Integer Reconstruction Public-Key Encryption

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Abstract In [AJPS17], Aggarwal, Joux, Prakash & Santha described an elegant public-key cryptosystem (AJPS-1) mimicking NTRU over the integers. This algorithm relies on the properties of Mersenne primes instead of polynomial rings.
A later ePrint [BCGN17] by Beunardeau et al. revised AJPS-1's initial security estimates. While lower than initially thought, the best known attack on AJPS-1 still seems to leave the defender with an exponential advantage over the attacker [dBDJdW17]. However, this lower exponential advantage implies enlarging AJPS-1's parameters. This, plus the fact that AJPS-1 encodes only a single plaintext bit per ciphertext, made AJPS-1 impractical. In a recent update, Aggarwal et al. overcame this limitation by extending AJPS-1's bandwidth. This variant (AJPS-ECC) modifies the definition of the public-key and relies on error-correcting codes.
This paper presents a different high-bandwidth construction. By opposition to AJPS-ECC, we do not modify the public-key, avoid using error-correcting codes and use backtracking to decrypt. The new algorithm is *orthogonal* to AJPS-ECC as both mechanisms may be concurrently used *in the same ciphertext* and cumulate their bandwidth improvement effects. Alternatively, we can increase AJPS-ECC's information rate by a factor of 26 for the parameters recommended in [AJPS17].
The obtained bandwidth improvement and the fact that encryption and decryption are reasonably efficient, make our scheme an interesting post-quantum candidate.

1 Introduction

In a recent paper [AJPS17], Aggarwal, Joux, Prakash, & Santha described an elegant public-key cryptosystem (AJPS-1) mimicking NTRU over the integers. AJPS-1 relies on a specific arithmetic property of Mersenne numbers[3] instead of polynomial rings.

A later ePrint [BCGN17] by Beunardeau et al. revised AJPS-1's initial security estimates. While lower than initially thought, the best known attack

[3] Recall that a Mersenne prime is a prime of the form $2^n - 1$ for $n \in \mathbb{N}$.

against AJPS-1's complexity assumption[4] still seems to leave the defender with an exponential advantage over the attacker [dBDJdW17]. However, this lower exponential advantage implies enlarging AJPS-1's parameters. This, plus the fact that AJPS-1 only encodes a single plaintext bit per ciphertext, made AJPS-1 impractical. In a recent update, Aggarwal et al. overcame this limitation by extending AJPS-1's bandwidth. This variant (AJPS-ECC) modifies the definition of the public-key and relies on error-correcting codes.

Our paper presents a different high-bandwidth construction. By opposition to AJPS-ECC, we do not modify the public-key, avoid the use of error-correcting codes and use backtracking to decrypt. The new algorithm is *orthogonal* to AJPS-ECC as both mechanisms may be concurrently implemented *in the same ciphertext* and cumulate their bandwidth improvement effects[5]. Alternatively, we can increase AJPS-ECC's information rate by a factor of 26 for the parameters recommended in [AJPS17].

The obtained bandwidth improvement and the fact that encryption and decryption are reasonably efficient, make our scheme an interesting post-quantum candidate.

1.1 The Original AJPS-1 Cryptosystem

We denote by $\|x\|$ the Hamming weight of x and let $\mathfrak{H}_{n,h}$ be the set of all n-bit strings of Hamming weight h.

The original AJPS-1 scheme is defined by the following sub-algorithms:

- **Setup**$(1^\lambda) \to \mathsf{pp}$. Chooses the public parameters $\mathsf{pp} = \{n, h\}$ so that $p = 2^n - 1$ is an n-bit Mersenne prime achieving some λ-bit security level.
- **KeyGen**$(\mathsf{pp}) \to \{\mathsf{sk}, \mathsf{pk}\}$. Picks $\{F, G\} \in_R \mathfrak{H}^2_{n,h}$ and returns:

$$\begin{cases} \mathsf{sk} & \leftarrow G \\ \mathsf{pk} & \leftarrow H = F/G \bmod p \end{cases}$$

- **Enc**$(\mathsf{pp}, \mathsf{pk}, m \in \{0,1\}) \to C$. Picks $\{A, B\} \in_R \mathfrak{H}^2_{n,h}$, and computes:

$$C \leftarrow (-1)^m (AH + B) \bmod p$$

- **Dec**$(\mathsf{pp}, \mathsf{sk}, C) \to \{\perp, 0, 1\}$, computes $d = \|GC \bmod p\|$ and returns:

$$\begin{cases} 0 & \text{if } d \leq 2h^2, \\ 1 & \text{if } d \geq n - 2h^2, \\ \perp & \text{otherwise} \end{cases}$$

[4] The Mersenne Low Hamming Ratio Assumption

[5] This requires that the $\{n, h\}$ parameters of both schemes coincide. We conjecture that such a meeting point exists.

The intuition behind the decryption formula is the observation that when $m = 0$ we get:

$$W = GC = G(AH + B) = FA + GB \Rightarrow W \text{ is of low Hamming weight}$$

We refer the reader to [AJPS17] for more details about this cryptosystem.

To increase bandwidth, Aggarwal et al. introduced the AJPS-ECC variant described hereafter.

1.2 The AJPS-ECC Cryptosystem

AJPS-ECC requires an ancillary error correction scheme $\{\mathcal{D}, \mathcal{E}\}$.

