\text{⚛}}_T(x)\right)}{f^i} + \left(\sum_{x \in \mathcal{U} \ s.t. \ \iota_T(x) = i} \ell^{\text{⚛}}_T(x)\right)\right)\right) \\
& + \sum_{x \in \mathcal{U} \ s.t. \ \iota_T(x) = \infty} \mathcal{B}^{\text{⚛}}_T(x) \\
= & \sum_{x \in \mathcal{U}, \iota_T(x) \in N^-} \left(P_A(x) \cdot \left(1 - \frac{e^{\varepsilon}}{f^{\iota_T(x)}}\right)\right) \\
& + \sum_{x \in \mathcal{U}, \iota_T(x) \in N^-} P_A(x) - e^{\varepsilon}\left(\frac{P_A(x)}{f^{\iota_T(x)}} + \ell^{\text{⚛}}_T(x)\right) \\
& + \sum_{x \in \mathcal{U} \ s.t. \ \iota_T(x) = \infty} P_A(x) \\
= & \sum_{x \in \mathcal{U}} \delta^{\text{⚛}}_T(x, \varepsilon)
\end{aligned}$$

We next distinguish the the four cases of the definition of $\delta^{\text{⚛}}_T(x, \varepsilon)$.

Case 1. This case occurs if $j_\varepsilon \leq \iota_T(x) \leq j_\varepsilon + u_T - 1$. By Lemma 6, we know the following

$$\begin{aligned}
P_A(x) \leq & f^{\iota_T(x)} P_B(x) \\
\Leftrightarrow \quad \frac{P_A(x)}{f^{\iota_T(x)}} \leq & P_B(x) \\
\Leftrightarrow \quad P_A(x) - e^{\varepsilon} \frac{P_A(x)}{f^{\iota_T(x)}} \geq & P_A(x) - e^{\varepsilon} P_B(x)
\end{aligned}$$

By definition of $\delta^{\text{⚛}}_T(x, \varepsilon)$, we get

$$\delta^{\text{⚛}}_T(x, \varepsilon) = P_A(x) - e^{\varepsilon} \frac{P_A(x)}{f^{\iota_T(x)}} \geq P_A(x) - e^{\varepsilon} P_B(x)$$

Moreover, as $\iota_T(x) >= j_\varepsilon$, we know that $e^\varepsilon \leq f^{\iota_T(x)}$. Hence, we also get

$$\delta_T^{\text{❀}}(x,\varepsilon) = \underbrace{P_A(x)}_{\geq 0} \cdot \left(1 - \underbrace{\frac{e^\varepsilon}{f^{\iota_T(x)}}}_{\leq 1} \right) \geq 0$$

Case 2. This case occurs if $\iota_T(x) \geq j_\varepsilon + u_T$, burt $\iota_T(x) \neq \infty$.

We first show that $\delta_T^{\text{❀}}(x,\varepsilon) \geq 0$. By Lemma 12 we know that $\ell_T^{\text{❀}}(x) \leq \frac{P_A(x)}{f^{\iota_T(x)-u_T}} - \frac{P_A(x)}{f^{\iota_T(x)}}$ holds; thus,

$$\begin{aligned}
& P_A(x) - e^\varepsilon \left(\frac{P_A(x)}{f^{\iota_T(x)}} + \ell_T^{\text{❀}}(x) \right) \\
\geq & P_A(x) - e^\varepsilon \left(\frac{P_A(x)}{f^{\iota_T(x)}} + \frac{P_A(x)}{f^{\iota_T(x)-u_T}} - \frac{P_A(x)}{f^{\iota_T(x)}} \right) \\
= & P_A(x) - e^\varepsilon \left(\frac{P_A(x)}{f^{\iota_T(x)-u_T}} \right) \\
\geq & P_A(x) - \frac{f^{j_\varepsilon}}{f^{\iota_T(x)-u_T}} P_A(x) \\
= & P_A(x) \cdot \left(1 - \frac{f^{j_\varepsilon}}{f^{\iota_T(x)-u_T}} \right) \\
\geq & 0,
\end{aligned}$$

as by assumption $\iota_T(x) \geq j_\varepsilon + u_T$. We now show that $\delta_T^{\text{❀}}(x,\varepsilon) \geq P_A(x) - e^\varepsilon P_B(x)$.

Note that from Lemma 10 we know that $\ell_T^{\text{❀}}(x) \leq \tilde{\ell}_T^{\text{❀}}(x)$.

$$\frac{P_A(x)}{f^{\iota_T(x)}} + \underbrace{\ell_T^{\text{❀}}(x)}_{\leq \tilde{\ell}_T^{\text{❀}}(x)} \leq \frac{P_A(x)}{f^{\iota_T(x)}} + \tilde{\ell}_T^{\text{❀}}(x) \overset{Lemma\ 11}{=} \frac{P_A(x)}{f^{\iota_T(x)}} + P_B(x) - \frac{P_A(x)}{f^{\iota_T(x)}} = P_B(x)$$

Thus,

$$\delta_T^{\text{❀}}(x,\varepsilon) = P_A(x) - e^\varepsilon \left(\frac{P_A(x)}{f^{\iota_T(x)}} + \ell_T^{\text{❀}}(x) \right) \geq P_A(x) - e^\varepsilon P_B(x)$$

We combine these results and get

$$\begin{aligned}
\delta_T^{\text{❀}}(x,\varepsilon) = & P_A(x) - e^\varepsilon \left(\frac{P_A(x)}{f^{\iota_T(x)}} + \ell_T^{\text{❀}}(x) \right) \\
\geq & \max(0, P_A(x) - e^\varepsilon P_B(x))
\end{aligned}$$

Case 3. This case occurs if $\iota_T(x) = \infty$. By definition we have $\delta_T^{\text{❀}}(x,\varepsilon) = P_A(x) > \max\left(0, P_A(x) - e^\varepsilon P_B(x)\right)$.
Case 4. This case occurs otherwise, i.e., if $\iota_T(x) < j_\varepsilon$. We calculate that

$$\begin{aligned}
& P_A(x) - e^\varepsilon P_B(x) \\
\overset{\text{since } e^\varepsilon \leq f^{j_\varepsilon}}{\leq} & P_A(x) - f^{j_\varepsilon} P_B(x) \\
\overset{\iota_T(x) < j_\varepsilon, P_B(x) \geq 0}{\leq} & P_A(x) - f^{\iota_T(x)} P_B(x) \overset{\text{Lemma 6}}{\leq} 0
\end{aligned}$$

and thus,

$$\delta_T^{\text{❀}}(x,\varepsilon) = 0 = \max\left(0, P_A(x) - e^\varepsilon P_B(x)\right)$$

□

4.5 Main result

We present our main technical theorem: for any $\varepsilon \geq 0$ and a value $\delta(\varepsilon)$, s.t. the distributions are tightly $(\varepsilon, \delta(\varepsilon))$-differentially private, the term δ_T in Definition 9 constitutes a sound upper bound on $\delta(\varepsilon)$ from Lemma 1 and δ_T^{low} a lower bound on $\delta(\varepsilon)$.

Definition 10 (Composition trees over distributions)**.** *Let X and Y be two distributions over the same universe $\mathcal{U}$. We call two composition trees T_1 and T_2* a pair of composition trees over the distributions X and Y *iff $A_{T_1} = B_{T_2} = X$ and $B_{T_1} = A_{T_2} = Y$.*

Theorem 2 (Buckets with EC terms are sound)**.** *Let A and B be two distributions and let T_1 and T_2 be a pair of composition trees over A and B as in Definition 10. Then for every $\varepsilon \geq 0$ and with $\delta^{\text{up}}(\varepsilon) = \max\left(\delta_{T_1}(\varepsilon), \delta_{T_2}(\varepsilon)\right)$ and $\delta^{\text{low}}(\varepsilon) = \min\left(\delta_{T_1}^{\text{low}}(\varepsilon), \delta_{T_2}^{\text{low}}(\varepsilon)\right)$ (see Definition 9), the distributions A and B are $(\varepsilon, \delta^{\text{up}}(\varepsilon))$-ADP and*

$$\delta^{\text{low}}(\varepsilon) \leq \delta(\varepsilon) \leq \delta^{\text{up}}(\varepsilon),$$

where $\delta(\varepsilon)$ is the tight δ as defined in Lemma 1.

Proof. Lemma 1 shows that A and B are tightly $(\varepsilon, \delta(\varepsilon))$-differentially private for

$$\begin{aligned}\delta(\varepsilon) = \max\Bigg(&\sum_{x \in \mathcal{U}} \max\left(P_A(x) - e^{\varepsilon} P_B(x), 0\right), \\ &\sum_{x \in \mathcal{U}} \max\left(P_B(x) - e^{\varepsilon} P_A(x), 0\right)\Bigg)\end{aligned}$$

and Lemma 13 proves that $\delta(\varepsilon) \leq \delta_{T_{A||B}}(\varepsilon)$ holds true (for any composition tree T and thus in particular for $T_{A||B}$).

Next, we show that $\delta^{\text{low}} \leq \delta(\varepsilon)$. We show the computation for $\delta_{A||B}^{\text{low}}$, the computation for $\delta_{B||A}^{\text{low}}$ follows analogously:

$$\begin{aligned}\delta_{A||B}^{\text{low}} &= \sum_{i \in \{j_\varepsilon, \ldots, n\}} \max\left(0, \mathcal{B}_{T_{A||B}}(i) - e^{\varepsilon}\left(\frac{\mathcal{B}_{T_{A||B}}(i)}{f^i} + \tilde{\ell}_{T_{A||B}}(i)\right)\right) \\ &\overset{\text{Lemma 8}}{=} \sum_{i \in \{j_\varepsilon, \ldots, n\}} \max\left(0, \sum_{x \in \mathcal{U}, \iota_T(x) = i} P_A(x) - e^{\varepsilon}\left(\frac{P_A(x)}{f^i} + \tilde{\ell}^{\text{⚛}}_{T_{A||B}}(x)\right)\right) \\ &\overset{\text{Lemma 11}}{=} \sum_{i \in \{-n, \ldots, n, \infty\}} \max\left(0, \sum_{x \in \mathcal{U}, \iota_T(x) = i} P_A(x) - e^{\varepsilon}\left(\frac{P_A(x)}{f^i} + \left(P_B(x) - \frac{P_A(x)}{f^{\iota_T(x)}}\right)\right)\right) \\ &= \sum_{i \in \{j_\varepsilon, \ldots, n\}} \max\left(0, \sum_{x \in \mathcal{U}, \iota_T(x) = i} P_A(x) - e^{\varepsilon} P_B(x)\right) \\ &\leq \sum_{i \in \{j_\varepsilon, \ldots, n\}} \sum_{x \in \mathcal{U}, \iota_T(x) = i} \max\left(0, P_A(x) - e^{\varepsilon} P_B(x)\right) \\ &\leq \sum_{x \in \mathcal{U}} \max\left(0, P_A(x) - e^{\varepsilon} P_B(x)\right)\end{aligned}$$

Hence, we conclude that

$$\delta_{A||B}^{\text{low}} \leq \sum_{x \in \mathcal{U}} \max\left(0, P_A(x) - e^{\varepsilon} P_B(x)\right) \leq \delta(\varepsilon).$$

the computation for $\delta_{B||A}^{\text{low}}$ follows analogously, ending with $\delta_{B||A}^{\text{low}} \leq \sum_{x \in \mathcal{U}} \max\left(0, P_B(x) - e^{\varepsilon} P_A(x)\right) \leq \delta(\varepsilon)$. □

As a result, the bounds calculated by privacy buckets constitute a sound over- and under-approximation of the precise differential privacy values. As discussed in Section 2, such pairs of distributions can be used to calculate differential privacy in a variety of applications. These worst-case distributions exist, e.g., in the presence of *worst-case* inputs that are independent of the random coins used by the mechanism in the previous rounds.

