The Discrete-Logarithm Problem with Preprocessing

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Abstract. This paper studies discrete-log algorithms that use *preprocessing*. In our model, an adversary may use a very large amount of precomputation to produce an "advice" string about a specific group (e.g., NIST P-256). In a subsequent online phase, the adversary's task is to use the preprocessed advice to quickly compute discrete logarithms in the group. Motivated by surprising recent preprocessing attacks on the discrete-log problem, we study the power and limits of such algorithms.

In particular, we focus on *generic* algorithms—these are algorithms that operate in every cyclic group. We show that any generic discrete-log algorithm with preprocessing that uses an S-bit advice string, runs in online time T, and succeeds with probability ϵ in a group of prime order N must satisfy $ST^2 = \widetilde{\Omega}(\epsilon N)$. Our lower bound, which is tight up to logarithmic factors, uses a synthesis of incompressibility techniques and classic methods for generic-group lower bounds. We apply our techniques to prove related lower bounds for the CDH, DDH, and multiple-discrete-log problems.

Finally, we demonstrate two new generic preprocessing attacks: one for the multiple-discrete-log problem and one for certain decisional-type problems in groups. This latter result demonstrates that, for generic algorithms with preprocessing, distinguishing tuples of the form $(g, g^x, g^{(x^2)})$ from random is much easier than the discrete-log problem.

1 Introduction

The problem of computing discrete logarithms in groups is fundamental to cryptography: it underpins the security of widespread cryptographic protocols for key exchange [27], public-key encryption [22,30], and digital signatures [40,47,62].

In the absence of an unconditional proof that computing discrete logarithms is hard, one fruitful research direction has focused on understanding the hardness of these problems against certain restricted classes of algorithms [6,54,66]. In particular, Shoup considered discrete-log algorithms that are *generic*, in the sense they only use the group operation as a black box [66]. Generic algorithms are useful in practice since they apply to every group. In addition, lower bounds against generic algorithms are meaningful because, in popular elliptic-curve groups, generic attacks are the best known [34,45].

The traditional notion of generic algorithms models *online-only attacks*, in which the adversary simultaneously receives the description of a cyclic group

$\mathbb{G} = \langle g \rangle$ and a problem instance $g^x \in \mathbb{G}$. In this model, when the attack algorithm begins executing, the attacker has no *a priori* information about the group $\mathbb{G}$. Shoup [66] showed that, in this online-only setting, every generic discrete-log algorithm that succeeds with good probability in a group of prime order N must run in time at least $N^{1/2}$.

In practice, however, an adversary may have access to the description of the group $\mathbb{G}$ long before it has to solve a discrete-log problem instance. In particular, the vast majority of real-world cryptosystems use one of a handful of groups, such as NIST P-256, Curve25519 [10], and the DSA groups. In this setting, a real-world adversary could potentially perform a *preprocessing attack* [24, 28, 39] relative to a popular group: In an offline phase, the adversary would compute and store a data structure ("advice string") that depends on the group $\mathbb{G}$. In a subsequent online phase, the adversary could use its precomputed advice to solve the discrete-log problem in the group $\mathbb{G}$ much more quickly than would be possible in an online-only attack.

In recent work, Mihalcik [51] and Bernstein and Lange [11] demonstrated the surprising power of preprocessing attacks against the discrete-log problem. In particular, they construct a *generic* algorithm with preprocessing that computes discrete logarithms in every group of order N using $N^{1/3}$ bits of group-specific advice and roughly $N^{1/3}$ online time. Since their algorithm is generic, it applies to every group, including popular elliptic-curve groups. In contrast, Shoup's result shows that, without preprocessing, every generic discrete-log algorithm requires at least $N^{1/2}$ time. The careful use of a large amount of preprocessing—roughly $N^{2/3}$ operations—is what allows the attack of Mihalcik, Bernstein, and Lange to circumvent this lower bound.

As of now, there is no reason to believe that the attack of Mihalcik, Bernstein, and Lange is the best possible. For example, we know of no results ruling out a generic attack that uses $N^{1/2}$ precomputation to build an advice string of size $N^{1/8}$, which can be used to compute discrete logs in online time $N^{1/8}$.

The existence of such an attack would—at the very least—shake our confidence in 256-bit elliptic-curve groups. An attacker who wanted to break NIST P-256, for example, could perform a one-time 2^{128} precomputation to compute a 2^{32}-bit advice string. Given this advice string, an attacker could compute discrete logarithms on the P-256 curve in online time 2^{32}. The precomputed advice string would essentially be a "trapdoor" that would allow its holder to compute discrete-logs on the curve in seconds.

The possibility of such devastating discrete-log preprocessing attacks, and the lack of lower-bounds for such algorithms, leads us to ask:

How helpful can preprocessing be to generic discrete-log algorithms?

In this paper, we extend the classic model of generic algorithms to capture preprocessing attacks. To do so, we introduce the notion of *generic algorithms with preprocessing* for computational problems in cryptographic groups. These algorithms make only black-box use of the group operation, but may perform a large number of group operations during a preprocessing phase. Following prior work on preprocessing attacks [24, 28, 31, 39], we measure the complexity of such algorithms by (a) the size of the advice string that the algorithm produces in the preprocessing phase, and (b) the running time of the algorithm's online phase.

These two standard cost metrics do not consider the preprocessing time required to compute the advice string. Ignoring the preprocessing cost only strengthens the resulting lower bounds, but it leaves open the question of how much preprocessing is really necessary to compute a useful advice string. Towards the end of this paper, we take up this question as well by extending our model to account for preprocessing time.

1.1 Our results

We prove new lower bounds on generic algorithms with preprocessing that relate the time, advice, and preprocessing complexity of generic discrete-log algorithms, and algorithms for related problems. We also introduce new generic preprocessing attacks for the multiple-discrete-log problem and for certain distinguishing problems in groups.

Lower bounds for discrete log and CDH. We prove in Theorem 2 that every generic algorithm that uses S bits of group-specific precomputed advice and that computes discrete logarithms in online time T with success probability ϵ must satisfy $ST^2 = \widetilde{\Omega}(\epsilon N)$, where the $\widetilde{\Omega}(\cdot)$ notation hides logarithmic factors in N. When $S = T$ the bound shows that, for constant ϵ, the best possible generic attack must use roughly $N^{1/3}$ bits of advice and runs in online time roughly $N^{1/3}$.

Our lower bound is tight, up to logarithmic factors, for the full range of parameters S, T, and ϵ, since the attack of Mihalcik [51] and Bernstein and Lange [11], which we summarize in Section 7.1, gives a matching upper bound. (These attacks sidestep Shoup's $N^{1/2}$-time lower bound for generic discrete-log algorithms [66] by using more than $N^{1/2}$ time in their preprocessing phase.) As a consequence, beating the preprocessing algorithm of Milahlcik, Bernstein, and Lange on the NIST P-256 curve, for example, would require developing a new non-generic attack.

Our lower bound extends naturally to the computational Diffie-Hellman problem, for which we also prove an $ST^2 = \widetilde{\Omega}(\epsilon N)$ lower bound (Theorem 6), and the M-instance multiple discrete-log problem, for which we prove an $ST^2/M + T^2 = \widetilde{\Omega}(\epsilon^{1/M} MN)$ lower bound (Theorem 8). The attacks of Section 7 show that these lower bounds are tight.

Lower bound for DDH with preprocessing. We also look at the more subtle case of distinguishing attacks. We show in Theorem 9, that every generic distinguisher with preprocessing that achieves advantage ϵ against the decisional Diffie-Hellman problem (DDH) must satisfy $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$. The quadratic dependence on the error probability makes this bound weaker than the previous ones. We know of no DDH distinguisher that matches this lower bound for all parameter ranges, and we leave the question of whether such a distinguisher exists as an open problem.

For the slightly different problem of distinguishing tuples of the form $(g, g^x, g^{(x^2)})$ from random, the same lower bound holds. While it is unclear whether our lower bound's quadratic dependence on ϵ is optimal for the DDH problem, for this problem, we give a matching attack that shows that our lower bound is tight.

Lower bound on preprocessing time. In addition, we prove lower bounds on the amount of computation required to produce the advice string in the preprocessing phase. We show in Theorem 10 that any generic discrete-log algorithm that uses preprocessing time P, online time T, and achieves success probability ϵ must satisfy: $PT + T^2 = \Omega(\epsilon N)$. Our lower bound matches the preprocessing time used by the discrete-log preprocessing attack of Mihalcik, Bernstein, and Lange, and essentially rules out the existence of very fast generic algorithms that also use modest amounts of preprocessing. For example, any generic algorithm that runs in online time $T = N^{1/8}$ must use close to $N^{7/8}$ preprocessing time to succeed with good probability—no matter how large of an advice string it uses.

New preprocessing attacks. Finally, in Theorem 11, we introduce a new preprocessing algorithm for the multiple-discrete-log problem that shows that our lower bound is tight for constant ϵ. In addition, we give a new algorithm in Theorem 13 for the problem of distinguishing tuples of the form $(g, g^x, g^{(x^2)})$ from random that satisfies $ST^2 = \widetilde{O}(\epsilon^2 N)$. The existence of such an algorithm is especially surprising because solving the $(g, g^x, g^{(x^2)})$ distinguishing problem is *as hard as* computing discrete logarithms for *online-only* algorithms. In contrast, our algorithm shows that this problem is *substantially easier* than computing discrete logarithms for *preprocessing* algorithms: computing discrete logarithms requires $S = T = 1/\epsilon = N^{1/4}$ while our new distinguishing attack requires $S = T = 1/\epsilon = N^{1/5}$.

1.2 Our techniques

The starting point of our lower bounds is an incompressibility argument, which is also at the heart of classic lower bounds against preprocessing algorithms (also known as "non-uniform algorithms") for inverting one-way permutations [37,71,72] and random functions [28]. At a high level, our approach is to show that if there exists a generic discrete-log algorithm $\mathcal{A}$ that (a) uses few bits of preprocessed advice and (b) uses few online group operations, then we can use such an algorithm $\mathcal{A}$ to compress a random permutation.

Incompressibility. The first technical challenge is that a straightforward application of incompressibility techniques does not suffice in the setting of generic groups. To explain the difficulty, let us sketch the argument that a random permutation oracle π is one-way, even against preprocessing adversaries [24, 37, 71, 72]. The argument builds a compression scheme by invoking $\mathcal{A}(x)$ on some point x in the image of π and answering $\mathcal{A}$'s queries to π. The key observation is that when $\mathcal{A}$ produces its output $y = \pi^{-1}(x)$, we have learned some extra information about π beyond the information that the query responses contain. In this way, each invocation of $\mathcal{A}$ yields some "profit," in terms of our knowledge of π. We can use this profit to compress π.

To apply this argument to generic groups, we could replace the random permutation oracle π by an oracle that implements the group operation for a random group. (We define the model precisely in Section 2.) The challenge is that a group-operation oracle has extra structure that a random permutation oracle does not. This extra structure fouls up the standard incompressibility argument, since the query responses that the compression routine must feed to $\mathcal{A}$ might themselves contain enough information to recover the discrete log that $\mathcal{A}$ will later output. If this happens, the compression scheme will not "profit" at all from invoking $\mathcal{A}$, and we will not be able to use $\mathcal{A}$ to compress the oracle.

To handle this case, we notice that this sort of compression failure only occurs when two distinct queries to the group oracle return the same string. By using a slightly more sophisticated compression routine, which notices and compensates for these "collision" events, we achieve compression even where the traditional incompressibility argument would have failed. (Dodis et al. [29] use a similar observation in their analysis of the RSA-FDH signature scheme.)

To keep track of when these collision events occur, we adopt an idea from Shoup's generic-group lower-bound proof [66], which does not use incompressibility at all. Shoup's idea is to keep a careful accounting of the information that the adversary's queries have revealed about the generic-group oracle at any point during the execution. Our compression scheme exploits a similar accounting strategy, which allows it to halt the adversary $\mathcal{A}$ as soon as the compressor notices that continuing to run $\mathcal{A}$ would be "unprofitable."

Handling randomized algorithms. The second technical challenge we face is in handling algorithms that succeed with arbitrarily small probability ϵ. The standard incompressibility methods invoke the algorithm $\mathcal{A}$ on many inputs, and the compression routine succeeds only if *all* of these executions succeed. If the algorithm $\mathcal{A}$ fails often, then we will fail to construct a useful compression scheme.

The naïve way around this problem would be to amplify $\mathcal{A}$'s success probability by having the compression scheme run the algorithm $\mathcal{A}$ many times on each input. The problem is that amplifying the success probability in this way decreases the "profit" that we gain from $\mathcal{A}$, since the compression scheme has to answer many more group-oracle queries in the amplified algorithm than in the unamplified algorithm. As a result, this naïve amplification strategy yields an $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$ lower bound that is loose in its dependence on the success probability ϵ.

Our approach is to leverage the observation, applied fruitfully to the random-permutation model by De et al. [24], that it is without loss of generality to assume that the compression and decompression algorithms share a common string of independent random bits. Rather than amplifying the success probability of $\mathcal{A}$ by iteration, the compression scheme simply finds a set of random bits in the shared random string that cause $\mathcal{A}$ to produce the correct output. The compression scheme then writes this pointer out as part of the compressed representation of the group oracle. This optimization yields the tight $ST^2 = \widetilde{\Omega}(\epsilon N)$ lower bound.

Along the way, we exploit the random self-reducibility of the discrete-log problem to transform an average-case discrete-log algorithm, which succeeds on a *random* instance with probability ϵ, to a worst-case algorithm, which succeeds on *every* instance with probability ϵ. Using the random self-reduction substantially simplifies the incompressibility argument, since it allows compression routine to invoke the algorithm $\mathcal{A}$ on arbitrary inputs.

Generalizing to decisional problems. The final technical challenge is to extend our core incompressibility argument to give lower bounds for the decisional Diffie-Hellman Problem (DDH). The difficulty with using a DDH algorithm to build a compression scheme is that each execution of the DDH distinguisher only produces a single bit of information. Furthermore, if the distinguishing advantage ϵ is small, the distinguisher produces only a fraction of a bit of information. The straightforward amplification would again work but would yield a very loose $ST^2 = \widetilde{\Omega}(\epsilon^4 N)$ bound.

To get around this issue, we execute the distinguisher on large batches of input instance. We judiciously choose the batch size to balance the profit from each batch with the probability that all runs in a batch succeed. Handling collision events in this case requires extra care. Putting these ingredients together, we achieve an $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$ lower bound for the DDH problem.

1.3 Related work

This paper builds upon two major lines of prior work: one on *preprocessing* lower bounds for *symmetric-key problems*, and the other on *online* lower bounds for *generic algorithms in groups*. We prove *preprocessing* lower bounds for *generic algorithms* and, indeed, our proofs use a combination of techniques from both prior settings.

Incompressibility methods. One prominent related area of research puts lower bounds on the efficiency of *preprocessing algorithms* for inverting random functions and random permutations. An early motivation was Hellman's preprocessing algorithm ("Hellman tables") for inverting random functions [39]. Fiat and Naor later extended the technique to allow inverting general functions [31] and Oechslin [56] proposed practical improvements to Hellman's construction.

Yao used an incompressibility argument [72] to show the optimality of Hellman's method for inverting random permutations. Gennaro and Trevisan [37] and Wee [71] proved related lower bounds, also using incompressibility methods. Barkan et al. [7] showed that, in a restricted model of computation, Hellman's method is optimal for inverting random functions (not just permutations).

De et al. [24] demonstrated how to use *randomized encodings*, essentially an incompressibility argument augmented with random oracles, to give alternative proofs of preprocessing lower bounds for the complexity inverting random permutations and breaking general pseduo-random generators. We adopt the powerful randomized encoding technique of De et al. in our proofs. Dodis et al. [28] applied this technique to show that salting [52] defeats preprocessing attacks against certain computational tasks (e.g., collision finding) in the random-oracle model [8].

Abusalah et al. [2] used the technique to construct proofs of space from random functions.

Unruh [69] gave an elegant framework for proving the hardness of computational problems in the random-oracle model against preprocessing adversaries (or against algorithms with "auxiliary input," in his terminology). He proves that if a computational problem is hard when a certain number of points of the random oracle are fixed ("presampled"), then the problem is hard in the random-oracle model against preprocessing adversaries using a certain amount of oracle-dependent advice. This presampling technique gives an often simpler alternative to incompressibility-based lower bounds. Coretti et al. [21] recently introduced new variants of Unruh's presampling technique that give tighter lower bounds against preprocessing adversaries for a broad set of problems.

Generic-group lower bounds. All of the aforementioned work studies precomputation attacks on one-way permutations and one-way functions, which are essentially symmetric-key primitives. In the setting of public-key cryptography, a parallel—and quite distinct—line of work studies lower bounds on algorithms for the discrete-log problem and related problems in generic groups. All of these lower bounds study online-only algorithms (i.e., that do not use preprocessing).

In particular, Shoup introduced the modern *generic-group model* [66] to capture algorithms that make black-box use of a group operation. In Shoup's model, which draws on earlier treatments of black-box algorithms for groups [6, 54], the discrete-logarithm problem in a group of prime order N requires time $\Omega(N^{1/2})$ to solve. Shoup's model captures many popular discrete-log algorithms, including Shank's baby-step giant-step [65] algorithm, Pollard's Rho and Kangaroo algorithms [61], and the Pohlig-Hellman algorithm [60]. For computing discrete logarithms on popular elliptic curves, variations of these algorithms are the best known [9, 35, 70, 75].

Subsequent works used Shoup's model to prove lower bounds against generic algorithms for RSA-type problems [23], knowledge assumptions [26], the multiple-discrete-log problem [74], assumptions in groups with pairings [13], and for algorithms with access to additional oracles [50]. A number of works also prove the security of specific cryptosystems in the generic-group model [19, 20, 25, 32, 43, 63, 67]. Other work studies computational problems in generic *rings*, to analyze generic algorithms for RSA-type problems [4, 49].