AJPS-ECC is formally defined by the following sub-algorithms:

- $\mathbf{Setup}(1^\lambda) \to \mathsf{pp}$. As in AJPS-1.
- $\mathbf{KeyGen}(\mathsf{pp}) \to \{\mathsf{sk}, \mathsf{pk}\}$. Picks $\{F, G\} \in_R \mathfrak{H}^2_{n,h}$, $R \in_R \{0,1\}^n$ and returns:
$$\begin{cases} \mathsf{sk} & \leftarrow F \\ \mathsf{pk} & \leftarrow \{R, T\} = \{R, F \times R + G \bmod p\} \end{cases}$$
- $\mathbf{Enc}(\mathsf{pp}, \mathsf{pk}, m \in \{0,1\}^\lambda) \to C$. Picks $\{A, B_1, B_2\} \in_R \mathfrak{H}^3_{n,h}$ and computes the ciphertext:
$$C = \begin{cases} C_1 & \leftarrow \ A \times R + B_1 \bmod p \\ C_2 & \leftarrow (A \times T + B_2 \bmod p) \oplus \mathcal{E}(m) \end{cases}$$
- $\mathbf{Dec}(\mathsf{pp}, \mathsf{sk}, C) \to \{\perp, m\}$ returns:

  $\mathcal{D}((F \times C_1 \bmod p) \oplus C_2)$.

For the sake of clarity, we keep the definition of C_1 unchanged but slightly depart from [AJPS17]'s original formulae by modifying the definitions of T and C_2 as follows:

$$\begin{aligned} T &\leftarrow \ F \times R - G \mod p \\ C_2 &\leftarrow (A \times T - B_2 \bmod p) \oplus \mathcal{E}(m) \end{aligned}$$

To understand the intuition behind **Dec** consider the quantity $W = FC_1 - C_2$ corresponding to the particular case $\mathcal{E}(m) = 0$:

$$W = FAR + FB_1 - AT + B_2 = FAR + FB_1 - A(FR - G) + B_2 = FB_1 + GA + B_2$$

As before, we see that $d = \|W\|$ is low. This means that the noise attached to $\mathcal{E}(m)$ after the clean-off operation $(F \times C_1 \bmod p) \oplus C_2$ is low and thus surmountable by the error-correcting code $\{\mathcal{E}, \mathcal{D}\}$.

The reader is referred, again, to [AJPS17] for more details about this cryptosystem and the parameter choices allowing successful decryption and sufficient security. Sticking only to the core idea, we purposely omit the hashing and re-encryption tests performed during the key de-encapsulation process.

2 The New Idea: Randomness Reconstruction

Our idea departs from AJPS-1 in a direction orthogonal to the above.

We set by design $m = 0$ in AJPS-1 or $\mathcal{E}(m) = 0$ in AJPS-ECC and attempt to recover *the randomness*[6] into which information (encapsulated keys and/or plaintext information) will be embedded.

The intuition is that the receiver might be able to recover the randomness if parameters are properly chosen *using his extra knowledge* of G, F and knowing that, in addition, the unknown randomness has a low Hamming weight.

We hence focus the rest of this paper on methods for solving the equations:

$$W = Fx + Gy \quad \text{or} \quad W = Fx + Gy + z \bmod p$$

Where all parameters[7] and unknowns are *randomly* chosen in $\mathfrak{H}_{n,h}$ and where a solution $\{x, y\}$ or $\{x, y, z\}$ *is known to exist*.

We do not introduce any modifications in **Setup** and **KeyGen**, nor do we modify pp or sp[8]. We thus focus on the encapsulation (encryption) and on the de-encapsulation (decryption) processes only.

In a non-KEM version, a plaintext m encoded in the unknowns (x, y or x, y, z) can be directly recovered upon decryption. Such an encryption mode must however be protected against active attacks using padding and randomization that we do not address here.

Note 1: It is tempting but inadvisable to create dependencies between the variables F, G and/or the unknowns x, y, z. Consider an AJPS-ECC where $m \in_R \{0,1\}^\lambda$ and $\{A, B_2\} \in_R \mathfrak{H}^2_{n,h}$ but where $B_1 \leftarrow \mathcal{H}(m)$ is obtained by hashing m into $\mathfrak{H}_{n,h}$. Given m, *anybody* can re-compute B_1 and algebraically infer A, B_2. We hence see in this example that A, B_2 do not add extra entropy as security solely rests upon m.

Note 2: We carefully distinguish between *security bandwidth* and *information rate*. An idea, unexplored in AJPS-ECC, may exploit Note 1 to transport more plaintext information in $\{C_1, C_2\}$ *without adding extra security*. To encrypt a τ-bit message μ, pick a key $m \in_R \{0,1\}^\lambda$ and encrypt $c \leftarrow \mathcal{F}_m(\mu)$ using a block cipher $\mathcal{F}$. Set $B_1 \leftarrow \mathcal{H}(m)$. Let $\mathcal{M}$ be any invertible public mapping $\mathcal{M} : \{0,1\}^\tau \rightarrow \mathfrak{H}^2_{n,h}$. Encode: $\{A, B_2\} \leftarrow \mathcal{M}(c)$ and form $\{C_1, C_2\}$ using AJPS-ECC. To decrypt, recover m using error-correction, recompute B_1, algebraically recover $\{A, B_2\}$ and retrieve the plaintext μ by:

[6] A, B or A, B_1, B_2

[7] Except W

[8] Note that in AJPS-1/ECC given G one can compute F and *vice versa*.

$$
\begin{aligned}
\mu &\leftarrow \mathcal{F}_m^{-1}(\mathcal{M}^{-1}(A, B_2)) \\
&= \mathcal{F}_m^{-1}\left(\mathcal{M}^{-1}(\frac{C_1 - B_1}{R} \bmod p, C_2 - \frac{(C_1 - B_1)T}{R} \bmod p)\right) \\
&= \mathcal{F}_m^{-1}\left(\mathcal{M}^{-1}(\frac{C_1 - \mathcal{H}(m)}{R} \bmod p, C_2 - \frac{(C_1 - \mathcal{H}(m))T}{R} \bmod p)\right)
\end{aligned}
$$

Because in AJPS-ECC $\{n, h\} = \{756839, 256\}$ the potential encoding capacity of $\mathcal{M}$ can be relatively high:

$$2 \log_2 \binom{756839}{256} = 6631 \text{ bits}$$

This increases AJPS-ECC's information rate by a factor of 26. Again, proper message padding may be necessary to resist active attacks (analysis underway). We stress, again, that this does not increase security bandwidth but information rate only. An attacker guessing m will determine μ.

