Encrypt-Augment-Recover: Function Private Predicate Encryption from Standard Assumptions in the Public-Key Setting

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Abstract. We present the first public-key predicate encryption schemes that are provably function private under standard computational assumptions. Existing function private predicate encryption schemes in the public-key setting are either secure only in the generic group model, or require strong assumptions such as indistinguishability obfuscation. Our framework for function privacy is indistinguishability-based in the sense that it requires a secret-key corresponding to a predicate sampled from a distribution with min-entropy super logarithmic in the security parameter λ, to be *computationally indistinguishable* from another secret-key corresponding to a uniformly and independently sampled predicate. Within this framework, we develop a novel approach, denoted as *encrypt-augment-recover*, that takes an existing predicate encryption scheme and transforms it into a computationally function private one while retaining its original data privacy guarantees. Our approach leads to public-key constructions for identity-based encryption (IBE) and inner-product encryption (IPE) that are fully data private and computationally function private under a family of weaker variants of the DLIN assumption. Our constructions are secure in the standard model, and avoid the need for strong assumptions such as indistinguishability obfuscation.

Keywords: Predicate Encryption, Public-Key, Function Privacy, Computational Indistinguishability, Min-Entropy, Identity-Based Encryption, Inner-Product Encryption

1 Introduction

Predicate encryption schemes [1–3] in the public-key setting allow a single public-key to be associated with multiple secret-keys, where each secret-key corresponds to a boolean predicate $f : \Sigma \longrightarrow \{0, 1\}$ over a pre-defined set of attributes Σ. A plaintext message in a predicate encryption system is an attribute-payload message pair $(I, M) \in \Sigma \times \mathcal{M}$, with $\mathcal{M}$ being the payload message space. A secret-key sk_f associated with a predicate f successfully decrypts a ciphertext C corresponding to a plaintext (I, M) and recovers the payload message M if and only if $f(I) = 1$. On the other hand, if $f(I) = 0$, attempting to decrypt C using sk_f returns the failure symbol $\perp$. A predicate encryption is said to be *attribute hiding* if the ciphertext C leaks no information about the underlying plaintext (I, M) to an adversary possessing benign secret-keys corresponding to predicates that do not trivially identify the attribute I.

Identity-Based Encryption. Identity-based encryption (IBE) [4–6] is the simplest sub-class of public-key predicate encryption. IBE supports a set of equality predicates of the form $f_{\mathsf{id}} : \Sigma \longrightarrow \{0, 1\}$ defined as $f_{\mathsf{id}}(x) = 1$ if and only if $x = \mathsf{id}$. The attribute space in this case is a set of identities $\mathcal{ID}$, and each identity $\mathsf{id} \in \mathcal{ID}$ is associated with its own secret-key $\mathsf{sk}_{\mathsf{id}}$.

Inner-Product Encryption. Inner-product encryption (IPE) [2, 3, 7, 8] is the most expressive sub-class of predicate encryption, supporting a set of predicates $f_{\overrightarrow{v}} : \Sigma \longrightarrow \{0, 1\}$ over a vector space of attributes $\Sigma = \mathbb{F}_q^n$ (q being a λ-bit prime). Of particular interest is a specific form of IPE called zero-IPE [3] where for $\overrightarrow{v}, \overrightarrow{x} \in \Sigma$, we have $f_{\overrightarrow{v}}(\overrightarrow{x}) = 1$ if and only if $\langle \overrightarrow{v}, \overrightarrow{x} \rangle = 0$, where $\langle \overrightarrow{v}, \overrightarrow{x} \rangle$ denotes the inner-product of two vectors $\overrightarrow{v}$ and $\overrightarrow{x}$. IPE is powerful enough to encompass IBE and many other predicate encryption systems [3].

Searchable Encryption and Function Privacy. Predicate encryption provides a generic framework for searchable encryption supporting a wide range of query predicates including conjunctive, disjunctive, range and subset queries [9, 1–3]. For instance, a predicate encryption system can be used to realize a mail gateway that follows some special instructions to route encrypted mails based on their header information (e.g. if the mail is from the boss and needs to be treated as urgent). The mail gateway is given the secret-key corresponding to the predicate *is-urgent*, the mail header serves as the attribute, while the routing instructions can be used as the payload message. Another application could be a payment gateway that flags encrypted payments if they correspond to amounts beyond some pre-defined threshold X. The payment gateway is given the secret-key corresponding to the predicate *greater-than-X*, the payment amount itself serves as the attribute, while the flag signal is encoded as the payload message. The attribute hiding property of the predicate encryption scheme ensures that neither gateway learns any information about the plaintext data from the entire operation.

A natural question now arises: should the gateways in the aforementioned examples be able to learn the underlying predicate from the secret-keys given to them? The answer in most scenarios is *no* - the secret-key sk_f should ideally reveal nothing about the predicate f beyond the absolute minimum. This notion of predicate hiding security is commonly referred to as *function privacy*, and predicate encryption scheme satisfying this notion of security are described as *function private*.

1.1 Function Private Predicate Encryption in the Public-Key Setting

As pointed out by Boneh, Raghunathan and Segev in [10, 11], formalizing a realistic notion of function privacy in the context of public-key predicate encryption is, in general, not straightforward. Consider, for example, an adversary against an IBE scheme who is given a secret-key $\mathsf{sk}_{\mathsf{id}}$ corresponding to an identity id and has access to an encryption oracle. As long as the adversary has some apriori information that the identity id belongs to a small set $\mathcal{S}$,(e.g. id is sampled distribution with min-entropy at most polynomial in the security parameter λ), it can fully recover id from $\mathsf{sk}_{\mathsf{id}}$: it can simply resort to encrypting a random message M under each identity in $\mathcal{S}$, and decrypting using $\mathsf{sk}_{\mathsf{id}}$ to check for a correct recovery. Consequently, [10, 11] consider a framework for function privacy under the minimal assumption that any predicate is sampled from a distribution with min-entropy at least super logarithmic in the security parameter λ. In this paper, we use the same framework for proving the computational function privacy of our proposed predicate encryption schemes.

Statistical Function Privacy. Boneh, Raghunathan and Segev introduced a a *statistical* notion of function privacy for equality predicates in [10], and subsequently generalized the same for subspace-membership predicates in [11]. Their approach may be briefly summarized as follows: instead of directly generating a secret-key sk_f for a predicate f, a strong randomness extractor Ext is first applied to f using a randomly chosen seed s, followed by the generation of the secret key sk_{f_s}, where $f_s = \mathsf{Ext}(f, s)$. The final secret-key for the predicate f is the pair (s, sk_{f_s}). Any such secret-key is thus *statistically indistinguishable* from random, as long the underlying predicates are sampled from sufficiently unpredictable distributions. In this paper, we focus on the alternative notion of computational function privacy, where the indistinguishability of the secret-keys from random can be based on standard computational assumptions, subject to the same constraint that the underlying predicates are sampled from sufficiently unpredictable distributions. Concretely realizing this computational notion of function privacy for public-key predicate encryption was left as an open problem in [11].

Function Privacy from Quasi-Strong Indistinguishability Obfuscation. A generic approach to achieving function privacy, proposed by Iovino et al. in [12], is to a use a quasi-strong indistinguishability obfuscation (Quasi-siO) scheme $\mathsf{Q\text{-}si}\mathcal{O}$ over the class of predicates $\mathcal{F}$ in the key-generation step of the predicate encryption scheme. In particular, given a predicate encryption system $\Pi = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$, one can construct a function-private $\Pi' = \big(\mathsf{Setup}, \mathsf{KeyGen}', \mathsf{Enc}, \mathsf{Dec}\big)$, such that:

$$\mathsf{KeyGen}'\left(msk, f\right) = \mathsf{KeyGen}\left(msk, \mathsf{Q\text{-}si}\mathcal{O}(f)\right)$$

While the above approach is generic and applies to a wide class of predicates (more specifically, to the class of all NC^1) circuits), it relies on the use of a Quasi-siO, the existence of which cannot be provably based on any standard computational assumption to the best of our knowledge.

Function Privacy in the Generic Group Model. Agrawal et al. have recently proposed a public-key construction for IPE in [13] that achieves a *wishful notion* of combined data and function privacy in a simulation-based security framework. Their construction is based on the use of bilinear pairings over prime order groups, and is secure in the generic group model. Indeed, known impossibilities [14] rule out the achievability of such a strong notion of security in the standard model. Interestingly, even while restricting to an indistinguishability-based security framework in the public-key setting, it has been an open question as to whether one can concretely achieve function private predicate encryption constructions in the standard model from known computational assumptions.

1.2 Our Contributions

In this paper, we present the first public-key encryption schemes that support a rich class of predicates and are provably function private under standard computational assumptions. Our framework for function privacy is indistinguishability-based in the sense that it requires a secret-key corresponding to a predicate sampled from a distribution with min-entropy super logarithmic in the security parameter λ, to be *computationally indistinguishable* from another secret-key corresponding to a uniformly and independently sampled predicate. Within this framework, we develop a novel approach, denoted as *encrypt-augment-recover*, that takes an existing predicate encryption scheme and transforms it into a computationally function private one while retaining its original data privacy guarantees. Our approach leads to the following constructions:

- In the standard model, we present a family of computationally function private identity-based encryption (IBE) schemes from bilinear pairings based on the anonymous IBE scheme proposed by Gentry in [4]. Our schemes retain the selective data privacy guarantees of the original scheme, and are additionally computationally function private under progressively weaker variants of the well-known DLIN assumption. The detailed constructions of these schemes, along with the proofs of data and function privacy, are presented in Section 4.

- We then extend our approach to achieve a family of computationally function private inner-product encryption (IPE) schemes based on the seminal scheme of Katz, Sahai and Waters [3]. Once again, our schemes retain the selectively attribute hiding property of the underlying scheme, and are computationally function private under progressively weaker variants of the DLIN assumption. The detailed constructions of these schemes, along with the proofs of data and function privacy, are presented in Section 5.

Additionally, both families of constructions avoid the need for strong assumptions such as indistinguishability obfuscation.

1.3 Overview of Our Approach: *Encrypt-Augment-Recover*

Our approach for achieving computationally function private predicate encryption schemes consists of three main steps - *encrypt*, *augment* and *recover*. We briefly describe the main ideas underlying each step, and exemplify them subsequently using a simple IBE scheme. Given a public-key predicate encryption scheme $\Pi = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$, and a CPA-secure public-key encryption algorithm $\mathsf{PKE} = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$, we create a function private predicate encryption scheme $\Pi' = (\mathsf{Setup}', \mathsf{KeyGen}', \mathsf{Enc}', \mathsf{Dec}')$ as follows:

- The modified setup algorithm Setup' invokes $\mathsf{PKE.KeyGen}$ and obtains the key pair (PK, SK). It also invokes $\Pi.\mathsf{Setup}$ and obtains the public parameters pp, along with master secret-key msk.

It outputs the modified public parameter $\mathsf{pp}' = (\mathsf{pp}, g(SK))$ and the modified master secret-key $\mathsf{msk}' = (\mathsf{msk}, PK)$, where g is a suitably chosen one way function.

- On input a predicate f and the augmented master secret-key $\mathsf{msk}' = (\mathsf{msk}, PK)$, the modified key-generation algorithm KeyGen' invokes $\Pi.\mathsf{KeyGen}$ to obtain the original secret-key sk_f. It then outputs an *encrypted* secret-key sk'_f as $\mathsf{PKE.Enc}(PK, \mathsf{sk}_f)$.

  This step allows us to base our function privacy arguments on the same computational assumption that guarantees the CPA security of the PKE scheme. More specifically, it ensures *adaptive* function privacy - the inherently random nature of the augmented key generation algorithm ensures that the function privacy guarantees hold even when the adversary is allowed to specify predicate distributions in an adaptive manner after seeing the public parameters of the scheme. We assume that any adversarially-chosen distribution of predicates is sufficiently unpredictable, so as to rule out a trivial breach of function privacy as mentioned earlier. This minimal assumption is thus sufficient to transform the original predicate encryption scheme into a computationally function private one.

- An even greater challenge is to synchronize the encryption and decryption algorithms in the modified scheme. This is achieved as follows. On input the public parameter $\mathsf{pp}' = (\mathsf{pp}, g(SK))$, and a message M corresponding to an attribute I, the modified encryption algorithm Enc' first obtains $C = \Pi.\mathsf{Enc}(\mathsf{pp}, I, M)$. It then outputs the *augmented* ciphertext $C' = (C, \sigma(C, SK))$, where σ is a function computable using the knowledge of C and $g(SK)$.

- Finally, Dec' cleverly uses the additional ciphertext component $\sigma(g(SK))$ in C' to remove the effect of PKE from the encrypted secret-key sk'_f, and *recover* the message M. Note that removal here is not same as decryption, since Dec' has access to only a one-way function of SK and not SK itself. It is, in fact, impossible to provide SK to Dec' in the clear without trivially compromising function privacy. The challenge is thus to ensure that Dec' can recover M without a complete decryption of sk'_f.

