

# Post-quantum IND-CCA-secure KEM without Additional Hash

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**Abstract.** With the gradual progress of NIST’s post-quantum cryptography standardization, several practical post-quantum secure key encapsulation mechanism (KEM) schemes have been proposed. Generally, an IND-CCA-secure KEM is usually achieved by introducing an IND-CPA-secure (or OW-CPA-secure) public-key encryption (PKE) scheme, then applying some generic transformations to it. All these generic transformations are constructed in the random oracle model (ROM). To fully assess the post-quantum security, security analysis in the quantum random oracle model (QROM) is preferred. However, current works either lacked a QROM security proof or just followed Targhi and Unruh’s proof technique (TCC-B 2016) and modified the original transformations by adding an additional hash to the ciphertext to achieve the QROM security.

In this paper, by using a novel proof technique, we present QROM security reductions for two widely used generic transformations without suffering any ciphertext overhead. Meanwhile, the security bounds are much tighter than the ones derived by utilizing Targhi and Unruh’s proof technique. Thus, our QROM security proofs not only provide a solid post-quantum security guarantee for previous KEM schemes, but also simplify the constructions and reduce the ciphertext sizes. We also provide QROM security reductions for Hofheinz-Hövelmanns-Kiltz modular transformations (TCC 2017), which can help to obtain a variety of combined transformations with different requirements and properties.

**Keywords:** quantum random oracle model · key encapsulation mechanism · IND-CCA security · generic transformation

## 1 Introduction

In December 2016, National Institute of Standards and Technology (NIST) launched a Post-Quantum Cryptography Project and published a call for submissions of quantum-resistant public-key cryptographic algorithms including

digital-signature, public-key encryption (PKE), and key encapsulation mechanism (KEM) (or key exchange) [1]. Triggered by that, there has been a rapid growth of interest in post-quantum cryptographic schemes.

As a foundational cryptography primitive, KEM is efficient and versatile. It can be used to construct, in a black-box manner, PKE (the KEM-DEM paradigm [2]), key exchange and authenticated key exchange [3, 4]. Compared with designing a full PKE scheme, the KEM construction is usually somewhat easier or more efficient. Recently, to make the KEM scheme secure against quantum computers, many researchers devoted to the KEM constructions based on the hardness of certain problems over lattices [5–11] and code theory [12, 13].

Indistinguishability against chosen-ciphertext attacks (IND-CCA) [14] is widely accepted as the standard security notion for many cryptography applications. However, the security is usually much more difficult to prove than IND-CPA (indistinguishability against chosen-plaintext attacks) security. Mostly, generic transformations [15, 16] are used to create an IND-CCA-secure KEM from some weakly secure (OW-CPA or IND-CPA) PKEs, see [5–9, 12].

In his “A Designer’s Guide to KEMs” paper [15], Dent provided several generic transformations from weakly secure PKE schemes to IND-CCA-secure KEMs. In particular, [15, Table 5] can be viewed as the KEM variant of Fujisaki-Okamoto (FO) transformation [17, 18], and is widely used in constructing post-quantum IND-CCA-secure KEMs, e.g., [6–8]. Recently, considering the drawbacks of previous analysis of FO transformation, such as a non-tight security reduction and the need for a perfectly correct scheme, Hofheinz, Hövelmanns and Kiltz [16] revisited the FO transformation and provided a fine-grained and modular toolkit of transformations. By combining these modular transformations, they obtained several variants of FO transformation. Subsequently, Bos et al. [5] and Barreto et al. [12] used one of these variants to construct IND-CCA-secure KEMs, Kyber (module-lattice-based) and CAKE (code-based), respectively. Specially, Kyber is part of the CRYSTALS (Cryptographic Suite for Algebraic Lattices) package that will be submitted to the NIST call for post-quantum standards.

Note that all above mentioned transformations are constructed in the random oracle model (ROM). When the KEM scheme is instantiated, the random oracle is usually replaced by a hash function, which a quantum adversary may evaluate on a quantum superposition of inputs. As a result, to fully assess post-quantum security, we should analyze security in the quantum random oracle model (QROM), as introduced in [19]. However, proving security in the QROM is quite challenging, as many classical ROM proof techniques will be invalid.

Among current works about post-quantum KEMs, they either lacked a QROM security proof [6, 8, 10] or just followed Targhi and Unruh’s proof idea [20, 21] and modified the original transformations by adding an additional hash to the ciphertext to achieve QROM security [5, 7, 11, 12, 16]. Intuitively, for 128-bit post-quantum security, such a modification merely increases the ciphertext size by 256 bits [22]. However, we note that the QROM security proof in [20, 21] requires the additional hash function to be length-preserving (that has the same domain

and range size). Thus, for some schemes where the message space is strictly larger than the output space of the hash function, the increase of the ciphertext size is significant. Hülsing et al. [7] tried several ways to circumvent this issue, unfortunately all straight forward approaches failed. For their specific NTRU-based KEM, additional 1128 bits are needed in the decapsulation, which accounts for 11% of the final encapsulation size.

In the ROM, this additional hash is clearly redundant for the constructions of IND-CCA-secure KEM [15, 16]. To use Targhi and Unruh’s proof technique to accomplish the QROM security proof, [5, 7, 11, 12, 16] deliberately introduced an additional length-preserving hash to the ciphertext, which increased the ciphertext size and complicated the implementation. Thus, a natural question is that: can we improve the QROM security proof without suffering any ciphertext overhead for these constructions? In this paper, we present a positive answer.

### 1.1 Our Contributions

1. We prove the QROM security of two generic transformations (variants of FO transformation) by reducing the IND-CCA security of KEM to the OW-CPA security of the underlying PKE.

One is the transformation  $\text{FO}^{\cancel{\leftarrow}}$  in [16], we denote such a construction by FO-I in our paper. In [16], Hofheinz et al. proved the security of FO-I in the ROM. When considering the QROM security, they followed Targhi and Unruh’s proof idea, and modified FO-I by adding an additional hash to the ciphertext. Kyber [5] and CAKE [12] were exactly constructed by using this modified transformation. Thus, with our security proof, Kyber and CAKE can be simplified by cutting off that additional hash, leading to performance improvement in terms of speed and sizes.

The other is  $\text{FO}_m^{\cancel{\leftarrow}}$  in [16], the transformation [15, Table 5] with implicit rejection (meaning that a pseudorandom key is returned when an invalid ciphertext is submitted to the decapsulation algorithm). We denote this transformation by FO-II in our paper. This transformation was widely used in [6–8]. But, these works either lacked a QROM security proof [6, 8] or just followed Targhi and Unruh’s work [21] and modified the original transformation by adding an additional hash to the ciphertext to achieve the QROM security [7]. Thus, our QROM security proof provides a solid post-quantum security guarantee for these KEM schemes without additional ciphertext overhead.

2. For our security reductions, the advantage of the adversary  $\mathcal{B}$  against the IND-CCA security of KEM  $\text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{B})$  is approximately bounded by  $q \cdot \sqrt{\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})}$ , which is much tighter than  $q^{\frac{3}{2}} \cdot [\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})]^{\frac{1}{4}}$  achieved by [5, 12, 16], where  $\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})$  is the advantage of the adversary  $\mathcal{A}$  against the OW-CPA security of PKE and  $q$  is the total number of  $\mathcal{B}$ ’s queries to various oracles.
3. We provide QROM security reductions for some fine-grained and modular transformations in [16]. Hofheinz et al. [16] provided seven fine-grained modular transformations  $T$ ,  $U^{\cancel{\leftarrow}}$ ,  $U^{\perp}$ ,  $U_m^{\cancel{\leftarrow}}$ ,  $U_m^{\perp}$ ,  $QU_m^{\cancel{\leftarrow}}$  and  $QU_m^{\perp}$ , which can be

used to obtain some combined transformations with different requirements and properties. But, they just presented QROM security proofs for the transformations  $T$ ,  $QU_m^\times$  and  $QU_m^\perp$ . Different from  $U^\times$ ,  $U^\perp$ ,  $U_m^\times$  and  $U_m^\perp$ , the transformations  $QU_m^\times$  and  $QU_m^\perp$  have an additional length-preserving hash in the ciphertext, thus the proof idea in [20, 21] can be used to prove the QROM security. As they pointed [21], such a proof technique quite relies on the additional hash. Therefore, QROM security reductions for  $U^\times$ ,  $U^\perp$ ,  $U_m^\times$  and  $U_m^\perp$  are missing in [16].

In this paper, we first define two new security notions, one-way against quantum plaintext checking attacks (OW-qPCA) and one-way against quantum plaintext and (classical) validity checking attacks (OW-qPVCA) (quantum plaintext checking attacks mean that the adversary can make quantum queries to the plaintext checking oracle). Then, we provide QROM security reductions for  $T$  from OW-qPCA to OW-CPA,  $U^\times$  from IND-CCA to OW-qPCA,  $U^\perp$  from IND-CCA to OW-qPVCA,  $U_m^\times$  from IND-CCA to OW-CPA and  $U_m^\perp$  from IND-CCA to OW-VA (one-way against validity checking attacks).

## 1.2 Techniques

As explained by Targhi and Unruh [21], their proof technique strongly relies on the additional hash. In their paper, they discussed the QROM security of a variant of FO transformation from OW-CPA-secure PKE to IND-CCA-secure PKE. To implement the security reduction, one needs to simulate the decryption oracle without possessing the secret key. In classical proof, a RO-query list is used to simulate such an oracle. In the QROM, the simulator has no way to learn the actual content of adversarial RO queries, therefore such a RO-query list does not exist. Targhi and Unruh circumvented this issue by adding an additional length-preserving hash (modeled as a RO) to the ciphertext. In the security reduction, this additional RO is simulated by a  $k$ -wise independent function. For every output of this RO, the simulator can recover the corresponding input by inverting this function. Thereby, the simulator can answer the decryption queries without a secret key.

When considering the generic transformations from weakly secure PKE scheme to IND-CCA-secure KEM, one needs to simulate the decapsulation oracle DECAPS without the secret key. Indeed, obviously, we can modify the scheme by adding an additional length-preserving hash to the ciphertext so that the simulator can carry out the decryption. Thus, using the key-derivation-function (KDF, modeled as a random oracle  $H$ ), he can easily simulate the DECAPS oracle.

In [19, Theorem 6], Boneh et al. proved the QROM security of a generic hybrid encryption scheme [23], built from an injective trapdoor function and symmetric key encryption scheme. Inspired by their proof idea, we present a novel approach to simulate the DECAPS oracle.

The high level idea is that we associate the random oracle  $H$  (KDF in the KEM) with a secret random function  $H'$  by setting  $H = H' \circ g$  such that  $H'(\cdot) =$

$\text{DECAPS}(sk, \cdot)$ . We demand that the function  $g$  should be indistinguishable from an injective function for any efficient quantum adversary. Thus, in the view of the adversary against the IND-CCA security of KEM,  $H$  is indeed a random oracle. Meanwhile, we can simulate the DECAPS oracle just by using  $H'$ . Note that in our simulation of the DECAPS oracle, we circumvent the decryption computation. Thereby, there is no need to read the content of adversarial RO queries, which makes it unnecessary to add an additional length-preserving hash to the ciphertext.

### 1.3 Discussion

**Tightness.** Having a tight security reduction is a desirable property for practice cryptography, especially in large-scale scenarios. A tight security reduction can ensure that breaking the scheme (within the respective adversarial model) is at least as hard as breaking the underlying hard computational problem. While, a non-tight security reduction requires to adapt the system parameters accordingly, which results in less efficient schemes.