2.1 The Bivariate Cryptosystem

- **Setup**$(1^\lambda) \rightarrow \mathsf{pp}$ and **KeyGen**$(\mathsf{pp}) \rightarrow \{\mathsf{sk}, \mathsf{pk}\}$ are identical to AJPS-1.
- **Enc**$(\mathsf{pp}, \mathsf{pk}) \rightarrow C$. Picks $\{A, B\} \in_R \mathfrak{H}_{n,h}^2$ and computes:

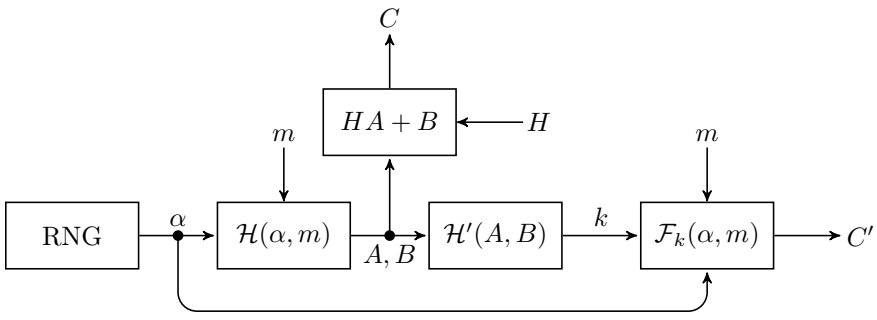
  $$C \leftarrow AH + B \bmod p$$

- **Dec**$(\mathsf{pp}, \mathsf{sk}, C) \rightarrow \{\perp, \{A, B\}\}$ returns:

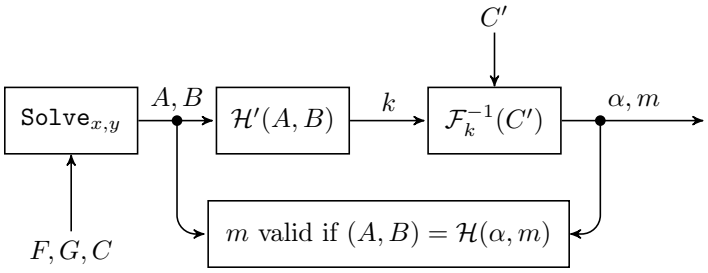
  $$\{A, B\} \leftarrow \mathtt{Solve}_{x,y}[GC = Fx + Gy \bmod p]$$

If $\{A, B\} \neq \perp$ use $\{A, B\}$ as KEM entropy for further encryption.

Example: Let μ be a plaintext and $\mathcal{R}$ a redundancy function. Compute $m \leftarrow \mathcal{R}(\mu, \rho)$ where ρ is random. A typical KEM[9] is shown here:



[9] Where $\mathcal{H}, \mathcal{H}'$s are hash functions and $\mathcal{F}$ is a block-cipher.



Retrieve m from $\mu \leftarrow \mathcal{R}^{-1}(m)$.

2.2 The Trivariate Cryptosystem

- **Setup**$(1^\lambda) \rightarrow \mathsf{pp}$ and **KeyGen**$(\mathsf{pp}) \rightarrow \{\mathsf{sk}, \mathsf{pk}\}$ are identical to AJPS-ECC but with the modified formula $T \leftarrow F \times R - G \bmod p$.
- **Enc**$(\mathsf{pp}, \mathsf{pk}) \rightarrow C$. Picks $\{A, B_1, B_2\} \in_R \mathfrak{H}_{n,h}^3$ and computes the ciphertext:

$$C = \begin{cases} C_1 & \leftarrow A \times R + B_1 \bmod p \\ C_2 & \leftarrow A \times T - B_2 \bmod p \end{cases}$$

- **Dec**$(\mathsf{pp}, \mathsf{sk}, C) \rightarrow \{\perp, \{A, B_1, B_2\}\}$ returns:

$$\{A, B_1, B_2\} \leftarrow \texttt{Solve}_{x,y,z}[FC_1 - C_2 = Fy + Gx + z \bmod p]$$

If $\{A, B_1, B_2\} \neq \perp$ use $\{A, B_1, B_2\}$ as KEM entropy for further encryption.

Note 3: As noted before, the trivariate version may accommodate in the encryption formula an *independent* $\mathcal{E}(m)$ and thus cumulate the bandwidth improvements due to both mechanisms. This requires that the $\{n, h\}$ values of both schemes coincide and the enforcement of the condition $n \leq 16h^2$, not addressed here. We conjecture that such a meeting point exists.

The following sections explain how to instantiate $\texttt{Solve}_{x,y}$. The routine $\texttt{Solve}_{x,y,z}$ is obtained *mutatis mutandis*.

