δ-subgaussian Random Variables in Cryptography

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Abstract. In the Ring-LWE literature, there are several works that use a statistical framework based on δ-subgaussian random variables. These were introduced by Miccancio and Peikert (Eurocrypt 2012) as a relaxation of subgaussian random variables. In this paper, we completely characterise δ-subgaussian random variables. In particular, we show that this relaxation from a subgaussian random variable corresponds only to the shifting of the mean. Next, we give an alternative noncentral formulation for a δ-subgaussian random variable, which we argue is more statistically natural. This formulation enables us to extend prior results on sums of δ-subgaussian random variables, and on their discretisation.

Keywords. Ring Learning with Errors, Subgaussian Random Variable.

1 Introduction

A subgaussian random variable [4] is a random variable that is bounded in a particular technical sense by a Normal random variable. Subgaussian random variables cover a wide class of random variables: for example is well known that any centred and bounded random variable is subgaussian [17]. They have many of the attractive properties of Normal random variables: for example, they form a linear space and their tails that are bounded by the tails of a Normal random variable [15]. Subgaussian random variables have been used widely in cryptography [2].

In [7], Micciancio and Peikert introduced the notion of a δ-subgaussian random variable, where δ can take a value $\delta \geq 0$, as a relaxation of a subgaussian random variable. In the formulation of [7], the case $\delta = 0$ gives a 0-subgaussian random variable, which is exactly a subgaussian random variable. Statistical arguments based on δ-subgaussian random variables have been used in Ring-LWE cryptography in many application settings including signature schemes [7], key exchange [10] and homomorphic encryption [6].

In this paper, we re-examine the relaxation in [7] of subgaussian random variables to give δ-subgaussian random variables. We completely characterise δ-subgaussian random variables by showing that this relaxation corresponds only to the shifting of the mean. This enables us to give a noncentral formulation for δ-subgaussian random variables which we argue is more statistically natural.

Amongst the prior literature using δ-subgaussian random variables, perhaps the prominent work is *A Toolkit for Ring-LWE Cryptography* [6]. This work gives an algebraic and statistical framework for Ring-LWE cryptography that is widely applicable. Using our noncentral formulation for δ-subgaussian random variables, we extend results presented in the *Toolkit* on sums of δ-subgaussian random variables, and on their discretisation.

1.1 Contributions

The first main contribution of this paper is to give a full and particularly simple characterisation of δ-subgaussian random variables. We show in Lemma 5 that any δ-subgaussian random variable with mean 0 must be a 0-subgaussian random variable. We then show in Lemma 6 that shifting a δ-subgaussian random variable by its mean gives a 0-subgaussian random variable. Finally, we show in Lemma 7 that any shift of a 0-subgaussian random variable is a δ-subgaussian random variable for some $\delta \geq 0$. These results give our main result in this section, Proposition 1, that the relaxation from 0-subgaussian random variables to δ-subgaussian random variables corresponds only to a shifting of the mean.

The second main contribution of this paper is to generalise results about δ-subgaussian random variables that have previously appeared in the literature. Firstly, we give an alternative noncentral formulation for a δ-subgaussian random variable which enables us in Theorem 1 to generalise the results in [10, 6] for sums of δ-subgaussian random variables. Secondly, in Theorem 2 we improve the result of the *Toolkit* [6] for the δ-subgaussian standard parameter of the coordinatewise randomised rounding discretisation (termed *CRR-discretisation* in our paper) of the *Toolkit* [6, Section 2.4.2] of a δ-subgaussian random variable.

1.2 Structure

We review the necessary background in Section 2. We analyse and characterise δ-subgaussian random variables in Section 3. We give a noncentral formulation for δ-subgaussian random variables in Section 4. We consider the discretisations of random variables arising in Ring-LWE in Section 5.

2 Background

2.1 Algebraic background

This section mainly follows [6]. We consider the ring $R = \mathbb{Z}[X]/(\Phi_m(X))$, where $\Phi_m(X)$ is the m^{th} cyclotomic polynomial of degree n, and we let R_a denote R/aR for an integer a. For simplicity, we only consider the case where m is a large prime, so $n = \phi(m) = m - 1$, though our arguments apply more generally.

Let ζ_m denote a (primitive) m^{th} root of unity, which has minimal polynomial $\Phi_m(X) = 1 + X + \ldots + X^n$. The m^{th} *cyclotomic number field* $K = \mathbb{Q}(\zeta_m)$ is the field extension of the rational numbers $\mathbb{Q}$ obtained by adjoining this m^{th} root of unity ζ_m, so K has degree n.

There are n *ring embeddings* $\sigma_1, \ldots, \sigma_n \colon K \to \mathbb{C}$ that fix every element of $\mathbb{Q}$. Such a ring embedding σ_k (for $1 \leq k \leq n$) is defined by $\zeta_m \mapsto \zeta_m^k$, so $\sum_{j=1}^n a_j \zeta_m^j \mapsto \sum_{j=1}^n a_j \zeta_m^{kj}$. The *canonical embedding* $\sigma : K \to \mathbb{C}^n$ is defined by

$$a \mapsto (\sigma_1(a), \ldots, \sigma_n(a))^T.$$

The *ring of integers* $\mathcal{O}_K$ of a number field is the ring of all elements of the number field which are roots of some monic polynomial with coefficients in $\mathbb{Z}$. The ring of integers of the m^{th} cyclotomic number field K is

$$R = \mathbb{Z}[\zeta_m] \cong \mathbb{Z}[x]/(\Phi_m).$$

The canonical embedding σ embeds R as a lattice $\sigma(R)$. The conjugate dual of this lattice corresponds to the embedding of the dual fractional ideal

$$R^\vee = \{a \in K \mid \mathrm{Tr}(aR) \subset \mathbb{Z}\}.$$

The ring embeddings $\sigma_1, \ldots, \sigma_n$ occur in conjugate pairs, and much of the analysis of Ring-LWE takes place in a space H of conjugate pairs of complex numbers. The conjugate pairs matrix T gives a basis for H that we call the T-basis.

Definition 1. The *conjugate pair matrix* is the $n \times n$ complex matrix T, so $T \colon \mathbb{C}^n \to \mathbb{C}^n$, given by

$$T = 2^{-\frac{1}{2}} \begin{pmatrix} 1 & 0 \ldots 0 & 0 \ldots & 0 & i \\ 0 & 1 \ldots 0 & 0 \ldots & i & 0 \\ \vdots & \vdots \ddots \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \ldots 1 & i \ldots & 0 & 0 \\ 0 & 0 \ldots 1 & -i \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 1 \ldots 0 & 0 \ldots & -i & 0 \\ 1 & 0 \ldots 0 & 0 \ldots & 0 & -i \end{pmatrix}. \qquad \square$$

Definition 2. The *complex conjugate pair space* H is given by $H = T(\mathbb{R}^n)$, where T is the conjugate pairs matrix. $\square$

Our results on discretisation will rely on the spectral norm of the basis for H being considered. We note that the spectral norm for the T-basis is 1.