An Example of Our Approach. We present an example of a computationally function private IBE scheme in the standard model achieved using our *encrypt-augment-decrypt* approach. A generalization of this scheme is presented in greater detail in Section 4, along with proofs for data and function privacy. Consider a public-key encryption scheme PKE with the key generation, encryption and decryption algorithms as described below:

- **KeyGen:** The key-generation algorithm samples $x_1, x_2, x_3 \xleftarrow{R} \mathbb{Z}_q^*$, where q is a λ-bit prime, and $g_1, g_2, g_3 \xleftarrow{R} \mathbb{G}$, where $\mathbb{G}$ is a cyclic group of prime order q. It outputs the secret-key SK and the public key PK as:

$$SK = (x_1, x_2, x_3) \ , \ PK = (g_1, g_2, g_3, (g_1^{x_1} \cdot g_3^{x_3}), (g_2^{x_2} \cdot g_3^{x_3}))$$

- **Enc:** The ciphertext C corresponding to a message $M \in \mathbb{G}$ is a tuple of the form:

$$C = \left(g_1^{y_1}, g_2^{y_2}, g_3^{y_1+y_2}, (g_1^{x_1} \cdot g_3^{x_3})^{y_1} \cdot (g_2^{x_2} \cdot g_3^{x_3})^{y_2} \cdot M\right)$$

  where $y_1, y_2 \xleftarrow{R} \mathbb{Z}_q^*$.

- **Dec:** The decryption algorithm, on input the ciphertext $C = (c_0, c_1, c_2, c_3)$ and the secret-key (x_1, x_2, x_3), recovers the message M as:

$$M = c_3 \Big/ (c_0^{x_1} \cdot c_1^{x_2} \cdot c_2^{x_3})$$

The above scheme is a simple variant of the Cramer-Shoup cryptosystem [15], and is CPA-secure under the DLIN assumption. We now present a computationally function private IBE scheme that is obtained by applying our *encrypt-augment-recover* approach to the anonymous IBE scheme proposed by Gentry [4], which is selectively data private under the decisional bilinear Diffie-Hellman exponent (BDHE) assumption in the standard model.

- **Setup:** The setup algorithm in Gentry's scheme samples $s \xleftarrow{R} \mathbb{Z}_q^*$, where q is a λ-bit prime. The public parameters are (g, g^s, h), where g and h are randomly sampled generators of a bilinear group $\mathbb{G}$ of prime order q, while the master secret-key is s. Our scheme additionally samples $x_1, x_2, x_3 \xleftarrow{R} \mathbb{Z}_q^*$ and $g_1, g_2, g_3 \xleftarrow{R} \mathbb{G}$. The augmented public parameter pp and master secret-key msk for our scheme are as follows:

$$\mathsf{pp} = \left(g, g^s, h, \boxed{g^{x_1}, g^{x_2}, g^{x_3}}, \boxed{g^{s \cdot x_1}, g^{s \cdot x_2}, g^{s \cdot x_3}}\right)$$
$$\mathsf{msk} = \left(s, \boxed{g_1, g_2, g_3, (g_1^{x_1} \cdot g_3^{x_3}), (g_2^{x_2} \cdot g_3^{x_3})}\right)$$

  Observe that the additional components in pp are one-way functions of (x_1, x_2, x_3) - the secret-key SK of the PKE scheme. Additionally, the modified msk contains the public-key PK of the PKE scheme.

- **KeyGen:** The key-generation algorithm in the Gentry's scheme computes a secret-key for an identity id as $\mathsf{sk}_{\mathsf{id}} = \left(y, (h \cdot g^{-y})^{1/(s-\mathsf{id})}\right)$, where $y \xleftarrow{R} \mathbb{Z}_q^*$. In our scheme, we augment the key generation process as follows. We additionally sample $y_1, y_2 \xleftarrow{R} \mathbb{Z}_q^*$, and output:

$$\mathsf{sk}_{\mathsf{id}} = \left(y, \boxed{g_1^{y_1}, g_2^{y_2}, g_3^{y_1+y_2}}, \boxed{(g_1^{x_1} \cdot g_3^{x_3})^{y_1} \cdot (g_2^{x_2} \cdot g_3^{x_3})^{y_2}} \cdot \left(h \cdot g^{-y}\right)^{1/(s-\mathsf{id})}\right)$$

  Observe that $\mathsf{sk}_{\mathsf{id}} = \mathsf{PKE.Enc}\left(PK, (h \cdot g^{-y})^{1/(s-\mathsf{id})}\right)$. This is an exemplification of the *encrypt* step of our approach described above.

- **Enc:** An encryption of a message M for an identity id in the Gentry's scheme is a tuple of the form $\left(g^{r \cdot (s-id)}, g\,(g,g)^r, M \cdot e\,(g,h)^{-r}\right)$, where $r \xleftarrow{R} \mathbb{Z}_q^*$. In our scheme, we augment the encryption process to produce the ciphertext:

$$C = \left(g^{r \cdot (s-id)}, \boxed{g^{r \cdot x_1 \cdot (s-id)}, g^{r \cdot x_2 \cdot (s-id)}, g^{r \cdot x_3 \cdot (s-id)}}, e\,(g,g)^r, M \cdot e\,(g,h)^{-r}\right)$$

  Note that the augmented ciphertext in our scheme retains unaltered the ciphertext of the original scheme.

- **Dec:** Our decryption algorithm, on input of a ciphertext $C = (c_0, c_1, c_2, c_3, c_4, c_5)$, and a secret-key $\mathsf{sk}_{\mathsf{id}} = (d_0, d_1, d_2, d_3, d_4)$, recovers the encrypted message M as:

$$\boxed{M = \frac{c_5 \cdot c_4^{d_0} \cdot e\,(d_4, c_0)}{e\,(d_1, c_1) \cdot e\,(d_2, c_2) \cdot e\,(d_3, c_3)}}$$

  Observe that at the core of the above computation is the original decryption procedure in the Gentry's scheme, with the additional components in the ciphertext and the secret-key canceling out each other to *recover the effect of the* PKE (the reader is referred to Section 4 for the detailed proof of correctness). It is important to note that this removal is different from directly decrypting $\mathsf{sk}_{\mathsf{id}}$, and in particular, does not require the knowledge of the secret-key of the PKE.

1.4 Other Related Work

Function Privacy in the Private-Key Setting. Computational function privacy for predicate encryption has been widely studied in the private-key setting [16, 17]. The inherent difficulty of achieving function privacy in the public-key setting does not apply to the private-key setting, where the encryptor and decryptor have a shared secret-key. In this setting, an adversary with access to a searching key cannot test the same on ciphertexts of its choice since it does not have access to the secret-key. Function privacy in the private-key setting is thus more natural to achieve. A general solution in this direction was proposed by Goldreich and Ostrovsky [18] in their construction of an oblivious RAM. More efficient constructions have been subsequently proposed for equality testing [19–23] and, more recently, for inner product testing [16, 24, 25]. In particular, the private-key IPE scheme proposed by Agrawal et al. in [26] achieves the strongest possible notion of combined data and function privacy from the DLIN assumption in a simulation based framework.

1.5 Paper Organization

The remainder of this paper is organized as follows. Section 2 presents background material on predicate encryption, and introduces several computational assumptions in bilinear groups. In Section 3, we formally define our framework for the computational function privacy of public-key predicate encryption. In Section 4, we present a family of adaptively data private and computationally function private IBE schemes in the random-oracle model. In Section 5, we present a family of selectively attribute hiding and computationally function private IPE schemes in the standard model. Finally, Section 7 concludes the paper and enumerates several open problems.

1.6 Notations Used

We write $x \xleftarrow{R} \chi$ to represent that an element x is sampled uniformly at random from a set $\mathcal{X}$. The output a of a deterministic algorithm $\mathcal{A}$ is denoted by $x \leftarrow \mathcal{A}$ and the output a' of a randomized algorithm $\mathcal{A}'$ is denoted by $x' \xleftarrow{R} \mathcal{A}'$. We refer to $\lambda \in \mathbb{N}$ as the security parameter, and denote by $\mathsf{exp}(\lambda)$, $\mathsf{poly}(\lambda)$ and $\mathsf{negl}(\lambda)$ any generic (unspecified) exponential function, polynomial function and negligible function in λ respectively. Note that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is said to be negligible in λ if for every positive polynomial p, $f(\lambda) < 1/p(\lambda)$ when λ is sufficiently large. Finally, for $a, b \in \mathbb{Z}$ such that $a \leq b$, we denote by $[a, b]$ the set of integers lying between a and b (both inclusive).

The min-entropy of a random variable Y is denoted as $\mathbf{H}_\infty(Y) = -\log\left(\max_y \Pr[Y = y]\right)$; a random variable Y is said to be a k-source if $\mathbf{H}_\infty(Y) \geq k$. A (T, k)-block-source is a random variable $\mathbf{Y} = (Y_1, \cdots, Y_T)$ where for each $i \in [1, T]$ and $y_1, \cdots, y_{i-1}$, it holds that:

$$\mathbf{H}_\infty(Y_i | Y_1 = y_1, \cdots, Y_{i-1} = y_{i-1}) \geq k$$

2 Preliminaries

2.1 Public-key Predicate Encryption

A public-key predicate encryption scheme for a class of predicates $\mathcal{F}$ over an attribute space Σ and a payload-message space $\mathcal{M}$ is a quadruple $\Pi = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ of probabilistic polynomial time algorithms. The Setup algorithm takes as input the security parameter λ, and generates the public parameter pp and the master secret-key msk for the system. The key-generation algorithm, KeyGen takes as input the master secret-key msk and a predicate $f \in \mathcal{F}$, and generates a secret-key sk_f corresponding to f. The Enc algorithm takes as input the public parameter pp, an attribute $I \in \Sigma$ and a payload-message $M \in \mathcal{M}$, and outputs the ciphertext $C = \mathsf{Enc}\left(\mathsf{pp}, I, M\right)$. The Dec algorithm takes as input the public parameter pp, a ciphertext C and a secret-key sk_f, and outputs either a payload-message $M \in \mathcal{M}$ or the symbol $\perp$.

Functional Correctness. A predicate encryption scheme $\Pi = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ is said to be functionally correct if for any security parameter λ, for any predicate $f \in \mathcal{F}$, for any attribute $I \in \Sigma$ and any payload-message $M \in \mathcal{M}$, the following hold with probability at least $1 - \mathsf{negl}(\lambda)$:

1. If $f(I) = 1$, we have $\mathsf{Dec}\left(\mathsf{pp}, \mathsf{Enc}\left(\mathsf{pp}, I, M\right), \mathsf{KeyGen}\left(\mathsf{msk}, f\right)\right) = M$.
2. If $f(I) = 0$, we have $\mathsf{Dec}\left(\mathsf{pp}, \mathsf{Enc}\left(\mathsf{pp}, I, M\right), \mathsf{KeyGen}\left(\mathsf{msk}, f\right)\right) = \perp$.

where the probability is taken over the internal randomness of the algorithms $\mathsf{Setup}, \mathsf{KeyGen}$, Enc, and Dec.

Data Privacy. We briefly recall the notion of indistinguishability-based data privacy for a predicate encryption scheme under an *adaptive* chosen-attribute chosen-payload-message attack. Data privacy of a functional encryption scheme guarantees that any probabilistic polynomial-time adversary can gain no information about either the attribute I nor the payload-message M associated with a ciphertext C from the knowledge of the public parameters pp. We denote this notion of security by DP throughout the rest of the paper.

Definition 2.1 (Adaptively Data Private Predicate Encryption). A predicate encryption scheme $\Pi = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ is said to be *adaptively data private* if for any probabilistic polynomial-time adversary $\mathcal{A}$, the following holds:

$$\mathbf{Adv}^{\mathsf{DP}}_{\Pi,\mathcal{A}}(\lambda) \stackrel{\text{def}}{=} \left| \Pr\left[\mathsf{Expt}^{(0)}_{\mathsf{DP},\Pi,\mathcal{A}}(\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{(1)}_{\mathsf{DP},\Pi,\mathcal{A}}(\lambda) = 1\right] \right| \leq \mathsf{negl}(\lambda)$$

where for each $\lambda \in \mathbb{N}$ and each $b \in \{0,1\}$, the experiment $\mathsf{Expt}^{(b)}_{\mathsf{DP},\Pi,\mathcal{A}}(\lambda)$ is defined as follows*:*

1. $(\mathsf{pp}, \mathsf{msk}) \xleftarrow{R} \mathsf{Setup}\left(1^{\lambda}\right)$.
2. $\left((I^*_0, M^*_0), (I^*_1, M^*_1), \mathsf{state}\right) \xleftarrow{R} \mathcal{A}^{\mathsf{KeyGen}(\mathsf{msk},\cdot)}\left(\mathsf{state}\right)$, where $I^*_0, I^*_1 \in \Sigma$ and $M^*_0, M^*_1 \in \mathcal{M}$, subject to the restriction that for each predicate f_i with which $\mathcal{A}$ queries $\mathsf{KeyGen}\left(\mathsf{msk}, \cdot\right)$, we have $f_i\left(I^*_0\right) = f_i\left(I^*_1\right)$.
3. $C^* \xleftarrow{R} \mathsf{Enc}\left(\mathsf{pp}, I^*_b, M^*_b\right)$.
4. $b' \xleftarrow{R} \mathcal{A}^{\mathsf{KeyGen}(\mathsf{msk},\cdot)}\left(C^*, \mathsf{state}\right)$, once again subject to the restriction that for each predicate f_i with which $\mathcal{A}$ queries $\mathsf{KeyGen}\left(\mathsf{msk}, \cdot\right)$, we have $f_i\left(I^*_0\right) = f_i\left(I^*_1\right)$.
5. Output b'.

We also consider a *selective* variant of the above security notion that requires the adversary to commit to the challenge pair of attributes before seeing the public parameters of the scheme. We denote this notion of security by sDP throughout the rest of the paper.