3 Instantiating $\texttt{Solve}_{x,y}$ Using Backtracking

The intuition behind $\texttt{Solve}_{x,y}$ is the following: assume that we are given the quantity $W = GC = AF + BG \bmod p$ where $\|W\| \cong 2h^2$. Because multiplication modulo p is (somewhat) weight-preserving, we can test the hypothesis that the i-th bit of A is equal to one by looking at the quantity Δ:

$$\Delta = \|W\| - \|W - 2^i F \bmod p\|$$

Intuitively, a good guess should result in a weight decrease of $\simeq h$ whereas a wrong guess should re-blur W by triggering random carry propagations. Evidently, because there may be false positives during this process, we must be able to backtrack. To reduce the false positive error probability, n must be large enough with respect to h. The exact same idea applies to $\texttt{Solve}_{x,y,z}$.

3.1 Prerequisites & Subroutines

We start by introducing three necessary prerequisites.

The ancillary function `Confirm`: Our algorithms require an ancillary function `Confirm`-ing a candidate solution $\{x, y\}$. e.g. given a candidate x, $\texttt{Confirm}(x)$ may solve $GC = Fx + Gy \bmod p$ for y and return $\{y, \mathsf{True}\}$ if $\|x\| = \|y\| = h$. Because in some cases several solutions may exist, a simpler implementation may just compare $\mathcal{H}(x, y)$ to a confirmation digest τ provided with the ciphertext and return $\{y, \mathsf{True}\}$ if the purported solution hashes into τ. If $\mathcal{H}(x, y) \neq \tau$ then $\texttt{Confirm}(x)$ returns $\{\perp, \mathsf{False}\}$.

Dealing with decoding failures: Because we may discard seemingly uninteresting (but actually promising) exploration paths, backtracking may fail to decode W. As it seems complex to formally compute the algorithm's success probability, we estimated it by simulation. To deal with decryption failures we re-attempt backtracking after index randomization *i.e.* pick t random permutations $\{\phi_0, \ldots, \phi_{t-1}\}$ of $\mathbb{Z}_k$ and re-run $\mathcal{B}_2(W, \emptyset, 0, \phi_j)$ t times hoping that at least one of the t runs will succeed[10]. A more brutal approach consists in sending t encapsulated keys to increase the probability exponentially. This (conjectured) exponential probability gain only handicaps the information rate by a constant factor[11]. A simple idea for (conjectured) squaring the failure probability consists in trying to backtrack on A and, upon failure, re-launch the algorithm to backtrack on B.

Determining the backtracking aperture Γ: Backtracking is parametrized by a constant Γ controlling the aperture of the exhausting process (*i.e.* the marginal tolerance allowing to exclude a search path from further investigation). Simulations indicate that for any given $\{n, h\}$ there is a Γ_{optimal} value minimizing the failure probability. We did not attempt a formal analysis of the dependency between $\{n, h\}$ and Γ_{optimal} but estimated Γ_{optimal} for various $\{n, h\}$ pairs using simulations as shown in Table 1 and Figure 1.

[10] Note that $\mathcal{B}_1(W, \emptyset, 0) = \mathcal{B}_2(W, \emptyset, 0, \text{ID})$.

[11] Link $B_0, \ldots, B_{t-1}$ in a way allowing the recovery of all the B_i if one of them is known (*e.g.* define $B_i = \mathcal{F}_i(\text{seed})$ where $\mathcal{F}_k(m)$ is a block-cipher encrypting into $\mathfrak{H}_{n,h}$). Use the A_i to transport entropy or information. One successful decryption reveals the seed $\Rightarrow$ open all the B_is $\Rightarrow$ all the t information containers A_i.

Γ	50	51	52	53	54	55	56	57	58	59	60
Probability	24%	20%	26%	30%	32%	20%	22%	18%	14%	9%	4%

Table 1: Backtracking success chances for $\{t, h, n\} = \{1, 72, 19937\}$. 50 decryption simulations per entry.

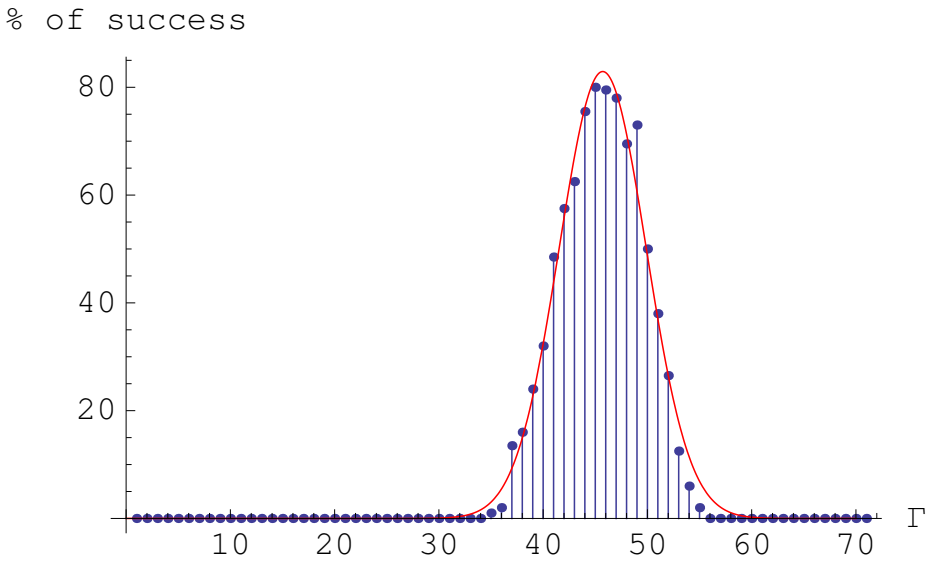


Figure 1: Backtracking success chances for $\{t, h, n\} = \{1, 65, 19937\}$. 200 decryption simulations per entry. Fitted with $82.922 \exp(-(\Gamma - 45.7122)^2/34.6815)$