Preprocessing attacks. The work most relevant to our new algorithms with preprocessing are Mihalcik's master's thesis [51], which surveys preprocessing attacks on the discrete-logarithm problem, and the paper of Bernstein and Lange [11], which demonstrated preprocessing attacks—both generic and non-generic—on a wide range of symmetric- and public-key primitives. We extend these attacks to the multiple-discrete-logarithm problem and to design a new preprocessing attack against a large class of distinguishing problems in groups.

In certain groups, such as $\mathbb{Z}_p^*$, there are non-generic discrete-log attacks that dramatically outperform the generic ones [41, 55]. Adrian et al. [3] show how to augment such an attack with preprocessing to compute discrete logarithms modulo a 512-bit prime in less than a minute of online time. In popular elliptic-curve groups, the generic preprocessing attacks discussed in this paper are still the best known.

Organization of this paper. In Section 2, we introduce notation, our model of computation, and a key lemma. In Section 3, we prove a lower bound on generic algorithms with preprocessing for the discrete-logarithm and CDH problems. In Sections 4 and 5, we extend these bounds to the multiple-discrete-logarithm and DDH problems. In Section 6, we investigate the amount of precomputation such generic preprocessing algorithms require. In Section 7, we introduce new generic preprocessing attacks. In Section 8, we conclude with open questions.

2 Background

In this section, we recall the standard model of computation in generic groups, we introduce our model of generic algorithms with preprocessing, and we recall an incompressibility lemma that will be essential to our proofs.

Notation. We use $\mathbb{Z}_N$ to denote the ring of integers modulo N, $[N]$ indicates the set $\{1, \dots, N\}$, and $\mathbb{Z}^+$ indicates the set of positive integers. Throughout this paper, we take N to be prime, so $\mathbb{Z}_N$ is also a field. We use the notation $x \leftarrow 5$ to indicate the assignment of a value to a variable and, when S is a finite set, the notation $x \xleftarrow{\text{R}} S$ indicates that x is a sample from the uniform distribution over S. For a probability distribution $\mathcal{D}$, $d \sim \mathcal{D}$ indicates that d is a random variable distributed according to $\mathcal{D}$. The statement $f(x) \overset{\text{def}}{=} x^2 - x$ indicates the definition of a function f. All logarithms are base two, unless otherwise noted.

We use the standard Landau notation $O(\cdot)$, $\Theta(\cdot)$, $\Omega(\cdot)$, and $o(\cdot)$ to indicate the asymptotics of a function. For example $f(N) = O(g(N))$ if there exists a constant $c > 0$ such that for all large enough N, $|f(N)| \leq c \cdot g(N)$. When there are many variables inside the big-O, as in $f(N) = O(\frac{N}{ST})$, all variables other than N are implicit functions of N. The tilde notation $\widetilde{O}(\cdot)$ and $\widetilde{\Omega}(\cdot)$ hides polylogarithmic factors in N. So, we can say for example that $S \log^2 N = \widetilde{O}(S)$.

Generic algorithms. Following Shoup [66], we model a generic group using a random injective function σ that maps the integers in $\mathbb{Z}_N$ (representing the set of discrete logarithms) to a set of labels $\mathcal{L}$ (representing the set of group elements). We then write the elements of an order-N group as $\{\sigma(1), \sigma(2), \dots, \sigma(N)\}$, instead of the usual $\{g, g^2, \cdots, g^N\}$. We often say that $i \in \mathbb{Z}_N$ is the "discrete log" of its label $\sigma(i) \in \mathcal{L}$.

The *generic group oracle* $\mathcal{O}_\sigma(\cdot, \cdot)$ for a labeling function σ takes as input two strings $s_i, s_j \in \mathcal{L}$ and responds as follows:

- If the arguments to the oracle are in the image of σ, then we can write $s_i = \sigma(i)$ and $s_j = \sigma(j)$. The oracle responds with $\sigma(i + j)$, where the addition is modulo the group order N.
- If either of the arguments to the oracle falls outside of the image of σ, the oracle returns $\perp$.

Given such an oracle and a label $\sigma(x)$, it is possible to compute $\sigma(\alpha x)$ for any constant $\alpha \in \mathbb{Z}_N$ using $O(\log N)$ oracle queries, by repeated squaring.

Some authors define the group oracle $\mathcal{O}_\sigma$ with a second functionality that maps labels $\sigma(x)$ to their inverses $\sigma(-x)$ in a single query. Our oracle can simulate this inversion oracle in at most $O(\log(N))$ queries. To do so: given an element $\sigma(x)$, compute the element $\sigma((N-1)x) = \sigma(-x)$. Since providing an inversion oracle can decrease a generic algorithm's running time by at most a logarithmic factor, we omit it for simplicity.

A *generic algorithm* for $\mathbb{Z}_N$ on $\mathcal{L}$ is a probabilistic algorithm that takes as input a list of labels $(\sigma(x_1), \ldots, \sigma(x_L))$, and has oracle access to $\mathcal{O}_\sigma$. We measure the time complexity of a generic algorithm by counting the number of queries it makes to the generic group oracle.

Although the generic algorithms we consider may be probabilistic, we require that for every choice of σ, inputs, and random tapes, every algorithm halts after a finite number of steps. In this way, for every group order $N \in \mathbb{Z}^+$, we can compute an upper bound on the number of random bits the algorithm uses by iterating over all possible labelings, inputs, and random tapes. For this reason, we need only consider finite probability spaces in our discussion.

Generic algorithms with preprocessing. A *generic algorithm with preprocessing* is a pair of generic algorithms $(\mathcal{A}_0, \mathcal{A}_1)$ for $\mathbb{Z}_N$ on $\mathcal{L}$ such that:

- Algorithm $\mathcal{A}_0$ takes the label $\sigma(1)$ as input, makes some number of queries to the oracle $\mathcal{O}_\sigma$ ("preprocessing queries"), and outputs an advice string st_σ.
- Algorithm $\mathcal{A}_1$ takes as input the advice string st_σ and a list of labels $(\sigma(x_1), \ldots, \sigma(x_L))$, makes some number of queries to the oracle $\mathcal{O}_\sigma$ ("online queries"), and produces some output.

We typically measure the complexity of the algorithm $(\mathcal{A}_0, \mathcal{A}_1)$ by (a) the size of the advice string st_σ that $\mathcal{A}_0$ outputs, and (b) the number of oracle queries that algorithm $\mathcal{A}_1$ makes.

In Section 6, we consider generic algorithms with preprocessing for which the running time of $\mathcal{A}_0$ (i.e., the preprocessing time) is also bounded. In all other sections, we put no running time bound on $\mathcal{A}_0$, so without loss of generality, we may assume in these sections that $\mathcal{A}_0$ is deterministic.

Incompressibility arguments. We use the following proposition of De et al. [24], which formalizes the notion that it is impossible to compress every element in a set $\mathcal{X}$ to a string less than $\log |\mathcal{X}|$ bits long, even relative to a random string.

Proposition 1 (De, Trevisan, and Tulsiani [24]). *Let $E : \mathcal{X} \times \{0,1\}^\rho \to \{0,1\}^m$ and $D : \{0,1\}^m \times \{0,1\}^\rho \to \mathcal{X}$ be randomized encoding and decoding procedures such that, for every $x \in \mathcal{X}$, $\Pr_{r \leftarrow \{0,1\}^\rho}\left[D(E(x,r),r) = x\right] \geq \delta$. Then $m \geq \log |\mathcal{X}| - \log 1/\delta$.*

Notice that the encoding and decoding algorithms of Proposition 1 take *the same* random string r as input. Additionally, that bound on the string length m is independent of the number of random bits that these routines take as input. As a consequence, Proposition 1 holds even when the algorithms E and D have access to a common random oracle.

3 Lower bound for discrete logarithms

In this section we prove that every generic algorithm that uses S bits of group-specific precomputed advice and that computes discrete logs in online time T with probability ϵ must satisfy $ST^2 = \widetilde{\Omega}(\epsilon N)$.

Theorem 2. *Let N be a prime. Let $(\mathcal{A}_0, \mathcal{A}_1)$ be a pair of generic algorithms for $\mathbb{Z}_N$ on $\mathcal{L}$, such that $\mathcal{A}_0$ outputs an S-bit state, $\mathcal{A}_1$ makes at most T oracle queries, and*

$$\Pr_{\sigma,x,\mathcal{A}_1}\left[\mathcal{A}_1^{\mathcal{O}_\sigma}(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)), \sigma(x)) = x\right] \geq \epsilon,$$

where the probability is taken over the uniformly random choice of the labeling σ, the instance $x \in \mathbb{Z}_N$, and the coins of $\mathcal{A}_1$. Then $ST^2 = \widetilde{\Omega}(\epsilon N)$.

Remark. The statement of Theorem 2 models the case in which the group generator $\sigma(1)$ is fixed, and the online algorithm must compute the discrete-log of the instance $\sigma(x)$ with respect to the fixed generator. Using a fixed generator is essentially without loss of generality, since an algorithm that computes discrete logarithms with respect to one generator can also be used to compute discrete logarithms with respect to any generator by increasing its running time by a factor of two. Because of this, we treat the generator as fixed throughout this paper.

Remark. Theorem 2 treats only prime-order groups. In the more general case of composite-order groups a similar result holds, except that the bound is $ST^2 = \widetilde{\Omega}(\epsilon p)$, where p is the largest prime factor of the group order. Since the techniques needed to arrive at this more general result are essentially the same as in the proof of Theorem 2, we focus on the prime-order case for simplicity.

We first give the idea behind the proof of Theorem 2 and then present a detailed proof.

Proof Idea for Theorem 2. Our proof uses an incompressibility argument. The basic idea is to compress the random labeling function σ using a discrete-log algorithm with preprocessing $(\mathcal{A}_0, \mathcal{A}_1)$. To do so, we write $\mathcal{A}_0$'s S-bit advice about σ into the compressed string. We then run $\mathcal{A}_1$ on many discrete-log instances $\sigma(x)$ and we write the T responses to $\mathcal{A}_1$'s queries into the compressed string. For each execution of $\mathcal{A}_1$, we only need to write T values of σ into the compressed string, but we get $T + 1$ values of σ back, since the output of $\mathcal{A}_1(\sigma(x))$ gives us the value of x "for free." If S and T are simultaneously small, then we can compress σ using this method, which yields a contradiction.

However, this naïve technique might never yield any compression at all. The problem is that the T responses to $\mathcal{A}_1$'s queries might contain "collision events," in which the response to one of $\mathcal{A}_1$'s queries is equal to a previously seen query response. For example, say that $\mathcal{A}_1$ makes a query of the form $\mathcal{O}_\sigma(\sigma(x), \sigma(3))$ and the oracle's response is a string $\sigma(7)$ that also appeared in response to a previous query. In this case, just seeing the queries of $\mathcal{A}_1$ and their responses are enough to conclude that $x + 3 = 7 \bmod N$, which immediately yields the discrete log $x = 4$. This is problematic because even if $\mathcal{A}_1$ eventually halts and outputs $x = 4$, we have not received any "profit" from $\mathcal{A}_1$ since the T query responses themselves already contain all of the information we need to conclude that $x = 4$.

To profit in spite of these collisions, our compression scheme halts the execution of $\mathcal{A}_1$ as soon as it finds such a collision, since every collision event yields the discrete log being sought. The profit comes from the fact that, as long as the list of previous query responses is not too long, encoding a pointer to the collision-causing response requires many fewer bits than encoding an arbitrary element in the range of σ.

Our lower bound needs to handle randomized algorithms $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ that succeed with arbitrarily small probability ϵ. Yet to use $\mathcal{A}$ to compress σ, the algorithm $\mathcal{A}_1$ must succeed with very high probability. That is because the compression routine may invoke $\mathcal{A}_1$ as many as N times, and each executions must succeed for the compression scheme to succeed. The random self-reducibility of the discrete-log problem allows us to convert an average-case algorithm that succeeds on an ϵ fraction of instances (for a given labeling σ) to a worst-case algorithm that succeeds with probability ϵ on every instance (for a given labeling σ).

We still need to handle the fact that ϵ may be quite small. The straightforward way to amplify the success probability of $\mathcal{A}_1$ would be to construct an algorithm $\mathcal{A}_1'$ that runs R independent executions of $\mathcal{A}_1$ and that succeeds with probability at least $1 - \epsilon^R$. We could then use the amplified algorithm $(\mathcal{A}_0, \mathcal{A}_1')$ to compress σ.

The problem in our setting is that this simple amplification strategy yields a loose lower bound: if we run $\mathcal{A}_1$ for R iterations, and each iteration makes T queries, our compression scheme ends up "paying" for RT queries instead of T queries for each bit of "profit" it gets (i.e., for each output of $\mathcal{A}_1'$). Carrying this argument through yields an $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$ bound, which is worse than our goal of $\widetilde{\Omega}(\epsilon N)$.

Our idea is to leverage the correlated randomness between the compressor and decompressor to our advantage. In our compression scheme, the compressor runs $\mathcal{A}_1$ using R sets of independent random coins, sampled from the random string shared with the decompressor. The compressor then writes into the compressed representation a $\log R$-bit pointer to a set of random coins (if one exists) that caused $\mathcal{A}_1$ to succeed. Using this strategy, instead of paying for RT queries per execution of $\mathcal{A}_1$, the compression scheme only pays for T queries, plus a small pointer. We can then choose R large enough to ensure that at least one of the R executions succeeds with extremely high probability. □

We now turn to the proof. Whereas Theorem 2 applies even to algorithms that succeed on given $\sigma(x)$ of a *random* point $x \xleftarrow{R} \mathbb{Z}_N$, the random self-reducibility of the discrete-log problem allows us to restrict ourselves to algorithms that succeed on *every* input point $x \in \mathbb{Z}_N$. This is formalized in the next lemma.

Lemma 3 (Adapted from Abadi, Feigenbaum, and Kilian [1]). *Let N be a prime. Let $(\mathcal{A}_0, \mathcal{A}_1)$ be a pair of generic algorithms for $\mathbb{Z}_N$ on $\mathcal{L}$ such that $\mathcal{A}_0$ outputs an S-bit advice string and $\mathcal{A}_1$ makes at most T oracle queries. Then, there* *exists a generic algorithm $\mathcal{A}_1'$ that makes at most $T + O(\log N)$ oracle queries and, for every σ from $\mathbb{Z}_N$ to $\mathcal{L}$, if* $\Pr_{x,\mathcal{A}_1}\left[\mathcal{A}_1^{\mathcal{O}_\sigma}(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)), \sigma(x)) = x\right] = \epsilon$, *then for every $x \in \mathbb{Z}_N$,* $\Pr_{\mathcal{A}_1'}\left[\mathcal{A}_1'^{\mathcal{O}_\sigma}(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)), \sigma(x)) = x\right] = \epsilon$.

Proof. On input $(\mathsf{st}_\sigma, \sigma(x))$, algorithm $\mathcal{A}_1'$ executes the following steps: First, it samples a random $r \xleftarrow{\text{R}} \mathbb{Z}_N$ and computes $\sigma(x + r)$, using $O(\log N)$ group operations. Then, it runs $\mathcal{A}_1(\mathsf{st}_\sigma, \sigma(x + r))$. Finally, when $\mathcal{A}_1$ outputs a discrete log x', algorithm $\mathcal{A}_1'$ outputs $x = x' - r \bmod N$.

Notice that $\mathcal{A}_1'$ invokes $\mathcal{A}_1$ on $\sigma(x + r)$, which is the image of a uniformly random point in $\mathbb{Z}_N$. Since $\mathcal{A}_1$ succeeds with probability at least ϵ over the random choice of $x \xleftarrow{\text{R}} \mathbb{Z}_N$ and its coins, $\mathcal{A}_1'$ succeeds with probability ϵ, only over the choice of its coins. □

To prove the theorem, we will use the generic algorithms $(\mathcal{A}_0, \mathcal{A}_1)$ to construct a randomized encoding scheme that compresses a good fraction of the labeling functions σ. The following lemma gives us such a scheme.

Lemma 4. *Let N be a prime. Let $G = \{\sigma_1, \sigma_2, \dots\}$ be a subset of the labeling functions from $\mathbb{Z}_N$ to $\mathcal{L}$. Let $(\mathcal{A}_0, \mathcal{A}_1)$ be a pair of generic algorithms for $\mathbb{Z}_N$ on $\mathcal{L}$ such that for every $\sigma \in G$ and every $x \in \mathbb{Z}_N$, $\mathcal{A}_0$ outputs an S-bit advice string, $\mathcal{A}_1$ makes at most T oracle queries, and $(\mathcal{A}_0, \mathcal{A}_1)$ satisfy*

$$\Pr_{\mathcal{A}_1}[\mathcal{A}_1^{\mathcal{O}_\sigma}(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)), \sigma(x)) = x] \geq \epsilon.$$

Then, there exists a randomized encoding scheme that compresses elements of G to bitstrings of length at most

$$\log \frac{|\mathcal{L}|!}{(|\mathcal{L}| - N)!} + S + 1 - \frac{\epsilon N}{6T(T+1)(\log N + 1)},$$

and succeeds with probability at least $1/2$.

We prove Lemma 4 in Section 3.1. Given the above two lemmas, we can prove Theorem 2.

Proof of Theorem 2. We say that a labeling σ is "good" if $(\mathcal{A}_0, \mathcal{A}_1)$ computes discrete logs with probability at least $\epsilon/2$ on σ. More precisely, a labeling σ is "good" if:

$$\Pr_{x,\mathcal{A}_1}[\mathcal{A}_1^{\mathcal{O}_\sigma}(\mathcal{A}_0^{\mathcal{O}_\sigma}(\mathcal{A}(\sigma(1)), \sigma(x)) = x] \geq \epsilon/2,$$

where the probability is taken over the choice of $x \in \mathbb{Z}_N$ as well as over the random tape of $\mathcal{A}_1$. Let G be the set of good labelings. A standard averaging argument [5, Lemma A.12] guarantees that an $\epsilon/2$ fraction of injective mappings from $\mathbb{Z}_N$ to $\mathcal{L}$ are good. Then $|G| \geq \epsilon/2 \cdot |\mathcal{L}|!/(|\mathcal{L}| - N)!$, where we've used the fact that the number of injective functions from $\mathbb{Z}_N$ to $\mathcal{L}$ is $|\mathcal{L}|!/(|\mathcal{L}| - N)!$.