Definition 2.2 (Selectively Data Private Predicate Encryption). A predicate encryption scheme $\Pi = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ is said to be *selectively data private* if for any probabilistic polynomial-time adversary $\mathcal{A}$, the following holds:

$$\mathbf{Adv}^{\mathsf{sDP}}_{\Pi,\mathcal{A}}(\lambda) \stackrel{\text{def}}{=} \left| \Pr\left[\mathsf{Expt}^{(0)}_{\mathsf{sDP},\Pi,\mathcal{A}}(\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{(1)}_{\mathsf{sDP},\Pi,\mathcal{A}}(\lambda) = 1\right] \right| \leq \mathsf{negl}(\lambda)$$

where for each $\lambda \in \mathbb{N}$ and each $b \in \{0,1\}$, the experiment $\mathsf{Expt}^{(b)}_{\mathsf{sDP},\Pi,\mathcal{A}}(\lambda)$ is defined as follows*:*

1. $\left(I^*_0, I^*_1, \mathsf{state}\right) \xleftarrow{R} \mathcal{A}\left(1^{\lambda}\right)$, where $I^*_0, I^*_1 \in \Sigma$.
2. $(\mathsf{pp}, \mathsf{msk}) \xleftarrow{R} \mathsf{Setup}\left(1^{\lambda}\right)$.
3. $\left(M^*_0, M^*_1, \mathsf{state}\right) \xleftarrow{R} \mathcal{A}^{\mathsf{KeyGen}(\mathsf{msk},\cdot)}\left(\mathsf{state}\right)$, where $M^*_0, M^*_1 \in \mathcal{M}$, subject to the restriction that for each predicate f_i with which $\mathcal{A}$ queries $\mathsf{KeyGen}\left(\mathsf{msk}, \cdot\right)$, we have $f_i\left(I^*_0\right) = f_i\left(I^*_1\right)$.
4. $C^* \xleftarrow{R} \mathsf{Enc}\left(\mathsf{pp}, I^*_b, M^*_b\right)$.
5. $b' \xleftarrow{R} \mathcal{A}^{\mathsf{KeyGen}(\mathsf{msk},\cdot)}\left(C^*, \mathsf{state}\right)$, once again subject to the restriction that for each predicate f_i with which $\mathcal{A}$ queries $\mathsf{KeyGen}\left(\mathsf{msk}, \cdot\right)$, we have $f_i\left(I^*_0\right) = f_i\left(I^*_1\right)$.
6. Output b'.

Identity-Based Encryption. An identity-based encryption scheme Π^{IBE} over an identity space $\mathcal{ID}$ and a message space $\mathcal{M}$ is a public-key predicate encryption scheme supporting the set of equality predicates $f_{\mathsf{id}} : \mathcal{ID} \longrightarrow \{0,1\}$ defined as $f_{\mathsf{id}}(\mathsf{id}') = 1$ if and only if $\mathsf{id}' = \mathsf{id}$. The secret-key associated with an identity $\mathsf{id} \in \mathcal{ID}$ is denoted as $\mathsf{sk}_{\mathsf{id}}$. The notions of anonymity and message indistinguishability security popularly associated with IBE are equivalent to the notion of adaptive data privacy as described above.

Inner-Product Encryption. An inner-product encryption scheme Π^{IPE} over an attribute space $\Sigma = \mathbb{F}_q^n$ (q being a λ-bit prime) and a payload message space $\mathcal{M}$ is a public-key predicate encryption scheme supporting the set of vector predicates $f_{\overrightarrow{v}} : \Sigma \longrightarrow \{0,1\}$. The secret-key associated with a vector $\overrightarrow{v} \in \Sigma$ is denoted as $\mathsf{sk}_{\overrightarrow{v}}$. Zero-IPE is a specific sub-class of IPE where for $\overrightarrow{v}, \overrightarrow{x} \in \Sigma$, we have $f_{\overrightarrow{v}}(\overrightarrow{x}) = 1$ if and only if $\langle \overrightarrow{v}, \overrightarrow{x} \rangle = 0$.

2.2 Computational Assumptions in Bilinear Groups

The weak decisional bilinear Diffie-Hellman Inversion (DBDHI) assumption. Let $\mathsf{GroupGen}(1^\lambda)$ be a probabilistic polynomial-time algorithm that takes as input a security parameter λ, and outputs the tuple $(\mathbb{G}, \mathbb{G}_T, q, g, e)$, where $\mathbb{G}$ and $\mathbb{G}_T$ are groups of order q (q being a λ-bit prime), g is a generator for $\mathbb{G}$ and $e : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}_T$ is an efficiently computable non-degenerate bilinear map. The group $\mathbb{G}$ is popularly referred to as a *bilinear group* [27]. The weak decisional bilinear Diffie-Hellman assumption, introduced by Boneh, Boyen and Goh in [28], is that the distribution ensembles:

$$\left\{ \left(g, h, g^a, e(g,h)^{1/a} \right) \right\}_{a \xleftarrow{R} \mathbb{Z}_q^*} \text{ and } \left\{ (g, h, g^a, Z) \right\}_{a \xleftarrow{R} \mathbb{Z}_q^*, Z \xleftarrow{R} \mathbb{G}_T}$$

are computationally indistinguishable, where $(\mathbb{G}, \mathbb{G}_T, q, g, e) \leftarrow \mathsf{GroupGen}(1^\lambda)$.

The decisional linear assumption (DLIN)[29]. Let $\mathbb{G}$ be a group of prime order q and let g_1, g_2, g_3 be arbitrary generators for $\mathbb{G}$. The decisional linear assumption is that the distribution ensembles:

$$\left\{ \left(g_1, g_2, g_3, g_1^{a_1}, g_2^{a_2}, g_3^{a_1+a_2} \right) \right\}_{a_1, a_2 \xleftarrow{R} \mathbb{Z}_q^*} \text{ and } \left\{ (g_1, g_2, g_3, g_1^{a_1}, g_2^{a_2}, g_3^{a_3}) \right\}_{a_1, a_2, a_3 \xleftarrow{R} \mathbb{Z}_q^*}$$

are computationally indistinguishable, where $g_1, g_2, g_3 \xleftarrow{R} \mathbb{G}$.

The DLIN assumption was introduced by Boneh, Boyen and Shacham [29], and was intended to take the place of the more standard decisional Diffie Hellman (DDH) assumption in groups where the DDH assumption does not hold. In particular, for bilinear groups as defined above, the DLIN assumption holds even if the DDH assumption does not, at least in the generic group model.

The generalized decisional k-linear assumption (k-DLIN) [30]. Let $\mathbb{G}$ be a group of prime order q and let $g_1, \cdots, g_k, g_{k+1}$ be arbitrary generators for $\mathbb{G}$. The generalized decisional k-linear assumption is that the distribution ensembles:

$$\left\{ \left(g_1, \cdots, g_k, g_{k+1}, g_1^{a_1}, \cdots, g_k^{a_k}, g_{k+1}^{\sum_{j=1}^{k} a_j} \right) \right\}_{a_1, \cdots, a_k \xleftarrow{R} \mathbb{Z}_q^*} \text{ and}$$
$$\left\{ \left(g_1, \cdots, g_k, g_{k+1}, g_1^{a_1}, \cdots, g_k^{a_k}, g_{k+1}^{a_{k+1}} \right) \right\}_{a_1, \cdots, a_k, a_{k+1} \xleftarrow{R} \mathbb{Z}_q^*}$$

are computationally indistinguishable, where $g_1, \cdots, g_{k+1} \xleftarrow{R} \mathbb{G}$.

Quite evidently, this assumption is a generalization of the DLIN assumption stated above. Note that the k-DLIN assumption implies the $(k+1)$-DLIN assumption for all $k \geq 1$, but the reverse is not necessarily true, implying that the k-DLIN assumption family is a family of progressively weaker assumptions [30].

3 Computational Function Privacy of Public-Key Predicate Encryption

We present our definitions for the computational function privacy of predicate encryption in the public-key setting. We consider adversaries that have access to the public parameters of the scheme, as well as a secret-key generation oracle. The adversary can also adaptively interact with a *real-or-random* function-privacy oracle $\mathsf{RoR}^{\mathsf{FP}}$. This oracle takes as input any adversarially-chosen distribution over the class of predicates $\mathcal{F}$, and outputs a secret-key either for a predicate sampled from the given distribution, or for an independently and uniformly sampled predicate. At the end of the interaction, the adversary should be able to distinguish between these *real* and *random* modes of operation of $\mathsf{RoR}^{\mathsf{FP}}$ with only negligible probability.

Formal Definitions. We now formally present the computational function privacy definitions for public-key predicate encryption.

Definition 3.1 (Real-or-Random Function Privacy Oracle). The real-or-random function privacy oracle $\mathsf{RoR}^{\mathsf{FP}}$ takes as input triplets of the form $(\mathsf{mode}, msk, \mathbf{F})$, where $\mathsf{mode} \in \{\mathsf{real}, \mathsf{rand}\}$, msk is the master secret-key, and $\mathbf{F}$ is a circuit representing a distribution over the class of predicates $\mathcal{F}$. If $\mathsf{mode} = \mathsf{real}$, the oracle samples $f \xleftarrow{R} \mathbf{F}$, while if $\mathsf{mode} = \mathsf{rand}$, it samples $f \xleftarrow{R} \mathcal{F}$. It then computes $\mathsf{sk}_f \xleftarrow{R} \mathsf{KeyGen}(msk, f)$ and responds with sk_f.

Definition 3.2 (Computational Function Privacy). A predicate encryption scheme $\Pi = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ is said to be *computationally function private* if for any probabilistic polynomial-time adversary $\mathcal{A}$, the following holds:

$$\mathbf{Adv}^{\mathsf{FP}}_{\Pi,\mathcal{A}}(\lambda) \stackrel{\text{def}}{=} \left| \Pr\left[\mathsf{Expt}^{\mathsf{real}}_{\mathsf{FP},\Pi,\mathcal{A}}(\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{rand}}_{\mathsf{FP},\Pi,\mathcal{A}}(\lambda) = 1\right] \right| \leq \mathsf{negl}(\lambda)$$

where for each $\lambda \in \mathbb{N}$ and each $\mathsf{mode} \in \{\mathsf{real}, \mathsf{rand}\}$, the experiment $\mathsf{Expt}^{\mathsf{mode}}_{\mathsf{FP},\Pi,\mathcal{A}}(\lambda)$ is defined as follows:

1. $(\mathsf{pp}, \mathsf{msk}) \xleftarrow{R} \mathsf{Setup}\left(1^\lambda\right)$.
2. $b \xleftarrow{R} \mathcal{A}^{\mathsf{RoR}^{\mathsf{FP}}(\mathsf{mode},\mathsf{msk},\cdot),\mathsf{KeyGen}(\mathsf{msk},\cdot)}\left(1^\lambda, \mathsf{pp}\right)$, subject to the restriction that each $\mathbf{F}_i$ with which $\mathcal{A}$ queries $\mathsf{RoR}^{\mathsf{FP}}(\mathsf{mode}, \mathsf{msk}, \cdot)$ represents a distribution with min-entropy $k = \omega(\log \lambda)$.
3. Output b.

Note that our definitions are generic, and may be suitably adopted for IBE, IPE and other classes of predicate encryption.

Min-Entropy Requirements. In our definitions for computational function privacy, the adversary is allowed to adaptively issue a polynomial number of queries to the $\mathsf{RoR}^{\mathsf{FP}}$ oracle, as long as the queries correspond to distributions with min-entropy $k = \omega(\log \lambda)$. As discussed in Section 1.3, such a restriction is *necessary* for any definition of function privacy to be meaningful in the public-key setting. In the context of IBE, for example, the adversary is allowed to query the real-or-random oracle with $\mathbf{ID}^* \in \mathcal{ID}$ only if $\mathbf{ID}^*$ represents a k-source such that $k = \omega(\log \lambda)$. In the context of IPE, on the other hand, and adversary can query the real-or-random oracle with $\mathbf{V}^* = (V_1^*, \cdots, V_n^*) \in \mathbb{Z}_N^n$ only of $\mathbf{V}^*$ is an (n, k)-block source such that $k = \omega(\log \lambda)$. Additionally, each component-wise distribution V_i^* for $i \in [1, n]$ should be completely uncorrelated with each of the other distributions in $\mathbf{V}^*$. This restriction is necessary to ensure that the adversary cannot carefully craft vectorial distributions with arbitrary inter-component correlations to trivially compromise function privacy (see [11] for a detailed explanation). Finally, note that within the purview of all predicate encryption schemes subsumed by IPE, our definitions are essentially equivalent to the left-or-right oracle based function privacy definitions proposed by Iovino et al. in [12].

Multi-Shot v/s Single-Shot Adversaries. Definition 3.1 considers *multi-shot* adversaries that are allowed to query the $\mathsf{RoR}^{\mathsf{FP}}$ oracle polynomially many times. However, it is polynomially equivalent to consider *single-shot* adversaries that can query the $\mathsf{RoR}^{\mathsf{FP}}$ at most once. This is easily established by a hybrid argument, where the hybrids are constructed such that only one query is forwarded to the $\mathsf{RoR}^{\mathsf{FP}}$ oracle, while the rest are answered by the key generation oracle.

3.1 Computational Function Privacy of Existing Predicate Encryption Schemes

Identity-Based Encryption. To the best of our knowledge, there exist no IBE schemes in literature that can be proven to be computationally function private under well-known cryptographic assumptions. The constructions in [27] and [31] have deterministic trapdoors, and are hence trivially not function private under Definition 3.1. For such schemes, an adversary could easily manufacture a circuit that uniformly samples id such that some function of the public parameters pp and the secret-key $\mathsf{sk}_{\mathsf{id}}$ is already known to the adversary, thus leading to a straightforward attack breaking function privacy [10]. While such straightforward attacks cannot be demonstrated on the IBE constructions proposed in [5, 32–34], we are not aware if their function privacy can be based on standard computationally intractable problems. Finally, the IBE constructions presented in [10] are secure under a different notion of function privacy called statistical function privacy, that is based on the statistical closeness of adversarially-chosen and random distributions. Once again, to the best of our knowledge, these constructions are not computationally function private to the best of our knowledge. In Section 4, we present a family of IBE constructions that are computationally function private under well-known cryptographic assumptions.