In the ROM, if we assume that the underlying PKE scheme in transformations FO-I and FO-II is IND-CPA-secure, we can obtain a tight reduction from IND-CCA security of KEM to IND-CPA security of PKE [16]. Specially, if the PKE scheme in FO-II is instantiated with a Ring-LWE-based PKE scheme [24], the IND-CCA security of KEM can be reduced to the security of the underlying Ring-LWE problem [6]. Albrecht et al. [6] pointed out that it is an important open problem whether one can achieve QROM security for a Dent-like KEM construction with a tight reduction and without suffering any ciphertext overhead. In our work, although we present a series of QROM security reductions for the Dent-like KEM constructions without suffering any ciphertext overhead, these reductions are non-tight like previous QROM security reductions [5, 7, 11, 12, 16, 19, 21]. For the tight ROM security reductions in [6, 16], all the simulators need to make an elaborate analysis of the RO-query inputs and determine which one of the query inputs can be used to break the one-way security of the underlying PKE scheme [16] or solve a decision Ring-LWE problem [6]. However, in the QROM, such a proof technique will be invalid for the reason that there is no way for the simulators to learn the RO-query inputs [25, 26]. Thus, in the QROM, it is still an important open problem that whether one can develop a novel proof technique to obtain a tight reduction for the KEM constructions discussed in this paper.

**Implicit rejection.** For most of the previous generic transformations from OW-CPA-secure (or IND-CPA-secure) PKE to IND-CCA-secure KEM, explicit rejection is adopted, i.e., an abnormal symbol  $\perp$  is returned when an invalid ciphertext is submitted to the decapsulation algorithm. In [16], Hofheinz et al. presented several transformations with implicit rejection (the decapsulation algorithm returns a pseudorandom key for the invalid ciphertext). These two different versions (explicit rejection and implicit rejection) have their own merits. The

transformation with implicit rejection [16] does not require the underlying PKE scheme to be  $\gamma$ -spread [17, 18] (meaning that the ciphertexts generated by the probabilistic encryption algorithm have sufficiently large entropy), which may allow choosing better system parameters for the same security level. Whereas, the ones with explicit rejection have a relatively simple decapsulation algorithm.

In our paper, we just give QROM security reductions for the transformations with implicit rejection. It is not obvious how to extend our QROM security proofs for the transformation with explicit rejection, since the simulator has no way to tell if the submitted ciphertext is valid. In classical ROM, we usually assume the underlying PKE scheme is  $\gamma$ -spread. Then, we can recognize invalid ciphertexts just by testing if they are in the RO-query list, as the probability that the adversary makes queries to the decapsulation oracle with a valid ciphertext which is not in the RO-query list is negligible [6, 16–18]. Unfortunately, in the QROM, the adversary makes quantum queries to the RO, above RO-query list does not exist. Thus, the ROM proof technique for the recognition of invalid ciphertexts is invalid in the QROM. Here, we leave it as an open problem to prove the QROM security of the transformations FO-I and FO-II with explicit rejection.

#### 1.4 Related Works

In concurrent and independent work, [27] gives a QROM security reduction from IND-CCA security of KEM to IND-CPA security of PKE with quadratic loss, i.e.,  $\text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{B}) \leq q \cdot \sqrt{\text{Adv}_{\text{PKE}}^{\text{IND-CPA}}(\mathcal{A})}$ . First, [27] presents a tight QROM reduction for  $U_m^\perp$  from the IND-CCA security of KEM to the PR-CPA security of the underlying PKE, where PR-CPA security<sup>5</sup> is a new security notion for a deterministic PKE scheme proposed by [27]. Then, [27] gives a transformation *THalf* (a variant of *T*) that converts an IND-CPA-secure PKE into PR-CPA-secure PKE, where the underlying IND-CPA-secure PKE is required to be perfectly correct and have sufficiently large plaintext space. And, the QROM security reduction for *THalf* suffers from loose reduction with quadratic loss. Thus, taking the transformations *THalf* and  $U_m^\perp$  together, [27] also obtains a QROM security reduction with quadratic loss from IND-CCA security of KEM to IND-CPA security of PKE.

## 2 Preliminaries

**Symbol description.** Denote  $\mathcal{K}$ ,  $\mathcal{M}$ ,  $\mathcal{C}$  and  $\mathcal{R}$  as key space, message space, ciphertext space and randomness space, respectively. For a finite set  $X$ , we denote the sampling of a uniform random element  $x$  by  $x \xleftarrow{\$} X$ , and we denote the

<sup>5</sup> A deterministic PKE scheme is PR-CPA-secure, if there exist an efficient fake key-generation algorithm and a fake encryption algorithm such that, (1) real and fake encryption keys are indistinguishable, (2) random real and fake ciphertexts on a fake key are indistinguishable, (3) the probability that a random fake ciphertext on a fake key falls in the range of a real ciphertext on the fake key is negligible.

sampling according to some distribution  $D$  by  $x \leftarrow D$ . By  $x =?y$  we denote the integer that is 1 if  $x = y$ , and otherwise 0.  $\Pr[P : G]$  is the probability that the predicate  $P$  holds true where free variables in  $P$  are assigned according to the program in  $G$ . Denote deterministic (probabilistic) computation of an algorithm  $A$  on input  $x$  by  $y := A(x)$  ( $y \leftarrow A(x)$ ).  $A^H$  means that the algorithm  $A$  gets access to the oracle  $H$ .

## 2.1 Quantum Random Oracle Model

In the ROM, we assume the existence of a random function  $H$ , and give all parties oracle access to this function. The algorithms comprising any cryptographic protocol can use  $H$ , as can the adversary. Thus we modify the security games for all cryptographic systems to allow the adversary to make random oracle queries.

When a random oracle scheme is implemented, some suitable hash function  $H$  is included in the specification. Any algorithm (including the adversary) replaces oracle queries with evaluations of this hash function. In quantum setting, because a quantum algorithm can evaluate  $H$  on an arbitrary superposition of inputs, we must allow the quantum adversary to make quantum queries to the random oracle. We call this the quantum random oracle model [19]. Unless otherwise specified, the queries to random oracles are quantum in our paper.

**Tools.** Next we state four lemmas that we will use throughout the paper. The first three lemmas have been proved in other works, and we prove the last one in Appendix B. Most of the background in quantum computation needed to understand this paper is just for above proof. Therefore, we present the necessary background in Appendix A. Here, we just recall two basic facts about quantum computation.

- Fact 1. Any classical computation can be implemented on a quantum computer.
- Fact 2. Any function that has an efficient classical algorithm computing it can be implemented efficiently as a quantum-accessible oracle.

**Lemma 1 (Simulating the random oracle [28, Theorem 6.1]).** *Let  $H$  be an oracle drawn from the set of  $2q$ -wise independent functions uniformly at random. Then the advantage any quantum algorithm making at most  $q$  queries to  $H$  has in distinguishing  $H$  from a truly random function is identically 0.*

**Lemma 2 (One-way to hiding [29, Lemma 6.2]).** *Let  $\Omega_H$  be the set of all functions  $H : \{0, 1\}^n \rightarrow \{0, 1\}^m$ .  $D^{\Omega_H}$  is a distribution on  $\Omega_H$ . Let  $H$  be an oracle sampled from  $D^{\Omega_H}$ . Consider an oracle algorithm  $A$  that makes at most  $q$  queries to  $H$ . Let  $B$  be an oracle algorithm that on input  $x$  does the following: pick  $i \xleftarrow{\$} \{1, \dots, q\}$  and  $y \xleftarrow{\$} \{0, 1\}^m$ , run  $A^H(x, y)$  until the  $i$ -th query, measure the argument of the query in the computational basis, and output the measurement outcome. (When  $A$  makes less than  $i$  queries,  $B$  outputs  $\perp \notin \{0, 1\}^n$ .)*

Let

$$\begin{aligned} P_A^1 &:= \Pr[b' = 1 : H \leftarrow D^{\Omega_H}, x \xleftarrow{\$} \{0, 1\}^n, b' \leftarrow A^H(x, H(x))] \\ P_A^2 &:= \Pr[b' = 1 : H \leftarrow D^{\Omega_H}, x \xleftarrow{\$} \{0, 1\}^n, y \xleftarrow{\$} \{0, 1\}^m, b' \leftarrow A^H(x, y)] \\ P_B &:= \Pr[x' = x : H \leftarrow D^{\Omega_H}, x \xleftarrow{\$} \{0, 1\}^n, x' \leftarrow B^H(x)]. \end{aligned}$$

Then

$$|P_A^1 - P_A^2| \leq 2q\sqrt{P_B}.$$

*Note.* In [29, Lemma 6.2],  $H$  is a random oracle, i.e.,  $H \xleftarrow{\$} \Omega_H$ . But, it is easy to verify that their proof can be extended to the generic case where  $H$  is sampled from any given distribution  $D^{\Omega_H}$ .

**Lemma 3 (Generic search problem [30, 31]).** *Let  $Z$  be a finite set.  $F : Z \rightarrow \{0, 1\}$  is the following function: For each  $z$ ,  $F(z) = 1$  with probability  $p_z$  ( $p_z \leq \gamma$ ), and  $F(z) = 0$  else. If an oracle algorithm  $A$  makes at most  $q$  classical queries to  $F$ , the probability  $\Pr[F(z) = 1 : z \leftarrow A^F]$  is at most  $q\gamma$ . When quantum queries are allowed, the upper bound of this probability is  $2q\sqrt{\gamma}$ .*

*Note.* [30, Lemma 37] and [31, Theorem 1] just consider the specific case where all  $p_z$ s are equal to  $\gamma$ . But in our security proof, we need to consider the case where  $p_z \leq \gamma$  and  $p_z$ s are independent from each other. Fortunately, it is not difficult to verify that the proof of [30, Lemma 37] can be extended to this generic case.

**Lemma 4.** *Let  $\Omega_H$  ( $\Omega_{H'}$ ) be the set of all functions  $H : \{0, 1\}^{n_1} \times \{0, 1\}^{n_2} \rightarrow \{0, 1\}^m$  ( $H' : \{0, 1\}^{n_2} \rightarrow \{0, 1\}^m$ ). Let  $H \xleftarrow{\$} \Omega_H$ ,  $H' \xleftarrow{\$} \Omega_{H'}$ . Consider an oracle algorithm  $A^H$  that makes at most  $q$  queries to  $H$ . Denote  $E_1$  as the event that  $A^H$  outputs 1. The probability of the event  $E_1$  is  $\Pr[E_1] = \Pr[b' = 1 : H \xleftarrow{\$} \Omega_H, x \xleftarrow{\$} \{0, 1\}^{n_1}, b' \leftarrow A^H(x, H(x, \cdot))]$ . Reprogram  $H$  at  $(x, \cdot)$  and let  $H(x, \cdot) = H'(\cdot)$ .  $x$  is chosen uniformly at random and independent from the  $A^H$ 's view. Let  $E_2$  be the event that  $A^H$  still outputs 1 even after the random oracle  $H$  is reprogrammed. The probability of the event  $E_2$  is  $\Pr[E_2] = \Pr[b' = 1 : H \xleftarrow{\$} \Omega_H, x \xleftarrow{\$} \{0, 1\}^{n_1}, H' \xleftarrow{\$} \Omega_{H'}, b' \leftarrow A^H(x, H'(\cdot))]$ . Then*

$$|\Pr[E_1] - \Pr[E_2]| \leq 2q \frac{1}{\sqrt{2^{n_1}}}.$$

*Proof.* See Appendix B. □

## 2.2 Cryptographic Primitives

**Definition 1 (Public-key encryption).** *A public-key encryption scheme PKE = (Gen, Enc, Dec) consists of a triple of polynomial time (in the security parameter  $\lambda$ ) algorithms and a finite message space  $\mathcal{M}$ . Gen, the key generation algorithm,*



is a probabilistic algorithm which on input  $1^\lambda$  outputs a public/secret key-pair  $(pk, sk)$ . The encryption algorithm  $Enc$ , on input  $pk$  and a message  $m \in \mathcal{M}$ , outputs a ciphertext  $c \leftarrow Enc(pk, m)$ . If necessary, we make the used randomness of encryption explicit by writing  $c := Enc(pk, m; r)$ , where  $r \xleftarrow{\$} \mathcal{R}$  ( $\mathcal{R}$  is the randomness space).  $Dec$ , the decryption algorithm, is a deterministic algorithm which on input  $sk$  and a ciphertext  $c$  outputs a message  $m := Dec(sk, c)$  or a special symbol  $\perp \notin \mathcal{M}$  to indicate that  $c$  is not a valid ciphertext. We follow the definition of correctness in [16]. The public-key encryption scheme PKE is  $\delta$ -correct if

$$E[\max_{m \in \mathcal{M}} \Pr[Dec(sk, c) \neq m : c \leftarrow Enc(pk, m)]] \leq \delta,$$

where the expectation is taken over  $(pk, sk) \leftarrow Gen$ .

