

Construction of Lightweight MDS Matrices from Generalized Feistel Structures

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Abstract. This paper investigates the construction of lightweight MDS matrices with generalized Feistel structures (GFS). The approach developed by this paper consists in deriving MDS matrices from the product of several sparser ones. This can be seen as a generalization to several matrices of the recursive construction which derives MDS matrices as the powers of a single companion matrix.

The first part of this paper gives some theoretical results on the iteration of GFS. In second part, using GFS and primitive matrices, we propose some types of sparse matrices that are called extended primitive GFS (EGFS) matrices. Then, by applying binary linear functions to several round of EGFS matrices, lightweight 4×4 , 6×6 and 8×8 MDS matrices are proposed which are implemented with 67, 156 and 260 XOR for 8-bit input, respectively. The results match the best known lightweight 4×4 MDS matrix and improve the best known 6×6 and 8×8 MDS matrices. Also, the proposed 6×6 MDS matrix is implemented with 114 XOR for 6-bit input.

Moreover, we propose an 6×6 MDS matrix such that the implementation cost of the proposed matrix is 90 XOR for 4-bit input. Furthermore, we determine the implementation cost of the inverses of the proposed matrices, since none of them are involutions. Generally, the construction presented in this paper is relatively general and can be applied for other matrix dimensions and finite fields as well.

Keywords: Lightweight cryptography · MDS matrix · Generalized Feistel Structures

1 Introduction

There are several approaches to construct MDS matrices which can be applied as diffusion layers for block ciphers and hash functions. The first method is based on the algebraic structures such as Cauchy matrix [You97, Gup13]. The next efficient method to be used in construction of MDS matrices is based on the recursive matrices [Guo11, Saj12, Aug14, Ber13]. The first two approaches are called local optimization, since these methods focus on the implementation cost of entries of an MDS matrix [Sar16, Sim15].

Recently, two new approaches are proposed that are called global optimization [Kra17, Bei16, Duv18]. The proposed method in [Kra17] is an application of heuristics algorithms to obtain a suitable implementation of previously known MDS matrices. The introduced technique in [Bei16, Duv18] is a type of search over formal matrices independently of binary linear functions (\mathbf{L}) and to instantiate \mathbf{L} later. In fact, the work of [Bei16, Duv18] is not an application of heuristics algorithms to obtain an efficient implementation of previously known MDS matrices, but a search for new lightweight MDS matrices starting from the implementation.

The space of matrix explored in this paper is a subspace of the space explored in [Duv18]. Although, the presented technique in this paper is exactly the same as [Duv18], the class of construction considered is different. In fact, the advantage of this work compared to [Duv18] is that the smaller search space can be used with larger dimensions.

In recursive or LFSR-based approach, we consider an $n \times n$ companion matrix \mathbf{A} such that the entries of the last row of \mathbf{A} are non-zero elements over a field \mathbb{F} . Then we check whether the n th power of \mathbf{A} , denoted with \mathbf{A}^n , is an MDS matrix over \mathbb{F} . The limitation to the recursive approach is that all entries of the last row of \mathbf{A} must be non-zero. In other words, the number of non-zero entries of \mathbf{A} must be at least $2n - 1$. Applying sparse matrices based on the Feistel structures is one of the best solutions for the mentioned limitation [Shi11, Wu12]. Actually, by applying Feistel structures, we can construct an $n \times n$ sparse matrix \mathbf{B} so that \mathbf{B}^n is an MDS matrix and also the number of non-zero entries of \mathbf{B} is less than $2n - 1$. The next limitation of construction an $n \times n$ MDS matrix based on the recursive approach or Feistel structures is that the n th power of a matrix is used as an MDS matrix. In fact, it is not possible to select an $n \times n$ companion matrix \mathbf{A} such that all entries of \mathbf{A}^k are non-zero provided that $k < n$. In addition, all known lightweight $n \times n$ MDS matrices which are derived from the Feistel structure are constructed from the n -th power of $n \times n$ sparse matrices [Toh18].

It is stated in [Duv18] that the XOR count may not be the best metric to estimate the true hardware cost of an implementation and because of this we consider the depth of the circuit of proposed MDS matrices. In other words, depth corresponds to time which implies that the lower depth results in faster operation. Therefore, we propose MDS matrices that are not only with low implementation cost, but also their circuits have low depth by terminology of the maximum number of gates in each path from all source to the sink [Duv18]. On the other hand, for software implementation, the depth factor is not considered and hence we propose MDS matrices with low implementation cost such that the proposed matrices are constructed from minimum number of binary linear functions.

Contribution of this work This paper follows a list of recent papers to design new MDS matrices with low implementation costs. Concerning the standard XOR count metric, it yields several new matrices having lower XOR cost than previous results. While for 4×4 matrices, the results match the best known lightweight 4×4 MDS matrix [Duv18] the results for 6×6 are slightly better and for 8×8 are substantially better. Compared with the recursive constructions, the matrices considered in this paper are generally sparse.

The structures of proposed lightweight MDS matrices are based on GFS. In addition, the proposed matrices are not only constructed from GFS, but also should be primitive matrices over \mathbb{R} , the field of real numbers. Moreover, binary linear functions are used in implementation of proposed matrices. First of all, we propose an 6×6 MDS matrix which is implemented with 156 XOR for 8-bit input such that the proposed matrix is constructed from six binary linear functions. Also, the proposed 6×6 MDS matrix is implemented with 114 XOR for 6-bit input. In addition, an 6×6 MDS matrix for 4-bit input is proposed such that its implementation cost is 90 XOR.

The next new result is the construction of three 8×8 lightweight MDS matrices for 8-bit input. The first 8×8 MDS matrix is implemented with 272 XOR such that the number of binary linear functions that are applied in construction of the proposed matrix is 16. The first proposed 8×8 matrix can be used in software implementation. The implementation cost of the second 8×8 MDS matrix is 260 XOR such that the depth of its circuit is 8. The second proposed 8×8 matrix can be applied in hardware implementation. The last proposed 8×8 MDS matrix is implemented with 264 XOR and also its depth is 11. The advantage of using the third 8×8 matrix than the second one is that the inverse of the third 8×8 MDS matrix is implemented with 264 XOR.

In general, using binary linear functions (\mathbf{L}) and applying GFS, we select $n \times n$ sparse matrices \mathbf{A}_i with $1 \leq i \leq k$ such that \mathbf{A}_i 's satisfy the following conditions. First \mathbf{A}_i 's have the same structure with respect to the location of their zero entries. Second, \mathbf{A}_i 's are primitive matrices over \mathbb{R} . Then, \mathbf{A}_i 's are non-singular matrices over $\mathbb{F}_2[\mathbf{L}]$. Moreover, the multiplication of \mathbf{A}_i 's is an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$. The last and the main condition is that \mathbf{A}_i 's can be implemented with the low implementation cost by terminology of

XOR counts. In the other words, our approach is equal to decomposition an MDS matrix into sparse matrices provided that these sparse matrices, first have the same and simple structure and then are non-singular over the ground field and finally can be implemented with the minimal XOR cost. Table 1 is provided for comparison our results with the best known results.

Outline of this paper The rest of paper is organized as follows. Definitions and notations are given in Section 2. In Section 3, it is motivated why primitive matrices are used in this paper. Definition of primitive GFS matrices is provided in Section 4. Moreover, by using primitive GFS matrices a probabilistic algorithm for construction lightweight MDS matrices are proposed in Section 4. In Section 5, by applying binary linear functions over primitive GFS matrices, 4×4 and 8×8 lightweight MDS matrices are proposed. Moreover in Section 5, the proposed MDS matrices and their inverse are implemented with 68 and 264 XOR. An extension of primitive GFS matrices, called EGFS matrices, is given in Section 6. Furthermore, the best result of this paper by applying binary linear functions over EGFS matrices are provided in Section 6. Finally, a conclusion is given in Section 7.

Table 1: Comparison our results with the best known results.

Type	Input Size	XOR _{1-bit}	Method	Depth	Reference
4×4 Matrices					
$\mathbb{F}_2[\mathbf{L}]$	4-bit	37	GFS St.	4	[Duv18]
\mathbb{F}_{2^4}	4-bit	36	Heuristics Al.	—	[Kra17]
$\mathbb{F}_{2^4}/0x19$	4-bit	36	GFS St.	6	Subsection 5.1
\mathbb{F}_{2^4}	4-bit	35	GFS St.	5	[Duv18]
\mathbb{F}_{2^4}	4-bit	35	GFS St.	5	Subsection 6.1
\mathbb{F}_{2^8}	8-bit	72	Subfield	—	[Kra17]
\mathbb{F}_{2^8}	8-bit	70	GFS St.	5	Subsection 6.1
$\mathbb{F}_2[\mathbf{L}]$	8-bit	69	GFS St.	4	[Duv18]
$\mathbb{F}_2[\mathbf{L}]$	8-bit	68	GFS St.	6	Subsection 5.1
$\mathbb{F}_2[\mathbf{L}]$	8-bit	67	GFS St.	5	[Duv18]
$\mathbb{F}_2[\mathbf{L}]$	8-bit	67	GFS St.	5	Subsection 6.1
6×6 Matrices					
$\mathbb{F}_2[\mathbf{L}]$	4-bit	150	COM St.	—	[Wu12]
$\mathbb{F}_{2^4}/0x13$	4-bit	90	GFS St.	9	Subsection 6.2
$\mathbb{F}_{2^6}/0x43$	6-bit	114	GFS St.	8	Subsection 6.2
$\mathbb{F}_{2^8}/0x1C3$	8-bit	186	DSI St.	—	[Toh18]
\mathbb{F}_{2^8}	8-bit	156	GFS St.	8	Subsection 6.2
8×8 Matrices					
\mathbb{F}_{2^8}	8-bit	392	Subfield	—	[Kra17]
$\mathbb{F}_{2^8}/0x1E7$	8-bit	272	GFS St.	9	Subsection 6.3
$\mathbb{F}_{2^8}/0x1A3$	8-bit	264	GFS St.	11	Subsection 5.2
$\mathbb{F}_{2^8}/0x187$	8-bit	260	GFS St.	8	Subsection 6.3

2 Definitions and Notations

Let \mathbf{A} be an $n \times n$ matrix over a field \mathbb{F}_q , the finite field with q elements. \mathbf{A} is called MDS over \mathbb{F}_q if all square submatrix of \mathbf{A} is nonsingular over \mathbb{F}_q [Bla99]. Moreover, a finite field with characteristic 2 is denoted with \mathbb{F}_{2^q} for some q . Furthermore, we present a finite field \mathbb{F}_{2^q} by hexadecimal representation. For instance, $\mathbb{F}_{2^8}/0x18D$ is the finite field \mathbb{F}_{2^8} which is

constructed from the primitive polynomial $\mathbf{f} = x^8 + x^7 + x^3 + x^2 + 1$. For simplicity, we use non-zero positions in each row of a binary matrix as a representation of the matrix.

As an example, $[[1, 2, 4], [1, 3], [2, 4], [3, 4]]$ is applied for $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

We denote a matrix with $(a_{i,j})$ where $a_{i,j}$ is the (i, j) th entry of the matrix. Consider an $n \times n$ matrix $\mathbf{A} = (a_{i,j})$ with $1 \leq i, j \leq n$ over \mathbb{R} , the field of real numbers. Then \mathbf{A} is called a positive matrix over \mathbb{R} provided that $a_{i,j} > 0$ for all $1 \leq i, j \leq n$.

Definition 1 ([Hor13]). Consider an $n \times n$ non-negative matrix \mathbf{A} over \mathbb{R} . The matrix \mathbf{A} is called a primitive matrix over \mathbb{R} if \mathbf{A}^k is a positive matrix, denoted $\mathbf{A}^k > 0$, for some integer $k \geq 1$. The primitive order \mathbf{A} is the minimum number k which satisfies $\mathbf{A}^k > 0$ and the matrix \mathbf{A} is called an k -primitive matrix over \mathbb{R} .

Moreover, an $n \times n$ non-negative matrix \mathbf{A} is called a non-primitive matrix over \mathbb{R} if there is no an integer number $k \geq 1$ such that $\mathbf{A}^k > 0$.

In this paper, the symbol $\mathbb{F}_2[\mathbf{L}]$ is considered as a set of all finite polynomials in the following form $\sum_{i=1}^n b_i \mathbf{L}^{t_i}$ where $b_i \in \mathbb{F}_2$ and t_i 's are integer numbers. Consider an $n \times n$ matrix \mathbf{A} over $\mathbb{F}_2[\mathbf{L}]$. Then $\mathbf{A} = (a_{i,j})$ is called a positive matrix over $\mathbb{F}_2[\mathbf{L}]$ if $a_{i,j} \neq 0$ for all $1 \leq i, j \leq n$. Moreover, \mathbf{A} is called an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$ if determinant of all square submatrices of \mathbf{A} are non-zero over $\mathbb{F}_2[\mathbf{L}]$.

Assume that \mathbf{r} is a set over $\mathbb{F}_2[\mathbf{L}]$. Then the set of all prime factors of \mathbf{r} is called the base set of \mathbf{r} . For instance, consider the set $\mathbf{r} = \{\mathbf{L}, \mathbf{L}^2, \mathbf{L} + 1, \mathbf{L}^2 + 1, (\mathbf{L}^2 + \mathbf{L} + 1)^2, \mathbf{L}^6 + \mathbf{L}^2 + 1\}$. Then the base set of \mathbf{r} is $\tilde{\mathbf{r}} = \{\mathbf{L}, \mathbf{L} + 1, \mathbf{L}^2 + \mathbf{L} + 1, \mathbf{L}^3 + \mathbf{L} + 1\}$, since we have $\mathbf{L}^2 + 1 = (\mathbf{L} + 1)^2$ and $\mathbf{L}^6 + \mathbf{L}^2 + 1 = (\mathbf{L}^3 + \mathbf{L} + 1)^2$ over $\mathbb{F}_2[\mathbf{L}]$. Now consider an $n \times n$ non-singular matrix \mathbf{A} over \mathbb{F}_2 . If \mathbf{A} , $\mathbf{A} + \mathbf{I}_n$, $\mathbf{A}^2 + \mathbf{A} + \mathbf{I}_n$ and $\mathbf{A}^3 + \mathbf{A} + \mathbf{I}_n$ are non-singular matrices over \mathbb{F}_2 , then we say the elements of $\tilde{\mathbf{r}}$ are non-singular matrices over \mathbb{F}_2 by applying \mathbf{A} . Notice that if by using an $n \times n$ matrix \mathbf{A} the elements of $\tilde{\mathbf{r}}$ are non-singular matrices over \mathbb{F}_2 , then it can be verified that the elements of \mathbf{r} are non-singular matrices over \mathbb{F}_2 using \mathbf{A} .

3 Relation between Primitive Matrices and Search Space

In this section, the role of primitive matrices for construction of proposed lightweight MDS matrices are explained. Let \mathbf{A} be an $n \times n$ non-negative matrix over \mathbb{R} . It is easy to see that if \mathbf{A} is a non-primitive matrix over \mathbb{R} then \mathbf{A} can not be a primitive matrix over $\mathbb{F}_2[\mathbf{L}]$. Next, we show that if \mathbf{A} is an k -primitive matrix over \mathbb{R} , then \mathbf{A} can be a non-primitive over $\mathbb{F}_2[\mathbf{L}]$ or \mathbf{A} be an k' -primitive matrix over $\mathbb{F}_2[\mathbf{L}]$ such that $k' \geq k$. In other words, the characteristic 2 in $\mathbb{F}_2[\mathbf{L}]$, puts limitation on order of primitive matrices.

Example 1. Consider the following three matrices

$$\mathbf{A}_1 = \begin{pmatrix} \mathbf{L} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{L} & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} \mathbf{L} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} \mathbf{L} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{L} & 0 & 0 & 0 \end{pmatrix} \quad (1)$$

Assume that \mathbf{L} is a positive integer in \mathbb{R} . Then it can be checked that \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_3 are 4-primitive matrices over \mathbb{R} . For instance, \mathbf{A}_1^4 over \mathbb{R} is in the following form.

$$\mathbf{A}_1^4 = \begin{pmatrix} \mathbf{L}^4 + 1 & \mathbf{L}^3 & 3\mathbf{L}^2 & 2\mathbf{L} \\ 2\mathbf{L} & 1 & \mathbf{L}^3 & \mathbf{L}^2 \\ 3\mathbf{L}^2 & 2\mathbf{L} & \mathbf{L}^4 + 1 & \mathbf{L}^3 \\ \mathbf{L}^3 & \mathbf{L}^2 & 2\mathbf{L} & 1 \end{pmatrix}.$$

In addition, there is no integer number $1 \leq k \leq 3$ such that \mathbf{A}_1^k be a positive matrix over \mathbb{R} . Now consider \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_3 over $\mathbb{F}_2[\mathbf{L}]$.

First of all, we prove \mathbf{A}_1 is a non-primitive matrix over $\mathbb{F}_2[\mathbf{L}]$. The characteristic polynomial of \mathbf{A}_1 over $\mathbb{F}_2[\mathbf{L}]$ is $x^4 + \mathbf{L}^2x^2 + 1$. Consider the equation $x^k = (x^4 + \mathbf{L}^2x^2 + 1)h(x) + r(x)$ where $r(x)$ is a polynomial of degree less than 4 over $\mathbb{F}_2[\mathbf{L}]$. Therefore, we get $\mathbf{A}_1^k = r(\mathbf{A}_1)$, since \mathbf{A}_1 satisfies its own characteristic equation [Hor13]. It can be verified that $r(x) = a_1 + a_2x^2$ when k is an even number and $r(x) = b_1x + b_2x^3$ when k is an odd number where a_i and b_i with $1 \leq i \leq 2$, are in $\mathbb{F}_2[\mathbf{L}]$. Hence, \mathbf{A}_1^k is a linear combination of $(\mathbf{I}_4$ and $\mathbf{A}_1^2)$ or $(\mathbf{A}_1$ and $\mathbf{A}_1^3)$ where \mathbf{I}_4 is the identity matrix of order 4. Moreover, \mathbf{A}_1 , \mathbf{A}_1^2 and \mathbf{A}_1^3 are not positive matrices over $\mathbb{F}_2[\mathbf{L}]$. Furthermore, it can be checked that $(\mathbf{I}_4$ and $\mathbf{A}_1^2)$ and $(\mathbf{A}_1$ and $\mathbf{A}_1^3)$ have zero entries in the same positions. Therefore, for any positive integer k , \mathbf{A}_1^k has at least one zero entry which results in \mathbf{A}_1 is a non-primitive matrix over $\mathbb{F}_2[\mathbf{L}]$. In addition, \mathbf{A}_2 and \mathbf{A}_3 are 7-primitive and 4-primitive matrices over $\mathbb{F}_2[\mathbf{L}]$.

Definition 2. Suppose that $\mathbf{A} = (a_{i,j})$ and $\mathbf{B} = (b_{i,j})$ with $1 \leq i, j \leq n$ are two $n \times n$ sparse matrices over \mathbb{R} . \mathbf{A} and \mathbf{B} are called with the same structure provided that if $a_{i,j} = 0$ then $b_{i,j} = 0$ and vice versa.

The 4×4 sparse matrices \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_3 , given in Example 1, are with the same structure. In the rest, by applying Definition 2 to primitive matrices, a method to construct lightweight MDS matrices are proposed.

Assume that \mathbf{A}_i with $1 \leq i \leq k$ are $n \times n$ sparse matrices over \mathbb{R} . Now based on the primitivity and structures of \mathbf{A}_i 's the following five cases are considered.

Case 1: \mathbf{A}_i 's are non-primitive matrices over \mathbb{R} and are with the same structure.

Case 2: \mathbf{A}_i 's are non-primitive matrices over \mathbb{R} and are not with the same structure.

Case 3: \mathbf{A}_i 's are primitive matrices over \mathbb{R} and are with the same structure.

Case 4: \mathbf{A}_i 's are primitive matrices over \mathbb{R} and are not with the same structure.