Inner-Product Encryption. While computational functional privacy with respect to IPE in the private key setting is well-studied [16, 24, 25], there exist, to the best of our knowledge, no equivalent public-key counterparts in literature that can be proven to be function private under well-known cryptographic assumptions. The authors of [11] present a generic technique to achieve statistical function privacy in the context of inner-product encryption in the public-key setting; however, they leave the construction of computationally function private IPE schemes in the public-key setting as an open problem. It also seems that the function privacy of existing IPE constructions, such as in [2, 3], cannot directly be based on standard computational assumptions without suitable modifications. In Section 5, we present a family of IPE constructions that are computationally function private under well-known cryptographic assumptions.

4 Computationally Function private Identity-Based Encryption

In this section, we apply our *encrypt-augment-recover* approach to the anonymous IBE scheme of Gentry [4] to achieve a family of computationally function private IBE schemes $\{\Pi_k^{\mathrm{IBE}}\}_{k\geq 1}$. The concrete scheme for $k = 1$, has already been introduced in Section 1.3. We present the generalized construction here, along with detailed proofs for data and function privacy.

A Generalized PKE Scheme. In keeping with our *encrypt-augment-recover* approach, at the core of Π_k^{IBE} is the following generalized version of the PKE scheme introduced in Section 1.3, which is CPA-secure under the $(k+1)$-DLIN assumption:

- **KeyGen:** The key-generation algorithm samples $x_1, \cdots, x_{k+2} \xleftarrow{R} \mathbb{Z}_q^*$, where q is a λ-bit prime, and $g_1, \cdots, g_{k+2} \xleftarrow{R} \mathbb{G}$, where $\mathbb{G}$ is a cyclic group of prime order q. It outputs the secret-key SK and the public-key PK as:

$$SK = (x_1, \cdots, x_{k+2}) \ , \ PK = \left(g_1, \cdots, g_{k+2}, \{\left(g_j^{x_j} \cdot g_{k+2}^{x_{k+2}}\right)\}_{j\in[1,k+1]}\right)$$

- **Enc:** The ciphertext C corresponding to a message $M \in \mathbb{G}$ is a tuple of the form:

$$C = \left(g_1^{y_1}, \cdots, g_{k+1}^{y_{k+1}}, g_{k+2}^{\sum_{j=1}^{k+1} y_j}, \left(\prod_{j=1}^{k+1} \left(g_j^{x_j} \cdot g_{k+2}^{x_{k+2}}\right)^{y_j}\right) \cdot M\right)$$

where $y_1, \cdots, y_{k+1} \xleftarrow{R} \mathbb{Z}_q^*$.

- **Dec:** The decryption algorithm, on input the ciphertext $C = (c_0, \cdots, c_{k+2})$ and the secret-key $(x_1, \cdots, x_{k+2})$, recovers the message M as:

$$M = c_{k+2} \Big/ \left(\prod_{j=1}^{k+2} c_{j-1}^{x_j} \right)$$

Gentry's IBE Scheme. We also briefly recall the original IBE scheme of Gentry [4] for clarity of presentation. Let $\mathsf{GroupGen}(1^\lambda)$ be a probabilistic polynomial-time algorithm that takes as input a security parameter λ, and outputs the tuple $(\mathbb{G}, \mathbb{G}_T, q, g, e)$, where $\mathbb{G}$ and $\mathbb{G}_T$ are groups of prime order $q = \mathcal{O}(2^\lambda)$, g is a generator for $\mathbb{G}$ and $e : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}_T$ is an efficiently computable non-degenerate bilinear map. Gentry's IBE scheme $\Pi_{\mathrm{G}}^{\mathrm{IBE}} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ is defined over the identity space $\mathcal{ID} = \mathbb{Z}_q^*$ and the message space $\mathcal{M} = \{\mathcal{M}_\lambda\}_{\lambda \in \mathbb{N}}$ as follows:

- **Setup:** The setup algorithm samples $(\mathbb{G}, \mathbb{G}_T, q, g, e) \xleftarrow{R} \mathsf{GroupGen}(1^\lambda)$ on input the security parameter 1^λ. It also samples $s \xleftarrow{R} \mathbb{Z}_q^*$ and $h \xleftarrow{R} \mathbb{G}$, and outputs the public parameter pp and the master secret-key msk as:

$$\mathsf{pp} = (g, g^s, h) \ , \ \mathsf{msk} = s$$

- **KeyGen:** On input the master secret-key msk and an identity $\mathsf{id} \in \mathcal{ID}$, the key generation algorithm samples $y \xleftarrow{R} \mathbb{Z}_q^*$ and outputs the secret-key $\mathsf{sk}_{\mathsf{id}} = (d_0, d_1)$ where:

$$d_0 = y \ , \ d_1 = \left(h \cdot g^{-y}\right)^{1/(s - \mathsf{id})}$$

- **Enc:** On input the public parameter pp, an identity $\mathsf{id} \in \mathcal{ID}$ and a message $M \in \mathcal{M}$, the encryption algorithm samples $r \xleftarrow{R} \mathbb{Z}_q^*$ and outputs the ciphertext $C = (c_0, c_1, c_2)$ where:

$$c_0 = g^{r \cdot (s - id)} \ , \ c_1 = e\left(g, g\right)^r \ , \ c_2 = M \cdot e\left(g, h\right)^{-r}$$

- **Dec:** On input a ciphertext $C = (c_0, c_1, c_2)$ and a secret-key $\mathsf{sk}_{\mathsf{id}} = (d_0, d_1)$, the decryption algorithm computes:

$$M' = c_1^{d_0} \cdot c_2 \cdot e\left(d_1, c_0\right)$$

If $M' \in \mathcal{M}$, the decryption algorithm outputs M', else it outputs $\perp$.

The above scheme is selectively data private under the weak decisional bilinear Diffie-Hellman exponent (DBDHE) assumption [4].

4.1 Our Function Private IBE Scheme $\Pi_k^{\mathbf{IBE}}$

We now present the construction for our function private IBE scheme Π_k^{IBE}, which is obtained via a combination of the PKE algorithm described above with Gentry's IBE scheme using our *encrypt-augment-recover* approach. For ease of understanding, we highlight the alterations made to Gentry's IBE scheme.

- **Setup:** The setup algorithm samples $(\mathbb{G}, \mathbb{G}_T, q, g, e) \xleftarrow{R} \mathsf{GroupGen}(1^\lambda)$ on input the security parameter 1^λ. It also samples $s, x_1, x_2, \cdots, x_{k+2} \xleftarrow{R} \mathbb{Z}_q^*$ as well as $g_1, g_2, \cdots, g_{k+2}, h \xleftarrow{R} \mathbb{G}$. It outputs the public parameter pp and the master secret-key msk as:

$$\mathsf{pp} = \left(g, g^s, h, \boxed{g^{x_1}, g^{x_2}, \cdots, g^{x_{k+2}}} \ , \ \boxed{g^{s \cdot x_1}, g^{s \cdot x_2}, \cdots, g^{s \cdot x_{k+2}}}\right)$$

$$\mathsf{msk} = \left(s, \boxed{g_1, \cdots, g_{k+2}, \left(g_1^{x_1} \cdot g_{k+2}^{x_{k+2}}\right), \cdots, \left(g_{k+1}^{x_{k+1}} \cdot g_{k+2}^{x_{k+2}}\right)}\right)$$

- **KeyGen:** On input the master secret-key msk and an identity $\mathsf{id} \in \mathcal{ID}$, the key generation algorithm samples $y, y_1, y_2 \cdots, y_{k+1} \xleftarrow{R} \mathbb{Z}_q^*$ and outputs the secret-key $\mathsf{sk}_{\mathsf{id}} = (d_0, d_1, \cdots, d_{k+3})$ where:

$$d_0 = y \ , \ \boxed{d_j = g_j^{y_j} \text{ for } j \in [1, k+1]}$$

$$\boxed{d_{k+2} = g_{k+2}^{\sum_{j=1}^{k+1} y_j}} \ , \ d_{k+3} = \boxed{\left(\prod_{j=1}^{k+1} \left(g_j^{x_j} \cdot g_{k+2}^{x_{k+2}} \right)^{y_j} \right)} \cdot \left(h \cdot g^{-y} \right)^{1/(s-\mathsf{id})}$$

  Observe that $\mathsf{sk}_{\mathsf{id}} = \mathsf{PKE.Enc}\left(PK, (h \cdot g^{-y})^{1/(s-\mathsf{id})} \right)$.

- **Enc:** On input the public parameter pp, an identity $\mathsf{id} \in \mathcal{ID}$ and a message $M \in \mathcal{M}$, the encryption algorithm samples $r \xleftarrow{R} \mathbb{Z}_q^*$ and outputs the ciphertext $C = (c_0, c_1, \cdots, c_{k+4})$ where:

$$c_0 = g^{r \cdot (s-id)} \ , \ \boxed{c_j = g^{r \cdot x_j \cdot (s-id)} \text{ for } j \in [1, k+2]} \ , \ c_{k+3} = e(g,g)^r \ , \ c_{k+4} = M \cdot e(g,h)^{-r}$$

- **Dec:** On input a ciphertext $C = (c_0, \cdots, c_{k+4})$ and a secret-key $\mathsf{sk}_{\mathsf{id}} = (d_0, \cdots, d_{k+3})$, the decryption algorithm computes:

$$\boxed{M = \frac{c_{k+4} \cdot c_{k+3}^{d_0} \cdot e(d_{k+3}, c_0)}{\prod_{j=1}^{k+2} e(d_j, c_j)}}$$

  If $M' \in \mathcal{M}$, the decryption algorithm outputs M', else it outputs $\perp$.

Correctness. First, consider a message $M \in \mathcal{M}$, a ciphertext $C = (c_0, \cdots, c_{k+4})$ corresponding to M under an identity $\mathsf{id} \in \mathcal{ID}$ and a secret-key $\mathsf{sk}_{\mathsf{id}} = (d_0, \cdots, d_{k+3})$ corresponding to id. Then, we have:

$$\begin{aligned} M' &= M \cdot \frac{e(g,h)^{-r} \cdot e(g,g)^{r \cdot y} \cdot e\left((h \cdot g^{-y})^{1/(s-\mathsf{id})}, g^{r \cdot (s-id)} \right) \cdot e\left(\left(\prod_{j=1}^{k+1} \left(g_j^{x_j} \cdot g_{k+2}^{x_{k+2}} \right)^{y_j} \right), g^{r \cdot (s-id)} \right)}{\prod_{j=1}^{k+1} e\left(g_j^{y_j}, (g^{x_j})^{r \cdot (s-id)} \right) \cdot e\left(g^{\sum_{j=1}^{k+1} y_j}, (g^{x_{k+2}})^{r \cdot (s-id)} \right)} \\ &= M \cdot \frac{\prod_{j=1}^{k+1} e\left(g_j^{y_j}, (g^{x_j})^{r \cdot (s-id)} \right) \cdot e\left(g^{\sum_{j=1}^{k+1} y_j}, (g^{x_{k+2}})^{r \cdot (s-id)} \right)}{\prod_{j=1}^{k+1} e\left(g_j^{x_j}, g^{r \cdot (s-id)} \right)^{y_j} \cdot e\left(g_{k+2}^{x_{k+2}}, g^{r \cdot (s-id)} \right)^{\sum_{j=1}^{k+1} y_j}} \\ &= M \end{aligned}$$

Therefore as long as the ciphertext and the secret-key correspond to the same identity, the message is recovered correctly. Again, when the ciphertext and the secret-key correspond to two different identities, say id and id' respectively, the decryption algorithm computes:

$$M' = M \cdot e(g,h)^{-r} \cdot e(g,g)^{r \cdot y} \cdot e\left(\left(h \cdot g^{-y} \right)^{1/\left(s-\mathsf{id}'\right)}, g^{r \cdot (s-id)} \right)$$

We may assume here that $\mathcal{M}$ is a small subset of $\mathbb{G}_T$, namely $|\mathcal{M}| < |\mathbb{G}_T|^{1/2}$. This is not very serious since the space of valid messages in reality is expected to be significantly smaller than $|\mathbb{G}_T|^{1/2}$. This restriction ensures that the probability of M' still lying in $\mathcal{M}$, for $y, r \xleftarrow{R} \mathbb{Z}_q^*$, is negligible in the security parameter λ. This completes the proof of correctness for our generalized IBE scheme Π_k^{IBE}.