Game OW-ATK	PCO( $m, c$ )	VAL( $c$ )
1: $(pk, sk) \leftarrow Gen$	1: <b>if</b> $m \notin \mathcal{M}$	1: $m := Dec(sk, c)$
2: $m^* \xleftarrow{\$} \mathcal{M}$	2: <b>return</b> $\perp$	2: <b>if</b> $m \in \mathcal{M}$
3: $c^* \leftarrow Enc(pk, m^*)$	3: <b>else return</b>	3: <b>return</b> 1
4: $m' \leftarrow \mathcal{A}^{O_{ATK}}(pk, c^*)$	4: $Dec(sk, c) = ?m$	4: <b>else return</b> 0
5: <b>return</b> $m' = ?m^*$		

Fig. 1: Games OW-ATK ( $ATK \in \{CPA, VA, qPCA, qPVCA\}$ ) for PKE, where  $O_{ATK}$  is defined in Definition 2. In games qPCA and qPVCA, the adversary  $\mathcal{A}$  can query the PCO oracle with quantum state.

We now define four security notions for public-key encryption: one-way against chosen plaintext attacks (OW-CPA), one-way against validity checking attacks (OW-VA), one-way against quantum plaintext checking attacks (OW-qPCA) and one-way against quantum plaintext and (classical) validity checking attacks (OW-qPVCA).

**Definition 2 (OW-ATK-secure PKE).** Let  $PKE = (Gen, Enc, Dec)$  be a public-key encryption scheme with message space  $\mathcal{M}$ . For  $ATK \in \{CPA, VA, qPCA, qPVCA\}$ , we define OW-ATK games as in Fig. 1, where

$$O_{ATK} := \begin{cases} \perp & ATK = CPA \\ VAL(\cdot) & ATK = VA \\ PCO(\cdot, \cdot) & ATK = qPCA \\ PCO(\cdot, \cdot), VAL(\cdot) & ATK = qPVCA. \end{cases}$$

Define the OW-ATK advantage function of an adversary  $\mathcal{A}$  against PKE as  $\text{Adv}_{PKE}^{\text{OW-ATK}}(\mathcal{A}) := \Pr[\text{OW-ATK}_{PKE}^{\mathcal{A}} = 1]$ .

*Remark.* We note that the security game OW-qPCA (OW-qPVCA) is the same as OW-PCA (OW-PVCA) except the adversary  $\mathcal{A}$ 's queries to the PCO oracle. In OW-qPCA (OW-qPVCA) game,  $\mathcal{A}$  can make quantum queries to the PCO oracle, while in OW-PCA (OW-PVCA) game only the classical queries are allowed. These two new security notations will be used in the modular analysis of FO transformation in Sec. 4.

Game IND-CCA	DECAPS( $sk, c$ )
1 : $(pk, sk) \leftarrow Gen$	1 : <b>if</b> $c = c^*$
2 : $b \xleftarrow{\$} \{0, 1\}$	2 : <b>return</b> $\perp$
3 : $(K_0^*, c^*) \leftarrow Encaps(pk)$	3 : <b>else return</b>
4 : $K_1^* \xleftarrow{\$} \mathcal{K}$	4 : $K := Decaps(sk, c)$
5 : $b' \leftarrow \mathcal{A}^{DECAPS}(c^*, K_b^*)$	
6 : <b>return</b> $b' =?b$	

Fig. 2: IND-CCA game for KEM.

**Definition 3 (Key encapsulation).** A key encapsulation mechanism KEM consists of three algorithms  $Gen$ ,  $Encaps$  and  $Decaps$ . The key generation algorithm  $Gen$  outputs a key pair  $(pk, sk)$ . The encapsulation algorithm  $Encaps$ , on input  $pk$ , outputs a tuple  $(K, c)$  where  $c$  is said to be an encapsulation of the key  $K$  which is contained in key space  $\mathcal{K}$ . The deterministic decapsulation algorithm  $Decaps$ , on input  $sk$  and an encapsulation  $c$ , outputs either a key  $K := Decaps(sk, c) \in \mathcal{K}$  or a special symbol  $\perp \notin \mathcal{K}$  to indicate that  $c$  is not a valid encapsulation.

We now define a security notion for KEM: indistinguishability against chosen ciphertext attacks (IND-CCA).

**Definition 4 (IND-CCA-secure KEM).** We define the IND-CCA game as in Fig. 2 and the IND-CCA advantage function of an adversary  $\mathcal{A}$  against KEM as  $\text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{A}) := |\Pr[\text{IND-CCA}_{\text{KEM}}^{\mathcal{A}} = 1] - \frac{1}{2}|$ .

We also define OW-ATK security of PKE and IND-CCA security of KEM in the QROM, where adversary  $A$  can make quantum queries to a random oracle  $H$ . Following the work [16], we also make the convention that the number  $q_H$  of the adversarial queries to  $H$  counts the total number of times  $H$  is executed in the experiment. That is, the number of  $\mathcal{A}$ 's explicit queries to  $H$  plus the number of implicit queries to  $H$  made by the experiment.

### 3 Security Proofs for Two Generic KEM Constructions in the QROM

In this section, we revisit two generic transformations from OW-CPA-secure PKE to IND-CCA-secure KEM. One is the transformation  $\text{FO}^{\not\leftarrow}$  in [16], which we call FO-I in our paper (see Fig. 3). The other is  $\text{FO}_m^{\not\leftarrow}$  in [16], the transformation [15, Table 5] with implicit rejection, which is denoted by FO-II (see Fig. 4). These two transformations are widely used in the post-quantum IND-CCA-secure KEM constructions [5–8, 12]. But, there are no QROM security proofs for them. To achieve QROM security, they followed Targhi and Unruh’s proof idea [20, 21] and modified FO-I [5, 12, 16] and FO-II [6, 7, 16] by adding an additional length-preserving hash function to the ciphertext. Here, we present two QROM security proofs for FO-I and FO-II respectively without suffering any ciphertext overhead.

$Gen'$	$Encaps(pk)$	$Decaps(sk', c)$
1: $(pk, sk) \leftarrow Gen$	1: $m \xleftarrow{\$} \mathcal{M}$	1: Parse $sk' = (sk, s)$
2: $s \xleftarrow{\$} \mathcal{R}$	2: $c = Enc(pk, m; G(m))$	2: $m' := Dec(sk, c)$
3: $sk' := (sk, s)$	3: $K := H(m, c)$	3: <b>if</b> $Enc(pk, m'; G(m')) = c$
4: <b>return</b> $(pk, sk')$	4: <b>return</b> $(K, c)$	4: <b>return</b> $K := H(m', c)$
		5: <b>else return</b>
		6: $K := H(s, c)$

Fig. 3: IND-CCA-secure KEM-I=FO-I[PKE,G,H]

To a public-key encryption scheme  $\text{PKE} = (Gen, Enc, Dec)$  with message space  $\mathcal{M}$  and randomness space  $\mathcal{R}$ , hash functions  $G : \mathcal{M} \rightarrow \mathcal{R}$ ,  $H : \{0, 1\}^* \rightarrow \{0, 1\}^n$  and a pseudorandom function  $f$  with key space  $\mathcal{K}^{prf}$ , we associate KEM-I=FO-I[PKE,G,H] and KEM-II=FO-II[PKE,G,H,f]<sup>6</sup> shown in Fig. 3 and Fig. 4, respectively. The following two theorems establish that IND-CCA securities of KEM-I and KEM-II can both reduce to the OW-CPA security of PKE, in the QROM.

<sup>6</sup> FO-II is the generic version of  $\text{FO}_m^{\not\leftarrow}$  in [16]. In their work, such a pseudorandom function  $f$  is instantiated with  $H(s, \cdot)$  ( $s$  is a random seed and contained in the secret key  $sk'$ ).

$Gen'$	$Encaps(pk)$	$Decaps(sk', c)$
1 : $(pk, sk) \leftarrow Gen$	1 : $m \xleftarrow{\$} \mathcal{M}$	1 : Parse $sk' = (sk, k)$
2 : $k \xleftarrow{\$} \mathcal{K}^{prf}$	2 : $c = Enc(pk, m; G(m))$	2 : $m' := Dec(sk, c)$
3 : $sk' := (sk, k)$	3 : $K := H(m)$	3 : <b>if</b> $Enc(pk, m'; G(m')) = c$
4 : <b>return</b> $(pk, sk')$	4 : <b>return</b> $(K, c)$	4 : <b>return</b> $K := H(m')$
		5 : <b>else return</b>
		6 : $K := f(k, c)$

Fig. 4: IND-CCA-secure KEM-II=FO-II[PKE,G,H,f]

**Theorem 1 (PKE OW-CPA  $\stackrel{QROM}{\Rightarrow}$  KEM-I IND-CCA).** *If PKE is  $\delta$ -correct, for any IND-CCA  $\mathcal{B}$  against KEM-I, issuing at most  $q_D$  queries to the decapsulation oracle DECAPS, at most  $q_G$  queries to the random oracle  $G$  and at most  $q_H$  queries to the random oracle  $H$ , there exists an OW-CPA adversary  $\mathcal{A}$  against PKE such that  $\text{Adv}_{\text{KEM-I}}^{\text{IND-CCA}}(\mathcal{B}) \leq 2q_H \frac{1}{\sqrt{\mathcal{M}}} + 4q_G \sqrt{\delta} + 2(q_G + q_H) \cdot \sqrt{\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})}$ .*

*Proof.* Let  $\mathcal{B}$  be an adversary against the IND-CCA security of KEM-I, issuing at most  $q_D$  queries to DECAPS, at most  $q_G$  queries to  $G$  and at most  $q_H$  queries to  $H$ . Denote  $\Omega_G, \Omega_H$  and  $\Omega_{H'}$  as the sets of all functions  $G : \mathcal{M} \rightarrow \mathcal{R}, H : \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{K}$  and  $H' : \mathcal{C} \rightarrow \mathcal{K}$ , respectively. Consider the games in Fig. 5.

GAME  $G_0$ . Since game  $G_0$  is exactly the IND-CCA game,

$$\left| \Pr[G_0^{\mathcal{B}} \Rightarrow 1] - \frac{1}{2} \right| = \text{Adv}_{\text{KEM-I}}^{\text{IND-CCA}}(\mathcal{B}).$$

GAME  $G_1$ . In game  $G_1$ , we change the DECAPS oracle that  $H_2(c)$  is returned instead of  $H(s, c)$  for an invalid encapsulation  $c$ . Apparently, such a change will be unnoticed by  $\mathcal{B}$  unless he makes queries to  $H$  on  $(s, \cdot)$ . However, in quantum setting, there is no well defined concept for the event that  $H(s, \cdot)$  is queried when  $\mathcal{B}$  can query  $H$  in superposition. Since  $\mathcal{B}$ 's view is independent of (the uniform secret)  $s$ , we can use Lemma 4 to obtain

$$|\Pr[G_0^{\mathcal{B}} \Rightarrow 1] - \Pr[G_1^{\mathcal{B}} \Rightarrow 1]| \leq 2q_H \cdot \frac{1}{\sqrt{\mathcal{M}}}.$$