Case 5: Let $\mathbf{I} = \{j_1, j_2, \dots, j_m\}$ with $1 \leq m < k$ be a subset of the set $\{1, 2, \dots, k\}$. Consider \mathbf{A}_{j_t} with $1 \leq t \leq m$ are primitive matrices over \mathbb{R} and \mathbf{A}_i for $i \notin \mathbf{I}$ are not primitive matrices over \mathbb{R} (the structures of \mathbf{A}_i 's can be chosen arbitrarily).

By considering the given five cases, let \mathbf{B} be the multiplication of \mathbf{A}_i 's, denoted with $\mathbf{B} = \prod_{i=1}^k \mathbf{A}_i$. Without loss of generality we may assume that all nonzero entries of \mathbf{A}_i 's are equal to 1. By considering the case 1, the matrix \mathbf{B} is not a primitive matrix over $\mathbb{F}_2[\mathbf{L}]$, since it can be checked that the primitivity of \mathbf{B} implies that \mathbf{A}_i 's are primitive matrices which is in contradiction to assumptions of the case 1. Moreover, \mathbf{B} is possibly a primitive matrix over $\mathbb{F}_2[\mathbf{L}]$ by considering the assumptions of cases 2,3,4 and 5 and hence the matrix \mathbf{B} possibly can be an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$. Therefore, to construct MDS matrices over $\mathbb{F}_2[\mathbf{L}]$, by applying search on sparse matrices, cases 2,3,4 and 5 can be used.

Performing a search using the third case has less complexity than cases 2,4 and 5, since \mathbf{A}_i 's are with the same structure. In fact, firstly we obtain an $n \times n$ k -primitive matrix \mathbf{C} over \mathbb{R} . Then we choose matrices \mathbf{A}_i with $1 \leq i \leq k'$ over $\mathbb{F}_2[\mathbf{L}]$ provided that $k' \geq k$ and \mathbf{A}_i 's are with the same structure as \mathbf{C} . Next, we check whether $\mathbf{B} = \prod_{i=1}^{k'} \mathbf{A}_i$ is an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$. Therefore, the main reason for using primitive matrices with the same structure is the issue of reducing the search space.

Example 2. Consider the following 4×4 sparse matrices over $\mathbb{F}_2[\mathbf{L}]$.

$$\tilde{\mathbf{A}}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tilde{\mathbf{A}}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & \mathbf{L} & 0 & 0 \end{pmatrix}, \tilde{\mathbf{A}}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{L} \\ 0 & 0 & 0 & 1 \\ \mathbf{L} & \mathbf{L} & 0 & 0 \end{pmatrix}, \tilde{\mathbf{A}}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The matrices $\tilde{\mathbf{A}}_i$ with $1 \leq i \leq 4$ are extracted from Appendix C.2, Figure 8 in [Duv18]. It is shown in [Duv18] the multiplication of $\tilde{\mathbf{A}}_i$'s, denoted with $\tilde{\mathbf{B}} = \prod_{i=1}^4 \tilde{\mathbf{A}}_i$, is an MDS

matrix over $\mathbb{F}_2[\mathbf{L}]$. Moreover, it can be checked that $\tilde{\mathbf{B}}$ is obtained from the Case 5, since $\tilde{\mathbf{A}}_2, \tilde{\mathbf{A}}_3$ and $\tilde{\mathbf{A}}_4$ are primitive matrices over \mathbb{R} and $\tilde{\mathbf{A}}_1$ is a non-primitive matrix over \mathbb{R} .

Consider the matrix \mathbf{A}_1 in Example 1. It is observed that \mathbf{A}_1 is a 4-primitive matrix over \mathbb{R} . Now we select matrices $\hat{\mathbf{A}}_i$ with $1 \leq i \leq 4$ provided that $\hat{\mathbf{A}}_i$'s are with the same structure as \mathbf{A}_1 . In addition, the multiplication of $\hat{\mathbf{A}}_i$'s, denoted with $\hat{\mathbf{B}} = \prod_{i=1}^4 \hat{\mathbf{A}}_i$, be an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$. The following matrices are derived from a simple search.

$$\hat{\mathbf{A}}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \hat{\mathbf{A}}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{L} & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \hat{\mathbf{A}}_3 = \begin{pmatrix} \mathbf{L} & \mathbf{L} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{L} & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \hat{\mathbf{A}}_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, by similar structural properties of $\tilde{\mathbf{A}}_i$ and $\hat{\mathbf{A}}_i$ with $1 \leq i \leq 4$ we conclude that $\tilde{\mathbf{B}}$ and $\hat{\mathbf{B}}$ can be implemented with the same XOR from hardware perspective.

In the next section, according to the technique given in [Duv18] and applying primitive matrices, a probabilistic algorithm for construction of lightweight MDS matrices are proposed. The output of the proposed algorithm are $n \times n$ sparse matrices \mathbf{A}_i with $1 \leq i \leq u$ over $\mathbb{F}_2[\mathbf{L}]$ with the following conditions. First, $u \leq n$ and \mathbf{A}_i 's are with the same structure. Second, \mathbf{A}_i 's are k -primitive matrices over \mathbb{R} such that $k \leq u$. Third, the multiplication of \mathbf{A}_i 's is an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$. The last and foremost condition is that using binary linear functions the implementation cost of \mathbf{A}_i 's is minimal.

4 Primitive GFS Matrices

There are two reasons why GFS is used in this paper. The first and most important reason is this fact the inverse of GFS is easy to compute as well. The second one is that combining primitive matrices and GFS reduces search space.

In this section, by applying GFS, a type of primitive sparse matrix is proposed which is called primitive GFS matrix. In Section 5 by using primitive GFS matrices, 4×4 and 8×8 MDS matrices are proposed such that the implementation cost of these matrices and their inverse are 68 and 264 XOR for 8-bit input, respectively. Consider the following 2×2 block-matrices

$$\mathbf{c}_1^{(m)} = \begin{pmatrix} \mathbf{L} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{c}_2^{(m)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{z}^{(m)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (2)$$

where \mathbf{L} is an $m \times m$ non-singular matrix over \mathbb{F}_2 . In addition, $\mathbf{1}$ and $\mathbf{0}$ are $m \times m$ identity and zero matrices, respectively. In Definition (3), by applying $\mathbf{c}_1^{(m)}$, $\mathbf{c}_2^{(m)}$ and $\mathbf{z}^{(m)}$ we propose primitive GFS matrices.

Definition 3 (Primitive GFS Matrices). Suppose that $\mathbf{p}_1 = \{a_1, a_2, \dots, a_n\}$ and $\mathbf{p}_2 = \{b_1, b_2, \dots, b_n\}$ are two permutations of integer numbers from 1 to n such that $a_i \neq b_i$ for all $1 \leq i \leq n$. Consider 2×2 block-matrices $\mathbf{c}_1^{(m)}$, $\mathbf{c}_2^{(m)}$ and $\mathbf{z}^{(m)}$ that are given in (2).

Suppose that the i th row of an $n \times n$ block-matrix $\mathbf{S} = (s_{i,j})$ with $1 \leq i, j \leq n$, based on the two permutations \mathbf{p}_1 and \mathbf{p}_2 , is filled in the following form.

$$s_{i,j} = \begin{cases} \mathbf{c}_1^{(m)} & j = a_i, \\ \mathbf{c}_2^{(m)} & j = b_i, \\ \mathbf{z}^{(m)} & j \notin \{a_i, b_i\}. \end{cases}$$

The block-matrix \mathbf{S} is called primitive GFS matrix if \mathbf{S} is a primitive matrix over \mathbb{R} .

First of all, notice that in Definition 3 to check whether \mathbf{S} is a primitive matrix over \mathbb{R} , we assume that $\mathbf{c}_1^{(m)}$, $\mathbf{c}_2^{(m)}$ and $\mathbf{z}^{(m)}$ are 2×2 matrices over \mathbb{R} and \mathbf{L} is a positive

integer. Moreover, for simplicity the block-matrix \mathbf{S} is denoted with $\mathbf{S}(n, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2])$. Furthermore, it follows from Definition 3 that a primitive GFS matrix can be a non-primitive matrix over $\mathbb{F}_2[\mathbf{L}]$. In addition, primitive GFS matrices are non-singular over $\mathbb{F}_2[\mathbf{L}]$, since it can be proved that $\det(\mathbf{S}) = 1$ over $\mathbb{F}_2[\mathbf{L}]$. In Example 3, it is observed that why the block-matrix \mathbf{S} in Definition 3 should be a primitive matrix over \mathbb{R} and not ask for \mathbf{S} to be primitive over $\mathbb{F}_2[\mathbf{L}]$.

Example 3. Suppose that in Definition 3, we assumed \mathbf{S} be a primitive matrix over $\mathbb{F}_2[\mathbf{L}]$. Consider the following $n \times n$ block-matrix that is constructed from two permutations $\mathbf{p}_1 = \{1, 2, \dots, n\}$ and $\mathbf{p}_2 = \{2, 3, \dots, n, 1\}$.

$$\mathbf{S}(n, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]) = \left(\begin{array}{cc|cc|cc|cc} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right).$$

For simplicity in representation set $\mathbf{A} = \mathbf{S}(n, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2])$. In the rest, we prove \mathbf{A} is a non-primitive matrix over $\mathbb{F}_2[\mathbf{L}]$. Suppose \mathbf{A} is a primitive matrix over $\mathbb{F}_2[\mathbf{L}]$ which implies that there is a positive integer k such that $\mathbf{A}^k > \mathbf{0}$ over $\mathbb{F}_2[\mathbf{L}]$. Hence, all entries of \mathbf{A}^k are equal to 1. But it is in contradiction to this fact that \mathbf{A} is a non-singular matrix over $\mathbb{F}_2[\mathbf{L}]$ but \mathbf{A}^k is a singular matrix over $\mathbb{F}_2[\mathbf{L}]$, since \mathbf{A}^k has two equal rows. Therefore, in order to use sparse block-matrices such as \mathbf{A} , the block-matrix \mathbf{S} in Definition 3 should be asked to be a primitive matrix over \mathbb{R} .

It follows from Appendix B that the order of a primitive GFS matrix $\mathbf{S}(n, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2])$ depends on two permutations \mathbf{p}_1 and \mathbf{p}_2 . Moreover, the implementation cost of proposed MDS matrices in Section 5, is directly related to the number of iterations required to reach full diffusion. Therefore, for construction of lightweight MDS matrices, the primitive GFS matrices with minimum order should be used. In Example 4, some types of primitive GFS matrices are made, which are used in Section 5 to construct lightweight 4×4 and 8×8 MDS matrices with the implementation cost 68 and 264 XOR for 8-bit input, respectively.

Example 4. For $n = 2$, the following primitive GFS matrix is used.

$$\mathbf{p}_1 = \{1, 2\}, \quad \mathbf{p}_2 = \{2, 1\}, \quad \mathbf{S}(2, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]) = \left(\begin{array}{cc|cc} \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right). \quad (3)$$

In Example 1, it is observed that $\mathbf{S}(2, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2])$ is an 4-primitive matrix over \mathbb{R} .

For $n = 3$, there are 12 permutations such as $\mathbf{p}_1 = \{a_1, a_2, a_3\}$ and $\mathbf{p}_2 = \{b_1, b_2, b_3\}$ from 1 to 3 so that $a_i \neq b_i$ for all $1 \leq i \leq 3$. Consider the following case

$$\mathbf{p}_1 = \{1, 3, 2\}, \quad \mathbf{p}_2 = \{2, 1, 3\}, \quad \mathbf{S}(3, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]) = \left(\begin{array}{ccc|cc} \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array} \right) \quad (4)$$

The primitive GFS matrix $\mathbf{S}(3, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2])$, is an 5-primitive matrix over \mathbb{R} .

For $n = 4$, there are 216 permutations $\mathbf{p}_1 = \{a_1, a_2, a_3, a_4\}$ and $\mathbf{p}_2 = \{b_1, b_2, b_3, b_4\}$ from 1 to 4 such that $a_i \neq b_i$ for $1 \leq i \leq 4$. These 216 permutations are divided into four cases, which are listed in Appendix B. It follows from Appendix B that the minimum order of a primitive GFS matrix such as $\mathbf{S}(4, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2])$ is 6. Therefore, we select two permutations \mathbf{p}_1 and \mathbf{p}_2 provided that $\mathbf{S}(4, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2])$ is an 6-primitive matrix over \mathbb{R} .

$$\mathbf{p}_1 = \{4, 3, 2, 1\}, \quad \mathbf{p}_2 = \{3, 2, 1, 4\}, \quad \mathbf{S}(4, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (5)$$

$$\widehat{\mathbf{p}}_1 = \{1, 3, 2, 4\}, \quad \widehat{\mathbf{p}}_2 = \{2, 1, 4, 3\}, \quad \mathbf{S}(4, \mathbf{L}_{(m)}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2]) = \begin{pmatrix} \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \quad (6)$$

Two primitive GFS matrices such as \mathbf{A}_1 and \mathbf{A}_2 are considered with the same structure if \mathbf{A}_1 and \mathbf{A}_2 are constructed with the same permutations. In the rest, we propose a probabilistic algorithm for construction of lightweight MDS matrices based on the primitive GFS matrices with the same structure. Actually, Algorithm 1 is a randomized algorithm, since with the proposed approach there is an extensive search space to construct lightweight MDS matrices for large dimensions.

Algorithm 1: Construction of Lightweight MDS Matrices based on the Primitive GFS Matrices with the same Structure

Input : Three positive integer n , m and r .

Output : An $2n \times 2n$ lightweight MDS matrix over m -bit input with $\leq r$ XOR.

- 1 Select two permutations \mathbf{p}_1 and \mathbf{p}_2 such that the order of primitive GFS matrix $\mathbf{C} = \mathbf{S}(n, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2])$ is minimal over \mathbb{R} .
 - 2 Let \mathbf{C} be an k -primitive matrix over \mathbb{R} and set $u = k$.
 - 3 Select primitive GFS matrices $\mathbf{A}_i = \mathbf{S}(n, \mathbf{f}_{(m)}^{(i)}, [\mathbf{p}_1, \mathbf{p}_2])$ with $1 \leq i \leq u$ such that $\mathbf{f}^{(i)} \in \{\mathbf{L}^{-2}, \mathbf{L}^{-1}, \mathbf{1}, \mathbf{L}, \mathbf{L}^{-2}\}$ and \mathbf{A}_i 's are constructed of minimal number of \mathbf{L} .
 - 4 Construct $\mathbf{B} = \prod_{i=1}^u \mathbf{A}_i$ over $\mathbb{F}_2[\mathbf{L}]$.
 - 5 If \mathbf{B} is an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$ then go in Step 8 end.
 - 6 If all cases in Step 3 are considered then set $u = u + 1$.
 - 7 If \mathbf{B} is not an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$ then go in Step 3 end.
 - 8 Get the base set of subdeterminants of \mathbf{B} over $\mathbb{F}_2[\mathbf{L}]$.
 - 9 Obtain an $m \times m$ non-singular binary matrix \mathbf{L} with the minimal implementation cost provided that elements of the base set are non-singular matrices over \mathbb{F}_2 by \mathbf{L} .
 - 10 If Step 9 fails to obtain a binary matrix \mathbf{L} then go in Step 3 end.
 - 11 Obtain the implementation cost of \mathbf{A}_i 's, denoted x , with respect to the cost of \mathbf{L} .
 - 12 If $x \leq r$ then go in Step 14 end.
 - 13 If $u \leq 2n$ then go in Step 3 else go in Step 1 end.
 - 14 Return \mathbf{B} , \mathbf{A}_i 's and the non-singular binary matrix \mathbf{L} .
-

Example 5. The best implementation of lightweight 6×6 MDS matrices for 8-bit input is 186 XOR [Toh18]. Set $n = 3$, $m = 8$ and $r = 186$ as input into Algorithm 1. It follows from (4) that $\mathbf{C} = \mathbf{S}(3, \mathbf{1}_{(8)}, [\mathbf{p}_1, \mathbf{p}_2])$ is an 5-primitive matrices over \mathbb{R} for two permutations $\mathbf{p}_1 = \{1, 3, 2\}$ and $\mathbf{p}_2 = \{2, 1, 3\}$. Using exhaustive search, it can be verified that there are no primitive GFS matrices $\mathbf{A}_i = \mathbf{S}(3, \mathbf{f}_{(m)}^{(i)}, [\mathbf{p}_1, \mathbf{p}_2])$ with $1 \leq i \leq 5$ such that $\prod_{i=1}^5 \mathbf{A}_i$ be an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$ provided that $\mathbf{f}^{(i)} \in \{\mathbf{L}^{-2}, \mathbf{L}^{-1}, \mathbf{1}, \mathbf{L}, \mathbf{L}^{-2}\}$. Now consider the following primitive GFS matrices \mathbf{A}_i with $1 \leq i \leq 6$.

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{R}(3, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]), & \mathbf{A}_2 &= \mathbf{R}(3, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]), & \mathbf{A}_3 &= \mathbf{R}(3, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]), \\ \mathbf{A}_4 &= \mathbf{R}(3, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]), & \mathbf{A}_5 &= \mathbf{R}(3, \mathbf{L}_{(m)}^2, [\mathbf{p}_1, \mathbf{p}_2]), & \mathbf{A}_6 &= \mathbf{R}(3, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]). \end{aligned}$$

It can be checked that $\mathbf{B} = \prod_{i=1}^6 \mathbf{A}_i$ is an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$. Moreover, applying exhaustive search, matrix \mathbf{B} is an optimal result with respect to the number of binary linear functions which are used to construct matrix \mathbf{B} (L, L, L, L^2, L^2 and L^2).

$$\mathbf{B} = \begin{pmatrix} \mathbf{L}^3 + 1 & \mathbf{L}^3 + 1 & \mathbf{L}^3 + \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L}^2 + 1 & 1 & \mathbf{L} \\ \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + 1 & \mathbf{L}^2 & \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^3 + 1 & \mathbf{L}^3 \\ 1 & \mathbf{L} & \mathbf{L}^3 + 1 & \mathbf{L}^3 + 1 & \mathbf{L}^3 + \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L}^2 + 1 \\ \mathbf{L}^3 + 1 & \mathbf{L}^3 & \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + 1 & \mathbf{L}^2 & \mathbf{L}^2 + \mathbf{L} \\ \mathbf{L}^3 + \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L}^2 + 1 & 1 & \mathbf{L} & \mathbf{L}^3 + 1 & \mathbf{L}^3 + 1 \\ \mathbf{L}^2 & \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^3 + 1 & \mathbf{L}^3 & \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + 1 \end{pmatrix}.$$

It is easy to verify that the number of subdeterminants of an $n \times n$ matrix is $\sum_{i=1}^n \binom{n}{i} \binom{n}{i} = \binom{2n}{n} - 1$. Therefore, \mathbf{B} has $\binom{12}{6} - 1 = 923$ subdeterminants. The base set of subdeterminants of \mathbf{B} is given in (7).

$$\begin{aligned} & \{L, L + 1, L^2 + L + 1, L^3 + L + 1, L^3 + L^2 + 1, L^4 + L + 1, L^4 + L^3 + 1, L^4 + L^3 + L^2 + L + 1, \\ & L^5 + L^2 + 1, L^5 + L^3 + 1, L^5 + L^3 + L^2 + L + 1, L^5 + L^4 + L^2 + L + 1, L^5 + L^4 + L^3 + L + 1, \\ & L^5 + L^4 + L^3 + L^2 + 1, L^6 + L + 1, L^6 + L^3 + 1, L^6 + L^4 + L^2 + L + 1, L^6 + L^4 + L^3 + L + 1, \\ & L^6 + L^5 + 1, L^6 + L^5 + L^2 + L + 1, L^6 + L^5 + L^3 + L^2 + 1, L^6 + L^5 + L^4 + L + 1, \\ & L^6 + L^5 + L^4 + L^2 + 1, L^7 + L^3 + L^2 + L + 1, L^7 + L^5 + L^3 + L + 1, L^7 + L^6 + 1, \\ & L^7 + L^6 + L^4 + L^2 + 1, L^7 + L^6 + L^5 + L^4 + 1, L^8 + L^6 + L^5 + L^3 + 1, \\ & L^8 + L^7 + L^3 + L + 1, L^8 + L^7 + L^6 + L^3 + L^2 + L + 1\}. \end{aligned} \quad (7)$$

Assume that $\#\mathbf{L}$ is the symbol for XOR cost of \mathbf{L} . Therefore, the implementation cost of \mathbf{B} for m -bit input is computed as follows

$$\underbrace{\#\mathbf{B}_1}_{(3m)} + \underbrace{\#\mathbf{B}_2}_{(3m + 3(\#\mathbf{L}))} + \underbrace{\#\mathbf{B}_3}_{(3m)} + \underbrace{\#\mathbf{B}_4}_{(3m)} + \underbrace{\#\mathbf{B}_5}_{(3m + 3(\#\mathbf{L}^2))} + \underbrace{\#\mathbf{B}_6}_{(3m)} = 18m + 3(\#\mathbf{L}) + 3(\#\mathbf{L}^2). \quad (8)$$

In the rest, we get 8×8 non-singular matrices \mathbf{L} over \mathbb{F}_2 such that the given elements in (7) are non-singular matrices over \mathbb{F}_2 by applying \mathbf{L} . For instance, the following non-singular binary matrices \mathbf{L}_j with $1 \leq j \leq 4$, are obtained.

$$\begin{aligned} \mathbf{L}_1 &= [[1, 8], [1], [2, 7], [3], [4], [5], [6], [7]], & \mathbf{L}_2 &= [[1, 8], [1], [2], [3], [4, 7], [5], [6], [7]], \\ \mathbf{L}_3 &= [[3, 8], [1], [2], [3], [4], [5, 6], [6], [7]], & \mathbf{L}_4 &= [[5, 8], [1], [2], [3], [4], [5], [6, 7], [7]]. \end{aligned} \quad (9)$$

The implementation cost of \mathbf{L}_j 's, given in (9), is two XOR. In addition \mathbf{L}_j^2 with $1 \leq j \leq 4$, can be implemented with four XOR. Therefore, it follows from (8) that the implementation cost of \mathbf{B} is $18 \times 8 + 3 \times 2 + 3 \times 4 = 162$ XOR. Moreover, \mathbf{B}^{-1} is implemented with 162 XOR, since $\mathbf{B}^{-1} = \prod_{i=1}^6 \mathbf{A}_{7-i}^{-1}$ and the implementation cost of \mathbf{A}_{7-i}^{-1} with $1 \leq i \leq 6$ is $18m + 3(\#\mathbf{L}) + 3(\#\mathbf{L}^2)$.