4.2 Security of Our IBE Scheme

Selective Data Privacy. We state the following theorem for the selective data privacy of Π_k^{IBE}:

Theorem 4.1 *Our IBE scheme Π_k^{IBE} is selectively data private in the standard model if Gentry's scheme $\Pi_{\mathrm{G}}^{\mathrm{IBE}}$ is selectively data private in the standard model.*

Proof. Let $\mathcal{A}$ be any probabilistic polynomial-time adversary such that:

$$\mathbf{Adv}^{\mathsf{DP}}_{\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda) = \left| \Pr\left[\mathsf{Expt}^{(0)}_{\mathsf{DP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{(1)}_{\mathsf{DP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda) = 1\right] \right| = \epsilon > \mathsf{negl}(\lambda)$$

We construct a polynomial-time algorithm $\mathcal{B}$ such that:

$$\mathbf{Adv}^{\mathsf{DP}}_{\Pi_{\mathrm{G}}^{\mathrm{IBE}},\mathcal{B}}(\lambda) = \left| \Pr\left[\mathsf{Expt}^{(0)}_{\mathsf{DP},\Pi_{\mathrm{G}}^{\mathrm{IBE}},\mathcal{B}}(\lambda) = 1\right] - \Pr\left[\mathsf{Expt}^{(1)}_{\mathsf{DP},\Pi_{\mathrm{G}}^{\mathrm{IBE}},\mathcal{B}}(\lambda) = 1\right] \right| = \epsilon$$

$\mathcal{B}$ interacts with $\mathcal{A}$ in the selective data privacy experiment as follows:

- ***Init:*** $\mathcal{A}$ commits to the challenge identity pair $(\mathsf{id}_0^*, \mathsf{id}_1^*)$. $\mathcal{B}$ also commits to the same identity pair.
- ***Setup:*** $\mathcal{B}$ begins by obtaining the public parameter $\mathsf{pp} = (g, g^s, h)$ for $\Pi_{\mathrm{G}}^{\mathrm{IBE}}$. It then samples $x_1, \cdots, x_{k+2} \xleftarrow{R} \mathbb{Z}_q^*$ and $g_1, g_2, \cdots, g_{k+2} \xleftarrow{R} \mathbb{G}$ and provides $\mathcal{A}$ with the modified public parameter:
$$\mathsf{pp}' = (g, g^s, h, g^{x_1}, g^{x_2}, \cdots, g^{x_{k+2}}, g^{s \cdot x_1}, g^{s \cdot x_2}, \cdots, g^{s \cdot x_{k+2}})$$
- ***Secret-Key Queries:*** When $\mathcal{A}$ issues a secret-key query for $\mathsf{id}_i \in \mathcal{ID}$, $\mathcal{B}$ forwards the query to the key-generation oracle for $\Pi_{\mathrm{G}}^{\mathrm{IBE}}$, and receives $\mathsf{sk}_{\mathsf{id}_i} = (d_0, d_1)$. It then samples $y_1, y_2, \cdots, y_{k+1} \xleftarrow{R} \mathbb{Z}_q^*$ and responds to $\mathcal{A}$ with the secret-key:
$$\mathsf{sk}'_{\mathsf{id}_i} = \left(d_0, g_1^{y_1}, g_2^{y_2}, \cdots, g_{k+1}^{y_{k+1}} g_{k+2}^{\sum_{j=1}^{k+1} y_j}, \left(\prod_{j=1}^{k+1} \left(g_j^{x_j} \cdot g_{k+2}^{x_{k+2}} \right)^{y_j} \right) \cdot d_1 \right)$$
- ***Challenge:*** $\mathcal{A}$ outputs the challenge message pair (M_0^*, M_1^*). $\mathcal{B}$ outputs the same challenge message pair and receives the challenge ciphertext $C^* = (c_0^*, c_1^*, c_2^*)$ for $\Pi_{\mathrm{G}}^{\mathrm{IBE}}$. It then computes the challenge ciphertext for $\mathcal{A}$ as:
$$C'^* = (c_0^*, (c_0^*)^{x_1}, (c_0^*)^{x_2} \cdots, (c_0^*)^{x_{k+2}}, c_1^*, c_2^*)$$
- ***Guess:*** At the end of the game, $\mathcal{A}$ outputs a bit b'. $\mathcal{B}$ outputs the same bit b'.

It is easy to see that $\mathcal{B}$'s simulation is perfect and hence, it has the same advantage ϵ as $\mathcal{A}$. This result, together with the fact that Gentry's IBE scheme is selectively data private under the decisional bilinear Diffie-Hellman exponent (DBDHE) assumption [4], completes the proof of selective data privacy for Π_k^{IBE}.

Extension to Adaptive Data Privacy. Our IBE scheme Π_k^{IBE} can be extended to achieve fully adaptive data privacy using the concept of *hybrid encryption*, introduced by Ananth et al. in [35]. Their technique embeds a hidden execution thread in the decryption keys of the underlying selectively data private scheme, to be activated within the proof of adaptive data privacy for the resulting scheme. This approach also does not require any additional assumptions such as obfuscation.

Computational Function Privacy. We state the following theorem for the computational function privacy of Π_k^{IBE}:

Theorem 4.2 *Our IBE scheme Π_k^{IBE} is computationally function private under the $(k+1)$-DLIN assumption for identities sampled uniformly from k-sources with $k = \omega(\log \lambda)$.*

Proof. We intend to prove the following claim:

Claim 4.1 *For any probabilistic polynomial-time adversary $\mathcal{A}$, the following holds:*

$$\left|\Pr\left[\mathsf{Expt}^{\mathsf{real}}_{\mathsf{FP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda)=1\right]-\Pr\left[\mathsf{Expt}^{\mathsf{rand}}_{\mathsf{FP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda)=1\right]\right|\leq \mathsf{negl}(\lambda)$$

To prove this claim, we assume the contrary. Let $\mathcal{A}$ be a probabilistic polynomial-time adversary such that:

$$\left|\Pr\left[\mathsf{Expt}^{\mathsf{real}}_{\mathsf{FP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda)=1\right]-\Pr\left[\mathsf{Expt}^{\mathsf{rand}}_{\mathsf{FP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda)=1\right]\right|=\epsilon>\mathsf{negl}(\lambda)$$

We assume that the adversary $\mathcal{A}$ issues a single query to the real-or-random oracle. As discussed in Section 3, such a single-shot adversary is polynomially equivalent to its multi-shot variant considered in Definition 3.1. We construct an algorithm $\mathcal{B}$ that solves an instance of the $(k+1)$-DLIN problem with non-negligible advantage $\epsilon'=\epsilon$. $\mathcal{B}$ is given $\left(g_1,\cdots,g_{k+2},g_1^{a_1},\cdots,g_{k+2}^{a_{k+2}}\right)$ and interacts with $\mathcal{A}$ as follows:

- ***Setup:*** $\mathcal{B}$ samples $s,x_1,\cdots,x_{k+2}\xleftarrow{R}\mathbb{Z}_q^*$ and $g_1,\cdots,g_{k+2},h\xleftarrow{R}\mathbb{G}$. It outputs the public parameter pp and the master secret-key msk as:

$$\mathsf{pp}=\left(g,g^s,h,g^{x_1},\cdots,g^{x_{k+2}}\right)$$
$$\mathsf{msk}=\left(s,g_1,\cdots,g_{k+2},\left(g_1^{x_1}\cdot g_{k+2}^{x_{k+2}}\right),\cdots,\left(g_{k+1}^{x_{k+1}}\cdot g_{k+2}^{x_{k+2}}\right)\right)$$

- ***Secret-Key Queries:*** Since $\mathcal{B}$ possesses the knowledge of the master secret-key msk, it can answer any secret-key query issued by $\mathcal{A}$ by invoking the KeyGen procedure.

- ***Real-or-Random Query:*** Suppose $\mathcal{A}$ queries the real-or-random oracle with $\mathbf{ID}^*$ - a circuit representing a k-source over the identity space $\mathcal{ID}$ such that $k=\omega\left(\log\lambda\right)$. $\mathcal{B}$ uniformly samples an identity $\mathsf{id}^*\xleftarrow{R}\mathbf{ID}^*$ and responds with the secret-key $\mathsf{sk}_{\mathsf{id}^*}$ as:

$$\mathsf{sk}_{\mathsf{id}^*}=\left(y,g_1^{a_1},\cdots,g_{k+2}^{a_{k+2}},\left(\prod_{j=1}^{k+2}\left(g_j^{a_j}\right)^{x_j}\right)\cdot\left(h\cdot g^{-y}\right)^{1/(s-\mathsf{id})}\right)$$

  where $g_1^{a_1},\cdots,g_{k+2}^{a_{k+2}}$ are part of its input instance and $y\xleftarrow{R}\mathbb{Z}_q^*$.

- ***Guess:*** At the end of the game, $\mathcal{A}$ outputs a bit b'. $\mathcal{B}$ outputs the same bit b'.

It is easy to see that when $a_{k+2}=\sum_{j=1}^{k+1}a_j$, the secret-key $\mathsf{sk}_{\mathsf{id}^*}$ is well-formed and identically distributed to the response of the real-or-random oracle in the experiment $\mathsf{Expt}^{\mathsf{real}}_{\mathsf{FP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda)$. On the other hand, when a_{k+2} is uniformly random in $\mathbb{Z}_q^*$, the secret-key $\mathsf{sk}_{\mathsf{id}^*}$ is uniformly random, and hence identically distributed to the response of the real-or-random oracle in the experiment $\mathsf{Expt}^{\mathsf{rand}}_{\mathsf{FP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda)$. Now, the advantage ϵ' of $\mathcal{B}$ in solving the $(k+1)$-DLIN instance (where the probability is taken over all possible choices of $a_1,\cdots,a_{k+2}\xleftarrow{R}\mathbb{Z}_q^*$ and all possible choices of $g_1,\cdots,g_{k+2}\xleftarrow{R}\mathbb{G}$) may be quantified as:

$$\begin{aligned}\epsilon'&=\left|\Pr\left[\mathcal{B}\left(g_1,\cdots,g_{k+2},g_1^{a_1},\cdots,g_{k+2}^{\sum_{j=1}^{k+1}a_j}\right)=1\right]-\Pr\left[\mathcal{B}\left(g_1,\cdots,g_{k+2},g_1^{a_1},\cdots,g_{k+2}^{a_{k+2}}\right)=1\right]\right|\\&=\left|\Pr\left[\mathsf{Expt}^{\mathsf{real}}_{\mathsf{FP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda)=1\right]-\Pr\left[\mathsf{Expt}^{\mathsf{rand}}_{\mathsf{FP},\Pi_k^{\mathrm{IBE}},\mathcal{A}}(\lambda)=1\right]\right|\\&=\epsilon\end{aligned}$$

This completes the proof of function privacy for Π_k^{IBE}.

5 Computationally Function private Inner-Product Encryption

In this section, we present a family of selectively data private zero-IPE schemes $\{\Pi_k^{\mathrm{IPE}}\}_{k\geq 1}$ that are also computationally function private under the generalized family of k-DLIN assumptions in the standard model. Our schemes are defined over the set of attributes $\Sigma=\mathbb{Z}_N^n$ (N being a product of three primes q_1, q_2 and q_3), and the class of vectorial predicates $\mathcal{F}=\{f_{\overrightarrow{v}}\mid\overrightarrow{v}\in\mathbb{Z}_N^n\}$, such that for $I=(I_1,\cdots,I_n)\in\mathbb{Z}_N^n$, we have $f_{\overrightarrow{v}}(I)=1$ if and only if $\langle\overrightarrow{v},I\rangle=0 \bmod N$. Once again, our constructions are obtained by applying our *encrypt-augment-recover* approach to the zero-IPE scheme of Katz, Sahai and Waters [3].

Construction Overview. Let $\mathbb{G}$ be a bilinear group of order $N = q_1 q_2 q_3$ (each of q_1, q_2 and q_3 being λ-bit primes), and let $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ denote the subgroups of $\mathbb{G}$ of order q_1, q_2 and q_3, respectively. Also, let $\hat{e} : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}_T$ be an efficiently computable non-degenerate bilinear map, where $\mathbb{G}_T$ is also a group of order N. Note that if g is the generator for $\mathbb{G}$, then the element $g_1 = g^{q_2 \cdot q_3}$ is a generator for $\mathbb{G}_1$, the element $g_2 = g^{q_1 \cdot q_3}$ is a generator for $\mathbb{G}_2$, and the element $g_3 = g^{q_1 \cdot q_2}$ is a generator for $\mathbb{G}_3$. Furthermore, for any elements $h_1 \in \mathbb{G}_1$, $h_2 \in \mathbb{G}_2$ and $h_3 \in \mathbb{G}_3$, we have $\hat{e}(h_1, h_2) = \hat{e}(h_2, h_3) = \hat{e}(h_1, h_3) = 1$. Also, let $\mathsf{GroupGen}'(1^\lambda)$ be a probabilistic polynomial-time algorithm that takes as input a security parameter λ, and outputs the tuple $(\mathbb{G}, \mathbb{G}_T, q_1, q_2, q_3, g_1, g_2, g_3, \hat{e})$. Finally, the payload message space $\mathcal{M}$ is assumed to be a small subset of $\mathbb{G}_T$, namely $|\mathcal{M}| < |\mathbb{G}_T|^{1/2}$. Our function private zero-IPE scheme uses the three subgroups for three distinct roles:

- The subgroup $\mathbb{G}_2$ is used to encode the vectors $\overrightarrow{v}$ and I in the secret-key and the ciphertexts, respectively, and to compute the inner product $\langle \overrightarrow{v}, I \rangle$ in the exponent of a bilinear map computation.

- The subgroup $\mathbb{G}_1$ serves a dual purpose in our scheme. On the one hand, it has the effect of *masking* the inner product computation in $\mathbb{G}_2$, and preventing the adversary from improperly manipulating the computation in any way to reveal information about the underlying attributes. In particular, it is pivotal in ensuring the non-malleability of the secret-keys and ciphertexts generated by the scheme. On the other hand, it is in the $\mathbb{G}_1$ subgroup that we incorporate our *encrypt-augment-recover* methodology to achieve computational function privacy.

- The subgroup $\mathbb{G}_3$ serves as an additional layer of masking for the other subgroups. In particular, random elements sampled from $\mathbb{G}_3$ are multiplied with various components in both the secret-keys as well as the ciphertexts to hide possible information leakages from the subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$.