Lemma 3 then implies that there exists a pair of generic algorithms $(\mathcal{A}_0, \mathcal{A}_1')$ such that for every $\sigma \in G$ and every $x \in \mathbb{Z}_N$, $\mathcal{A}_1'^{\mathcal{O}_\sigma}(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)), \sigma(x))$ makes at most $T' = T + O(\log N)$ queries, and outputs x with probability at least $\epsilon/2$.

Lemma 4 then implies that we can use $(\mathcal{A}_0, \mathcal{A}_1')$ to compress any labeling $\sigma \in G$ to a string of bitlength at most

$$\log \frac{|\mathcal{L}|!}{(|\mathcal{L}| - N)!} + S + 1 - \frac{(\epsilon/2)N}{6T'(T'+1)(\log N + 1)}, \tag{1}$$

where the encoding scheme works with probability at least 1/2. By Proposition 1, this length must be at least $\log |G| - \log 2$. Thus, it must hold that

$$\log \frac{|\mathcal{L}|}{(|\mathcal{L}| - N)!} + S + 1 - \frac{\epsilon N}{12T'(T'+1)(\log N + 1)} \geq \log \frac{|\mathcal{L}|!}{(|\mathcal{L}| - N)!} - \log \frac{4}{\epsilon}.$$

Overall, we have

$$S \geq \frac{\epsilon N}{O(T^2) \cdot \mathrm{polylog}(N)} - \log \frac{8}{\epsilon},$$

which implies that $ST^2 = \tilde{\Omega}(\epsilon N)$.

□

3.1 Proof of Lemma 4

Recall that a randomized encoding scheme consists of an encoding and a decoding routine, such that both routines take the same string r of random bits as input. The encoding scheme we construct for the purposes of Lemma 4 operates on labelings σ. That is, the encoding routine takes a labeling $\sigma \in G$ and the random bits r, and constructs a compressed representation of σ. Correspondingly, the decoding routine takes this compressed representation and the *same* random bits r, and reconstructs σ.

While the encoding routine runs, it builds up a table of pairs $(f, \sigma(i)) \in (\mathbb{Z}_N[X] \times \mathcal{L})$. The decoder constructs a similar table during its execution. At any point during the encoding process, the table contains a representation of the information about σ that the encoder has communicated to the decoder up to the current point in the encoding process. The indeterminate X that appears in this table represents a discrete log value $x \in \mathbb{Z}_N$, which the decoder does not know. Once the decoder has enough information to determine x, each of the routines replaces every non-constant polynomial $f(X)$ in the table with its evaluation $f(x)$ at the point x. Subsequently, both routines can introduce a new variable X into the table, which represents a different unknown discrete logarithm in $\mathbb{Z}_N$. Therefore, at any point during the execution, there is at most a single indeterminate X in the table. Finally, when each of the routines completes, the table contains only constant polynomials, and the table fully determines σ.

We stress that the table is not part of the compressed representation of σ, but is part of the internal state of both routines.

Simulating $\mathcal{A}_1$'s random tape. Since the algorithm $\mathcal{A}_1$ is randomized, each time the encoder (or decoder) runs the algorithm $\mathcal{A}_1$, it must provide $\mathcal{A}_1$ with a fresh random tape. Both routines take as input a common random bitstring, and the encoder can reserve a substring of it to feed to each invocation of $\mathcal{A}_1$

as that algorithm's random tape. Since $\mathcal{A}_1$ always terminates, the encoder can determine an upper bound on the number of random bits that $\mathcal{A}_1$ will need for a given group size N and can partition the common random string accordingly.

The decoder follows the same process, and the fact that the encoder and decoder take the same random string r as input ensures that $\mathcal{A}_1$ behaves identically during the encoding and decoding processes.

Encoding routine. The encoding routine, on input σ, uses two parameters $d, R \in \mathbb{Z}^+$, which we will set later, and proceeds as follows:

1. Compute $\mathsf{st}_\sigma \leftarrow \mathcal{A}_0(\sigma(1))$. The encoder can respond to all of the algorithm's oracle queries since the encoder knows all of σ. Write the S-bit output st_σ into the encoding.
2. Encode the image of σ as a subset of $\mathcal{L}$ using $\log \binom{|\mathcal{L}|}{N}$ bits, and append it to the encoding.
3. Initialize the table of pairs to an empty list.
4. Repeat d times:
   (a) Choose the lexicographically first string in the image of σ that does not yet appear in the table. Call this string $\sigma(x)$ and add the pair $(X, \sigma(x))$ to the table.
   (b) Run $\mathcal{A}_1(\mathsf{st}_\sigma, \sigma(x))$ up to R times using independent randomness from the encoder's random string in each run. The encoder answers all of $\mathcal{A}_1$'s oracle queries using its knowledge of σ. If $\mathcal{A}_1$ fails on all R executions, abort the entire encoding routine. Otherwise, write into the encoding the index $r^* \in [R]$ of the successful execution (using $\log R$ bits).
   (c) Write a placeholder of $\log T$ zeros into the encoding. (The routine overwrites these zeros with a meaningful value once this execution of $\mathcal{A}_1$ terminates.)
   (d) Rerun $\mathcal{A}_1(\mathsf{st}_\sigma, \sigma(x))$ using the r^*-th random tape. While $\mathcal{A}_1$ is running, it makes a number of queries and then outputs its guess of the discrete log x. The encoding routine processes each of $\mathcal{A}_1$'s queries $(\sigma(i), \sigma(j))$ as follows:
      i. If either of the query arguments is outside of the range of σ, reply $\perp$ and continue to the nextquery.
      ii. If either (or both) of the arguments is missing from the table, then this is an "unexpected" query input. Add each such input, together with its discrete log, to the table and append the discrete log value i to the encoding, using $\log(N - |\mathsf{Table}|)$ bits.
      iii. Otherwise, look up the linear polynomials f_i, f_j representing $\sigma(i)$, $\sigma(j)$ in the table, and compute the linear polynomial $f_i + f_j$ representing the response $\sigma(i + j)$. We then distinguish between three cases:
         A. If $(f_i + f_j, \sigma(i + j))$ is already in the table, simply reply with $\sigma(i + j)$.
         B. If $\sigma(i + j)$ does not appear in the table, then add $\sigma(i + j)$ to the encoding, using $\log(N - |\mathsf{Table}|)$ bits, and reply with $\sigma(i + j)$.
         C. If $\sigma(i + j)$ appears in the table but its discrete log in the table is a polynomial f_k such that $f_k \neq f_i + f_j$, encode the reply to this

query as a ($\log |\mathsf{Table}|$)-bit pointer to the table entry $(f_k, \sigma(i+j))$ and add this pointer the encoding. Stop this execution of $\mathcal{A}_1$, and indicate this "early stop" by writing the actual number of queries $t \leq T$ into its placeholder above.

   (e) When the execution $\mathcal{A}_1(\mathsf{st}_\sigma, \sigma(x))$ outputs x, evaluate all of the polynomials in the table at the point x.

5. Append the remaining values that do not yet appear in the table to the encoding in lexicographic order.

Decoding routine. The decoder proceeds analogously to the encoder. A key property of our randomized encoding scheme is that each position in the encoded string corresponds to the same state of the table in both the encoding and the decoding routines. In other words, when the decoding routine reads a certain position in the encoded string, its internal table is identical to the internal table the encoding routine had when it wrote to that position in the encoded string. The table allows the decoder to correctly classify each query to the correct category.

Note that in the case of a collision query (case C above), the decoder can use the collision to recover the value x of the indeterminate X. Specifically, for a query (u, v) where $u, v \in \mathcal{L}$, the decoder reads the reply $w \in \mathcal{L}$ from the encoding string, looks up the polynomials f_u, f_v, and f_w in the table, and solves for X the equation $f_w = f_u + f_v \bmod N$. This equation always has a unique solution, since N is a prime and f_u, f_v and f_w are linear polynomials in X such that $f_u + f_v$ is not identical to f_w.

The full description of the decoder appears in Appendix A.

Encoding Length. The encoding contains:

- the advice to the algorithm about the labeling σ (S bits),
- the encoding of the image of σ ($\log \binom{|\mathcal{L}|}{N}$ bits),
- for each of the d invocations of $\mathcal{A}_1$, the index r^* of the random tape on which it succeeded ($d \cdot \log R$ bits in total)
- for the i-th entry added to the table ($0 \leq i < N$), if the entry was added
  - as the result of resolving a collision within the table, $\log i$ bits,
  - from the output of $\mathcal{A}_1$, 0 bits,
  - otherwise, $\log(N - i)$ bits,
- a counter indicating the number of queries for which to run each execution ($d \cdot \log T$ bits in total).

Observe that each of the d executions of $\mathcal{A}_1$ saves $\log(N - i)$ bits compared to the straightforward encoding (either due to $\mathcal{A}_1$ successfully computing the discrete log of its input, or finding a collision), but incurs an additional cost of at most $\log R + \log T + \log |\mathsf{Table}|$ bits. Since each execution of $\mathcal{A}_1$ adds at most $3T + 1$ rows to the table (T replies plus $2T$ unexpected inputs and either one collision or one output of $\mathcal{A}_1$) we have that $|\mathsf{Table}| \leq d \cdot (3T + 1)$. Setting $d = \lfloor N/((2RT + 1)(3T + 1)) \rfloor$ guarantees that each of the d executions results in a net profit of

$$\log \frac{N - |\mathsf{Table}|}{RT|\mathsf{Table}|} \geq \log \frac{N - d(3T+1)}{RdT(3T+1)} \geq \log \frac{1 - \frac{1}{2RT+1}}{\frac{RT}{2RT+1}} = \log 2 = 1$$

bit. In this case, the total bitlength of the encoding is at most

$$\begin{aligned} S + \log \binom{|\mathcal{L}|}{N} + \sum_{i=0}^{N-1} \log(N-i) - d &= \log \frac{|\mathcal{L}|!}{(|\mathcal{L}|-N)!} + S - d \\ &\leq \log \frac{|\mathcal{L}|!}{(|\mathcal{L}|-N)!} + S - \frac{N}{(2RT+1)(3T+1)} + 1 \\ &\leq \log \frac{|\mathcal{L}|!}{(|\mathcal{L}|-N)!} + S - \frac{N}{6RT(T+1)} + 1\,. \end{aligned}$$

We need to choose R large enough to ensure that the encoding routine fails with probability at most $1/2$. If we choose $R = (1 + \log N)/\epsilon$, then the probability that R invocations of $\mathcal{A}_1$ all fail is, by a union bound, at most $(1-\epsilon)^R \leq e^{-\epsilon R} \leq 2^{-\epsilon R} \leq 2^{-1-\log N} \leq 1/(2N)$. The encoding scheme invokes $\mathcal{A}_1$ on at most N different inputs, so by a union bound, the probability that any invocation fails is at most $1/2$. Overall, the encoding length is at most:

$$\log \frac{|\mathcal{L}|!}{(|\mathcal{L}|-N)!} + S + 1 - \frac{\epsilon N}{6T(T+1)(\log N + 1)} \text{ bits,}$$

which completes the proof of Lemma 4. □

3.2 Discrete logarithms in short intervals

When working in groups of large order N, it is common to rely on the hardness of the *short-exponent discrete-log problem*, rather than the standard discrete-log problem [38, 46, 57, 59]. In the usual discrete-log problem, a problem instance is a pair of the form $(g, g^x) \in \mathbb{G}^2$ for $x \xleftarrow{\text{R}} \mathbb{Z}_N$. The short-exponent problem is identical, except that x is sampled at random from $\{1, \dots, W\} \subset \mathbb{Z}_N$, for some interval width parameter $W < N$. Using short exponents speeds up the Diffie-Hellman key-agreement protocol when it is feasible to set the interval width W to be much smaller than the group order N [57]. A variant of Pollard's "Lambda Method" [36, 61] solves the short-exponent discrete-log problem in every group in time $O(W^{1/2})$, so W cannot be too small.

The following corollary of Theorem 2 shows that the short-exponent problem is no easier for generic algorithms than computing a discrete-logarithm in an order-W group.

Corollary 5 (Informal). *Let $\mathcal{A}$ be a generic algorithm with preprocessing that solves the short-exponent discrete-log problem in an interval of width W. If $\mathcal{A}$ uses S bits of group-specific advice, runs in online time T, and succeeds with probability ϵ, then $ST^2 = \widetilde{\Omega}(\epsilon W)$.*

Proof. We claim that the algorithm $\mathcal{A}$ of the corollary solves the standard discrete-log problem with probability $\epsilon' = \epsilon \cdot (W/N)$. The reason is that a standard discrete-log instance g^x for $x \overset{\text{R}}{\leftarrow} \mathbb{Z}_N$ has a short exponent (i.e., $x \in [W]$) with probability W/N. Algorithm $\mathcal{A}$ solves these short instances with probability ϵ. By Theorem 2, $ST^2 = \widetilde{\Omega}(\epsilon' N) = \widetilde{\Omega}(\epsilon W)$. □

As an application: decryption in the Boneh-Goh-Nissim cryptosystem [16] requires solving a short-exponent discrete-log problem in an interval of width W, for a polynomially large width W. The designers of that system suggest using a size-W table of precomputed discrete logs (i.e., $S = \widetilde{O}(W)$) to enable decryption in constant time. Corollary 5 shows that the best generic decryption algorithm that uses a size-S table requires roughly $\sqrt{W/S}$ time.

3.3 The computational Diffie-Hellman problem

A generic algorithm for the computational Diffie-Hellman problem takes as input a triple of labels $(\sigma(1), \sigma(x), \sigma(y))$ and must output the label $\sigma(xy)$. The following theorem demonstrates that in generic groups—even allowing for preprocessing—the computational Diffie-Hellman problem is as hard as computing discrete logarithms.

Theorem 6 (Informal). *Let $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ be a generic algorithm with preprocessing for the computational Diffie-Hellman problem in a group of prime order N. If $\mathcal{A}$ uses S bits of group-specific advice, runs in online time T, and succeeds with probability ϵ, then $ST^2 = \widetilde{\Omega}(\epsilon N)$.*

We present only the proof idea, since the structure of the proof is very similar to that of Theorem 2.

Proof Idea. The primary difference from the proof of Theorem 2, is that, we run $\mathcal{A}_1$ on pairs of labels $(\sigma(x), \sigma(y))$, and a successful run of $\mathcal{A}_1$ produces the CDH value $\sigma(xy)$. Since we run $\mathcal{A}_1$ on two labels at once, the encoder's table now has two formal variables: X and Y.

In this case, whenever the encoder encounters a collision, it gets a single linear relation on X and Y modulo the group order N. Since there are at most N solutions (x_0, y_0) to a linear relation in X and Y over $\mathbb{Z}_N$, the encoder can describe the solution to the decoder using $\log(N - |\mathsf{Table}|)$ bits. The encoder gets some profit, in terms of encoding length, since it will get two discrete logs for the cost of one discrete log and one pointer into the table (of length $\log |\mathsf{Table}|$ bits).

The rest of the proof is as in Theorem 2. □

3.4 Lower bounds for families of groups

The lower bound of Theorem 2 suggests that one way to mitigate the risk of generic preprocessing attacks is to increase the group size. Doubling the size of group elements from $\log N$ to $2 \log N$ recovers the same level of security as if the attacker could not do any preprocessing. The downside of this mitigation strategy is that increasing the group size also increases the cost of each group operation and requires using larger cryptographic keys (e.g., when using the group for Diffie-Hellman key exchange [27]).

One might ask whether it would be possible to defend against preprocessing attacks without having to pay the price of using longer keys. One now-standard method to defend against preprocessing attacks when using a common cryptographic hash function H is to use “salts” [52]. When using salts, each user u of the hash function H chooses a random salt value s_u from a large space of possible salts. User u then uses the salted function $H_u(x) \stackrel{\text{def}}{=} H(s_u, x)$ as her hash function. Even if every user makes her salt value public, Dodis et al. [28] show that salting defends against precomputation attacks as long as the salt space is large enough.

The analogue to salting in generic groups would be to have a large family of groups (e.g., of elliptic-curve groups) $\{\mathbb{G}_k\}_{k=1}^{K}$ indexed by a key k. Rather than having all users share a single group—as is the case today with NIST P-256—different users and systems could use different groups $\mathbb{G}_k$ sampled from this large family. In particular, pairs of users executing the Diffie-Hellman key-exchange protocol could first jointly sample a group $\mathbb{G}_k$ from this large family and then perform their key exchange in $\mathbb{G}_k$.

We show that using group families in this way effectively defends against generic preprocessing attacks, as long as the family contains a large enough number of groups.

To model group families, we replace the labeling function $\sigma : \mathbb{Z}_N \to \mathcal{L}$ with a keyed family of labeling functions $\sigma_{\text{key}} : [K] \times \mathbb{Z}_N \to \mathcal{L}$. The keyed generic-group oracle $\mathcal{O}_{\sigma_{\text{key}}}(\cdot,\cdot,\cdot)$ then takes a key k and two labels $\sigma_1, \sigma_2 \in \mathcal{L}$ and returns $\sigma_{\text{key}}(k, x+y)$ if there exist $x, y \in \mathbb{Z}_N$ such that $\sigma_{\text{key}}(k, x) = \sigma_1$ and $\sigma_{\text{key}}(k, y) = \sigma_2$. The oracle returns $\perp$ otherwise. In addition, when fed the pair $(k, \star)$, for a key $k \in [K]$ and a special symbol $\star$, the oracle returns the identity element in the kth group: $\sigma(k, 1)$.