Definition 3. Suppose that the lattice Λ has (column) basis matrix B. The *Gram matrix* of the basis matrix B is $B^\dagger B$, where $B^\dagger = \overline{B}^T$ is the complex conjugate of B. The *spectral norm* $\lambda(B) > 0$ of the basis matrix B is the square root of largest eigenvalue of the Gram matrix $B^\dagger B$. $\square$

2.2 The Ring-LWE problem

The Learning with Errors (LWE) problem [13, 14] has become a standard hard problem in cryptology that is at the heart of lattice-based cryptography [8, 11]. The Ring Learning with Errors (Ring-LWE) problem [16, 5] is a generalisation of the LWE problem from the ring of integers to certain other number field rings. Both the LWE problem and the Ring-LWE problem are related to well-studied lattice problems that are believed to be hard [1, 5, 6, 9, 13, 12].

Definition 4 ([16, 5]). Let R be the ring of integers of a number field K. Let $q \geq 2$ be an integer modulus. Let $R^\vee$ be the dual fractional ideal of R. Let $R_q = R/qR$ and $R_q^\vee = R^\vee/qR^\vee$. Let $K_\mathbb{R} = K \otimes_\mathbb{Q} \mathbb{R}$.

Let χ be a distribution over $K_\mathbb{R}$. Let $s \in R_q^\vee$ be a secret. A sample from the *Ring-LWE distribution* $A_{s,\chi}$ over $R_q \times K_\mathbb{R}/qR^\vee$ is generated by choosing $a \leftarrow R_q$ uniformly at random, choosing $e \leftarrow \chi$ and outputting

$$(a, b = (a \cdot s)/q + e \mod qR^\vee).$$

Let Ψ be a family of distributions over $K_\mathbb{R}$. The *Search Ring-LWE* problem is defined as follows: given access to arbitrarily many independent samples from $A_{s,\chi}$ for some arbitrary $s \in R_q^\vee$ and $\chi \in \Psi$, find s.

Let Υ be a distribution over a family of error distributions, each over $K_\mathbb{R}$. The *average-case Decision Ring-LWE* problem is to distinguish with non-negligible advantage between arbitrarily many independent samples from $A_{s,\chi}$ for a random choice of $(s, \chi) \leftarrow \mathcal{U}\left(R_q^\vee\right) \times \Upsilon$, and the same number of uniformly random samples from $R_q \times K_\mathbb{R}/qR^\vee$. □

2.3 Moment generating functions

The moment generating function is a basic tool of probability theory, and we first give a definition for a univariate random variable.

Definition 5. The *moment generating function* M_W of a real-valued univariate random variable W is the function from a subset of $\mathbb{R}$ to $\mathbb{R}$ defined by

$$M_W(t) = \mathbf{E}\left(\exp(tW)\right) \quad \text{for } t \in \mathbb{R} \text{ whenever this expectation exists.} \qquad \square$$

Fundamental results underlying the utility of the moment generating function are given in Lemma 1.

Lemma 1 ([3]). If M_W is the moment generating function of a real-valued univariate random variable W, then M_W is a continuous function within its radius of convergence and the k^{th} moment of W is given by $\mathbf{E}(W^k) = M_W^{(k)}(0)$ when the k^{th} derivative of the moment generating function exists at 0. In particular, (i) $M_W(0) = 1$, (ii) $\mathbf{E}(W) = M_W'(0)$ and (iii) $\text{Var}(W) = M_W''(0) - M_W'(0)^2$, where these derivatives exist. □

More generally, the statistical properties of a random variable W can be determined from its moment generating function M_W, and in particular from the behaviour of this moment generating function M_W in a neighbourhood of 0 as its Taylor series expansion (where it exists) is given by

$$\begin{aligned} M_W(t) &= 1 + M'_W(0)\, t + \tfrac{1}{2} M''_W(0)\, t^2 + \ldots + \tfrac{1}{k!} M^{(k)}_W(0)\, t^k + \ldots \\ &= 1 + \ \mathbf{E}(W)\, t \ + \tfrac{1}{2}\mathbf{E}(W^2)\, t^2 + \ldots + \ \tfrac{1}{k!}\mathbf{E}(W^k)\, t^k \ + \ldots \ . \end{aligned}$$

The definition of a moment generating function for a real-valued univariate random variable generalises to multivariate random variables and to random variables on H, and the above results also generalise in the appropriate way.

Definition 6. The *moment generating function* M_W of a multivariate random variable W on $\mathbb{R}^l$ is the function from a subset of $\mathbb{R}^l$ to $\mathbb{R}$ defined by

$$M_W(t) = \mathbf{E}\left(\exp(\langle t, W\rangle\right) = \mathbf{E}\left(\exp\left(t^T W\right)\right) \text{ whenever this expectation exists. } \square$$

Definition 7. The *moment generating function* M_W of a multivariate random variable W on H is the function from a subset of H to $\mathbb{R}$ defined by

$$M_W(t) = \mathbf{E}\left(\exp(\langle t, W\rangle\right) = \mathbf{E}\left(\exp\left(t^\dagger W\right)\right) \text{ whenever this expectation exists. } \square$$

2.4 Subgaussian random variables

In Lemma 2 we recall the standard result for the moment generating function of a Normal random variable with mean 0.

Lemma 2 ([3]). If $W \sim \mathrm{N}(0, b^2)$ is a Normal random variable with mean 0 and standard deviation $b \geq 0$, then W has moment generating function

$$M_W(t) = \mathbf{E}\left(\exp(tW)\right) = \exp(\tfrac{1}{2} b^2 t^2) \qquad \text{for all } t \in \mathbb{R}. \qquad \square$$

Lemma 2 gives rise to the idea of considering random variables with mean 0 whose moment generating function is dominated everywhere by the moment generating function of an appropriate Normal random variable with mean 0. Such a random variable is known as a *subgaussian* random variable [15] and is specified in Definition 8.

Definition 8. A real-valued random variable W is *subgaussian* with *standard parameter* $b \geq 0$ if its moment generating function M_W satisfies

$$M_W(t) = \mathbf{E}(\exp(tW)) \leq \exp(\tfrac{1}{2} b^2 t^2) \qquad \text{for all } t \in \mathbb{R}. \qquad \square$$

An example of a subgaussian random variable is illustrated in Figure 1, which shows the moment generating function $M_X(t) = \cosh t$ for the subgaussian random variable X taking values ± 1 with probability $\frac{1}{2}$ (so $\mathbf{E}(X) = 0$ and $\mathrm{Var}(X) = 1$), together with its corresponding bounding function $\exp(\frac{1}{2}t^2)$, which is the moment generating function of a standard Normal $\mathrm{N}(0, 1)$ random variable having the same mean and variance.