GAME  $G_2$ . Note that in game  $G_1$ ,  $H(m, c) = H_3(m, c)$ . In game  $G_2$ , if  $H$ -query input  $(m, c)$  satisfies  $g(m) = c$  ( $g(\cdot) = Enc(pk, \cdot; G(\cdot))$ ), the response is replaced by  $H_1^g(m) = H_1 \circ g(m) = H_1(g(m)) = H_1(c)$ . If the function  $g$  is injective, the output distribution of  $H$  is the same as the one in  $G_1$ . Note that  $\mathcal{B}$  cannot distinguish  $g$  from an injective function unless he can find a collision that  $g(m_1) = g(m_2)$  ( $m_1 \neq m_2$ ). A collision implies that  $m_1$ 's or  $m_2$ 's ciphertext

can not be decrypted correctly with the deterministic algorithm  $Dec$ . Define  $E$  as the event that  $\mathcal{B}$  finds a plaintext  $m$  such that  $Dec(sk, g(m)) \neq m$ . Note that  $\mathcal{B}$  can get access to  $g$  at most  $q_G$  times. Thus, in classical ROM, by using the union bound, we can directly obtain that  $\Pr[E] \leq q_G \delta$ . For quantum setting<sup>7</sup>, we define  $g' : \mathcal{M} \rightarrow \{0, 1\}$  such that  $g'(m) = 0$  if  $Dec(sk, g(m)) = m$ , and otherwise  $g'(m) = 1$ . And, if  $\mathcal{B}$  can find a plaintext  $m$  such that  $Dec(sk, g(m)) \neq m$  with at most  $q_G$  quantum queries to  $g$ , we can easily construct another adversary  $\mathcal{B}'$  who can find a plaintext  $m$  such that  $g'(m) = 1$  with at most  $q_G$  quantum queries to  $g'$ . Considering that the PKE scheme is  $\delta$ -correct, we can derive the upper bound of  $\Pr[E]$  by utilizing Lemma 3,  $\Pr[E] \leq 2q_G \sqrt{\delta}$ . Then,

$$|\Pr[G_1^{\mathcal{B}} \Rightarrow 1] - \Pr[G_2^{\mathcal{B}} \Rightarrow 1]| \leq 2q_G \sqrt{\delta}.$$

GAMES $G_0 - G_4$	$H(m, c)$
1: $(pk, sk') \leftarrow Gen'; G \xleftarrow{\$} \Omega_G$	1: <b>if</b> $Enc(pk, m; G(m)) = c$ // $G_2 - G_4$
2: $H_1, H_2 \xleftarrow{\$} \Omega_{H'}; H_3 \xleftarrow{\$} \Omega_H$	2: <b>return</b> $H_1(c)$ // $G_2 - G_4$
3: $m^* \xleftarrow{\$} \mathcal{M}$	3: <b>return</b> $H_3(m, c)$
4: $r^* := G(m^*)$ // $G_0 - G_3$	<u>DECAPS (<math>c \neq c^*</math>) // <math>G_0 - G_2</math></u>
5: $r^* \xleftarrow{\$} \mathcal{R}$ // $G_4$	1: Parse $sk' = (sk, s)$
6: $c^* := Enc(pk, m^*; r^*)$	2: $m' := Dec(sk, c)$
7: $k_0^* := H(m^*, c^*)$	3: <b>if</b> $Enc(pk, m'; G(m')) = c$
8: $k_0^* \xleftarrow{\$} \mathcal{K}$ // $G_4$	4: <b>return</b> $K := H(m', c)$
9: $k_1^* \xleftarrow{\$} \mathcal{K}$	5: <b>else return</b>
10: $b \xleftarrow{\$} \{0, 1\}$	6: $K := H(s, c)$ // $G_0$
11: $b' \leftarrow B^{G, H, DECAPS}(pk, c^*, k_b^*)$	7: $K := H_2(c)$ // $G_1 - G_2$
12: <b>return</b> $b' = ?b$	<u>DECAPS (<math>c \neq c^*</math>) // <math>G_3 - G_4</math></u>
	1: <b>return</b> $K := H_1(c)$

Fig. 5: Games  $G_0$ - $G_4$  for the proof of Theorem 1

GAME  $G_3$ . In game  $G_3$ , the DECAPS oracle is changed that it makes no use of the secret key  $sk'$  any more. When  $\mathcal{B}$  queries the DECAPS oracle on  $c$  ( $c \neq c^*$ ),

<sup>7</sup> In quantum query model, distinguishing the function  $g$  from an injective function is also equivalent to detecting a collision in  $g$ , see [32]. And, finding a collision for  $g$  by making quantum queries is a well-defined and widely-studied problem [33, 34]. Note that a collision implies an incorrect decryption. Therefore, we can bound  $|\Pr[G_1^{\mathcal{B}} \Rightarrow 1] - \Pr[G_2^{\mathcal{B}} \Rightarrow 1]|$  by the probability of finding an incorrect decryption.

$K := H_1(c)$  is returned as the response. Let  $m' := Dec(sk, c)$  and consider the following two cases.

**Case 1:**  $Enc(pk, m'; G(m')) = c$ . In this case,  $H(m', c) = H_1(c)$ . Thus, both DECAPS oracles in games  $G_2$  and  $G_3$  return the same value.

**Case 2:**  $Enc(pk, m'; G(m')) \neq c$ . Random values  $H_2(c)$  and  $H_1(c)$  are returned in game  $G_2$  and game  $G_3$  respectively. In game  $G_2$ ,  $H_2$  is a random function independent of the oracles  $G$  and  $H$ , thus  $H_2(c)$  is uniform at random in  $\mathcal{B}$ 's view. In game  $G_3$ ,  $\mathcal{B}$ 's queries to  $H$  can only help him get access to  $H_1$  at  $\hat{c}$  such that  $Enc(pk, \hat{m}; G(\hat{m})) = \hat{c}$  for some  $\hat{m}$ . Consequently, if  $\mathcal{B}$  can not find a  $m''$  such that  $Enc(pk, m''; G(m'')) = c$ ,  $H_1(c)$  is also a fresh random key just like  $H_2(c)$  in his view. Since  $m'' \neq m'$ , finding such a  $m''$  is exactly the event  $E$ . That is, in this case, if  $E$  does not happen, the output distributions of the DECAPS oracles in  $G_2$  and  $G_3$  are same in  $\mathcal{B}$ 's view.

As a result,  $G_2$  and  $G_3$  only differ when  $E$  happens. Therefore,

$$|\Pr[G_2^{\mathcal{B}} \Rightarrow 1] - \Pr[G_3^{\mathcal{B}} \Rightarrow 1]| \leq \Pr[E] \leq 2q_G \sqrt{\delta}.$$

GAME  $G_4$ . In game  $G_4$ ,  $r^*$  and  $k_0^*$  are chosen uniformly at random from  $\mathcal{R}$  and  $\mathcal{K}$ , respectively. In this game, bit  $b$  is independent from  $\mathcal{B}$ 's view. Hence,

$$\Pr[G_4^{\mathcal{B}} \Rightarrow 1] = \frac{1}{2}.$$

GAMES $G_5$									
1:	$i \xleftarrow{\$} \{1, \dots, q_G + q_H\}, (pk, sk) \leftarrow Gen, G \xleftarrow{\$} \Omega_G$								
2:	$H_1 \xleftarrow{\$} \Omega_{H'}, H_3 \xleftarrow{\$} \Omega_H$								
3:	$m^* \xleftarrow{\$} \mathcal{M}$								
4:	$r^* \xleftarrow{\$} \mathcal{R}$								
5:	$c^* := Enc(pk, m^*; r^*)$								
6:	$k^* \xleftarrow{\$} \mathcal{K}$								
7:	run $B^{G, H, DECAPS}(pk, c^*, k^*)$ until the $i$ -th query to $G \times H_1^q$								
8:	measure the argument $\hat{m}$								
9:	<b>return</b> $\hat{m} = ? m^*$								
<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="text-align: left; border-bottom: 1px solid black; padding: 5px;"><math>H(m, c)</math></th> <th style="text-align: left; border-bottom: 1px solid black; padding: 5px;">DECAPS (<math>c \neq c^*</math>)</th> </tr> </thead> <tbody> <tr> <td style="padding: 5px;">1: <b>if</b> <math>Enc(pk, m; G(m)) = c</math></td> <td style="padding: 5px;">1: <b>return</b> <math>K := H_1(c)</math></td> </tr> <tr> <td style="padding: 5px;">2:     <b>return</b> <math>H_1(c)</math></td> <td></td> </tr> <tr> <td style="padding: 5px;">3: <b>else return</b> <math>H_3(m, c)</math></td> <td></td> </tr> </tbody> </table>		$H(m, c)$	DECAPS ( $c \neq c^*$ )	1: <b>if</b> $Enc(pk, m; G(m)) = c$	1: <b>return</b> $K := H_1(c)$	2: <b>return</b> $H_1(c)$		3: <b>else return</b> $H_3(m, c)$	
$H(m, c)$	DECAPS ( $c \neq c^*$ )								
1: <b>if</b> $Enc(pk, m; G(m)) = c$	1: <b>return</b> $K := H_1(c)$								
2: <b>return</b> $H_1(c)$									
3: <b>else return</b> $H_3(m, c)$									

Fig. 6: Game  $G_5$  for Theorem 1

Note that in this game we reprogram the oracles  $G$  and  $H$  on inputs  $m^*$  and  $(m^*, c^*)$  respectively. Similarly, in classical setting, this will be unnoticed unless the event QUERY that  $\mathcal{B}$  queries  $G$  on  $m^*$  or  $H$  on  $(m^*, c^*)$  happens. Then we can argue that  $G_2$  and  $G_3$  are indistinguishable until QUERY happens. In quantum setting, due to the quantum queries to  $G$  and  $H$ , the case is complicated. Note that  $(m^*, c^*)$  is a valid plaintext-ciphertext pair, i.e.,  $g(m^*) = c^*$ . Therefore,  $H(m^*, c^*) = H_1(c^*) = H_1(g(m^*))$ . Actually, we just reprogram  $G$  and  $H_1^g$  ( $H_1^g(\cdot) = H_1(g(\cdot))$ ) at input  $m^*$ . As a result, we use Lemma 2 to obtain an upper bound for  $|\Pr[G_3^{\mathcal{B}} \Rightarrow 1] - \Pr[G_4^{\mathcal{B}} \Rightarrow 1]|$ .

Use the random oracle  $G \times H_1^g$  (where  $(G \times H_1^g)(m) = (G(m), H_1^g(m))$ ) to answer the queries made by  $\mathcal{B}$  to  $G$  and  $H_1^g$ . Note that  $H_1^g$  and  $H_3$  are internal random oracles that  $\mathcal{B}$  can have access to only by querying the oracle  $H$ . Then, the number of total queries to  $G \times H_1^g$  is at most  $q_G + q_H$ . Define game  $G_5$  as following: pick  $i \xleftarrow{\$} \{1, \dots, q_G + q_H\}$ , run the game  $G_4$  until  $\mathcal{B}$ 's  $i$ -th query to  $G \times H_1^g$ , measure the argument of the query in the computational basis, output the measurement outcome (when  $\mathcal{B}$  makes less than  $i$  queries, output  $\perp$ ). The game  $G_5$  is shown in Fig. 6. Then, by using Lemma 2, we obtain

$$|\Pr[G_3^{\mathcal{B}} \Rightarrow 1] - \Pr[G_4^{\mathcal{B}} \Rightarrow 1]| \leq 2(q_G + q_H) \sqrt{\Pr[G_5^{\mathcal{B}} \Rightarrow 1]}.$$

Next, we construct an adversary  $\mathcal{A}$  against the OW-CPA security of the PKE scheme such that  $\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A}) = \Pr[G_5^{\mathcal{B}} \Rightarrow 1]$ . The adversary  $\mathcal{A}$  on input  $(1^\lambda, pk, c)$  does the following:

1. Run the adversary  $\mathcal{B}$  in Game  $G_5$ .
2. Use a  $2q_G$ -wise independent function and two different  $2q_H$ -wise independent functions to simulate the random oracles  $G$ ,  $H_1$  and  $H_3$  respectively. The random oracle  $H$  is simulated in the same way as the one in game  $G_5$ .
3. Answer the decapsulation queries by using the DECAPS oracle in Fig. 6.
4. Select  $k^* \xleftarrow{\$} \mathcal{K}$  and respond to  $\mathcal{B}$ 's challenge query with  $(c, k^*)$ .
5. Select  $i \xleftarrow{\$} \{1, \dots, q_G + q_H\}$ , measure the argument  $\hat{m}$  of  $i$ -th query to  $G \times H_1^g$  and output  $\hat{m}$ .