Encrypt-Augment-Recover. We apply our *encrypt-augment-recover* approach to the zero-IPE scheme of Katz, Waters and Sahai to achieve computational function privacy. At the core of our approach is the public-key encryption algorithm $\mathsf{PKE} = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ that is CPA-secure under the $(k+1)$-DLIN assumption (the reader is referred to Section 4 for recalling the PKE scheme). The PKE essentially operates in the subgroup $\mathbb{G}_1$ of prime order q_1, and its outputs are suitably masked before being incorporated in our scheme. We modify the algorithms of the original scheme as follows:

- The modified setup algorithm runs $(SK, PK) \xleftarrow{R} \mathsf{PKE.KeyGen}$. It incorporates PK in the master secret-key of the original scheme, and modifies the public parameter to include a one way function of SK.

- The original zero-IPE scheme of Katz, Sahai and Waters comprises of secret-keys of the form $\mathsf{sk}_{\overrightarrow{v}} = \big(d_0, \{d_{1,i}, d_{2,i}\}_{i \in [1,n]}\big)$. The modified key-generation algorithm in our scheme generates a secret-key of the form $\mathsf{sk}'_{\overrightarrow{v}} = \Big(d_0, \{d^j_{1,i}, d^j_{2,i}\}_{i \in [1,n], j \in [0,k+2]}\Big)$ such that for $i \in [1, n]$, we have:

$$\Big(\{d^j_{1,i}\}_{j \in [0,k+2]}\Big) = \mathsf{PKE.Enc}\,(PK, d_{1,i})$$
$$\Big(\{d^j_{2,i}\}_{j \in [0,k+2]}\Big) = \mathsf{PKE.Enc}\,(PK, d_{2,i})$$

  along with suitable masking as necessary. Observe that this naturally ensures that each component of the modified secret-key is independent and identically distributed. In the proof of function privacy, we argue the indistinguishability of a well-formed secret-key component from a uniformly random one by relating it to the hardness of solving a $(k+1)$-DLIN instance in $\mathbb{G}_1$.

- The modified encryption algorithm generates an *augmented* ciphertext that retains the ciphertext of the original scheme unaltered as one of its components. The additional ciphertext components are used by the modified decryption algorithm subsequently to remove the effect of PKE and recover the

payload message M. The additional components are also in the group $\mathbb{G}_1$, and are suitably masked using uniformly random elements from $\mathbb{G}_3$. The masking ensures that the data privacy guarantees of the original scheme are not weakened.

5.1 Construction Details for Our Zero-IPE Scheme Π_k^{IPE}

We now present the construction for Π_k^{IPE} in details.

- **Setup:** The setup algorithm samples $(\mathbb{G}, \mathbb{G}_T, q_1, q_2, q_3, g_1, g_2, g_3, \hat{e}) \xleftarrow{R} \mathsf{GroupGen}'(1^\lambda)$. It also samples $\{x_{1,j}, x_{2,j} \xleftarrow{R} \mathbb{Z}_{q_1}^*\}_{j\in[1,k+2]}$, $\{g_{1,j}, g_{2,j} \xleftarrow{R} \mathbb{G}_1\}_{j\in[1,k+2]}$, $\{h_{1,i}, h_{2,i} \xleftarrow{R} \mathbb{G}_1\}_{i\in[1,n]}$ and $\{R_{1,i}^j, R_{2,i}^j \xleftarrow{R} \mathbb{G}_3\}_{i\in[1,n],j\in[0,k+2]}$. It additionally samples $h \xleftarrow{R} \mathbb{G}_1$, $\gamma \xleftarrow{R} \mathbb{Z}_{q_1}^*$ and $R_3 \xleftarrow{R} \mathbb{G}_3$, and sets:

$$
\begin{aligned}
Q &= g_2 \cdot R_3 \\
S_{1,i}^0 &= h_{1,i} \cdot R_{1,i}^0 \ , \ S_{2,i}^0 = h_{2,i} \cdot R_{2,i}^0 \text{ for } i \in [1,n] \\
S_{1,i}^j &= h_{1,i}^{x_{1,j}} \cdot R_{1,i}^j \ , \ S_{2,i}^j = h_{2,i}^{x_{2,j}} \cdot R_{2,i}^j \text{ for } i \in [1,n], j \in [1,k+2]
\end{aligned}
$$

  It outputs the public parameter pp and the master secret-key msk as:

$$
\begin{aligned}
\mathsf{pp} &= \left(g_1, g_3, Q, \{S_{1,i}^j, S_{2,i}^j\}_{i\in[1,n],j\in[0,k+2]}, \hat{e}\left(g_1, h\right)^\gamma\right) \\
\mathsf{msk} &= \left(q_1, q_2, q_3, g_2, \{g_{1,j}, g_{2,j}\}_{j\in[1,k+2]}, \{g_{1,j}^{x_{1,j}} \cdot g_{1,k+2}^{x_{1,k+2}}, g_{2,j}^{x_{2,j}} \cdot g_{2,k+2}^{x_{2,k+2}}\}_{j\in[1,k+1]}, \{h_{1,i}, h_{2,i}\}_{i\in[1,n]}, h^\gamma\right)
\end{aligned}
$$

- **KeyGen:** On input the master secret-key msk and a vector $\overrightarrow{v} = (v_1, \cdots, v_n)$, the key generation algorithm samples $\{z_{1,i}, z_{2,i} \xleftarrow{R} \mathbb{Z}_{q_1}^*\}_{i\in[1,n]}$, $\{y_{1,i}^j, y_{2,i}^j \xleftarrow{R} \mathbb{Z}_{q_1}^*\}_{i\in[1,n],j\in[1,k+1]}$, $Q_4 \xleftarrow{R} \mathbb{G}_2$, $R_5 \xleftarrow{R} \mathbb{G}_3$ and $f_1, f_2 \xleftarrow{R} \mathbb{Z}_{q_2}^*$. It then sets $d_0 = Q_4 \cdot R_5 \Big/ \left(h^\gamma \cdot \prod_{i=1}^n h_{1,i}^{z_{1,i}} \cdot h_{2,i}^{z_{2,i}}\right)$. It also sets:

$$
\begin{aligned}
d_{1,i}^0 &= g_1^{z_{1,i}} \cdot g_2^{f_1 \cdot v_i} \Big/ \left(\prod_{j=1}^{k+1} \left(g_{1,j}^{x_{1,j}} \cdot g_{1,k+2}^{x_{1,k+2}}\right)^{y_{1,i}^j}\right) \text{ for } i \in [1,n] \\
d_{2,i}^0 &= g_1^{z_{2,i}} \cdot g_2^{f_2 \cdot v_i} \Big/ \left(\prod_{j=1}^{k+1} \left(g_{2,j}^{x_{2,j}} \cdot g_{2,k+2}^{x_{2,k+2}}\right)^{y_{2,i}^j}\right) \text{ for } i \in [1,n]
\end{aligned}
$$

  Finally, it sets the following additional components:

$$
\begin{aligned}
d_{1,i}^j = g_{1,j}^{y_{1,i}^j} \ , \ d_{2,i}^j = g_{2,j}^{y_{2,i}^j} \text{ for } i \in [1,n], j \in [1,k+1] \\
d_{1,i}^{k+2} = g_{1,k+2}^{\sum_{j=1}^{k+1} y_{1,i}^j} \ , \ d_{2,i}^{k+2} = g_{2,k+2}^{\sum_{j=1}^{k+1} y_{2,i}^j} \text{ for } i \in [1,n]
\end{aligned}
$$

  and outputs the secret-key $\mathsf{sk}_{\overrightarrow{v}}$ as:

$$
\mathsf{sk}_{\overrightarrow{v}} = \left(d_0, \{d_{1,i}^j, d_{2,i}^j\}_{i\in[1,n],j\in[0,k+2]}\right)
$$

- **Enc:** On input the public parameter pp, an attribute $I = (I_1, \cdots, I_n) \in \mathbb{Z}_N^n$ and a payload message $M \in \mathcal{M}$, the encryption algorithm samples $r, \alpha, \beta \xleftarrow{R} \mathbb{Z}_N^*$ and $\{R_{6,i}^j, R_{7,i}^j \xleftarrow{R} \mathbb{G}_3\}_{i\in[1,n],j\in[0,k+2]}$. It then sets $c_0 = g_1^r$. It also sets :

$$
\begin{aligned}
c_{1,i}^0 = \left(S_{1,i}^0\right)^r \cdot Q^{\alpha \cdot I_i} \cdot R_{6,i}^0 \ , \ c_{2,i}^0 = \left(S_{2,i}^0\right)^r \cdot Q^{\beta \cdot I_i} \cdot R_{7,i}^0 \text{ for } i \in [1,n] \\
c_{1,i}^j = \left(S_{1,i}^j\right)^r \cdot R_{6,i}^j \ , \ c_{2,i}^j = \left(S_{2,i}^j\right)^r \cdot R_{7,i}^j \text{ for } i \in [1,n], j \in [1,k+2]
\end{aligned}
$$

  Finally, it sets $c_3 = M \cdot \left(\hat{e}\left(g_1, h\right)^\gamma\right)^r$ and outputs the ciphertext C as:

$$
C = \left(c_0, \{c_{1,i}^j, c_{2,i}^j\}_{i\in[1,n],j\in[0,k+2]}, c_3\right)
$$

- **Dec:** On input a ciphertext $C = \left(c_0, \{c_{1,i}^j, c_{2,i}^j\}_{i\in[1,n],j\in[0,k+2]}\right)$ and a secret-key $\mathsf{sk}_{\overrightarrow{v}} = \left(d_0, \{d_{1,i}^j, d_{2,i}^j\}_{i\in[1,n],j\in[0,k+2]}\right)$, the decryption algorithm computes:

$$M' = c_3 \cdot \hat{e}\left(d_0, c_0\right) \cdot \left(\prod_{i=1}^{n}\prod_{j=0}^{k+2} \hat{e}\left(d_{1,i}^j, c_{1,i}^j\right) \cdot \hat{e}\left(d_{2,i}^j, c_{2,i}^j\right)\right)$$

If $M' \in \mathcal{M}$, the decryption algorithm outputs M', else it outputs $\perp$.

Correctness. To see that correctness holds for our zero-IPE scheme, let C and $\mathsf{sk}_{\overrightarrow{v}}$ be as described in Section 5. Then we have:

$$\begin{aligned}
M' &= c_3 \cdot \hat{e}\left(d_0, c_0\right) \cdot \left(\prod_{i=1}^{n}\prod_{j=0}^{k+2} \hat{e}\left(d_{1,i}^j, c_{1,i}^j\right) \cdot \hat{e}\left(d_{2,i}^j, c_{2,i}^j\right)\right) \\
&= M \cdot \left(\hat{e}\left(g_1, h\right)^{\gamma}\right)^r \cdot \left(\frac{\prod_{i=1}^{n} \hat{e}\left(g_1^{z_{1,i}}, h_{1,i}^r\right) \cdot \hat{e}\left(g_1^{z_{2,i}}, h_{2,i}^r\right)}{\hat{e}\left(h^{\gamma}, g_1^r\right) \cdot \hat{e}\left(\prod_{i=1}^{n} h_{1,i}^{z_{1,i}} \cdot h_{2,i}^{z_{2,i}}, g_1^r\right)}\right) \cdot \left(\prod_{i=1}^{n} \hat{e}\left(g_2^{f_1 \cdot v_i}, g_2^{\alpha \cdot I_i}\right) \cdot \hat{e}\left(g_2^{f_2 \cdot v_i}, g_2^{\beta \cdot I_i}\right)\right) \\
&\cdot \prod_{i=1}^{n} \left(\frac{\prod_{j=1}^{k+1} \left(\hat{e}\left(g_{1,j}^{y_{1,i}^j}, \left(h_{1,i}^{x_{1,j}}\right)^r\right) \cdot \hat{e}\left(g_{2,j}^{y_{2,i}^j}, \left(h_{2,i}^{x_{2,j}}\right)^r\right)\right) \cdot \hat{e}\left(g_{1,k+2}^{\sum_{j=1}^{k+1} y_{1,i}^j}, \left(h_{1,i}^{x_{1,k+2}}\right)^r\right) \cdot \hat{e}\left(g_{2,k+2}^{\sum_{j=1}^{k+1} y_{2,i}^j}, \left(h_{2,i}^{x_{2,k+2}}\right)^r\right)}{\hat{e}\left(\prod_{j=1}^{k+1} \left(g_{1,j}^{x_{1,j}} \cdot g_{1,k+2}^{x_{1,k+2}}\right)^{y_{1,i}^j}, h_{1,i}^r\right) \cdot \hat{e}\left(\prod_{j=1}^{k+1} \left(g_{2,j}^{x_{2,j}} \cdot g_{2,k+2}^{x_{2,k+2}}\right)^{y_{2,i}^j}, h_{2,i}^r\right)}\right) \\
&= M \cdot \prod_{i=1}^{n} \hat{e}\left(g_2, g_2\right)^{(\alpha f_1 + \beta f_2) \cdot v_i \cdot I_i} \\
&= M \cdot \hat{e}\left(g_2, g_2\right)^{(\alpha f_1 + \beta f_2 \bmod q_2) \cdot \langle \overrightarrow{v}, I \rangle}
\end{aligned}$$

where α, β are uniformly random in $\mathbb{Z}_N^*$ and f_1, f_2 are uniformly random in $\mathbb{Z}_{q_2}^*$. If $\langle \overrightarrow{v}, I \rangle = 0 \bmod N$, then we have $M' = M$. If $\langle \overrightarrow{v}, I \rangle \neq 0 \bmod N$, there are two cases: if $\langle \overrightarrow{v}, I \rangle \neq 0 \bmod q_2$, then with all but negligible probability (over random choice of α, β, f_1, f_2), M' does not lie in $\mathbb{G}_T$ (since $\mathcal{M}$ is a small subset of $\mathbb{G}_T$). Otherwise, we have $\langle \overrightarrow{v}, I \rangle = 0 \bmod q_2$, in which case M' will always be equal to M; however, this would reveal a non-trivial factor of N, and so this too occurs with negligible probability. In fact, the data privacy property of this zero-IPE construction relies on a set of assumptions in bilinear groups that imply the hardness of finding a non-trivial factor of N.