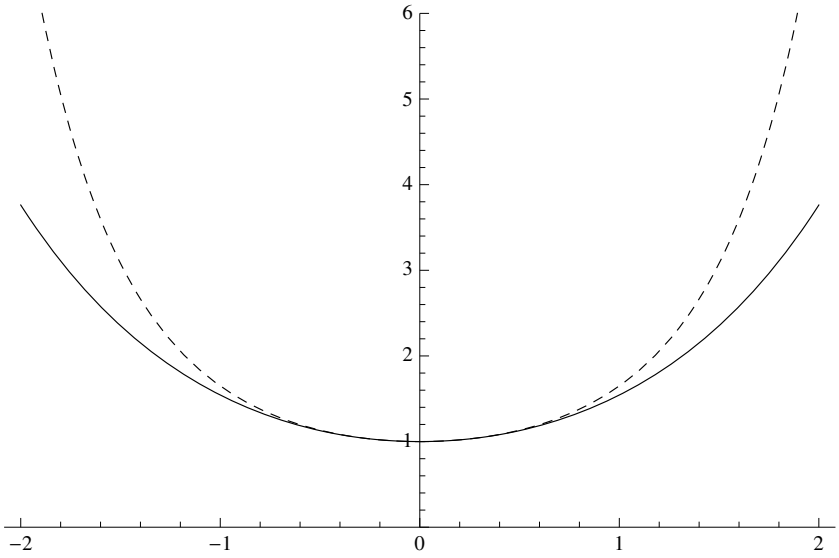

Fig. 1. Moment generating function $M_X(t) = \cosh t$ for the random variable X taking values ± 1 with probability $\frac{1}{2}$ (solid line) and subgaussian bounding function $\exp(\frac{1}{2}t^2)$ (dashed line).

3 δ-subgaussian random variables

In this section, we give a complete and particularly simple characterisation of δ-subgaussian random variables. Statistical arguments based on δ-subgaussian random variables have been widely used in Ring-LWE [7, 10, 6], as noted in Section 1. Our main result, Proposition 1, shows that a δ-subgaussian random variable (for $\delta \geq 0$) is simply a translation of some 0-subgaussian random variable.

3.1 Defining a δ-subgaussian random variable

A δ-subgaussian random variable is a generalisation of a subgaussian random variable in the following sense: δ is allowed to be any value $\delta \geq 0$, and taking the case $\delta = 0$ gives a subgaussian random variable. In other words, what is termed a 0-subgaussian random variable for example in [7, 6] is exactly a subgaussian random variable.

We now give two definitions for a univariate δ-subgaussian random variable to make this generalisation precise. Definition 9 corresponds with the usual probability theory of moment generating functions [3]. Definition 10 is used for example in [6]. Lemma 3 shows that these definitions are equivalent.

Definition 9. A real-valued random variable W is *δ-subgaussian* ($\delta \geq 0$) with *standard parameter* $b \geq 0$ if its moment generating function M_W satisfies

$$M_W(t) = \mathbf{E}(\exp(tW)) \leq \exp(\delta)\exp(\tfrac{1}{2}b^2t^2) \qquad \text{for all } t \in \mathbb{R}. \qquad \square$$

Definition 10. A real-valued random variable W is *δ-subgaussian* ($\delta \geq 0$) with *scaled parameter* $s \geq 0$ if its moment generating function M_W satisfies

$$M_W(2\pi t) = \mathbf{E}(\exp(2\pi t W)) \leq \exp(\delta)\exp(\pi s^2 t^2). \qquad \text{for all } t \in \mathbb{R}. \qquad \square$$

Lemma 3. A real-valued univariate random variable is δ-subgaussian with standard parameter b if and only if it is δ-subgaussian with scaled parameter $(2\pi)^{\frac{1}{2}}b$.

The definition of a univariate δ-subgaussian random variable generalises to a multivariate δ-subgaussian random variable and a δ-subgaussian random variable on H in the obvious way.

Definition 11. A multivariate random variable W on $\mathbb{R}^l$ is *δ-subgaussian* ($\delta \geq 0$) with *standard parameter* $b \geq 0$ if its moment generating function M_W satisfies

$$M_W(t) = \mathbf{E}\left(\exp\left(t^T W\right)\right) \leq \exp(\delta)\exp(\tfrac{1}{2}b^2|t|^2) \qquad \text{for all } t \in \mathbb{R}^l. \qquad \square$$

Definition 12. A random variable W on H is *δ-subgaussian* ($\delta \geq 0$) with *standard parameter* $b \geq 0$ if its moment generating function M_W satisfies

$$M_W(t) = \mathbf{E}\left(\exp\left(t^\dagger W\right)\right) \leq \exp(\delta)\exp(\tfrac{1}{2}b^2|t|^2) \qquad \text{for all } t \in H. \qquad \square$$

3.2 Characterisation of univariate δ-subgaussian random variables

In this section, we give a complete characterisation of a univariate δ-subgaussian random variable. We show that the relaxation of the 0-subgaussian condition to give the δ-subgaussian condition for a univariate random variable does not correspond to any relaxation in the fundamental statistical conditions on the random variable except for the location of its mean.

We firstly recall in Lemma 4 a property of 0-subgaussian random variables proved in [15], namely that their mean is 0. This can be heuristically explained as follows. Lemma 1(i) shows that any moment generating function must pass through $(0, 1)$. However, a 0-subgaussian bounding function $\exp(\frac{1}{2}b^2t^2)$ also passes through $(0, 1)$ and has derivative 0 at 0. Thus any moment generating function bounded by $\exp(\frac{1}{2}b^2t^2)$ must have derivative 0 at 0. Lemma 1(ii) then shows that such a 0-subgaussian random variable with moment generating function bounded by $\exp(\frac{1}{2}b^2t^2)$ must have mean 0.

Lemma 4 ([15]). If W is a univariate real-valued 0-subgaussian random variable, then $\mathbf{E}(W) = 0$. $\square$

We now give some results to show that the relaxation of the 0-subgaussian condition to the δ-subgaussian condition (for $\delta \geq 0$) corresponds exactly to the relaxation of the condition that the mean of the random variable is 0. These results are illustrated in Figure 2 for a random variable with mean 1.