3.2 The Backtracking Algorithms

The deterministic backtracking algorithm $\mathcal{B}_1$ subtracts left-shifted Fs from W to obtain candidate ws having smaller and smaller weights. $\mathcal{B}_1$ maintains a set of integers R containing the bit positions of x discovered so far. The deterministic algorithm is called by $\{A, B\} \leftarrow \mathcal{B}_1(W, \emptyset, 0)$ and the randomized version is called by $\mathcal{B}_2(W, \emptyset, 0, \phi_j)$ where ϕ_js are random permutations of $\mathbb{Z}_k$. Code is available from the authors upon request.

Algorithm 1
Backtracking $\mathcal{B}_1(w, R, e)$

Input: w, R, e. The values $G, F, h, n, p = 2^n - 1, C$ are global and invariant.
Output: $\{x, y\} = \{A, B\}$ such that $W = CG = Fx + Gy \bmod p$ or Failure.

if $e = n$ **then return** Failure
else
  if $\#R = h$ **then**
    $x \leftarrow \sum_{i \in R} 2^i$
    $\{s, y\} \leftarrow \texttt{Confirm}(x)$
    if s **then return** $\{x, y\}$
$\overline{w} \leftarrow w - 2^e \times F \bmod p$
if $|\|\overline{w}\| - \|w\| + h| \leq \Gamma$ **then**
  $\mathcal{B}_1(\overline{w}, R \cup \{e\}, e + 1)$
else
  $\mathcal{B}_1(w, R, e + 1)$

Algorithm 2
Backtracking $\mathcal{B}_2(w, R, e, \phi)$

Input: w, R, e, ϕ. The values $G, F, h, n, p = 2^n - 1, C$ are global and invariant.
Output: $\{x, y\} = \{A, B\}$ such that $W = CG = Fx + Gy \bmod p$ or Failure.

if $e = n$ **then return** Failure
else
  if $\#R = h$ **then**
    $x \leftarrow \sum_{i \in R} 2^{\phi(i)}$
    $\{s, y\} \leftarrow \texttt{Confirm}(x)$
    if s **then return** $\{x, y\}$
$\overline{w} \leftarrow w - 2^{\phi(e)} \times F \bmod p$
if $|\|\overline{w}\| - \|w\| + h| \leq \Gamma$ **then**
  $\mathcal{B}_2(\overline{w}, R \cup \{e\}, e + 1, \phi)$
else
  $\mathcal{B}_2(w, R, e + 1, \phi)$

Note 4: We conjecture that working with a fixed Γ during the entire backtracking process handicaps the algorithm. When the process starts the weight of W is high, hence the probability to strike-out h bits by subtraction is high. However as subtractions make w sparser aperture should intuitively decrease. It may hence make sense to explore algorithms in which the constant Γ_{optimal} is replaced by a function $\Gamma(\|w\|, \|\overline{w}\|, n, h)$.

Note 5 (Best candidate search): $\mathcal{B}_1$ and $\mathcal{B}_2$ explore all the paths starting by an *a priori* promising Δ. However, $\mathcal{B}_1$ and $\mathcal{B}_2$ do not explore the most promising paths first. A more complex backtracking strategy ($\mathcal{B}_3$) trying with priority the

paths starting by a Δ as close as possible to h was developed as well (information available from the authors upon request). We do not include this algorithm here for the sake of concision.

Note 6 (Information leakage from decryption failures): Because decryption may fail, a possible cryptanalysis (that we did not investigate) might be to analyze, possibly adaptively, the ciphertexts causing failures and thereby extract information on $\{F, G\}$. We do not regard this as a major problem for the following reasons:

- Failure is highly dependent on the backtracking algorithm chosen by the receiver. The backtracking procedures that we give here are one possibility amongst many.
- An empirical protection consists in randomizing the backtracking process, *e.g.*, assume all the ϕ_i to be *randomly drawn per decryption and secret*.
- Another protection is to purposely fail decryption with some probability ϵ to prevent the cryptanalyst from identifying true failures. Note that the random tape used to simulate false failures must be derived from a fixed secret and *the ciphertext itself* to avoid replays and majority votes.

Note 7 (Protection against side-channel attacks): It is reasonable to assume that, like most encryption schemes, the algorithms described in this paper are vulnerable to timing and side-channel attacks, an aspect that we did not investigate here.

3.3 Eccentric Reconstruction Strategies

Backtracking might be improved in a variety of ways. As examples, we list here a few research ideas that we did not explore in detail.

