Unbounded ABE via Bilinear Entropy Expansion, Revisited

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Abstract. We present simpler and improved constructions of unbounded attribute-based encryption (ABE) schemes with constant-size public parameters under static assumptions in bilinear groups. Concretely, we obtain:

- a simple and adaptively secure unbounded ABE scheme in composite-order groups, improving upon a previous construction of Lewko and Waters (Eurocrypt '11) which only achieves selective security;
- an improved adaptively secure unbounded ABE scheme based on the k-linear assumption in prime-order groups with shorter ciphertexts and secret keys than those of Okamoto and Takashima (Asiacrypt '12);
- the first adaptively secure unbounded ABE scheme for arithmetic branching programs under static assumptions.

At the core of all of these constructions is a "bilinear entropy expansion" lemma that allows us to generate any polynomial amount of entropy starting from constant-size public parameters; the entropy can then be used to transform existing adaptively secure "bounded" ABE schemes into unbounded ones.

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1 Introduction

Attribute-based encryption (ABE) [25, 13] is a generalization of public-key encryption to support fine-grained access control for encrypted data. Here, ciphertexts and keys are associated with descriptive values which determine whether decryption is possible. In a key-policy ABE (KP-ABE) scheme for instance, ciphertexts are associated with attributes like '(author:Waters), (inst:UT), (topic:PK)' and keys with access policies like '((topic:MPC) OR (topic:Qu)) AND (NOT(inst:CWI))', and decryption is possible only when the

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attributes satisfy the access policy. A ciphertext-policy (CP-ABE) scheme is the dual of KP-ABE with ciphertexts associated with policies and keys with attributes.

Over past decade, substantial progress has been made in the design and analysis of ABE schemes, leading to a large families of schemes that achieve various trade-offs between efficiency, security and underlying assumptions. Meanwhile, ABE has found use as a tool for providing and enhancing privacy in a variety of settings from electronic medical records to messaging systems and online social networks.

As institutions grow and with new emerging and more complex applications for ABE, it became clear that we need ABE schemes that can readily accommodate the addition of new roles, entities, attributes and policies. This means that the ABE set-up algorithm should put no restriction on the length of the attributes or the size of the policies that will be used in the ciphertexts and keys. This requirement was introduced and first realized in the work of Lewko and Waters [21] under the term *unbounded ABE*. Their constructions have since been improved and extended in several subsequent works [18, 23, 24, 2, 17, 3, 5, 12, 1] (cf. Fig 1, 2).

In this work, we put forth new ABE schemes that simultaneously:

(1) are unbounded (the set-up algorithm is independent of the length of the attributes or the size of the policies);
(2) can be based on faster asymmetric prime-order bilinear groups;
(3) achieve adaptive security;
(4) rely on simple hardness assumptions in the standard model.

All four properties are highly desirable from both a practical and theoretical stand-point and moreover, properties (1) – (3) are crucial for many real-world applications of ABE. Indeed, properties (2), (3) and (4) are by now standard cryptographic requirements pertaining to speed and efficiency, strong security guarantees under realistic and natural attack models, and minimal hardness assumptions. Property (2) is additionally motivated by the fact that pairing-based schemes are currently more widely implemented and deployed than lattice-based ones. There is now a vast body of works (e.g. [19, 22, 27, 2, 6, 3]) showing how to achieve properties (2) – (4) for "bounded" ABE where the set-up time and public parameters grow with the attributes or policies, culminating in unifying frameworks that provide a solid understanding of the design and analysis of these schemes. Unbounded ABE, on the other hand, has received comparatively much less attention in the literature; this is in part because the schemes and proofs remain fairly complex and delicate. Amongst these latter works, only the work of Okamato and Takashima (OT) [23] simultaneously achieved (1) – (4).

Our results. We present simpler and more modular constructions of unbounded ABE that realize properties (1) – (4) with better efficiency and expressiveness than was previously known.

(i) We present new adaptively secure, unbounded KP-ABE schemes for monotone span programs –which capture access policies computable by monotone Boolean formulas– whose ciphertexts are 42% smaller and our keys are 8% smaller than the state-of-the-art in [23] (with even more substantial savings with our SXDH-based scheme), as well as CP-ABE schemes with similar savings, cf. Fig 3.
(ii) Our constructions generalize to the larger class of arithmetic span programs [15], which capture many natural computational models, such as monotone Boolean formulas, as well as Boolean and arithmetic branching programs; this yields the first adaptively secure, unbounded KP-ABE for arithmetic span programs. Prior to this work, we do not even know any selectively secure, unbounded KP-ABE for arithmetic span programs.

Moreover, our constructions generalize readily to the k-Lin assumption.

At the core of all of these constructions is a "bilinear entropy expansion" lemma [17] that allows us to generate any polynomial amount of entropy starting from constant-size public parameters; the entropy can then be used to transform existing adaptively secure bounded ABE schemes into unbounded ones in a *single* shot. The fact that we only need to invoke our entropy expansion lemma *once* yields both quantitative and

reference	adaptive	assumption	standard model
OT12 [23]	✓	2-Lin	✓
RW13 [24]		q-type	✓
Att16 [3]	✓	q-type + k-Lin	✓
AC17 [1]	✓	k-Lin, $k \geq 2$	
ours	✓	k-Lin, $k \geq 1$ ✓	✓

Fig. 1. Summary of unbounded KP-ABE schemes for monotone span programs from *prime-order* groups with $O(1)$-size mpk.

reference	|mpk|	adaptive	assumption
LW11 [21]	$O(1)$✓		static ✓
Att14 [2]	$O(1)$✓	✓	q-type
KL15 [17]	$O(\log n)$	✓	static ✓
ours	$O(1)$✓	✓	static ✓

Fig. 2. Summary of unbounded KP-ABE schemes for monotone span programs with n-bit attributes (i.e. universe $[n]$) from *composite-order* groups.

	reference	|mpk|	|sk|	|ct|	assumption
KP-ABE	OT12 [23]	$79\|G_1\|+\|G_T\|$	$14n+5$	$14n+5$	DLIN
	Ours	$9\|G_1\|+\|G_T\|$	$8n$	$5n+3$	SXDH
		$28\|G_1\|+2\|G_T\|$	$13n$	$8n+5$	DLIN
		$(5k^2+4k)\|G_1\|+k\|G_T\|$	$(5k+3)n$	$(3k+2)n+2k+1$	k-LIN
CP-ABE	OT12 [23]	$79\|G_1\|+\|G_T\|$	$14n+5$	$14n+5$	DLIN
	Ours	$11\|G_1\|+\|G_T\|$	$5n+5$	$7n+3$	SXDH
		$32\|G_1\|+2\|G_T\|$	$9n+9$	$12n+6$	DLIN
		$(7k^2+4)\|G_1\|+k\|G_T\|$	$(4k+1)(n+1)$	$(5k+2)n+3k$	k-LIN

Fig. 3. Summary of adaptively secure, unbounded ABE schemes for read-once monotone span programs with n-bit attributes (i.e. universe $[n]$) from *prime-order* groups. The columns |sk| and |ct| refer to the number of group elements in G_2 and G_1 respectively (minus a $|G_T|$ contribution in ct).

qualitative advantages over prior works [23, 17]: (i) we achieve security loss $O(n+Q)$ for n-bit attributes (i.e. universe $[n]$) and Q secret key queries, improving upon $O(n \cdot Q)$ in [23] and $O(\log n \cdot Q)$ in [17] and (ii) there is clear delineation between entropy expansion and the analysis of the underlying bounded ABE schemes, whereas prior works interweave both techniques in a more complex nested manner.

Following the recent literature on adaptively secure bounded ABE, we first describe our constructions in the simpler setting of composite-order bilinear groups, and then derive our final prime-order schemes by building upon and extending previous frameworks in [6, 11, 7]. Along the way, we also present a simple adaptively secure unbounded KP-ABE scheme in composite-order groups whose hardness relies on standard, static assumptions (cf. Fig 2).

1.1 Technical overview

We will start with asymmetric composite-order bilinear groups (G_N, H_N, G_T) whose order N is the product of three primes p_1, p_2, p_3. Let g_i, h_i denote generators of order p_i in G_N and H_N, for $i = 1,2,3$.

Warm-up. We begin with the LOSTW KP-ABE for monotone span programs [19]; this is a bounded, adaptively secure scheme that uses composite-order groups. Here, ciphertexts $\mathsf{ct}_{\mathbf{x}}$ are associated with attribute vector[5] $\mathbf{x} \in \{0,1\}^n$ and keys $\mathsf{sk}_{\mathbf{M}}$ with read-once monotone span programs $\mathbf{M}$.[6]

$$\mathsf{mpk} := (g_1, g_1^{v_1}, \ldots, g_1^{v_n}, e(g_1, h_1)^{\alpha}) \tag{1}$$

[5] Some works associate ciphertexts with a set $S \subseteq [n]$ where $[n]$ is referred to as the attribute universe, in which case $\mathbf{x} \in \{0,1\}^n$ corresponds to the characteristic vector of S.