Intuitively, relaxing the constraint that $\delta = 0$ in the δ-subgaussian bounding function $\exp(\delta)\exp(\frac{1}{2}b^2t^2)$ essentially shifts the bounding function "up the

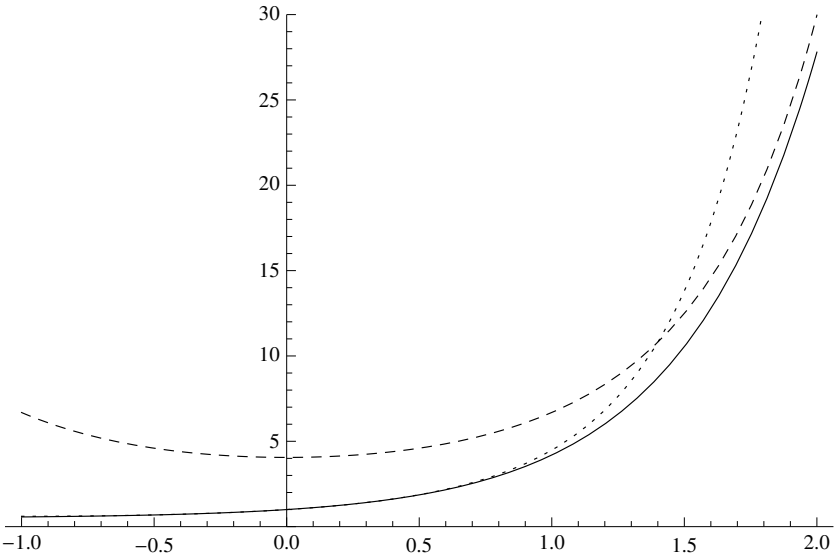

Fig. 2. Moment generating function $M_{X+1}(t) = \frac{1}{2}(1+\exp(2t))$ for the random variable $X + 1$ (for $X \sim \text{Uni}(\{-1, 1\})$ of Figure 1) taking values 0 and 2 with probability $\frac{1}{2}$ and having mean 1 (solid line), δ-subgaussian bounding function $\exp(\frac{7}{5} + \frac{1}{2}t^2)$ (dashed line), and "noncentral" subgaussian bounding function $\exp(t + \frac{1}{2}t^2)$ (dotted line).

y-axis", and in particular away from the point $(0, 1)$. However, a moment generating function must pass through the point $(0, 1)$. This relaxation essentially permits us to "tilt" the moment generating function of a 0-subgaussian random variable, pivoting about the point $(0, 1)$, so that the moment generating function has a nonzero derivative at 0. This allows random variables with nonzero mean potentially to be δ-subgaussian random variables.

We now make the intuition described above and illustrated by Figure 2 more precise in a number of ways. First, Lemma 5 shows that any δ-subgaussian random variable with mean 0 must be a 0-subgaussian random variable.

Lemma 5. If W is a univariate real-valued δ-subgaussian random variable ($\delta \geq 0$) with mean $\mathbf{E}(W) = 0$, then W is a 0-subgaussian random variable. □

Proof. The δ-subgaussian bounding function $\exp(\delta)\exp(\frac{1}{2}b^2t^2)$ is bounded above and away from 1 when $\delta > 0$. However, the moment generating function M_W of W is continuous at 0 with $M_W(0) = 1$, so the δ-subgaussian bounding function $\exp(\delta)\exp(\frac{1}{2}b^2t^2)$ is neccesarily always a redundant bounding function for any moment generating function in some open neighbourhood of 0. The proof therefore proceeds by considering the moment generating function M_W of W in two separate regions: an open neighbourhood containing 0 and the region away from this open neighbourhood.

We first consider a region that is some open neighbourhood of 0. Taylor's Theorem (about 0) shows that the moment generating function M_W of W can

be expressed in this open neighbourhood of 0 as

$$\begin{aligned} M_W(t) = \mathbf{E}\left(\exp(tW)\right) &= 1 + \mathbf{E}(W)t + \tfrac{1}{2}\mathbf{E}\left(W^2\right)t^2 + o(t^2) \\ &= 1 + \tfrac{1}{2}\mathbf{E}\left(W^2\right)t^2 + o(t^2), \end{aligned}$$

where a function $g(t) = o(t^2)$ in the infinitesimal sense near 0 if $t^{-2}g(t) \to 0$ as $t \to 0$. Similarly we can write $\exp(\frac{1}{2}c^2t^2) = 1 + \frac{1}{2}c^2t^2 + o(t^2)$, so we have

$$\frac{M_W(t) - \exp(\frac{1}{2}c^2t^2)}{t^2} = \tfrac{1}{2}\left(\mathbf{E}\left(W^2\right) - c^2\right) + \frac{o(t^2)}{t^2}.$$

Thus for values of c such that $c^2 > \mathbf{E}\left(W^2\right)$ we have

$$\lim_{t\to 0} \frac{M_W(t) - \exp(\frac{1}{2}c^2t^2)}{t^2} = \tfrac{1}{2}\left(\mathbf{E}\left(W^2\right) - c^2\right) < 0,$$

in which case there exists an open neighbourhood $(-\nu, \nu)$ of 0 ($\nu > 0$) such that

$$\frac{M_W(t) - \exp(\frac{1}{2}c^2t^2)}{t^2} < 0$$

in this neighbourhood, so

$$M_W(t) \leq \exp(\tfrac{1}{2}c^2t^2) \quad [|t| < \nu].$$

We now consider the complementary region away from the open neighbourhood $(-\nu, \nu)$ of 0. If W is δ-subgaussian with standard parameter $b \geq 0$, then its moment generating function satisfies $M_W(t) \leq \exp(\delta)\exp(\frac{1}{2}b^2t^2)$ for all $t \in \mathbb{R}$, and in particular for $|t| \geq \nu$. If we let $d^2 = b^2 + 2\nu^{-2}\delta$, then in this other region the moment generating function M_W of W satisfies

$$\begin{aligned} M_W(t) &\leq \exp(\delta)\exp(\tfrac{1}{2}b^2t^2) = \exp(\delta)\exp(\tfrac{1}{2}d^2t^2)\exp(-\delta\nu^{-2}t^2) \\ &\leq \exp(\delta(1-\nu^{-2}t^2))\exp(\tfrac{1}{2}d^2t^2) \leq \exp(\tfrac{1}{2}d^2t^2) \qquad [|t| \geq \nu]. \end{aligned}$$

Taking the two regions together shows that the moment generating function M_W of W satisfies

$$M_W(t) \leq \exp(\ \tfrac{1}{2}\max\{c^2, d^2\}\ t^2\ ) \qquad \text{for all } t \in \mathbb{R}.$$

Thus W is a 0-subgaussian random variable. □

Next, Lemma 6 shows that shifting a δ-subgaussian random variable by its mean results in a 0-subgaussian random variable.