Lemma 1 follows a condition which can be used to reduce the complexity of step 9 in Algorithm 1 when we run Algorithm 1 to construct MDS matrices with large dimensions.

Lemma 1. Let $\mathbf{r} = \{f_1, f_2, \dots, f_k\}$ be a set of pairwise distinct irreducible polynomials over \mathbb{F}_2 for some positive integer k . Suppose that \mathbf{A} is an $n \times n$ binary matrix such that its characteristic polynomial over \mathbb{F}_2 is the polynomial g^m for some positive integer m where g is an irreducible polynomial. If $g \notin \mathbf{r}$ then matrices $f_i(\mathbf{A})$ with $1 \leq i \leq k$ are non-singular matrices over \mathbb{F}_2 .

Proof. First, we prove the matrices $f_i(\mathbf{A})$ with $1 \leq i \leq k$ are nonzero matrices over \mathbb{F}_2 . It follows from assumptions that g is minimal polynomial of \mathbf{A} . Suppose that $f_i(\mathbf{A})$ is equal to zero matrix for some $1 \leq i \leq k$. Then we get a contradiction to our assumptions, since f_i 's are irreducible polynomials over \mathbb{F}_2 such that $f_i \neq g$ for all $1 \leq i \leq k$. Now assume that $f_i(\mathbf{A})$ is equal to a singular matrix for some $1 \leq i \leq k$ which implies that the determinant of $f_i(\mathbf{A})$ is equal to zero. From linear algebra [Hor13] we know that $\det(f_i(\mathbf{A})) = \prod_{j=1}^n f_i(\lambda_j)$ where λ_j 's are eigenvalues of \mathbf{A} . Therefore, there is at least an integer $1 \leq j \leq n$ such that $f_i(\lambda_j) = 0$ and this is in contradiction to this fact that two distinct irreducible polynomials have no common roots. \square

It follows from Lemma 1 that in step 9 in Algorithm 1, it is sufficient to obtain an $m \times m$ non-singular binary matrix such that its minimal polynomial over \mathbb{F}_2 is not an element of the base set of the matrix \mathbf{B} . Moreover, in Lemma 1 the implementation cost of \mathbf{A} can be considered as the implementation cost of a root of polynomial g over \mathbb{F}_2 [Bei16].

5 Construction of Lightweight MDS Matrices by Applying Primitive GFS Matrices

In this section, by applying primitive GFS matrices, we propose lightweight 4×4 and 8×8 MDS matrices for 8 bit input. The proposed 4×4 and 8×8 MDS matrices are implemented with 68 and 264 XOR. Moreover, the given results in this section are optimal according to the terminology of XOR cost and the number of binary linear functions. In other words, it is not possible to achieve better results by using the proposed approach in Section 4. In fact, in order to obtain the results of this section, an exhaustive search has been applied to Algorithm 1. The point of the proposed 4×4 and 8×8 MDS matrices is the inverse of these matrices which can be implemented with 68 and 264 XOR, respectively.

5.1 Lightweight 4×4 MDS Matrices from Primitive GFS Matrices

Consider the primitive GFS matrix $\mathbf{C} = \mathbf{R}(2, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2])$, over two permutations $\mathbf{p}_1 = \{1, 2\}$ and $\mathbf{p}_2 = \{2, 1\}$. It follows from (3) that \mathbf{C} is an 4-primitive matrix over \mathbb{R} . In the rest, based on the structure of matrix \mathbf{C} , the following primitive GFS matrices are used to construct a lightweight 4×4 MDS matrix for m -bit input.

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{R}(2, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]), & \mathbf{A}_2 &= \mathbf{R}(2, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]), \\ \mathbf{A}_3 &= \mathbf{R}(2, \mathbf{L}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]), & \mathbf{A}_4 &= \mathbf{R}(2, \mathbf{1}_{(m)}, [\mathbf{p}_1, \mathbf{p}_2]). \end{aligned}$$

It can be checked that $\mathbf{B} = \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 \mathbf{A}_4$, given in (10), is an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$.

$$\begin{aligned} \mathbf{B} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{L} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{L} & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{L} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{L} & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{L}^2 + 1 & \mathbf{L}^2 & 1 & \mathbf{L} + 1 \\ \mathbf{L} + 1 & 1 & \mathbf{L}^2 & \mathbf{L}^2 \\ 1 & \mathbf{L} + 1 & \mathbf{L}^2 + 1 & \mathbf{L}^2 \\ \mathbf{L}^2 & \mathbf{L}^2 & \mathbf{L} + 1 & 1 \end{pmatrix}. \end{aligned} \quad (10)$$

Then the implementation cost of the matrix \mathbf{B} for m -bit input is equal to

$$\underbrace{\#\mathbf{A}_1}_{(2m)} + \underbrace{\#\mathbf{A}_2}_{(2m + 2(\#\mathbf{L}))} + \underbrace{\#\mathbf{A}_3}_{(2m + 2(\#\mathbf{L}))} + \underbrace{\#\mathbf{A}_4}_{(2m)} = 8m + 4(\#\mathbf{L}). \quad (11)$$

There are $\binom{8}{4} - 1 = 69$ subdeterminants in \mathbf{B} . The base set of these subdeterminants is:

$$\{\mathbf{L}, \mathbf{L} + 1, \mathbf{L}^2 + \mathbf{L} + 1, \mathbf{L}^3 + \mathbf{L} + 1, \mathbf{L}^3 + \mathbf{L}^2 + 1, \mathbf{L}^4 + \mathbf{L} + 1\} \quad (12)$$

For $m = 4$, consider the following 4×4 non-singular matrices over \mathbb{F}_2 . It can be verified that by applying \mathbf{L}_i with $1 \leq i \leq 4$, the given elements in (12) are non-singular matrices over \mathbb{F}_2 . Moreover, the implementation cost of \mathbf{L}_i 's is one XOR.

$$\mathbf{L}_1 = [[1, 4], [1], [2], [3]], \quad \mathbf{L}_2 = [[4], [1, 2], [2], [3]], \quad \mathbf{L}_3 = [[4], [1], [2, 3], [3]], \quad \mathbf{L}_4 = [[4], [1], [2], [3, 4]].$$

Hence, by applying \mathbf{L}_i 's and relation (11), \mathbf{B} is implemented with $8 \times 4 + 4 \times 1 = 36$ XOR for 4-bit input. Moreover, $\mathbf{L}^4 + \mathbf{L}^3 + 1$ is not an element of (12). Therefore, \mathbf{B} is

implemented with 36 XOR over $\mathbb{F}_{2^4}/0x19$. Actually, \mathbf{L}_4 is the binary matrix representation of the root of $0x19$. Furthermore, by using \mathbf{L}_i with $1 \leq i \leq 4$, the implementation cost of \mathbf{B}^{-1} is 36 XOR, since we have

$$\mathbf{B}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \mathbf{L} \\ 0 & 1 & 0 & 0 \\ 0 & \mathbf{L} & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \mathbf{L} \\ 0 & 1 & 0 & 0 \\ 0 & \mathbf{L} & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad (13)$$

For $m = 8$, consider 8×8 non-singular matrices \mathbf{L}_j with $1 \leq j \leq 7$ in (14). It can be verified that by applying \mathbf{L}_j 's the given elements in (12) are non-singular matrices over \mathbb{F}_2 .

$$\begin{aligned} \mathbf{L}_1 &= [[2, 8], [1], [2], [3], [4], [5], [6], [7]], & \mathbf{L}_2 &= [[8], [1, 3], [2], [3], [4], [5], [6], [7]], \\ \mathbf{L}_3 &= [[8], [1], [2, 4], [3], [4], [5], [6], [7]], & \mathbf{L}_4 &= [[8], [1], [2], [3, 5], [4], [5], [6], [7]], \\ \mathbf{L}_5 &= [[8], [1], [2], [3], [4, 6], [5], [6], [7]], & \mathbf{L}_6 &= [[8], [1], [2], [3], [4], [5, 7], [6], [7]], \\ \mathbf{L}_7 &= [[8], [1], [2], [3], [4], [5], [6, 8], [7]]. \end{aligned} \quad (14)$$

The implementation cost of \mathbf{L}_j 's is one XOR. Therefore, by using \mathbf{L}_j 's and relation (11), \mathbf{B} is implemented with $8 \times 8 + 4 \times 1 = 68$ XOR for 8-bit input. Moreover, it follows from (13) that the implementation cost of \mathbf{B}^{-1} is 68 XOR for 8-bit input. Furthermore, \mathbf{B} can be implemented with at least $8 \times 8 + 4 \times 2 = 72$ XOR over \mathbb{F}_{2^8} , since some roots of irreducible polynomials of degree 8 require only two XOR [Bei16].

5.2 Lightweight 8×8 MDS Matrices from Primitive GFS Matrices

First of all, we select two permutations $\widehat{\mathbf{p}}_1 = \{1, 3, 2, 4\}$ and $\widehat{\mathbf{p}}_2 = \{2, 1, 4, 3\}$. Then we construct the primitive GFS matrix $\mathbf{C} = \mathbf{R}(4, \mathbf{1}_{(m)}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2])$. It follows from relation (6) that \mathbf{C} is an 6-primitive GFS matrix over \mathbb{R} . In fact, based on the Appendix B, the GFS matrix \mathbf{C} has minimum primitive order. Moreover, it follows from exhaustive search that there are no primitive GFS matrices $\mathbf{A}_i = \mathbf{S}(4, \mathbf{f}_{(m)}^{(i)}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2])$ with $1 \leq i \leq 6$ such that $\prod_{i=1}^6 \mathbf{A}_i$ be an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$ provided that $\mathbf{f}^{(i)} \in \{\mathbf{L}^{-2}, \mathbf{L}^{-1}, \mathbf{1}, \mathbf{L}, \mathbf{L}^{-2}\}$. Now consider the following primitive GFS matrices \mathbf{A}_i with $1 \leq i \leq 7$.

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{R}(4, \mathbf{L}_{(m)}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2]), & \mathbf{A}_2 &= \mathbf{R}(4, \mathbf{1}_{(m)}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2]), & \mathbf{A}_3 &= \mathbf{R}(4, \mathbf{1}_{(m)}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2]), & \mathbf{A}_4 &= \mathbf{R}(4, \mathbf{L}_{(m)}^2, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2]), \\ \mathbf{A}_5 &= \mathbf{R}(4, \mathbf{L}_{(m)}^{-1}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2]), & \mathbf{A}_6 &= \mathbf{R}(4, \mathbf{L}_{(m)}^{-1}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2]), & \mathbf{A}_7 &= \mathbf{R}(4, \mathbf{1}_{(m)}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2]). \end{aligned}$$

It can be verified that $\mathbf{B} = \prod_{i=1}^7 \mathbf{A}_i$ is an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$. Moreover, using exhaustive search, \mathbf{B} is one of the optimal results in relation to the number of binary linear functions (\mathbf{L}). Actually, it is not possible to construct an MDS matrix \mathbf{B} by applying primitive GFS matrices $\mathbf{A}_i = \mathbf{S}(4, \mathbf{f}_{(m)}^{(i)}, [\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2])$ with $1 \leq i \leq 7$ provided that \mathbf{A}_i 's are constructed from less than twenty \mathbf{L} . The matrix structure \mathbf{B} is as follows

$$\begin{aligned} \mathbf{B} &= \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 \mathbf{A}_4 \mathbf{A}_5 \mathbf{A}_6 \mathbf{A}_7 \\ &= \begin{pmatrix} \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L} \\ \mathbf{L}^2 + \mathbf{L}^{-2} + 1 & \mathbf{L}^2 + \mathbf{L}^{-2} & \mathbf{L}^2 + \mathbf{L}^{-1} + 1 & \mathbf{L}^2 + 1 \\ \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L} & \mathbf{L}^{-2} + 1 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-2} + 1 \\ \mathbf{L}^2 + \mathbf{L}^{-1} + 1 & \mathbf{L}^2 + 1 & \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L}^{-1} + 1 \\ \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} \\ \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^{-1} + 1 & \mathbf{L}^2 + \mathbf{L}^{-2} + 1 & \mathbf{L}^2 + \mathbf{L}^{-2} \\ \mathbf{L}^{-2} + 1 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-2} + 1 & \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 \\ \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L}^{-1} + 1 & \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^{-1} + 1 \end{pmatrix} \quad (15) \\ &= \begin{pmatrix} \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^{-2} + 1 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-2} + 1 \\ \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^{-1} + 1 & \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L}^{-1} + 1 \\ \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 \\ \mathbf{L}^2 + \mathbf{L}^{-2} + 1 & \mathbf{L}^2 + \mathbf{L}^{-2} & \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^{-1} + 1 \\ \mathbf{L}^{-2} + 1 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-2} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L} \\ \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L}^{-1} + 1 & \mathbf{L}^2 + \mathbf{L}^{-1} + 1 & \mathbf{L}^2 + 1 \\ \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L} & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} \\ \mathbf{L}^2 + \mathbf{L}^{-1} + 1 & \mathbf{L}^2 + 1 & \mathbf{L}^2 + \mathbf{L}^{-2} + 1 & \mathbf{L}^2 + \mathbf{L}^{-2} \end{pmatrix}. \end{aligned}$$

It follows from (15) that the implementation cost of \mathbf{B} for m -bit input is equal to

$$\begin{aligned} & \overbrace{(4m + 4(\#\mathbf{L}))}^{\#\mathbf{A}_1} + \overbrace{(4m)}^{\#\mathbf{A}_2} + \overbrace{(4m)}^{\#\mathbf{A}_3} + \overbrace{(4m + 4(\#\mathbf{L}^2))}^{\#\mathbf{A}_4} + \overbrace{(4m + 4(\#\mathbf{L}^{-1}))}^{\#\mathbf{A}_5} + \overbrace{(4m + 4(\#\mathbf{L}^{-1}))}^{\#\mathbf{A}_6} + \overbrace{(4m)}^{\#\mathbf{A}_7} \\ & = 28m + 4(\#\mathbf{L}) + 4(\#\mathbf{L}^2) + 8(\#\mathbf{L}^{-1}) \end{aligned} \quad (16)$$

The base set of subdeterminants of \mathbf{B} has 380 elements which are given in Appendix D. In this base set, there are all irreducible polynomials of degree 4 and 8, except for the irreducible polynomial $0x1A3$. It can be checked that the elements in Appendix D are non-singular matrices over \mathbb{F}_2 by applying the following non-singular 8×8 binary matrices \mathbf{L}_i with $1 \leq i \leq 10$.

$$\begin{aligned} \mathbf{L}_1 &= [[3], [7], [2], [5], [1, 8], [4], [6], [4, 8]], & \mathbf{L}_2 &= [[6], [3], [1, 4], [2, 4], [7], [5], [8], [2]], \\ \mathbf{L}_3 &= [[7], [1, 5], [2], [8], [3, 5], [4], [6], [3]], & \mathbf{L}_4 &= [[1, 5], [7], [2], [8], [6], [1, 4], [5], [3]], \\ \mathbf{L}_5 &= [[2], [8], [2, 3], [5], [7], [1], [6], [3, 4]], & \mathbf{L}_6 &= [[7], [3], [6], [2], [1, 8], [5], [4], [6, 8]], \\ \mathbf{L}_7 &= [[5], [3], [8], [4, 5], [7], [1], [2, 4], [6]], & \mathbf{L}_8 &= [[8], [2, 4], [1], [6], [3], [2, 7], [5], [4]], \\ \mathbf{L}_9 &= [[1, 2], [8], [2], [5], [6], [3], [4], [1, 7]], & \mathbf{L}_{10} &= [[5], [4], [8], [1], [3], [3, 6], [2], [6, 7]]. \end{aligned} \quad (17)$$

First notice that the implementation cost of \mathbf{L}_i with $1 \leq i \leq 10$ is two XOR. In addition, \mathbf{L}_i^{-1} and \mathbf{L}_i^2 with $1 \leq i \leq 10$ can be implemented with two and four XOR, respectively. Therefore, by applying \mathbf{L}_i 's and relation (16), \mathbf{B} is implemented with $28 \times 8 + 4 \times 2 + 4 \times 4 + 8 \times 2 = 264$ XOR for 8-bit input. Moreover, it can be verified that the implementation cost of \mathbf{B}^{-1} is 264 XOR for 8-bit input. Furthermore, it follows from Lemma 1 that the matrix \mathbf{B} is implemented with 264 XOR over $\mathbb{F}_{2^8}/0x1A3$, since the minimal polynomial of \mathbf{L}_i 's over \mathbb{F}_2 is $0x1A3$.

6 Construction of Lightweight MDS Matrices by Applying Extended Primitive GFS Matrices

In this section, using an extension of Definition 3, we define some type of sparse matrices called EGFS matrices. Then, by applying EGFS matrices we propose 4×4 , 6×6 and 8×8 lightweight MDS matrices with implementation cost 67, 156 and 260 XOR for 8-bit input, respectively. Moreover, we implement the proposed 6×6 MDS matrix with 114 XOR for 6-bit input which can be used in hardware implementation. Furthermore, we propose an 6×6 MDS matrix for 4-bit input such that its implementation cost is 90 XOR. The proposed MDS matrices are not only suitable by terminology of implementation cost, but also are efficient with respect to the number of binary linear functions that are used in construction of these matrices.