The following theorem demonstrates that using a large keyed family of groups effectively defends against generic preprocessing attacks:

Theorem 7. *Let N be a prime. Let $(\mathcal{A}_0, \mathcal{A}_1)$ be a pair of generic algorithms for $[K] \times \mathbb{Z}_N$ on $\mathcal{L}$, such that $\mathcal{A}_0$ outputs an S-bit state, $\mathcal{A}_1$ makes at most T oracle queries, and*

$$\Pr_{\sigma,k,x,\mathcal{A}_1}\left[\mathcal{A}_1^{\mathcal{O}_{\sigma_{\text{key}}}}\left(\mathcal{A}_0^{\mathcal{O}_{\sigma_{\text{key}}}}(), k, \sigma(k,x)\right) = x\right] \geq \epsilon,$$

where the probability is taken over the uniformly random choice of the labeling σ_{key}, the key $k \in [K]$, the instance $x \in \mathbb{Z}_N$, and the coins of $\mathcal{A}_1$. Then $ST^2 = \widetilde{\Omega}(\epsilon K N)$.

The proof of Theorem 7 appears in Appendix B. The structure of the proof follows that of Theorem 2, except that we need some extra care to handle the fact that an adversary may query the oracle at many different values of k in a single execution.

4 Lower bound for computing many discrete logarithms

A natural extension of the standard discrete-log problem is the *multiple*-discrete-log problem [33, 42, 48, 73, 74], in which the adversary's task is to solve M discrete-log problems at once. This problem arises in the setting of multiple-instance security of discrete-log-based cryptosystems. If an adversary has a list of M public keys $(g^{x_1}, \ldots, g^{x_M})$ in some group $\mathbb{G} = \langle g \rangle$ of prime order N, we would like to understand the cost to the adversary of recovering all M secret keys $x_1, \ldots, x_M \in \mathbb{Z}_N$.

Solving the multiple-discrete-log problem cannot be harder than solving M instances of the standard discrete-log problem independently using $\widetilde{O}(M\sqrt{N})$ time overall. One can however do better: generic algorithms due to Kuhn and Struik [48] and Foque, Joux, and Mavromati [33] solve it in time $\widetilde{O}(\sqrt{MN})$. These algorithms achieve a speed-up over solving M discrete-log instances in sequence by reusing some of the work between instances. Yun [74] showed that in the generic-group model, these algorithms are optimal up to logarithmic factors by proving an $\Omega(\sqrt{NM})$-time lower bound for online-only algorithms, subject to the natural restriction that $M = o(N)$.

Our methods give the more general $ST^2 = \widetilde{\Omega}(\epsilon^{1/M} NM)$ generic lower bound for the M-instance multiple-discrete-log problem with preprocessing. For the special case of algorithms without preprocessing, our bound gives $T = \widetilde{\Omega}(\sqrt{NM})$, which matches the above upper and lower bounds. An additional benefit of our analysis it that it handles arbitrarily small success probabilities ϵ, whereas Yun's bound applies only to the $\epsilon = \Omega(1)$ case.

Let $\bar{x} = (x_1, \ldots, x_M) \in \mathbb{Z}_N^M$ and, for a labeling $\sigma : \mathbb{Z}_N \to \mathcal{L}$, define the vector $\sigma(\bar{x}) = (\sigma(x_1), \ldots, \sigma(x_M)) \in \mathcal{L}^M$. We restrict ourselves to the case of $M \leq T$, as otherwise the algorithm cannot even afford to perform a group operation on each of its inputs.

Theorem 8. *Let N be a prime. Let $(\mathcal{A}_0, \mathcal{A}_1)$ be a pair of generic algorithms for $\mathbb{Z}_N$ on $\mathcal{L}$ such that $\mathcal{A}_0$ outputs an S-bit advice string, $\mathcal{A}_1$ makes at most T oracle queries,*

$$\Pr_{\sigma, \bar{x}, \mathcal{A}_1}\left[\mathcal{A}_1^{\mathcal{O}_\sigma}\left(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)),\, \sigma(\bar{x}) \right) = \bar{x} \right] \geq \epsilon,$$

where the probability is taken over the random choice of the labeling σ, a random input vector $\bar{x} \in \mathbb{Z}_N^M$ (for $M \leq T$), and the coins of $\mathcal{A}_1$. Then

$$ST^2/M + T^2 = \widetilde{\Omega}(\epsilon^{1/M} NM).$$

We prove this theorem in Appendix C.

The proof follows the proof of Theorem 2, except the encoder now runs $\mathcal{A}_1$ on M labels at a time. The encoder and decoder keep a table in M formal variables $(X_1, \ldots, X_M)$, representing the M discrete logs being sought. With every "collision event," we show that the number of formal variables in the table can decrease by one until either (a) $\mathcal{A}_1$ outputs the M discrete logs, or (b) the table has no more formal variables and the encoder halts $\mathcal{A}_1$.

5 The decisional Diffie-Hellman problem

The decisional Diffie-Hellman problem [12] (DDH) is to distinguish tuples of the form (g, g^x, g^y, g^{xy}) from tuples of the form (g, g^x, g^y, g^z), for random $x, y, z \in \mathbb{Z}_N$. In this section, we show that every generic distinguisher with preprocessing for the decisional Diffie-Hellman problem that achieves advantage ϵ must satisfy $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$. More formally:

Theorem 9. *Let N be a prime. Let $(\mathcal{A}_0, \mathcal{A}_1)$ be a pair of generic algorithms for $\mathbb{Z}_N$ on $\mathcal{L}$, such that $\mathcal{A}_0$ outputs an S-bit state, $\mathcal{A}_1$ makes at most T oracle queries, and*

$$\Bigl|\Pr\Bigl[\mathcal{A}_1^{\mathcal{O}_\sigma}\Bigl(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)), \sigma(x), \sigma(y), \sigma(xy)\Bigr) = 1\Bigr] \\ - \Pr\Bigl[\mathcal{A}_1^{\mathcal{O}_\sigma}\Bigl(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)), \sigma(x), \sigma(y), \sigma(z)\Bigr) = 1\Bigr]\Bigr| \geq \epsilon\,,$$

where the probabilities are over the choice of the label σ, the values $x, y, z \in \mathbb{Z}_N$, and the randomness of $\mathcal{A}_1$. Then $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$.

The proof of Theorem 9 appears in Appendix D.

While the proof uses an incompressibility argument, extending the technique of Theorem 2 to give lower bounds for decisional-type problems requires overcoming additional technical challenges. Consider a DDH distinguisher with preprocessing $(\mathcal{A}_0, \mathcal{A}_1)$ that achieves advantage ϵ. The difficulty with using such an algorithm to build a scheme for compression σ is that each execution of $\mathcal{A}_1$ only produces a single bit of output. When $\epsilon < 1$, each execution of $\mathcal{A}_1$ produces even less—a fraction of a bit of useful information.

To explain why getting only a single bit of output from $\mathcal{A}_1$ is challenging: the encoder of Theorem 2 derandomized $\mathcal{A}_1$ by writing a pointer $r^* \in [R]$ to a "good" set of random coins for $\mathcal{A}_1$ into the encoding, thus turning a faulty randomized algorithm into a correct deterministic algorithm at the cost of slightly increasing the encoding length. This derandomization technique does not apply immediately here, since the $\log R$-bit value required to point to the "good" set of random coins eliminates any profit in encoding length that we would have gained from the fraction of a bit that $\mathcal{A}_1$ produces as output.

A straightforward amplification strategy—building an algorithm $\mathcal{A}_1'$ that calls $\mathcal{A}_1$ many times and takes the majority output—would circumvent this problem, but would yield an $ST^2 = \widetilde{\Omega}(\epsilon^4 N)$ lower bound that is loose in ϵ.

To achieve a tighter $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$ bound, our strategy is to use $\mathcal{A}_1$ to construct an algorithm $\mathcal{A}_1^{\times B}$ that executes $\mathcal{A}_1$ on a batch of B independent DDH problem instances (one at a time), for some batch size parameter $B \in \mathbb{Z}^+$. The algorithm $\mathcal{A}_1^{\times B}$ now produces B bits of output and succeeds with probability ϵ^B. If we now choose R such that $\log R < B$, we can now apply our prior derandomization technique, since each execution of $\mathcal{A}_1^{\times B}$ will yield some profit in our compression scheme.

Handling collisions in this case involves additional technicalities, since there might (or might not) be a collision in each of the B sub-executions of $\mathcal{A}_1^{\times B}$ and we need to be able to identify which execution encountered collided without squandering the small profit that $\mathcal{A}_1^{\times B}$ yields.

Putting everything together, we achieve an $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$ lower bound for the DDH problem.

6 Lower bounds with limited preprocessing

Up to this point, we have measured the cost of a discrete-log algorithm with preprocessing by (a) number of bits of preprocessed advice it requires and (b) its online running time. In this section, we explore the preprocessing cost—the time require to compute the advice string—and we prove tight lower bounds on the preprocessing cost of generic discrete-log algorithms.

Let $(\mathcal{A}_0, \mathcal{A}_1)$ be a generic discrete-log algorithm with preprocessing, as defined in Section 2. For this section, we allow $\mathcal{A}_0$ to be randomized. We say that $(\mathcal{A}_0, \mathcal{A}_1)$ uses P preprocessing queries and T online queries if $\mathcal{A}_0$ makes P oracle queries and $\mathcal{A}_1$ makes T oracle queries. In this section, we do not put any restriction on the size of the state that $\mathcal{A}_0$ outputs—we are only interested in understanding the relationship between the preprocessing time P and the online time T.

Remark. When $P = \Theta(N)$, there is a trivial discrete log algorithm with preprocessing $(\mathcal{A}_0, \mathcal{A}_1)$ that uses $T = 0$ online queries and succeeds with constant probability. In the preprocessing step, $\mathcal{A}_0$ computes a table of $\Theta(N)$ distinct pairs of the form $(i, \sigma(i)) \in \mathbb{Z}_N \times \mathcal{L}$. On receiving a discrete log instance $\sigma(x)$, the online algorithm $\mathcal{A}_1$ looks to see if $\sigma(x)$ is already stored in its precomputed table and outputs the discrete log x if so. This algorithm succeeds with probability $\epsilon = P/N = \Omega(1)$.

Remark. When $P = o(\sqrt{N})$, we can rule out algorithms that run in online time $T = o(\sqrt{N})$ and succeed with constant probability. To do so, we observe that every generic discrete log algorithm that uses P preprocessing queries and T online queries can be converted into an algorithm that uses *no* preprocessing queries and $T' = (P + T)$ online queries, such that both algorithms achieve the same success probability.

Shoup's lower bound [66] states that every generic discrete log algorithm *without preprocessing* that runs in time T' succeeds with probability at most $\epsilon = O(T'^2/N)$. This implies that any algorithm *with preprocessing* P and online time T succeeds with probability at most $\epsilon = O((T + P)^2/N)$.

Put another way: Shoup's result implies a lower bound of $(T + P)^2 = \Omega(\epsilon N)$. So any algorithm that makes only $P = o(\sqrt{N})$ preprocessing queries must use $T = \Omega(\sqrt{N})$ online queries to succeed with constant probability. Thus, an algorithm that uses $o(\sqrt{N})$ preprocessing queries cannot asymptotically outperform a online algorithm.

Given these two remarks, the remaining parameter regime of interest is when $\sqrt{N} < P < N$. We prove:

Theorem 10. *Let $(\mathcal{A}_0, \mathcal{A}_1)$ be a generic discrete log algorithm with preprocessing for $\mathbb{Z}_N$ on $\mathcal{L}$ that makes at most P preprocessing queries and T online queries. If $x \in \mathbb{Z}_N$ and a labeling function σ are chosen at random, then $\mathcal{A}$ succeeds with probability $\epsilon = O((PT + T^2)/N)$.*

As a corollary, we find that every algorithm that succeeds with probability ϵ must satisfy $PT + T^2 = \Omega(\epsilon N)$. For example, an algorithm that uses $P = O(N^{2/3})$ preprocessing queries must use online time at least $T = \Omega(N^{1/3})$ to succeed with constant probability.

The full proof appears in Appendix E, and we sketch the proof idea here.

Proof Idea for Theorem 10. We prove the theorem using a pair of probabilistic experiments, following the general strategy of Shoup's now-classic proof technique [66].

In both experiments, the adversary interacts with a challenger, who plays the role of the generic group oracle $\mathcal{O}_\sigma$. The challenger defines the labeling function $\sigma(\cdot)$ lazily in response to the adversary's queries. Both experiments follow similar steps:

1. The challenger sends a label $s_1 \in \mathcal{L}$, representing $\sigma(1)$, to the adversary.
2. The adversary makes P preprocessing group oracle queries to the challenger.
3. The challenger sends the discrete log instance $s_x \in \mathcal{L}$, representing $\sigma(x)$, to the adversary.
4. The adversary makes T online queries and outputs a guess x' of x.

The difference between two experiments is in how the challenger defines the discrete log of the instance $s_x \in \mathcal{L}$.

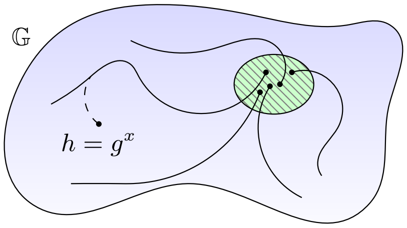
In Experiment 0, the challenger chooses the discrete log $x \in \mathbb{Z}_N$ of s_x *before* the adversary makes any online queries. The challenger in Experiment 0 is thus a faithful (or honest) oracle.

In Experiment 1, the challenger chooses the discrete log x of σ_x *after* the adversary has made all of its online queries. In this latter case, the challenger is essentially "cheating" the adversary, since all of the challenger's query responses are independent of x and the adversary cannot recover x with probability better than random guessing. To complete the argument, we show that unless the adversary makes many queries, it can only rarely distinguish between the two experiments.

A detailed description of the experiments and their analysis appears in Appendix E. □

The lower bound is tight. From Theorem 2, we know that a discrete log algorithm that succeeds with constant probability must use advice S and online time T such that $ST^2 = \widetilde{\Omega}(N)$. From Theorem 10, we know that any such algorithm must also use preprocessing P such that $PT + T^2 = \Omega(N)$. The best tradeoff we could hope for, ignoring the constants and logarithmic factors, is $PT + T^2 = ST^2$, or $P = ST$. Indeed, the known upper bound with preprocessing (Section 7.1) matches this lower bound, disregarding low-order terms.

Fig. 1. The discrete-log algorithm with preprocessing (Section 7.1) uses a random function F to define a walk on the elements of $\mathbb{G}$. The preprocessed advice consists of the discrete logs of S points that lie at the end of length-$\Theta(T)$ disjoint paths on the walk. In the online phase, the algorithm walks from the input point until hitting a stored endpoint, which occurs with good probability.



7 Preprocessing attacks on discrete-log problems

In this section, we recall the known generic discrete-log algorithm with preprocessing and we introduce two new generic attacks with preprocessing. Specifically, we show an attack on the multiple-discrete-log problem that matches our lower bound of Theorem 8, and we show an attack on certain decisional problems in groups that matches our lower bound of Theorem 9.

These attacks are all generic, so they apply to every group, including popular elliptic-curve groups. Our preprocessing attacks are *not* polynomial-time attacks—indeed our lower bounds rule out such attacks—but they yield better-than-known exponential-time attacks on these problems.

The analysis of the algorithms in these sections rely on the attacker having access to a random function (i.e., a random oracle [8]), which the attacker could instantiate with a standard cryptographic hash function, such as SHA-256. Removing the attacks' reliance on a truly random function remains a useful task for future work.

7.1 The existing discrete-log algorithm with preprocessing

For the reader's reference, we describe a variation of the discrete-log algorithm with preprocessing, introduced by Mihalcik [51] and Bernstein and Lange [11], with a slightly more detailed analysis. This discrete-log algorithm shows that the lower bound of Theorem 2 is tight. Our algorithm for the multiple-discrete-log problem (Section 7.2) and for distinguishing pseudo-random generators (Section 7.3) use ideas from this algorithm.

The algorithm computes discrete logs in a group $\mathbb{G}$ of prime order N with generator g. The algorithm takes as input parameters $S, T \in \mathbb{Z}^+$ such that $ST^2 \leq N$. The algorithm uses $\widetilde{O}(S)$ bits of precomputed advice about the group $\mathbb{G}$, uses $\widetilde{O}(T)$ group operations in the online phase, and succeeds with probability $\epsilon = \Omega(ST^2/N)$.

Let $F : \mathbb{G} \to \mathbb{Z}_N$ be a random function, which we can instantiate in practice using a standard hash function. We use the function F to define a walk on the elements of $\mathbb{G}$. Given a point $h \in \mathbb{G}$, the walk computes $\alpha \leftarrow F(h)$ and moves to the point $g^{\alpha} h \in \mathbb{G}$.

Given these preliminaries, the algorithm works as follows:

- *Preprocessing phase.* Repeat S times: pick $r \xleftarrow{\text{R}} \mathbb{Z}_N$ and, starting at $g^r \in \mathbb{G}$, take the walk defined by F for $T/2$ steps. Store the endpoint of the walk $g^{r'}$ and its discrete log r' in a table: $(r', g^{r'})$.

  At the end of the preprocessing phase, the algorithm stores this table of S group elements along with their discrete logs, using $O(S \log N)$ bits.
- *Online phase.* Given a discrete-log instance $h = g^x$, the algorithm takes T steps along the random walk defined by F, starting from the point h (see Figure 1). If the walk hits one of the S points stored in the precomputed table, this collision yields a linear relation on x in the exponent: $g^{r'} = (g^x)^{\alpha_1 + \cdots + \alpha_k} \in \mathbb{G}$. Solving this linear relation for $x \in \mathbb{Z}_N$ reveals the desired discrete log.

The algorithm uses $\widetilde{O}(S)$ bits of group-specific advice and runs in online time $\widetilde{O}(T)$. The remaining task is to analyze its success probability.

We first claim that, with good probability, the S walks in the preprocessing touch at least $ST/4$ distinct points. To this end, observe that for every walk in the preprocessing phase, the probability that it touches $T/2$ new points is at least $(1 - ST/(2N))^{T/2} \geq 1 - ST^2/(4N)$, by Bernoulli's inequality. Since $ST^2 \leq N$, we have that $1 - ST^2/(4N) \geq 1 - 1/4 = 3/4$. Therefore, in expectation, each walk touches at least $3T/8$ new points and by linearity of expectation, the overall expected number of touched points is at least $3ST/8$. The number of touched points is at most $ST/2$ and is at least $3ST/8$, in expectation. We can apply Markov's inequality to an auxiliary random variable to conclude that the number of touched points is greater than $ST/4$ with probability at least $1/2$.