[6] All known adaptively secure ABE for monotone span programs under static assumptions in the standard model (even in the bounded setting and even with composite-order groups) have a read-once restriction [19, 22, 27, 2, 6, 3].

$$\begin{aligned} \mathsf{ct}_{\mathbf{x}} &:= (g_1^s, \{g_1^{sv_j}\}_{x_j=1}, e(g_1,h_1)^{\alpha s}\cdot m) \\ \mathsf{sk}_{\mathbf{M}} &:= (\{h_1^{\alpha_j + r_j v_j}, h_1^{r_j}\}_{j\in[n]}) \end{aligned}$$

where $\alpha_1,\ldots,\alpha_n$ are shares of α w.r.t. the span program $\mathbf{M}$; the shares satisfy the requirement that for any $\mathbf{x}\in\{0,1\}^n$, the shares $\{\alpha_j\}_{x_j=1}$ determine α if $\mathbf{x}$ satisfies $\mathbf{M}$, and reveal nothing about α otherwise. For decryption, observe that we can compute $\{e(g_1,h_1)^{\alpha_j s}\}_{x_j=1}$, from which we can compute the blinding factor $e(g_1,h_1)^{\alpha s}$. The proof of security relies on Waters' dual system encryption methodology [26, 20, 27, 2], in the most basic setting at the core of which is an information-theoretic statement about α_j, v_j.

Towards our unbounded ABE. The main challenge in building an unbounded ABE lies in "compressing" $g_1^{v_1},\ldots,g_1^{v_n}$ in mpk down to a constant number of group elements. The first idea following [21, 23] is to generate $\{v_j\}_{j\in[n]}$ via a pairwise-independent hash function as $w_0 + j\cdot w_1$, as in the Lewko-Waters IBE. Simply replacing v_j with $w_0 + j\cdot w_1$ leads to natural malleability attacks on the ciphertext, and instead, we would replace sv_j with $s_j(w_0+j\cdot w_1)$, where $s_1,\ldots,s_n$ are fresh randomness used in encryption. Next, we need to bind the $s_j(w_0+j\cdot w_1)$'s together via some common randomness s; it suffices to use $sw+s_j(w_0+j\cdot w_1)$ in the ciphertext. That is, we start with the scheme in (1) and we perform the substitutions (*) for each $j\in[n]$:

$$\begin{aligned} &\text{ciphertext:} && (s, sv_j) \mapsto (s, sw + s_j(w_0 + j\cdot w_1), s_j) \\ &\text{secret key:} && (\alpha_j + v_j r_j, r_j) \mapsto (\alpha_j + r_j w, r_j, r_j(w_0 + j\cdot w_1)) \end{aligned} \tag{*}$$

This yields the following scheme:

$$\begin{aligned} \mathsf{mpk} &:= (g_1, g_1^w, g_1^{w_0}, g_1^{w_1}, e(g_1,h_1)^{\alpha}) \\ \mathsf{ct}_{\mathbf{x}} &:= (g_1^s, \{g_1^{sw+s_j(w_0+j\cdot w_1)}, g_1^{s_j}\}_{x_j=1}, e(g_1,h_1)^{\alpha s}\cdot m) \\ \mathsf{sk}_{\mathbf{M}} &:= (\{h_1^{\alpha_j+r_j w}, h_1^{r_j}, h_1^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]}) \end{aligned} \tag{2}$$

As a sanity check for decryption, observe that we can compute $\{e(g_1,h_1)^{\alpha_j s}\}_{x_j=1}$ and then $e(g_1,h_1)^{\alpha s}$ as before. We note that the ensuing scheme is similar to Attrapadung's unbounded KP-ABE in [2, Section 7.1], except the latter requires q-type assumptions.[7]

Our proof strategy. To analyze our scheme in (2), we follow a very simple and natural proof strategy: we would "undo" the substitutions described in (*) to recover ciphertext and keys similar to those in the LOSTW KP-ABE, upon which we could apply the analysis for the bounded setting from the prior works. That is, we want to computationally replace each $w_0 + j\cdot w_1$ with a fresh u_j:

$$\left\{ \begin{array}{l} g_1^s,\ \{g_1^{sw+s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j\in[n]} \\ \{h_1^{\alpha_j+r_j w},\ h_1^{r_j},\ h_1^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]} \end{array} \right\} \overset{\text{hopefully}}{\approx_c} \left\{ \begin{array}{l} g_1^s,\ \{g_1^{sw+s_j u_j},\ g_1^{s_j}\}_{j\in[n]} \\ \{h_1^{\alpha_j+r_j w},\ h_1^{r_j},\ h_1^{r_j u_j}\}_{j\in[n]} \end{array} \right\} \tag{3}$$

Unfortunately, once we give out $g_1^{w_0}, g_1^{w_1}$ in mpk, the above distributions are trivially distinguishable by using the relation $e(g_1, h_1^{r_j(w_0+j\cdot w_1)}) = e(g_1^{w_0+j\cdot w_1}, h_1^{r_j})$. Furthermore, the above statement does not yield a scheme similar to LOSTW when applied to our scheme in (2); for that, we would need to also replace w on the RHS in (3) with fresh v_j as described by

$$(g_1^{sw+s_j u_j}, h_1^{\alpha_j + r_j w}) \mapsto (g_1^{s\underline{v_j}+s_j u_j}, h_1^{\alpha_j + r_j \underline{v_j}})$$

in order to match up with the LOSTW KP-ABE in (1).

[7] Attrapadung's unbounded KP-ABE does have the advantage that there is no read-once restriction on the span programs, but even with the read-once restriction, the proof still requires q-type assumptions.

1.2 Bilinear entropy expansion

The core of our analysis is a *(bilinear) entropy expansion lemma* [17] that captures the spirit of the above statement in (3), namely, it allows us to generate fresh independent randomness starting from the correlated randomness, albeit in a new subgroup of order p_2 generated by g_2, h_2.

More formally, given public parameters $(g_1, g_1^w, g_1^{w_0}, g_1^{w_1}, h_1, h_1^w, h_1^{w_0}, h_1^{w_1})$, we show that

$$\left\{ \begin{array}{l} g_1^s,\ \{g_1^{sw+s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j\in[n]} \\ \{h_1^{r_j w},\ h_1^{r_j},\ h_1^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]} \end{array} \right\} \approx_c \text{—}\cdot \boxed{\left\{ \begin{array}{l} g_2^s,\ \{g_2^{sv_j+s_j u_j},\ g_2^{s_j}\}_{j\in[n]} \\ \{h_2^{r_j v_j},\ h_2^{r_j},\ h_2^{r_j u_j}\}_{j\in[n]} \end{array} \right\}} \tag{4}$$

where "—" is short-hand for duplicating the terms on the LHS, so that the g_1, h_1-components remain unchanged. That is, starting with the LHS, we replaced (i) $w_0 + j \cdot w_1$ with fresh u_j, and (ii) w with fresh v_j, both in the p_2-subgroup. We also omitted the α_j's from (3). We clarify that the trivial distinguisher on (3) fails here because $e(g_1, h_2) = 1$.

Prior work. We clarify that the name "bilinear entropy expansion" was introduced in the prior work of Kowalczyk and Lewko (KL) [17], which also proved a statement similar to (3), with three notable differences: (i) our entropy expansion lemma starts with 3 units of entropy (w, w_0, w_1) whereas KL uses $O(\log n)$ units of entropy; (ii) the KL statement does not account for the public parameters, and therefore (unlike our lemma) cannot serve as an immediate bridge from the unbounded ABE to the bounded variant; (iii) our entropy expansion lemma admits an analogue in prime-order groups, which in turn yields an unbounded ABE scheme in prime-order groups, whereas the composite-order ABE scheme in KL does not have an analogue in prime-order setting (an earlier prime-order construction was retracted on June 1, 2016). In fact, the "consistent randomness amplification" techniques used in the unbounded ABE schemes of Okamoto and Takashima (OT) [23] also seem to yield an entropy expansion lemma with $O(1)$ units of entropy in prime-order groups. As noted earlier in the introduction, our approach is also different from both KL and OT in the sense that we only need to invoke our entropy expansion lemma once when proving security of the unbounded ABE.

Proof overview. We provide a proof overview of our entropy expansion lemma in (4). The proof proceeds in two steps: (i) replacing $w_0 + j \cdot w_1$ with fresh u_j, and then (ii) replacing w with fresh v_j.