The proposed 4×4 MDS matrix is obtained by exhaustive search. But a random search is applied to achieve the proposed 6×6 and 8×8 MDS matrices. Actually, there is an extensive search space to obtain lightweight 6×6 and 8×8 MDS matrices using EGFS matrices. Therefore, by performing a full search on EGFS matrices, better implementation results may be achieved to construct 6×6 and 8×8 lightweight MDS matrices.

Assume that $\mathbf{1}$ and $\mathbf{0}$ are $m \times m$ identity and zero matrices over \mathbb{F}_2 , respectively. Consider \mathbf{L}_j with $1 \leq j \leq 3n$ such that \mathbf{L}_j 's are $m \times m$ non-singular matrices over \mathbb{F}_2 . In Definition 4, the following 2×2 block-matrices with $1 \leq i \leq n$ are used.

$$\mathbf{c}_i^{(1,m)} = \begin{pmatrix} \mathbf{L}_{3i-2} & \mathbf{L}_{3i-1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{c}_i^{(2,m)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{L}_{3i} & \mathbf{0} \end{pmatrix}, \quad \mathbf{z}^{(m)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \quad (18)$$

Definition 4 (Extended Primitive GFS Matrices). Suppose that $\mathbf{p}_1 = \{a_1, a_2, \dots, a_n\}$ and $\mathbf{p}_2 = \{b_1, b_2, \dots, b_n\}$ are two permutations from 1 to n such that $a_i \neq b_i$ for $1 \leq i \leq n$. Consider 2×2 block-matrices $\mathbf{c}_i^{(1,m)}$, $\mathbf{c}_i^{(2,m)}$ and $\mathbf{z}^{(m)}$ with $1 \leq i \leq n$ that are given in (18).

Assume that by using two permutations \mathbf{p}_1 and \mathbf{p}_2 , the i th row of an $n \times n$ block-matrix $\mathbf{E} = (e_{i,j})$ with $1 \leq i, j \leq n$, is filled in the following form.

$$e_{i,j} = \begin{cases} \mathbf{c}_i^{(1,m)} & j = a_i, \\ \mathbf{c}_i^{(2,m)} & j = b_i, \\ \mathbf{z}^{(m)} & j \notin \{a_i, b_i\}. \end{cases}$$

The block-matrix \mathbf{E} is called an extended primitive GFS matrix, denoted with EGFS matrix, if \mathbf{E} be a primitive matrix over \mathbb{R} .

In Definition 4 to check whether \mathbf{E} is a primitive matrix over \mathbb{R} , we suppose that $\mathbf{c}_i^{(1,m)}$, $\mathbf{c}_i^{(2,m)}$ and $\mathbf{z}^{(m)}$ with $1 \leq i \leq n$ are 2×2 matrices over \mathbb{R} and also \mathbf{L}_j with $1 \leq j \leq 3n$ are positive integer numbers over \mathbb{R} .

Moreover, it can be verified that $\det(\mathbf{E}) = \prod_{i=1}^n \mathbf{L}_{3i-1} \mathbf{L}_{3i}$ over $\mathbb{F}_2[\mathbf{L}]$. Therefore, $\det(\mathbf{E}) \neq 0$ over $\mathbb{F}_2[\mathbf{L}]$, since it is assumed that \mathbf{L}_j with $1 \leq j \leq 3n$ are non-singular matrices over \mathbb{F}_2 . In other words, EGFS matrices are non-singular matrices over $\mathbb{F}_2[\mathbf{L}]$.

Example 6. For $n = 2$, consider two permutations $\mathbf{p}_1 = \{1, 2\}$ and $\mathbf{p}_2 = \{2, 1\}$. Then the following two EGFS matrices \mathbf{E}_1 and \mathbf{E}_2 are 4-primitive matrices over \mathbb{R} .

$$\begin{aligned} \mathbf{c}_1^{(1,m)} &= \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \mathbf{c}_2^{(1,m)} = \begin{pmatrix} \mathbf{L} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, & \mathbf{c}_1^{(1,m)} &= \begin{pmatrix} \mathbf{L} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \mathbf{c}_2^{(1,m)} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \\ \mathbf{c}_1^{(2,m)} &= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \mathbf{c}_2^{(2,m)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{L}^{-1} & \mathbf{0} \end{pmatrix}, & \mathbf{c}_1^{(2,m)} &= \mathbf{c}_2^{(2,m)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \\ \mathbf{E}_1 &= \left(\begin{array}{cc|cc} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} \\ \mathbf{L}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right). & \mathbf{E}_2 &= \left(\begin{array}{cc|cc} \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right). \end{aligned}$$

For $n = 3$, the given EGFS matrix in (19) is an 6-primitive matrix over \mathbb{R} .

$$\begin{aligned} \mathbf{c}_1^{(1,m)} &= \mathbf{c}_2^{(1,m)} = \mathbf{c}_3^{(1,m)} = \begin{pmatrix} \mathbf{L} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \\ \mathbf{c}_1^{(2,m)} &= \mathbf{c}_2^{(2,m)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \mathbf{c}_3^{(2,m)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{L}^{-1} & \mathbf{0} \end{pmatrix}, & \mathbf{E} &= \left(\begin{array}{cc|cc|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{L}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right). \\ \mathbf{p}_1 &= \{3, 1, 2\}, \quad \mathbf{p}_2 = \{2, 3, 1\}. \end{aligned} \tag{19}$$

6.1 Construction of Lightweight 4×4 MDS Matrices from EGFS

In this subsection, we propose four EGFS matrices \mathbf{E}_i with $1 \leq i \leq 4$ such that \mathbf{E}_i 's are 4-primitive matrices over \mathbb{R} . Moreover, proposed EGFS matrices are constructed of two permutations $\mathbf{p}_1 = \{1, 2\}$ and $\mathbf{p}_2 = \{2, 1\}$. In other words, \mathbf{E}_i 's are with the same structure. In the rest, by applying \mathbf{E}_i 's we obtain an MDS matrix \mathbf{H} over $\mathbb{F}_2[\mathbf{L}]$ such that the implementation cost of \mathbf{H} over 4 and 8-bit input are 35 and 67 XOR, respectively. It follows from the full search that the matrix \mathbf{H} can be obtained from one of the following two cases. Although, the structure of \mathbf{H}_i with $1 \leq i \leq 2$ are different, the base set of

subdeterminants of \mathbf{H}_i 's are the same.

$$\begin{aligned}
\mathbf{H}_1 &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{L} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{L} & 1 \\ \mathbf{L}^{-1} & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{L}+1 & \mathbf{L} & 1 & \mathbf{L}+1 \\ \mathbf{L}^{-1}+1 & \mathbf{L}^{-1} & \mathbf{L} & \mathbf{L} \\ \mathbf{L}^{-1} & \mathbf{L}^{-1}+1 & \mathbf{L}+1 & \mathbf{L} \\ \mathbf{L} & \mathbf{L} & \mathbf{L}+1 & 1 \end{pmatrix}, \\
\mathbf{H}_2 &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{L} & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{L} & 1 & 0 & 0 \\ 0 & 0 & \mathbf{L}^{-1} & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{L}+1 & \mathbf{L} & \mathbf{L}^{-1} & \mathbf{L}^{-1}+1 \\ \mathbf{L}+1 & 1 & \mathbf{L} & \mathbf{L} \\ 1 & \mathbf{L}+1 & \mathbf{L}+1 & \mathbf{L} \\ \mathbf{L} & \mathbf{L} & \mathbf{L}^{-1}+1 & \mathbf{L}^{-1} \end{pmatrix}.
\end{aligned}$$

For instance, consider $\mathbf{H}_1 = \mathbf{E}_1 \cdot \mathbf{E}_2 \cdot \mathbf{E}_3 \cdot \mathbf{E}_4$, then the implementation cost of \mathbf{H}_1 for m -bit input is equal to

$$\underbrace{(2m)}_{\#\mathbf{E}_1} + \underbrace{(2m + \#\mathbf{L})}_{\#\mathbf{E}_2} + \underbrace{(2m + \#\mathbf{L} + \#(\mathbf{L}^{-1}))}_{\#\mathbf{E}_3} + \underbrace{(2m)}_{\#\mathbf{E}_4} = 8m + 2(\#\mathbf{L}) + \#(\mathbf{L}^{-1}). \quad (20)$$

The base set of subdeterminants of \mathbf{H}_1 is:

$$\{\mathbf{L}, \mathbf{L} + 1, \mathbf{L}^2 + \mathbf{L} + 1, \mathbf{L}^3 + \mathbf{L} + 1, \mathbf{L}^3 + \mathbf{L}^2 + 1\}. \quad (21)$$

For $m = 4$, consider 4×4 non-singular binary matrices \mathbf{L}_i with $1 \leq i \leq 6$ in (22). It can be checked that by using \mathbf{L}_i 's the given elements in (21) are non-singular matrices over \mathbb{F}_2 . In addition, the implementation cost of \mathbf{L}_i and \mathbf{L}_i^{-1} with $1 \leq i \leq 6$ are one XOR.

$$\begin{aligned}
\mathbf{L}_1 &= [[1, 4], [1], [2], [3]], & \mathbf{L}_2 &= [[3, 4], [1], [2], [3]], & \mathbf{L}_3 &= [[4], [1, 2], [2], [3]], \\
\mathbf{L}_4 &= [[4], [1, 4], [2], [3]], & \mathbf{L}_5 &= [[4], [1], [2, 3], [3]], & \mathbf{L}_6 &= [[4], [1], [2], [3, 4]].
\end{aligned} \quad (22)$$

Therefore, by applying \mathbf{L}_i 's and relation (20), the implementation cost of \mathbf{H}_1 is $8 \times 4 + 2 \times 1 + 1 \times 1 = 35$ XOR for 4-bit input. Moreover, it follows from (21) that the matrix \mathbf{H}_1 can be implemented with 35 XOR over \mathbb{F}_{2^4} , since two irreducible polynomials $0x13$ and $0x19$ are not elements of the set (21).

For $m = 8$, the next 8×8 non-singular binary matrices \mathbf{L}_j with $1 \leq j \leq 10$ are obtained such that the given elements in (21) are non-singular matrices over \mathbb{F}_2 by applying \mathbf{L}_j 's. Moreover, the implementation cost of \mathbf{L}_j and \mathbf{L}_j^{-1} with $1 \leq j \leq 10$ are one XOR.

$$\begin{aligned}
\mathbf{L}_1 &= [[2, 8], [1], [2], [3], [4], [5], [6], [7]], & \mathbf{L}_2 &= [[6, 8], [1], [2], [3], [4], [5], [6], [7]], \\
\mathbf{L}_3 &= [[8], [1, 3], [2], [3], [4], [5], [6], [7]], & \mathbf{L}_4 &= [[8], [1, 7], [2], [3], [4], [5], [6], [7]], \\
\mathbf{L}_5 &= [[8], [1], [2, 4], [3], [4], [5], [6], [7]], & \mathbf{L}_6 &= [[8], [1], [2, 8], [3], [4], [5], [6], [7]], \\
\mathbf{L}_7 &= [[8], [1], [2], [3, 5], [4], [5], [6], [7]], & \mathbf{L}_8 &= [[8], [1], [2], [3], [4, 6], [5], [6], [7]], \\
\mathbf{L}_9 &= [[8], [1], [2], [3], [4], [5, 7], [6], [7]], & \mathbf{L}_{10} &= [[8], [1], [2], [3], [4], [5], [6, 8], [7]].
\end{aligned} \quad (23)$$

Hence, it follows from (20) that the implementation cost of \mathbf{H}_1 by applying \mathbf{L}_j 's is $8 \times 8 + 2 \times 1 + 1 \times 1 = 67$ XOR. Moreover, some roots of irreducible polynomials of degree 8 require only two XOR [Bei16]. Therefore, \mathbf{H}_1 can be implemented with at least $8 \times 8 + 2 \times 2 + 1 \times 2 = 70$ XOR over \mathbb{F}_{2^8} , since there are no irreducible polynomials in \mathbb{F}_2 of degree 8 in (21).

6.2 Construction of Lightweight 6×6 MDS Matrices from EGFS

In this section, we propose lightweight 6×6 MDS matrices for 4,6 and 8-bit input such that the implementation cost of the proposed matrices are equal to the given matrices.

6.2.1 A Lightweight 6×6 MDS Matrix for 4-bit input

In this subsection, by applying distinct EGFS matrices, we propose a lightweight 6×6 MDS matrix which is implemented with 90 XOR for 4-bit input. In fact, we could not obtain 6×6 EGFS matrices \mathbf{E}_i with $1 \leq i \leq 6$ provided that \mathbf{E}_i 's satisfy the following conditions. First, \mathbf{E}_i 's are with the same structure. Second, the multiplication of \mathbf{E}_i 's denoted with $\mathbf{H} = \prod_{i=1}^6 \mathbf{E}_i$, is an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$. Finally, at least one of the two irreducible polynomials $0x13$ and $0x19$ are not elements of the base set of \mathbf{H} . Therefore, we used distinct EGFS matrices to get a lightweight 6×6 MDS matrix for 4-bit input. Consider the following EGFS matrices \mathbf{E}_i with $1 \leq i \leq 6$.

$$\begin{aligned} \mathbf{E}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, & \mathbf{E}_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & \mathbf{L}^{-6} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{L}^{-6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{L}^{-6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ \mathbf{E}_3 &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, & \mathbf{E}_4 &= \begin{pmatrix} 0 & 0 & \mathbf{L}^{-1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{L}^{-1} & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{L}^{-1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{E}_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{L}^{-1} & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{L}^{-1} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{L}^{-1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, & \mathbf{E}_6 &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Two EGFS matrices \mathbf{E}_1 and \mathbf{E}_5 are with the same structure and are constructed from two permutations $\mathbf{p}_1 = \{3, 2, 1\}$ and $\mathbf{p}_2 = \{2, 1, 3\}$ and also are 5-primitive matrices over \mathbb{R} . The EGFS matrix \mathbf{E}_2 is constructed from $\mathbf{p}_1 = \{3, 1, 2\}$ and $\mathbf{p}_2 = \{1, 2, 3\}$ and also is an 6-primitive matrix over \mathbb{R} . Moreover, EGFS matrices \mathbf{E}_3 and \mathbf{E}_4 are 6-primitive matrices and \mathbf{E}_6 is an 5-primitive matrix over \mathbb{R} . Next, applying EGFS matrices \mathbf{E}_i with $1 \leq i \leq 6$, the proposed lightweight 6×6 MDS matrix \mathbf{H} is given by

$$\begin{aligned} \mathbf{H} &= \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_4 \mathbf{E}_5 \mathbf{E}_6 \\ &= \begin{pmatrix} \mathbf{L}^{-2} + \mathbf{L}^{-6} + \mathbf{L}^{-7} + 1 & \mathbf{L}^{-2} + \mathbf{L}^{-6} & \mathbf{L}^{-2} + 1 \\ 1 & \mathbf{L}^{-1} + 1 & \mathbf{L}^{-8} + 1 \\ \mathbf{L}^{-2} + 1 & \mathbf{L}^{-1} + \mathbf{L}^{-2} + 1 & \mathbf{L}^{-8} \\ \mathbf{L}^{-8} + 1 & \mathbf{L}^{-8} & \mathbf{L}^{-2} + \mathbf{L}^{-6} + \mathbf{L}^{-7} \\ \mathbf{L}^{-8} & \mathbf{L}^{-1} + \mathbf{L}^{-8} + 1 & \mathbf{L}^{-2} + \mathbf{L}^{-6} + \mathbf{L}^{-7} + 1 \\ \mathbf{L}^{-2} + \mathbf{L}^{-6} + \mathbf{L}^{-7} & \mathbf{L}^{-2} + \mathbf{L}^{-6} & 1 \\ \mathbf{L}^{-1} + \mathbf{L}^{-2} + 1 & \mathbf{L}^{-8} & \mathbf{L}^{-1} + \mathbf{L}^{-8} + 1 \\ \mathbf{L}^{-8} & \mathbf{L}^{-2} + \mathbf{L}^{-6} + \mathbf{L}^{-7} & \mathbf{L}^{-2} + \mathbf{L}^{-6} \\ \mathbf{L}^{-1} + \mathbf{L}^{-8} + 1 & \mathbf{L}^{-2} + \mathbf{L}^{-6} + \mathbf{L}^{-7} + 1 & \mathbf{L}^{-2} + \mathbf{L}^{-6} \\ \mathbf{L}^{-2} + \mathbf{L}^{-6} & 1 & \mathbf{L}^{-1} + 1 \\ \mathbf{L}^{-2} + \mathbf{L}^{-6} & \mathbf{L}^{-2} + 1 & \mathbf{L}^{-1} + \mathbf{L}^{-2} + 1 \\ \mathbf{L}^{-1} + 1 & \mathbf{L}^{-8} + 1 & \mathbf{L}^{-8} \end{pmatrix}. \end{aligned}$$

The implementation cost of \mathbf{H} for m -bit input is equal to

$$\begin{aligned} \#\mathbf{H} &= \underbrace{\#\mathbf{E}_1}_{(3m)} + \underbrace{\#\mathbf{E}_2}_{(3m + 3(\#\mathbf{L}^{-6}))} + \underbrace{\#\mathbf{E}_3}_{(3m)} + \underbrace{\#\mathbf{E}_4}_{(3m + 3(\#\mathbf{L}^{-1}))} + \underbrace{\#\mathbf{E}_5}_{(3m + 3(\#\mathbf{L}^{-1}))} + \underbrace{\#\mathbf{E}_6}_{(3m)} \\ &= 18m + 6(\#\mathbf{L}^{-1}) + 3(\#\mathbf{L}^{-6}). \end{aligned} \quad (24)$$

The base set of subdeterminants of \mathbf{H} has 149 elements that are given in (25). For simplicity

in representation, the elements of the base set are presented by hexadecimal values.

$$\{0x2, 0x3, 0x7, 0xB, 0xD, 0x19, 0x1F, 0x25, 0x29, 0x2F, 0x37, 0x3B, 0x3D, 0x43, 0x49, 0x61, 0x57, 0x5B, 0x67, 0x6D, 0x73, 0x75, 0x89, 0x91, 0xC1, 0x8F, 0x9D, 0xA7, 0xAB, 0xB9, 0xCB, 0xD5, 0xF1, 0xBF, 0xEF, 0xF7, 0xFD, 0x11D, 0x12B, 0x165, 0x171, 0x18B, 0x18D, 0x1A3, 0x1B1, 0x13F, 0x19F, 0x1CF, 0x1DD, 0x1F3, 0x1F5, 0x1F9, 0x203, 0x301, 0x21B, 0x22D, 0x233, 0x313, 0x323, 0x361, 0x3A1, 0x27D, 0x2F9, 0x36B, 0x3D9, 0x3FB, 0x40F, 0x463, 0x4A9, 0x4C9, 0x50B, 0x685, 0x4AF, 0x557, 0x597, 0x5C7, 0x5E5, 0x65B, 0x7A9, 0x5F7, 0x5FB, 0x945, 0xB03, 0x8EB, 0x973, 0xAAB, 0xB59, 0xCD5, 0xE27, 0xB6F, 0xCF7, 0xF79, 0x1081, 0x1201, 0x1A21, 0x1C05, 0x131D, 0x1371, 0x15C5, 0x1A55, 0x1B0B, 0x1C4D, 0x1A5F, 0x1CED, 0x1E3D, 0x2027, 0x2289, 0x2819, 0x3281, 0x252D, 0x2E07, 0x31E1, 0x369B, 0x2EFD, 0x31FF, 0x6601, 0x7081, 0x5159, 0x644D, 0x6781, 0x51DB, 0x51ED, 0x6A5B, 0x51FF, 0x7A77, 0x7DE9, 0xA023, 0xD009, 0xA323, 0xF089, 0x9E47, 0xCCB3, 0xEB3B, 0x1248D, 0x15C05, 0x1B083, 0x1632D, 0x137D5, 0x25003, 0x3344B, 0x40201, 0x48BD1, 0x8F145, 0xC29DB, 0x1407ED, 0xA43081, 0xA73081, 0xA7E001, 0x1A42251\}$$
(25)

For $m = 4$, consider the next 4×4 non-singular matrices over \mathbb{F}_2 . It can be verified that using \mathbf{L}_i with $1 \leq i \leq 2$, the given elements in (25) are non-singular matrices over \mathbb{F}_2 . Moreover, the implementation cost of \mathbf{L}_i and \mathbf{L}_i^{-1} with $1 \leq i \leq 2$ are one XOR.

$$\mathbf{L}_1 = [[3, 4], [1], [2], [3]], \quad \mathbf{L}_2 = [[4], [1, 4], [2], [3]].$$

In addition, \mathbf{L}_i^{-6} with $1 \leq i \leq 2$ can be implemented with four XOR. For instance, consider $\mathbf{x} = [x_1, x_2, x_3, x_4]$. Then we get

$$\begin{aligned} \mathbf{L}_1^{-6} \cdot \mathbf{x}^T &= [x_1 + x_2 + x_3 + x_4, x_1 + x_2 + x_3, x_2 + x_3 + x_4, x_1 + x_3]^T, \\ u_1 &= x_1 + x_3, \quad u_2 = u_1 + x_2, \quad u_3 = u_2 + x_4, \quad u_4 = u_3 + x_1. \end{aligned}$$

Next, by applying \mathbf{L}_i 's and relation (24), \mathbf{H} is implemented with $18 \times 4 + 6 \times 1 + 3 \times 4 = 90$ XOR for 4-bit input. Moreover, $0x13$ ($\mathbf{L}^4 + \mathbf{L} + 1$) is not an element of (25). Therefore, \mathbf{H} is implemented with 90 XOR over $\mathbb{F}_{2^4}/0x13$. Furthermore, it is easy to check that \mathbf{H}^{-1} can be implemented with 90 XOR for 4-bit input.