Definition 11 (Worst-case inputs). *Inputs x_0, x_1 are worst-case inputs for a given sensitivity s and a mechanism M if $\forall S, \Pr[M(x_0) \in S] \leq e^{\varepsilon} \Pr[M(x_1) \in S] + \delta$, implies M is (ε, δ)-ADP for all inputs with a sensitivity of at most s.*

These worst-case inputs commonly exist when differential privacy is applied (see Section 2.1) and they enable us to directly derive the relevant distributions.

As a corollary to Theorem 2, we see that we can compute upper and lower bounds for a sequence of privacy-enhancing mechanisms, where each of the mechanisms may be different from the others (but known in advance; see below for adaptive composition).

Corollary 1. *Let $M_1, \ldots, M_r$ be privacy-enhancing mechanisms for which there exist worst-case inputs $(x_{0,1}, x_{1,1}), \ldots (x_{0,r}, x_{1,r})$. Let $M_i(b)$ be the output distribution of the mechanism M_i on input $x_{b,i}$. Let T_1 and T_2 be a pair of composition trees over $\prod_{i=1}^{r} M_i(0)$ and $\prod_{i=1}^{r} M_i(1)$ as in Definition 10. Then for every $\varepsilon \geq 0$ and with $\delta^{\text{up}}(\varepsilon) = \max\left(\delta_{T_1}(\varepsilon), \delta_{T_2}(\varepsilon)\right)$ and $\delta^{\text{low}}(\varepsilon) = \min\left(\delta^{\text{low}}_{T_1}(\varepsilon), \delta^{\text{low}}_{T_2}(\varepsilon)\right)$, A and B are $(\varepsilon, \delta^{\text{up}}(\varepsilon))$-ADP and $\delta^{\text{low}}(\varepsilon) \leq \delta(\varepsilon) \leq \delta^{\text{up}}(\varepsilon)$, where $\delta(\varepsilon)$ is the tight δ as defined in Lemma 1.*

Proof. Consider the reduction that replaces all inputs of the attacker that have a sensitivity of s with the worst case inputs $((x_{0,1}, x_{1,1}), \ldots, (x_{0,r}, x_{1,r}))$ for sensitivity s. Theorem 2 gives a bound for product distributions $\prod_{i=1}^{r} M_i(x_{0,i})$ and $\prod_{i=1}^{r} M_i(x_{1,i}$. By the definition of worst-case inputs, we know that the result holds for any other sequence of inputs $((x'_{0,1}, x'_{1,1}), \ldots, (x'_{0,r}, x'_{1,r}))$. □

Heterogenous adaptive r-fold composition Bounds for (heterogenous) adaptive r-fold composition classically only restrict mechanisms to the class of all (ε, δ)-ADP mechanisms. Thus, by choosing worst-case mechanisms $M_{\varepsilon,\delta}$ (see Section 2) for each step, we get a bound on for heterogeneous adaptive r-fold composition as in [14].

When the class of mechanisms is restricted further, e.g., the structure of the mechanisms is partially known, we suggest to derive (and prove sound) tighter worst-case mechanisms or distributions for which we can then give significantly better results.

4.6 Implementation

We implemented the computation of the upper and lower bounds $\delta^{\text{up}}(\varepsilon)$ and $\delta^{\text{low}}(\varepsilon)$ from Theorem 2, and our implementation has been re-implemented by David Sommer in Python in 405 LoC.[8] Given a bucket factor and a number of buckets n, the implementation constructs privacy buckets from any given histogram / distribution with a limed number of events. For Laplacian noise and Gaussian noise we have implemented special constructors that create privacy buckets for those distributions. Our implementation adaptively decides whether or not to perform the ▼-operation, i.e., to rebase the factor depending on whether the bucket with index ∞ would otherwise grow too much. Empirically, we found that an increase of weight of the ∞ bucket by more than a factor of 2.2 is a good indicator that squaring should be performed. Additionally, we include a parameter `free_infty_budget` that disables squaring as long as the $\mathcal{B}(\infty)$ is below this parameter, which is important for cases where $\mathcal{B}(\infty)$ is initially zero or very small.

The complexity of our algorithm is dominated by the evaluation of the composition nodes, which requires a constant amount of convolutions per node. Convolutions can be done in $O(n \log n)$ steps (using FFT-convolution), for n buckets. The implementation does not use FFT-convolutions, due to numerical challenges, and needs $O(n^2)$ steps. For computing r-fold ADP $O(\log r)$ composition computations are necessary, totalling $O(\log r \cdot n \log n)$, for r-fold ADP with n privacy buckets and $O(\log r \cdot n^2)$ in the prototype.

On a Lenovo ThinkPad X250 (2.6 GHz Intel Core i5 with and 8 GB RAM) computing $2^{18} = 262,144$ compositions with $100,000$ buckets took around 3 minutes and 57 seconds by using repeated squaring (13 seconds

[8] The implementation is available here, which includes an FAQ about further practical aspects [19]: https://github.com/dabingo/privacybuckets

per composition operation): we compose the bucket distribution with itself in each round, thus calculating 2^r compositions in r composition steps. For $10,000$ buckets we achieve tight bounds for 2^9 compositions and only need 1.2 seconds, i.e., 0.13 seconds per composition operation. The implementation adaptively decides whether or not to perform the squaring operation if $\mathcal{B}(\infty)$ would otherwise grow significantly, as described in Figure 10.

5 Evaluation and comparison

We compute results for several distributions (modeling the Laplacian mechanism, the Gauss mechanism, real-world leakage data from CoverUp [24], the randomized response, and the stochastic gradient descent mechanism [1]) and compare our results with bounds from previous work, such as Kairouz, Oh and Viswanath's composition theorem and methods based on Rényi divergence.

We consider the following other bounds in our evaluation: Kairouz, Oh and Viswanath's composition theorem [14] (KOV), the moments accountant [1] (MA), which derives exactly the same bounds as Rényi differential privacy [20], and concentrated differential privacy [7, 2].

In these evaluations, we illustrate two advantages of our approach: (i) (ε, δ)-graphs for a fixed number of compositions highlight that we achieve significantly reduced δ-bounds for very small e^{ε}; (ii) plots about e^{ε}-bounds for a fixed bound on δ but a growing number of compositions illustrate that we achieve significantly reduced e^{ε}-bounds.

We now discuss how to embed the mechanisms into our buckets and then visualize the difference between the different bounds in one $(\varepsilon,\ \delta)$-graph each. Each such graph shows which values for δ can be achieved for a given value of ϵ after a number of compositions.

5.1 Embedding the Laplace mechanism

We analyze the Laplace mechanism, the classical mechanism to achieve DP, by comparing two distributions of Laplace noise with means 0 and 1 respectively. This case corresponds to many applications of the Laplace mechanism for DP, such as counting queries for databases with sensitivity 1. We choose in our case study a Laplace distribution with mean $\mu = 0$ and scale factor $\gamma = 200$, denoted as $\mathrm{LP}(\mu, \gamma)$. As a result, an attacker either makes observations from $\mathrm{LP}(0, 200)$ or from $\mathrm{LP}(1, 200)$ (as the sensitivity is 1). We consider truncated Laplace distributions, since that corresponds closer to real-world applications. If not mentioned otherwise, we truncate at $\mu - 2500$ and $\mu + 2500$.

We want to give strong evidence that both Kairouz et al.'s composition theorem and our privacy buckets are tight for the bounds of the Laplace mechanism. As a consequence, we carefully embed the Laplace mechanism in a way that has a small discretization error. The bucket method introduced in Definition 7 iterates over all atomic events in the support of the distributions. For modeling the Laplace distribution, or rather, two Laplace distributions A and B, we consider the quotients of the probability mass functions and integrate distribution A over the range of events that fall into each bucket: for $\mathcal{B}(i)$ we integrate over all events x such that $f^i < p_A(x)/p_B(x) \leq f^{i+1}$. This technique can also be applied to other distributions with an infinitely large support, where all areas where B has a probability of zero naturally contribute to the bucket $\mathcal{B}_{\infty}$.

Recall the probability density function for the Laplace distribution with mean μ and scale parameter γ as $\mathrm{Laplace}(x) := \frac{1}{2\gamma} e^{\frac{-|x-\mu|}{\gamma}}$. For differential privacy we often compare two such distributions with the same scale parameter γ and different medians μ_1 and μ_2, where the means are the real values to which we add Laplace noise with scale parameter γ. We know that without composition, we get $(\varepsilon, 0)$-ADP with $\varepsilon = \frac{1}{\gamma}$. Consequently, we can describe the quotient f at each point x as We calculate the quotient $f(x) = \frac{\mathrm{Laplace}_{\mu_1}(x)}{\mathrm{Laplace}_{\mu_2}(x)}$ depending on the relation between the values for x, μ_1 and μ_2:

- $x \leq \min(\mu_1, \mu_2)$: $f(x) = e^{-(\mu_1 - x)\varepsilon} / e^{-(\mu_2 - x)\varepsilon} = e^{(-\mu_1 + x - x + \mu_2)\varepsilon} = e^{(\mu_2 - \mu_1)\varepsilon}$
- $\mu_1 \geq x \geq \mu_2$: $f(x) = e^{-(\mu_1 - x)\varepsilon} / e^{-(x - \mu_2)\varepsilon} = e^{(-\mu_1 + x + x - \mu_2)\varepsilon} = e^{(-\mu_1 - \mu_2 + 2x)\varepsilon}$
- $\mu_1 \leq x \leq \mu_2$: $f(x) = e^{-(x - \mu_1)\varepsilon} / e^{-(\mu_2 - x)\varepsilon} = e^{(-x + \mu_1 + \mu_2 - x)\varepsilon} = e^{(\mu_1 + \mu_2 - 2x)\varepsilon}$

- $x \geq \max(\mu_1, \mu_2)$: $f(x) = e^{-(x-\mu_1)\varepsilon}/e^{-(x-\mu_2)\varepsilon} = e^{(\mu_1 - x + x - \mu_2)\varepsilon} = e^{(\mu_1-\mu_2)\varepsilon}$

It turns out that for a pair of Laplace distributions the quotient in the region $\min(\mu_1, \mu_2) \leq x \leq \max(\mu_1, \mu_2)$ is either monotonically increasing or monotonically decreasing. For any x smaller than $\min(\mu_1, \mu_2)$, the quotient is stable at $e^{-\varepsilon}$ and for any x larger than $\max(\mu_1, \mu_2)$ the quotient is stable at e^{ε}. Recall that our buckets capture a *range of quotients*: bucket i captures all x such that $f^i < p_A(E)/p_B(E) \leq f^{i+1}$. As a result, each bucket i contains contiguous points and defines an interval on the $x - axis$. For each interval we define the *bucket borders*, i.e., for the bucket with index i, we call the value x with $f(x) = f^{i-1}$ the *left bucket border* $\mathrm{lbb}(i)$ and the value x with $f(x) = f^i$ the *right bucket border* $\mathrm{rbb}(i)$.