(i) We replace $w_0 + j \cdot w_1$ with fresh u_j; that is,

$$\left\{ \begin{array}{l} \{g_1^{s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j\in[n]} \\ \{h_1^{r_j},\ h_1^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]}, \end{array} \right\} \approx_c \text{—}\cdot \boxed{\left\{ \begin{array}{l} \{g_2^{s_j u_j},\ g_2^{s_j}\}_{j\in[n]} \\ \{h_2^{r_j},\ h_2^{r_j u_j}\}_{j\in[n]} \end{array} \right\}} \tag{5}$$

where we suppressed the terms involving w; moreover, this holds even given $g_1, g_1^{w_0}, g_1^{w_1}$. Our first observation is that we can easily adapt the proof of Lewko-Waters IBE [20, 8] to show that for each $i \in [n]$,

$$\left\{ \begin{array}{l} g_1^{s_i(w_0+i\cdot w_1)},\ g_1^{s_i} \\ \{h_1^{r_j},\ h_1^{r_j(w_0+j\cdot w_1)}\}_{j\neq i} \end{array} \right\} \approx_c \text{—}\cdot \boxed{\left\{ \begin{array}{l} g_2^{s_i u_i},\ g_2^{s_i} \\ \{h_2^{r_j},\ h_2^{r_j u_j}\}_{j\neq i} \end{array} \right\}} \tag{6}$$

The idea is that the first term on the LHS corresponds to an encryption for the identity i, and the next $n-1$ terms correspond to secret keys for identities $j \neq i$; on the right, we have the corresponding "semi-functional entities". At this point, we can easily handle $(h_2^{r_i}, h_2^{r_i(w_0+i\cdot w_1)})$ via a statistical argument, thanks to the entropy in $w_0 + i \cdot w_1 \bmod p_2$. Next, we need to get from a single $(g_1^{s_i(w_0+i\cdot w_1)},\ g_1^{s_i})$ on the LHS in (6) to n such terms on the LHS in (5). This requires a delicate "two slot" hybrid argument over $i \in [n]$ and the use of an additional subgroup; similar arguments also appeared in [23, 14]. This is where we used the fact that N is a product of three primes, whereas the Lewko-Waters IBE and the statement in (6) works with two primes in the asymmetric setting.

(ii) Next, we replace w with fresh v_j; that is,

$$\left\{\begin{array}{l} g_2^s,\ \{g_2^{sw+s_ju_j},\ g_2^{s_j}\}_{j\in[n]} \\ \{h_2^{r_jw},\ h_2^{r_j},\ h_2^{r_ju_j}\}_{j\in[n]} \end{array}\right\} \approx_c \left\{\begin{array}{l} g_2^s,\ \{g_2^{sv_j+s_ju_j},\ g_2^{s_j}\}_{j\in[n]} \\ \{h_2^{r_jv_j},\ h_2^{r_j},\ h_2^{r_ju_j}\}_{j\in[n]} \end{array}\right\}$$

Intuitively, this should follow from the DDH assumption in the p_2-subgroup, which says that $(h_2^{r_jw}, h_2^{r_j}) \approx_c (h_2^{r_jv_j}, h_2^{r_j})$. The actual proof is more delicate since w also appears on the other side of the pairing as $g_2^{sw+s_ju_j}$; fortunately, we can treat u_j as a one-time pad that masks w.

Completing the proof of unbounded ABE. We return to a proof sketch of our unbounded ABE in (2). Let us start with the simpler setting where the adversary makes only a single key query. Upon applying our entropy expansion lemma[8], we have that the ciphertext/key pair $(\mathsf{ct}_{\mathbf{x}}, \mathsf{sk}_{\mathbf{M}})$ satisfies

$$\left\{\begin{array}{l} g_1^s, \{g_1^{sw+s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{x_j=1} \\ \{h_1^{\alpha_j+r_jw}, h_1^{r_j}, h_1^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]} \end{array}\right\} \approx_c \cdots \boxed{\left\{\begin{array}{l} g_2^s, \{g_1^{sv_j+s_ju_j},\ g_2^{s_j}\}_{x_j=1} \\ \{h_2^{\alpha_j+r_jv_j}, h_2^{r_j}, h_2^{r_ju_j}\}_{j\in[n]} \end{array}\right\}}$$

with $e(g_1,h_1)^{\alpha s}\cdot m$ omitted. Note that the boxed term on the RHS is *exactly* the LOSTW KP-ABE ciphertext/key pair in (1) over the p_2-subgroup, once we strip away the terms involving u_j, s_j.

Finally, to handle the general setting where the ABE adversary makes Q key queries, we simply observe that thanks to self-reducibility, our entropy expansion lemma extends to a Q-fold setting (with Q copies of $\{r_j\}_{j\in[n]}$) without any additional security loss:

$$\left\{\begin{array}{l} g_1^s,\ \{g_1^{sw+s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j\in[n]} \\ \{h_1^{r_{j,\kappa}w},\ h_1^{r_{j,\kappa}},\ h_1^{r_{j,\kappa}(w_0+j\cdot w_1)}\}_{j\in[n],\kappa\in[Q]} \end{array}\right\} \approx_c \cdots \boxed{\left\{\begin{array}{l} g_2^s,\ \{g_2^{sv_j+s_ju_j},\ g_2^{s_j}\}_{j\in[n]} \\ \{h_2^{r_{j,\kappa}v_j},\ h_2^{r_{j,\kappa}},\ h_2^{r_{j,\kappa}u_j}\}_{j\in[n],\kappa\in[Q]} \end{array}\right\}}$$

At this point, we can rely on the (adaptive) security for the LOSTW KP-ABE for the setting with a single challenge ciphertext and Q key queries.

1.3 Our prime-order scheme

To obtain prime-order analogues of our composite-order schemes, we build upon and extend the previous framework of Chen et al. [6, 11] for simulating composite-order groups in prime-order ones. Along the way, we present a more general framework that provides prime-order analogues of the static assumptions used in the security proof for our composite-order ABE. Moreover, we show that these prime-order analogues follow from the standard k-Linear assumption (and more generally, the MDDH assumption [9]) in prime-order bilinear groups.

Our KP-ABE. Let (G_1, G_2, G_T) be a bilinear group of prime order p. Following [6, 11], we start with our composite-order KP-ABE scheme in (2), sample $\mathbf{A}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times k}, \mathbf{B} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(k+1)\times k}$, and carry out the following

8 and a subgroup assumption to introduce the $h_2^{\alpha_j}$'s.

substitutions:

$$\begin{array}{llll} g_1 & \mapsto [\mathbf{A}_1]_1, & h_1 & \mapsto [\mathbf{B}]_2 \\ \alpha & \mapsto \mathbf{k} \in \mathbb{Z}_p^{2k+1} & w, w_0, w_1 & \mapsto \mathbf{W}, \mathbf{W}_0, \mathbf{W}_1 \in \mathbb{Z}_p^{(2k+1)\times(k+1)} \\ s, s_j & \mapsto \mathbf{s}, \mathbf{s}_j \in \mathbb{Z}_p^k, & r_j & \mapsto \mathbf{r}_j \in \mathbb{Z}_p^k \\ g_1^s & \mapsto [\mathbf{s}^\top \mathbf{A}_1^\top]_1, & h_1^{r_j} & \mapsto [\mathbf{B}\mathbf{r}_j]_2 \\ g_1^{ws} & \mapsto [\mathbf{s}^\top \mathbf{A}_1^\top \mathbf{W}]_1, & h_1^{wr_j} & \mapsto [\mathbf{W}\mathbf{B}\mathbf{r}_j]_2 \end{array} \tag{7}$$

where $[\cdot]_1, [\cdot]_2$ correspond respectively to exponentiations in the prime-order groups G_1, G_2. This yields the following prime-order KP-ABE scheme for monotone span programs:

$$\begin{aligned} \mathsf{mpk} &:= (\, [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1,\ e([\mathbf{A}_1^\top]_1, [\mathbf{k}]_2)\,), \\ \mathsf{ct}_{\mathbf{x}} &:= (\, [\mathbf{s}^\top \mathbf{A}_1^\top]_1, \{[\mathbf{s}^\top \mathbf{A}_1^\top \mathbf{W} + \mathbf{s}_j^\top \mathbf{A}_1^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1)]_1, [\mathbf{s}_j^\top \mathbf{A}_1^\top]_1\}_{x_j=1}, e([\mathbf{s}^\top \mathbf{A}_1^\top]_1, [\mathbf{k}]_2) \cdot m\,) \\ \mathsf{sk}_{\mathbf{M}} &:= (\, \{[\mathbf{k}_j + \mathbf{W}\mathbf{B}\mathbf{r}_j]_2, [\mathbf{B}\mathbf{r}_j]_2, [(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{B}\mathbf{r}_j]_2\}_{j\in[n]}\,) \end{aligned}$$

where $\mathbf{k}_j$ is the j'th share of $\mathbf{k}$. Decryption proceeds as before by first computing $\{e([\mathbf{s}^\top \mathbf{A}_1^\top]_1, [\mathbf{k}_j]_2)\}_{x_j=1}$ and relies on the associativity relations $\mathbf{A}_1^\top \mathbf{W} \cdot \mathbf{B} = \mathbf{A}_1^\top \cdot \mathbf{W}\mathbf{B}$ (ditto $\mathbf{W}_0 + j \cdot \mathbf{W}_1$) [7].