6.2.2 A Lightweight 6×6 MDS Matrix for 6-bit and 8-bit input

In this subsection, using EGFS matrices, we propose a new lightweight 6×6 MDS matrix which is implemented with 156 XOR for 8-bit input. Moreover, the proposed 6×6 MDS matrix is implemented with 114 XOR for 6-bit input. The result of this subsection is not only efficient from hardware perspective, but also are suitable by software terminology. In fact, in order to construct the proposed 6×6 MDS matrix for 8-bit input, six binary linear functions (\mathbf{L}) are used such that the cost of \mathbf{L} is very small compared to the total XOR.

Consider the following EGFS matrices \mathbf{E}_i with $1 \leq i \leq 2$ such that \mathbf{E}_1 is constructed from two permutations $\mathbf{p}_1 = \{1, 2, 3\}$ and $\mathbf{p}_2 = \{3, 1, 2\}$ and also \mathbf{E}_2 is obtained from $\mathbf{p}_1 = \{3, 2, 1\}$ and $\mathbf{p}_2 = \{2, 1, 3\}$. It can be checked that \mathbf{E}_1 and \mathbf{E}_2 are 6-primitive and 5-primitive matrices over \mathbb{R} , respectively.

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{L} & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{L} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{L} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now based on the EGFS matrices \mathbf{E}_i with $1 \leq i \leq 2$, the proposed lightweight 6×6 MDS

matrix \mathbf{H} is constructed as follows.

$$\mathbf{H} = \mathbf{E}_1^2 \times \mathbf{E}_2^2 \times \mathbf{E}_1^2 = \begin{pmatrix} \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^2 + 1 & \mathbf{L}^2 & 1 & \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 \\ \mathbf{L} + 1 & \mathbf{L} & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^2 + 1 \\ \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^2 + 1 & \mathbf{L}^2 & 1 \\ \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^2 + 1 & \mathbf{L} + 1 & \mathbf{L} & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L} + 1 \\ \mathbf{L}^2 & 1 & \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^2 + 1 \\ \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^2 + 1 & \mathbf{L} + 1 & \mathbf{L} \end{pmatrix}.$$

The implementation cost of \mathbf{H} for m -bit input is equal to

$$\underbrace{2(\#\mathbf{E}_1)}_{2(3m)} + \underbrace{\#2(\mathbf{E}_2)}_{2(3m + 3(\#\mathbf{L}))} + \underbrace{2(\#\mathbf{E}_1)}_{2(3m)} = 18m + 6(\#\mathbf{L}) \quad (26)$$

The base set of subdeterminants of \mathbf{H} has 20 elements that are listed in (27).

$$\{0x2, 0x3, 0x7, 0xB, 0xD, 0x13, 0x19, 0x1F, 0x25, 0x29, 0x2F, \\ 0x37, 0x3B, 0x3D, 0x61, 0x57, 0x5B, 0x67, 0x6D, 0x75\} \quad (27)$$

Consider 8×8 non-singular binary matrices \mathbf{L}_i with $1 \leq i \leq 32$ that are given in (28). It can be verified that by applying \mathbf{L}_i 's, the given elements in (27) are non-singular matrices over \mathbb{F}_2 . Moreover, the implementation cost of \mathbf{L}_i and \mathbf{L}_i^{-1} with $1 \leq i \leq 32$ are two XOR.

$$\begin{aligned} \mathbf{L}_1 &= [[1, 8], [1], [2, 5], [3], [4], [5], [6], [7]], & \mathbf{L}_2 &= [[1, 8], [1], [2, 7], [3], [4], [5], [6], [7]], \\ \mathbf{L}_3 &= [[1, 8], [1], [2], [3, 6], [4], [5], [6], [7]], & \mathbf{L}_4 &= [[1, 8], [1], [2], [3], [4, 7], [5], [6], [7]], \\ \mathbf{L}_5 &= [[2, 8], [1], [2], [3, 6], [4], [5], [6], [7]], & \mathbf{L}_6 &= [[2, 8], [1], [2], [3], [4, 7], [5], [6], [7]], \\ \mathbf{L}_7 &= [[3, 8], [1], [2], [3], [4, 5], [5], [6], [7]], & \mathbf{L}_8 &= [[3, 8], [1], [2], [3], [4, 6], [5], [6], [7]], \\ \mathbf{L}_9 &= [[3, 8], [1], [2], [3], [4], [5, 6], [6], [7]], & \mathbf{L}_{10} &= [[3, 8], [1], [2], [3], [4], [5, 7], [6], [7]], \\ \mathbf{L}_{11} &= [[3, 8], [1], [2], [3], [4], [5], [6, 7], [7]], & \mathbf{L}_{12} &= [[5, 8], [1], [2], [3], [4], [5], [6, 7], [7]], \\ \mathbf{L}_{13} &= [[8], [1, 2], [2], [3, 6], [4], [5], [6], [7]], & \mathbf{L}_{14} &= [[8], [1, 2], [2], [3, 8], [4], [5], [6], [7]], \\ \mathbf{L}_{15} &= [[8], [1, 2], [2], [3], [4, 7], [5], [6], [7]], & \mathbf{L}_{16} &= [[8], [1, 2], [2], [3], [4], [5, 8], [6], [7]], \\ \mathbf{L}_{17} &= [[8], [1, 3], [2], [3], [4, 7], [5], [6], [7]], & \mathbf{L}_{18} &= [[8], [1, 3], [2], [3], [4], [5, 8], [6], [7]], \\ \mathbf{L}_{19} &= [[8], [1, 4], [2], [3], [4], [5, 6], [6], [7]], & \mathbf{L}_{20} &= [[8], [1, 4], [2], [3], [4], [5, 7], [6], [7]], \\ \mathbf{L}_{21} &= [[8], [1, 4], [2], [3], [4], [5], [6, 7], [7]], & \mathbf{L}_{22} &= [[8], [1, 4], [2], [3], [4], [5], [6, 8], [7]], \\ \mathbf{L}_{23} &= [[8], [1, 4], [2], [3], [4], [5], [6], [7, 8]], & \mathbf{L}_{24} &= [[8], [1, 6], [2], [3], [4], [5], [6], [7, 8]], \\ \mathbf{L}_{25} &= [[8], [1], [2, 3], [3], [4, 7], [5], [6], [7]], & \mathbf{L}_{26} &= [[8], [1], [2, 3], [3], [4], [5, 8], [6], [7]], \\ \mathbf{L}_{27} &= [[8], [1], [2, 4], [3], [4], [5, 8], [6], [7]], & \mathbf{L}_{28} &= [[8], [1], [2, 5], [3], [4], [5], [6, 7], [7]], \\ \mathbf{L}_{29} &= [[8], [1], [2, 5], [3], [4], [5], [6, 8], [7]], & \mathbf{L}_{30} &= [[8], [1], [2, 5], [3], [4], [5], [6], [7, 8]], \\ \mathbf{L}_{31} &= [[8], [1], [2], [3, 4], [4], [5, 8], [6], [7]], & \mathbf{L}_{32} &= [[8], [1], [2], [3, 6], [4], [5], [6], [7, 8]]. \end{aligned} \quad (28)$$

Therefore, by applying \mathbf{L}_i 's and relation (26), \mathbf{H} is implemented with $18 \times 8 + 6 \times 2 = 156$ XOR for 8-bit input. Moreover, \mathbf{H} can be implemented with 156 XOR over \mathbb{F}_{2^8} , since some roots of irreducible polynomials of degree 8 require only two XOR [Bei16] and there are no irreducible polynomials of degree 8 in (27). Furthermore, it is easy to verify that \mathbf{H}^{-1} is implemented with 156 XOR over \mathbb{F}_{2^8} .

Finally, we implement \mathbf{H} for 6-bit input with low XOR cost. In the base set of \mathbf{H} , given in (27), there are all irreducible polynomials of degree 6 except two irreducible polynomials $0x43$ and $0x49$. In the rest, we search on 6×6 binary matrices provided that the implementation cost of these matrices are one XOR and their minimal polynomials are $0x43$ or $0x49$ over \mathbb{F}_2 . For instance, consider the following two 6×6 matrices.

$$\mathbf{L}_1 = [[5, 6], [1], [2], [3], [4], [5]], \quad \mathbf{L}_2 = [6], [1, 4], [2], [3], [4], [5]].$$

The implementation cost of \mathbf{L}_i with $1 \leq i \leq 2$ is one XOR. Moreover, the minimal polynomial of \mathbf{L}_1 and \mathbf{L}_2 are $0x43$ and $0x49$, respectively. Therefore, by applying \mathbf{L}_i 's the given elements in (27) are non-singular matrices over \mathbb{F}_2 . Hence, using \mathbf{L}_i 's in (26), \mathbf{H} and \mathbf{H}^{-1} are implemented with $18 \times 6 + 6 \times 1 = 114$ XOR for 6-bit input. Moreover, based on the Lemma 1 the implementation cost of \mathbf{H} over $\mathbb{F}_{2^6}/0x43$ and $\mathbb{F}_{2^6}/0x49$ is 114 XOR.

6.3 Construction of Lightweight 8×8 MDS Matrices from EGFS

In this section, we propose two lightweight 8×8 MDS matrices for 8-bit input. The first proposed matrix is implemented with 272 XOR and the second new lightweight 8×8 MDS matrix is implemented with 260 XOR for 8-bit input. The first proposed matrix can be used in software implementation, since the proposed matrix is constructed from 16 binary linear functions. The second matrix can be applied in hardware implementation, since has low XOR cost and also the depth of its circuit is 8.

6.3.1 Construction of a Lightweight 8×8 MDS Matrix with 272 XOR Cost

First of all, by applying the Appendix B, we tried to obtain 8×8 EGFS matrices \mathbf{E}_i with $1 \leq i \leq 6$ such that \mathbf{E}_i 's satisfy the following conditions. First, \mathbf{E}_i 's are with the same structure and are 6-primitive matrices over \mathbb{R} . Second, the multiplication of \mathbf{E}_i 's denoted with $\mathbf{H} = \prod_{i=1}^6 \mathbf{E}_i$, is an MDS matrix over $\mathbb{F}_2[\mathbf{L}]$. Finally, the implementation cost of \mathbf{H} is less than 392 XOR for 8-bit input. But we could not get EGFS matrices under the stated conditions. Therefore, we increased the number of 8×8 EGFS matrices.

Consider 8×8 EGFS matrices \mathbf{E}_i with $1 \leq i \leq 5$, given in (29). The structures of \mathbf{E}_i 's are the same, since \mathbf{E}_i 's are constructed from two permutations $\mathbf{p}_1 = \{4, 3, 2, 1\}$ and $\mathbf{p}_2 = \{3, 2, 1, 4\}$. Moreover, using Appendix B, it can be checked that EGFS matrices \mathbf{E}_i with $1 \leq i \leq 5$ are 6-primitive matrices over \mathbb{R} .

$$\begin{array}{|l}
 \mathbf{c}_1^{(1,m)} = \dots = \mathbf{c}_4^{(1,m)} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \\
 \mathbf{c}_1^{(2,m)} = \dots = \mathbf{c}_4^{(2,m)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
 \mathbf{E}_1 = \left(\begin{array}{cc|cc|cc|cc}
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
 \end{array} \right) \\
 \hline
 \mathbf{c}_1^{(1,m)} = \dots = \mathbf{c}_4^{(1,m)} = \begin{pmatrix} 1 & \mathbf{L} \\ 0 & 0 \end{pmatrix}, \\
 \mathbf{c}_1^{(2,m)} = \dots = \mathbf{c}_4^{(2,m)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
 \mathbf{E}_2 = \left(\begin{array}{cc|cc|cc|cc}
 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{L} \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & \mathbf{L} & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 1 & \mathbf{L} & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & \mathbf{L} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
 \end{array} \right), \\
 \mathbf{E}_3 = \left(\begin{array}{cc|cc|cc|cc}
 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{L} & 1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \mathbf{L} & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & \mathbf{L} & 1 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \mathbf{L} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
 \end{array} \right) \\
 \hline
 \mathbf{c}_1^{(1,m)} = \dots = \mathbf{c}_4^{(1,m)} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \\
 \mathbf{c}_1^{(2,m)} = \dots = \mathbf{c}_4^{(2,m)} = \begin{pmatrix} 0 & 0 \\ \mathbf{L}^{-1} & 0 \end{pmatrix}, \\
 \mathbf{E}_4 = \left(\begin{array}{cc|cc|cc|cc}
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & \mathbf{L}^{-1} & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & \mathbf{L}^{-1} & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 \mathbf{L}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{L}^{-1} & 0
 \end{array} \right), \\
 \mathbf{E}_5 = \left(\begin{array}{cc|cc|cc|cc}
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & \mathbf{L} & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & \mathbf{L} & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 \mathbf{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{L} & 0
 \end{array} \right) \\
 \hline
 \mathbf{c}_1^{(1,m)} = \dots = \mathbf{c}_4^{(1,m)} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \\
 \mathbf{c}_1^{(2,m)} = \dots = \mathbf{c}_4^{(2,m)} = \begin{pmatrix} 0 & 0 \\ \mathbf{L} & 0 \end{pmatrix}, \\
 \mathbf{E}_5 = \left(\begin{array}{cc|cc|cc|cc}
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & \mathbf{L} & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & \mathbf{L} & 0 & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 \mathbf{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{L} & 0
 \end{array} \right)
 \end{array}
 \tag{29}$$

Now based on the EGFS matrices \mathbf{E}_i with $1 \leq i \leq 5$, we propose an MDS matrix \mathbf{H} over $\mathbb{F}_2[\mathbf{L}]$ such that \mathbf{H} is constructed from seven EGFS matrices.

$$\mathbf{H} = \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_3 \mathbf{E}_4 \mathbf{E}_5 \mathbf{E}_1$$

$$= \begin{pmatrix} \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L} \\ \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L}^{-1} \\ \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^2 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} \\ \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L}^{-1} & \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L} \\ \mathbf{L}^2 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} \\ \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L} & \mathbf{L}^2 & \mathbf{L}^2 + 1 \\ \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 \\ \mathbf{L}^2 & \mathbf{L}^2 + 1 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L} + 1 \end{pmatrix} \quad (30)$$

$$\begin{pmatrix} \mathbf{L}^2 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} \\ \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L} & \mathbf{L}^2 & \mathbf{L}^2 + 1 \\ \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 \\ \mathbf{L}^2 & \mathbf{L}^2 + 1 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L} + 1 \\ \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + 1 & \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L} \\ \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L} + 1 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L}^{-1} \\ \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^2 & \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} \\ \mathbf{L}^2 + \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^2 + \mathbf{L}^{-1} & \mathbf{L}^3 + \mathbf{L} + 1 & \mathbf{L}^3 + \mathbf{L} \end{pmatrix}.$$

It follows from (30) that the implementation cost of matrix \mathbf{H} for m -bit input is equal to

$$\begin{aligned} & \overbrace{(4m)}^{\#\mathbf{E}_1} + \overbrace{(4m + 4(\#\mathbf{L}))}^{\#\mathbf{E}_2} + \overbrace{(4m)}^{\#\mathbf{E}_1} + \overbrace{(4m + 4(\#\mathbf{L}))}^{\#\mathbf{E}_3} + \overbrace{(4m + 4(\#\mathbf{L}^{-1}))}^{\#\mathbf{E}_4} + \overbrace{(4m + 4(\#\mathbf{L}))}^{\#\mathbf{E}_5} \\ & + \overbrace{(4m)}^{\#\mathbf{E}_1} = 28m + 12(\#\mathbf{L}) + 4(\#\mathbf{L}^{-1}). \end{aligned} \quad (31)$$

The base set of subdeterminants of \mathbf{H} has 285 elements that are provided in Appendix E. In this base set, there are all irreducible polynomials of degrees 4 and 8, except for the primitive polynomial $0x1E7$. It can be checked that the given elements in the Appendix E, are non-singular matrices over \mathbb{F}_2 by applying the following non-singular 8×8 binary matrices \mathbf{L}_i with $1 \leq i \leq 24$.

$$\begin{aligned} \mathbf{L}_1 &= [[1, 8], [1, 3, 7], [2], [3], [4], [5], [6], [7]], & \mathbf{L}_2 &= [[1, 8], [1, 3], [2, 8], [3], [4], [5], [6], [7]], \\ \mathbf{L}_3 &= [[1, 8], [1, 7], [2, 4], [3], [4], [5], [6], [7]], & \mathbf{L}_4 &= [[1, 8], [1, 7], [2], [3, 5], [4], [5], [6], [7]], \\ \mathbf{L}_5 &= [[1, 8], [1, 7], [2], [3], [4, 6], [5], [6], [7]], & \mathbf{L}_6 &= [[1, 8], [1, 7], [2], [3], [4], [5, 7], [6], [7]]. \end{aligned} \quad (32)$$

Moreover, the implementation cost of \mathbf{L}_i 's, given in (32), is three XOR. Although, the implementation cost of \mathbf{L}_i^{-1} with $1 \leq i \leq 24$ are not three XOR, the inverse of \mathbf{L}_i 's can be implemented with three XOR. For instance, consider $\mathbf{x} = [x_1, x_2, \dots, x_8]$. Then we have

$$\begin{aligned} \mathbf{L}_2^{-1} \cdot \mathbf{x}^T &= [x_2 + x_4, x_1 + x_2 + x_3 + x_4, x_4, x_5, x_6, x_7, x_8, x_1 + x_2 + x_4]^T, \\ u_1 &= x_2 + x_4, \quad u_2 = u_1 + x_1, \quad u_3 = u_2 + x_3. \end{aligned}$$

Hence, using \mathbf{L}_i with $1 \leq i \leq 24$ and relation (31), \mathbf{H} is implemented by $28 \times 8 + 12 \times 3 + 4 \times 3 = 272$ XOR for 8-bit input. Moreover, it can be verified that the minimal polynomial of \mathbf{L}_i 's is the irreducible polynomial $0x1E7$. Therefore, by applying Lemma 1 \mathbf{H} is implemented with 272 XOR over $\mathbb{F}_{2^8}/0x1E7$.