The Need for Two Sub-Systems. Note that our zero-IPE scheme uses two parallel sub-systems (in the key generation and encryption algorithms) that are apparently redundant since they perform the same functions. Indeed, our scheme inherits this feature from the original zero-IPE scheme of Katz, Sahai and Waters [3]. Eliminating one of the sub-systems from our scheme would retain functional correctness as well as computational function privacy, while also improving performance and efficiency. However, the proof methodology in [3] for data privacy relies on the existence of the parallel sub-systems in an essential way. Since our aim is to retain the same data privacy guarantees as in the original scheme, we stick to the use of two parallel sub-systems in our augmented zero-IPE scheme.

5.2 Security of Our IPE Scheme

Data Privacy. We state the following theorem for the data privacy of Π_k^{IPE}:

Theorem 5.1 *Our zero-IPE scheme Π_k^{IPE} retains the selective data privacy guarantees of the original zero-IPE scheme of Katz, Sahai and Waters* [3].

Proof Overview. We provide a brief overview of the proof technique for our scheme, which essentially follows the proof technique presented in [3]. We consider a probabilistic polynomial-time adversary that tries to determine whether the *challenge ciphertext* is associated with either of the two attributes I_0 or I_1. The proof proceeds via a sequence of hybrid games in which an entire attribute used in the challenge ciphertext is changed in one step, instead of changing them component by component for reasons mentioned in the proof of the original scheme in [3]. This is facilitated by the presence of the two parallel sub-systems, which allows the hybrid games to use *ill-formed* ciphertexts that are encrypted with respect to two *different* attributes I and I' and in the two sub-systems. Let such a ciphertext be denoted informally as (I, I'). The proof establishes indistinguishability between the well-formed ciphertexts (I_0, I_0) and (I_1, I_1) via a sequence of intermediate hybrid games using the ill-formed ciphertexts $\left(I_0, \overrightarrow{0}\right)$, (I_0, I_1) and $\left(\overrightarrow{0}, I_1\right)$. The zero vector is used since it orthogonal to any other vector. The simulator in our proof works in one sub-system independent of what happens in the other one. In each hybrid game, the simulator embeds a subgroup-decision like assumption in the challenge ciphertext, and the structure of the challenge determines whether a sub-system embeds a given vector or a zero vector. This is essentially an adoption of the proof technique originally presented in [3] to our function private scheme.

An additional requirement in our proof is that the simulator should be able to embed the $(k+1)$-DLIN instances when responding to the key generation queries from the adversary. As demonstrated in the proof of Theorem 4.1, this is straightforward to achieve: since the simulator in the data privacy game is allowed to set up the $(k+1)$-DLIN instances entirely on its own, it can easily augment the secret-key generation process in the proof of the original scheme by appropriately embedding these instances where necessary. Moreover, since the $(k+1)$-DLIN instances are sampled uniformly at random, the resulting distribution of secret-keys is exactly as in the real world from the point of view of the adversary. Similarly, in the challenge phase, the simulator generates the additional components in the augmented ciphertext uniformly at random, without altering the nature of the ciphertext distribution from the adversary's point of view.

Extension to Adaptive Data Privacy. Once again, our zero-IPE scheme Π_k^{IPE} can also be extended to achieve fully adaptive data privacy using the concept of *hybrid encryption*, introduced by Ananth et al. in [35].

Computational Function Privacy. We state the following theorem for the computational function privacy of Π_k^{IPE}:

Theorem 5.2 *Our zero-IPE scheme Π_k^{IPE} is computationally function private under the $(k+1)$-DLIN assumption for predicate vectors sampled uniformly from (n,k)-block sources with $k = \omega(\log \lambda)$.*

Proof. We present a proof for the above theorem. Our aim is to show that any probabilistic poly-time adversary $\mathcal{A}$ cannot distinguish between the real and random modes of operation of the function privacy oracle, provided that the oracle is queried with circuits that sample sufficiently unpredictable distributions over the space of predicates. In particular, such distributions should be (n,k)-block sources over $\mathbb{Z}_N^n$, such that each component of a vector $\overrightarrow{v}$ sampled from an adversarially chosen distribution has a min-entropy of $k = \omega(\log \lambda)$, and is uncorrelated with all other components. Additionally, the proof shows that the simulator $\mathcal{B}$ can additionally simulate the function privacy encryption oracle, and that the real and random modes of operation of the function privacy oracle are indistinguishable even in the presence of the encryption oracle.

We define a series of hybrid experiments $\mathsf{Expt}^{\mathsf{mode},m}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)$ for $\mathsf{mode} \in \{\mathsf{real}, \mathsf{rand}\}$ and $m \in [0,n]$ as follows:

- $\mathsf{Expt}^{\mathsf{mode},0}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)$ is exactly identical to $\mathsf{Expt}^{\mathsf{mode}}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)$.

- $\mathsf{Expt}^{\mathsf{mode},m}_{\mathsf{FP},\Pi^{\mathrm{IPE}}_k,\mathcal{A}}(\lambda)$ for $m \in [1,n]$ is identical to $\mathsf{Expt}^{\mathsf{mode}}_{\mathsf{FP},\Pi^{\mathrm{IPE}}_k,\mathcal{A}}(\lambda)$ except that the secret-key $\mathsf{sk}_{\overrightarrow{v^*}} = \left(d^*_0, \{{d^*}^j_{1,i}, {d^*}^j_{2,i}\}_{i\in[1,n],j\in[0,k+2]}\right)$ generated by the real-or-random oracle is such that the set of components $\{{d^*}^j_{1,i}, {d^*}^j_{2,i}\}_{i\in[1,m],j\in[0,k+2]}$ are uniformly random and independent of the underlying vector $\overrightarrow{v^*}$. In addition, the ciphertext C^* for a message M generated by the function privacy encryption oracle is uniformly random and independent of $\overrightarrow{v^*}$; it, however, produces M upon decryption using the $\mathsf{sk}_{\overrightarrow{v^*}}$ generated by the real-or-random oracle.

Quite evidently, the following holds:

$$\left|\Pr\left[\mathsf{Expt}^{\mathsf{real},n}_{\mathsf{FP},\Pi^{\mathrm{IPE}}_k,\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{rand},n}_{\mathsf{FP},\Pi^{\mathrm{IPE}}_k,\mathcal{A}}(\lambda)=1\right]\right| = 0$$

We now state and prove the following claim:

Claim 5.1 *For any probabilistic polynomial-time adversary $\mathcal{A}$, for $\mathsf{mode} \in \{\mathsf{real},\mathsf{rand}\}$ and for $m \in [0,n-1]$, the following holds:*

$$\left|\Pr\left[\mathsf{Expt}^{\mathsf{mode},m}_{\mathsf{FP},\Pi^{\mathrm{IPE}}_k,\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{mode},m+1}_{\mathsf{FP},\Pi^{\mathrm{IPE}}_k,\mathcal{A}}(\lambda)=1\right]\right| \leq \mathsf{negl}(\lambda)$$

To prove this claim, we assume the contrary. Let $\mathcal{A}$ be a probabilistic polynomial-time adversary such that:

$$\left|\Pr\left[\mathsf{Expt}^{\mathsf{mode},m}_{\mathsf{FP},\Pi^{\mathrm{IPE}}_k,\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{mode},m+1}_{\mathsf{FP},\Pi^{\mathrm{IPE}}_k,\mathcal{A}}(\lambda)=1\right]\right| = \epsilon > \mathsf{negl}(\lambda)$$

for some $m \in [0,n-1]$. Also, let $\mathbb{G}$ be a bilinear group of order $N = q_1q_2q_3$ (each of q_1, q_2 and q_3 being λ-bit primes), and let $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ denote the subgroups of $\mathbb{G}$ of order q_1, q_2 and q_3, respectively. Also, let $\hat{e}: \mathbb{G}\times\mathbb{G} \longrightarrow \mathbb{G}_T$ be an efficiently computable non-degenerate bilinear map, where $\mathbb{G}_T$ is also a group of order N. We construct an algorithm $\mathcal{B}$ such that:

$$\left|\Pr\left[\mathcal{B}\left(\left(g_{1,1},\cdots,g_{1,k+2},g_{1,1}^{a_1},\cdots,g_{1,k+2}^{\sum_{j=1}^{k+1}a_j}\right),\left(g_{2,1},\cdots,g_{2,k+2},g_{2,1}^{a'_1},\cdots,g_{2,k+2}^{\sum_{j=1}^{k+1}a'_j}\right)\right)=1\right] - \right.$$
$$\left.\Pr\left[\mathcal{B}\left(\left(g_{1,1},\cdots,g_{1,k+2},g_{1,1}^{a_1},\cdots,g_{1,k+2}^{a_{k+2}}\right),\left(g_{2,1},\cdots,g_{2,k+2},g_{2,1}^{a'_1},\cdots,g_{2,k+2}^{a'_{k+2}}\right)\right)=1\right]\right| = \epsilon$$

where the probability is over random choice of $\{a_j, a'_j \xleftarrow{R} \mathbb{Z}^*_{q_1}\}_{j\in[1,k+2]}$, and over random choice of $\{g_{1,j}, g_{2,j} \xleftarrow{R} \mathbb{G}_1\}_{j\in[1,k+2]}$. Observe that $\mathcal{B}$ can in turn be trivially used to construct another algorithm that has advantage at least ϵ in solving a given instance of the $(k+1)$-DLIN problem in the group $\mathbb{G}_1$.

- ***Setup:*** $\mathcal{B}$ uniformly samples $\{x_{1,j}, x_{2,j} \xleftarrow{R} \mathbb{Z}^*_{q_1}\}_{j\in[1,k+2]}$, $\{h_{1,i}, h_{2,i} \xleftarrow{R} \mathbb{G}_1\}_{i\in[1,n]}$ and $\{R^j_{1,i}, R^j_{2,i} \xleftarrow{R} \mathbb{G}_3\}_{i\in[1,n],j\in[0,k+2]}$. It additionally samples $h \xleftarrow{R} \mathbb{G}_1$, $\gamma \xleftarrow{R} \mathbb{Z}^*_{q_1}$ and $R_3 \xleftarrow{R} \mathbb{G}_3$, and sets:

$$Q = g_2 \cdot R_3$$
$$S^0_{1,i} = h_{1,i}\cdot R^0_{1,i}\ ,\ S^0_{2,i} = h_{2,i}\cdot R^0_{2,i} \text{ for } i\in[1,n]$$
$$S^j_{1,i} = h^{x_{1,j}}_{1,i}\cdot R^j_{1,i}\ ,\ S^j_{2,i} = h^{x_{2,j}}_{2,i}\cdot R^j_{2,i} \text{ for } i\in[1,n], j\in[1,k+2]$$

  Finally, it sets the public parameter pp and the master secret-key msk as:

$$\mathsf{pp} = \left(g_1, g_3, Q, \{S^j_{1,i}, S^j_{2,i}\}_{i\in[1,n],j\in[0,k+2]}, \hat{e}\,(g_1,h)^\gamma\right)$$
$$\mathsf{msk} = \left(q_1,q_2,q_3,g_2,\{g_{1,j},g_{2,j}\}_{j\in[1,k+2]},\{g_{1,j}^{x_{1,j}}\cdot g_{1,k+2}^{x_{1,k+2}}, g_{2,j}^{x_{2,j}}\cdot g_{2,k+2}^{x_{2,k+2}}\}_{j\in[1,k+1]},\{h_{1,i},h_{2,i}\}_{i\in[1,n]},h^\gamma\right)$$

  $\mathcal{B}$ provides pp to $\mathcal{A}$. Observe that pp is distributed exactly as in the real world.

- ***Secret-Key Queries:*** When $\mathcal{A}$ issues a secret-key query for $\overrightarrow{v} \in \mathbb{Z}_N^n$, $\mathcal{B}$ responds with $\mathsf{sk}_{\overrightarrow{v}} = \mathsf{KeyGen}(\mathsf{msk}, \overrightarrow{v})$.