Dimensions of $\mathbf{A}_1, \mathbf{B}$. It is helpful to compare the dimensions of $\mathbf{A}_1, \mathbf{B}$ to those of the CGW prime-order analogue of LOSTW in [6]; once we fix the dimensions of $\mathbf{A}_1, \mathbf{B}$, the dimensions of $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1$ are also fixed. In all of these constructions, the width of $\mathbf{A}_1, \mathbf{B}$ is always k, for constructions based on the k-linear assumption. CGW uses a shorter $\mathbf{A}_1$ of dimensions $(k+1) \times k$, and a $\mathbf{B}$ of the same dimensions $(k+1) \times k$. Roughly speaking, increasing the height of $\mathbf{A}_1$ by k plays the role of adding a subgroup in our composite-order scheme; in particular, the LOSTW KP-ABE uses a group of order $p_1 p_2$ in the asymmetric setting, whereas our unbounded ABE uses a group of order $p_1 p_2 p_3$.

We note that the direct adaptation of the prior techniques in [11] would yield $\mathbf{A}_1$ of height $3k$ and $\mathbf{B}$ of height $k+1$, and reducing the height of $\mathbf{A}_1$ down to $2k+1$ is the key to our efficiency improvements over the prime-order unbounded KP-ABE scheme in [23]. To accomplish this, we need to optimize on the static assumptions used in the composite-order bilinear entropy expansion lemma, and thereafter, carefully transfer these optimizations to the prime-order setting, building upon and extending the recent prime-order IBE schemes in [11].

Bilinear entropy expansion lemma. In the rest of this overview, we motivate the prime-order analogue of our bilinear entropy expansion lemma in (4), and defer a more accurate treatment to Section 5. Upon our substitutions in (7), we expect to prove a statement of the form:

$$\left\{ \begin{array}{l} [\mathbf{s}^\top \mathbf{A}_1^\top]_1,\ \{[\mathbf{s}^\top \mathbf{A}_1^\top \mathbf{W} + \mathbf{s}_j^\top \mathbf{A}_1^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1)]_1,\ [\mathbf{s}_j^\top \mathbf{A}_1^\top]_1\}_{j\in[n]} \\ \{[\mathbf{W}\mathbf{B}\mathbf{r}_j]_2,\ [\mathbf{B}\mathbf{r}_j]_2,\ [(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{B}\mathbf{r}_j]_2\}_{j\in[n]} \end{array} \right\} \tag{8}$$

$$\overset{\text{roughly}}{\approx_c} \ \cdot \boxed{\left\{ \begin{array}{l} [\hat{\mathbf{s}}^\top \mathbf{A}_2^\top]_1,\ \{[\hat{\mathbf{s}}^\top \mathbf{A}_2^\top \mathbf{V}_j + \hat{\mathbf{s}}_j^\top \mathbf{A}_2^\top \mathbf{U}_j]_1,\ [\hat{\mathbf{s}}_j^\top \mathbf{A}_2^\top]_1\}_{j\in[n]} \\ \{[\mathbf{V}_j \mathbf{B}\mathbf{r}_j]_2,\ [\mathbf{0}]_2,\ [\mathbf{U}_j \mathbf{B}\mathbf{r}_j]_2\}_{j\in[n]} \end{array} \right\}}$$

given also the public parameters $[\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1$. Here, $\mathbf{A}_2 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times k}$ is an additional matrix that plays the role of g_2, whereas $\mathbf{U}_j, \mathbf{V}_j$ play the roles of the fresh entropy u_j, v_j. Note that we do not introduce additional terms that correspond to those involving h_2 on the RHS, and can therefore keep $\mathbf{B}$ of dimensions $(k+1) \times k$. To prevent a trivial distinguishing attack based on the associativity relation

$\mathbf{A}_1^\top \mathbf{W} \cdot \mathbf{B} = \mathbf{A}_1^\top \cdot \mathbf{WB}$, we need to sample random $\mathbf{U}_j, \mathbf{V}_j$ subject to the constraints $\mathbf{A}_1^\top \mathbf{U}_j = \mathbf{A}_1^\top \mathbf{V}_j = \mathbf{0}$. In the proof of the entropy expansion lemma, we will show that the k-Lin assumption implies

$$(\mathbf{A}_1, \mathbf{A}_1^\top \mathbf{W}, \{[\mathbf{WBr}_j]_2, [\mathbf{Br}_j]_2\}_{j\in[n]}) \approx_c (\mathbf{A}_1, \mathbf{A}_1^\top \mathbf{W}, \{[(\mathbf{W} + \boxed{\mathbf{U}_j})\mathbf{Br}_j]_2, [\mathbf{Br}_j]_2\}_{j\in[n]}).$$

To complete the proof of the unbounded ABE, we proceed as before in the composite-order setting, and observe that the boxed term in (8) above (once we strip away the terms involving $\mathbf{U}_j$ and $\hat{\mathbf{s}}_j$) correspond to the prime-order variant of the LOSTW KP-ABE in CGW, as given by:

$$\begin{aligned} \mathsf{ct}_{\mathbf{x}} &:= (\, [\hat{\mathbf{s}}^\top \mathbf{A}_2^\top]_1, \{[\hat{\mathbf{s}}^\top \mathbf{A}_2^\top \mathbf{V}_j]_1\}_{x_j=1}, e([\hat{\mathbf{s}}^\top \mathbf{A}_2^\top]_1, [\mathbf{k}]_2) \cdot m\,) \\ \mathsf{sk}_{\mathbf{M}} &:= (\, \{[\mathbf{k}_j + \mathbf{V}_j \mathbf{Br}_j]_2, [\mathbf{Br}_j]_2\}_{j\in[n]}\,) \end{aligned}$$

As in the composite-order setting, we need to first extend our bilinear entropy expansion lemma to a Q-fold setting via random self-reducibility. We may then carry out the analysis in CGW to complete the proof of our unbounded ABE.

1.4 Extensions

We briefly sketch two extensions: CP-ABE for monotone span programs, and KP-ABE for arithmetic span programs.

CP-ABE. Here, we start with the LOSTW CP-ABE for monotone span programs [19], which basically reverses the structures of the ciphertexts and keys. This means that we will need a variant of our entropy expansion lemma that accommodates a similar reversal. The statement adapts naturally to this setting, and so does the proof, except we need to make some changes to step two, which requires that we start with a taller $\mathbf{A}_1 \in \mathbb{Z}_q^{3k\times k}$. This gives rise to the following prime-order CP-ABE:

$$\begin{aligned} \mathsf{mpk} &:= (\, [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_1^\top \mathbf{U}_0]_1\; e([\mathbf{A}_1^\top]_1, [\mathbf{k}]_2)\,), \\ \mathsf{ct}_{\mathbf{M}} &:= (\, [\mathbf{s}^\top \mathbf{A}_1^\top]_1, \{\, [\mathbf{c}_{0,j}^\top + \mathbf{s}_j^\top \mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{s}_j^\top \mathbf{A}_1^\top]_1, [\mathbf{s}_j^\top \mathbf{A}_1^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1)]_1 \,\}_{j\in[n]}, e([\mathbf{s}^\top \mathbf{A}_1^\top]_1, [\mathbf{k}]_2) \cdot m\,) \\ \mathsf{sk}_{\mathbf{x}} &:= (\, [\mathbf{k} + \mathbf{U}_0 \mathbf{Br}]_2, [\mathbf{Br}]_2, \{\, [\mathbf{WBr} + (\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{Br}_j]_2, [\mathbf{Br}_j]_2 \,\}_{x_j=1}\,) \end{aligned}$$

where $\mathbf{c}_{0,j}$ is the j'th share of $\mathbf{c}_0 := \mathbf{s}^\top \mathbf{A}_1^\top \mathbf{U}_0$ w.r.t. $\mathbf{M}$. Decryption proceeds by first computing $\{e([\mathbf{c}_{0,j}^\top]_1, [\mathbf{Br}]_2)\}_{x_j=1}$ and then $e([\mathbf{c}_0^\top]_1, [\mathbf{Br}]_2)$.

Arithmetic span programs. In arithmetic span programs, the attributes $\mathbf{x}$ come from $\mathbb{Z}_p^n$ instead of $\{0,1\}^n$, which enable richer and more expressive arithmetic computation. The analogue of the LOSTW KP-ABE for arithmetic span programs [6, 15] will then have ciphertexts:

$$\mathsf{ct}_{\mathbf{x}} := (g_1^s, \{g_1^{(v_j + x_j v_j')s}\}_{j\in[n]}, e(g_1, h_1)^{\alpha s} \cdot m).$$

That is, we replaced $g_1^{x_j v_j s}$ in (1) with $g_1^{(v_j + x_j v_j')s}$. In the unbounded setting, we will need to generate $\{v_j\}_{j\in[n]}$ and $\{v_j'\}_{j\in[n]}$ via two different pairwise-independent hash functions, given by $w_0 + j w_1$ and $w_0' + j w_1'$ respectively. Our entropy expansion lemma generalizes naturally to this setting.