For $\mu_1 > \mu_2$, the right bucket border $\mathrm{rbb}(i)$ is the x such that

$$\begin{aligned}
e^{(2x-\mu_1-\mu_2)\varepsilon} =& f^i = e^{(i\varepsilon/\mathrm{gr})} =: e^j \\
\Leftrightarrow (2x-\mu_1-\mu_2)\varepsilon =& j \\
\Leftrightarrow (2x-\mu_1-\mu_2) =& j/\varepsilon \\
\Leftrightarrow 2x =& \mu_1+\mu_2+j/\varepsilon \\
\Leftrightarrow x =& (\mu_1+\mu_2+j/\varepsilon)/2 \\
\Leftrightarrow x =& (\mu_1+\mu_2+\frac{(i\varepsilon/\mathrm{gr})}{\varepsilon})/2 \\
\Leftrightarrow x =& (\mu_1+\mu_2+i/\mathrm{gr})/2 \\
\implies \mathrm{rbb}(i) =& 1/2(\mu_1+\mu_2+i/\mathrm{gr}) \\
\implies \mathrm{rbb}(i-1) =& 1/2(\mu_1+\mu_2+i/\mathrm{gr}-1/\mathrm{gr}) \\
=& \mathrm{rbb}(i) - 1/(2\mathrm{gr}) \\
=& \mathrm{lbb}(i)
\end{aligned}$$

For $\mu_1 < \mu_2$, the right bucket border $\mathrm{rbb}(i)$ is the x such that

$$\begin{aligned}
e^{(-2x+\mu_1+\mu_2)\varepsilon} =& f^i = e^{(i\varepsilon/\mathrm{gr})} =: e^j \\
\Leftrightarrow (-2x+\mu_1+\mu_2)\varepsilon =& j \\
\Leftrightarrow (-2x+\mu_1+\mu_2) =& j/\varepsilon \\
\Leftrightarrow 2x =& \mu_1+\mu_2-j/\varepsilon \\
\Leftrightarrow x =& (\mu_1+\mu_2-j/\varepsilon)/2 \\
\Leftrightarrow x =& (\mu_1+\mu_2-\frac{(i\varepsilon/\mathrm{gr})}{\varepsilon})/2 \\
\Leftrightarrow x =& (\mu_1+\mu_2-i/\mathrm{gr})/2 \\
\implies \mathrm{rbb}(i) =& 1/2(\mu_1+\mu_2-i/\mathrm{gr}) \\
\implies \mathrm{rbb}(i-1) =& 1/2(\mu_1+\mu_2-i/\mathrm{gr}+1/\mathrm{gr}) \\
=& \mathrm{rbb}(i) + 1/(2\mathrm{gr}) \\
=& \mathrm{lbb}(i)
\end{aligned}$$

As a result, the bucket i has the value $\int_{\mathrm{lbb}(i)}^{\mathrm{rbb}(i)} \mathrm{Laplace}(\mu_1, 1/\epsilon)$.

We compute the error correction term as $\ell(i) := \int_{\mathrm{lbb}(i)}^{\mathrm{rbb}(i)} \left(B(x) - \frac{A(x)}{f^i}\right)$ and we can directly compute the virtual error from this term.

For the buckets with index $\pm i$ s.t. $f^i = e^{\varepsilon}$ we integrate over the respective remaining areas $\mathcal{B}(-i) = \int_{-\infty}^{\mathrm{rbb}(-i)} \mathrm{Laplace}(\mu_1, 1/\epsilon)$ and to $\mathcal{B}(i)$ we add $\int_{\mathrm{rbb}(i)}^{\infty} \mathrm{Laplace}(\mu_1, 1/\epsilon)$. As we chose f to fit e^{ε} the events in these regions exactly have the respective quotient of the bucket and we don't have errors for these integrals. Consequently, the error terms for bucket $\mathcal{B}(-i)$ are zero and the error terms for bucket $\mathcal{B}(i)$ are composed of the error terms for the values x with $\mathrm{lbb}(i) < x < \mathrm{rbb}(i)$.

Truncated Laplace distributions. The truncation of each of either of these functions, causes the quotient of a region to be either 0 or to have 0 in the denominator, which we treat as infinity. The regions are captured by the outer buckets with indexes $-n$ and ∞ respectively.

5.2 Embedding the Gauss mechanism

The truncated Gauss mechanism is also an often-used mechanism in privacy-preserving applications. It works similar to the Laplace mechanism insofar as it convolves the input (e.g., a query response) with a Gaussian distribution. In this work, we use a mean $\mu = 0$ and a standard deviation $\sigma = 200\sqrt{2}$ (to achieve the same variance as $\mathrm{LP}(0, 200)$), denoted as $\mathrm{GS}(\mu, \sigma^2)$, and we truncate these distributions at $\mu - 2500$ and $\mu + 2500$, if not mentioned otherwise. For the truncated Gauss mechanism, we do not use a precise embedding but rather produce a histogram for each of the two distributions, using SciPy's `scipy.stat.norm` function. Then, we use the normal interface of our bucketing implementation that parses a pair of histograms and produces a bucketlist vector, a real error vector, and a virtual error vector. We accept that this implementation may produce discretization artifacts that, however, should be both small w.r.t. the values concerned and should not lead to a significantly different shape of the distributions under composition.

5.3 Embedding CoverUp's data

Classical anonymous communication networks (ACN) have the goal of hiding the IP address of the sender and the recipient of a communication. Such ACNs do however not hide the participation time, i.e., whether, when, and for how long a party uses an ACN. This participation time can be used for long-term attacks (e.g., intersection attacks) and can raise suspicion national state-level adversaries. Sommer et al. [24] propose a system, called CoverUp, that has the goal of hiding this participation time leakage. CoverUp assumes a collaborating popular web service with a significant amount of regular visitors. This webpage would be incorporated into the usage of an ACN and trigger all its visitors to produce cover traffic. This web page would serve an iFrame that loads content from a trusted server, which in turn would serve a piece of JavaScript code that executes a dummy client for the ACN on the visitors browser. ACN users would act as a normal visitor, receive the JS code, but additionally have a dedicated CoverUp browser extension installed. The browser extension would enable a communication channel to an external application by replacing the dummy messages from the dummy client with actual messages from an external application and by forwarding all messages from the network to the external application. For CoverUp to properly hide the participation time ACN users (called *active* participants) and normal website visitors (called *passive* participants) have to be indistinguishable. While both execute the same piece of JS code, the active participants perform additional computations. As a consequence, the response time of the active participants differs by a few milliseconds from the response time of the passive participants. CoverUp remedies this timing leakage by adding random delays in the JS code, i.e., for active and passive participants.

The CoverUp paper presents an analysis of this timing leakage (after adding the noise) and aims for a high degree of privacy after more than 250k observations. The CoverUp authors experimentally measured the timing delays of active and passive participants in the lab and produced histograms of these timing delays. These histograms are used as a model for the timing delays of active and passive participants to assess the timing leakage of CoverUp. We apply our algorithm to these histograms of timing delays, to illustrate that and how well our approach works on measured data. We use data from the CoverUp project, which is openly available online.[9]

In this comparison, we only consider those measured delays on a Linux system that are observable while the webpage is loading, called the "linux loading active" measurements in the CoverUp paper.

5.4 Embedding the Stochastic Gradient Descent

Abadi et al. devised a privacy-preserving method for training training deep neural networks [1]. They prove that for their mechanism a pair of worst-case distributions exists: a Gaussian mixture model $(1-q)\mathrm{GS}(0, \sigma^2) + q\mathrm{GS}(1, \sigma^2)$ and a Gaussian distribution $\mathrm{GS}(0, \sigma^2)$, for some q that depends on the training parameters. We construct privacy buckets for these two distributions and compute the composition with $100,000$ buckets.

[9] Available under `http://coverup.tech`.

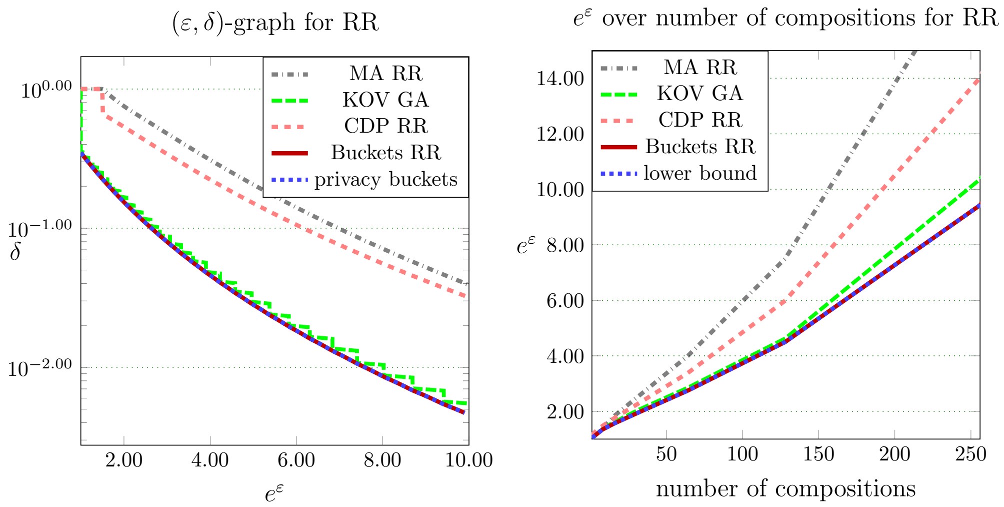


Figure 11: Comparison of bounds for randomized response (RR) mechanism with $p = 0.51$. Left: (ε, δ)-graph for 512 compositions. Right: growth of e^ε over the number of compositions for $\delta \leq 10^{-4}$.

To improve comparability, we use the same parameters as in their paper, $\sigma = 4, q = 0.01, \delta = 10^{-5}$ and 640 epochs ($r = 2^{16}$); however, we plot on the y-axis e^ε and not ε.

5.5 Embedding the Randomized Response Mechanism

The randomized response mechanism $\mathrm{RR}_{p,f}$ is parametric in a bias p and is defined for a binary predicate $f : X \to \{0, 1\}$. For an input database D, if $f(D)$ the mechanism $\mathrm{RR}_{p,f}(D)$ outputs 1 with probability p and 0 with probability $1 - p$. If $f(D)$ is not true, $\mathrm{RR}_{p,f}(D)$ outputs 0 with probability p and 1 with probability $1 - p$. It can be easily shown that the pair (A, B) of distributions $A(0) = p, A(1) = 1 - p$ and $B(0) = 1 - p$ and $B(1) = p$ is a worst case distribution, which corresponds to the case where two databases D_0, D_1 are compared with $f(D_0) = 0$ and $f(D_1) = 1$.[10] In particular, since differential privacy considers worst-case inputs, we do not need to explicitly specify the predicate f as long as there are at least two databases D_0 and D_1 for which $f(D_0) \neq f(D_1)$, i.e., as long as f is not constantly true or false on all databases. Figure 11 shows that privacy buckets clearly lead to tighter ADP-bounds, followed by the CDP and then the RDP bounds. As for the Gauss mechanism, CDP yields tighter bounds than RDP since the translation to ADP is tighter.