6.3.2 Construction of a Lightweight 8×8 MDS Matrix with Depth 8

The given result in this subsection is the major work of this paper. In fact, we propose an 8×8 MDS matrix \mathbf{H} such that the proposed matrix is implemented with 260 XOR for 8-bit input. Moreover, the depth of its circuit is 8 which means \mathbf{H} can be considered as a candidate for 8×8 lightweight MDS matrices in the lightweight cryptography. Consider

8×8 EGFS matrices \mathbf{E}_i with $1 \leq i \leq 4$, given in (33) which are constructed from two permutations $\mathbf{p}_1 = \{4, 3, 2, 1\}$ and $\mathbf{p}_2 = \{3, 2, 1, 4\}$.

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{E}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{L}^3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{L}^3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{L}^3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{L}^3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbf{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{L} & 0 \end{pmatrix}, \quad \mathbf{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{L}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{L}^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbf{L}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{L}^{-1} & 0 \end{pmatrix}. \quad (33)$$

Next, using EGFS matrices \mathbf{E}_i with $1 \leq i \leq 4$, the proposed lightweight 8×8 MDS matrix \mathbf{H} is presented by

$$\mathbf{H} = \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_1 \mathbf{E}_4 \mathbf{E}_1$$

$$= \begin{pmatrix} \mathbf{L}^4 + \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^4 + \mathbf{L}^2 + 1 & \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} + 1 \\ \mathbf{L}^3 + \mathbf{L}^{-1} & \mathbf{L}^{-1} + 1 & 1 & \mathbf{L}^3 + 1 \\ \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L}^4 & \mathbf{L}^3 \\ 1 & \mathbf{L}^3 + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} \\ \mathbf{L}^4 & \mathbf{L}^3 & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^4 + \mathbf{L}^3 + \mathbf{L}^{-1} \\ \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} & \mathbf{L}^2 & \mathbf{L}^2 + 1 \\ \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^4 + \mathbf{L}^3 + \mathbf{L}^{-1} & \mathbf{L}^4 + \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^4 + \mathbf{L}^2 + 1 \\ \mathbf{L}^2 & \mathbf{L}^2 + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} & \mathbf{L}^{-1} + 1 \end{pmatrix} \quad (34)$$

$$\begin{pmatrix} \mathbf{L}^4 & \mathbf{L}^3 & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^4 + \mathbf{L}^3 + \mathbf{L}^{-1} \\ \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} & \mathbf{L}^2 & \mathbf{L}^2 + 1 \\ \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^4 + \mathbf{L}^3 + \mathbf{L}^{-1} & \mathbf{L}^4 + \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^4 + \mathbf{L}^2 + 1 \\ \mathbf{L}^2 & \mathbf{L}^2 + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} & \mathbf{L}^{-1} + 1 \\ \mathbf{L}^4 + \mathbf{L}^2 + \mathbf{L} & \mathbf{L}^4 + \mathbf{L}^2 + 1 & \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} + 1 \\ \mathbf{L}^3 + \mathbf{L}^{-1} & \mathbf{L}^{-1} + 1 & 1 & \mathbf{L}^3 + 1 \\ \mathbf{L} + \mathbf{L}^{-1} & \mathbf{L}^3 + \mathbf{L} + \mathbf{L}^{-1} + 1 & \mathbf{L}^4 & \mathbf{L}^3 \\ 1 & \mathbf{L}^3 + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} + 1 & \mathbf{L}^3 + \mathbf{L}^{-1} \end{pmatrix}.$$

By applying (34) the implementation cost of matrix \mathbf{H} for m -bit input is equal to

$$\underbrace{\#\mathbf{E}_1}_{(4m)} + \underbrace{\#\mathbf{E}_2}_{(4m + 4(\#\mathbf{L}))} + \underbrace{\#\mathbf{E}_3}_{(4m + 4(\#\mathbf{L}^3))} + \underbrace{\#\mathbf{E}_1}_{(4m)} + \underbrace{\#\mathbf{E}_1}_{(4m)} + \underbrace{\#\mathbf{E}_4}_{(4m + 4(\#\mathbf{L}^{-1}))} + \underbrace{\#\mathbf{E}_1}_{(4m)}$$

$$= 28m + 4(\#\mathbf{L}) + 4(\#\mathbf{L}^{-1}) + 4(\#\mathbf{L}^3). \quad (35)$$

The base set of subdeterminants of \mathbf{H} has 324 elements that are given in Appendix A. In this base set, there are all irreducible and primitive polynomials of degree 4 and 8, except for the primitive polynomial $0x187$. Now using the following non-singular 8×8 binary matrices \mathbf{L}_i with $1 \leq i \leq 6$, it can be verified that the elements of Appendix A are non-singular matrices over \mathbb{F}_2 .

$$\mathbf{L}_1 = [[1, 8], [1, 7], [2], [3], [4], [5], [6], [7]], \quad \mathbf{L}_2 = [[1, 8], [1], [2, 8], [3], [4], [5], [6], [7]],$$

$$\mathbf{L}_3 = [[6, 8], [1], [2], [3], [4], [5], [6, 7], [7]], \quad \mathbf{L}_4 = [[6, 8], [1], [2], [3], [4], [5], [6], [7, 8]],$$

$$\mathbf{L}_5 = [[8], [1, 2], [2, 8], [3], [4], [5], [6], [7]], \quad \mathbf{L}_6 = [[8], [1, 7], [2], [3], [4], [5], [6], [7, 8]].$$

The implementation cost of \mathbf{L}_i with $1 \leq i \leq 6$ is two XOR. Moreover, \mathbf{L}_i^{-1} and \mathbf{L}_i^3 with $1 \leq i \leq 6$ can be implemented with two and five XOR, respectively. For instance, set

$\mathbf{x} = [x_1, x_2, \dots, x_8]$. Then the implementation cost of \mathbf{L}_6^3 can be reduced to five XOR.

$$\mathbf{L}_6^3 \cdot \mathbf{x}^T = [x_6 + x_7 + x_8, x_5 + x_7 + x_8, x_6 + x_8, x_1 + x_7, x_2, x_3, x_4, x_5 + x_6 + x_7 + x_8]^T, \\ u_1 = x_1 + x_7, \quad u_2 = x_6 + x_8, \quad u_3 = u_2 + x_7, \quad u_4 = u_3 + x_5, \quad u_5 = u_4 + x_6.$$

Therefore, by applying \mathbf{L}_i 's and relation (35), \mathbf{H} is implemented with $28 \times 8 + 4 \times 2 + 4 \times 2 + 4 \times 5 = 260$ XOR for 8-bit input. Furthermore, it can be checked that the minimal polynomial of \mathbf{L}_i 's is the irreducible polynomial $0x187$. Hence, using Lemma 1 the matrix \mathbf{H} is implemented with 260 XOR over $\mathbb{F}_{2^8}/0x187$. A summary of results of this paper is presented in Table 2.

Table 2: A summary of results of this paper.

Iteration	Implementation Cost	Total Cost	Inverse Cost	Depth	Fig.
Lightweight 4×4 MDS Matrices for 4-bit					
4 Round	8 XOR _{4-bit} , 4 L	36 XOR _{1-bit}	36 XOR _{1-bit}	6	1
4 Round	8 XOR _{4-bit} , 3 L	35 XOR _{1-bit}	36 XOR _{1-bit}	5	2
Lightweight 4×4 MDS Matrices for 8-bit					
4 Round	8 XOR _{8-bit} , 4 L	68 XOR _{1-bit}	68 XOR _{1-bit}	6	1
4 Round	8 XOR _{8-bit} , 3 L	67 XOR _{1-bit}	68 XOR _{1-bit}	5	2
Lightweight 6×6 MDS Matrices for 4, 6, and 8-bit					
6 Round	18 XOR _{4-bit} , 24 L	90 XOR _{1-bit}	90 XOR _{1-bit}	9	4
6 Round	18 XOR _{6-bit} , 6 L	114 XOR _{1-bit}	114 XOR _{1-bit}	8	3
6 Round	18 XOR _{8-bit} , 6 L	156 XOR _{1-bit}	156 XOR _{1-bit}	8	3
Lightweight 8×8 MDS Matrices for 8-bit					
7 Round	28 XOR _{8-bit} , 16 L	272 XOR _{1-bit}	296 XOR _{1-bit}	9	5
7 Round	28 XOR _{8-bit} , 20 L	264 XOR _{1-bit}	264 XOR _{1-bit}	11	6
7 Round	28 XOR _{8-bit} , 20 L	260 XOR _{1-bit}	276 XOR _{1-bit}	8	7

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7 Conclusion

This paper proposes a construction heuristic method to design MDS matrices with low hardware implementation cost based on the generalized Feistel structures (GFS). Feistel-based structures such as GFS are suitable choices to construct MDS matrices, since their inverses can be implemented with simplicity. First of all, using GFS, some types of sparse matrices, called primitive GFS matrices, are proposed. Then, by applying binary linear functions to primitive GFS matrices we proposed 4×4 and 8×8 MDS matrices. The proposed 4×4 and 8×8 matrices and their inverse are implemented with 68 and 264 XOR for 8-bit input, respectively. Next, using an extension of primitive GFS matrices we defined another type of sparse matrices called EGFS matrices. Then based on the EGFS matrices, 4×4 , 6×6 and 8×8 lightweight MDS matrices are implemented with 67, 156 and 260 XOR for 8-bit input, respectively. Moreover, we proposed an 6×6 MDS matrix such that the implementation cost of the proposed matrix and its inverse for 4-bit input is 90 XOR. One of the important features of this work is that the proposed MDS matrices are not only efficient by hardware terminology, but also are suitable by software perspective. In fact, the proposed 6×6 MDS matrix is constructed from six binary linear functions. Moreover, the depth of the circuit of the proposed 8×8 MDS matrix is eight.

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Appendix A

{0x2, 0x3, 0x7, 0xB, 0xD, 0x13, 0x19, 0x1F, 0x25, 0x29, 0x2F, 0x37, 0x3B, 0x3D, 0x43, 0x49, 0x61, 0x57, 0x5B, 0x67, 0x6D, 0x73, 0x75, 0x83, 0x89, 0x91, 0xC1, 0x8F, 0x9D, 0xA7, 0xAB, 0xB9, 0xCB, 0xD3, 0xD5, 0xE5, 0xF1, 0xBF, 0xEF, 0xF7, 0xFD, 0x11B, 0x11D, 0x12B, 0x12D, 0x139, 0x14D, 0x163, 0x165, 0x169, 0x171, 0x18B, 0x18D, 0x1A3, 0x1A9, 0x1B1, 0x1C3, 0x13F, 0x15F, 0x177, 0x17B, 0x19F, 0x1BD, 0x1CF, 0x1D7, 0x1DD, 0x1E7, 0x1F3, 0x1F5, 0x1F9, 0x203, 0x221, 0x301, 0x217, 0x21B, 0x233, 0x24B, 0x259, 0x265, 0x269, 0x287, 0x295, 0x299, 0x2A3, 0x2A5, 0x2D1, 0x313, 0x315, 0x323, 0x331, 0x349, 0x361, 0x385, 0x3A1, 0x25F, 0x26F, 0x277, 0x27D, 0x2AF, 0x2B7, 0x2BD, 0x2CF, 0x2DB, 0x2F5, 0x2F9, 0x31F, 0x33B, 0x34F, 0x35B, 0x36D, 0x373, 0x38F, 0x3B5, 0x3B9, 0x3C7, 0x3CB, 0x3D5, 0x3E3, 0x3E9, 0x37F, 0x3FB, 0x409, 0x481, 0x40F, 0x41B, 0x41D, 0x427, 0x42D, 0x435, 0x447, 0x499, 0x4C9, 0x50B, 0x50D, 0x523, 0x531, 0x543, 0x561, 0x5A1, 0x615, 0x631, 0x685, 0x689, 0x6C1, 0x711, 0x721, 0x781, 0x46F, 0x4AF, 0x4D7, 0x51F, 0x567, 0x56B, 0x597, 0x59B, 0x5AB, 0x5B9, 0x5C7, 0x637, 0x6A7, 0x6AD, 0x6B5, 0x6D3, 0x70F, 0x739, 0x747, 0x74D, 0x759, 0x763, 0x793, 0x4FF, 0x5FB, 0x67F, 0x6BF, 0x6DF, 0x6FD, 0x77B, 0x77D, 0x7DB, 0x7FF, 0x805, 0xA01, 0x817, 0x863, 0x871, 0x8D1, 0x90D, 0x913, 0x929, 0x945, 0x949, 0xA29, 0xA61, 0xC0B, 0xD03, 0xE21, 0x87B, 0x89F, 0xAB5, 0xAE3, 0xB33, 0xB4B, 0xB65, 0xB95, 0xBC9, 0xC73, 0xC75, 0xCCD, 0xCD3, 0xD27, 0xD2D, 0xDC9, 0xDE1, 0xE2B, 0xE33, 0xE4B, 0xE55, 0xEC9, 0xF19, 0xF83, 0xF91, 0x97F, 0xAEF, 0BAF, 0BBB, 0xBED, 0xCF7, 0xD9F, 0xDBB, 0xE7B, 0xE9F, 0xECF, 0xEDD, 0xF79, 0xFA7, 0xFAD, 0xFD3, 0x1161, 0x1431, 0x1823, 0x1883, 0x18A1, 0x1905, 0x103F, 0x10ED, 0x123D, 0x144F, 0x16A5, 0x1715, 0x186D, 0x192D, 0x1935, 0x1A2B, 0x1B19, 0x1CC9, 0x1DC1, 0x147F, 0x1B2F, 0x1B57, 0x1B8F, 0x1D2F, 0x1D67, 0x1DB3, 0x1E8F, 0x1EB9, 0x1ED3, 0x1EF1, 0x1F39, 0x1FC3, 0x1FC9, 0x17BF, 0x1FAF, 0x1FBD, 0x1FFF, 0x201B, 0x2189, 0x2461, 0x24C1, 0x2603, 0x21E3, 0x290F, 0x2AA9, 0x2E15, 0x320F, 0x324B, 0x3293, 0x3299, 0x344D, 0x3817, 0x3C23, 0x22EF, 0x274F, 0x2A7D, 0x2D37, 0x2EAB, 0x32B7, 0x32DB, 0x36D3, 0x3967, 0x3B63, 0x3F91, 0x2F5F, 0x2FF5, 0x37DD, 0x3BE7, 0x41D9, 0x5271, 0x612D, 0x6549, 0x6CC1, 0x7119, 0x6C3D, 0x6DA3, 0x7FD7, 0x8B3B, 0x94F5, 0x974D, 0xB0F3, 0xB84F, 0xE28F, 0xF40F, 0xB797, 0x1594B, 0x26B0B}

Appendix B

Case	\mathbf{p}_1	\mathbf{p}_2	Order
1	{1, 2, 3, 4}	{2, 1, 4, 3}	non
2	{1, 2, 3, 4}	{2, 3, 4, 1}	8
3	{1, 2, 3, 4}	{2, 4, 1, 3}	8
4	{1, 2, 3, 4}	{3, 1, 4, 2}	8
5	{1, 2, 3, 4}	{3, 4, 1, 2}	non
6	{1, 2, 3, 4}	{3, 4, 2, 1}	8
7	{1, 2, 3, 4}	{4, 1, 2, 3}	8
8	{1, 2, 3, 4}	{4, 3, 1, 2}	8
9	{1, 2, 3, 4}	{4, 3, 2, 1}	non
10	{1, 2, 4, 3}	{2, 1, 3, 4}	non
11	{1, 2, 4, 3}	{2, 3, 1, 4}	7
12	{1, 2, 4, 3}	{2, 4, 3, 1}	7
13	{1, 2, 4, 3}	{3, 1, 2, 4}	7
14	{1, 2, 4, 3}	{3, 4, 1, 2}	6
15	{1, 2, 4, 3}	{3, 4, 2, 1}	7
16	{1, 2, 4, 3}	{4, 1, 3, 2}	7
17	{1, 2, 4, 3}	{4, 3, 1, 2}	7
18	{1, 2, 4, 3}	{4, 3, 2, 1}	6
19	{1, 3, 2, 4}	{2, 1, 4, 3}	6
20	{1, 3, 2, 4}	{2, 4, 1, 3}	7
21	{1, 3, 2, 4}	{2, 4, 3, 1}	7
22	{1, 3, 2, 4}	{3, 1, 4, 2}	7
23	{1, 3, 2, 4}	{3, 2, 4, 1}	7
24	{1, 3, 2, 4}	{3, 4, 1, 2}	6
25	{1, 3, 2, 4}	{4, 1, 3, 2}	7
26	{1, 3, 2, 4}	{4, 2, 1, 3}	7
27	{1, 3, 2, 4}	{4, 2, 3, 1}	non
28	{1, 3, 4, 2}	{2, 1, 3, 4}	7
29	{1, 3, 4, 2}	{2, 4, 1, 3}	6
30	{1, 3, 4, 2}	{2, 4, 3, 1}	6
31	{1, 3, 4, 2}	{3, 1, 2, 4}	6
32	{1, 3, 4, 2}	{3, 2, 1, 4}	7
33	{1, 3, 4, 2}	{3, 4, 2, 1}	6
34	{1, 3, 4, 2}	{4, 1, 2, 3}	6
35	{1, 3, 4, 2}	{4, 2, 1, 3}	6
36	{1, 3, 4, 2}	{4, 2, 3, 1}	7
37	{1, 4, 2, 3}	{2, 1, 3, 4}	7
38	{1, 4, 2, 3}	{2, 3, 1, 4}	6
39	{1, 4, 2, 3}	{2, 3, 4, 1}	6
40	{1, 4, 2, 3}	{3, 1, 4, 2}	6
41	{1, 4, 2, 3}	{3, 2, 1, 4}	7
42	{1, 4, 2, 3}	{3, 2, 4, 1}	6
43	{1, 4, 2, 3}	{4, 1, 3, 2}	6
44	{1, 4, 2, 3}	{4, 2, 3, 1}	7
45	{1, 4, 2, 3}	{4, 3, 1, 2}	6
46	{1, 4, 3, 2}	{2, 1, 4, 3}	6
47	{1, 4, 3, 2}	{2, 3, 1, 4}	7
48	{1, 4, 3, 2}	{2, 3, 4, 1}	7
49	{1, 4, 3, 2}	{3, 1, 2, 4}	7
50	{1, 4, 3, 2}	{3, 2, 1, 4}	non
51	{1, 4, 3, 2}	{3, 2, 4, 1}	7
52	{1, 4, 3, 2}	{4, 1, 2, 3}	7
53	{1, 4, 3, 2}	{4, 2, 1, 3}	7
54	{1, 4, 3, 2}	{4, 3, 2, 1}	6