2 Preliminaries

Notation. We denote by $s \leftarrow_{\mathrm{R}} S$ the fact that s is picked uniformly at random from a finite set S. By PPT, we denote a probabilistic polynomial-time algorithm. Throughout this paper, we use 1^λ as the security parameter. We use lower case boldface to denote (column) vectors and upper case boldcase to denote matrices. We use $\equiv$ to denote two distributions being identically distributed, and $\approx_c$ to denote two distributions being computationally indistinguishable. For any two finite sets (also including spaces and groups) S_1 and S_2, the notation "$S_1 \approx_c S_2$" means the uniform distributions over them are computationally indistinguishable.

2.1 Monotone span programs

We define (monotone) span programs [16].

Definition 1 (span programs [4, 16]). *A* (monotone) span program *for attribute universe* $[n]$ *is a pair* $(\mathbf{M}, \rho)$ *where* $\mathbf{M}$ *is a* $\ell \times \ell'$ *matrix over* $\mathbb{Z}_p$ *and* $\rho : [\ell] \to [n]$*. Given* $\mathbf{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$*, we say that*

$$\mathbf{x} \textit{ satisfies } (\mathbf{M}, \rho) \textit{ iff } \mathbf{1} \in \mathsf{span}\langle \mathbf{M}_{\mathbf{x}} \rangle,$$

Here, $\mathbf{1} := (1, 0, \ldots, 0)^\top \in \mathbb{Z}^{1 \times \ell'}$ *is a row vector;* $\mathbf{M}_{\mathbf{x}}$ *denotes the collection of vectors* $\{\mathbf{M}_j : x_{\rho(j)} = 1\}$ *where* $\mathbf{M}_j$ *denotes the* j*'th row of* $\mathbf{M}$*; and* span *refers to linear span of collection of (row) vectors over* $\mathbb{Z}_p$*.*

That is, $\mathbf{x}$ satisfies $(\mathbf{M}, \rho)$ iff there exists constants $\omega_1, \ldots, \omega_\ell \in \mathbb{Z}_p$ such that

$$\sum_{j : x_{\rho(j)} = 1} \omega_j \mathbf{M}_j = \mathbf{1} \tag{9}$$

Observe that the constants $\{\omega_j\}$ can be computed in time polynomial in the size of the matrix $\mathbf{M}$ via Gaussian elimination. Like in [19, 6], we need to impose a one-use restriction, that is, ρ is a permutation and $\ell = n$. By re-ordering the rows of $\mathbf{M}$, we may assume WLOG that ρ is the identity map, which we omit in the rest of this section.

Lemma 1 (statistical lemma [6, Appendix A.6]). *For any* $\mathbf{x}$ *that does not satisfy* $\mathbf{M}$*, the distributions*

$$\left(\{v_j\}_{j : x_j = 1}, \{\mathbf{M}_j \binom{\alpha}{\mathbf{u}} + r_j v_j,\ r_j\}_{j \in [n]}\right)$$

perfectly hide α*, where the randomness is taken over* $v_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p$*,* $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell' - 1}$*, and for any fixed* $r_j \neq 0$*.*

2.2 Attribute-based encryption

An attribute-based encryption (ABE) scheme for a predicate $\mathsf{P}(\cdot, \cdot)$ consists of four algorithms $(\mathsf{Setup}, \mathsf{Enc}, \mathsf{KeyGen}, \mathsf{Dec})$:

$\mathsf{Setup}(1^\lambda, \mathcal{X}, \mathcal{Y}, \mathcal{M}) \to (\mathsf{mpk}, \mathsf{msk})$. The setup algorithm gets as input the security parameter λ, the attribute universe $\mathcal{X}$, the predicate universe $\mathcal{Y}$, the message space $\mathcal{M}$ and outputs the public parameter mpk, and the master key msk.

$\mathsf{Enc}(\mathsf{mpk}, x, m) \to \mathsf{ct}_x$. The encryption algorithm gets as input mpk, an attribute $x \in \mathcal{X}$ and a message $m \in \mathcal{M}$. It outputs a ciphertext ct_x. Note that x is public given ct_x.

$\mathsf{KeyGen}(\mathsf{mpk}, \mathsf{msk}, y) \to \mathsf{sk}_y$. The key generation algorithm gets as input msk and a value $y \in \mathcal{Y}$. It outputs a secret key sk_y. Note that y is public given sk_y.

$\mathsf{Dec}(\mathsf{mpk}, \mathsf{sk}_y, \mathsf{ct}_x) \to m$. The decryption algorithm gets as input sk_y and ct_x such that $\mathsf{P}(x, y) = 1$. It outputs a message m.

Correctness. We require that for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ such that $\mathsf{P}(x, y) = 1$ and all $m \in \mathcal{M}$,

$$\Pr[\mathsf{Dec}(\mathsf{mpk}, \mathsf{sk}_y, \mathsf{Enc}(\mathsf{mpk}, x, m)) = m] = 1,$$

where the probability is taken over $(\mathsf{mpk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(1^\lambda, \mathcal{X}, \mathcal{Y}, \mathcal{M})$, $\mathsf{sk}_y \leftarrow \mathsf{KeyGen}(\mathsf{mpk}, \mathsf{msk}, y)$, and the coins of Enc.

Security definition. For a stateful adversary $\mathcal{A}$, we define the advantage function

$$\mathsf{Adv}^{\text{ABE}}_{\mathcal{A}}(\lambda) := \Pr\left[b = b' : \begin{array}{l} (\mathsf{mpk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(1^\lambda, \mathcal{X}, \mathcal{Y}, \mathcal{M}); \\ (x^*, m_0, m_1) \leftarrow \mathcal{A}^{\mathsf{KeyGen}(\mathsf{msk},\cdot)}(\mathsf{mpk}); \\ b \leftarrow_{\mathrm{R}} \{0,1\}; \mathsf{ct}_{x^*} \leftarrow \mathsf{Enc}(\mathsf{mpk}, x^*, m_b); \\ b' \leftarrow \mathcal{A}^{\mathsf{KeyGen}(\mathsf{msk},\cdot)}(\mathsf{ct}_{x^*}) \end{array} \right] - \frac{1}{2}$$

with the restriction that all queries y that $\mathcal{A}$ makes to $\mathsf{KeyGen}(\mathsf{msk}, \cdot)$ satisfies $\mathsf{P}(x^*, y) = 0$ (that is, sk_y does not decrypt ct_{x^*}). An ABE scheme is *adaptively secure* if for all PPT adversaries $\mathcal{A}$, the advantage $\mathsf{Adv}^{\text{ABE}}_{\mathcal{A}}(\lambda)$ is a negligible function in λ.

Unbounded ABE. An ABE scheme is *unbounded* [21] if the running time of Setup only depends on λ; otherwise, we say that it is bounded.

3 Bilinear Entropy Expansion, Revisited

In this section, we present our (bilinear) entropy expansion lemma in composite-order groups.

3.1 Composite-order bilinear groups and computational assumptions

A generator $\mathcal{G}$ takes as input a security parameter λ and outputs $\mathbb{G} := (G_N, H_N, G_T, e)$, where N is product of three primes p_1, p_2, p_3 of $\Theta(\lambda)$ bits, G_N, H_N and G_T are cyclic groups of order N and $e : G_N \times H_N \to G_T$ is a non-degenerate bilinear map. We require that the group operations in G_N, H_N and G_T as well the bilinear map e are computable in deterministic polynomial time with respect to λ. We assume that a random generator g (resp. h) of G_N (resp. H_N) is always contained in the description of bilinear groups. For every divisor n of N, we denote by G_n the subgroup of G_N of order n. We use g_1, g_2, g_3 to denote random generators of the subgroups $G_{p_1}, G_{p_2}, G_{p_3}$ respectively. We define h_1, h_2, h_3 random generators of the subgroups $H_{p_1}, H_{p_2}, H_{p_3}$ analogously.

Computational assumptions. We review two static computational assumptions in the composite-order group, used e.g. in [20, 8].