Next, observe that if at any of its first $T/2$ steps, the online walk hits any of the points touched by one of the preprocessed walks, in the remaining $T/2$ steps it will hit the stored endpoint of that preprocessed walk. It will then successfully compute the discrete log. Moreover, as long as the online walk does not hit any of these points, its steps are independent random points in $\mathbb{G}$. If the number points touched during preprocessing is at least $ST/4$, then the online walk succeeds with probability at least $1-(1-(ST/(4N))^{T/2} \geq 1-\exp(-ST^2/(8N)) \geq ST^2/(16N)$. Overall, the probability of success ϵ is at least $1/2 \cdot ST^2/(16N) = \Omega(ST^2/N)$.

7.2 Multiple discrete logarithms with preprocessing

We now demonstrate that a similar technique allows solving the multiple-discrete-log problem more quickly using preprocessing. The algorithm is a modification to the attack of Foque et al. [33] to allow for precomputation, in the spirit of the algorithm of Section 7.1.

This upper bound matches the lower bound of Theorem 8 for a constant ϵ, up to logarithmic factors, which shows that the lower bound is tight for constant ϵ. To recall, an instance of the multiple-discrete-log problem is a vector $(g^{x_1}, \ldots, g^{x_M})$

for random $x_i \in \mathbb{Z}_N$. The solution is the vector $(x_1, \ldots, x_M)$. Then we have the following theorem:

Theorem 11. *There exists a generic algorithm with preprocessing for the M-instance multiple-discrete-log problem in a group of prime order N that makes use of a random function, uses $\widetilde{O}(S)$ bits of group-specific advice, runs in time $\widetilde{O}(T)$, succeeds with constant probability, and satisfies $ST^2/M + T^2 = O(MN)$.*

We prove the theorem in Appendix F.

7.3 Distinguishers with preprocessing

In this section, we give a new distinguishing algorithm for certain decisional problems in groups.

For concreteness, we first demonstrate how to use preprocessing to attack the *square decisional Diffie-Hellman problem* (sqDDH) [44], which is the problem of distinguishing tuples of the form (g, g^x, g^y) from tuples of the form $(g, g^x, g^{(x^2)})$ for random $x, y \in \mathbb{Z}_N$. In groups for which DDH is hard, the best known attack against this assumption requires solving the discrete-log problem.

Definition 12. We say that a generic algorithm $\mathcal{A}$ has *advantage* ϵ at distinguishing distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ if $\left|\Pr[\mathcal{A}^{\mathcal{O}_\sigma}(d_1) = 1] - \Pr[\mathcal{A}^{\mathcal{O}_\sigma}(d_2) = 1]\right| = \epsilon$, where the probability is over the random choice of the group oracle and samples $d_1 \sim \mathcal{D}_1$ and $d_2 \sim \mathcal{D}_2$.

An argument analogous to Theorem 9 shows that any generic sqDDH distinguisher that uses S bits of group-specific advice, runs in time T, and achieves distinguishing advantage ϵ must satisfy $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$. Our attack against shows that this bound is tight. Later on, we show how to generalize the attack to a larger family of natural decisional assumptions in groups.

Theorem 13. *There is a sqDDH distinguisher that makes use of a random function, uses $\widetilde{O}(S)$ bits of group-specific advice, runs in time $\widetilde{O}(T)$, and achieves distinguishing advantage ϵ whenever $ST^2 = \Omega(\epsilon^2 N)$.*

Remark. A simple sqDDH distinguisher takes as input a sample $(h_0, h_1) \in \mathbb{G}^2$, computes the discrete-log $x = \log_g(h_0)$ of the first group element and checks whether $h_1 = g^{(x^2)} \in \mathbb{G}$. Theorem 2 indicates that such a distinguisher using advice S and time T and achieving advantage ϵ must satisfy $ST^2 = \widetilde{\Omega}(\epsilon N)$. So, this attack allows the parameter setting $S = T = 1/\epsilon = N^{1/4}$. In contrast, the distinguisher of Theorem 13 allows the better running time and advice complexity roughly $S = T = 1/\epsilon = N^{1/5}$.

Remark. To see the cryptographic significance of Theorem 13, consider the pseudo-random generator $P(x) \stackrel{\text{def}}{=} (g^x, g^{(x^2)})$ that maps $\mathbb{Z}_N$ to $\mathbb{G}^2$. Theorem 13 shows that, for generic algorithms with preprocessing, it is significantly easier to distinguish this PRG from random than it is to compute discrete logs.

Proof Sketch of Theorem 13. The attack that proves the theorem combines two technical tools. The first tool is a general method to use preprocessing to distinguish PRG outputs from random, which we adopt from Bernstein and Lange [11]. (De et al. [24] rigorously analyze a more nuanced PRG distinguisher with preprocessing.) The second tool, adopted from the attack of Section 7.1, is the idea of taking a walk on the elements of the group, and applying the PRG distinguisher only to the set of points that lie at the end of long walks.

The attack works because a walk that begins at a point of the form $(g^x, g^{(x^2)})$ is likely to hit one of the precomputed endpoints quickly and applying the PRG distinguisher yields an ϵ-biased output value. In contrast, an attack that begins at a point of the form (g^x, g^y) will never hit a precomputed point and applying the distinguisher yields a relatively unbiased output.

The algorithm takes as input parameters $S, T \in \mathbb{Z}^+$.

As in the attack of Section 7.1, we use a random function to define a walk on a graph. In this case, the vertices of the graph are *pairs* of group elements—so every vertex is an element of $\mathbb{G}^2$. We also define the subset of vertices $\mathcal{Y} = \{(g^x, g^{(x^2)}) \mid x \in \mathbb{Z}_N\} \subset \mathbb{G}^2$ that correspond to “yes” instances of the sqDDH problem. The subset $\mathcal{Y}$ is very small relative to the set of all vertices $\mathbb{G}^2$, since $|\mathbb{G}^2| = N^2$, while $|\mathcal{Y}| = N$.

To define the walk on the vertices of this graph, we use a random function F that maps $\mathbb{G}^2 \to \mathbb{Z}_N$. Given a point $(h_0, h_1) \in \mathbb{G}^2$, the walk computes $\alpha \leftarrow F(h_0, h_1)$ and moves to the point $(h_0^\alpha, h_1^{(\alpha^2)}) \in \mathbb{G}^2$. Observe that if the walk starts in $\mathcal{Y}$ (i.e., at a “yes” point), the walk remains inside of $\mathcal{Y}$. If the walk starts at a point outside of $\mathcal{Y}$, the walk remains outside of $\mathcal{Y}$.

Out of the N^2 total vertices in the graph, we choose a set of distinguished or “marked” points $\mathcal{M}$, by marking each point independently at random with probability $1/T$. (In practice, we can choose the set of marked points using a hash function.) To each point in $\mathcal{M}$, we assign one of S different “colors,” again using a hash function. So there are roughly $N^2/(ST)$ points each with color $1, 2, \dots, S$.

Given these preliminaries, the algorithm works as follows:

- *Preprocessing phase.* Choose a $N/3T^2$ random points in $\mathcal{Y}$. From each of these points, take $2T$ steps of the walk on $\mathbb{G}^2$ that F defines. Halt the walk upon reaching a marked point $m \in \mathcal{M}$. If the walk hits a marked point, store the marked point along with its color c in a table.

  Group the endpoints of the walks by color. For each of the colors $c \in [S]$, find the prefix string $p_c \in \{0,1\}^{\log N}$ that maximizes the sum $\sum H(p_c, m)$, where $H : \{0,1\}^{\log N} \times \mathbb{G}^2 \to \{0,1\}$ is a random function and the sum is taken over the stored marked points m of color c.

  Store the prefix strings $(p_1, \dots, p_S)$ as the distinguisher's advice.
- *Online phase.* Given a sqDDH challenge $(h_0, h_1) \in \mathbb{G}^2$ as input, perform at most $10T$ steps of the walk on $\mathbb{G}^2$ that the function F defines. As soon as the walk hits a marked point $m \in \mathcal{M}$ of color c, return the value $H(p_c, m)$ as output. If the walk never hits a marked point, output “0”.

The distinguisher uses $\widetilde{O}(S)$ bits of group-specific advice and runs in time $\widetilde{O}(T)$ as desired. So all we must argue is that the algorithm achieves distinguishing advantage $\epsilon = \Omega(\sqrt{ST^2/N})$. We argue this last step in Appendix G. □

Attacking more general problems. The distinguishing attack of Theorem 13 applies to a general class of decisional problems in cyclic groups. Let $(f_1, \dots, f_\ell)$ be k-variate polynomials and let $\bar{x} = (x_1, \dots, x_k)$. Then we can define the problem of distinguishing, tuples of the form

$$(g^{x_1}, \dots, g^{x_k}, g^{f_1(\bar{x})}, \dots, g^{f_\ell(\bar{x})}) \quad \text{from} \quad (g^{x_1}, \dots, g^{x_k}, g^{r_1}, \dots, g^{r_\ell}),$$

for uniformly random $x_1, \dots, x_k, r_1, \dots, r_\ell \in \mathbb{Z}_N$.

The attack of Theorem 13 applies whenever there exists an index i, a linear function $L : \mathbb{G}^{k+\ell} \to \mathbb{G}$, and a constant $c > 1$ such that $L(\bar{x}, f_1(\bar{x}), \dots, f_\ell(\bar{x})) = x_i^c$. To apply the attack, first apply $L(\cdot)$ "in the exponent" to the challenge to get a pair $(g^{x_i}, g^{x_i^c}) \in \mathbb{G}^2$ and then run the distinguisher on this pair of elements.

As an example, this attack can distinguish tuples of the form $(g^{x_1}, g^{x_2}, g^{(x_1^2)}, g^{x_1 x_2}, g^{(x_2^2)})$ from random. The attack uses $i = 1$, $L(z_1, z_2, z_3, z_4, z_5) = z_3$, and $c = 2$. Note that this assumption is very closely related to the standard DDH assumption, except that the challenge tuple includes the extra elements $g^{(x_1^2)}$ and $g^{(x_2^2)}$.

Remark. Somewhat surprising is that the distinguishing attack of Theorem 13 *does not* imply the existence of a distinguisher for the DDH problem. So the lower bound of Theorem 9 may not be tight for the DDH problem, even though it is tight for related distinguishing problems. The attack of Theorem 13 implies that the problem of distinguishing $(g, g^x, g^{(x^2)})$ from random cannot be as hard as DDH unless there is a better DDH distinguishing attack with preprocessing.

8 Conclusion

We studied the limits of generic group algorithms with preprocessing for the discrete-logarithm problem and related computational tasks.

In almost all cases, our lower bounds match the best known attacks up to logarithmic factors in group order. The one exception is our lower bound for the decisional Diffie-Hellman problem, in which our lower bound is $ST^2 = \widetilde{\Omega}(\epsilon^2 N)$, but the attack requires computing a discrete logarithm with $ST^2 = \widetilde{O}(\epsilon N)$. When the success probability ϵ is constant, these bounds match. For intermediate values of ϵ, such as $\epsilon = N^{-1/4}$, it is not clear which bound is correct.

One useful task for future work would be to generalize our lower bounds to more complex assumptions, such as Diffie-Hellman assumptions on pairing-equipped groups [15], q-type assumptions [13], or the "uber" assumptions [14, 18].

In addition, our upper bounds of Section 7 make use of a public random function. Making the attacks fully constructive by removing this heuristic, in the spirit of Fiat and Naor [31] and De et al. [24], would be valuable as well.

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A Decoder for the proof of Lemma 4

Given the encoded string, the decoding routine recovers σ as follows.

1. Read the S-bit advice string st_σ from the encoding.
2. Read $\log \binom{|\mathcal{L}|}{N}$ bits from the encoding and decode the image of σ.
3. Initialize the table of pairs Table to the empty list.
4. Read $\sigma(1)$ from the encoding and add $(1, \sigma(1))$ to the table.
5. Repeat d times:
   (a) Choose the lexicographically first string s in the image of σ that does not yet appear in the table and add the pair (X, s) to the table.
   (b) Read $\log R$ bits from the encoding and decode the value $r^* \in [R]$.
   (c) Read $\log T$ bits from the encoding and decode the value $t \in [T]$.
   (d) Run $\mathcal{A}_1(\mathsf{st}_\sigma, s)$ for t queries using the r^*-th random tape allocated for this instance of $\mathcal{A}_1$. Reply to each query (u, v) that $\mathcal{A}_1$ makes as follows.
      i. If either u or v is outside the image of σ, reply $\perp$.
      ii. If either u or v is in the image of σ, yet it does not appear in the table, read $\log(N - |\mathsf{Table}|)$ bits from the encoding for each of them and decode $\sigma^{-1}(u)$ and/or $\sigma^{-1}(v)$.
      iii. Look up in the table the polynomials f_u, f_v representing $\sigma^{-1}(u)$, $\sigma^{-1}(v)$ in the table, and compute the linear polynomial $f_u + f_v$ representing the reply to the query. We then distinguish between the same three cases as above:
         A. If $(f_u + f_v, w)$ is already in the table, simply reply with w and continue to the next query.
         B. If this is not the last (t-th) query of this execution, read $\log(N - |\mathsf{Table}|)$ bits from the encoding, decode the bits as an element w of $\sigma(\mathbb{Z}_N)$ that is not in the table, and reply with w. Add $(f_u + f_v, w)$ to the table and continue to the next query.
         C. Finally if this is the t-th query, read a $(\log |\mathsf{Table}|)$-bit pointer from the encoding and look up the entry in the table (f_w, w) that it points to. Use this collision to recover the value x of the indeterminate X, by solving for X the equation $f_w = f_u + f_v \bmod N$. This equation always has a unique solution, since N is a prime and f_u, f_v and f_w are linear polynomials in X such that $f_u + f_v$ is not identical to f_w.

iv. If the execution $\mathcal{A}_1(\mathsf{st}_\sigma, s)$ completes and outputs x, evaluate all of the polynomials in the table at the point x.

6. Read the remaining values that do not yet appear in the table from the encoding.

B Proof of Theorem 7

Proof. As in the previous sections, we begin by using an averaging argument to get a subset G of all keyed group families $\sigma_{\text{key}} : [K] \times \mathbb{Z}_N \to \mathcal{L}$ for which

$$\Pr_{k,x,\mathcal{A}_1}\left[\mathcal{A}_1\left(\mathcal{A}_0^{\mathcal{O}_{\sigma_{\text{key}}}}(), k, \sigma(k,x)\right) = x\right] \geq \epsilon/2\,,$$

and

$$|G| \geq \frac{\epsilon}{2} \cdot \frac{||\mathcal{L}|!}{(|\mathcal{L}|-N)!}\,.$$

Then, we can use the random self-reduction of Lemma 3 to construct an algorithm $\mathcal{A}_1'$ that makes at most $T + O(\log N)$ oracle queries such that for every $x \in \mathbb{Z}_N$ it holds that

$$\Pr_{k,\mathcal{A}_1}\left[\mathcal{A}_1\left(\mathcal{A}_0^{\mathcal{O}_{\sigma_{\text{key}}}}(), k, \sigma(k,x)\right) = x\right] \geq \epsilon/2\,.$$

Note that the random self-reduction allows us to reduce only to instances having the same key, so even though the algorithm $\mathcal{A}_1$ is a *worst-case* algorithm over the choice of input $x \in \mathbb{Z}_N$, it is still only an *average-case* algorithm over the choice of key.

We can now use $\mathcal{A}_1'$ to construct a randomized encoding scheme for all $\sigma_{\text{key}} \in G$. Note that one cannot simply encode $\sigma_{\text{key}}(k,\cdot)$ independently for each key $k \in [K]$ because in general $\mathcal{A}_1(k, \sigma_{\text{key}}(k,x))$ may query the group oracle on other keys $k' \neq k$.

For a fixed labeling $\sigma_{\text{key}} : [K] \times \mathbb{Z}_N \to \mathcal{L}$, let $\{\epsilon_k\}_{k\in[K]}$ be the success probabilities of $\mathcal{A}_1$ for all possible keys. By definition, it holds that $\sum_{k=1}^{K} \epsilon_k/K \geq \epsilon/2$. Moreover, we may assume without the loss of generality that $\epsilon_k \geq 1/N$. Let

$$R_k = \frac{\log(KN)+1}{\epsilon_k}\ , \qquad D_k = \frac{N}{(2R_kT+1)} - 3T\ , \qquad D = \sum_{k=1}^{K} D_k.$$

The encoder then proceeds as follows:

1. Compute $\mathsf{st}_{\sigma_{\text{key}}} \leftarrow \mathcal{A}_0^{\mathcal{O}_{\sigma_{\text{key}}}}()$ and write this value into the encoding.
2. For each key $k \in [K]$, write the images $\sigma_{\text{key}}(k,\cdot)$ into the encoding.
3. Compute $\{\epsilon\}_{k\in[K]}$ by means of trivial derandomization of $\mathcal{A}_1$ (note that the encoder is not computationally bounded). Write the values $\{R_k\}_{k\in[K]}$ into the encoding, using $K(\log N + \log(\log(KN)+1))$ bits.
4. Initialize K empty tables of discrete logs $\{\mathsf{Table}_k\}_{k\in[K]}$.
5. While $\sum_{k=1}^{K} |\mathsf{Table}_k| \leq D$:

(a) Take the lexicographically first $k \in [K]$ such that $|\mathsf{Table}_k| \leq D_k$.

(b) Take the lexicographically first string s in the image of $\sigma_{\text{key}}(k, \cdot)$ which is still not in Table_k. Add (X, s) to Table_k, where X is a formal variable representing the discrete log of s.

(c) Execute $\mathcal{A}_1(\mathcal{A}_0, k, s))$ up to R_k times, using independent random coins each time (e.g., by means of partitioning the shared random tape as in Section 3). If all executions fail to compute the discrete log of s, abort. Otherwise, write the index of the first successful execution to the encoding.