5.6 Computing Kairouz et al.'s composition theorem

Kairouz et al. proved a composition theorem [14] that significantly improves on the standard and advanced composition theorem. This composition theorem [14] provides a composition result where each ε, δ pair after r-fold composition is solely derived from one ε, δ pair of the original pair of distributions. Hence, this composition result does take the entire shape of the distribution into account. In other words, the resulting epsilon and delta bounds are not necessarily tight in the sense of Definition 1.

We directly implement the bounds from Kairouz et al.'s theorem. We do not use any statements specific to Gauss or Laplace, as those are simplified and provide worse bounds.

Theorem 3 ([14])**.** *For any $\varepsilon \geq 0$ and $\delta \in [0, 1]$, the class of (ε, δ)-ADP mechanisms satisfies (ε', δ')-ADP under r-fold composition, for all $i \in \{0, \dots, \lfloor r/2 \rfloor\}$ where $\varepsilon' = (r - 2i)\varepsilon$ and $\delta' = 1 - (1 - \delta)^r (1 - \delta_i)$*

$$\delta_i = \frac{\sum_{\ell=0}^{i-1} \binom{r}{\ell} \left(e^{(r-\ell)\varepsilon} - e^{(r-2i+\ell)\varepsilon} \right)}{(1 + e^\varepsilon)^r}$$

[10] For arbitrary D_0', D_1', the reduction checks whether $f(D_0') = f(D_1')$. If so, the reduction draws a p-biased coin and outputs the result. If $f(D_0') = 1$ and $f(D_1') = 0$, the reduction flips the output of the game with the worst-case distributions, and if $f(D_0') = 1$ and $f(D_1') = 0$ the reduction simply forwards the result of the game with the worst-case distributions.

We compute for a given number r of compositions the epsilon-delta graph by looking up for a fine-grid of ε values the corresponding δ value of the original pair of distributions and then computing and storing all (ε', δ') pairs according to the theorem above, i.e., for all $i \in \{0, \dots, \lfloor r/2 \rfloor\}$. From these stored (ε', δ') pairs, we remove all pairs for which we have stored lower $(\varepsilon'', \delta'')$ pairs, i.e., pairs such that $\varepsilon'' \leq \varepsilon'$ and $\delta'' \leq \delta'$. We output the remaining list of (ε', δ') pairs, which form a monotonically decreasing (ε, δ)-graph. Due to our direct implementation of δ_i, we can only evaluate the composition theorem up to $r = 512$ before the intermediate computation results (in particular, the $e^{O(k)}$-terms) become too large.

In our computation, the granularity of the grid of ε values of the original pair of distributions naturally leads to an imprecision. We use a fine grid of

$$e^{\varepsilon} \in \{(1 + 10^{-14})^{1.1^j} \mid j \in \{0, \dots, n\}\},$$

where we choose n as a point where the (ε, δ) after r-fold composition becomes stationary. While we concede that it might be possible to obtain a slightly lower bound from the composition theorem, we are confident that, due to this fine grid, the resulting graphs for Kairouz et al.'s composition theorem that we compute are representative.

5.7 Computing bounds based on Rényi divergence

In contrast to KOV's general purpose bounds, the recently introduced notions of concentrated differential privacy (CDP) [7, 2], Rényi differential privacy (RDP) [20], and moments accountant [1] introduced mechanism-aware bounds for differential privacy by using the so-called Rényi divergence, which can be expressed[11] as the higher log-moments of our privacy buckets distributions (for $\alpha > 0$):

$$D_{\alpha+1} = \frac{1}{\alpha} \log \sum_i \mathcal{B}(i) \cdot f^{\alpha i}$$

A pair of distributions A, B satisfies (ξ, ρ)-Concentrated DP if the Rényi divergence is bounded by an affine linear function: $D_\alpha \leq \xi + \rho\alpha$ (for all $\alpha \geq 0$). Rényi differential privacy directly characterizes the privacy bound by the Rényi divergences: $(\alpha, D_\alpha)_\alpha$. Rényi differential privacy can be translated to (ε, δ)-ADP as follows: whenever $(\alpha, D_\alpha)_\alpha$, then also $(\varepsilon, \alpha D_\alpha - \alpha\varepsilon)$-ADP holds. The moments accountant uses the same characterization and proposes $(\varepsilon, \min_\alpha(\alpha D_\alpha - \alpha\varepsilon))$ as ADP bounds. Consequently, we search for the value α that minimizes this bound.

In our implementation, we approximated the Rényi divergence by using our buckets. Figure 22 (in the appendix) compares this approximation with the original implementation of the DP-SGD paper [1] and shows that the two implementations coincide.

5.8 Comparison results

Figure 12 and Figure 13 graphically depicts our results. We used in the computation of all these graphs 100,000 buckets. We consistently see that our bounds are tighter than prior bounds. We use our results to evaluate other bounds. For the Laplace mechanism and randomized response, KOV is tight. As a side-note, this observation suggests that there is no mechanism with the same, or smaller, initial ε and δ values that composes significantly worse than the Laplace mechanism. For the Gauss mechanism, KOV is far away from our bounds and, for large values of e^{ε} is even outperformed by both CDP and MA. Similar to the Gauss mechanism, MA outperforms KOV for large values of e^{ε}. For the analysis of the stochastic gradient, we omit the KOV bound, since the large number of $r = 2^{16}$ observations makes a calculation of the KOV bound difficult.

In summary, we see that previous bounds have their strengths and weaknesses: KOV is tight for Laplace, but not for other scenarios; MA outperforms KOV for large values of e^{ε}, but not for smaller values; CDP is significantly tighter than MA whenever it can be applied, but requires the fitting of a Gaussian over the privacy loss variable, which often isn't tight and sometimes is impossible. Figure 13 shows that the KOV bound becomes inaccurate with an increasing number of compositions. In all cases our bounds derived from the privacy buckets are tight.

[11] This theoretical characterization assumes a sufficiently high number of buckets such that no approximations need to be made.

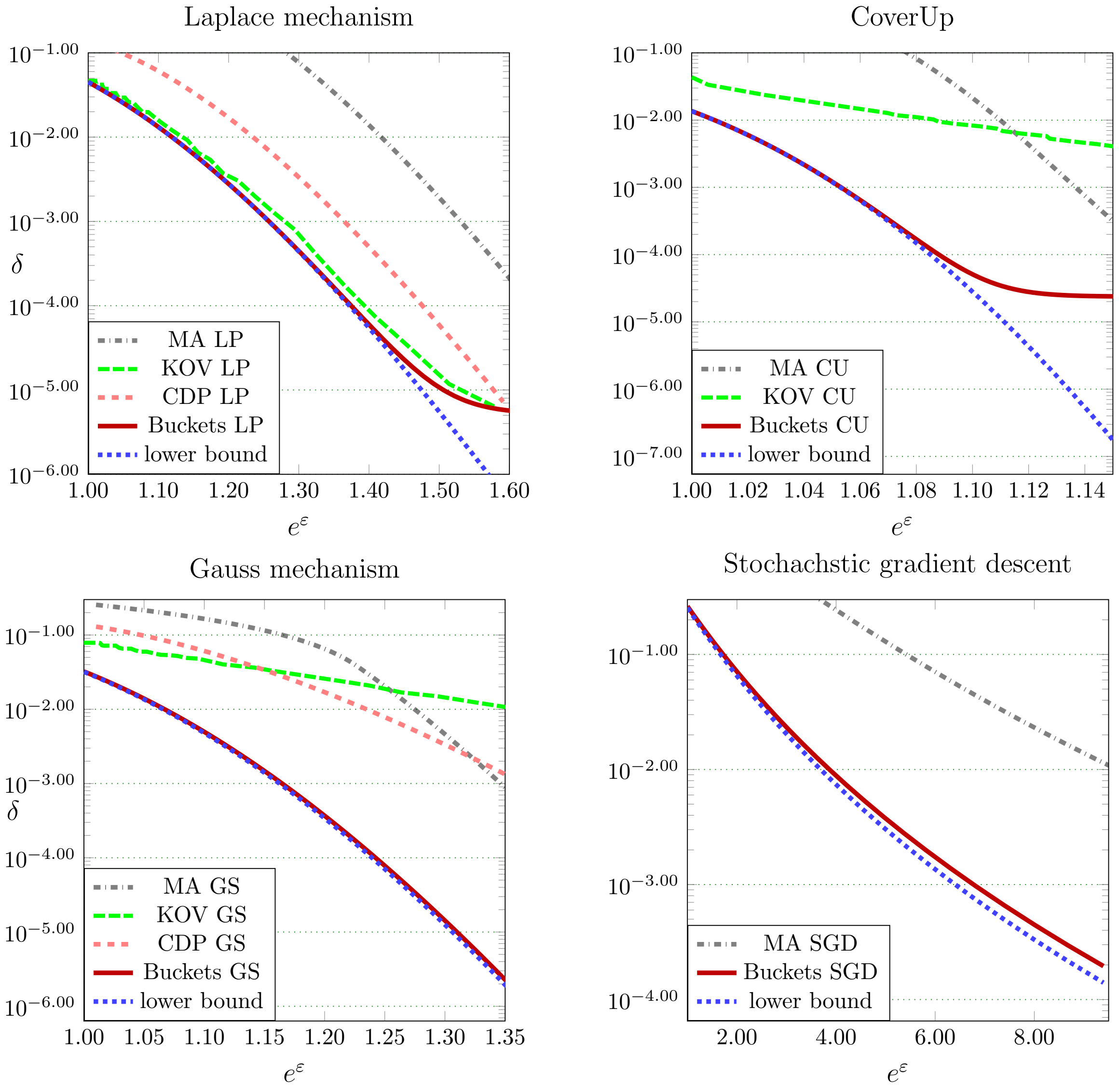


Figure 12: A group of (ε, δ)-graphs for the Laplace mechanism, the Gauss mechanism, the CoverUp mechanism and the stochastic gradient descent mechanism. The first three are for $r = 512$, the latter for $r = 2^{16}$.

6 Comparison of the Gaussian and the Laplace mechanism

As we have seen in Section 5.8, Kairouz et al.'s composition theorem is fairly tight for the Laplace mechanism but not for the Gauss mechanism. Figure 14 (upper two graphs) compares a truncated Laplace and a truncated Gauss mechanism and find that for the same variance the Gauss mechanism provides a significantly higher degree of privacy.[12] For a fixed variance of $80,000$, a sensitivity of 1 ($mu_1 = 0$ and $\mu_2 = 2$), and a truncation at -2500 and 2500 for μ_1 (and -2499 and 2501 for μ_2), the upper left graph in Figure 14 depicts how, for different but fixed epsilon values, the delta increases over the course of 512 evaluations. The graph clearly shows that in the course of 512 compositions, the reduced leakage of the Gauss mechanism becomes visible. The lower left graph in Figure 14 shows the full epsilon-delta graphs of a Gaussian and a Laplace mechanism after 512 compositions, where the two mechanisms use noise that has the same variance $(80,000)$. In particular, the delta-value where the (ε, δ) graph levels out is 4 orders of magnitude lower for Gaussian noise than it is for Laplace noise, since the Gaussian distribution falls much steeper than Laplace

[12] All computations have been conducted with $100,000$ buckets.