- ***Real-or-Random Query:*** The crux of the proof lies in how $\mathcal{B}$ embeds its input $(k+1)$-DLIN instance in its response to the real-or-random query issued by $\mathcal{A}$. Suppose $\mathcal{A}$ queries the real-or-random oracle with an (n,k)-block source $\mathbf{V}^* = (V_1^*, \cdots, V_n^*)$ over $\mathbb{Z}_N^n$ such that $k = \omega(\log \lambda)$. $\mathcal{B}$ samples $\mathsf{mode} \xleftarrow{R} \{\mathsf{real}, \mathsf{rand}\}$. For each $i \in [1,n]$, $\mathcal{B}$ samples $v_i^* \xleftarrow{R} V_i^*$ if $\mathsf{mode} = \mathsf{real}$, or $v_i^* \xleftarrow{R} \mathbb{Z}_N$ if $\mathsf{mode} = \mathsf{rand}$. The vector $\overrightarrow{v^*} = (v_1^*, \cdots, v_n^*)$ is the challenge vector that $\mathcal{B}$ uses to respond to the query from $\mathcal{A}$. $\mathcal{B}$ now sets the various components of the secret-key $\mathsf{sk}_{\overrightarrow{v^*}}$ as follows:

  1. The secret-key elements corresponding to the first m components of $\overrightarrow{v^*}$ are crafted by $\mathcal{B}$ to be uniformly random, while the elements corresponding to the last $n-m-1$ components of $\overrightarrow{v^*}$ are crafted to be well-formed. We present the details of how this may be achieved. $\mathcal{B}$ samples $\{z_{1,i}, z_{2,i} \xleftarrow{R} \mathbb{Z}_{q_1}^*\}_{i\in[1,m]}$, $\{y_{1,i}^j, y_{2,i}^j \xleftarrow{R} \mathbb{Z}_{q_1}^*\}_{i\in[1,m], j\in[1,k+1]}$, $Q_4 \xleftarrow{R} \mathbb{G}_2$, $R_5 \xleftarrow{R} \mathbb{G}_3$ and $f_1, f_2 \xleftarrow{R} \mathbb{Z}_{q_2}^*$. It then sets the following:

$$d_0^* = Q_4 \cdot R_5 \Big/ \left(\prod_{i=1}^{n} h_{1,i}^{z_{1,i}} \cdot h_{2,i}^{z_{2,i}} \right)$$

$$d^{*0}_{1,i} = g_1^{z_{1,i}} \cdot g_2^{f_1 \cdot v_i} \Big/ \left(\prod_{j=1}^{k+1} \left(g_{1,j}^{x_{1,j}} \cdot g_{1,k+2}^{x_{1,k+2}} \right)^{y_{1,i}^j} \right) \text{ for } i \in [1,n] \setminus \{m+1\}$$

$$d^{*0}_{2,i} = g_1^{z_{2,i}} \cdot g_2^{f_2 \cdot v_i} \Big/ \left(\prod_{j=1}^{k+1} \left(g_{2,j}^{x_{2,j}} \cdot g_{2,k+2}^{x_{2,k+2}} \right)^{y_{2,i}^j} \right) \text{ for } i \in [1,n] \setminus \{m+1\}$$

     Now, $\mathcal{B}$ additionally samples $\{y_{1,i}^{k+2}, y_{2,i}^{k+2} \xleftarrow{R} \mathbb{Z}_{q_1}^*\}_{i\in[1,m]}$, and sets:

$$d^{*j}_{1,i} = g_{1,j}^{y_{1,i}^j} \;\;,\;\; d^{*j}_{2,i} = g_{2,j}^{y_{2,i}^j} \text{ for } i \in [1,n] \setminus \{m+1\}, j \in [1,k+1]$$
$$d^{*k+2}_{1,i} = g_{1,k+2}^{y_{1,i}^{k+2}} \;\;,\;\; d^{*k+2}_{2,i} = g_{2,k+2}^{y_{2,i}^{k+2}} \text{ for } i \in [1,m]$$
$$d^{*k+2}_{1,i} = g_{1,k+2}^{\sum_{j=1}^{k+1} y_{1,i}^j} \;\;,\;\; d^{*k+2}_{2,i} = g_{2,k+2}^{\sum_{j=1}^{k+1} y_{2,i}^j} \text{ for } i \in [m+2,n]$$

     Observe that $\mathcal{B}$ uses a randomly sampled $y_{1,i}^{k+2}$ instead of $\sum_{j=1}^{k+1} y_{1,i}^j$, and a randomly sampled $y_{2,i}^{k+2}$ instead of $\sum_{j=1}^{k+1} y_{2,i}^j$, for $i \in [1,m]$. This step ensures that secret-key elements corresponding to the first m components of $\overrightarrow{v^*}$ are indeed uniformly random, as desired. Also, it is straightforward to observe that the secret-key elements corresponding to the last $(n-m-1)$ components are well-formed.

  2. $\mathcal{B}$ now embeds its input $(k+1)$-DLIN-instance pair in the secret key elements corresponding to the $(m+1)^{\text{th}}$ component of $\overrightarrow{v^*}$. In particular, it sets:

$$d^{*0}_{1,m+1} = g_1^{z_{1,m+1}} \cdot g_1^{f_1 \cdot v_{m+1}} \Big/ \left(\prod_{j=1}^{k+2} \left(g_{1,j}^{a_j} \right)^{x_{1,j}} \right)$$

$$d^{*0}_{2,m+1} = g_1^{z_{2,m+1}} \cdot g_2^{f_2 \cdot v_{m+1}} \Big/ \left(\prod_{j=1}^{k+2} \left(g_{2,j}^{a'_j} \right)^{x_{2,j}} \right)$$

$$d^{*j}_{1,m+1} = g_{1,j}^{a_j} \;\;,\;\; d^{*j}_{2,m+1} = g_{2,j}^{a'_j} \text{ for } j \in [1,k+2]$$

     where $\{g_{1,j}^{a_j}\}_{j\in\{1,k+2\}}$ and $\{g_{2,j}^{a'_j}\}_{j\in\{1,k+2\}}$ are parts of its input instances.

$\mathcal{B}$ finally responds to $\mathcal{A}$ with the secret-key $\mathsf{sk}_{\overrightarrow{v^*}}$ as:

$$\mathsf{sk}_{\overrightarrow{v^*}} = \left(d_0^*, \{d^{*j}_{1,i}, d^{*j}_{2,i}\}_{i\in[1,n],j\in[0,k+2]}\right)$$

- ***Guess:*** At the end of the game, $\mathcal{A}$ outputs a bit b'. $\mathcal{B}$ outputs the same bit b'.

It is easy to see that when $a_{k+2} = \sum_{j=1}^{k+1} a_j$ and $a'_{k+2} = \sum_{j=1}^{k+1} a'_j$, the secret-key $\mathsf{sk}_{\overrightarrow{v^*}}$ is identically distributed to the response of the real-or-random oracle in the experiment $\mathsf{Expt}^{\mathsf{mode},m}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)$. On the other hand, when either or both of a_{k+2} and a'_{k+2} are uniformly random in $\mathbb{Z}_q^*$, the secret-key $\mathsf{sk}_{\overrightarrow{v^*}}$ is identically distributed to the response of the real-or-random oracle in the experiment $\mathsf{Expt}^{\mathsf{mode},m+1}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)$. It follows readily that $\mathcal{B}$ has the same advantage ϵ as $\mathcal{A}$ in solving its input instance pair. This completes the proof of Claim 5.1.

We now make the following observation:

$$\begin{aligned}
&\left|\Pr\left[\mathsf{Expt}^{\mathsf{mode}}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{mode},n}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right]\right| \\
&\quad\le \sum_{m=0}^{n-1}\left|\Pr\left[\mathsf{Expt}^{\mathsf{mode},m}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{mode},m+1}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right]\right| \\
&\quad\le \mathsf{negl}(\lambda) \text{ (from Claim 5.1) for } n=\mathsf{poly}(\lambda)
\end{aligned}$$

Consequently, for $\mathsf{mode} \xleftarrow{R} \{\mathsf{real},\mathsf{rand}\}$, we have:

$$\begin{aligned}
\mathbf{Adv}^{\mathsf{FP}}_{\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda) &= \left|\Pr\left[\mathsf{Expt}^{\mathsf{real}}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{rand}}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right]\right| \\
&\le \left|\Pr\left[\mathsf{Expt}^{\mathsf{real}}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{real},n}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right]\right| \\
&\quad + \left|\Pr\left[\mathsf{Expt}^{\mathsf{rand}}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{rand},n}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right]\right| \\
&\quad + \left|\Pr\left[\mathsf{Expt}^{\mathsf{real},n}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{rand},n}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right]\right| \\
&\le 2\left|\Pr\left[\mathsf{Expt}^{\mathsf{mode}}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right] - \Pr\left[\mathsf{Expt}^{\mathsf{mode},n}_{\mathsf{FP},\Pi_k^{\mathrm{IPE}},\mathcal{A}}(\lambda)=1\right]\right| \text{ (from Equation 1)} \\
&= 2\epsilon' \le \mathsf{negl}(\lambda)
\end{aligned}$$

This hybrid argument completes the proof of enhanced function privacy for Π_k^{IPE}.

6 Feasible Extensions and Applications of Our Techniques

Our work proposes the first public-key predicate encryption schemes that are provably indistinguishable function private under standard computational assumptions. Existing function private predicate encryption schemes in the public-key setting are either secure only in the generic group model, or require strong assumptions such as indistinguishability obfuscation. We develop a novel approach, denoted as *encrypt-augment-recover*, that takes an existing predicate encryption scheme and transforms it into a computationally function private one while retaining its original data privacy guarantees. Our approach yields constructions for IBE in the random oracle model that are function private under weaker variants of the DLIN assumption. Our approach also yields public-key IPE constructions in the standard model that are also function private under the same family of assumptions. Our constructions do not require additional assumptions such as indistinguishability obfuscation. In this section, we present some feasible extensions and applications of our techniques.

Function Privacy for Private-Key Predicate Encryption. Our methodology is equally applicable for achieving computationally function private predicate encryption schemes in the private-key setting, even when the underlying predicates are not necessarily sampled from distributions with at least super-logarithmic min-entropy. In particular, the core function privacy arguments for our constructions presented in this paper do not essentially rely on the unpredictability of the predicate distributions; this assumption is additionally made to rule out trivial attacks in the public-key setting. Consequently, our approach anticipates an expansion to the existing body of work in designing function private predicate encryption schemes in the private-key setting.

Function Privacy for Multi-Input Predicate Encryption. Multi-input predicate encryption (MIPE) introduced by Goldwasser et al. [36] is a generalization of functional encryption to the setting of multi-input predicates. An MIPE scheme has several encryption slots and each decryption key sk_f for a multi-input predicate f jointly decrypts the ciphertexts $\mathsf{Enc}(I_1), ..., \mathsf{Enc}(I_n)$ for all slots to obtain $f(I_1, ..., I_n)$ without revealing anything more about the encrypted attributes. In particular, this provides a framework to evaluate *bounded-norm* multi-input IPE: each predicate is specified by a collection of vectors $\overrightarrow{v}_1, \cdots, \overrightarrow{v}_n$, and takes as input a collection of vectors $\overrightarrow{x}_1, \cdots, \overrightarrow{x}_n$ to output $f_{\overrightarrow{v}_1, \cdots, \overrightarrow{v}_n}(\overrightarrow{x}_1, \cdots, \overrightarrow{x}_n) = \sum_{i=1}^{n} \langle \overrightarrow{v}_i, \overrightarrow{x}_i \rangle$.

We point out that our technique can be easily generalized to obtain function private IPE schemes in the multi-input setting as follows: we first use our technique to obtain a function private IPE construction in the single-input setting, and then run n independent copies of this construction. The i^{th} copy is used to encrypt $\overrightarrow{x}_i$ in the i^{th} slot, while the new secret-key is the ensemble of the n secret-keys corresponding to $\overrightarrow{v}_1, \cdots, \overrightarrow{v}_n$. The decryption algorithm computes each inner product individually, and returns their sum. Although this means that the adversary also learns each individual inner product, this is an inherent leakage in the public-key setting and does not weaken the security guarantees. The data privacy guarantees of the underlying scheme ensure no further leakage, while the function privacy guarantees of the underlying scheme continue to hold as long as each $\overrightarrow{v}_i$ is sampled from block sources with sufficient min-entropy, and is independent of the other $n - 1$ vectors.

7 Open Problems

In this section, we discuss some interesting open problems that arise from our work.

Hidden Vector Encryption and Polynomial Evaluation. Boneh and Waters [1] proposed hidden vector encryption (HVE), a pre-cursor to IPE, that supports search using conjunctive, range and comparison-based query predicates. In HVE, attributes correspond to vectors over an alphabet Σ, while secret-keys correspond to predicate vectors over the augmented alphabet $\Sigma_\star = \Sigma \cup \{\star\}$ containing the wild card character $\star$. Decryption succeeds if the attribute matches the predicate vector in every coordinate that is not $\star$. We note that although IPE can be used to realize HVE [3], our computational function privacy definitions do not naturally extend to HVE. In particular, the presence of the wild card character $\star$ in the predicate vectors of HVE trivially violates our min-entropy requirements, making it difficult to *hide* their presence in the secret-key. It is still an open problem, however, to formalize a stronger function privacy definition for HVE, and to realize function private constructions satisfying this definition. This would also provide insight into the limits of function privacy for searchable encryption schemes supporting comparison and range queries. Finally, it is also open to formalize security definitions and realize constructions for function private encryption schemes that support arbitrary polynomial evaluation predicates [3].

Generalization of Our Approach. In this work, we have applied our *encrypt-augment-recover* approach to transform certain existing public-key predicate encryption schemes that are not function private, into computationally function private ones. An interesting open problem is to explore whether our approach can be generalized for *any* public-key predicate encryption scheme, or if there are any specific

properties of existing predicate encryption schemes that make them amenable to transformation using our approach. A starting point in this direction could be to explore the applicability of our approach to public-key predicate encryption schemes based on lattices, such as the IPE scheme in [2]. It would also be interesting to explore if our approach can be used to design public-key encryption schemes supporting a set of predicates beyond inner-products. Iovino et al. [12] have demonstrated that computational function privacy from standard assumptions seems unattainable for a very generalized class of predicates, such as the class of all NC^1) circuits. This motivates exploring the limits of our techniques in terms of the range of predicates for which they are applicable.

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