According to Lemma 1,  $\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A}) = \Pr[G_5^{\mathcal{B}} \Rightarrow 1]$ . Finally, combing this with the bounds derived above, we can conclude that

$$\text{Adv}_{\text{KEM-I}}^{\text{IND-CCA}}(\mathcal{B}) \leq 2q_H \frac{1}{\sqrt{\mathcal{M}}} + 4q_G \sqrt{\delta} + 2(q_G + q_H) \cdot \sqrt{\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})}.$$

□

**Theorem 2 (PKE OW-CPA  $\stackrel{QROM}{\Rightarrow}$  KEM-II IND-CCA).** *If PKE is  $\delta$ -correct, for any IND-CCA  $\mathcal{B}$  against KEM – II, issuing at most  $q_D$  classical queries to the decapsulation oracle DECAPS and at most  $q_G$  ( $q_H$ ) queries to random oracle  $G$  ( $H$ ), there exists a quantum OW-CPA adversary  $\mathcal{A}$  against PKE and an adversary  $\mathcal{A}'$  against the security of PRF with at most  $q_D$  classical queries such that  $\text{Adv}_{\text{KEM-II}}^{\text{IND-CCA}}(\mathcal{B}) \leq \text{Adv}_{\text{PRF}}(\mathcal{A}') + 4q_G \sqrt{\delta} + 2(q_H + q_G) \cdot \sqrt{\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})}$ .*

*Proof.* Let  $\mathcal{B}$  be an adversary against the IND-CCA security of KEM-II, issuing at most  $q_D$  classical queries to DECAPS, at most  $q_G$  queries to  $G$  and at most  $q_H$  queries to  $H$ . Consider the sequence of games given in Fig. 7. Let  $\Omega_{H''}$  be the set of all functions  $H'' : \mathcal{M} \rightarrow \mathcal{K}$  and we follow the same notations  $\Omega_G$ ,  $\Omega_H$  and  $\Omega_{H'}$  in Theorem 1.

GAMES $G_0 - G_4$	$H(m)$
1 : $(pk, sk') \leftarrow Gen'$	1 : <b>return</b> $H_1(m)$ // $G_0 - G_1$
2 : $G \xleftarrow{\$} \Omega_G, H_1 \xleftarrow{\$} \Omega_{H''}$	2 : $g(\cdot) := Enc(pk, \cdot; G(\cdot))$ // $G_2 - G_4$
3 : $H_2, H_3 \xleftarrow{\$} \Omega_{H'}$	3 : <b>return</b> $H_2(g(m))$ // $G_2 - G_4$
4 : $m^* \xleftarrow{\$} \mathcal{M}$	
5 : $r^* := G(m^*)$ // $G_0 - G_3$	DECAPS ( $c \neq c^*$ ) // $G_0 - G_2$
6 : $r^* \xleftarrow{\$} \mathcal{R}$ // $G_4$	1 : Parse $sk' = (sk, k)$
7 : $c^* := Enc(pk, m^*; r^*)$	2 : $m' := Dec(sk, c)$
8 : $k_0^* := H(m^*)$ // $G_0 - G_3$	3 : <b>if</b> $Enc(pk, m'; G(m')) = c$
9 : $k_0^* \xleftarrow{\$} \mathcal{K}$ // $G_4$	4 : $K := H(m')$
10 : $k_1^* \xleftarrow{\$} \mathcal{K}$	5 : <b>else return</b>
11 : $b \xleftarrow{\$} \{0, 1\}$	6 : <b>return</b> $K := f(k, c)$ // $G_0$
12 : $b' \leftarrow B^{G, H, DECAPS}(pk, c^*, k_b^*)$	7 : <b>return</b> $K := H_3(c)$ // $G_1 - G_2$
13 : <b>return</b> $b' = ? b$	
	DECAPS ( $c \neq c^*$ ) // $G_3 - G_4$
	1 : <b>return</b> $K := H_2(c)$

Fig. 7: Games  $G_0 - G_4$  for the proof of Theorem 2

GAME  $G_0$ . Game  $G_0$  is exactly the IND-CCA game,

$$\left| \Pr[G_0^{\mathcal{B}} \Rightarrow 1] - \frac{1}{2} \right| = \text{Adv}_{\text{KEM-II}}^{\text{IND-CCA}}(\mathcal{B}).$$

GAME  $G_1$ . In game  $G_1$ , the DECAPS oracle is changed that the pseudorandom function  $f$  is replaced by a random function  $H_3$ . Thus, the private key  $k$ , contained in the secret key  $sk'$ , is never used in  $G_1$ . Because  $\mathcal{B}$ 's queries to DECAPS are just classical,  $\mathcal{B}$  can make classical queries to  $f$  at most  $q_D$  times.  $\mathcal{B}$ 's views in  $G_0$  and  $G_1$  are same unless there exists some adversary  $\mathcal{A}'$  who can distinguish  $f$  from the random function  $H_3$  with at most  $q_D$  classical queries to  $f$ . Then,

$$|\Pr[G_0^{\mathcal{B}} \Rightarrow 1] - \Pr[G_1^{\mathcal{B}} \Rightarrow 1]| \leq \text{Adv}_{\text{PRF}}(\mathcal{A}').$$



GAME  $G_2$ . In game  $G_2$ ,  $H_1$  is substituted with  $H_2 \circ g$  ( $g(\cdot) := \text{Enc}(pk, \cdot; G(\cdot))$ ). If the function  $g$  is injective,  $H_2 \circ g$  is a perfect random function. Note that  $\mathcal{B}$  can not distinguish  $g$  from an injective function unless he can find a collision that  $g(m_1) = g(m_2)$  ( $m_1 \neq m_2$ ). A collision implies that the event  $E$  that  $\mathcal{B}$  finds a plaintext  $m$  such that  $\text{Dec}(sk, g(m)) \neq m$  happens. Using the same method in Theorem 1, we obtain  $\Pr[E] \leq 2q_G \sqrt{\delta}$ . Thus,

$$|\Pr[G_1^{\mathcal{B}} \Rightarrow 1] - \Pr[G_2^{\mathcal{B}} \Rightarrow 1]| \leq 2q_G \sqrt{\delta}.$$

GAME  $G_3$ . In game  $G_3$ , the DECAPS oracle is changed that it makes no use of the secret key  $sk'$  any more. When  $\mathcal{B}$  queries the DECAPS oracle on  $c$  ( $c \neq c^*$ ),  $K := H_2(c)$  is returned as the response. Using the same analysis in Theorem 1, we know that  $G_2$  and  $G_3$  only differ when  $E$  happens. Hence,

$$|\Pr[G_2^{\mathcal{B}} \Rightarrow 1] - \Pr[G_3^{\mathcal{B}} \Rightarrow 1]| \leq 2q_G \sqrt{\delta}.$$

GAME  $G_4$ . In game  $G_4$ ,  $r^*$  and  $k_0^*$  are chosen uniformly at random from  $\mathcal{R}$  and  $\mathcal{K}$ , respectively. In this game, bit  $b$  is independent from  $\mathcal{B}$ 's view. Hence,

$$\Pr[G_4^{\mathcal{B}} \Rightarrow 1] = \frac{1}{2}.$$

GAMES $G_5$	
1: $i \xleftarrow{\$} \{1, \dots, q_G + q_H\}$	<u><math>H(m)</math></u>
2: $(pk, sk') \leftarrow \text{Gen}'$	1: $g(\cdot) := \text{Enc}(pk, \cdot; G(\cdot))$
3: $G \xleftarrow{\$} \Omega_G$	2: <b>return</b> $H_2(g(m))$
4: $H_2 \xleftarrow{\$} \Omega'_H$	
5: $m^* \xleftarrow{\$} \mathcal{M}$	<u>DECAPS (<math>c \neq c^*</math>)</u>
6: $r^* \xleftarrow{\$} \mathcal{R}$	1: <b>return</b> $K := H_2(c)$
7: $c^* := \text{Enc}(pk, m^*; r^*)$	
8: $k^* \xleftarrow{\$} \mathcal{K}$	
9: run $B^{G, H, \text{DECAPS}}(pk, c^*, k^*)$	
10: until the $i$ -th query to $G \times H$	
11: measure the argument $\hat{m}$	
12: <b>return</b> $\hat{m} =? m^*$	

Fig. 8: Game  $G_5$  for Theorem 2

Next, we use Lemma 2 to obtain an upper bound for  $|\Pr[G_3^{\mathcal{B}} \Rightarrow 1] - \Pr[G_4^{\mathcal{B}} \Rightarrow 1]|$ . Use the random oracle  $G \times H$  (where  $(G \times H)(m) = (G(m), H(m))$ ) to answer

the queries made by  $\mathcal{B}$  to  $G$  and  $H$ . Then, the number of total queries to  $G \times H$  is at most  $q_G + q_H$ . Define game  $G_5$  as following: pick  $i \xleftarrow{\$} \{1, \dots, q_G + q_H\}$ , run the game  $G_4$  until  $\mathcal{B}$ 's  $i$ -th query to  $G \times H$ , measure the argument of the query in the computational basis, output the measurement outcome (when  $\mathcal{B}$  makes less than  $i$  queries, output  $\perp$ ). The game  $G_5$  is shown in Fig. 8. Then, by using Lemma 2, we obtain

$$|\Pr[G_3^{\mathcal{B}} \Rightarrow 1] - \Pr[G_4^{\mathcal{B}} \Rightarrow 1]| \leq 2(q_G + q_H) \sqrt{\Pr[G_5^{\mathcal{B}} \Rightarrow 1]}.$$

Then, we construct an adversary  $\mathcal{A}$  against the OW-CPA security of PKE such that  $\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A}) = \Pr[G_5^{\mathcal{B}} \Rightarrow 1]$ . The adversary  $\mathcal{A}$  on input  $(1^\lambda, pk, c)$  does the following:

1. Run the adversary  $\mathcal{B}$  in game  $G_5$ .
2. Use a  $2q_G$ -wise independent function and a  $2q_H$ -wise independent function to simulate random oracles  $G$  and  $H_2$  respectively. The random oracle  $H$  is simulated by  $H_2 \circ g$ . Use  $G \times H$  to answer  $\mathcal{B}$ 's queries to both  $G$  and  $H$ .
3. Answer the decapsulation queries by using the DECAPS oracle in Fig. 8.
4. Select  $k^* \xleftarrow{\$} \mathcal{K}$  and respond to  $\mathcal{B}$ 's challenge query with  $(c, k^*)$ .
5. Select  $i \xleftarrow{\$} \{1, \dots, q_G + q_H\}$ , measure the argument  $\hat{m}$  of the  $i$ -th query to  $G \times H$  and output  $\hat{m}$ .

It is obvious that  $\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A}) = \Pr[G_5^{\mathcal{B}} \Rightarrow 1]$ . Combing this with the bounds derived above, we can conclude that

$$\text{Adv}_{\text{KEM-II}}^{\text{IND-CCA}}(\mathcal{B}) \leq \text{Adv}_{\text{PRF}}(\mathcal{A}') + 4q_G \cdot \sqrt{\delta} + 2(q_H + q_G) \cdot \sqrt{\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})}.$$

□

*Remark.* For the reduction from the IND-CCA security of KEM to the OW-CPA security of PKE, we inevitably reprogram the quantum random oracles  $G$  and  $H$ . Lemma 2 (one-way to hiding, O2H) is a practical tool to argue the indistinguishability between games where the random oracles are reprogrammed. [16] analyzed the QROM security of  $QFO_m^{\neq}$  (a Targhi-Unruh variant of FO-II) by two steps. First, they presented a QROM security reduction from the OW-PCA security of an intermediate scheme  $\text{PKE}'$  to the OW-CPA security of the underlying PKE. In this step, the random oracle  $G$  was reprogrammed, thus by using the O2H lemma they obtained<sup>8</sup> that  $\text{Adv}_{\text{PKE}'}^{\text{OW-PCA}}(\mathcal{C}) \leq q \sqrt{\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})}$ . In the second step, they reduced the IND-CCA security of KEM to OW-PCA security of  $\text{PKE}'$ , where the random oracles  $H$  and  $H'$  (the additional hash) were reprogrammed. Again, by

<sup>8</sup> The bounds here are informal. Concretely, the negligible terms and constant coefficients are not considered and the numbers of adversarial queries to different oracles are replaced by the total number  $q$  of adversarial queries to various oracles.

using the O2H lemma, they gained  $\text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{B}) \leq q \sqrt{\text{Adv}_{\text{PKE}'}^{\text{OW-PCA}}(\mathcal{C})}$ . Finally, combing above two bounds, they obtained the security bound of KEM,

$$\text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{B}) \leq q^{\frac{3}{2}} \cdot [\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})]^{\frac{1}{4}}. \quad (1)$$

Direct combination of the modular analyses leads to twice utilization of O2H lemma, which makes security bound highly non-tight. In our security reductions for FO-I and FO-II, we just reduce the IND-CCA security of KEM to OW-CPA security of underlying PKE scheme directly without introducing the intermediate scheme  $\text{PKE}'$ . Specifically, the quantum random oracles  $G$  and  $H$  are reprogrammed simultaneously, thus the O2H lemma is used just once in our reductions. Our derived security bound is approximately  $\text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{B}) \leq q \cdot \sqrt{\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})}$ , which is much tighter than the bound (1).