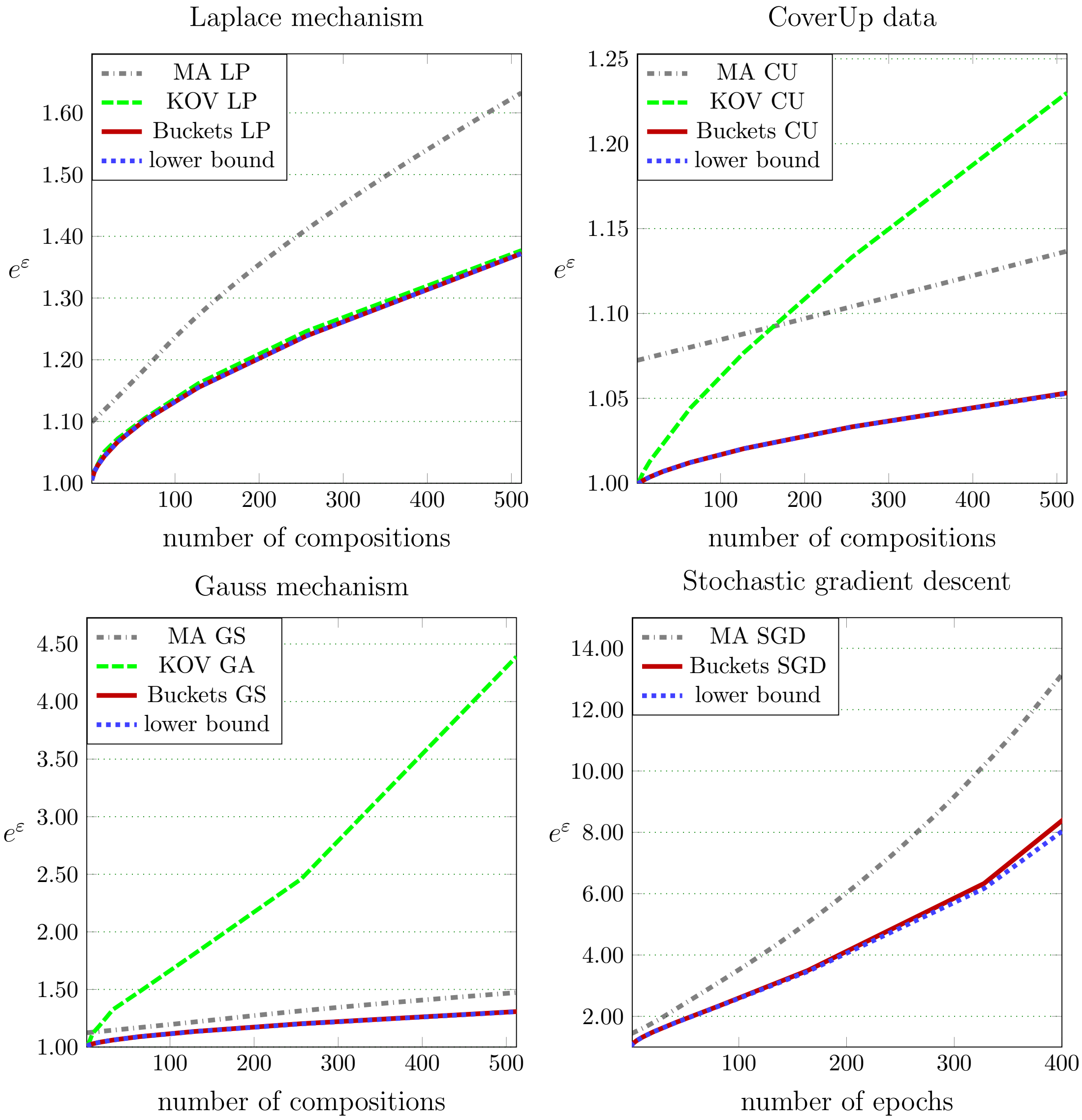
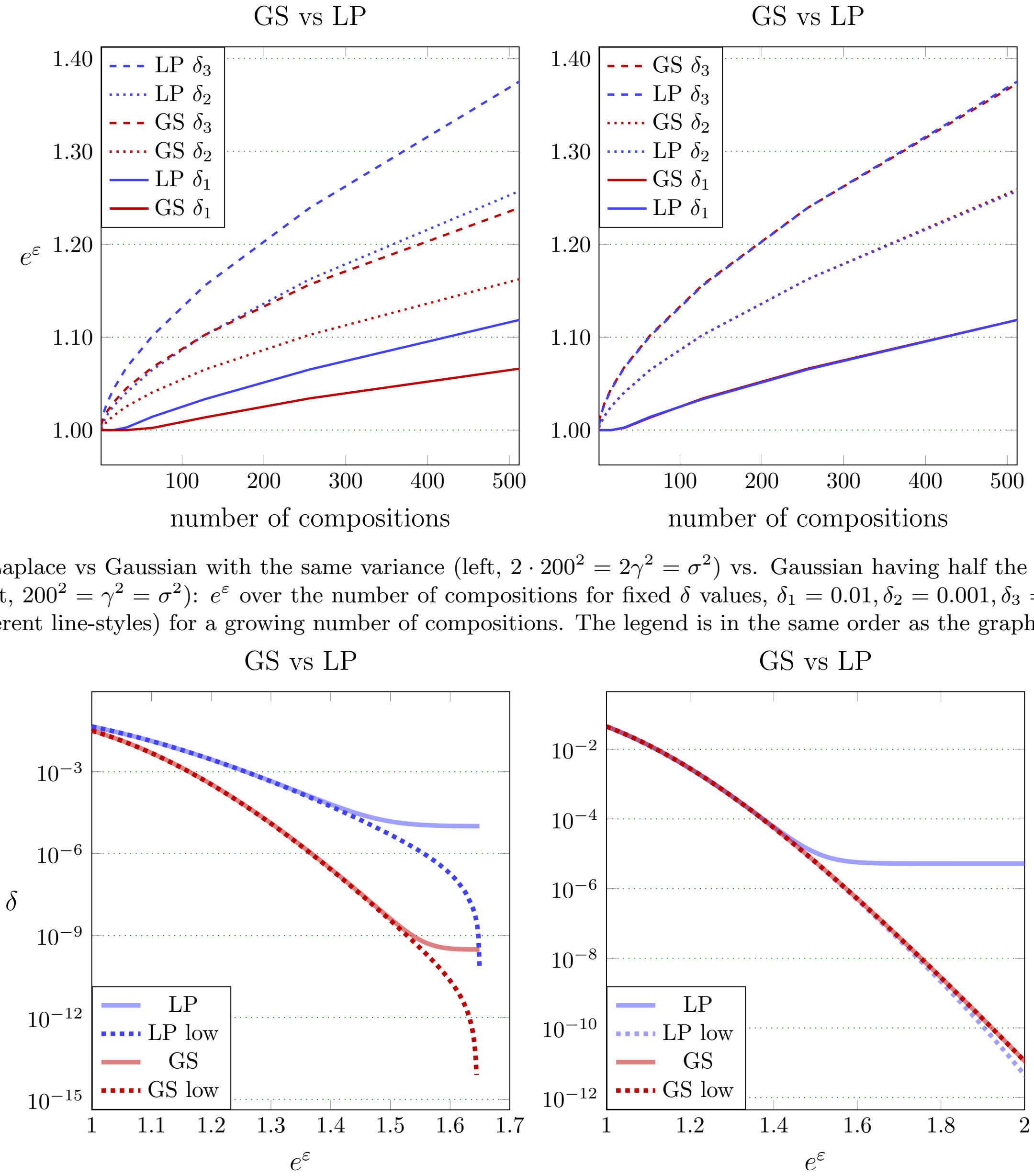


Figure 13: Comparison with the bounds from Rényi Privacy and CDP for the Gauss mechanism, i.e., the distributions $\mathrm{GS}(0, 2 * 200^2)$ and $\mathrm{GS}(1, 2 * 200^2)$. Left: (ε, δ)-graph for 512 compositions. Right: growth of e^ε over the number of compositions for $\delta \leq 10^{-5}$.

distribution. This difference of the Gaussian and the Laplace mechanisms becomes even more pronounced in our analysis and improvement of the Vuvuzela protocol in Section 7. The analysis of Vuvuzela also illustrates that the steepness of the Gaussian distribution enables a much tighter truncation, i.e., the distribution can be truncated much earlier than a Laplace distribution without sacrificing privacy. This tighter truncation, in turn, leads to a smaller range of noise that is required to achieve the same privacy goals as with Laplace noise.

Additionally, we found evidence that the epsilon-delta graph of the Laplace mechanism converges toward the epsilon-delta graph of a Gauss mechanism with half the variance of the Laplace mechanism. For the same sensitivity, and truncations as above, the two right graphs in Figure 14 illustrate that after 512 compositions these two graphs converge toward each other. The upper right graph in Figure 14 depicts how, for different but fixed epsilon values, the delta increases over the course of 512 evaluations. The graph clearly shows how in the course of 512 compositions, the delta values of the Laplace mechanism converge toward the delta



(a) Laplace vs Gaussian with the same variance (left, $2 \cdot 200^2 = 2\gamma^2 = \sigma^2$) vs. Gaussian having half the variance (right, $200^2 = \gamma^2 = \sigma^2$): e^ε over the number of compositions for fixed δ values, $\delta_1 = 0.01, \delta_2 = 0.001, \delta_3 = 0.0001$ (different line-styles) for a growing number of compositions. The legend is in the same order as the graphs.

(b) The ε, δ graphs (upper and lower bounds) after $k = 512$ compositions applied to a Gaussian and a Laplace mechanism with δ on the y-axis and e^ε on the x-axis.

Figure 14: Truncated Gauss mechanisms (red) vs. truncated Laplace mechanism (blue) both with sensitivity $= 1$. For both mechanism truncation is at $\mu_i - 2500$ and $\mu_i + 2500$ ($\mu_1 = 0$ and $\mu_2 = 1$). At twice the variance the Laplace mechanism converges towards the Gauss mechanism, so much that the blue lines almost completely cover the red lines.

values of the Gauss mechanism. The lower right graph in Figure 14 shows the full epsilon-delta graphs of a Gaussian and a Laplace mechanism after 512 compositions, where the Laplace mechanism has twice the variance $(80,000)$ of the Gauss mechanism $(40,000)$. This figure shows how close the two epsilon-delta graphs are and that they almost only differ due to their different y-values at the point where they have been truncated. This difference, however, is crucial. As explained above, it is caused by the steepness of the Gaussian distribution and enables a much tighter truncation, which in turn can lead to significantly less noise overhead, as we illustrate in our analysis of Vuvuzela.

Dwork and Rothblum [7] presented a related result. They characterized the composition behavior of a mechanisms (i.e., a pair of distributions A, B) for which the privacy loss distribution $e\mathcal{L}_{(A||B)}$ is Subgaussian, i.e., the moment-generating function (MGF) is smaller than the MGF of a Gaussian. They show that such for mechanisms the privacy loss distribution after r-fold composition can be bounded by a Gaussian. Moreover, showed that the privacy loss distribution, i.e., the privacy buckets distribution, of the Gauss mechanism is a Gaussian distribution. Complementarily, our findings show that the privacy loss distribution of a Laplace mechanism converges towards a Gaussian, already after 512 compositions. We leave it for future work to investigate the connection between the Laplace distribution and a Gaussian distribution with half the variance.