Case	\mathbf{p}_1	\mathbf{p}_2	Order
55	{2, 1, 3, 4}	{1, 2, 4, 3}	non
56	{2, 1, 3, 4}	{1, 3, 4, 2}	7
57	{2, 1, 3, 4}	{1, 4, 2, 3}	7
58	{2, 1, 3, 4}	{3, 2, 4, 1}	7
59	{2, 1, 3, 4}	{3, 4, 1, 2}	6
60	{2, 1, 3, 4}	{3, 4, 2, 1}	7
61	{2, 1, 3, 4}	{4, 2, 1, 3}	7
62	{2, 1, 3, 4}	{4, 3, 1, 2}	7
63	{2, 1, 3, 4}	{4, 3, 2, 1}	6
64	{2, 1, 4, 3}	{1, 2, 3, 4}	non
65	{2, 1, 4, 3}	{1, 3, 2, 4}	6
66	{2, 1, 4, 3}	{1, 4, 3, 2}	6
67	{2, 1, 4, 3}	{3, 2, 1, 4}	6
68	{2, 1, 4, 3}	{3, 4, 1, 2}	non
69	{2, 1, 4, 3}	{3, 4, 2, 1}	8
70	{2, 1, 4, 3}	{4, 2, 3, 1}	6
71	{2, 1, 4, 3}	{4, 3, 1, 2}	8
72	{2, 1, 4, 3}	{4, 3, 2, 1}	non
73	{2, 3, 1, 4}	{1, 2, 4, 3}	7
74	{2, 3, 1, 4}	{1, 4, 2, 3}	6
75	{2, 3, 1, 4}	{1, 4, 3, 2}	7
76	{2, 3, 1, 4}	{3, 1, 4, 2}	6
77	{2, 3, 1, 4}	{3, 2, 4, 1}	6
78	{2, 3, 1, 4}	{3, 4, 2, 1}	6
79	{2, 3, 1, 4}	{4, 1, 2, 3}	6
80	{2, 3, 1, 4}	{4, 1, 3, 2}	6
81	{2, 3, 1, 4}	{4, 2, 3, 1}	7
82	{2, 3, 4, 1}	{1, 2, 3, 4}	8
83	{2, 3, 4, 1}	{1, 4, 2, 3}	7
84	{2, 3, 4, 1}	{1, 4, 3, 2}	7
85	{2, 3, 4, 1}	{3, 1, 2, 4}	7
86	{2, 3, 4, 1}	{3, 2, 1, 4}	7
87	{2, 3, 4, 1}	{3, 4, 1, 2}	8
88	{2, 3, 4, 1}	{4, 1, 2, 3}	non
89	{2, 3, 4, 1}	{4, 1, 3, 2}	7
90	{2, 3, 4, 1}	{4, 2, 1, 3}	7
91	{2, 4, 1, 3}	{1, 2, 3, 4}	8
92	{2, 4, 1, 3}	{1, 3, 2, 4}	7
93	{2, 4, 1, 3}	{1, 3, 4, 2}	7
94	{2, 4, 1, 3}	{3, 1, 2, 4}	7
95	{2, 4, 1, 3}	{3, 1, 4, 2}	non
96	{2, 4, 1, 3}	{3, 2, 4, 1}	7
97	{2, 4, 1, 3}	{4, 1, 3, 2}	7
98	{2, 4, 1, 3}	{4, 2, 3, 1}	7
99	{2, 4, 1, 3}	{4, 3, 2, 1}	8
100	{2, 4, 3, 1}	{1, 2, 4, 3}	7
101	{2, 4, 3, 1}	{1, 3, 2, 4}	7
102	{2, 4, 3, 1}	{1, 3, 4, 2}	6
103	{2, 4, 3, 1}	{3, 1, 2, 4}	6
104	{2, 4, 3, 1}	{3, 1, 4, 2}	6
105	{2, 4, 3, 1}	{3, 2, 1, 4}	7
106	{2, 4, 3, 1}	{4, 1, 2, 3}	6
107	{2, 4, 3, 1}	{4, 2, 1, 3}	6
108	{2, 4, 3, 1}	{4, 3, 1, 2}	6

Case	p_1	p_2	Order
109	{3, 1, 2, 4}	{1, 2, 4, 3}	7
110	{3, 1, 2, 4}	{1, 3, 4, 2}	6
111	{3, 1, 2, 4}	{1, 4, 3, 2}	7
112	{3, 1, 2, 4}	{2, 3, 4, 1}	6
113	{3, 1, 2, 4}	{2, 4, 1, 3}	6
114	{3, 1, 2, 4}	{2, 4, 3, 1}	6
115	{3, 1, 2, 4}	{4, 2, 1, 3}	6
116	{3, 1, 2, 4}	{4, 2, 3, 1}	7
117	{3, 1, 2, 4}	{4, 3, 1, 2}	6
118	{3, 1, 4, 2}	{1, 2, 3, 4}	8
119	{3, 1, 4, 2}	{1, 3, 2, 4}	7
120	{3, 1, 4, 2}	{1, 4, 2, 3}	7
121	{3, 1, 4, 2}	{2, 3, 1, 4}	7
122	{3, 1, 4, 2}	{2, 4, 1, 3}	non
123	{3, 1, 4, 2}	{2, 4, 3, 1}	7
124	{3, 1, 4, 2}	{4, 2, 1, 3}	7
125	{3, 1, 4, 2}	{4, 2, 3, 1}	7
126	{3, 1, 4, 2}	{4, 3, 2, 1}	8
127	{3, 2, 1, 4}	{1, 3, 4, 2}	7
128	{3, 2, 1, 4}	{1, 4, 2, 3}	7
129	{3, 2, 1, 4}	{1, 4, 3, 2}	non
130	{3, 2, 1, 4}	{2, 1, 4, 3}	6
131	{3, 2, 1, 4}	{2, 3, 4, 1}	7
132	{3, 2, 1, 4}	{2, 4, 3, 1}	7
133	{3, 2, 1, 4}	{4, 1, 2, 3}	7
134	{3, 2, 1, 4}	{4, 1, 3, 2}	7
135	{3, 2, 1, 4}	{4, 3, 2, 1}	6
136	{3, 2, 4, 1}	{1, 3, 2, 4}	7
137	{3, 2, 4, 1}	{1, 4, 2, 3}	6
138	{3, 2, 4, 1}	{1, 4, 3, 2}	7
139	{3, 2, 4, 1}	{2, 1, 3, 4}	7
140	{3, 2, 4, 1}	{2, 3, 1, 4}	6
141	{3, 2, 4, 1}	{2, 4, 1, 3}	6
142	{3, 2, 4, 1}	{4, 1, 2, 3}	6
143	{3, 2, 4, 1}	{4, 1, 3, 2}	6
144	{3, 2, 4, 1}	{4, 3, 1, 2}	6
145	{3, 4, 1, 2}	{1, 2, 3, 4}	non
146	{3, 4, 1, 2}	{1, 2, 4, 3}	6
147	{3, 4, 1, 2}	{1, 3, 2, 4}	6
148	{3, 4, 1, 2}	{2, 1, 3, 4}	6
149	{3, 4, 1, 2}	{2, 1, 4, 3}	non
150	{3, 4, 1, 2}	{2, 3, 4, 1}	8
151	{3, 4, 1, 2}	{4, 1, 2, 3}	8
152	{3, 4, 1, 2}	{4, 2, 3, 1}	6
153	{3, 4, 1, 2}	{4, 3, 2, 1}	non
154	{3, 4, 2, 1}	{1, 2, 3, 4}	8
155	{3, 4, 2, 1}	{1, 2, 4, 3}	7
156	{3, 4, 2, 1}	{1, 3, 4, 2}	7
157	{3, 4, 2, 1}	{2, 1, 3, 4}	7
158	{3, 4, 2, 1}	{2, 1, 4, 3}	8
159	{3, 4, 2, 1}	{2, 3, 1, 4}	7
160	{3, 4, 2, 1}	{4, 1, 3, 2}	7
161	{3, 4, 2, 1}	{4, 2, 1, 3}	7
162	{3, 4, 2, 1}	{4, 3, 1, 2}	non

Case	p_1	p_2	Order
163	{4, 1, 2, 3}	{1, 2, 3, 4}	8
164	{4, 1, 2, 3}	{1, 3, 4, 2}	7
165	{4, 1, 2, 3}	{1, 4, 3, 2}	7
166	{4, 1, 2, 3}	{2, 3, 1, 4}	7
167	{4, 1, 2, 3}	{2, 3, 4, 1}	non
168	{4, 1, 2, 3}	{2, 4, 3, 1}	7
169	{4, 1, 2, 3}	{3, 2, 1, 4}	7
170	{4, 1, 2, 3}	{3, 2, 4, 1}	7
171	{4, 1, 2, 3}	{3, 4, 1, 2}	8
172	{4, 1, 3, 2}	{1, 2, 4, 3}	7
173	{4, 1, 3, 2}	{1, 3, 2, 4}	7
174	{4, 1, 3, 2}	{1, 4, 2, 3}	6
175	{4, 1, 3, 2}	{2, 3, 1, 4}	6
176	{4, 1, 3, 2}	{2, 3, 4, 1}	6
177	{4, 1, 3, 2}	{2, 4, 1, 3}	6
178	{4, 1, 3, 2}	{3, 2, 1, 4}	7
179	{4, 1, 3, 2}	{3, 2, 4, 1}	6
180	{4, 1, 3, 2}	{3, 4, 2, 1}	6
181	{4, 2, 1, 3}	{1, 3, 2, 4}	7
182	{4, 2, 1, 3}	{1, 3, 4, 2}	6
183	{4, 2, 1, 3}	{1, 4, 3, 2}	7
184	{4, 2, 1, 3}	{2, 1, 3, 4}	7
185	{4, 2, 1, 3}	{2, 3, 4, 1}	6
186	{4, 2, 1, 3}	{2, 4, 3, 1}	6
187	{4, 2, 1, 3}	{3, 1, 2, 4}	6
188	{4, 2, 1, 3}	{3, 1, 4, 2}	6
189	{4, 2, 1, 3}	{3, 4, 2, 1}	6
190	{4, 2, 3, 1}	{1, 3, 2, 4}	non
191	{4, 2, 3, 1}	{1, 3, 4, 2}	7
192	{4, 2, 3, 1}	{1, 4, 2, 3}	7
193	{4, 2, 3, 1}	{2, 1, 4, 3}	6
194	{4, 2, 3, 1}	{2, 3, 1, 4}	7
195	{4, 2, 3, 1}	{2, 4, 1, 3}	7
196	{4, 2, 3, 1}	{3, 1, 2, 4}	7
197	{4, 2, 3, 1}	{3, 1, 4, 2}	7
198	{4, 2, 3, 1}	{3, 4, 1, 2}	6
199	{4, 3, 1, 2}	{1, 2, 3, 4}	8
200	{4, 3, 1, 2}	{1, 2, 4, 3}	7
201	{4, 3, 1, 2}	{1, 4, 2, 3}	7
202	{4, 3, 1, 2}	{2, 1, 3, 4}	7
203	{4, 3, 1, 2}	{2, 1, 4, 3}	8
204	{4, 3, 1, 2}	{2, 4, 3, 1}	7
205	{4, 3, 1, 2}	{3, 1, 2, 4}	7
206	{4, 3, 1, 2}	{3, 2, 4, 1}	7
207	{4, 3, 1, 2}	{3, 4, 2, 1}	non
208	{4, 3, 2, 1}	{1, 2, 3, 4}	non
209	{4, 3, 2, 1}	{1, 2, 4, 3}	6
210	{4, 3, 2, 1}	{1, 4, 3, 2}	6
211	{4, 3, 2, 1}	{2, 1, 3, 4}	6
212	{4, 3, 2, 1}	{2, 1, 4, 3}	non
213	{4, 3, 2, 1}	{2, 4, 1, 3}	8
214	{4, 3, 2, 1}	{3, 1, 4, 2}	8
215	{4, 3, 2, 1}	{3, 2, 1, 4}	6
216	{4, 3, 2, 1}	{3, 4, 1, 2}	non

Appendix C

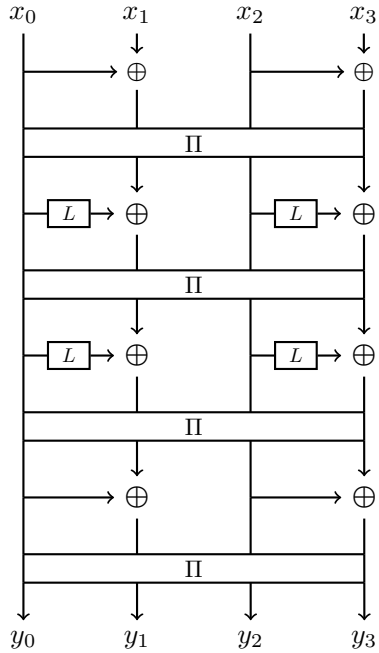


Figure 1: 4×4 MDS matrix with depth 6:
 $\Pi(0, 1, 2, 3) = (3, 0, 1, 2)$

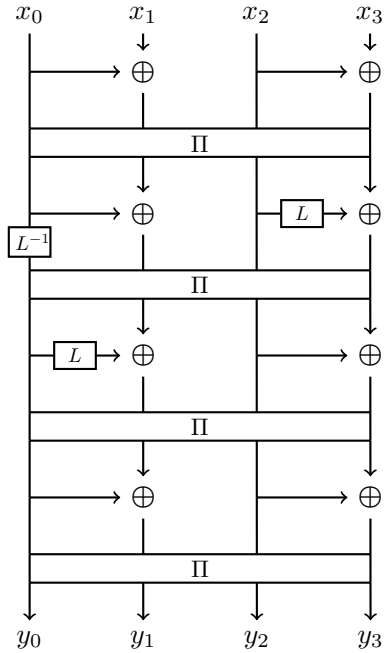


Figure 2: 4×4 MDS matrix with depth 5:
 $\Pi(0, 1, 2, 3) = (3, 0, 1, 2)$

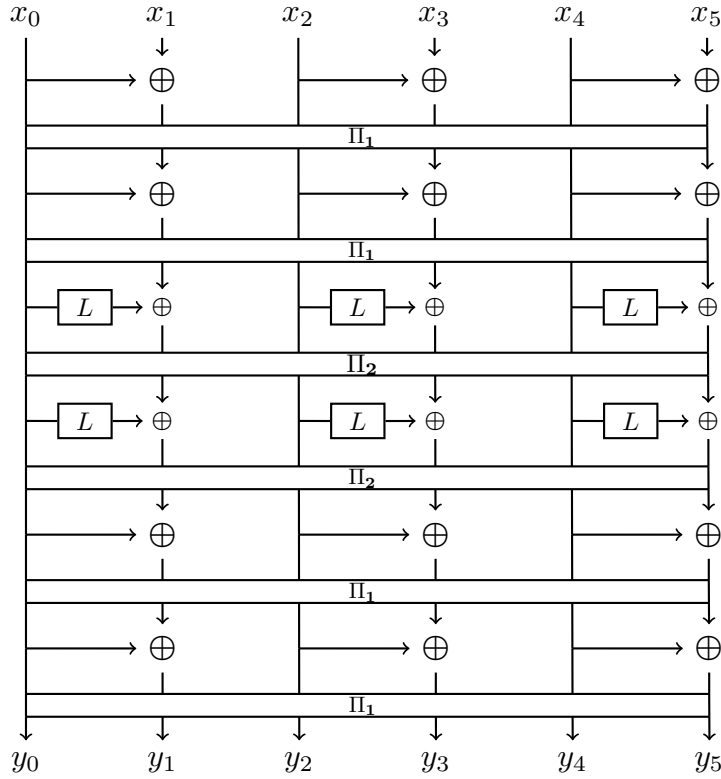


Figure 3: 6×6 MDS matrix with depth 8:
 $\Pi_1(0, 1, 2, 3, 4, 5, 6) = (3, 0, 5, 2, 1, 4)$
 $\Pi_2(0, 1, 2, 3, 4, 5, 6) = (3, 4, 1, 2, 5, 0)$

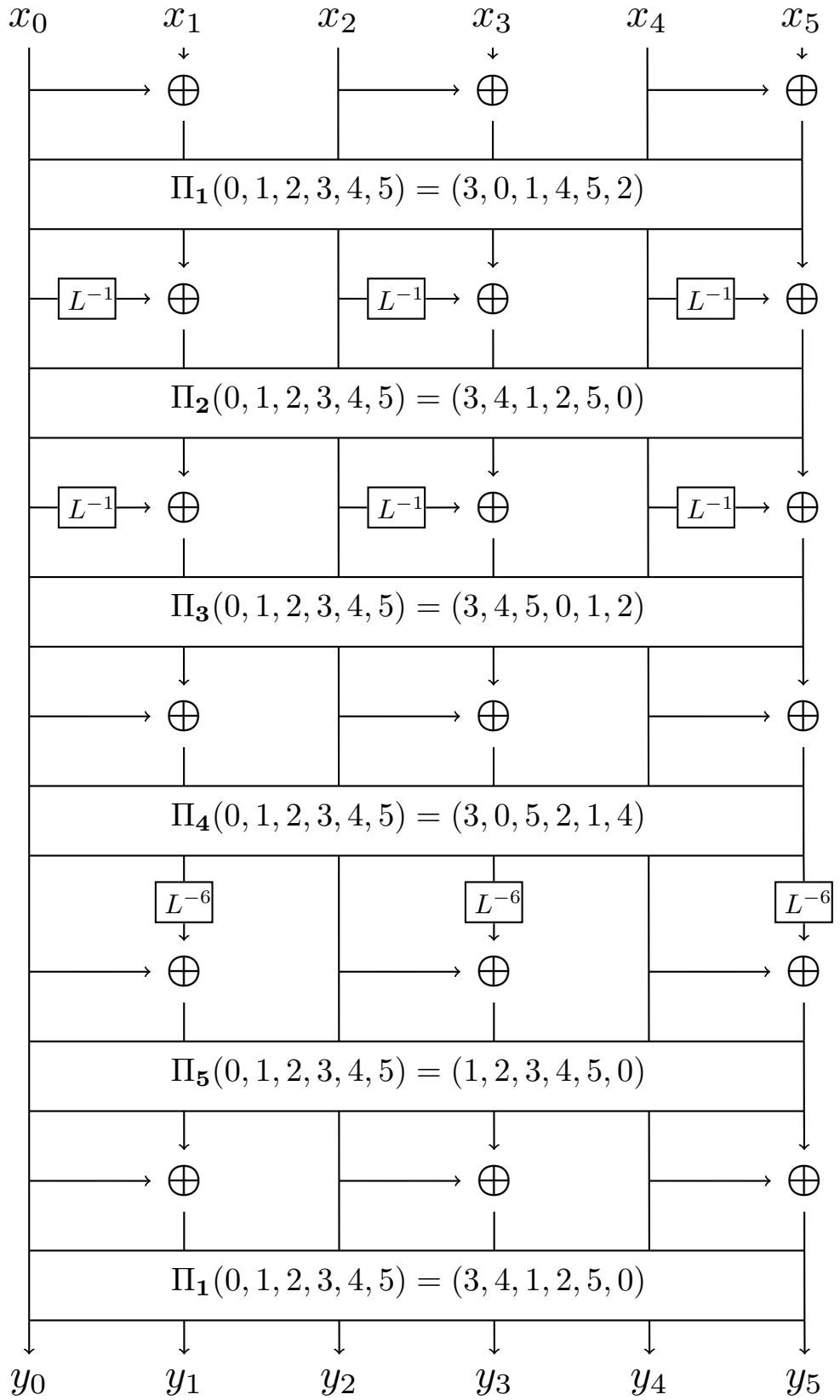


Figure 4: 6×6 MDS matrix with depth 9

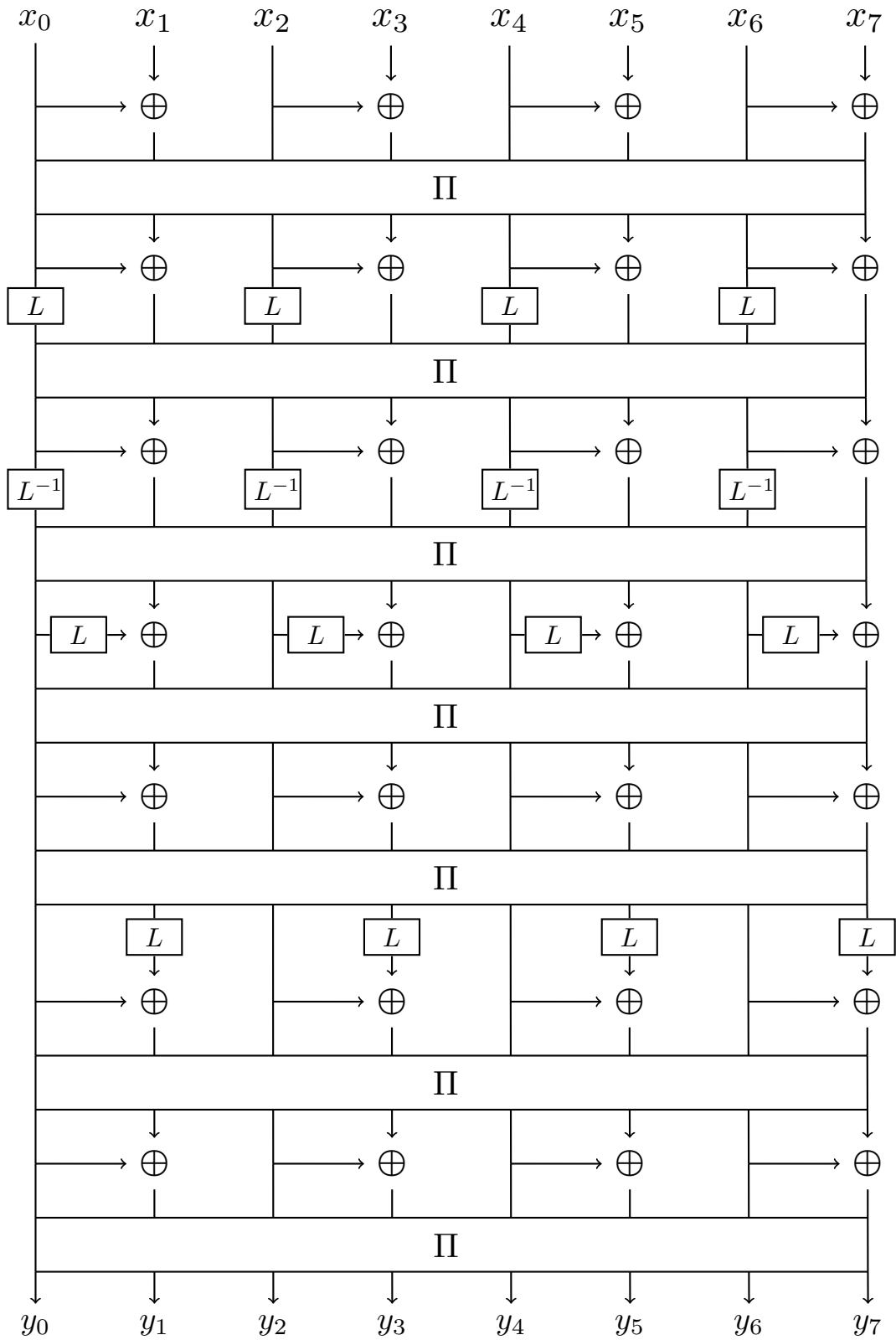


Figure 5: 8×8 MDS matrix with depth 9:
 $\Pi(0, 1, 2, 3, 4, 5, 6, 7) = (5, 6, 3, 4, 1, 2, 7, 0)$