## 4 Modular Analysis of FO transformation in the QROM

In [16], Hofheinz et al. introduced seven modular transformations  $T$ ,  $U^{\times}$ ,  $U^{\perp}$ ,  $U_m^{\times}$ ,  $U_m^{\perp}$ ,  $QU_m^{\times}$  and  $QU_m^{\perp}$ . But, they just presented QROM security reductions for the transformations  $T$ ,  $QU_m^{\times}$  and  $QU_m^{\perp}$ . Different from the transformations  $U^{\times}$ ,  $U^{\perp}$ ,  $U_m^{\times}$  and  $U_m^{\perp}$ , the transformations  $QU_m^{\times}$  and  $QU_m^{\perp}$  have an additional length-preserving hash in the ciphertext, thus they can follow the proof technique in [20, 21] to give QROM security reductions for them. As they pointed [21], their QROM security reductions quite rely on this additional hash. And, QROM security reductions for  $U^{\times}$ ,  $U^{\perp}$ ,  $U_m^{\times}$  and  $U_m^{\perp}$  are missing in [16].

In this section, we revisit the transformations  $U^{\times}$ ,  $U^{\perp}$ ,  $U_m^{\times}$  and  $U_m^{\perp}$ , and argue their QROM security without any modification to the constructions. [16] has shown that the transformation  $T$  can turn OW-CPA-secure PKE into OW-PCA-secure PKE in the QROM. In Section 4.1, we first show that the resulting PKE scheme by applying  $T$  to OW-CPA-secure PKE is also OW-qPCA-secure. The QROM security reduction for  $U^{\times}$  ( $U^{\perp}$ ) from IND-CCA security of KEM to OW-qPCA (OW-qPVCA) security of PKE is given in Section 4.2 (4.3). In Section 4.4, we show that  $U_m^{\times}$  ( $U_m^{\perp}$ ) transforms any OW-CPA-secure (OW-VA-secure) deterministic PKE into an IND-CCA-secure KEM in the QROM.

### 4.1 $T$ : from OW-CPA to OW-qPCA in the QROM

To a public-key encryption  $\text{PKE}=(\text{Gen}, \text{Enc}, \text{Dec})$  with message space  $\mathcal{M}$  and randomness space  $R$ , and a hash function  $G : \mathcal{M} \rightarrow \mathcal{R}$ , we associate  $\text{PKE}' = T[\text{PKE}, G]$ . The algorithms of  $\text{PKE}'=(\text{Gen}, \text{Enc}', \text{Dec}')$  are defined in Fig. 9.

**Theorem 3 (PKE OW-CPA  $\stackrel{\text{QROM}}{\Rightarrow}$  PKE' OW-qPCA).** *If PKE is  $\delta$ -correct, for any OW-qPCA  $\mathcal{B}$  against PKE', issuing at most  $q_G$  quantum queries to the random oracle  $G$  and at most  $q_P$  quantum queries to the plaintext checking oracle  $\text{PCO}$ , there exists an OW-CPA adversary  $\mathcal{A}$  against PKE such that*

$$\text{Adv}_{\text{PKE}'}^{\text{OW-qPCA}}(\mathcal{B}) \leq 2q_G \cdot \sqrt{\delta} + (1 + 2q_G) \cdot \sqrt{\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A})}.$$

The proof is essentially the same as the one of [16, Theorem 4.4] except the argument about the difference in  $\mathcal{B}$ 's success probability between game  $G_0$  and game  $G_1$ . Game  $G_0$  is exactly the original OW-qPCA game. In game  $G_1$ , the PCO oracle is replaced by a simulation that  $Enc(pk, m; G(m)) = ?c$  is returned for the query input  $(m, c)$ . As  $pk$  is public and  $G$  is a quantum random oracle, such a PCO simulation can be queried on a quantum superposition of inputs. Note that  $\mathcal{B}$ 's views in game  $G_0$  and game  $G_1$  are totally identical unless he can find a plaintext  $m$  such that  $Dec(sk, Enc(pk, m, G(m))) \neq m$ . Thus, using Lemma 3, we can obtain that  $|\Pr[G_0^{\mathcal{B}} \Rightarrow 1] - \Pr[G_1^{\mathcal{B}} \Rightarrow 1]| \leq 2q_G \cdot \sqrt{\delta}$ . Then, following the security reduction for [16, Theorem 4.4], we can easily prove Theorem 3.

$Enc'(pk, m)$	$Dec'(sk, c)$
1 : $c = Enc(pk, m; G(m))$	1 : $m' := Dec(sk, c)$
2 : <b>return</b> $c$	2 : <b>if</b> $Enc(pk, m'; G(m')) = c$
	3 : <b>return</b> $m'$
	4 : <b>else return</b> $\perp$

Fig. 9: OW-qPCA-secure  $PKE' = T[PKE, G]$

#### 4.2 $U^{\mathcal{A}}$ : from OW-qPCA to IND-CCA in the QROM

To a public-key encryption  $PKE' = (Gen', Enc', Dec')$  and a hash function  $H$ , we associate  $KEM - III = U^{\mathcal{A}}[PKE', H]$ . The algorithms of  $KEM - III = (Gen, Encaps, Decaps)$  are defined in Fig. 10.

$Gen$	$Encaps(pk)$	$Decaps(sk', c)$
1 : $(pk, sk) \leftarrow Gen'$	1 : $m \xleftarrow{\$} \mathcal{M}$	1 : Parse $sk' = (sk, s)$
2 : $s \xleftarrow{\$} \mathcal{M}$	2 : $c \leftarrow Enc'(pk, m)$	2 : $m' := Dec'(sk, c)$
3 : $sk' := (sk, s)$	3 : $K := H(m, c)$	3 : <b>if</b> $m' = \perp$
4 : <b>return</b> $(pk, sk')$	4 : <b>return</b> $(K, c)$	4 : <b>return</b> $K := H(s, c)$
		5 : <b>else return</b>
		6 : $K := H(m', c)$

Fig. 10: IND-CCA-secure  $KEM - III = U^{\mathcal{A}}[PKE', H]$

**Theorem 4 (PKE' OW-qPCA  $\stackrel{QROM}{\Rightarrow}$  KEM-III IND-CCA).** *If PKE' is  $\delta$ -correct, for any IND-CCA  $\mathcal{B}$  against KEM - III, issuing at most  $q_D$  (classical) queries to the decapsulation oracle DECAPS and at most  $q_H$  queries to*

the quantum random oracle  $H$ , there exists a quantum OW-qPCA adversary  $\mathcal{A}$  against PKE' that makes at most  $q_H$  queries to the PCO oracle such that  $\text{Adv}_{\text{KEM-III}}^{\text{IND-CCA}}(\mathcal{B}) \leq 2q_H \frac{1}{\sqrt{\mathcal{M}}} + 2q_H \cdot \sqrt{\text{Adv}_{\text{PKE}'}^{\text{OW-qPCA}}(\mathcal{A})}$ .

*Proof.* Let  $\mathcal{B}$  be an adversary against the IND-CCA security of KEM-III, issuing at most  $q_D$  queries to DECAPS and at most  $q_H$  queries to  $H$ . We follow the notations  $\Omega_G$ ,  $\Omega_H$  and  $\Omega_{H'}$  in Theorem 1. Consider the games in Fig. 11.

GAMES $G_0 - G_4$	$H(m, c)$
1 : $(pk, sk') \leftarrow \text{Gen}'; G \xleftarrow{\$} \Omega_G$	1 : <b>if</b> $\text{PCO}(m, c) = 1$ // $G_2 - G_4$
2 : $H_1, H_2 \xleftarrow{\$} \Omega_{H'}; H_3 \xleftarrow{\$} \Omega_H$	2 : <b>return</b> $H_1(c)$ // $G_2 - G_4$
3 : $m^* \xleftarrow{\$} \mathcal{M}$	3 : <b>return</b> $H_3(m, c)$
4 : $c^* \leftarrow \text{Enc}(pk, m^*)$	DECAPS ( $c \neq c^*$ ) // $G_0 - G_2$
5 : $k_0^* := H(m^*, c^*)$	1 : Parse $sk' = (sk, s)$
6 : $k_0^* \xleftarrow{\$} \mathcal{K}$ // $G_4$	2 : $m' := \text{Dec}'(sk, c)$
7 : $k_1^* \xleftarrow{\$} \mathcal{K}$	3 : <b>if</b> $m' \neq \perp$ <b>return</b> $K := H(m', c)$
8 : $b \xleftarrow{\$} \{0, 1\}$	4 : <b>else return</b>
9 : $b' \leftarrow B^{G, H, \text{DECAPS}}(pk, c^*, k_b^*)$	5 : $K := H(s, c)$ // $G_0$
10 : <b>return</b> $b' =? b$	6 : $K := H_2(c)$ // $G_1 - G_2$
	DECAPS ( $c \neq c^*$ ) // $G_3 - G_4$
	1 : <b>return</b> $K := H_1(c)$

Fig. 11: Games  $G_0$ - $G_4$  for the proof of Theorem 4

GAME  $G_0$ . Since game  $G_0$  is exactly the IND-CCA game,

$$\left| \Pr[G_0^{\mathcal{B}} \Rightarrow 1] - \frac{1}{2} \right| = \text{Adv}_{\text{KEM-III}}^{\text{IND-CCA}}(\mathcal{B}).$$

GAME  $G_1$ . In game  $G_1$ , the DECAPS oracle is changed that  $H_2(c)$  is returned instead of  $H(s, c)$  for the invalid encapsulation  $c$ . Considering that  $\mathcal{B}$ 's view is independent from (the uniform secret)  $s$ , we can use Lemma 4 to obtain

$$|\Pr[G_0^{\mathcal{B}} \Rightarrow 1] - \Pr[G_1^{\mathcal{B}} \Rightarrow 1]| \leq 2q_H \cdot \frac{1}{\sqrt{\mathcal{M}}}.$$

GAME  $G_2$ . In game  $G_2$ ,  $H$  is changes that  $H_1(c)$  is returned instead of  $H_3(m, c)$  when  $(m, c)$  satisfies  $\text{PCO}(m, c) = 1$  (i.e.,  $\text{Dec}'(sk, c) = m$ ). Note that it is impossible that  $\text{PCO}(m_1, c) = \text{PCO}(m_2, c) = 1$  for  $m_1 \neq m_2$  because  $\text{Dec}'$  is

a deterministic algorithm. Further, as  $H_1$  is a random function independent of  $H_3$ ,  $H$  in game  $G_2$  is also a uniformly random function like the one in game  $G_1$ . Thus,

$$\Pr[G_1^{\mathcal{B}} \Rightarrow 1] = \Pr[G_2^{\mathcal{B}} \Rightarrow 1].$$

**GAME  $G_3$ .** In game  $G_3$ , the DECAPS oracle is changed that it makes no use of the secret key  $sk'$  any more. When  $\mathcal{B}$  queries the DECAPS oracle on  $c$  ( $c \neq c^*$ ),  $K := H_1(c)$  is returned as the response. In order to show that the output distributions of DECAPS are identical in games  $G_2$  and  $G_3$ , we consider the following cases for a fixed ciphertext  $c$  and  $m' := Dec'(sk, c)$ .

**Case 1:**  $m' \neq \perp$ . Note that  $H(m', c) = H_1(c)$  on account of  $PCO(m', c) = 1$ . Therefore, the two DECAPS oracles in games  $G_2$  and  $G_3$  return the same value.