7 Application to Vuvuzela

In this section, we show how aiming for tight bounds in a privacy analysis can significantly improve the bandwidth overhead of a protocol. As a case study, we use the Vuvuzela [26] protocol, which is an anonymous communication system tailored towards messengers. Vuvuzela uses Laplace noise to achieve strong privacy properties. Using the insights from Section 6, we not only estimate tighter bounds for the Laplace noise but also propose to change the shape of the noise distribution to Gaussian noise. With our bucketing approach, we show that already 5 to 10 times less noise[13] suffices to achieve the same strong privacy properties. [14]

We refer to the original Vuvuzela paper for a full presentation and restrict our presentation to the bare bones that are needed to understand the noise messages that Vuvuzela uses to achieve strong privacy properties.

We stress that our work contributes to improving the epsilon-delta bounds and thus to improve a given privacy analysis. This work is not meant to help in finding a suitable attacker model, a suitable definition or accurate usage profiles. Hence, we stick to Vuvuzela's privacy analysis, as it was presented in the original paper.

7.1 Protocol overview

Vuvuzela clients communicate by deposing their encrypted messages in virtual locations in the one of the mixes (the locations are called *dead drops*). For agreeing on such a dead drops, Vuvuzela deploys a dialing protocol where the dialer sends the ID of a dead drop to dedicated invitation dead drops. This ID is encrypted with the peer's public key with an encryption schemes that is designed to hide the recipient's identity. On the dialer's side directly the conversation protocol is started where the client regularly retrieves the chat messages from and deposits chat messages to the dead drop from the invitation. If the recipient receives and accepts the invitation, the recipient also starts the conversation protocol.

Privacy analysis Vuvuzela assumes a global network-level attacker that is additionally able to compromise some mixes. To achieve strong resistance against compromised servers, each path in Vuvuzela traverses all nodes. To counter traffic correlation attacks, Vuvuzela clients produce dummy traffic at a constant rate. The Vuvuzela paper argues that the only remaining source of leakage is the patterns of registering invitations and patterns of access requests to these dead drops: single requests to dead drops, corresponding to dummy messages or messages before the peer accepted the conversation, and pairs of requests to the same dead drop, corresponding to an active conversation.

Privacy-enhancing measures Vuvuzela reduces the information that an attacker can learn by triggering each mix to produce cover stories for potentially communicating parties. For the dialing protocol, the mixes produce cover stories (i) by sending dummy invitation registrations and invitation requests to the dedicated invitation dead drops. The number of these dummy registrations and dummy requests is in each round

[13] The more observations are estimated, the higher the error of the advanced composition result, which is used in the original analysis from the Vuvuzela paper; hence, in those cases the tightness of our bounds leads to a more significant improvement.

[14] We acknowledge that for the analysis of the Laplace noise previous results [14] would already yield tight results, but for the Gaussian noise our approach yields much tighter results (see Section 6).

drawn from the truncated Laplace distribution $\lceil \max(0, \text{Laplace}(\gamma_d, \mu_d)) \rceil$ for some system parameters γ_d and μ_d. For the conversation protocol, the mixes produce cover stories (*ii*) for idle parties, by sending pairs of dummy access requests to uniform-randomly chosen dead drops, and (*iii*) for (bi-directionally) communicating parties, by sending (single) dummy access requests to uniform-randomly chosen dead drops. The number of (single) dummy access requests (*ii*) is in each round drawn from the truncated Laplace distribution $\lceil \max(0, \text{Laplace}(\gamma_c, \mu_c)) \rceil$ for system parameters γ_c and μ_c, and the number of pairs of dummy access requests (*iii*) is in each round drawn from the truncated Laplace distribution $\lceil \max(0, \text{Laplace}(\mu_c/2, \gamma_c/2)) \rceil$. The system parameters $\mu_d, \mu_c, \gamma_d, \gamma_c$ determine how much noise-overhead the protocol produces and how much privacy it will offer.

Privacy-impact of the dummy requests The goal of the these dummy requests and invitations is to produce a cover stories for dialing parties (*i*), for idle parties (*ii*), and for conversing (*iii*). The Vuvuzela paper separately conducts a privacy analysis for the dialing protocol ((*i*)) and the conversation protocol ((*ii*) and (*iii*) combined). For the dialing protocol, the paper concludes that it suffices to bound the r-fold (ϵ, δ) differential privacy of $\max(0, \text{Laplace}(\mu_d, \gamma_d))$ and $\max(0, \text{Laplace}(\mu_d + 2, \gamma_d))$, i.e., the (ϵ, δ) differential privacy of the product distributions $\max(0, \text{Laplace}(\mu_d, \gamma_d))^r$ and $\max(0, \text{Laplace}(\mu_d + 2, \gamma_d))^r$. The parameter r indicates the number of rounds at which that the attacker conducts an observation. For the conversation protocol, the paper concludes that it suffices to estimate the r-fold (ϵ, δ) differential privacy of $\max(0, \text{Laplace}(\mu_c, \gamma_c)) + \max(0, \text{Laplace}(\mu_c/2, \gamma_c/2))$ and $\max(0, \text{Laplace}(\mu_c + 2, \gamma_c)) + \max(0, \text{Laplace}(\mu_c/2 + 1, \gamma_c/2))$. The Vuvuzela paper uses the advanced composition theorem for differential privacy [8] to bound ϵ and δ. The paper analyzes for the conversation protocol three system parameters: $\mu = 150k, \gamma = 7.5k$, $\mu = 300k, \gamma = 13.8k$, and $\mu = 450k, \gamma = 20k$. We show that the resulting bounds can be significantly improved and we indicate all new bounds with a "$*$" sign in the respective figures.

We apply our method to estimate tighter ε and δ bounds for Vuvuzela, and to reduce the recommended noise. Recall that we observed in Section 6 that Gaussian noise for the same variance behaves better under composition than Laplacian noise. This section studies how much our tighter bounds enable us to reduces the noise in the case that Gaussian noise is used or that Laplace noise is used, and this section studies how much the originally recommended amount of noise improves the degree of privacy, in case Gaussian noise is used or Laplace noise is used. We stress that while in the case of Vuvuzela there is no utility function that we have to preserve other than to minimize the bandwidth overhead, our approach is also suited for applications where a utility function has to be preserved. In those cases, we would probably reduce the variance to an appropriate level and then compute tight bounds.

7.2 Tighter privacy analysis for the dialing protocol

For the dialing protocol, we show that with Gaussian noise the noise rate can be reduced by a factor of almost 5 while still meeting the privacy requirements, and for the conversation protocol the noise rate can be reduced by a factor of 10 while still meeting the privacy requirements. With Laplace noise the noise rate can be reduced by a factor of 2 and for the conversation protocol by a factor of 4. We refer to Figures 20 and 21, placed in the appendix. As the conversation protocol produces more observations (i.e., more compositions) and the looseness of the bounds that the original Vuvuzela paper used amplifies more heavily for a high the number of observations, the tightness of our bounds is more pronounced for the conversation protocol.

For comparability, we depict in Figure 15 the original graphs from the Vuvuzela analysis, which show the epsilon graph and the delta graph with increasing r, respectively, for the dialing protocol and estimated with the advanced composition result. We extend those Figures with the lowest, magenta graphs (marked with a $*$) that show the performance of our proposed Gaussian noise that uses nearly 5 times less noise and is computed with our bucketing approach.[15] As our method computes not only one ε, δ pair for each number of observations r but an entire ε, δ graph, we chose representative ϵ values that are close to (and even below) the epsilon values for the highest noise configuration $\text{LP}(20k, 1130)$ from the original Vuvuzela paper. The figure shows that our bounds with the reduced noise and with using Gaussian noise $\text{GS}(4.1k, 833^2)$ are below the previous bounds for the highest noise configuration $\text{LP}(20k, 1130)$, proving that a noise reduction of nearly a factor of 5 still yields for the dialing protocol to achieve the privacy requirements of $e^\varepsilon \leq 2$ and $\delta \leq 10^{-4}$.

[15] All computations have been conducted with $100,000$ buckets.

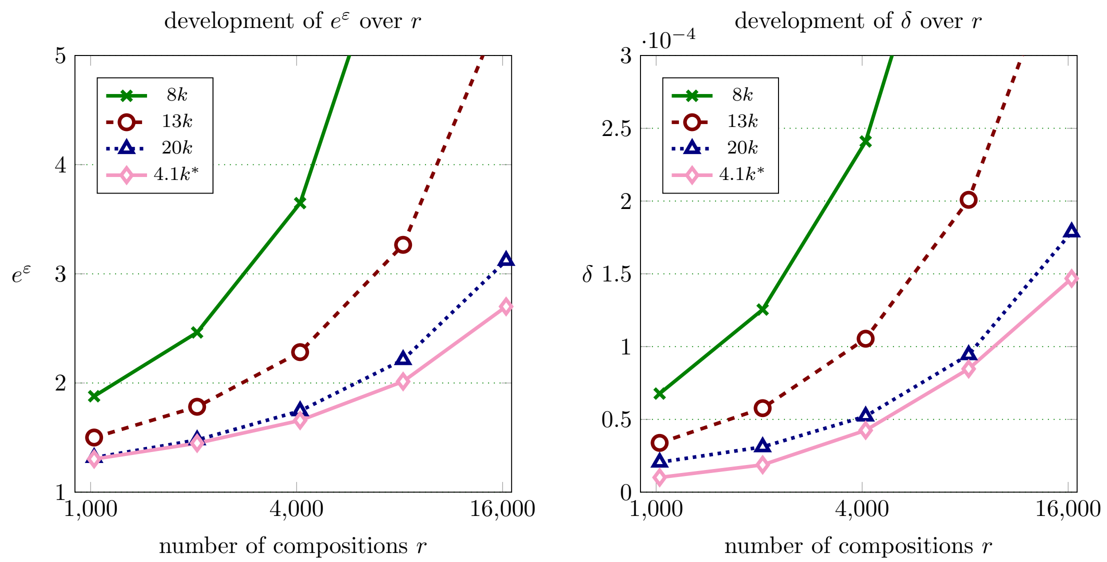


Figure 15: The privacy bounds for Vuvuzela's dialing protocol. The left graph shows the e^ε-values on the y-axis and the number of observations r on the x-axis (i.e., r-fold composition) in log-scale and the right graph shows the corresponding δ-values on the y-axis. The solid green ($\mu = 8k, \gamma = 500$), the dashed red ($\mu = 13k, \gamma = 770k$), and the dotted blue line ($\mu = 20k, \gamma = 1130$) are from the original Vuvuzela paper, and the solid magenta line (Gaussian noise, $\mu = 4.1k^*, \sigma = 320$) is computed with this work's technique.