Brittle encryption formulae: We may modify the bivariate encryption formula to $C \leftarrow HA + 3B$ and enforce by design that H, A, B do not contain the binary sequence `11`. This means that the bit positions representing $3B$ will be "colored" by a pattern `11` making their isolation and identification easier. If $n \gg h$ we may even attempt to brutally reset all the isolated ones in W and divide the result by $3G$ to directly obtain B. For the trivariate version one may use:

$$C = \begin{cases} C_1 & \leftarrow AR + B_1 \mod p \\ C_2 & \leftarrow AT - 3B_2 \mod p \end{cases}$$

resulting in the decryption formula $W = FC_1 - C_2 = FB_1 + GA + 3B_2$. Here as well, we banish the pattern `11` from G, F, A, B_1, B_2. We may thus attempt to identify in W the binary patterns `11`, hinting the probable presence of B_2 to ease

decoding. Note that the pattern `11` may result naturally from the multiplication, the addition or the reduction and hence mislead the decoder (backtrack). Similarly, an `11` due to $3B_2$ may disappear due to addition (backtrack). Note that marking B with `11`s makes backtracking more efficient as this increases the SNR. *e.g.* if we replace each `1` in B by a `1111` we increase overall weight of W to $5h^2$ but a correct guess will cause a weight decrease of $\simeq 4h$ instead of h. In other words, while requiring a larger n, this improves the SNR:

$$\text{from} \;\; \text{SNR} = \frac{h}{2h^2} = \frac{1}{2h} \;\; \text{to} \;\; \text{SNR} = \frac{4h}{5h^2} = \frac{4}{5h}$$

Dye tracing: In hydrogeology, *dye tracing* is a technique for tracking various flows using dye added to the water source. In other words, dye tracing uses dye as a flow tracer. It is an evolution of the ages-known float tracing method, which consists of throwing a buoyant object into a waterflow to see where it emerges. To simulate the effect of dye tracing, we inject into F's digits a few low-weight binary patterns and track their appearance in W. For instance (toy example), generate an F of weight $h - 10$ not containing any of the ten sequences ℓ_i:

`11`	`101`	`111`	`1001`	`1011`	`1101`	`10001`	`11001`	`10101`	`10011`

randomly insert those ten ℓ_is into F's blank spaces (insert each ℓ_i once, this will increase the weight of F to $h - 10 + \sum \|\ell_i\| = h - 10 + 26 = h + 16$ and the weight of W to $\simeq 2h^2 + 16h$). To retrieve A, isolate the 10 dyes tracers in W and use majority voting on bit offsets to infer the probable positions of A's bits.

Demodulation: We can attempt to "travel back in time" and infer $\omega = FA + GB \in \mathbb{Z}$ from W, or at least estimate the probability that a candidate bit in W originates from the number's pre-reduced upper half. Given $\omega \in \mathbb{Z}$ decryption[12] is immediate because:

$$A = \omega F^{-1} \bmod G = (W \text{ demod } p) \times F^{-1} \bmod G$$

To demodulate W we work modulo $p = 2^n - 3$ that "colors" the folded MSBs by turning them into LSB `11`s. The process is error-prone[13] but *actually works* for parameters that are large enough. We implemented the idea very brutally, by simply translating each `11` in W into a `1` in the MSB of ω without taking any further precautions. 100 demodulation attempts for $\{n = 6 \times 10^7, h = 55\}$ resulted in 29 successes. Although n is huge, the resulting information rate is not "that" catastrophic as we can pack:

[12] Take F, G coprime in **Setup**.

[13] Again, "natural" `11`s may be already present in the LSBs of ω, `11` + `01` may destroy an `11`, `10` + `01` may create fake `11`s etc.

$$2\log_2\binom{6\times10^7}{55} = 2356 \text{ plaintext bits into the ciphertext.}$$

In other words, each plaintext bit claims 25461 ciphertext bits and is successfully transmitted with probability 29%.

While $h = 55$ is not very large and $n = 6 \times 10^7$ is extremely large, our simulation shows that it is definitely possible to make ingredients meet at a workable parameter combination. We conjecture that with proper analysis and refined demodulation strategies k might be reduced by at least two orders of magnitude. It may also be possible to work modulo $2^n - \pi$ with a more distinguishable color $\pi \neq 3$ despite an extra weight due to a more complex π. $\pi = -1$ is interesting as well as -1 turns folded bits into long chains of `1`s.

Note 8 (important): One of the features preventing lattice-based attacks in AJPS-1/ECC is the emergence of parasitic short vectors due to working modulo $2^n - 1$ (section 5.1.[AJPS17] [14]). We did not evaluate the impact of $\pi \neq 1$ on the number and the norm of parasitic short vectors *and hence on security*.

Pattern identification: Another idea consists in exploiting the fact that $W = AF + BG \bmod p$ will naturally contain binary sequences of the form:

$$v_\ell = \underbrace{\texttt{0},\ldots,\texttt{0}}_{\ell \text{ zeros}}|1|\underbrace{\texttt{0},\ldots,\texttt{0}}_{\ell+1 \text{ zeros}}$$

Let m be an ℓ-bit encapsulated key[15] and define $C = m(AH + B) \bmod p$ we get:

$$CG = m(AF + GB) = mW = m \times (w'|v_\ell|w) = u'|m|u \bmod p$$

m can thus be read[16] on $CG \bmod p$. It remains to identify m. To do so, we can generate $\{A, B\} \leftarrow \mathcal{H}(m)$ and hence confirm proper decryption using n re-hashings and re-encryptions. This workload may be considerably reduced by sacrificing a few bits of m, *e.g.* 16 bits, to display a specific pattern (*e.g.* `0xFFFF`) allowing a quick identification of m. This divides the number of hashings and re-encryptions by 2^{16}. As a numerical example, $\{n, h\} = \{75 \times 10^4, 100\}$ corresponds to an $\ell \simeq 200$. If we sacrifice 20 bits for an identification pattern we can hope to decapsulate a $\simeq 160$-bit key using one re-encryption only.

ℓ has a low variance as it is essentially determined by a max-min over the differences between the positions s_i of the bits equal to one in W:

$$\ell \cong \max_i \min(s_i - s_{i-1}, s_{i+1} - s_i)$$

[14] ePrint version 20170530:072202.
[15] We consider m to be beyond exhaustive search, typically 160 bits.
[16] Note that reading is circular *i.e.* wrapping around $CG \bmod p$.