Lemma 6. If W is a univariate real-valued δ-subgaussian random variable ($\delta \geq 0$), then the centred random variable $W_0 = W - \mathbf{E}(W)$ is a 0-subgaussian random variable. □

Proof. If W is a δ-subgaussian random variable with standard parameter b, then its moment generating function M_W satisifies

$$M_W(t) \leq \exp(\delta)\exp(\tfrac{1}{2}b^2t^2) \qquad \text{for all } t \in \mathbb{R}.$$

The centred random variable $W_0 = W - \mathbf{E}(W)$ with mean $\mathbf{E}(W_0) = 0$ has moment generating function M_{W_0} given by

$$\begin{aligned} M_{W_0}(t) &= \mathbf{E}\left(\exp(tW_0)\right) = \mathbf{E}\left(\exp(t(W - \mathbf{E}(W)))\right) \\ &= \exp(-\mathbf{E}(W)t)\ \mathbf{E}\left(\exp(tW)\right) \\ &= \exp(-\mathbf{E}(W)t)\ M_W(t). \end{aligned}$$

The required result can be obtained by noting that for $c > b > 0$, the inequality

$$\left(\delta + \left(\tfrac{1}{2}b^2t^2 - \mathbf{E}(W)t\right)\right) \ \leq\ \left(\left(\delta + \frac{1}{2}\frac{\mathbf{E}(W)^2}{c^2 - b^2}\right) + \tfrac{1}{2}c^2t^2\right)$$

holds, which can be demonstrated as

$$\left(\left(\delta + \frac{1}{2}\frac{\mathbf{E}(W)^2}{c^2 - b^2}\right) + \tfrac{1}{2}c^2t^2\right) - \left(\delta + \left(\tfrac{1}{2}b^2t^2 - \mathbf{E}(W)t\right)\right) = \frac{c^2 - b^2}{2}\left(t + \frac{\mathbf{E}(W)}{c^2 - b^2}\right)^2$$

is non-negative for $c > b > 0$. This inequality means that the moment generating function M_{W_0} of W_0 satisfies

$$\begin{aligned} M_{W_0}(t) &= \exp(-\mathbf{E}(W)t)\ M_W(t) \\ &\leq \exp(-\mathbf{E}(W)t)\exp(\delta)\exp(\tfrac{1}{2}b^2t^2) \\ &\leq \exp\left(\delta + (\tfrac{1}{2}b^2t^2 - \mathbf{E}(W)t)\right) \\ &\leq \exp\left(\delta + \frac{1}{2}\frac{\mathbf{E}(W)^2}{c^2 - b^2}\right)\exp(\tfrac{1}{2}c^2t^2). \end{aligned}$$

Thus W_0 is a $\left(\delta + \frac{1}{2}\frac{\mathbf{E}(W)^2}{c^2 - b^2}\right)$-subgaussian random variable. As W_0 has mean $\mathbf{E}(W_0) = 0$, Lemma 5 therefore shows that $W_0 = W - \mathbf{E}(W)$ is a 0-subgaussian random variable. □

Finally, Lemma 7 shows that any shift of a δ_0-subgaussian random variable with mean 0 is a δ-subgaussian random variable for some $\delta \geq 0$.

Lemma 7. If W_0 is a univariate real-valued δ_0-subgaussian random variable with mean $\mathbf{E}(W_0) = 0$, then for $\beta \in \mathbb{R}$ the real-valued shifted random variable $W = W_0 + \beta$ is a δ-subgaussian random variable for some $\delta \geq 0$. □

Proof. If W_0 is a δ_0-subgaussian random variable with mean 0, then Lemma 5 shows that W_0 is a 0-subgaussian random variable with some standard parameter $c \geq 0$. The moment generating function M_{W_0} of W_0 is therefore bounded as $M_{W_0}(t) \leq \exp(\frac{1}{2}c^2t^2)$. If $b > c \geq 0$ and $\delta \geq \dfrac{\beta^2}{2(b^2 - c^2)}$, then we note that

$$(\tfrac{1}{2}b^2t^2 + \delta) - (\tfrac{1}{2}c^2t^2 + \beta t) = \frac{(b^2 - c^2)}{2}\left(t - \frac{\beta}{b^2 - c^2}\right)^2 + \delta - \frac{\beta^2}{2(b^2 - c^2)} \geq 0.$$

In this case, the moment generating function M_W of $W = W_0 + \beta$ satisfies

$$M_W(t) = \exp(\beta t) M_{W_0}(t) \leq \exp(\tfrac{1}{2}c^2t^2 + \beta t) \leq \exp(\delta)\exp(\tfrac{1}{2}b^2t^2).$$

Thus $W = W_0 + \beta$ is δ-subgaussian with standard parameter b. □

Lemmas 5, 6 and 7 collectively give the main result Proposition 1 of this section. Proposition 1 precisely characterises δ-subgaussian random variables as shifts of 0-subgaussian random variables, which must have mean 0.

Proposition 1. A real-valued univariate δ-subgaussian random variable can essentially be described in terms of a 0-subgaussian random variable (which must have mean 0) as:

$$\delta\text{-}\textit{subgaussian univariate RV} = 0\text{-}\textit{subgaussian univariate RV} + \textit{constant}. \quad \square$$

3.3 Properties of δ-subgaussian random variables

In this section, we give some basic properties of δ-subgaussian random variables. These are analogous to well-known properties of subgaussian random variables, given for example in [15].

Lemma 8. Suppose that W is a univariate real-valued δ-subgaussian random variable ($\delta \geq 0$) with standard parameter $b \geq 0$. Such a random variable W satisfies: (a) $\mathrm{Var}(W) \leq b^2$, (b) $\mathbf{P}\left(|W - \mathbf{E}(W)| > \alpha\right) \leq 2\exp\left(-\frac{1}{2}b^{-2}\alpha^2\right)$ and (c) $\mathbf{E}\left(\exp(a(W - \mathbf{E}(W))^2)\right) \leq 2$ for some $a > 0$. □

Lemma 9. The set of δ-subgaussian random variables form a linear space. □

Lemma 10. If W is a bounded univariate real-valued random variable, then W is a δ-subgaussian random variable for some $\delta \geq 0$. □

Proof. If W is a bounded random variable, then $W_0 = W - \mathbf{E}(W)$ is a bounded random variable with mean 0. However, Theorem 2.5 of [15] or Theorem 9.9 of [17] shows that a bounded random variable with mean 0, such as W_0, is a 0-subgaussian random variable. Thus Lemma 7 shows that $W = W_0 + \mathbf{E}(W)$ is a δ-subgaussian random variable for some $\delta \geq 0$. □

4 Noncentral Subgaussian Random Variables

Proposition 1 shows that the class of δ-subgaussian random variables are precisely those random variables that can be obtained as shifts of 0-subgaussian random variables. In this section, we use this characterisation to give an alternative noncentral formulation for a δ-subgaussian random variable. We then use this formulation to analyse sums and products of δ-subgaussian random variables. Our main result is Theorem 1, which generalises a result of [6] on sums of δ-subgaussian random variables.

4.1 A noncentral formulation for δ-subgaussian random variables

Proposition 1 enables us to see a δ-subgaussian random variable as a shifted 0-subgaussian random variable. This motivates the following definition.