(d) As in Section 3, add the execution trace of the successful execution to the encoding. The trace ends either when $\mathcal{A}_1$ makes the first non-trivial collision query or when it successfully outputs the correct value of the discrete log of s. In both cases, the decoder will be able to use this trace to recover the value of X. Note that a non-trivial collision may only occur on the keys k, since there is only a single indeterminate X at all times, and in each execution this indeterminate appears in the table Table_k, where k is the key for the input of that execution.
Also note that the execution trace may contain replies to oracle queries for all possible keys. It may also contain queries on unexpected labels, which do not appear in the table at the time of query. If that happens, the encoder adds discrete log of that label to the trace.

6. Add the values missing in each of the tables $\{\mathsf{Table}_k\}_{k\in[K]}$ in lexicographic order.

The Chernoff bound implies that all R_k attempts of a single execution fail with probability at most $1/(2KN)$, and taking a union bound over at most KN executions guarantees that the encoding routine fails with probability at most $1/2$.

Encoding Length. Non-colliding queries, as well as unexpected queries may belong to keys different than the one $\mathcal{A}_1$ is given as input, and for those tables it might happen that $\mathsf{Table}_{k'} > D_k$. However, regardless of the table size, the encoding of the replies to those queries will be of the same length as in the standard encoding: $\log(N - |\mathsf{Table}_{k'}|)$ bits for each value.

Each execution of $\mathcal{A}_1$ adds to the collection of tables $\{\mathsf{Table}_k\}_{k\in[K]}$ at most $3T + 1$ entries overall: the input to the execution, T replies and at most $2T$ unexpected query labels. Therefore the total number of executions is at least

$$\begin{aligned} d = \frac{D}{3T+1} &= \sum_{k=1}^{K} \frac{N}{(2R_kT+1)(3T+1)} - K\frac{3T}{3T+1} \geq \sum_{k=1}^{K} \frac{N}{6R_kT(T+1)} - K \\ &\geq \frac{\sum_{k=1}^{K} \epsilon_k N}{6T(T+1)(\log N + 1)} - K = \frac{\epsilon KN}{12T(T+1)(\log N+1)} - K\,. \end{aligned}$$

Moreover, each execution is guaranteed to save the $\log(N - |\mathsf{Table}_k|)$ bits needed to encode the discrete log of its input, either directly or due to a collision. This profit comes at a cost of at most $\log R_k + \log T + \log|\mathsf{Table}_k|$ additional bits.

Since at the beginning of each iteration, the table corresponding to the input key is bounded, we know that

$$\log R_k + \log T + \log |\mathsf{Table}_k| \leq \log \frac{R_k T N}{2R_k T + 1} \leq \log N - 1\,,$$

where we have used the fact that the iteration itself can add at most $3T$ additional entries to the table Table_k on top of the D_k entries at the beginning of the iteration. The last inequality implies that each of the d iterations saves at least one bit from the encoding.

We therefore obtain a randomized encoding of set G of length at most

$$S + K \log \binom{|\mathcal{L}|}{N} + K(\log N + \log(\log(KN) + 1)) + K \sum_{i=1}^{N-1} \log(N-i) - d$$

$$= K \log \frac{|\mathcal{L}||}{(|\mathcal{L}| - N)!} + S + O(K \log N) - \frac{\widetilde{\Omega}(\epsilon K N)}{T^2}\,.$$

Proposition 1 then gives us the bound

$$ST^2 = \widetilde{\Omega}(\epsilon K N)\,.$$

□

C Proof of Theorem 8

As in the proof of Theorem 2, we prove the theorem using a randomized encoding scheme.

As in Theorem 2, we first define the set of "good" labelings G and argue that it must have size at least $\epsilon/2 \cdot |\mathcal{L}|!/(|\mathcal{L}| - N)!$. We then apply the random self-reducibility of discrete log to convert a time-T average-case discrete log algorithm into a time $T' = T + O(M \log N)$ worst-case discrete log algorithm. We then construct a randomized encoding scheme that compresses every element of G, which is a generalization of the scheme constructed in Lemma 4.

We make two major changes to the encoder from Lemma 4. First, the encoding scheme runs $\mathcal{A}_1$ on batches of M labels at a time: $(\sigma(x_1), \ldots, \sigma(x_M))$. At any point during the encoder's execution, there are at most M formal variables $X_1, \ldots, X_M$ in the table, which represent the discrete logs of the M labels on which $\mathcal{A}_1$ is executing. (In contrast, the encoder of Lemma 4 uses a table that contains at most a single formal variable X at any one time.)

Second, the encoder runs each execution $\mathcal{A}_1$ until the latter either (a) outputs the discrete logs $(x_1, \ldots, x_M) \in \mathbb{Z}_N^M$, or (b) causes M "collision events." As before, we define a collision event to be one in which $\mathcal{A}_1$ makes an oracle query $(\sigma(i), \sigma(j))$ such that:

- the discrete logs of $\sigma(i)$ and $\sigma(j)$ are represented by polynomials f_i and f_j in the encoder's table,

– the response string $\sigma(i+j)$ is in the encoder's table, and

– the discrete logarithm of the response string stored in the table is a polynomial unequal to $f_i + f_j$.

The encoding scheme obeys the invariant that after the c-th such collision event, there are at most $M - c$ formal variables in the table. With each collision event, the number of free variables in the table decreases by one, and after M collision events, the encoder is able to resolve all of the free variables in the table to constants. The encoder then halts $\mathcal{A}_1$.

As before, the encoder handles the fact that algorithm $\mathcal{A}_1$ succeeds only with probability ϵ by running it $R = (1 + \log N)/\epsilon$ times on each input (using fresh randomness in each run), and including in the encoding the index r^* of the first successful execution. This ensures that the encoding scheme succeeds with probability at least $1/2$.

The encoder, which we describe in detail in Appendix C.1, runs $\mathcal{A}_1$ on d batches of M instances. The encoder then writes the values of σ that are still undetermined directly into the encoded string using a naïve encoding.

The source of the "profit" for the encoding (in terms of its length) is that each of the d iterations encodes M discrete logs using a string that is shorter than than the standard encoding. Specifically, each iteration encodes M discrete logs using at most $\log R$ bits to indicate the index r^* of the good random tape, $\log \binom{T}{M}$ bits to indicate the colliding queries, and $M \log |\mathsf{Table}|$ bits to indicate the replies to the colliding queries within the table of previous replies. Replies to non-colliding queries, unexpected query inputs, and the values remaining after the d iterations, are all encoded using the standard encoding.

Each iteration therefore results in a profit of at least

$$\begin{aligned} & M \log(N - |\mathsf{Table}|) - \log R - \log \binom{T}{M} - M \log |\mathsf{Table}| \\ & = M \left(\log(N - |\mathsf{Table}|) - \log R^{\frac{1}{M}} - \log \frac{3T}{M} - \log |\mathsf{Table}| \right) \\ & = M \log \frac{(N - |\mathsf{Table}|)M}{3R^{1/M} T |\mathsf{Table}|} \ \text{ bits,} \end{aligned}$$

where we applied the inequality $\binom{T}{M} \le (eT/M)^M \le (3T/M)^M$.

Starting with a table that contains only $\sigma(1)$, each iteration adds at most $3T + M$ values to the table (via M inputs, at most $2T$ unexpected query inputs and T query outputs). After d iterations the table grows to a size of at most $d(3T + M) + 1 \le 5T$, where the last inequality follows from the fact that $M \le T$. Setting

$$d = \left\lfloor \frac{NM}{35 R^{\frac{1}{M}} T^2} \right\rfloor \ge \frac{\epsilon^{\frac{1}{M}} NM}{35(1 + \log N)^{\frac{1}{M}} T^2} - 1$$

guarantees that

$$|\mathsf{Table}| \le \frac{NM}{7 R^{\frac{1}{M}} T}$$

and subsequently

$$\log \frac{(N - |\mathsf{Table}|)M}{3R^{1/M}T|\mathsf{Table}|} \geq \log \frac{(6N/7)M}{3NM/7} \geq 1\,.$$

The profit from each iteration is therefore at least M bits, and the overall profit of d iterations is $dM - S$ bits.

The encoding scheme succeeds with probability $1/2$, and it compresses all labelings in the set G, which is a set of size $\epsilon/2$ of all labelings. By Proposition 1, the profit of such an encoding scheme can be at most $\log \frac{2}{\epsilon} + 1$, which implies

$$dM - S \leq \log \frac{2}{\epsilon} + 1\,,$$

or

$$\left(\frac{\epsilon^{\frac{1}{M}} NM}{35(1 + \log N)^{\frac{1}{M}} T^2} - 1 \right) M - S \leq 2 + \log(1/\epsilon)\,.$$

Using the fact that $\log(1/\epsilon) \leq \log N$, we get

$$ST^2/M + T^2 = \widetilde{\Omega}(\epsilon^{1/M} NM)\,. \quad \square$$

We now describe the encoding and decoding routines in detail.

C.1 Detailed description of the encoding scheme

Encoding routine. The encoding routine, on input σ, uses two parameters $d, R \in \mathbb{Z}^+$, which we will set later. The encoder keeps a table of pairs in $\mathbb{Z}_N[X_1, \ldots, X_M] \times \mathcal{L}$ as its state. The encoder proceeds as follows:

1. Compute $\mathsf{st}_\sigma \leftarrow \mathcal{A}_0(\sigma(1))$. The encoder can respond to all of the algorithm's oracle queries since the encoder knows all of σ. Write the S-bit output st_σ into the encoding.
2. Encode the image of σ as a subset of $\mathcal{L}$ using $\log \binom{|\mathcal{L}|}{N}$ bits, and append it to the encoding.
3. Initialize the table of pairs to an empty list.
4. Repeat d times:
   (a) Select the lexicographically next M strings in the image of σ that do not yet appear in the table. Call these strings $\sigma(x_1), \ldots, \sigma(x_M)$. Add the M pairs $(X_1, \sigma(x_1)), \ldots, (X_M, \sigma(x_M))$ to the table.
   (b) Run $\mathcal{A}_1(\mathsf{st}_\sigma, \sigma(x_1), \ldots, \sigma(x_M))$ up to R times using independent randomness from the encoder's random string r in each run. The encoder answers all of $\mathcal{A}_1$'s oracle queries using its knowledge of σ. If $\mathcal{A}_1$ fails on all R executions, abort the entire encoding routine. Otherwise, write into the encoding the index $r^* \in [R]$ of the successful execution (using $\log R$ bits).
   (c) Write a placeholder value of $\log \binom{T}{M}$ zero bits into the encoding. (The routine overwrites these zeros with a meaningful values as the execution of $\mathcal{A}_1$ terminates.)

(d) Rerun the algorithm $\mathcal{A}_1(\mathsf{st}_\sigma, \sigma(x_1), \ldots, \sigma(x_M))$ using the r^*th random tape. While the algoritm is running, it makes oracle queries and then outputs its guess of $(x_1, \ldots, x_M)$. The encoding routine processes each of $\mathcal{A}_1$'s queries $(\sigma(i), \sigma(j))$ as follows:

   i. If either of the query arguments is outside of the range of σ, reply $\perp$ and continue to the next query.

   ii. If either (or both) of the arguments is missing from the table, then this is an "unexpected" query input. Add each such input, together with its discrete log, to the table and append the discrete log value i to the encoding, using $\log(N - |\mathsf{Table}|)$ bits.

   iii. Look up the linear polynomials f_i and f_j, representing $\sigma(i)$ and $\sigma(j)$, in the table. Compute the linear polynomial $f_i + f_j$ representing the response $\sigma(i + j)$. We then distinguish between three cases:

      A. If $(f_i + f_j, \sigma(i + j))$ is already in the table, simply reply with $\sigma(i + j)$.

      B. If $\sigma(i + j)$ does not appear in the table, then add $\sigma(i + j)$ to the encoding, using $\log(N - |\mathsf{Table}|)$ bits, and reply with $\sigma(i + j)$.

      C. If $\sigma(i + j)$ appears in the table but its discrete log stored in the table is $f_k \neq f_i + f_j$, encode the reply to this query as a $(\log|\mathsf{Table}|)$-bit pointer into the table entry $(f_k, \sigma(i + j))$ and add this pointer to the encoding.

      At the cth such collision event, the table has $M' = M - c$ free variables in it. Without loss of generality, assume that these variables are $X_1, \ldots, X_{M'}$. We then have a linear relation of the form $f_i + f_j = f_k \in \mathbb{Z}_N[X_1, \ldots, X_{M'}]$, or

$$\alpha_1 X_1 + \cdots + \alpha_{M'} X_{M'} = \beta_1 X_1 + \cdots + \beta_{M'} X_{M'} \quad \in \mathbb{Z}_N. \quad (2)$$

      Since the left- and right-hand polynomials are unequal, there must be at least one $m^* \in [M']$ for which the coefficients α_{m^*} and β_{m^*} are unequal. Choose the first such m^*, and use (2) to write X_{m^*} as a linear function of the other variables:

$$X_{m^*} = (\beta_{m^*} - \alpha_{m^*})^{-1} \cdot \sum_{m \neq m^*} (\beta_m - \alpha_m) X_m.$$

      Then replace every instance of X_m in the table with this function f_m. We have now decreased the number of free variables in the table by one.

      Store the time t_c at which this "collision event" occurred and, if there are no more variables in the table, halt this execution of $\mathcal{A}_1$.

(e) When the execution $\mathcal{A}_1(\mathsf{st}_\sigma, \sigma(x_1), \ldots, \sigma(x_M))$ outputs $(x_1, \ldots, x_M)$, evaluate all of the polynomials in the table at the point $(x_1, \ldots, x_M)$.

(f) After each execution of $\mathcal{A}_1$ completes, write the times $t_1, \ldots, t_M$ at which the M collisions occurred into the placeholder using $\log \binom{T}{M}$ bits.

5. Append the remaining values that do not yet appear in the table to the encoding in lexicographic order.

Decoding routine. Given the encoded string, the decoding routine recovers σ as follows.

1. Read the S-bit advice string st_σ from the encoding.
2. Read $\log \binom{|\mathcal{L}|}{N}$ bits from the encoding and decode the image of σ.
3. Initialize the table of pairs Table to the empty list.
4. Read $\sigma(1)$ from the encoding and add $(1, \sigma(1))$ to the table.
5. Repeat d times:
   (a) Choose the M lexicographically first strings $s_1, \ldots, s_M$ in the image of σ that do not yet appear in the table and add the pairs $(X_1, s_1), \ldots, (X_M, s_M)$ to the table.
   (b) Read $\log R$ bits from the encoding and decode $r^* \in [R]$.
   (c) Read $\log \binom{T}{M}$ bits from the encoding and decode $t_1, \ldots, t_M \in [T]$.
   (d) Run $\mathcal{A}_1(\mathsf{st}_\sigma, s_1, \ldots, s_M)$ for t queries using the r^*th random tape allocated for this instance of $\mathcal{A}_1$. Reply to each query (u, v) that $\mathcal{A}_1$ makes as follows.
      i. If either u or v is outside the image of σ, reply $\perp$.
      ii. If either u or v is in the image of σ, yet it does not appear in the table, read $\log(N - |\mathsf{Table}|)$ bits from the encoding for each of them and decode $\sigma^{-1}(u)$ and $\sigma^{-1}(v)$.
      iii. Look up in the table the polynomials f_u, f_v representing $\sigma^{-1}(u)$, $\sigma^{-1}(v)$ in the table, and compute the linear polynomial $f_u + f_v$ representing the reply to the query. We then distinguish between the same three cases as above:
         A. If $(f_u + f_v, w)$ is already in the table, simply reply with w and continue to the next query.
         B. If the index t of this query is in $\{t_1, \ldots, t_M\}$, read a $(\log |\mathsf{Table}|)$-bit pointer from the encoding and look up the entry in the table (f_w, w) that it points to.
         Following the same principle as described in the encoder, we can use this collision event to reduce the number of free variables in the table by one. If there are no more free variables in the table after this step, halt this execution of $\mathcal{A}_1$.
         C. Otherwise, read $\log(N - |\mathsf{Table}|)$ bits from the encoding, decode the bits as an element w of $\sigma(\mathbb{Z}_N)$ that is not in the table, and reply with w. Add $(f_u + f_v, w)$ to the table and continue to the next query.
      iv. If the execution $\mathcal{A}_1(\mathsf{st}_\sigma, s_1, \ldots, s_M)$ completes and outputs $(x_1, \ldots, x_M)$, evaluate all of the polynomials in the table at the point $(x_1, \ldots, x_M)$.
6. Read the remaining values that do not yet appear in the table from the encoding.

D Proof of Theorem 9

The core idea of the proof is in the construction of the following randomized encoding scheme for labeling functions.

Lemma D.1. *Let N be a prime and let $G = \{\sigma_1, \sigma_2, \dots\}$ be a set of labeling functions from $\mathbb{Z}_N$ to $\mathcal{L}$. Let $(\mathcal{A}_0, \mathcal{A}_1)$ be a pair of generic algorithms such that algorithm $\mathcal{A}_0$ outputs an S-bit state, algorithm $\mathcal{A}_1$ makes at most T oracle queries, and for every $\sigma \in G$ and every $x, y, z \in \mathbb{Z}_N$, $\mathcal{A}(\sigma(x), \sigma(y), \sigma(z))$ agrees with a DDH oracle with probability $1/2 + \epsilon$ over the coins of $\mathcal{A}_1$. Then, there exists a randomized encoding scheme that compresses elements of G to bitstrings of length at most*

$$\log \frac{|\mathcal{L}|!}{(|\mathcal{L}| - N)!} + S - \widetilde{\Omega}\left(\frac{\epsilon^2 N}{T^2}\right).$$

and fails with probability at most $1/2$.

Before proving Lemma D.1, we show how to use the lemma to prove Theorem 9.

Proof of Theorem 9. Without loss of generality, we may assume that algorithm $\mathcal{A}_0$ is deterministic, as it is not bounded in its running time or its number of queries, and thus can be trivially derandomized.