For a subtle technical reason ℓ is actually higher than this crude estimate. To understand why we refer the reader to Figure 2 where we illustrate the expected distribution of $\hbar = 2h^2$ bits amongst n potential positions. The least significant 1-bit $\bullet$ is expected to appear at $\gamma = (n-1)/(\hbar+1)$. Similarly, the most significant 1-bit position $\bullet$ is expected at $\simeq \hbar\gamma$. The reason why two other points are singled-out by $\bullet$ and $\bullet$ will be clarified later. Now, because arithmetics modulo p wrap everything that overflows 2^n on 2^0 the actual gap between $\bullet$ and $\bullet$ is not γ but 2γ. This is illustrated in Figure 3. In other words, the primary formula for ℓ should be corrected to:

$$\ell \cong \max\left(\underline{\alpha}, \overline{\alpha}, \max_i \min(s_i - s_{i-1}, s_{i+1} - s_i)\right)$$

Where:

$$\underline{\alpha} = \min([\bullet, \bullet], [\bullet, \bullet]) \overset{\mathrm{u}}{=} [\bullet, \bullet] \simeq \gamma$$

and

$$\overline{\alpha} = \min([\bullet, \bullet], [\bullet, \bullet]) \overset{\mathrm{u}}{=} [\bullet, \bullet] \simeq \gamma$$

Where $\overset{\mathrm{u}}{=}$ denotes "usually (or frequently) equal to". We therefore see that wrapping due to modular arithmetic has an unexpected favorable effect on ℓ.

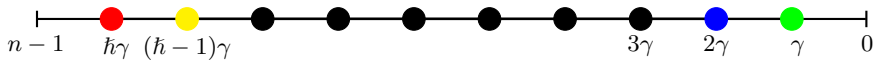


Figure 2: Dots show the expected positions of $\hbar$ bits picked randomly amongst n positions. Here $\gamma = \frac{n-1}{\hbar+1}$.

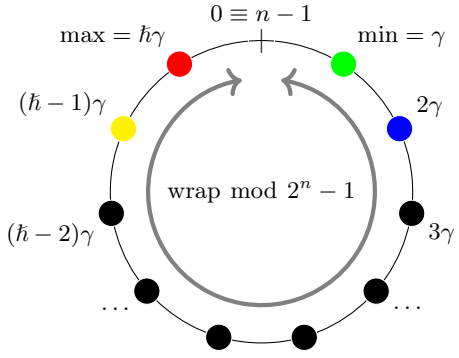


Figure 3: The interval $[\min, \max] = [\bullet, \bullet]$ of size 2γ created by wrapping mod$p$.

Pattern identification is somewhat homomorphic but with a very fast increasing noise: encode a zero as $m = \texttt{01}$, a one as $m = \texttt{11}$ and read the plaintext on the most significant bit of that encoding.

Prime embedding: A variant of the above, mostly of theoretical interest, is the following: because there is a number of natural leading and tailing zeros in $\eta = AF + GB \bmod p$ we can encode $C = m(AH + B) \bmod p$, recover $m(AF + GB) \bmod p$ and, provided that m is short enough, hope that η is small enough to get[17] $\omega = m\eta \in \mathbb{Z}$. It remains to extract m from ω. To do so, pick m as a product of, say 64-bit random primes. The receiver can pull-out those primes from ω using ECM factorization[18]. When a candidate m was formed, confirm it using hashing and re-encryption as before. $\{n, h\} = \{2.5 \times 10^6, 100\}$ gives an average expected margin of $\simeq 240$ bits for encoding m. The main problem with this variant would be the high variance in the size of m embeddable into the ciphertext which would make decryption very uncertain.

Caveat

Brittle encryption, dye tracing, demodulation, pattern identification and prime embedding are only illustrative research directions that we consider interesting or curious but that we do not claim nor conjecture to be secure.

Note 9 (research note): To the above we add two ideas that *we conjecture to be insecure* (by opposition to the previous ideas that we do not conjecture to be secure).

Variant 9.1. Correlated As: To ease backtracking we wish to give the decoder several Δs generated from the same B. Generate t independent keys $\{F_i, G_i, H_i\}$ (possibly modulo different p_is). Pick $t-1$ public random permutations $\phi_1, \ldots, \phi_{t-1}$ of $\mathbb{Z}_n$. Generate $\{B_0, \ldots, B_{t-1}, A_0\} \in_R \mathfrak{H}_{n,h}^{t+1}$. Let

$$A_0 = \sum_{i=0}^{n-1} 2^i a_i$$

and form t ciphertexts $\{C_0, \ldots, C_{t-1}\}$:

$$C_j = A_j H_j + B_j \bmod p_j \quad \text{where} \quad A_j = \sum_{i=0}^{n-1} 2^{\phi_j(i)} a_i$$

Simultaneous backtracking on the A_js will reveal more information per bit guess to the decoder.

Variant 9.2. Correlated Hs: Set G and define $H_i = F_i/G$ for $i \geq 1$. We illustrate the idea with two H_is. Encrypt $C = A_0 H_0 + A_1 H_1 + B$. We see that $W = GC = F_0 A_0 + F_1 A_1 + BG$. Linking A_0 and A_1 as in note 9.1 we see that the SNR[19] in the case of a successful guess increases:

[17] the possibly rotated

[18] Accidental similar-size prime factors may come from $AF + GB$ as well, but those are few and hence easy to filter.