Definition 13. A random variable Z (on $\mathbb{R}^l$ or H) is a *noncentral subgaussian* random variable with *standard parameter* $d \geq 0$ if the centred random variable $Z - \mathbf{E}(Z)$ is a 0-subgaussian random variable with standard parameter d. □

Lemma 11 establishes the equivalence of the δ-subgaussian and noncentral subgaussian definitions. Lemma 11 also gives a basic property of noncentral subgaussian random variables, which follows from Lemma 9.

Lemma 11. A noncentral subgaussian random variable Z (on $\mathbb{R}^l$ or H) is a δ-subgaussian random variable and vice versa, and the set of noncentral subgaussian random variables (on $\mathbb{R}^l$ or H) is a linear space. □

4.2 Motivation for the noncentral formulation

In this section, we motivate the alternative noncentral formulation. We begin by specifying a noncentral subgaussian random variable in terms of its moment generating function.

Lemma 12. The random variable Z is a noncentral subgaussian random variable (on $\mathbb{R}^l$ or H) with standard parameter d if and only if the moment generating function M_Z of Z satisfies $M_Z(t) \leq \exp\left(\langle t, \mathbf{E}(Z)\rangle\right) \exp(\frac{1}{2}d^2|t|^2)$. □

Proof. If Z is a noncentral subgaussian random variable, then $Z - \mathbf{E}(Z)$ is a 0-subgaussian random variable with standard parameter d and so has moment generating function $M_{Z-\mathbf{E}(Z)}$ satisfying $M_{Z-\mathbf{E}(Z)}(t) \leq \exp(\frac{1}{2}d^2|t|^2)$. Thus M_Z satisfies $M_Z(t) = M_{\mathbf{E}(Z)}(t)\, M_{Z-\mathbf{E}(Z)}(t) \leq \mathbf{E}(\exp(\langle t, \mathbf{E}(Z)\rangle))\ \exp(\frac{1}{2}d^2|t|^2)$.

Conversely, if $M_Z(t) \leq \exp\left(\langle t, \mathbf{E}(Z)\rangle\right) \exp(\frac{1}{2}d^2|t|^2) = M_{\mathbf{E}(Z)}(t) \exp(\frac{1}{2}d^2|t|^2)$, then $Z - \mathbf{E}(Z)$ has moment generating function $M_{Z-\mathbf{E}(Z)} = M_Z M_{-\mathbf{E}(Z)}$ satisfying $M_{Z-\mathbf{E}(Z)}(t) = M_{\mathbf{E}(Z)}(t) \exp(\frac{1}{2}d^2|t|^2) M_{-\mathbf{E}(Z)}(t) \leq \exp(\frac{1}{2}d^2|t|^2)$. Thus $Z - \mathbf{E}(Z)$ is a 0-subgaussian random variable with standard parameter d, and so Z is a noncentral subgaussian random variable with standard parameter d. □

We now argue that the noncentral subgaussian formulation is more natural from a statistical point of view, for the following reasons.

Firstly, the bounding function of Lemma 12 allows us to directly compare such a noncentral subgaussian random variable with a corresponding Normal random variable. Figure 2 illustrates an example of a noncentral subgaussian bounding function and a δ-subgaussian bounding function. It can be seen that this noncentral subgaussian bounding function is a tight bounding function to the moment generating function at 0, and hence captures better the behaviour at 0. Moreover, the noncentral subgaussian bounding function is actually a moment generating function of some Normal random variable.

Secondly, the standard parameter of a noncentral subgaussian random variable is invariant under translation of the random variable, mirroring a fundamental property of standard deviation. By contrast, in Example 1 we show that the standard parameter of a δ-subgaussian random variable is not necessarily invariant under translation.

Example 1. Suppose that $W \sim \mathrm{N}(0, \sigma^2)$ is a Normal random variable with mean 0 and variance σ^2, so has moment generating function $M_W(t) = \exp(\frac{1}{2}\sigma^2 t^2)$. In terms of Definiton 13, it is clear that W is a noncentral subgaussian random variable with mean 0 and standard parameter σ. Similarly, the translated random variable $W + a \sim \mathrm{N}(a, \sigma^2)$ is by definition a noncentral random variable with mean a and standard parameter σ.

In terms of Definition 9, W is a 0-subgaussian random variable with standard parameter σ. If $W+a$ is a δ-subgaussian random variable with the same standard parameter σ, then $M_{W+a}(t) = \exp(\frac{1}{2}\sigma t^2 + at) \leq \exp(\delta + \frac{1}{2}\sigma^2 t^2)$ so $at \leq \delta$ for all t, which is impossible for $a \neq 0$. Thus even though $W + a$ is a Normal random variable with standard deviation σ, it is not a δ-subgaussian random variable with standard parameter σ when $a \neq 0$. □

4.3 Sums of univariate noncentral subgaussian random variables

In this section, we give our main result, Theorem 1, on sums of noncentral subgaussian (equivalently δ-subgaussian) random variables. This a far more general result than previous results [10, 6] on sums of δ-subgaussian random variables, which apply only in restricted settings. For example, [10, Fact 2.1] applies when the summands are independent, and [6, Claim 2.1] applies in a martingale-like setting.

Theorem 1. Suppose that $W_1, \ldots, W_l$ are noncentral subgaussian, or equivalently δ-subgaussian, random variables where W_j has standard parameter $d_j \geq 0$ for $j = 1, \ldots, l$.
(i) The sum $\sum_{j=1}^{l} W_j$ is a noncentral subgaussian random variable with mean $\sum_{j=1}^{l} \mathbf{E}(W_j)$ and standard parameter $\sum_{j=1}^{l} d_j$.
(ii) If $W_1, \ldots, W_l$ are independent, then the standard parameter of the sum $\sum_{j=1}^{l} W_j$ can be improved to $\left(\sum_{j=1}^{l} d_j^2\right)^{\frac{1}{2}}$. □

Proof. If W_j is a noncentral subgaussian random variable with standard parameter $d_j \geq 0$, then $W_j' = W_j - \mathbf{E}(W_j)$ is a 0-subgaussian random variable with standard parameter d_j. Theorem 2.7 of [15] therefore shows that $\sum_{j=1}^{l} W_j' = \sum_{j=1}^{l} W_j - \sum_{j=1}^{l} \mathbf{E}(W_j)$ is a 0-subgaussian random variable with standard parameter $\sum_{j=1}^{l} d_j$. Thus $\sum_{j=1}^{l} W_j$ is a noncentral subgaussian random variable with mean $\sum_{j=1}^{l} \mathbf{E}(W_j)$ and standard parameter $\sum_{j=1}^{l} d_j$. The second (independence) result similarly follows from the independence result of Theorem 2.7 of [15]. □

5 Discretisation

Discretisation is a fundamental part of Ring-LWE cryptography in which a point is "rounded" to a nearby point in a lattice coset. In fact, such a discretisation process usually involves randomisation, so discretisation typically gives rise to a random variable on the elements of the coset. We consider the coordinate-wise randomised rounding method of discretisation [6, Section 2.4.2] or *CRR-discretisation*, as an illustration of a discretisation process, though most of our comments apply more generally.