**Case 2:**  $m' = \perp$ . Random values  $H_2(c)$  and  $H_1(c)$  in  $\mathcal{K}$  are returned in games  $G_2$  and  $G_3$ , respectively. In game  $G_2$ ,  $H_2$  is a random function independent of  $G$  and  $H$ . In game  $G_3$ ,  $\mathcal{B}$ 's queries to  $H$  can only help him get access to  $H_1$  at  $c$  such that  $Dec'(sk, c) = \hat{m}$  for some  $\hat{m} \neq \perp$ . Therefore,  $\mathcal{B}$  never sees  $H_1(c)$  by querying  $G$  and  $H$ . Hence, in  $\mathcal{B}$ 's view,  $H_1(c)$  is totally uniform at random like  $H_2(c)$ . As a result, the DECAPS oracle in game  $G_3$  has the same output distribution as the one in game  $G_2$ .

We have shown that  $\mathcal{B}$ 's views are identical in both games and

$$\Pr[G_2^{\mathcal{B}} \Rightarrow 1] = \Pr[G_3^{\mathcal{B}} \Rightarrow 1].$$

**GAME  $G_4$ .** In game  $G_4$ ,  $k_0^*$  is chosen uniformly at random from  $\mathcal{K}$ . In this game, bit  $b$  is independent from  $\mathcal{B}$ 's view. Hence,

$$\Pr[G_4^{\mathcal{B}} \Rightarrow 1] = \frac{1}{2}.$$

Next, we use Lemma 2 to bound  $|\Pr[G_3^{\mathcal{B}} \Rightarrow 1] - \Pr[G_4^{\mathcal{B}} \Rightarrow 1]|$ . Define game  $G_5$  as following: pick  $i \xleftarrow{\$} \{1, \dots, q_H\}$ , run the game  $G_4$  until  $\mathcal{B}$ 's  $i$ -th query to  $H$ , measure the argument of the query in the computational basis, output the measurement outcome (when  $\mathcal{B}$  makes less than  $i$  queries, output  $\perp$ ). The game  $G_5$  is shown in Fig. 12. Then, by using Lemma 2, we obtain

$$|\Pr[G_3^{\mathcal{B}} \Rightarrow 1] - \Pr[G_4^{\mathcal{B}} \Rightarrow 1]| \leq 2q_H \sqrt{\Pr[G_5^{\mathcal{B}} \Rightarrow 1]}.$$

GAMES $G_5$									
1 :	$i \xleftarrow{\$} \{1, \dots, q_G + q_H\}, (pk, sk) \leftarrow Gen$								
2 :	$H_1 \xleftarrow{\$} \Omega_{H'}, H_3 \xleftarrow{\$} \Omega_H$								
3 :	$m^* \xleftarrow{\$} \mathcal{M}$								
4 :	$c^* \leftarrow Enc(pk, m^*)$								
5 :	$k^* \xleftarrow{\$} \mathcal{K}$								
6 :	run $B^{G, H, \text{DECAPS}}(pk, c^*, k^*)$ until the $i$ -th query to $H$								
7 :	measure the argument $\hat{m} \parallel \hat{c}$								
8 :	<b>return</b> $\hat{m} =? m^* \wedge \hat{c} =? c^*$								
<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="text-align: left; border-bottom: 1px solid black; padding: 5px;"><math>H(m, c)</math></th> <th style="text-align: left; border-bottom: 1px solid black; padding: 5px;">DECAPS (<math>c \neq c^*</math>)</th> </tr> </thead> <tbody> <tr> <td style="padding: 5px;">1 : <b>if</b> PCO(<math>m, c</math>) = 1</td> <td style="padding: 5px;">1 : <b>return</b> <math>K := H_1(c)</math></td> </tr> <tr> <td style="padding: 5px;">2 :     <b>return</b> <math>H_1(c)</math></td> <td></td> </tr> <tr> <td style="padding: 5px;">3 : <b>else return</b> <math>H_3(m, c)</math></td> <td></td> </tr> </tbody> </table>		$H(m, c)$	DECAPS ( $c \neq c^*$ )	1 : <b>if</b> PCO( $m, c$ ) = 1	1 : <b>return</b> $K := H_1(c)$	2 : <b>return</b> $H_1(c)$		3 : <b>else return</b> $H_3(m, c)$	
$H(m, c)$	DECAPS ( $c \neq c^*$ )								
1 : <b>if</b> PCO( $m, c$ ) = 1	1 : <b>return</b> $K := H_1(c)$								
2 : <b>return</b> $H_1(c)$									
3 : <b>else return</b> $H_3(m, c)$									

Fig. 12: Game  $G_5$  for Theorem 4

Then, we construct an adversary  $\mathcal{A}$  against the OW-qPCA security of the PKE' scheme such that  $\text{Adv}_{\text{PKE}'}^{\text{OW-qPCA}}(\mathcal{A}) = \Pr[G_5^{\mathcal{B}} \Rightarrow 1]$ . The adversary  $\mathcal{A}$  on input  $(1^\lambda, pk, c)$  does the following:

1. Run the adversary  $\mathcal{B}$  in game  $G_5$ .
2. Use two different  $2q_H$ -wise independent functions to simulate the random oracles  $H_1$  and  $H_3$  respectively. The random oracle  $H$  is simulated<sup>9</sup> in the same way as the one in game  $G_5$ .
3. Answer the decapsulation queries by using the DECAPS oracle in Fig. 12.
4. Select  $k^* \xleftarrow{\$} \mathcal{K}$  and respond to  $\mathcal{B}$ 's challenge query with  $(c, k^*)$ .
5. Select  $i \xleftarrow{\$} \{1, \dots, q_H\}$ , measure the argument  $\hat{m} \parallel \hat{c}$  of the  $i$ -th query to  $H$  and output  $\hat{m}$ .

According to Lemma 1,  $\text{Adv}_{\text{PKE}'}^{\text{OW-qPCA}}(\mathcal{A}) = \Pr[G_5^{\mathcal{B}} \Rightarrow 1]$ . Finally, combing this with the bounds derived above, we can conclude that

$$\text{Adv}_{\text{KEM-III}}^{\text{IND-CCA}}(\mathcal{B}) \leq 2q_H \frac{1}{\sqrt{\mathcal{M}}} + 2q_H \cdot \sqrt{\text{Adv}_{\text{PKE}'}^{\text{OW-qPCA}}(\mathcal{A})}.$$

□

<sup>9</sup> To simulate the quantum random oracle  $H$ , we need to make quantum queries to the PCO oracle. This is the reason why we require the scheme PKE' to be OW-qPCA-secure.

### 4.3 $U^\perp$ : from OW-qPVCA to IND-CCA in the QROM

To a public-key encryption  $\text{PKE}' = (\text{Gen}', \text{Enc}', \text{Dec}')$  and a hash function  $H$ , we associate  $\text{KEM-IV} = U^\perp[\text{PKE}', H]$ . We remark that  $U^\perp$  is essentially the transformation [15, Table 2], a KEM variant of the REACT/GEM transformations [35, 36]. The algorithms of  $\text{KEM-IV} = (\text{Gen}, \text{Encaps}, \text{Decaps}^\perp)$  are defined in Fig. 13.

$\text{Gen}$	$\text{Encaps}(pk)$	$\text{Decaps}^\perp(sk, c)$
1: $(pk, sk) \leftarrow \text{Gen}'$	1: $m \xrightarrow{\$} \mathcal{M}$	1: $m' := \text{Dec}'(sk, c)$
2: <b>return</b> $(pk, sk)$	2: $c \leftarrow \text{Enc}'(pk, m)$	2: <b>if</b> $m' = \perp$
	3: $K := H(m, c)$	3: <b>return</b> $\perp$
	4: <b>return</b> $(K, c)$	4: <b>else return</b>
		5: $K := H(m', c)$

Fig. 13: IND-CCA-secure  $\text{KEM-IV} = U^\perp[\text{PKE}', H]$

**Theorem 5 (PKE' OW-qPVCA  $\xrightarrow{\text{QROM}}$  KEM-IV IND-CCA).** *If PKE' is  $\delta$ -correct, for any IND-CCA  $\mathcal{B}$  against KEM-IV, issuing at most  $q_D$  (classical) queries to the decapsulation oracle DECAPS and at most  $q_H$  queries to the quantum random oracle  $H$ , there exists an OW-qPVCA adversary  $\mathcal{A}$  against PKE' that makes at most  $q_H$  queries to the PCO oracle and at most  $q_D$  queries to the VAL oracle such that  $\text{Adv}_{\text{KEM-IV}}^{\text{IND-CCA}}(\mathcal{B}) \leq 2q_H \cdot \sqrt{\text{Adv}_{\text{PKE}'}^{\text{OW-qPVCA}}(\mathcal{A})}$ .*

The only difference between KEM-III and KEM-IV is the response to the invalid ciphertext in the decapsulation algorithm. When the ciphertext  $c$  is invalid, the decapsulation algorithm in KEM-III returns a random key related to  $c$ . In this way, whatever the ciphertext (valid or invalid) is submitted, the return values have the same distribution. As a result,  $\mathcal{A}$  can easily simulate the decapsulation oracle DECAPS without recognition of the invalid ciphertexts. While the decapsulation algorithm in KEM-IV returns  $\perp$  when the submitted  $c$  is invalid. Thus, in order to simulate DECAPS,  $\mathcal{A}$  needs to judge if the ciphertext  $c$  is valid. As we assume that the scheme PKE' is OW-qPVCA-secure,  $\mathcal{A}$  can query the VAL oracle to fulfill such a judgement. Then, it is easy to verify that by using the same proof method in Theorem 4 we can obtain the desired security bound.

### 4.4 $U_m^\perp/U_m^\perp$ : from OW-CPA/OW-VA to IND-CCA for Deterministic Encryption in the QROM

The transformation  $U_m^\perp(U_m^\perp)$  is a variant of  $U^\perp(U^\perp)$  that derives the KEM key as  $K = H(m)$  instead of  $K = H(m, c)$ . To a deterministic public-key encryption scheme  $\text{PKE}' = (\text{Gen}', \text{Enc}', \text{Dec}')$  with message space  $\mathcal{M}$ , a hash function



$H : \mathcal{M} \rightarrow \mathcal{K}$ , and a pseudorandom function  $f$  with key space  $\mathcal{K}^{prf}$ , we associate  $\text{KEM-V} = U_m^\neq[\text{PKE}', H, f]$  and  $\text{KEM-VI} = U_m^\perp[\text{PKE}', H]$  shown in Fig. 14 and Fig. 15, respectively.

We note that for a deterministic PKE scheme the OW-PCA security is equivalent to the OW-CPA security as we can simulate the PCO oracle via re-encryption during the proof. Thus, combing the proofs of Theorem 2, Theorem 4 and Theorem 5, we can easily obtain the following two theorems.