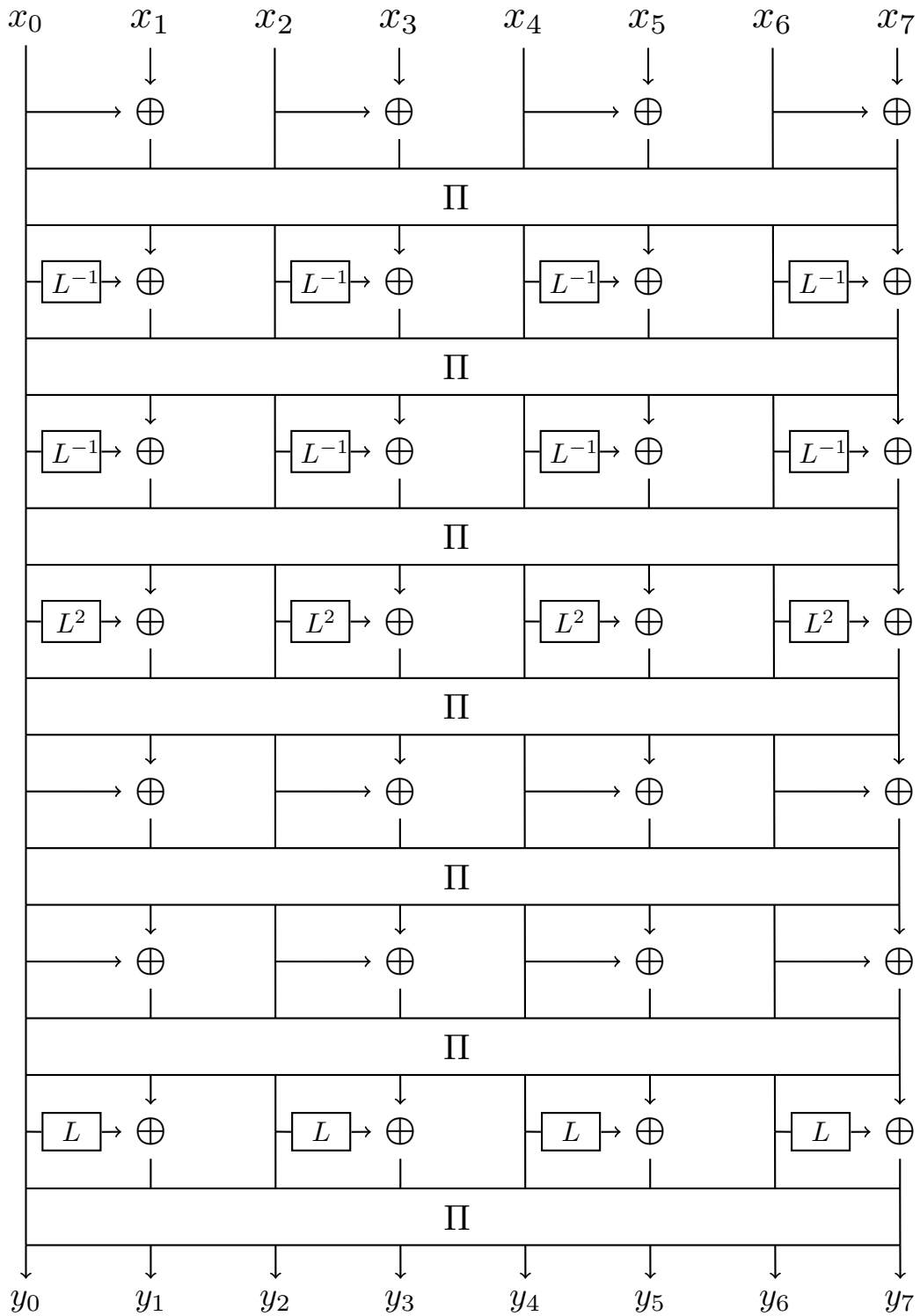


Figure 6: 8×8 MDS matrix with depth 11:
 $\Pi(0, 1, 2, 3, 4, 5, 6, 7) = (3, 0, 1, 4, 7, 2, 5, 6)$

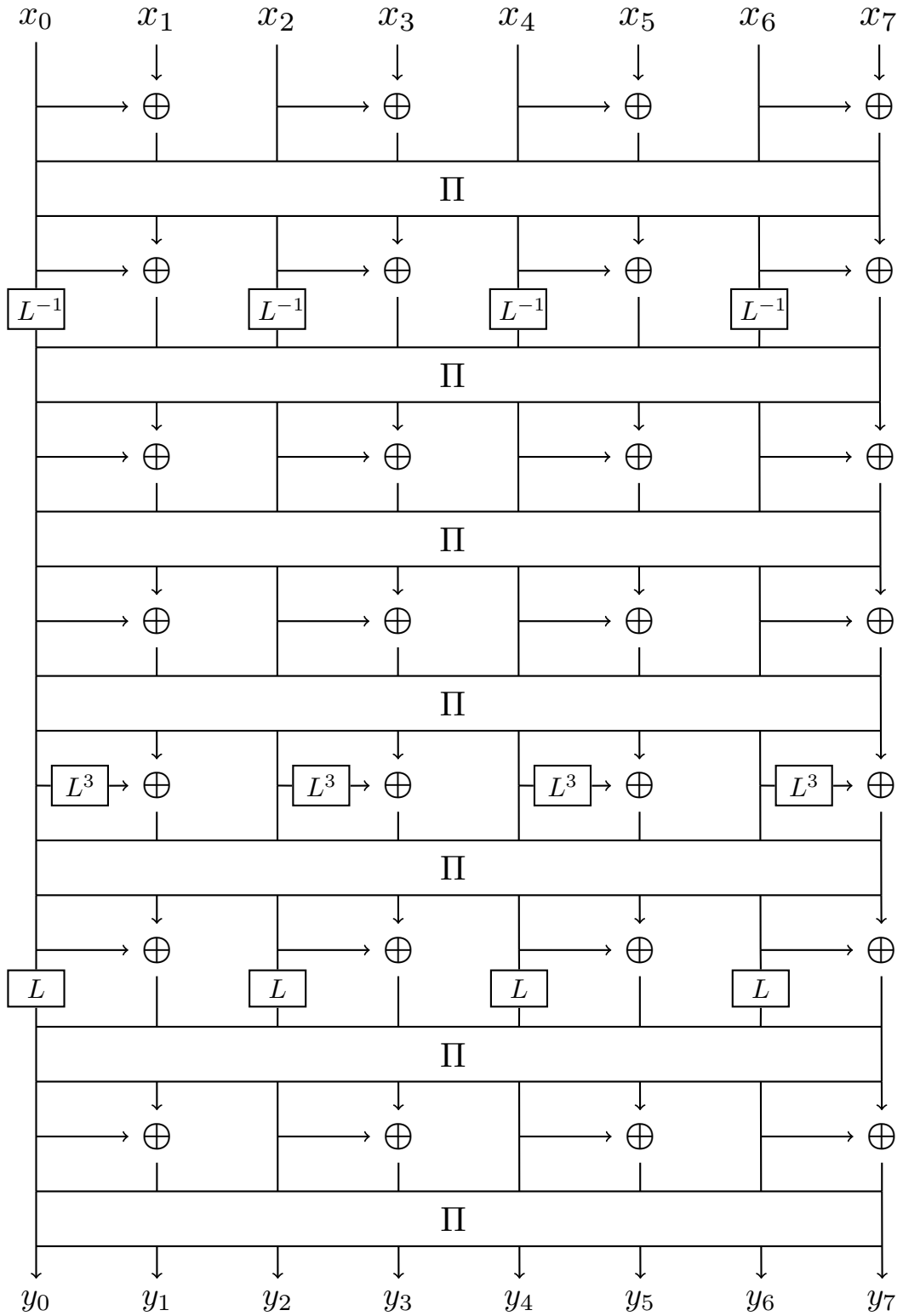


Figure 7: 8×8 MDS matrix with depth 8:
 $\Pi(0, 1, 2, 3, 4, 5, 6, 7) = (5, 6, 3, 4, 1, 2, 7, 0)$

Appendix D

```
{0x2, 0x3, 0x7, 0xB, 0xD, 0x13, 0x19, 0x1F, 0x25, 0x29, 0x2F, 0x37, 0x3B, 0x3D, 0x43, 0x49,
0x61, 0x57, 0x5B, 0x67, 0x6D, 0x73, 0x75, 0x83, 0x89, 0x8F, 0x91, 0x9D, 0xA7, 0xAB, 0xB9,
0xCB, 0xD3, 0xD5, 0xE5, 0xF1, 0xBF, 0xEF, 0xF7, 0xFD, 0x11B, 0x11D, 0x12B, 0x12D, 0x139, 0x14D,
0x163, 0x165, 0x169, 0x171, 0x187, 0x18B, 0x18D, 0x1A9, 0x1B1, 0x1C3, 0x13F, 0x15F, 0x177,
0x17B, 0x19F, 0x1BD, 0x1CF, 0x1D7, 0x1DD, 0x1E7, 0x1F3, 0x1F5, 0x1F9, 0x203, 0x211, 0x301,
0x217, 0x21B, 0x22D, 0x233, 0x24B, 0x259, 0x269, 0x287, 0x295, 0x299, 0x2A3, 0x2A5, 0x2D1,
0x313, 0x315, 0x323, 0x331, 0x349, 0x361, 0x385, 0x3A1, 0x25F, 0x26F, 0x277, 0x27D, 0x2AF,
0x2BD, 0x2CF, 0x2DB, 0x2F5, 0x31F, 0x33B, 0x34F, 0x35B, 0x36D, 0x373, 0x3B5, 0x3B9, 0x3C7,
0x3CB, 0x3CD, 0x3D5, 0x3D9, 0x3E3, 0x3E9, 0x37F, 0x3FB, 0x409, 0x481, 0x40F, 0x41B, 0x41D,
0x427, 0x447, 0x465, 0x48B, 0x4C5, 0x4C9, 0x50D, 0x519, 0x543, 0x561, 0x5A1, 0x5C1, 0x613,
0x615, 0x623, 0x625, 0x643, 0x651, 0x685, 0x689, 0x6C1, 0x711, 0x721, 0x4AF, 0x4D7, 0x4E7,
0x4ED, 0x4F3, 0x51F, 0x53D, 0x557, 0x56B, 0x597, 0x59B, 0x5B9, 0x5C7, 0x5E5, 0x637, 0x65B,
0x679, 0x6A7, 0x6AD, 0x6B5, 0x6CD, 0x6D3, 0x70F, 0x71D, 0x72B, 0x735, 0x739, 0x747, 0x74D,
0x759, 0x763, 0x793, 0x7A9, 0x4FF, 0x5F7, 0x5FB, 0x67F, 0x6DF, 0x6F7, 0x77D, 0x7DB, 0x7F3,
0x7F9, 0x805, 0xA01, 0x82D, 0x847, 0x863, 0x865, 0x895, 0x8A9, 0x8C3, 0x8E1, 0x913, 0x951,
0xA07, 0xA13, 0xA15, 0xA61, 0xB11, 0xB41, 0xC31, 0xC89, 0xE05, 0x87B, 0x8F5, 0x937, 0x93B,
0x9C7, 0x9D9, 0xA6D, 0xA79, 0xAA7, 0xAAB, 0xAE3, 0xAF1, 0xB4B, 0xB65, 0xB93, 0xC1F, 0xC57,
0xCC7, 0xCCD, 0xD1D, 0xD27, 0xD47, 0xD55, 0xD63, 0xD71, 0xD93, 0xDA9, 0xDE1, 0xE2B, 0xE33,
0xE4B, 0xE93, 0xEC9, 0xF91, 0x97F, 0x9EF, 0xB6F, 0xB7D, 0xBBD, 0xBDB, 0xBDD, 0xBED, 0xD6F,
0xD9F, 0xDBD, 0xE7B, 0xEDD, 0xF5D, 0xF6B, 0xF79, 0xFA7, 0xFF7, 0xFFB, 0x1009, 0x1407,
0x1431, 0x1641, 0x180D, 0x1909, 0x1981, 0x114F, 0x1157, 0x1237, 0x123D, 0x126D, 0x12CB,
0x130F, 0x131D, 0x1365, 0x13A3, 0x14A7, 0x150F, 0x154D, 0x161B, 0x1659, 0x1719, 0x181F,
0x18CB, 0x1965, 0x1A4D, 0x1A69, 0x1A8B, 0x1BC1, 0x1C0F, 0x1CC9, 0x1F11, 0x15BB, 0x15DD,
0x174F, 0x17B3, 0x18FB, 0x196F, 0x19B7, 0x1B57, 0x1BCB, 0x1CF3, 0x1D5B, 0x1E3D, 0x1E97,
0x1F71, 0x17DF, 0x1FAF, 0x1FBD, 0x208B, 0x2219, 0x240B, 0x2605, 0x20BB, 0x2355, 0x246D,
0x2595, 0x2875, 0x28AD, 0x290F, 0x2AA5, 0x3253, 0x3315, 0x34A5, 0x3625, 0x276B, 0x27B9,
0x2CD7, 0x2E4F, 0x2E6B, 0x2E97, 0x2EB9, 0x36D5, 0x3967, 0x3B2B, 0x3BBB, 0x3BFF, 0x4941,
0x6841, 0x42CB, 0x4D31, 0x6545, 0x7123, 0x44DF, 0x50FB, 0x617D, 0x64F5, 0x6EC3, 0x78D9,
0x79C9, 0x7AA9, 0x7AE1, 0x7B45, 0x7C95, 0x47EF, 0x57D7, 0x79D7, 0x7DD5, 0x73FF, 0x81B9,
0xB889, 0xF141, 0xA16F, 0xA3D5, 0xAE2D, 0xB077, 0xB4BF, 0x11361, 0x15D15, 0x19437,
0x39967, 0x3BBC7, 0x3DF2D, 0x3FCCD}
```

Appendix E

```
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0x57, 0x5B, 0x61, 0x67, 0x6D, 0x73, 0x75, 0x83, 0x89, 0x8F, 0x91, 0x9D, 0xA7, 0xAB, 0xB9,
0xBF, 0xC1, 0xCB, 0xD3, 0xD5, 0xE5, 0xEF, 0xF1, 0xF7, 0xFD, 0x11B, 0x11D, 0x12B, 0x12D,
0x139, 0x13F, 0x14D, 0x15F, 0x163, 0x165, 0x169, 0x171, 0x177, 0x17B, 0x187, 0x18B,
0x18D, 0x19F, 0x1A3, 0x1A9, 0x1B1, 0x1BD, 0x1C3, 0x1CF, 0x1D7, 0x1DD, 0x1F3, 0x1F5,
0x1F9, 0x203, 0x21B, 0x221, 0x22D, 0x233, 0x24B, 0x25F, 0x265, 0x269, 0x277, 0x27D,
0x287, 0x295, 0x299, 0x2A3, 0x2A5, 0x2B7, 0x2CF, 0x2DB, 0x2F5, 0x2F9, 0x313, 0x315,
0x31F, 0x331, 0x33B, 0x34F, 0x35B, 0x361, 0x36B, 0x36D, 0x373, 0x37F, 0x385, 0x3A1,
0x3B9, 0x3C7, 0x3CB, 0x3CD, 0x3D5, 0x3D9, 0x3E3, 0x3E9, 0x3FB, 0x409, 0x41B, 0x435,
0x447, 0x453, 0x465, 0x46F, 0x481, 0x4A9, 0x4C5, 0x4E7, 0x4F3, 0x4FF, 0x523, 0x53D,
0x543, 0x557, 0x58F, 0x59B, 0x5A1, 0x5AB, 0x5C7, 0x5F7, 0x615, 0x623, 0x631, 0x637,
0x64F, 0x65B, 0x679, 0x67F, 0x685, 0x689, 0x6A7, 0x6AD, 0x6B5, 0x6C1, 0x6CD, 0x711,
0x717, 0x71D, 0x721, 0x72B, 0x735, 0x755, 0x759, 0x77B, 0x77D, 0x781, 0x787, 0x7B1,
0x7C5, 0x7DB, 0x7F3, 0x7F9, 0x7FF, 0x805, 0x82D, 0x88D, 0x8A9, 0x8C3, 0x8CF, 0x8D1,
0x8E7, 0x93B, 0x949, 0x951, 0x973, 0x975, 0x9E5, 0x9EF, 0xA07, 0xA13, 0xA15, 0xA6D,
0xA79, 0xA7F, 0xAD5, 0xADF, 0xB11, 0xB33, 0xB3F, 0xB87, 0xB95, 0xBAF, 0xBBD, 0xBC9,
0xC0D, 0xC97, 0xCBF, 0xCC7, 0xD0F, 0xD1D, 0xD27, 0xD93, 0xDBB, 0xDC9, 0xDD7, 0xE27,
0xE2B, 0xE7B, 0xEA3, 0xEC9, 0xEFCF, 0xEF9, 0xF0B, 0xF19, 0xF6B, 0x1069, 0x1077,
0x10D1, 0x11EF, 0x1219, 0x13A9, 0x14B5, 0x154D, 0x1593, 0x15BB, 0x15C5, 0x16E7,
0x17FB, 0x1823, 0x1879, 0x197B, 0x19CF, 0x19F9, 0x1A69, 0x1BFD, 0x1C03, 0x1C27,
0x1CBB, 0x1CED, 0x1E3D, 0x1F11, 0x1F1B, 0x1FAF, 0x1FC3, 0x1FE1, 0x227F, 0x232B,
0x2429, 0x25BD, 0x2B2F, 0x2B97, 0x2F5F, 0x329F, 0x33E5, 0x3499, 0x3A61, 0x3FB5,
0x49E1, 0x4A17, 0x549F, 0x5585, 0x6A6B, 0x6D05, 0x7327, 0x74C7, 0x7BB9, 0x7CA3,
0xA6C7, 0xA7D1, 0xAF2F, 0xB08D, 0xB24F, 0xB909, 0xBB15, 0xD91B, 0xE05F, 0xEB97, 0x14ABF}
```