A standard averaging argument implies that there exists a set G such that for every $\sigma \in G$, it holds that

$$\left| \Pr_{x,y,z,\mathcal{A}} \left[\mathcal{A}_1^{\mathcal{O}_\sigma}\left(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)), \sigma(x), \sigma(y), \sigma(xy)\right) = 1\right] \right.$$
$$\left. - \Pr_{x,y,z,\mathcal{A}} \left[\mathcal{A}_1^{\mathcal{O}_\sigma}\left(\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1)), \sigma(x), \sigma(y), \sigma(z)\right) = 1\right] \right| \geq \epsilon/2\,,$$

and

$$|G| \geq \frac{\epsilon}{2} \log \frac{|\mathcal{L}|!}{(|\mathcal{L}| - N)!}\,.$$

Next, we use the random self-reducibility of the DDH problem [53, 68] to convert the average-case algorithm to a worst-case one. Specifically, using a random self-reduction we construct an algorithm $\mathcal{A}_1'$ that makes $T' = T + O(\log N)$ oracle queries, and for every $\sigma \in G$ and every $x, y, z \in \mathbb{Z}_N$, it holds that, for $\mathsf{st}_\sigma \leftarrow \mathcal{A}_0(\sigma(1))$, $\mathcal{A}_1'(\mathsf{st}_\sigma, \sigma(x), \sigma(y), \sigma(z))$ agrees with a DDH oracle with probability $1/2 + \epsilon/2$, over the choice of its coins only.

Using Lemma D.1, we can then obtain a randomized encoding scheme for every $\sigma \in G$, that succeeds with probability $1/2$ and produces encodings of bitlength at most

$$\log \frac{|\mathcal{L}|!}{(|\mathcal{L}| - N)!} + S - \widetilde{\Omega}\left(\frac{\epsilon^2 N}{T^2}\right).$$

By Proposition 1, it must hold that $ST^2 \geq \widetilde{\Omega}\left(\epsilon^2 N\right)$. □

Proof of Lemma D.1

As before, we use algorithms $(\mathcal{A}_0, \mathcal{A}_1)$ to produce a short encoding of a labeling function σ. We begin the encoding with the advice string $\mathcal{A}_0^{\mathcal{O}_\sigma}(\sigma(1))$ and an encoding of the image of σ as a subset of $\mathcal{L}$ using $\log \binom{|\mathcal{L}|}{N}$ bits.

The DDH algorithm in this section outputs only a single bit, rather than the discrete log of any of its inputs, so the encoding scheme must leverage $\mathcal{A}_1$'s output to save one bit from the encoding length of σ. The basic idea is that the encoder will encode the two product terms $\sigma(x_1 \cdot y_1)$ and $\sigma(x_2 \cdot y_2)$ of two DDH triplets $(\sigma(x_1), \sigma(y_1), \sigma(x_1 \cdot y_1))$ and $(\sigma(x_2), \sigma(y_2), \sigma(x_2 \cdot y_2))$ as an *unordered set* $\{\sigma(x_1 \cdot y_1), \sigma(x_2 \cdot y_2)\}$ and the decoder will recover their correct order by running the DDH algorithm $\mathcal{A}_1$. Encoding the two elements as an unordered set saves one bit for every two DDH triplets, which is the source of profit for our encoding.

At each iteration, the encoder takes the four lexicographically-next unencoded labels $\sigma(x_1), \sigma(y_1), \sigma(x_2), \sigma(y_2)$ and computes $s_1 = \sigma(x_1 \cdot y_1), s_2 = \sigma(x_2 \cdot y_2)$. As before, both the encoder and the decoder build up a table of pairs $(f, s) \in \mathbb{Z}[X_1, Y_1, X_2, Y_2] \times \mathcal{L}$, that represents the partial information about σ that has been encoded or decoded up to that point. During each iteration, the indeterminates correspond to the unknown discrete logs of the two DDH triplets of that iteration.

The encoder first executes $\mathcal{A}_1$ on the DDH triplet $(\sigma(x_1), \sigma(y_1), s)$, where s is the lexicographically-first label out of s_1 and s_2. We distinguish between the following two cases.

Case I: Execution with collisions. As in the proof of Theorem 2, we define a "collision event" to be a case in which (a) the response to one of the adversary's queries is a label that already appears in the encoder's table and (b) the label's discrete log in the table is represented by a different formal polynomial than the query response.

A collision occurs during execution on the triplet $(\sigma(x_1), \sigma(y_1), s)$ if the algorithm $\mathcal{A}_1$ issues a query that collides with an element that already appears in the table at the time of query. In the DDH encoder, we must handle queries that collide with the DDH challenge elements s_1 and s_2. Such collisions could happen if as soon as the encoder picks x_1, y_1, x_2, y_2, the value $s_1 = \sigma(x_1 \cdot y_1)$ (or similarly $\sigma(x_2 \cdot y_2)$) already appears in the table.

If any collision occurs, the encoder proceeds as following:

1. Write to the encoding the "time of collision" as a number t between 1 and $T + 2$, where the first two time-slots represent the elements s_1 and s_2.
2. Write to the encoding the execution trace of $\mathcal{A}_1$ on the DDH triplet $(\sigma(x_1), \sigma(x_2), s)$, where s is the lexicographically-first label out of s_1 and s_2. The execution trace starts with the two elements s_1 and s_2 and ends after t queries. with their corresponding table entries $(X_1 \cdot Y_1, s_1)$ and $(X_2 \cdot Y_2, s_2)$. Moreover, as in Section 3.1, the trace contains also the discrete logs of "unexpected" queries.
3. Indicate the entry in the table with which the t-th query collides.
4. The collision induces a degree-two equation in four variables X_1, Y_1, X_2, Y_2. By the Schwartz-Zippel Lemma [64, 76], this equation has at most $2(N -$

$|\mathsf{Table}|)^3$ solutions over the set of unused labels (which is a set of size $N - |\mathsf{Table}|$). The encoder indicates the actual solution x_1, y_1, x_2, y_2 using $3\log(N - |\mathsf{Table}|) + 1$ bits.

Case II: Collision-free execution. If the execution of algorithm $\mathcal{A}_1$ on the DDH triplet $(\sigma(x_1), \sigma(x_2), s)$ does not contain any collision, we encode it differently. In this case, the encoder writes to the encoding the discrete logs x_1, y_1, x_2, y_2 and the *unordered set* $\{s_1, s_2\}$. It then writes to the encoding the execution trace of $\mathcal{A}_1$ on this DDH triplet.

When the decoder reads this part of the encoding, it recovers the table entries $(x_1, \sigma(x_1))$, $(y_1, \sigma(y_1))$, $(x_2, \sigma(x_2))$, $(y_2, \sigma(y_2))$ as well as two elements $s, s' \in \mathcal{L}$, yet it does not know the correct mapping between $x_1 \cdot y_1, x_2 \cdot y_2$ and s, s'. The decoder then runs the DDH algorithm $\mathcal{A}_1$ on the DDH triplet $(\sigma(x_1), \sigma(y_1), s)$ and replies to its oracle queries using the trace that the encoder has provided it with in the encoding. When $\mathcal{A}_1$ completes, it outputs a bit, which the decoder uses to recover the correct mapping between $x_1 \cdot y_1, x_2 \cdot y_2$ and s, s'.

Batching. Until now, we have assumed that the execution of $\mathcal{A}_1$ on $(\sigma(x_1), \sigma(x_2), s)$ outputs the correct bit. However, algorithm $\mathcal{A}_1$ is only guaranteed to be correct with probability $1/2+\epsilon$. One option to deal with this issue would have been to use standard error reduction, namely execute the algorithm $O(1/\epsilon^2)$ times on each input, using independent randomness each time and then take the majority vote on its outputs. However, this would increase the algorithm's effective running time to $O(T/\epsilon^2)$, which would weaken the resulting lower bound.

Instead, we group our executions into batches of size B, where each execution within the batch handles two DDH triplets. We run each batch R times, using independent randomness each time, until every one of the R executions within the batch either (a) successfully outputs the correct DDH bit or (b) makes a collision query. If none of the R attempts is successful, the entire encoding routine fails.

For each batch, the encoding algorithm, writes the following information into the encoding:

1. The index $r^* \in [R]$ of the successful attempt.
2. The number of executions within this attempt that have collisions, written as a *unary* string, terminated with a single '0' bit. For example, if there were five collisions, write "111110" into the encoding. If there were no collisions, just write "0."
3. The indices of the executions that have collisions.
4. The encodings of each of the executions within the batch.

Overall, the encoder processes d batches, after which it encodes the remaining values of σ using the standard encoding.

Success probability. The encoding succeeds if we find a good random tape r^* in one of the R attempts we make for each batch. Each attempt of the entire batch succeeds with probability at least $(1/2+\epsilon)^B$. Therefore, at least one of R attempts succeeds with probability $1 - (1 - (1/2+\epsilon)^B)^R > 1 - e^{-R(1/2+\epsilon)^B}$.

Taking

$$R = (1 + \log N)/(1/2 + \epsilon)^B \tag{3}$$

implies this probability is at least $1 - 1/(2N)$. The number of batches cannot be more than N, so by a union bound the encoding succeeds with probability at least $1/2$.

Encoding length. An execution that does not contain a collision encodes the labels $s_1 = \sigma(x_1 \cdot y_1)$, $s_2 = \sigma(x_2 \cdot y_2)$ using $\log \binom{N-|\mathsf{Table}|}{2}$ bits instead of $\log(N - |\mathsf{Table}|) + \log(N - |\mathsf{Table}| - 1)$ bits. Such an execution therefore saves one bit compared to the standard encoding.

An execution that contains a collision includes encodes the four discrete logs x_1, y_1, x_2, y_2 by including the index of the execution within the batch, the index of the collision query within the execution, the answer to the collision query within the table of existing values, and the index of the solution to the equation within the set of all solutions. We also let it account for one of the bits in the unary encoding of the number of collisions. The overall cost is

$$\log B + \log T + \log |\mathsf{Table}| + (3\log(N - |\mathsf{Table}|) + 1) + 1$$

bits instead of $\log(N - |\mathsf{Table}|) + \log(N - |\mathsf{Table}| - 1) + \log(N - |\mathsf{Table}| - 2) + \log(N - |\mathsf{Table}| - 3)$ bits. Therefore, each execution results in a profit of

$$\log \frac{(N - |\mathsf{Table}|)(N - |\mathsf{Table}| - 1)(N - |\mathsf{Table}| - 2)(N - |\mathsf{Table}| - 3)}{4BT|\mathsf{Table}|(N - |\mathsf{Table}|)^3} \tag{4}$$

bits. To bound the size of the table, note that each of the d batches contains B executions, each of which adds to the table at most $3T + 6$ entries: two DDH triplets, up to T query replies and up to $2T$ unexpected query inputs. Therefore the table grows to a size of at most $6dB(T + 2)$. Setting

$$d = \frac{\frac{N}{12BT} - 3}{6B(T + 2)} \tag{5}$$

implies that the table grows to a size of at most $N/(12BT) - 3$, which in its turn implies that $N - |\mathsf{Table}| - 3 \geq 11N/12$. This means that the profit in Expression 4 is at least

$$\log \frac{(11N/12)^4}{4BT \cdot N/(12BT) \cdot N^3} \geq \log \frac{11^4}{4 \cdot 12^3} \geq 1\,.$$

So far, we have set our parameters such that each of the B executions within a batch results in a profit of at least one bit. However, the encoding of each batch also includes r^*, written using $\log R$ bits, which we must take into account. Additionally, we need to account for the final bit of the unary encoding of the number of collisions. Using our choice of R from Equation 3, we get that the profit of each batch is

$$\begin{aligned} B - \log R - 1 &\geq B - \log\log(2N) + B\log(1/2 + \epsilon) - 1 \\ &\geq -\log\log(2N) + \epsilon B - 1\,, \end{aligned}$$

where we've used the fact that $\log(1/2 + \epsilon) = -1 + \log(1 + 2\epsilon) \geq -1 + \epsilon$. Setting

$$B = \frac{\log\log(2N) + 2}{\epsilon}$$

implies that each batch results in a profit of at least 1 bit. Plugging this into Equation 5, we get that the number of batches is

$$d = \varOmega\left(\frac{\epsilon^2 N}{T^2 \log\log N}\right) ,$$

and the overall length of our encoding is

$$S + \log\binom{|\mathcal{L}|}{N} + \sum_{i=0}^{N-1} \log(N - i) - d = \log\frac{|\mathcal{L}|!}{(|\mathcal{L}| - N)!} + S - d$$
$$= \log\frac{|\mathcal{L}|!}{(|\mathcal{L}| - N)!} + S - \widetilde{\varOmega}\left(\frac{\epsilon^2 N}{T^2}\right) .$$

E Proof of Theorem 10

Proof. The proof uses a pair of probabilistic experiments. As the experiments proceed, the challenger maintains a table representing all of the information that the adversary has about σ, given the challenger's query responses.

In particular, fix a formal variable X, representing the value of the discrete log that the adversary is trying to recover. Each row in the challenger's table contains a label $\ell_i \in \mathcal{L}$ and a linear polynomial $\mathbb{Z}_N[X]$ representing the label's discrete log as a function of X. After the t-th query, the table consists of elements $(f_1, \ell_1), \ldots, (f_t, \ell_t) \in (\mathbb{Z}_N[X] \times \mathcal{L})$.

In Experiment 0, the values $f_1, \ldots, f_t$ are all constant polynomials in $\mathbb{Z}_N[X]$ (i.e., the f_is are all constants in $\mathbb{Z}_N$). In Experiment 1, the values $f_1, \ldots, f_t$ may be linear polynomials in the formal variable X.

Experiment 0: Interaction with real oracle σ. At the start of the preprocessing phase, the challenger sets $s_1 \xleftarrow{\text{R}} \mathcal{L}$, representing the value $\sigma(1)$, and sends s_1 to the adversary $\mathcal{A}_0$. The challenger's adds the single pair $(1, s_1)$ to its table.

Preprocessing phase. The adversary $\mathcal{A}_0$ makes P preprocessing queries. Upon receiving a query, the challenger processes it as follows:

- First, the challenger checks if both query arguments are in the list of labels $\ell_1, \ldots, \ell_t$ in the table. If either argument is not in the table, we say that it is an "unexpected" label.
- For each unexpected label, the challenger defines the label's discrete log to be a random value in $\mathbb{Z}_N$ consistent with the simulation so far. Let there be t elements in the table, and let ℓ_{t+1} be the unexpected label. The challenger samples a random constant $f_{t+1} \xleftarrow{\text{R}} \mathbb{Z}_N$ such that f_{t+1} is different from all other constants in the table. The challenger then adds the pair (f_{t+1}, ℓ_{t+1}) to the table.

- Again, let t be the number of pairs in the table (t might have increased if we had to handle unexpected arguments). Now the query is of the form $(\ell_\alpha, \ell_\beta)$, such that both arguments appear in the table. Define $f_{t+1} \leftarrow f_\alpha + f_\beta \in \mathbb{Z}_N[X]$.
  - If $f_{t+1} = f_i$ for some $i \in \{1, \dots, t\}$, set $\ell_{t+1} \leftarrow \ell_i$.
  - Otherwise, sample $\ell_{t+1} \overset{\text{R}}{\leftarrow} \mathcal{L} \setminus \{\ell_1, \dots, \ell_t\}$.
- Respond with ℓ_{t+1} and add the pair (f_{t+1}, ℓ_{t+1}) to the table.

Online phase. After the P preprocessing queries, the challenger chooses $x \overset{\text{R}}{\leftarrow} \mathbb{Z}_N$.

- If x is already defined in the table, the challenger reads its label s_x from the table.
- Otherwise, the challenger sets s_x to be a random label in $\mathcal{L}$ different from all labels in the table and adds the pair (x, s_x) to the table. Note here that x is a constant in $\mathbb{Z}_N$, *not* a formal variable. So, in Experiment 0, the table maps only constants in $\mathbb{Z}_N$ to labels in $\mathcal{L}$.

The challenger sends the label s_x to the adversary $\mathcal{A}_1$ as the discrete-log instance. The adversary now makes T additional queries and we respond to the queries as in the preprocessing phase. The adversary then outputs a guess x' of x and we say that the adversary wins if $x = x'$.

Experiment 1: Interaction with oblivious oracle $\mathcal{O}_\sigma$. Experiment 1 is identical to Experiment 0, except that (a) the challenger adds the pair (X, s_x) to the table at the start of the experiment, where X is a formal variable, and (b) the challenger chooses $x \overset{\text{R}}{\leftarrow} \mathbb{Z}_N$ independently and uniformly at random *after* the adversary has made its T online queries. Thus, in Experiment 1, the oracle is oblivious of x until the end of the experiment, and the adversary's view is independent of x.

First, notice that the challenger in Experiment 0 exactly implements a group operation oracle for a random labeling function $\sigma : \mathbb{Z}_N \to \mathcal{L}$. So the adversary's success probability in Experiment 0 is exactly the adversary's discrete-log success probability. Next, notice that the adversary's view in Experiment 1 is independent of the discrete log x that the challenger has chosen. Because of this, the adversary's success probability in Experiment 1 can be no higher than $1/N$, the probability of guessing x at random.

Let W_b the probability that the adversary wins in Experiment $b \in \{0, 1\}$. We know that $W_1 \leq 1/N$. To prove the theorem, we need only show that $|\Pr[W_0] - \Pr[W_1]| \leq O((PT + T^2)/N)$.

Towards this goal, we define a collision event C on the probability space shared by both experiments. We say that C occurs if there are two unequal polynomials f_i and f_j in the challenger's table such that $f_i(x) = f_j(x) \in \mathbb{Z}_N$, where x is the discrete-log value chosen during the experiment. There are two types of collisions:

1. A collision between a polynomial added to the table during preprocessing and a polynomial added during the online phase.