<i>Gen</i>	<i>Encaps(pk)</i>	<i>Decaps(sk', c)</i>
1: $(pk, sk) \leftarrow \text{Gen}'$	1: $m \xleftarrow{\$} \mathcal{M}$	1: Parse $sk' = (sk, k)$
2: $k \xleftarrow{\$} \mathcal{K}^{prf}$	2: $c := \text{Enc}'(pk, m)$	2: $m' := \text{Dec}'(sk, c)$
3: $sk' := (sk, k)$	3: $K := H(m)$	3: <b>if</b> $m' \neq \perp$
4: <b>return</b> $(pk, sk')$	4: <b>return</b> $(K, c)$	4: <b>return</b> $K := H(m')$
		5: <b>else return</b>
		6: $K := f(k, c)$

Fig. 14: IND-CCA-secure  $\text{KEM-V} = U_m^\neq[\text{PKE}', H, f]$

<i>Gen</i>	<i>Encaps(pk)</i>	<i>Decaps(sk, c)</i>
1: $(pk, sk) \leftarrow \text{Gen}'$	1: $m \xleftarrow{\$} \mathcal{M}$	1: $m' := \text{Dec}(sk, c)$
2: <b>return</b> $(pk, sk)$	2: $c := \text{Enc}'(pk, m)$	2: <b>if</b> $m' \neq \perp$
	3: $K := H(m)$	3: <b>return</b> $K := H(m')$
	4: <b>return</b> $(K, c)$	4: <b>else return</b> $\perp$

Fig. 15: IND-CCA-secure  $\text{KEM-VI} = U_m^\perp[\text{PKE}', H]$

**Theorem 6 (PKE' OW-CPA  $\xrightarrow{QROM}$  KEM-V IND-CCA).** *If PKE' is  $\delta$ -correct and deterministic, for any IND-CCA  $\mathcal{B}$  against KEM-V, issuing at most  $q_E$  quantum queries to the encryption oracle<sup>10</sup>, at most  $q_D$  (classical) queries to the decapsulation oracle DECAPS and at most  $q_H$  quantum queries to the random oracle  $H$ , there exists a quantum OW-CPA adversary  $\mathcal{A}$  against PKE' and an adversary  $\mathcal{A}'$  against the security of PRF with at most  $q_D$  classical queries such that  $\text{Adv}_{\text{KEM-V}}^{\text{IND-CCA}}(\mathcal{B}) \leq \text{Adv}_{\text{PRF}}(\mathcal{A}') + 4q_E\sqrt{\delta} + 2q_H \cdot \sqrt{\text{Adv}_{\text{PKE}'}^{\text{OW-CPA}}(\mathcal{A})}$ .*

<sup>10</sup> For the deterministic scheme PKE', given public key  $pk$ , quantum adversary  $\mathcal{B}$  can execute the encryption algorithm  $\text{Enc}'$  in a quantum computer.

**Theorem 7 (PKE' OW-VA  $\stackrel{QROM}{\Rightarrow}$  KEM-VI IND-CCA).** *If PKE' is  $\delta$ -correct and deterministic, for any IND-CCA  $\mathcal{B}$  against KEM – VI, issuing at most  $q_E$  quantum queries to the encryption oracle, at most  $q_D$  (classical) queries to the decapsulation oracle DECAPS and at most  $q_H$  quantum queries to the random oracle  $H$ , there exists a quantum OW-VA adversary  $\mathcal{A}$  against PKE' who makes at most  $q_D$  queries to the VAL oracle such that  $\text{Adv}_{\text{KEM-VI}}^{\text{IND-CCA}}(\mathcal{B}) \leq 2q_E\sqrt{\delta} + 2q_H \cdot \sqrt{\text{Adv}_{\text{PKE}'}^{\text{OW-VA}}(\mathcal{A})}$ .*

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## A Quantum Computation

We give a short introduction to quantum computation. For a more thorough discussion, please see [37].

A quantum system  $A$  is a complex Hilbert space  $\mathcal{H}$  with an inner product  $\langle \cdot | \cdot \rangle$ . The state of a quantum system is given by a vector  $|\Psi\rangle$  of unit norm ( $\langle \Psi | \Psi \rangle = 1$ ). Given quantum systems  $A$  and  $B$  over spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively, we define the joint or composite quantum system through the tensor product  $\mathcal{H}_A \otimes \mathcal{H}_B$ . The product state of  $|\varphi_A\rangle \in \mathcal{H}_A$  and  $|\varphi_B\rangle \in \mathcal{H}_B$  is denoted by  $|\varphi_A\rangle \otimes |\varphi_B\rangle$  or simply  $|\varphi_A\rangle |\varphi_B\rangle$ . A  $n$ -qubit system lives in the joint quantum system of  $n$  two-dimensional Hilbert spaces. The standard orthonormal computational basis  $B = \{|x\rangle\}$  for such a system is given by  $|x_1\rangle \otimes \cdots \otimes |x_n\rangle$  for  $x = x_1 \cdots x_n$ . Any (classical) bit string  $x$  is encoded into a quantum state by  $|x\rangle$ . Denote  $TD(|\Psi\rangle, |\varphi\rangle)$  as the trace distance between quantum states  $|\Psi\rangle$  and  $|\varphi\rangle$ .

*Quantum measurement.* Given a state  $|\varphi\rangle$ , we can measure  $|\varphi\rangle$  in the basis  $B$ , obtaining the value  $x$  with probability  $|\langle x | \varphi \rangle|^2$ . Thus, to each  $|\varphi\rangle$ , we associate a distribution  $D_\varphi$  where  $D_\varphi(x) = |\langle x | \varphi \rangle|^2$ . The normalization constant and the fact that  $B$  is an orthonormal basis ensure that  $D_\varphi$  is exactly a valid distribution. After measurement, the system is in state  $|x\rangle$ .

*Quantum algorithm.* A quantum algorithm  $A$  over a Hilbert space  $\mathcal{H}$  with a standard orthonormal basis  $B$  is specified by unitary transformation  $U$ . The input to  $A$  is the initial state  $|x_0\rangle$ . Then  $U$  is applied to the system, and the final state is obtained  $|\varphi\rangle = U|x_0\rangle$ . At last,  $A$ 's output is obtained by performing a measurement on  $|\varphi\rangle$ .

Quantum algorithm usually operates on a product space  $S \otimes K \otimes V$ , where  $S$  represents the work space,  $K$  the input space, and  $V$  the output space. Given a function  $H : K \rightarrow V$ , define the standard orthonormal basis  $B$  as the set  $|s, k, v\rangle$  for  $s \in S$ ,  $k \in K$ , and  $v \in V$ . Define the unitary transformation  $O_H$  over the Hilbert space spanned by  $B$  as the transformation that takes  $|s, k, v\rangle$  into  $|s, k, v \oplus H(k)\rangle$ .  $O_H$  is unitary, its own inverse, and Hermitian.

A quantum algorithm  $A$  making  $q$  quantum queries to  $H$  is then specified by a sequence of unitary transformations  $U_0, \dots, U_q$ . The evaluation of  $A$  then consists of alternately applying  $U_i$  and  $O_H$  to the initial state  $U_0|x_0\rangle$ . The final state of the algorithm is

$$U_q O_H \dots U_1 O_H U_0 |x_0\rangle.$$

We say that a quantum algorithm is efficient if  $q$  is a polynomial, and all the  $U_i$ s are composed of a polynomial number of universal basis gates (the Hadamard, CNOT, and phase shift gates are commonly used).

## B Proof of Lemma 4

*Proof.* Assume that  $A$  uses three quantum systems  $S$ ,  $K$  and  $V$  for its state, oracle input and oracle output, where  $K$  has two subsystems  $K = K_1 \otimes K_2$ .  $K_1$ ,  $K_2$  and  $V$  have  $n_1$ ,  $n_2$  and  $m$  qubits respectively. Then an execution of  $A$  leads to the final state  $(UO_H)^q |\Psi_{xH'}\rangle$ , where  $|\Psi_{xH'}\rangle$  is the initial state,  $O_H |s, k_1 \otimes k_2, v\rangle := |s, k_1 \otimes k_2, v \oplus H(k_1, k_2)\rangle$ , and  $U$  is  $A$ 's state transition operation. We assume that all the transition operations  $U_i$  are identical and equal to  $U$  (the proof in the general case is essentially identical).  $A$ 's output is produced by applying a measurement  $M$  to  $A$ 's final state.

Define  $|\Psi_{HxH'}^i\rangle := (UO_H)^i |\Psi_{xH'}\rangle$ . Then, we can obtain

$$\Pr[E_1] = \sum_{HxH'} \alpha b_{HxH'},$$

where  $b_{HxH'} = \Pr[M \text{ outputs } 1 \text{ on state } |\Psi_{HxH'}^q\rangle]$ ,  $\alpha = 2^{-m2^{(n_1+n_2)} - n_1 - m2^{n_2}}$ .

Reprogram  $H$  at  $(x, \cdot)$ . Denote  $H_{xH'}$  as the function that  $H_{xH'}(x, \cdot) = H'(\cdot)$  and  $H_{xH'} = H$  everywhere else. Thus,

$$\Pr[E_2] = \sum_{HxH'} \alpha b_{H_{xH'}xH'}.$$

According to [37, Theorem 9.1], we know that

$$|\Pr[E_1] - \Pr[E_2]| \leq \sum_{HxH'} \alpha |b_{HxH'} - b_{H_{xH'}xH'}| \leq \sum_{HxH'} \alpha D_q, \quad (2)$$

where  $D_i := TD(|\Psi_{HxH'}^i\rangle, |\Psi_{HxH'xH'}^i\rangle)$  is the trace distance between quantum states  $|\Psi_{HxH'}^i\rangle$  and  $|\Psi_{HxH'xH'}^i\rangle$ .

Note that  $D_0 = TD(|\Psi_{xH'}\rangle, |\Psi_{xH'}\rangle) = 0$  and

$$\begin{aligned} D_i &= TD(UO_H|\Psi_{HxH'}^{i-1}\rangle, UO_{HxH'}|\Psi_{HxH'xH'}^{i-1}\rangle) \\ &\leq TD(UO_H|\Psi_{HxH'}^{i-1}\rangle, UO_{HxH'}|\Psi_{HxH'}^{i-1}\rangle) + TD(UO_{HxH'}|\Psi_{HxH'}^{i-1}\rangle, UO_{HxH'}|\Psi_{HxH'xH'}^{i-1}\rangle) \\ &\leq D_{i-1} + TD(O_H|\Psi_{HxH'}^{i-1}\rangle, O_{HxH'}|\Psi_{HxH'}^{i-1}\rangle). \end{aligned}$$

Hence,

$$D_q \leq \sum_{i=1}^q TD(O_H|\Psi_{HxH'}^{i-1}\rangle, O_{HxH'}|\Psi_{HxH'}^{i-1}\rangle) \quad (3)$$

Let  $O_{H'}|a, k_1 \otimes k_2, v\rangle := |a, k_1 \otimes k_2, v \oplus H'(k_2)\rangle$ .  $Q_x$  is the projector projecting  $K_1$  onto  $|x\rangle$  (i.e.,  $Q_x = I \otimes |x\rangle\langle x| \otimes I$ ). Then,  $O_{HxH'} = O_H(1 - Q_x) + O_{H'}Q_x$ . By using [38, Lemma 12], we can get that

$$\begin{aligned} &TD(O_H|\Psi_{HxH'}^{i-1}\rangle, O_{HxH'}|\Psi_{HxH'}^{i-1}\rangle) \\ &= TD(O_H(1 - Q_x)|\Psi_{HxH'}^{i-1}\rangle + O_HQ_x|\Psi_{HxH'}^{i-1}\rangle, O_H(1 - Q_x)|\Psi_{HxH'}^{i-1}\rangle + O_{H'}Q_x|\Psi_{HxH'}^{i-1}\rangle) \\ &\leq 2\|O_HQ_x|\Psi_{HxH'}^{i-1}\rangle\| = 2\|Q_x|\Psi_{HxH'}^{i-1}\rangle\|. \end{aligned} \quad (4)$$

Combing the equations (2, 3, 4), we obtain that

$$|\Pr[E_1] - \Pr[E_2]| \leq \sum_{HxH'i} 2\alpha \|Q_x|\Psi_{HxH'}^{i-1}\rangle\| \stackrel{(*)}{\leq} 2q \sqrt{\sum_{HxH'i} \frac{\alpha}{q} \|Q_x|\Psi_{HxH'}^{i-1}\rangle\|^2},$$

where (\*) uses Jensen's inequality.

Define algorithm  $B$  as follows: pick  $i \stackrel{\$}{\leftarrow} \{1, \dots, q\}$ , measure the quantum system  $K_1$  of  $A$ 's  $i$ -th query state  $|\Psi_{HxH'}^{i-1}\rangle$ , obtain  $\hat{x}$  and output  $\hat{x} = ?x$ . Thus,  $\Pr[B \Rightarrow 1]$  is exactly  $\sum_{HxH'i} \frac{\alpha}{q} \|Q_x|\Psi_{HxH'}^{i-1}\rangle\|^2$ . Because  $x$  is chosen uniformly at random and independent from  $A$ 's view,  $\Pr[B \Rightarrow 1] = \frac{1}{2^{n_1}}$ . Therefore,

$$|\Pr[E_1] - \Pr[E_2]| \leq 2q \frac{1}{\sqrt{2^{n_1}}}.$$

□