Assumption 1 ($\mathrm{SD}^{G_N}_{p_1 \mapsto p_1 p_2}$) *We say that $(p_1 \mapsto p_1 p_2)$-subgroup decision assumption, denoted by $\mathrm{SD}^{G_N}_{p_1 \mapsto p_1 p_2}$, holds if for all PPT adversaries $\mathcal{A}$, the following advantage function is negligible in λ.*

$$\mathsf{Adv}^{\mathrm{SD}^{G_N}_{p_1 \mapsto p_1 p_2}}_{\mathcal{A}}(\lambda) := |\Pr[\mathcal{A}(\mathbb{G}, D, T_0) = 1] - \Pr[\mathcal{A}(\mathbb{G}, D, T_1) = 1]|$$

where

$$D := (g_1, g_2, g_3, h_1, h_3, h_{12}), \quad h_{12} \leftarrow_{\mathrm{R}} H_{p_1 p_2}$$
$$T_0 \leftarrow_{\mathrm{R}} \boxed{G_{p_1}},\ T_1 \leftarrow_{\mathrm{R}} \boxed{G_{p_1 p_2}}.$$

Assumption 2 ($\mathrm{DDH}^{H_N}_{p_1}$) *We say that p_1-subgroup Diffie-Hellman assumption, denoted by $\mathrm{DDH}^{H_N}_{p_1}$, holds if for all PPT adversaries $\mathcal{A}$, the following advantage function is negligible in λ.*

$$\mathsf{Adv}^{\mathrm{DDH}^{H_N}_{p_1}}_{\mathcal{A}}(\lambda) := |\Pr[\mathcal{A}(\mathbb{G}, D, T_0) = 1] - \Pr[\mathcal{A}(\mathbb{G}, D, T_1) = 1]|$$

where

$$D := (g_1, g_2, g_3, h_1, h_2, h_3),$$
$$T_0 := (h_1^x, h_1^y, \boxed{h_1^{xy}}),\ T_1 := (h_1^x, h_1^y, \boxed{h_1^{xy+z}}), \quad x, y, z \leftarrow_{\mathrm{R}} \mathbb{Z}_N.$$

By symmetry, one may permute the indices for subgroups and/or exchange the roles of G_N and H_N, and define $\mathrm{SD}^{G_N}_{p_1 \mapsto p_1 p_3}$, $\mathrm{SD}^{G_N}_{p_3 \mapsto p_3 p_2}$, $\mathrm{SD}^{H_N}_{p_1 \mapsto p_1 p_2}$, $\mathrm{SD}^{H_N}_{p_1 \mapsto p_1 p_3}$ and $\mathrm{DDH}^{H_N}_{p_2}, \mathrm{DDH}^{H_N}_{p_3}$ analogously.

3.2 Lemma in Composite-order groups

We state our entropy expansion lemma in composite-order groups as follows.

Lemma 2 (Bilinear entropy expansion lemma). *Under the $SD^{H_N}_{p_1 \mapsto p_1 p_2}$, $SD^{H_N}_{p_1 \mapsto p_1 p_3}$, $SD^{G_N}_{p_1 \mapsto p_1 p_2}$, $DDH^{H_N}_{p_2}$, $SD^{G_N}_{p_1 \mapsto p_1 p_3}$, $DDH^{H_N}_{p_3}$, $SD^{G_N}_{p_3 \mapsto p_3 p_2}$ assumptions, we have*

$$\left\{\begin{array}{r} \mathsf{aux}: g_1, g_1^w, g_1^{w_0}, g_1^{w_1} \\ \mathsf{ct}: g_1^s, \{g_1^{sw+s_j(w_0+j\cdot w_1)}, g_1^{s_j}\}_{j\in[n]} \\ \mathsf{sk}: \{h_1^{r_j w}, h_1^{r_j}, h_1^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]} \end{array}\right\} \approx_c \left\{\begin{array}{r} \mathsf{aux}: g_1, g_1^w, g_1^{w_0}, g_1^{w_1} \\ \mathsf{ct}: g_1^s \cdot \boxed{g_2^s}, \{g_1^{sw+s_j(w_0+j\cdot w_1)} \cdot \boxed{g_2^{sv_j+s_j u_j}}, g_1^{s_j} \cdot \boxed{g_2^{s_j}}\}_{j\in[n]} \\ \mathsf{sk}: \{h_1^{r_j w} \cdot \boxed{h_2^{r_j v_j}}, h_1^{r_j} \cdot \boxed{h_2^{r_j}}, h_1^{r_j(w_0+j\cdot w_1)} \cdot \boxed{h_2^{r_j u_j}}\}_{j\in[n]} \end{array}\right\}$$

where

$$w, w_0, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ v_j, u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ s, s_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N.$$

Concretely, the distinguishing advantage $\mathsf{Adv}^{\mathrm{ExpLem}}_{\mathcal{A}}(\lambda)$ *is at most*

$$\mathsf{Adv}^{\mathrm{SD}^{H_N}_{p_1\mapsto p_1p_2}}_{\mathcal{B}}(\lambda) + \mathsf{Adv}^{\mathrm{SD}^{H_N}_{p_1\mapsto p_1p_3}}_{\mathcal{B}'}(\lambda) + \mathsf{Adv}^{\mathrm{SD}^{G_N}_{p_1\mapsto p_1p_2}}_{\mathcal{B}''}(\lambda) + \mathsf{Adv}^{\mathrm{SD}^{H_N}_{p_1\mapsto p_1p_3}}_{\mathcal{B}'''}(\lambda) + \mathsf{Adv}^{\mathrm{DDH}^{H_N}_{p_2}}_{\mathcal{B}_0}(\lambda)$$
$$+\ n\cdot\big(\mathsf{Adv}^{\mathrm{SD}^{G_N}_{p_1\mapsto p_1p_3}}_{\mathcal{B}_1}(\lambda) + \mathsf{Adv}^{\mathrm{DDH}^{H_N}_{p_3}}_{\mathcal{B}_2}(\lambda) + \mathsf{Adv}^{\mathrm{SD}^{G_N}_{p_3\mapsto p_3p_2}}_{\mathcal{B}_4}(\lambda) + \mathsf{Adv}^{\mathrm{DDH}^{H_N}_{p_3}}_{\mathcal{B}_6}(\lambda) + \mathsf{Adv}^{\mathrm{SD}^{G_N}_{p_1\mapsto p_1p_3}}_{\mathcal{B}_7}(\lambda)\big) + \mathsf{Adv}^{\mathrm{DDH}^{H_N}_{p_2}}_{\mathcal{B}_8}(\lambda)$$

where $\mathsf{Time}(\mathcal{B})$, $\mathsf{Time}(\mathcal{B}')$, $\mathsf{Time}(\mathcal{B}'')$, $\mathsf{Time}(\mathcal{B}''')$, $\mathsf{Time}(\mathcal{B}_0)$, $\mathsf{Time}(\mathcal{B}_1)$, $\mathsf{Time}(\mathcal{B}_2)$, $\mathsf{Time}(\mathcal{B}_4)$, $\mathsf{Time}(\mathcal{B}_6)$, $\mathsf{Time}(\mathcal{B}_7)$, $\mathsf{Time}(\mathcal{B}_8) \approx \mathsf{Time}(\mathcal{A})$.

We will prove the lemma in two steps via the following two lemmas.

Lemma 3 (Bilinear entropy expansion lemma (step one)). *Under the $DDH^{H_N}_{p_2}$, $SD^{G_N}_{p_1 \mapsto p_1 p_3}$, $DDH^{H_N}_{p_3}$, $SD^{G_N}_{p_3 \mapsto p_3 p_2}$ assumptions, we have*

$$\left\{\begin{array}{r} \mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1}, g_2 \\ \mathsf{ct}: \{g_1^{s_j(w_0+j\cdot w_1)}, g_1^{s_j}\}_{j\in[n]} \\ \mathsf{sk}: \{h_{123}^{r_j}, h_{123}^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]} \end{array}\right\} \approx_c \left\{\begin{array}{r} \mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1}, g_2 \\ \mathsf{ct}: \{g_1^{s_j(w_0+j\cdot w_1)} \cdot \boxed{g_2^{s_j u_j}}, g_1^{s_j} \cdot \boxed{g_2^{s_j}}\}_{j\in[n]} \\ \mathsf{sk}: \{h_{13}^{r_j} \cdot \boxed{h_2^{r_j}}, h_{13}^{r_j(w_0+j\cdot w_1)} \cdot \boxed{h_2^{r_j u_j}}\}_{j\in[n]} \end{array}\right\}$$

where

$$w_0, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ s_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N.$$

Concretely, the distinguishing advantage $\mathsf{Adv}^{\mathrm{Step1}}_{\mathcal{A}}(\lambda)$ *is at most*

$$\mathsf{Adv}^{\mathrm{DDH}^{H_N}_{p_2}}_{\mathcal{B}_0}(\lambda) + n\cdot\big(\mathsf{Adv}^{\mathrm{SD}^{G_N}_{p_1\mapsto p_1p_3}}_{\mathcal{B}_1}(\lambda) + \mathsf{Adv}^{\mathrm{DDH}^{H_N}_{p_3}}_{\mathcal{B}_2}(\lambda) + \mathsf{Adv}^{\mathrm{SD}^{G_N}_{p_3\mapsto p_3p_2}}_{\mathcal{B}_4}(\lambda) + \mathsf{Adv}^{\mathrm{DDH}^{H_N}_{p_3}}_{\mathcal{B}_6}(\lambda) + \mathsf{Adv}^{\mathrm{SD}^{G_N}_{p_1\mapsto p_1p_3}}_{\mathcal{B}_7}(\lambda)\big)$$

where $\mathsf{Time}(\mathcal{B}_0)$, $\mathsf{Time}(\mathcal{B}_1)$, $\mathsf{Time}(\mathcal{B}_2)$, $\mathsf{Time}(\mathcal{B}_4)$, $\mathsf{Time}(\mathcal{B}_6)$, $\mathsf{Time}(\mathcal{B}_7) \approx \mathsf{Time}(\mathcal{A})$.