We begin by giving a formal definition of CRR-discretisation in terms of a Balanced Reduction function. This allows us to establish general results about the CRR-discretisation of δ-subgaussian random variables. In particular, our main result is Theorem 2, which improves prior results [6] for the δ-subgaussian standard parameter of the CRR-discretisation of a δ-subgaussian random variable.

5.1 Coordinate-wise Randomised Rounding Discretisation

In this section we describe the coordinate-wise randomised rounding discretisation method of the first bullet point of [6, Section 2.4.2], which we term CRR-discretisation. We first introduce the Balanced Reduction function in Definition 14, and give its basic properties in Lemma 13.

Definition 14. The univariate *Balanced Reduction* function $\mathcal{R}$ on $\mathbb{R}$ is the random function with support on $[-1, 1]$ given by

$$\mathcal{R}(a) = \begin{cases} 1 - (\lceil a \rceil - a) & \text{with probability} \quad \lceil a \rceil - a \\ -(\lceil a \rceil - a) & \text{with probability} \quad 1 - (\lceil a \rceil - a). \end{cases}$$

The multivariate *Balanced Reduction* function $\mathcal{R}$ on $\mathbb{R}^l$ with support on $[-1, 1]^l$ is the random function $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_l)$ with component functions $\mathcal{R}_1, \ldots, \mathcal{R}_l$ that are independent univariate Balanced Reduction functions. □

Lemma 13. The random variable $\mathcal{R}(a) + (\lceil a \rceil - a) \sim \text{Bern}(\lceil a \rceil - a)$ has a Bernoulli distribution for any $a \in \mathbb{R}$, and the random variable $\mathcal{R}(a)$ satisifies (i) $\mathbf{E}(\mathcal{R}(a)) = 0$, (iii) $\text{Var}(\mathcal{R}(a)) \leq \frac{1}{4}$ and (iii) $a - \mathcal{R}(a) \in \{\lfloor a \rfloor, \lceil a \rceil\} \subset \mathbb{Z}$. □

We are now in a position to define CRR-discretisation in terms of the Balanced Reduction function.

Definition 15. Suppose B is a (column) basis matrix for the n-dimensional lattice Λ in H. If $\mathcal{R}$ is the Balanced Reduction function, then the *coordinate-wise randomised rounding discretisation* or *CRR-discretisation* $\lfloor X \rceil^B_{\Lambda+c}$ of the random variable X to the lattice coset $\Lambda + c$ with respect to the basis matrix B is the random variable

$$\lfloor X \rceil^B_{\Lambda+c} = X + B\,\mathcal{R}\left(B^{-1}(c - X)\right). \qquad \square$$

In Lemma 14 we show that the specification of coordinate-wise randomised rounding in Definition 15 is well-defined.

Lemma 14. The CRR-discretisation $\lfloor X \rceil^B_{\Lambda+c}$ of the random variable X with resect to the (column) basis B is (i) a random variable on the lattice coset $\Lambda + c$, (ii) is *valid* (does not depend on the chosen coset representative c) and (iii) has mean $\mathbf{E}(\lfloor X \rceil^B_{\Lambda+c}) = \mathbf{E}(X)$. □

Proof. For part (i), the CRR-discretisation can be expressed as

$$\begin{aligned}\lfloor X \rceil^B_{\Lambda+c} &= X + B\mathcal{R}\left(B^{-1}(c-X)\right) = B\left(B^{-1}X + \mathcal{R}\left(B^{-1}(c-X)\right)\right)\\ &= c - B\left(B^{-1}(c-X) - \mathcal{R}\left(B^{-1}(c-X)\right)\right)\\ &\in \Lambda + c,\end{aligned}$$

as Lemma 13(iii) shows that $B^{-1}(c-X) - \mathcal{R}\left(B^{-1}(c-X)\right)$ is a random variable on $\mathbb{Z}^n$. For part (ii), if $c' \in \Lambda + c$, so $c - c' \in \Lambda$, then there exists an integer vector z such that $c - c' = Bz$, so $B^{-1}(c-X) - B^{-1}(c'-X) = z$, that is to say $B^{-1}(c-X)$ and $B^{-1}(c'-X)$ differ by an integer vector. Thus $\mathcal{R}\left(B^{-1}(c-X)\right)$ and $\mathcal{R}\left(B^{-1}(c'-X)\right)$ have identical distributions. The distribution of $\lfloor X \rceil^B_{\Lambda+c}$ on the lattice coset $\Lambda + c$ does not therefore depend on the chosen coset representative c, and so the discretisation is *valid*. Finally, for part (iii), Lemma 13(i) shows that $\mathbf{E}(\lfloor X \rceil^B_{\Lambda+c}) = \mathbf{E}(X) + B\mathbf{E}\left(\mathcal{R}\left(B^{-1}(c-X)\right)\right) = \mathbf{E}(X)$. □

5.2 The CRR-Discretisation of δ-Subgaussian Random Variables

In this section we examine the subgaussian properties of the CRR-discretisation of a noncentral subgaussian random variable. Our main result is Theorem 2, which gives a subgaussian standard parameter for such a CRR-discretisation arising in Ring-LWE, that is to say discretisation for a lattice in H. Theorem 2 uses a factor of $\frac{1}{2}$ with the standard parameter of a random variable obtained by such a CRR-discretisation. By contrast, any comparable result in [6] uses a factor of 1 (see for example the first bullet point of [6, Section 2.4.2]). Thus the results of this Section improve and extend any comparable result in [6] about a CRR-discretisation of a δ-subgaussian random variable.

We first give in Lemma 15 the subgaussian property of the (multivariate) Balanced Reduction function.