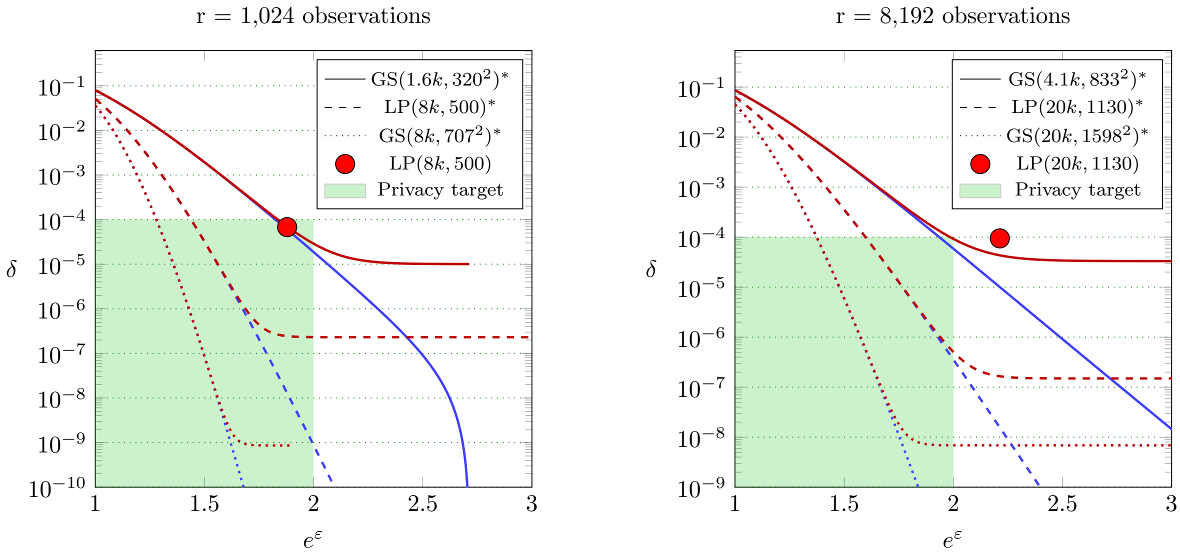
Next, we illustrate that our method computes bounds that are several orders of magnitude better than Vuvuzela's original bounds. For $r = 8,192$ observations, Figure 16b illustrates that using the highest noise configuration with Laplace noise $\mathrm{LP}(20k, 1130)$ results in a privacy bound that is almost 3 orders of magnitude lower, in terms of the delta, and with Gaussian noise $\mathrm{GS}(20k, 1598^2)$ more than 4 orders of magnitude. The figure depicts the ε, δ graphs computed by our approach for the highest noise configuration $\mathrm{LP}(20k, 1130)$, for the corresponding Gaussian noise $\mathrm{GS}(20k, 1598^2)$, for the configuration that we propose $\mathrm{GS}(4.1k, 833^2)$), and compares it against Vuvuzela's previous bounds $\mathrm{LP}(20k, 1130)$. We additionally depict the respective lower bounds, which show that our bounds are quite tight in the sense that there is not much room for improvement. Moreover, due to the more comprehensive view that a full ε, δ graph provides, we can see that the the highest noise configuration with Gaussian noise $\mathrm{GS}(20k, 1598^2)$ even achieves the privacy requirements ($\delta \leq 10^{-4}$) for less than $e^\varepsilon = 1.5$ after $8,192$ observations.[16]

We would like to stress that the lower bounds show that our result is tight up to $\delta \geq 10^{-4}$ for $\mathrm{GS}(4.1k, 833^2)$, $\delta \geq 10^{-6}$ for $\mathrm{LP}(20k, 1130)$, and $\mathrm{GS}(20k, 1598^2)$ for $\delta \geq 10^{-8}$. This tightness is solely a scalability issue and ultimately only depends on the number (and hence granularity) of the buckets. A more optimized implementation (e.g., based on GPUs) would be able to significantly increase the number of buckets, thus achieving even tighter upper and lower bounds.

For completeness, we also show in Figure 16a the ε, δ graphs for the dialing protocol for low r: $r = 1024$ and the recommended parameters $\mu = 8k, \gamma = 500$. Here, we can see that our bound is 2 orders of magnitude lower than Vuvuzela's previous bounds for the noise level. The figure also shows that reducing the noise by a factor of 5, i.e., $\mathrm{GS}(1.6k, 320)$, still achieves the privacy requirements ($e^\varepsilon \leq 2$ and $\delta \leq 10^{-4}$).

As a comparison, using Laplace noise only enables a noise reduction of a factor of 2, as shown in Figure 21 in the appendix. Interestingly, the reduced Laplace noise achieves the same privacy bounds as the reduced Gaussian noise if the Laplace noise has twice the variance as the Gaussian noise (i.e., $\gamma = \sigma$) but a 2.5 times wider range, as indicated in Section 6. This shows what a significant effect the steepness of the Gaussian noise can have in practice.

[16] Recall that the variance of $\mathrm{GS}(\mu, (\sqrt{2}x)^2) = 2x^2$ equals the variance of $\mathrm{LP}(\mu, x) = 2x^2$.



(a) After $r = 1,024$ observations with Gaussian noise with $\mu = 1.6k$ and $\sigma = 320$ (solid), Laplace noise $\mu = 8k, \gamma = 500$ (dashed), and Gaussian noise with $\mu = 8k$ and $\sigma = 707$ (dotted), and the red dot represents the ε, δ combination for $\mu = 8k, \gamma = 500$ from the original Vuvuzela paper.

(b) After $r = 8,192$ observations with Gaussian noise with $\mu = 4.1k$ and $\sigma = 833$ (solid), Laplace noise $\mu = 20k, \gamma = 1130$ (dashed), and Gaussian noise with $\mu = 20k$ and $\sigma = 1598$ (dotted), and the red dot represents the ε, δ combination for $\mu = 20k, \gamma = 1130$ from the original Vuvuzela paper.

Figure 16: The (ε, δ) graphs (y-axis and x axis, respectively, y-axis in $\log_{10}$-scale) from our method in comparison with the bound from the original Vuvuzela paper (for the dialing protocol). The figure depicts upper (red) and a lower bounds (blue) and Vuvuzela's privacy target (green, $\delta \leq 10^{-4}, e^\varepsilon \leq 2$).

name	μ	γ for LP	σ^2 (used for GS)
new_1	$15k$	—	$2.5k^2$
new_2	$45k$	—	$7.5k^2$
low	$150k$	$7.3k$	$2 \cdot 7.3k^2$
medium	$300k$	$13.8k$	—
high	$450k$	$20k$	$2 \cdot 20k^2$

Figure 17: Parameters of our Vuvuzela analysis.

7.3 Tighter privacy analysis for the conversation protocol

Figure 18 depicts the epsilon graph and the delta graph with increasing r, respectively, for the conversation protocol. We compare Gaussian noise GS-new_2 with the previous bounds for the recommended noise configurations. We see that although GS-new_2 adds significantly less noise, the bounds outperform the ones from the original analysis.

For $r = 524,288$ observations, Figure 19b shows that using LP-high results in bounds for δ that are almost 4 orders of magnitude lower, and for the corresponding Gaussian noise GS-high more than 6 orders of magnitude in comparison to their original result. Also, Figure 19b shows the corresponding lower bounds. We can see that our bounds are tight for reasonably small values of ε. Furthermore, we can see that even GS-new_1 meets the privacy requirements of $e^\varepsilon \leq 2$ and $\delta \leq 10^{-4}$ for $r = 524,288$ observations.

For completeness, we also show in Figure 19a the ε, δ graphs for the conversation protocol for $r = 65,536$. Here, we can also see the tightness of our bound for reasonably small ε. We can see that GS-low is more than 7 orders of magnitude lower than Vuvuzela's previous bounds for the same noise level. Moreover, we can see that even GS-new_2 meets the privacy requirements of $e^\varepsilon \leq 2$ and $\delta \leq 10^{-4}$ for $r = 65,536$ observations.

As a comparison, using Laplace noise only enables a noise reduction of a factor of 4, as shown in Figure 20.

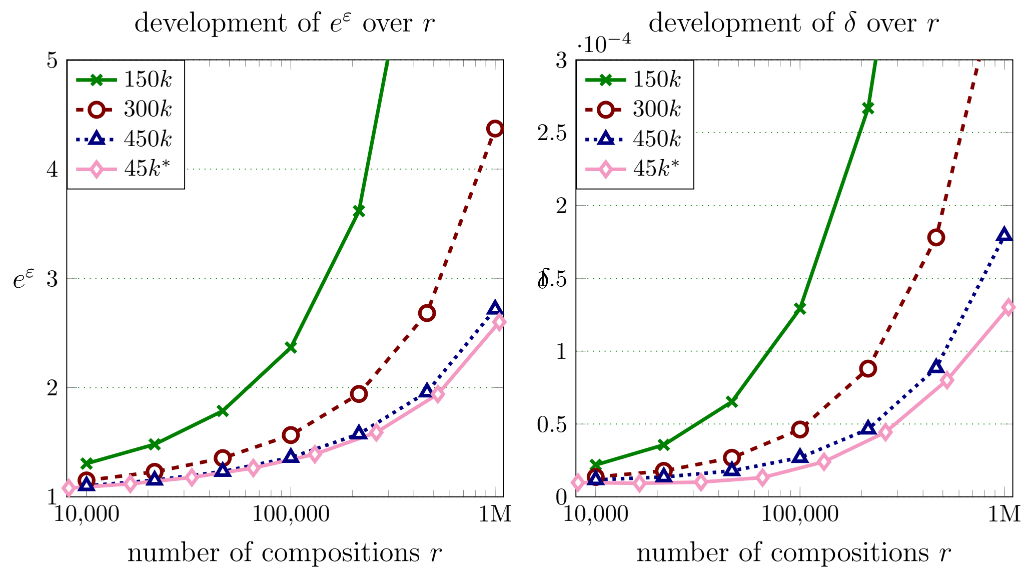
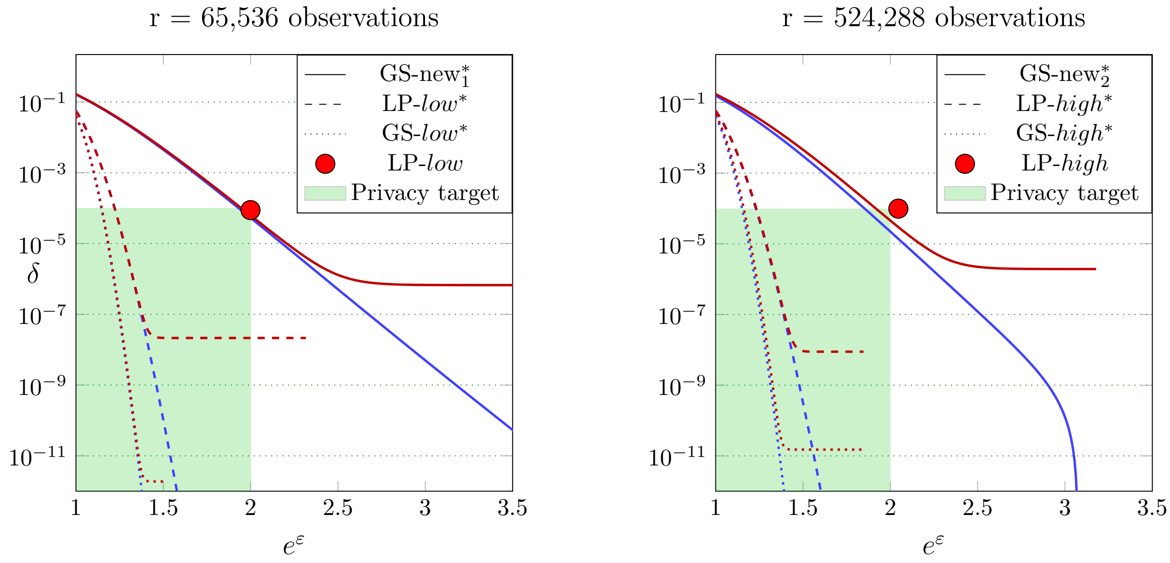


Figure 18: Vuvuzela conversation protocol: bounds on ε and δ over r (log-scale). Originally recommended mechanisms with $150k$, $300k$, $450k$ messages overhead per round, analyzed with previous bounds [8], vs our recommended mechanism with $45k$ overhead, analyzed using privacy buckets.