Note that sk in the LHS of this lemma has an extra h_{23}-component, which we may introduce using the $\mathrm{SD}^{H_N}_{p_1 \mapsto p_1 p_2}$ and $\mathrm{SD}^{H_N}_{p_1 \mapsto p_1 p_3}$ assumption. The proof of this lemma is fairly involved, and we defer the proof to Section 3.3.

Lemma 4 (Bilinear entropy expansion lemma (step two)). *Under the* $\mathrm{DDH}_{p_2}^{H_N}$ *assumption, we have*

$$\left\{\begin{array}{l}\mathsf{aux}: g_1, g_1^w, h_1, h_1^w \\ \quad \mathsf{ct}: g_2^s,\ \{\, g_2^{sw}\cdot g_2^{s_j u_j},\ g_2^{s_j}\,\}_{j\in[n]} \\ \quad \mathsf{sk}: \{h_2^{r_j w},\ h_2^{r_j},\ h_2^{r_j u_j}\}_{j\in[n]}\end{array}\right\} \approx_c \left\{\begin{array}{l}\mathsf{aux}: g_1, g_1^w, h_1, h_1^w \\ \quad \mathsf{ct}: g_2^s,\ \{\,\boxed{g_2^{s v_j}}\cdot g_2^{s_j u_j},\ g_2^{s_j}\,\}_{j\in[n]} \\ \quad \mathsf{sk}: \{\boxed{h_2^{r_j v_j}},\ h_2^{r_j},\ h_2^{r_j u_j}\}_{j\in[n]}\end{array}\right\}$$

where

$$w \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ v_j, u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ s, s_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N.$$

Concretely, the distinguishing advantage $\mathsf{Adv}_{\mathcal{A}}^{\mathrm{STEP2}}(\lambda)$ *is at most* $\mathsf{Adv}_{\mathcal{B}_8}^{\mathrm{DDH}_{p_2}^{H_N}}(\lambda)$ *where* $\mathsf{Time}(\mathcal{B}_8) \approx \mathsf{Time}(\mathcal{A})$.

Proof. This follows from the $\mathrm{DDH}_{p_2}^{H_N}$ assumption, which tells us that

$$\{h_2^{r_j}, h_2^{r_j w}\}_{j\in[n]} \approx_c \{h_2^{r_j}, \boxed{h_2^{r_j v_j}}\}_{j\in[n]}.$$

The adversary $\mathcal{B}_8$ on input $\{h_2^{r_j}, T_j\}_{j\in[n]}$ along with g_1, g_2, h_1, h_2, samples $\tilde{w}, s, s_j, \tilde{u}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ (and implicitly sets $u_j = \frac{1}{s_j}(\tilde{u}_j - sw)$), then runs $\mathcal{A}$ on input

$$\left\{\begin{array}{l}\mathsf{aux}: g_1, g_1^{\tilde{w}}, h_1, h_1^{\tilde{w}} \\ \quad \mathsf{ct}: g_2^s,\ \{\, g_2^{\tilde{u}_j},\ g_2^{s_j}\,\}_{j\in[n]} \\ \quad \mathsf{sk}: \{T_j,\ h_2^{r_j},\ (h_2^{r_j})^{\frac{\tilde{u}_j}{s_j}}\cdot T_j^{-\frac{s}{s_j}}\}_{j\in[n]}\end{array}\right\}.$$

By the Chinese Remainder Theorem, we have $(g_1^w, h_1^w, g_2^w, h_2^w) \equiv (g_1^{\tilde{w}}, h_1^{\tilde{w}}, g_2^w, h_2^w)$, where $w, \tilde{w} \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. Next, observe that

- When $T_j = g^{r_j w}$ and if we write $r_j u_j = r_j \cdot \frac{\tilde{u}_j}{s_j} + r_j w \cdot (-\frac{s}{s_j})$, then $\tilde{u}_j = sw + s_j u_j$ and the distribution we feed to $\mathcal{A}$ is exactly that of the left distribution.
- When $T_j = g^{r_j v_j}$ and if we write $r_j u_j = r_j \cdot \frac{\tilde{u}_j}{s_j} + r_j v_j \cdot (-\frac{s}{s_j})$, then $\tilde{u}_j = s v_j + s_j u_j$ and the distribution we feed to $\mathcal{A}$ is exactly that of the right distribution.

This completes the proof. □

We may now complete the proof of Lemma 2 using Lemmas 3 and 4.

Proof (of Lemma 2). The statement follows readily from the following hybrid argument:

$$\left\{\begin{array}{l}\mathsf{aux}: g_1, g_1^w, g_1^{w_0}, g_1^{w_1} \\ \quad \mathsf{ct}: g_1^s,\ \{g_1^{sw+s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j\in[n]} \\ \quad \mathsf{sk}: \{h_1^{r_j w},\ h_1^{r_j},\ h_1^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]}\end{array}\right\} \quad \text{LHS in Lemma 2}$$

$$\overset{p_1 \mapsto p_1p_2p_3}{\approx_c} \left\{\begin{array}{l}\mathsf{aux}: g_1, g_1^w, g_1^{w_0}, g_1^{w_1} \\ \quad \mathsf{ct}: g_1^s,\ \{g_1^{sw+s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j\in[n]} \\ \quad \mathsf{sk}: \{\boxed{h_{123}}^{r_j w},\ \boxed{h_{123}}^{r_j},\ \boxed{h_{123}}^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]}\end{array}\right\}$$

$$\overset{p_1 \mapsto p_1p_2}{\approx_c} \left\{\begin{array}{l}\mathsf{aux}: g_1, g_1^w, g_1^{w_0}, g_1^{w_1} \\ \quad \mathsf{ct}: g_1^s\cdot\boxed{g_2^s},\ \{g_1^{sw}\cdot\boxed{g_2^{sw}}\cdot g_1^{s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j\in[n]} \\ \quad \mathsf{sk}: \{h_{123}^{r_j w},\ h_{123}^{r_j},\ h_{123}^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]}\end{array}\right\}$$

$$
\overset{\text{Lemma 3}}{\approx_c}
\left\{
\begin{array}{l}
\mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1} \\
\mathsf{ct}: g_1^{s}\cdot g_2^{s},\ \{g_1^{sw}\cdot g_2^{sw}\cdot g_1^{s_j(w_0+j\cdot w_1)}\cdot \boxed{g_2^{s_j u_j}},\ g_1^{s_j}\cdot \boxed{g_2^{s_j}}\}_{j\in[n]} \\
\mathsf{sk}: \{h_{13}^{r_j w}\cdot \boxed{h_2^{r_j w}},\ h_{13}^{r_j}\cdot \boxed{h_2^{r_j}},\ h_{13}^{r_j(w_0+j\cdot w_1)}\cdot \boxed{h_2^{r_j u_j}}\}_{j\in[n]}
\end{array}
\right\}
$$

$$
\overset{p_1\mapsto p_1p_3}{\approx_c}
\left\{
\begin{array}{l}
\mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1} \\
\mathsf{ct}: g_1^{s}\cdot g_2^{s},\ \{g_1^{sw}\cdot g_2^{sw}\cdot g_1^{s_j(w_0+j\cdot w_1)}\cdot g_2^{s_j u_j},\ g_1^{s_j}\cdot g_2^{s_j}\}_{j\in[n]} \\
\mathsf{sk}: \{\boxed{h_1}^{r_j w}\cdot h_2^{r_j w},\ \boxed{h_1}^{r_j}\cdot h_2^{r_j},\ \boxed{h_1}^{r_j(w_0+j\cdot w_1)}\cdot h_2^{r_j u_j}\}_{j\in[n]}
\end{array}
\right\}
$$

$$
\overset{\text{Lemma 4}}{\approx_c}
\left\{
\begin{array}{l}
\mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1} \\
\mathsf{ct}: g_1^{s}\cdot g_2^{s},\ \{g_1^{sw}\cdot \boxed{g_2^{sv_j}}\cdot g_1^{s_j(w_0+j\cdot w_1)}\cdot g_2^{s_j u_j},\ g_1^{s_j}\cdot g_2^{s_j}\}_{j\in[n]} \\
\mathsf{sk}: \{h_1^{r_j w}\cdot \boxed{h_2^{r_j v_j}},\ h_1^{r_j}\cdot h_2^{r_j},\ h_1^{r_j(w_0+j\cdot w_1)}\cdot h_2^{r_j u_j}\}_{j\in[n]}
\end{array}
\right\}
\quad \text{RHS in Lemma 2}
$$