Lemma 15. The (multivariate) Balanced Reduction $\mathcal{R}(v)$ (Definition 14) is a 0-subgaussian random variable with standard parameter $\frac{1}{2}$ for all $v \in \mathbb{R}^l$. □

Proof. We first consider the univariate random variable $R_j = \mathcal{R}(p)$ given by the Balanced Reduction of the constant p, where $0 \leq p \leq 1$ without loss of generality. Thus R_j takes the value p with probability $1-p$ and the value $p-1$ with probability p, so has moment generating function

$$M_{R_j}(t) = \mathbf{E}(\exp(tR_j)) = (1-p)\exp(pt) + p\exp((p-1)t) = \exp(pt)h(t),$$

where $h(t) = (1-p) + p\exp(-t)$. We consider the logarithm of the moment generating function given by the function

$$g(t) = \log M_{R_j}(t) = pt + \log h(t).$$

The first three derivatives of g are given by

$$g'(t) = \frac{p(1-p)(1-\exp(-t))}{h(t)}, \qquad g''(t) = \frac{p(1-p)\exp(-t)}{h(t)^2}$$
$$\text{and } g'''(t) = \frac{-p(1-p)\exp(-t)\,((1-p)-p\exp(-t))}{h(t)^3}.$$

We see that $g''(t) \geq 0$ and that solving $g'''(t) = 0$ shows that the maximum of g'' occurs at $t_0 = \log\left(\frac{p}{1-p}\right)$ with a maximum value of $g''(t_0) = \frac{1}{4}$, so $0 \leq g''(t) \leq \frac{1}{4}$ for all $t \in \mathbb{R}$, and we also note that $g(0) = g'(0) = 0$. The Lagrange remainder form of Taylor's Theorem shows that there exists ξ between 0 and t such that $g(t) = \frac{1}{2}g''(\xi)t^2$, so $0 \leq g(t) \leq \frac{1}{8}t^2$. Thus $M_{R_j}(t) = \exp(g(t)) \leq \exp(\frac{1}{2}(\frac{1}{2})^2t^2)$, so R_j is a 0-subgaussian random variable with standard parameter $\frac{1}{2}$.

We now consider the multivariate random variable $R = (R_1, \ldots, R_l)^T$ given by the Balanced Reduction of a vector, which has moment generating function M_R satisfying

$$\begin{aligned} M_R(t) &= \mathbf{E}(\exp(t^T R)) = \mathbf{E}\left(\exp\left(\textstyle\sum_{j=1}^l t_j R_j\right)\right) = \mathbf{E}\left(\prod_{j=1}^l \exp(t_j R_j)\right) \\ &= \prod_{j=1}^{l} \mathbf{E}(\exp(t_j R_j)) = \prod_{j=1}^{l} M_{R_j}(t_j) \\ &\leq \prod_{j=1}^{l} \exp(\tfrac{1}{2}(\tfrac{1}{2})^2 t_j^2) = \exp\left(\tfrac{1}{2}(\tfrac{1}{2})^2 \textstyle\sum_{j=1}^l t_j^2\right) = \exp(\tfrac{1}{2}(\tfrac{1}{2})^2|t|^2). \end{aligned}$$

Thus R is a 0-subgaussian random variable with standard parameter $\frac{1}{2}$. □

We now give in Theorem 2 a subgaussian standard parameter for a CRR-discretisation. The details of the CRR-discretisation depend on the lattice basis used, and in particular on the spectral norm of a lattice basis matrix.

Theorem 2. Suppose that B is a (column) basis matrix for a lattice Λ in H with spectral norm $\lambda(B)$. If Z is a noncentral subgaussian random variable with standard parameter b, then its CRR-discretisation $\lfloor Z \rceil_{\Lambda+c}^B$ is a noncentral subgaussian random variable with mean $\mathbf{E}(Z)$ and standard parameter $\left(b^2 + (\frac{1}{2}\lambda(B))^2\right)^{\frac{1}{2}}$. □

Proof. Lemma 14(iii) shows that $\lfloor Z \rceil_{\Lambda+c}^B = Z + B\mathcal{R}(B^{-1}(c-Z))$ has mean $\mathbf{E}(Z)$. For $v \in H$, Lemma 15 allows us to bound the relevant conditional expectation

as

$$\begin{aligned}
\mathbf{E}\left(\exp\left(v^\dagger \lfloor Z \rceil^B_{\Lambda+c}\right)\middle|\, Z = z\right) &= \mathbf{E}\left(\exp\left(v^\dagger\left(z + B\mathcal{R}(B^{-1}(c-z))\right)\right)\right) \\
&= \exp(v^\dagger z)\ \mathbf{E}\left(\exp\left(v^\dagger B\mathcal{R}(B^{-1}(c-z))\right)\right) \\
&= \exp(v^\dagger z)\ \mathbf{E}\left(\exp\left((B^\dagger v)^\dagger \mathcal{R}(B^{-1}(c-z))\right)\right) \\
&= \exp(v^\dagger z)\ M_{\mathcal{R}(B^{-1}(c-z))}\left(B^\dagger v\right) \\
&\leq \exp(v^\dagger z)\ \exp\left(\tfrac{1}{2}(\tfrac{1}{2})^2 |B^\dagger v|^2\right) \\
&\leq \exp(v^\dagger z)\ \exp\left(\tfrac{1}{2}(\tfrac{1}{2}\lambda(B))^2 |v|^2\right),
\end{aligned}$$

so the corresponding conditional expectation random variable is bounded as

$$\mathbf{E}\left(\exp\left(v^\dagger \lfloor Z \rceil^B_{\Lambda+c}\right)\middle|\, Z\right) \leq \exp(v^\dagger Z)\ \exp\left(\tfrac{1}{2}(\tfrac{1}{2}\lambda(B))^2 |v|^2\right).$$

Thus the Law of Total Expectation shows that the moment generating function $M_{\lfloor Z \rceil^B_{\Lambda+c}}$ of the discretisation $\lfloor Z \rceil^B_{\Lambda+c}$ is bounded by

$$\begin{aligned}
M_{\lfloor Z \rceil^B_{\Lambda+c}}(v) &= \mathbf{E}\left(\exp\left(v^\dagger \lfloor Z \rceil^B_{\Lambda+c}\right)\right) = \mathbf{E}\left(\mathbf{E}\left(\exp\left(v^\dagger \lfloor Z \rceil^B_{\Lambda+c}\right)\middle|\, Z\right)\right) \\
&\leq \exp\left(\tfrac{1}{2}(\tfrac{1}{2}\lambda(B))^2 |v|^2\right)\ \mathbf{E}\left(\exp(v^\dagger Z)\right) \\
&= \exp\left(\tfrac{1}{2}(\tfrac{1}{2}\lambda(B))^2 |v|^2\right) M_Z(v) \\
&\leq \exp\left(\tfrac{1}{2}(\tfrac{1}{2}\lambda(B))^2 |v|^2\right) \exp(v^\dagger \mathbf{E}(Z)) \exp(\tfrac{1}{2} b^2 |v|^2) \\
&\leq \exp(v^\dagger \mathbf{E}(Z)) \exp\left(\tfrac{1}{2}\left(b^2 + (\tfrac{1}{2}\lambda(B))^2\right)\right)
\end{aligned}$$

as Z is a noncentral subgaussian random variable with standard parameter b. Thus its discretisation $\lfloor Z \rceil^B_{\Lambda+c}$ is a noncentral subgaussian random variable with standard parameter $\left(b^2 + (\frac{1}{2}\lambda(B))^2\right)^{\frac{1}{2}}$. □

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