[19] Note that as backtracking proceeds the SNR improves. In this paper SNR stands for the SNR at the beginning of the backtracking process.

$$\text{from}\ \ \text{SNR} = \frac{h}{2h^2} = \frac{1}{2h}\ \ \text{to}\ \ \text{SNR} = \frac{2h}{3h^2} = \frac{2}{3h}$$

This modifies the complexity assumption as well.

Note 10 (a broken variant): We close this paper by attracting the reader's attention to the broken variant given in the appendix, that we mention as a target for fixing.

4 Security & Parameter Sizes

This work did not cover the security of the proposed constructions and focused on the textbook modes in which data is encoded and decoded. Parameter sizes were not recommended and numerical examples are given for illustrative purposes.

A careful balance must be established between ① the security, ② the decodability (backtracking failure probability) and ③ the efficiency of the various `Solve` processes. So far, simulations indicate that there are ways to practically satisfy those three constraints at once.

5 Acknowledgments

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A Appendix: A Broken Scheme Target for Repair

While designing the algorithms in this paper we broke the following variant that we mention here as a fixing target. Let R be a secret n-bit number of the form: $R \leftarrow \text{random}|v_\ell|\text{random}$ where v_ℓ is defined as in Pattern identification, and define the auxiliary public-key $L \leftarrow R/G \bmod p$. Note that this modifies the complexity assumption on which the scheme rests.

Encrypt by $C \leftarrow AH + B + Lm \bmod p$ and decrypt by $W \leftarrow GC = AF + GB + Rm \bmod p$. This offers a (noisy) visibility window on m allowing to extract and error-correct m.

The problem here is the equation defining L from which G can be revealed using LLL. It is unclear if this can be fixed using modifications in the encryption process and/or in parameter sizes. It is also interesting to determine if secure H-less variants[20] can be designed.

Assuming that the above could be repaired, the following is for readers fond of dangerous games.

A dangerous game, that we do not recommend, reduces noise in the visibility window. If A, B, G, F are generated in a biased way by shifting more Hamming weight into the MSBs and the LSBs as shown in Figure 4, then the result of the multiplication modulo p of two such numbers results in a number of the form shown in Figure 5. Those densities are illustrated in Figures 7 and 8. This reduces the noise in the reading window and allows an easier recovery of m. It must be stressed that this increases the vulnerability of the public-key to the partition attacks of [BCGN17] as the attacker can better zoom on the information-rich part of F and G. This might be compensated by a higher h. Note that weight shifting does not necessarily need to be similar in all four variables A, B, F, G and that several weight shifting schemes are circularly equivalent because of the multiplication modulo p.

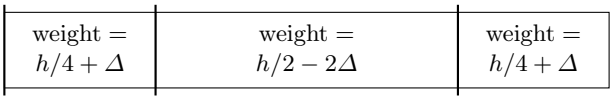

Figure 4: Unbalanced weight of A, B, F, G before multiplication.

[20] *i.e.* where the sender encrypts by $C \leftarrow Lm + B \bmod p$ (this is insecure) or a similar trick.

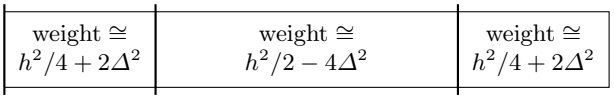

Figure 5: Unbalanced weight of $AF \bmod p$ and $BG \bmod p$ after multiplication.

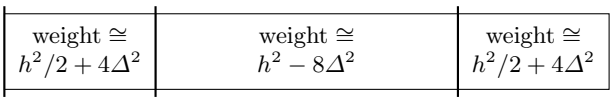

Figure 6: Unbalanced weight of $AF + BG \bmod p$ after addition.

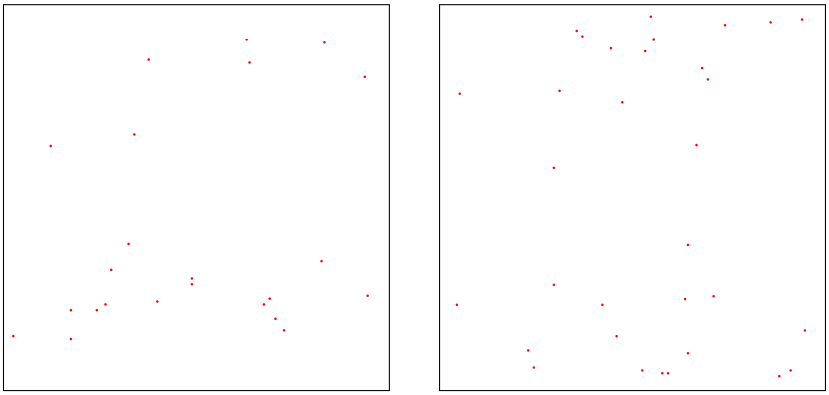

Figure 7: Unbalanced A and F for $\{n, h, \Delta\} = \{2^{14}, 32, 6\}$.

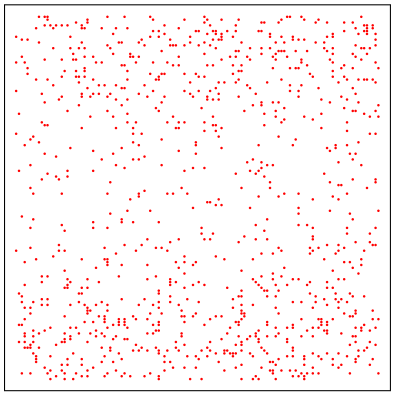

Figure 8: Unbalanced $AF \bmod p$ for $\{n, h, \Delta\} = \{2^{14}, 32, 6\}$. Note the relatively lower dot density at the middle of the diagram.