2. A collision between two polynomials added during the online phase.

(The polynomials added to the table during preprocessing cannot collide, since they are all distinct constants.) For a fixed pair of unequal polynomials, the probability that they collide, over a choice of a random $z \in \mathbb{Z}_N$, is at most $1/N$, by the Schwartz-Zippel Lemma [64, 76]. Each preprocessing query adds at most three constant polynomials to the table, and each online query adds at most two constant polynomials and one linear polynomial to the table. There are then at most $3P + 2T$ distinct constant polynomials and T distinct linear polynomials in the table. There are then $(3P + 2T)T + \binom{T}{2} = O(PT + T^2)$ possible colliding pairs (f_i, f_j) in the table. A union bound yields that $\Pr[C] \leq O(PT + T^2)/N$.

We claim that $W_0 \wedge \bar{C}$ occurs if and only if $W_1 \wedge \bar{C}$ occurs. To see why: if the table never contains a pair of colliding polynomials then the behavior of the challenger is identical in both games. By the Difference Lemma [17, Theorem 4.5], this implies that $|\Pr[W_0] - \Pr[W_1]| \leq \Pr[C] \leq O((PT + T^2)/N)$. □

F Proof of Theorem 11

Proof Sketch of Theorem 11. The algorithm takes as input parameters $S, T \in \mathbb{Z}^+$ such that $ST^2/M + T^2 = \Theta(MN)$. As in the algorithm of Section 7.1, we use a random function $F : \mathbb{G} \to \mathbb{Z}_N$ to define a walk on the elements of the group $\mathbb{G}$.

Given these preliminaries, the algorithm works as follows:

– *Preprocessing phase.* Pick S distinct elements of $\mathbb{G}$ at random. For each of the elements, take a walk of length $T/(2M)$ on $\mathbb{G}$ that F defines. Store the S endpoints of these walks along with their discrete logs.

– *Online phase.* Let the problem instance be $(h_1, \ldots, h_M) = (g^{x_1}, \ldots, g^{x_M})$.

  The algorithm repeats the following steps $O(M)$ times:

  1. The algorithm computes a product of the group elements: $h = \prod_i h_i^{r_i}$, for random $r_i \in \mathbb{Z}_N$.
  2. Starting at the point h, the algorithm follows the walk on $\mathbb{G}$ determined by F for T/M steps.
  3. If the walk ever collides with a previous walk or a stored precomputed point, the collision yields a random linear relation on $(x_1, \ldots, x_M)$. The algorithm stores this relation. (*Note:* In practice, an implementation would detect collisions using the distinguished-points technique [58].)

  If the relations collected during the $O(M)$ walks yield a full-rank linear system, solving the system yields the desired discrete logs. Otherwise, the algorithm fails. To avoid linear dependence between relations, one can eliminate one variable after each step and apply the random reduction of step 1 above only to the uneliminated variables).

To analyze the complexity of the algorithm: The online algorithm takes $O(M)$ walks, each of T steps. Each step requires a logarithmic number of queries, so the total running time is $\widetilde{O}(T)$. The space usage is $\widetilde{O}(S)$ for the discrete logs of the S stored points.

Finally, we compute the success probability. Consider first the case in which $T^2 = \Theta(MN)$. In this case, any two of the $O(M)$ walks in the online phase, each of which is of length T/M, have a probability $T^2/(M^2N) = \Omega(1/M)$ to collide. We have $\Omega(M^2)$ pairs, so we will have M collisions in expectation, for an appropriate choice of the constants, giving M linear relations.

Next consider the case where $ST^2/M = \Theta(MN)$. In particular, assume that $S \leq NM^2/T^2$. We observe that the S walks in the preprocessing phase must touch at least $ST/(eM)$ distinct points in expectation. To see this, consider any such walk. The probability that it does not hit any of the previous walks is at least $(1 - \frac{ST}{MN})^{(T/M)} \approx \exp(-\frac{ST^2}{M^2N}) \geq 1/e$. Such a walk touches T/M new points. By linearity of expectation, the expected number of touched points is at least $ST/(eM)$.

Consider now any of the online walks, each of length T/M. If in one of its first $T/(2M)$ steps such a walk hits any of the $ST/(eM)$ points touched in the preprocessing phase, then in the remaining $T/(2M)$ of its steps it will hit a stored precomputed point. This is because any touched point is at a distance of at most $T/(2M)$ from a stored point. Furthermore, as long as the online walk does not hit a touched point, its next step is an independent random point. Therefore, the probability of an online walk to collide with a preprocessed walk is at least $1-(1-ST/(eMN))^{T/(2M)} \geq 1-(1-ST^2/(eM^2N)) = ST^2/(eM^2N) = \Omega(1)$. Therefore, the $O(M)$ online walks will result in an expected number of M collisions. □

G Companion analysis to proof of Theorem 13

Analysis of preprocessing phase. We say that a walk in the preprocessing phase is unsuccessful if (i) it hits either a marked point or a point on any of the previous walks in fewer than T steps, or (ii) it does not hit a marked point within $2T$ steps.

To compute the probability of bad event (i), consider one of the walks in the preprocessing phase. There are N/T marked points in $\mathcal{Y}$. In addition, all previous walks in the preprocessing phase could have explored at most $2T \cdot N/(3T^2) = 2N/(3T)$ points. So there are at most $(1 + 2/3)N/T = 5N/(3T)$ points that a walk must avoid in its first T steps. The probability of bad event (i) is then at most $1 - (1 - 5/(3T))^T \approx 1 - e^{-5/3}$.

To compute the probability of bad event (ii): for a walk to be more than $2T$ steps long, it must avoid the set of N/T marked points for $2T$ consecutive steps. The probability of bad event (ii) is then at most $(1 - 1/T)^{2T} \approx e^{-2}$.

By a union bound, the probability of either bad event occurring is at most $1 - e^{-5/3} + e^{-2} < 0.95$. So each walk in the preprocessing phase succeeds with probability at least $1/20$. Therefore, by linearity of expectation, the expected number of successful walks is at least $N/(60T^2)$. A successful walk touches at least T new points, and so, in expectation, the total number of points touched by successful walks is at least $N/(60T)$. Note that the total number points touched by any of the walks—successful or not—is at most $2N/(3T)$.

Analysis of online phase ("yes" case). To calculate the distinguishing advantage ϵ of our algorithm $\mathcal{A}$, we first compute the probability that $\mathcal{A}$ outputs "1" on a "yes" instance $h_{\text{yes}} \in \mathcal{Y}$.

Towards analyzing $\Pr[\mathcal{A}(h_{\text{yes}}) = 1]$, consider a walk that starts at a random point in the set $\mathcal{Y}$ and continues for $10T$ steps. We consider three cases:

1. *Good marked.* The online walk hits a preprocessed marked point before hitting any non-preprocessed marked point.
2. *Bad marked.* The online walk hits a non-preprocessed marked point before hitting any preprocessed marked point.
3. *No marked.* The online walk does not hit a marked point at all.

To end at a good marked point, it is sufficient for the online walk to hit, during its first $8T$ steps, any of the (at least) $N/(60T)$ points touched by the successful preprocessed walks, before hitting any of the (at most) $2N/(3T)$ points on an unsuccessful walk, or the (at most) N/T non-preprocessed marked points. We can decompose this event into $8T$ disjoint sub-events $(E_1, \dots, E_{8T})$ as following: for $1 \le i \le 8T$, event E_i occurs if during its first $i-1$ steps, the walk avoids all marked points as well as points touched by preprocessed walks, and in step i, the walk hits a point touched by the successful preprocessed walks. Note that as long as the walk avoids points touched by preprocessed walks, each step moves to an independent random point. We have that

$$\begin{aligned}
\Pr[E_i] &\ge \Pr\left[\begin{matrix}\text{Avoid set of } 5N/(3T) \\ \text{points in first } i-1 \text{ steps}\end{matrix}\right] \cdot \Pr\left[\begin{matrix}\text{Hit set of } N/(60T) \\ \text{points in } i\text{th step}\end{matrix}\right] \\
&\ge (1 - 5/(3T))^{i-1} \cdot \left(\frac{1}{60T}\right).
\end{aligned}$$

The probability that the walk hits a good marked point first is therefore:

$$\begin{aligned}
\Pr[\text{Good marked}] &\ge \Pr\left[\bigcup_{i=1}^{8T} E_i\right] \\
&\ge \frac{1}{60T} \cdot \sum_{i=1}^{8T} (1 - 5/(3T))^{i-1} \\
&= \frac{1 - (1 - 5/(3T))^{8T}}{60T(1 - (1 - 5/(3T)))} \\
&= \frac{1 - (1 - 5/(3T))^{8T}}{100} \\
&= 1/100(1 - e^{-40/3}) \\
&\ge 1/101.
\end{aligned}$$

To compute the probability of never hitting a marked point, note that the number of marked points in $\mathcal{Y}$ is distributed as a Binomial random variable with N trials and success probability $1/T$. The expected number of marked points in $\mathcal{Y}$ is then N/T and the probability, over the choice of hash function, that the number of marked points is less than its expectation by $100\sqrt{N/T}$ (i.e., by many standard deviations) is vanishingly small. So the probability that the walk never hits a marked point is at most

$$\begin{aligned}\Pr[\text{No marked}] &\leq \left(1 - \frac{N/T - 100\sqrt{N/T}}{N}\right)^{10T} \\ &\leq \exp\left(-10 + 1000\sqrt{\frac{T}{N}}\right).\end{aligned}$$

We now compute the probability that $\mathcal{A}$ outputs "1".

In the good case: the number of preprocessed chain endpoints is at most $N/(3T^2)$. In the preprocessing phase, we chose the prefix string p_c to maximize the probability that $H(p_c, \cdot)$ outputs "1" when fed a preprocessed point. For each choice of p_c, the value $H(p_c, m)$ follows the Binomial distribution with $n_{\text{good}} \leq N/(3ST^2)$ trials and success probability $p = 1/2$. With overwhelming probability there exists a prefix string p_c that causes $\sum H(p_c, m)$ to achieve at least one standard deviation $\sqrt{n_{\text{good}}}/2$ above the mean of $n_{\text{good}}/2$. Then

$$\Pr[H(p_c, m) = 1 \mid \text{Good marked}\,] \geq \frac{n_{\text{good}}/2 + \sqrt{n_{\text{good}}}/2}{n_{\text{good}}} \geq 1/2 + \sqrt{\frac{3ST^2}{4N}}.$$

In the bad case: there are $N/(ST)$ marked points of each color in $\mathcal{Y}$ and at most $N/(3ST^2)$ of these are preprocessed. So there are $n_{\text{bad}} = \frac{N}{S}(\frac{1}{T} - \frac{1}{2T^2}) \approx N/(ST)$ marked points of each color in $\mathcal{Y}$ that are not preprocessed. For each color c, the value of the sum $\sum H(p_c, m)$ over the non-preprocessed marked points in $\mathcal{Y}$ of color c is distributed as a binomial random variable of n_{bad} trials with success probability $1/2$.

The probability that this value is less than its mean by more than, say, 100 times its standard deviation is vanishingly small. Then, with overwhelming probability (over the choice of H), we have

$$\Pr[H(p_c, m) = 1 \mid \text{Bad marked}] \geq 1/2 - 50\sqrt{\frac{ST}{N}}.$$

Since in the unmarked case, $\mathcal{A}$ always outputs "0", the overall probability that $\mathcal{A}$ outputs "1" on a random point h_{yes} in $\mathcal{Y}$ is:

$$\begin{aligned}\Pr[\mathcal{A}(h_{\text{yes}}) = 1] &= \Pr[\mathcal{A}(h_{\text{yes}}) = 1 \mid \text{Good marked}] \cdot \Pr[\text{Good marked}] \\ &\quad + \Pr[\mathcal{A}(h_{\text{yes}}) = 1 \mid \text{Bad marked}] \cdot \Pr[\text{Bad marked}] \\ &\geq \left(\frac{1}{2} + \sqrt{\frac{3ST^2}{4N}}\right) \cdot \frac{1}{101} + \left(\frac{1}{2} - 50\sqrt{\frac{ST}{N}}\right) \cdot \left(1 - \exp\left(-10 + 1000\sqrt{\frac{T}{N}}\right) - 1/101\right) \\ &\geq \left(1 - e^{-10}e^{1000\sqrt{T/N}}\right)\frac{1}{2} + \frac{1}{101} \cdot \sqrt{\frac{3ST^2}{4N}} - O\left(\sqrt{\frac{ST}{N}}\right) \\ &\geq \left(1 - e^{-10}e^{1000\sqrt{T/N}}\right)\frac{1}{2} + \Omega\left(\sqrt{\frac{ST^2}{N}}\right).\end{aligned}$$

Analysis of online phase ("no" case). To compute the probability that $\mathcal{A}$ outputs "1" on a random point point h_{no} in $\mathbb{G}^2 \setminus \mathcal{Y}$, we first compute the probability that a $2T$-step walk outside of $\mathcal{Y}$ ends at a marked point.

The number of marked points outside of $\mathcal{Y}$ will, with overwhelming probability, be at most $N^2/T + 100\sqrt{N^2/T}$. Then, whenever a walk starts at a point outside of $\mathcal{Y}$, it reaches a marked after at most $10T$ steps with probability at most

$$\begin{aligned} \Pr[\text{Hits marked}] &\le 1 - \left(1 - \frac{N^2/T + 100N/\sqrt{T}}{N^2 - N}\right)^{10T} \\ &\approx 1 - \exp\left(-10 - 1000\sqrt{T}/N\right). \end{aligned}$$

There are $(N^2 - N)/T$ marked points outside of $\mathcal{Y}$, and there are roughly $(N^2 - N)/(ST)$ marked points of each color. For each color c, the number of marked points m outside of $\mathcal{Y}$ that have $H(p_c, m) = 1$ is a binomial random variable for $n_{\text{no}} = (N^2 - N)/ST$ trials with success probability $1/2$. The standard deviation of this value is $\sqrt{n_{\text{no}}}/2$, so with very high probability over the choice of H, we will have

$$\Pr\left[H(p_c, m) = 1 \mid \text{Hits marked}\right] \le \frac{1}{2} + 50\sqrt{\frac{ST}{N^2 - N}} = \frac{1}{2} + O\left(\sqrt{\frac{ST}{N^2}}\right)$$

when the walk begins at a "no" instance.

Then

$$\begin{aligned} \Pr[\mathcal{A}(h_{\text{no}}) = 1] &\le (1 - e^{-10}e^{-1000\sqrt{T}/N}) \cdot \left(\frac{1}{2} + O\left(\sqrt{\frac{ST}{N^2}}\right)\right) \\ &\le (1 - e^{-10}e^{-1000\sqrt{T}/N})\frac{1}{2} + O\left(\sqrt{\frac{ST}{N^2}}\right). \end{aligned}$$

Now, the distinguishing advantage ϵ that we care about is the difference between the probability that $\mathcal{A}$ outputs "1" when fed either a "yes" instance $h_{\text{yes}} \xleftarrow{R} \mathcal{Y}$ or a random instance in $h_{\text{rand}} \xleftarrow{R} \mathbb{G}^2$: $|\Pr[\mathcal{A}(h_{\text{yes}}) = 1] - \Pr[\mathcal{A}(h_{\text{rand}}) = 1]|$. To compute the second probability:

$$\begin{aligned} \Pr[\mathcal{A}(h_{\text{rand}}) = 1] &= \Pr[\mathcal{A}(h_{\text{rand}}) = 1 \mid h_{\text{rand}} \in \mathcal{Y}] \cdot \Pr[h_{\text{rand}} \in \mathcal{Y}] \\ &\qquad + \Pr[\mathcal{A}(h_{\text{rand}}) = 1 \mid h_{\text{rand}} \notin \mathcal{Y}] \cdot \Pr[h_{\text{rand}} \notin \mathcal{Y}] \\ &\le \Pr[h_{\text{rand}} \in \mathcal{Y}] + \Pr[\mathcal{A}(h_{\text{rand}}) = 1 \mid h_{\text{rand}} \notin \mathcal{Y}] \\ &= (1/N) + \Pr_{h_{\text{no}} \xleftarrow{R} \mathbb{G}^2 \setminus \mathcal{Y}}[\mathcal{A}(h_{\text{no}}) = 1] \\ &\le (1/N) + (1 - e^{-10}e^{-1000\sqrt{T}/N})\frac{1}{2} + O\left(\sqrt{\frac{ST}{N^2}}\right) \\ &\le (1 - e^{-10}e^{-1000\sqrt{T}/N})\frac{1}{2} + O\left(\sqrt{\frac{ST}{N^2}}\right). \end{aligned}$$

Distinguishing advantage. Then:

$$\begin{aligned}
\epsilon &= \Big| \Pr[\mathcal{A}(h_{\text{yes}}) = 1] - \Pr[\mathcal{A}(h_{\text{rand}}) = 1] \Big| \\
&\geq \left((1 - e^{-10} e^{1000\sqrt{T/N}}) \frac{1}{2} + \mathit{\Omega}\left(\sqrt{\frac{ST^2}{N}} \right) \right) - \left((1 - e^{-10} e^{-1000\sqrt{T}/N}) \frac{1}{2} + O\left(\sqrt{\frac{ST}{N^2}} \right) \right) \\
&\geq \Big((1 - e^{-10} e^{1000\sqrt{T/N}}) - (1 - e^{-10} e^{-1000\sqrt{T}/N}) \Big) \frac{1}{2} + \mathit{\Omega}\left(\sqrt{\frac{ST^2}{N}} \right) \\
&\geq \Big(e^{-1000\sqrt{T}/N} - e^{1000\sqrt{T/N}} \Big) \frac{e^{-10}}{2} + \mathit{\Omega}\left(\sqrt{\frac{ST^2}{N}} \right) \\
&\approx \Big((1 - 1000\sqrt{T}/N) - (1 + 1000\sqrt{T/N}) \Big) \frac{e^{-10}}{2} + \mathit{\Omega}\left(\sqrt{\frac{ST^2}{N}} \right) \\
&\geq \Big(1000\sqrt{T/N} - 1000\sqrt{T}/N \Big) \frac{e^{-10}}{2} + \mathit{\Omega}\left(\sqrt{\frac{ST^2}{N}} \right) \\
&\geq \mathit{\Omega}\left(\sqrt{\frac{ST^2}{N}} \right).
\end{aligned}$$