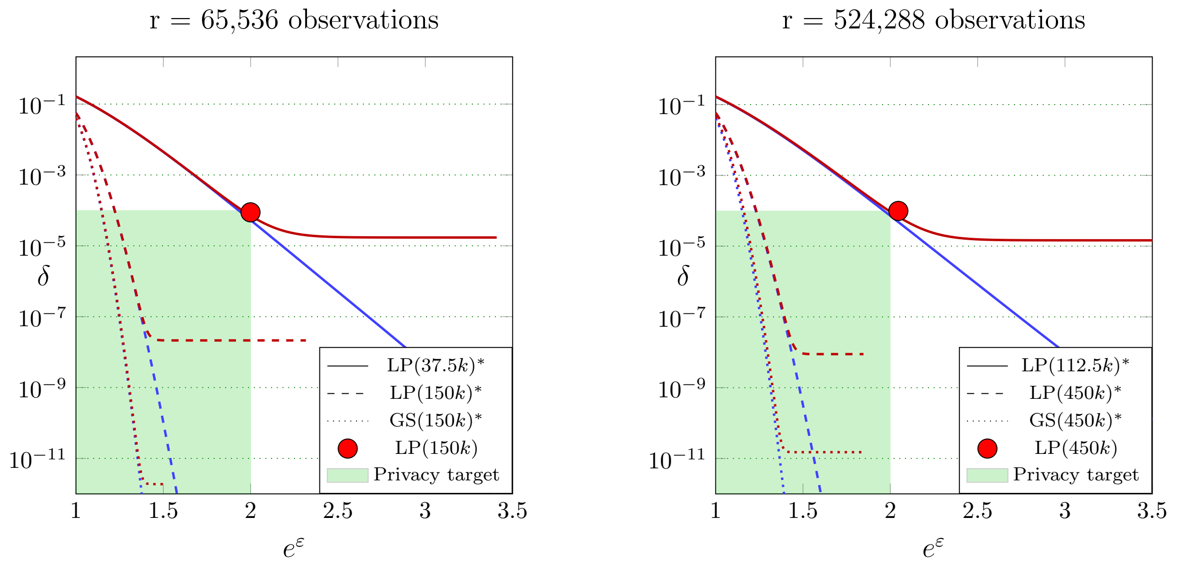


(a) After $r = 65,536$ observations with Gaussian noise with $\mu = 15k$ and $\sigma = 2.5k$ (solid), Laplace noise $\mu = 150k, \gamma = 7.3k$ (dashed), and Gaussian noise with $\mu = 150k$ and $\sigma = 10.3k$ (dotted), and the red dot represents the ε, δ bound for $\mu = 150k, \gamma = 7.3k$ from the original Vuvuzela paper.

(b) After $r = 524,288$ observations with Gaussian noise with $\mu = 45k$ and $\sigma = 7.5k$ (solid), Laplace noise $\mu = 450k, \gamma = 20k$ (dashed), and Gaussian noise with $\mu = 450k$ and $\sigma = 28.2k$ (dotted), and the red dot represents the ε, δ bound for $\mu = 450k, \gamma = 20k$ from the original Vuvuzela paper.

Figure 19: The (ε, δ) graphs (y-axis and x axis, respectively, y-axis in $\log_{10}$-scale) from our method in comparison with the bound from the original Vuvuzela paper (for the conversation protocol). The figure depicts upper (red) and a lower bounds (blue) and Vuvuzela's privacy target (green area, $\delta \leq 10^{-4}, e^\varepsilon \leq 2$).

Also here, we can observe that the Laplace noise has twice the variance of the Gaussian noise and has a 2.5 times wider range, illustrating the advantages of Gaussian noise in practice.



(a) After $r = 65,536$ observations with Laplace noise with $\mu = 37.5k$ and $\sigma = 2.3k$ (solid), Laplace noise $\mu = 150k, \gamma = 7.3k$ (dashed), and Gaussian noise with $\mu = 150k$ and $\sigma = 10.3k$ (dotted), and the red dot represents the ε, δ bound for $\mu = 150k, \gamma = 7.3k$ from the original Vuvuzela paper.

(b) After $r = 524,288$ observations with Laplace noise with $\mu = 112.5k$ and $\sigma = 6.9k$ (solid), Laplace noise $\mu = 450k, \gamma = 20k$ (dashed), and Gaussian noise with $\mu = 450k$ and $\sigma = 28.2k$ (dotted), and the red dot represents the ε, δ bound for $\mu = 450k, \gamma = 20k$ from the original Vuvuzela paper.

Figure 20: The (ε, δ) graphs (y-axis and x axis, respectively, y-axis in $\log_{10}$-scale) from our method in comparison with the bound from the original Vuvuzela paper (for the conversation protocol). The figure depicts upper (red) and a lower bounds (blue) and Vuvuzela's privacy target (green area, $\delta \leq 10^{-4}, e^{\varepsilon} \leq 2$).

8 Conclusion and future work

We have presented *privacy buckets*, a sound numerical approach for computing upper and lower bounds for approximate differential privacy after r-fold composition. We illustrate that our method can derive upper and lower bounds for differentially private mechanisms with worst-case distributions. Our method can be applied in a variety of cases, including adaptive composition, evolving sequences of distributions, and static distributions. Our bounds have been proven to be sound, and (empirically) illustrated to be tight.

We compared our approach to the Kairouz, Oh and Viswanath's (KOV) composition theorem, as well as to the moments accountant (MA) bounds and bounds derived via concentrated differential privacy (CDP). We found that the KOV theorem provides reasonably tight bounds for the Laplace mechanism but not for other distributions, such as the Gauss mechanism or for a pair of histograms of timing-leakage measurements from the CoverUp system. Our bounds significantly improve over MA bounds and CDP bounds, which is particularly relevant for smaller values of e^{ε}. We also observed that the Gauss mechanism behaves much better under a high number of compositions than a Laplace mechanism with the same variance, and we found evidence that the (ε, δ)-graph of a Laplace mechanism converges to the (ε, δ)-graph of a Gauss mechanism with half the variance. By improving the analysis of the anonymity network Vuvuzela we show that tighter bounds can have a significant impact on actual protocols. Our analysis can help to devise better protocols, e.g., to exchange the Laplace noise with Gaussian noise, for which better results can be achieved.

We encourage the application of our privacy buckets to other ADP mechanisms, such as to the optimal ADP mechanisms [10, 15] (e.g., comparing their composition behavior to the Gauss mechanism), to measure the impact of our bounds on precision and recall of privacy-preserving ML methods by using less noise, and to improve more existing privacy analyses. We consider it interesting for future work to explore the relationship between the Gauss mechanism and the Laplace mechanism and to analyze the development of ADP under composition for other noise distributions.

9 Acknowledgement

This work has been partially supported by the European Commission through H2020-DS-2014-653497 PANORAMIX, the EPSRC Grant EP/M013-286/1, and the Zurich Information Security Center (ZISC). We would like to thank the anonymous reviewers of S&P'18 and CCS'18 for their helpful comments and David Sommer for our discussions and the efficient re-implementation of privacy buckets.

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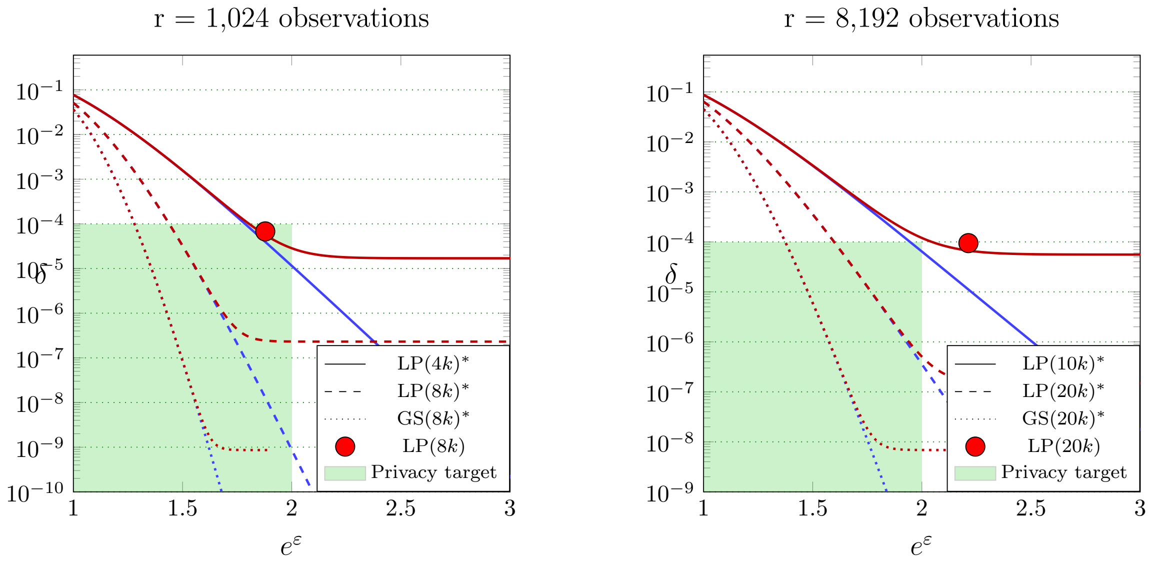

(a) After $r = 1,024$ observations with Laplace noise with $\mu = 4k$ and $\sigma = 330$ (solid), Laplace noise $\mu = 8k, \gamma = 500$ (dashed), and Gaussian noise with $\mu = 8k$ and $\sigma = 707$ (dotted), and the red dot represents the ε, δ combination for $\mu = 8k, \gamma = 500$ from the original Vuvuzela paper.

(b) After $r = 8,192$ observations with Laplace noise with $\mu = 10k$ and $\sigma = 827$ (solid), Laplace noise $\mu = 20k, \gamma = 1130$ (dashed), and Gaussian noise with $\mu = 20k$ and $\sigma = 1598$ (dotted), and the red dot represents the ε, δ combination for $\mu = 20k, \gamma = 1130$ from the original Vuvuzela paper.

Figure 21: The (ε, δ) graphs (y-axis and x axis, respectively, y-axis in $\log_{10}$-scale) from our method in comparison with the bound from the original Vuvuzela paper (for the dialing protocol). The figure depicts upper (red) and a lower bounds (blue) and Vuvuzela's privacy target (green, $\delta \leq 10^{-4}, e^{\varepsilon} \leq 2$).

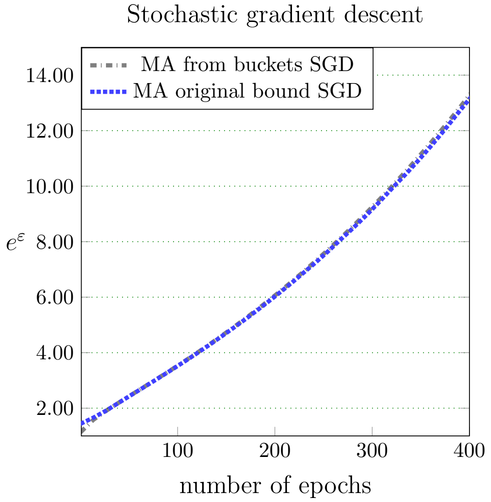

Figure 22: Comparison of bounds for the moments accountant computation.