It is direct to justify every transitions as follows:

- The first transition follows from

$$\{h_1^{r_j}\}_{j\in[n]} \approx_c \{h_{123}^{r_j}\}_{j\in[n]} \ \text{ given } g_1$$

  where h_1 and h_{123} are random generators of H_{p_1} and H_N, which is implied by the $\mathrm{SD}^{H_N}_{p_1\mapsto p_1p_2}$ and $\mathrm{SD}^{H_N}_{p_1\mapsto p_1p_3}$ assumption. In the reduction, we sample $w, w_0, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ and simulate $\mathsf{aux}, \mathsf{ct}$ honestly.
- The second transition is ensured by the $\mathrm{SD}^{G_N}_{p_1\mapsto p_1p_2}$ assumption which tells us that

$$g_1^{s} \approx_c g_1^{s}\cdot g_2^{s} \ \text{ given } g_1$$

  where g_1 and g_2 are random generators of G_{p_1} and G_{p_2}. In the reduction, we sample $w, w_0, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ and simulate aux and sk honestly.
- The third transition follows from Lemma 3. In the reduction, we sample $w \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ and simulate g_1^{w} in aux honestly. We then use generators g_1 and g_2 to simulate $(g_1^{s}\cdot g_2^{s}, g_1^{sw}\cdot g_2^{sw})$ in ct and use $h_{123}^{r_j}$ (or $h_{13}^{r_j}\cdot h_2^{r_j}$) to simulate $h_{123}^{r_j w}$ in sk.
- The fourth transition follows from the $\mathrm{SD}^{H_N}_{p_1\mapsto p_1p_3}$ assumption asserting

$$\{h_1^{r_j}\}_{j\in[n]} \approx_c \{h_{13}^{r_j}\}_{j\in[n]} \ \text{ given } g_1, g_2, h_2$$

  where h_1 and h_{13} are random generators of H_{p_1} and $H_{p_1p_3}$. In the reduction, we sample $w, w_0, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ and $u_j, v_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$, and simulate aux using g_1 and ct from g_1, g_2.
- The final transition follows from Lemma 4. In the reduction, we sample $w_0, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ and simulate $g_1^{w_0}$ and $g_1^{w_1}$ in aux honestly. By $g_1, g_1^{w}, h_1, h_1^{w}$, we can simulate all g_1-components in ct (i.e., g_1^{s} and $\{g_1^{sw+s_j(w_0+j\cdot w_1)}, g_1^{s_j}\}_{j\in[n]}$) and all h_1-components in sk (i.e., $\{h_1^{r_j w}, h_1^{r_j}, h_1^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]}$). □

3.3 Entropy expansion lemma: Step one

Proof overview. First, we note that we can adapt the proof of the Lewko-Waters IBE [20, 8][9] to show that under $\mathrm{SD}^{G_N}_{p_1\mapsto p_1p_3}$ and $\mathrm{DDH}^{H_N}_{p_3}$ assumptions, we have that for each $i \in [n]$:

$$
\left\{
\begin{array}{l}
\mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1} \\
\mathsf{ct}: \{g_1^{s_i(w_0+i\cdot w_1)},\ g_1^{s_i}\} \\
\mathsf{sk}: \{h_{13}^{r_j},\ h_{13}^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]}
\end{array}
\right\}
\approx_c
\left\{
\begin{array}{l}
\mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1} \\
\mathsf{ct}: \{g_1^{s_i(w_0+i\cdot w_1)}\cdot \boxed{g_3^{s_i u_i}},\ g_1^{s_i}\cdot \boxed{g_3^{s_i}}\} \\
\mathsf{sk}: \{h_1^{r_j}\cdot \boxed{h_3^{r_j}},\ h_1^{r_j(w_0+j\cdot w_1)}\cdot \boxed{h_3^{r_j u_j}}\}_{j\in[n]}
\end{array}
\right\}.
$$

[9] with two main differences: (i) we are in the selective setting which allows for a much simpler proof, (ii) we allow $j = i$ in sk.

We can then use the $\mathrm{SD}^{G_N}_{p_3 \mapsto p_2 p_3}$ assumption to argue that

$$(g_3^{s_i}, g_3^{s_i u_i}) \approx_c (g_3^{s_i} \cdot \boxed{g_2^{s_i}}, g_3^{s_i u_i} \cdot \boxed{g_2^{s_i u_i}})$$

Roughly speaking, we will then repeat the above argument n times for each $i \in [n]$ (see $\mathsf{Sub\text{-}Game}_{i,1}$ through $\mathsf{Sub\text{-}Game}_{i,4}$ below). Here, there is an additional complication arising from the fact that in order to invoke the $\mathrm{SD}^{G_N}_{p_1 \mapsto p_1 p_3}$ assumption, we need to simulate sk given only h_1, h_{13}, h_2. To do this, we need to switch sk back to $\{h_{13}^{r_j}, h_{13}^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]}$, which we do in $\mathsf{Sub\text{-}Game}_{i,5}$ through $\mathsf{Sub\text{-}Game}_{i,7}$.

At this point, we are almost done, except we still need to introduce a $(h_2^{r_j}, h_2^{r_j u_j})$-component into sk. We will handle this at the very beginning of the proof (cf. $\mathsf{Game}_{0'}$). Fortunately, we can carry out the above argument even with the extra $(h_2^{r_j}, h_2^{r_j u_j})$-component in sk.

Actual proof. We prove step one of the entropy expansion lemma in Lemma 3 via the following game sequence, which is summarized in Fig 4. By ct_j (resp. sk_j), we denote the j'th tuple of ct (resp. sk).

$\underline{\mathsf{Game}_0.}$ This is the left distribution in Lemma 3:

$$\left\{ \begin{array}{l} \mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1}, g_2 \\ \quad \mathsf{ct}: \{g_1^{s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j\in[n]} \\ \quad \mathsf{sk}: \{h_{123}^{r_j},\ h_{123}^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]} \end{array} \right\}.$$

$\underline{\mathsf{Game}_{0'}.}$ We modify sk as follows:

$$\mathsf{sk}: \{h_{13}^{r_j} \cdot \boxed{h_2^{r_j}},\ h_{13}^{r_j(w_0+j\cdot w_1)} \cdot \boxed{h_2^{r_j u_j}}\}_{j\in[n]}$$

where $u_1, \ldots, u_n \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. We claim that $\mathsf{Game}_0 \approx_c \mathsf{Game}_{0'}$. This follows from the $\mathrm{DDH}^{H_N}_{p_2}$ assumption, which tells us that

$$\{h_2^{r_j}, h_2^{r_j w_0}\}_{j\in[n]} \approx_c \{h_2^{r_j}, \boxed{h_2^{r_j u'_j}}\}_{j\in[n]} \ \text{ given } \ g_1, g_2, h_{13}$$

where $u'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ and we will then implicitly set $u_j = u'_j + j \cdot w_1$ for all $j \in [n]$. In the security reduction, we use the fact that $\mathsf{aux}, \mathsf{ct}$ leak no information about $w_0 \bmod p_2$. See Lemma 22 for details.

$\underline{\mathsf{Game}_i\ (i = 1, \ldots, n+1).}$ We modify ct as follows:

$$\begin{array}{ll} \mathsf{ct}: \{g_1^{s_j(w_0+j\cdot w_1)} \cdot \boxed{g_2^{s_j u_j}},\ g_1^{s_j} \cdot \boxed{g_2^{s_j}}\}_{j<i} \\ \qquad \{g_1^{s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j\ge i} \end{array}$$

where $u_1, \ldots, u_{i-1}$ are defined as in $\mathsf{Game}_{0'}$. It is easy to see that $\mathsf{Game}_{0'} \equiv \mathsf{Game}_1$. To show that $\mathsf{Game}_i \approx_c \mathsf{Game}_{i+1}$, we will require another sequence of sub-games.

$\underline{\mathsf{Sub\text{-}Game}_{i,1}.}$ Identical to Game_i except that we modify ct_i as follows:

$$\mathsf{ct}_i: \{g_1^{s_i(w_0+i\cdot w_1)} \cdot \boxed{g_3^{s_i(w_0+i\cdot w_1)}},\ g_1^{s_i} \cdot \boxed{g_3^{s_i}}\}$$

We claim that $\mathsf{Game}_i \approx_c \mathsf{Sub\text{-}Game}_{i,1}$. This follows from the $\mathrm{SD}^{G_N}_{p_1 \mapsto p_1 p_3}$ assumption, which tells us that

$$g_1^{s_i} \approx_c g_1^{s_i} \cdot \boxed{g_3^{s_i}} \text{ given } g_1, g_2, h_{13}, h_2$$

In the reduction, we will sample $w_0, w_1, u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ and use g_1, g_2 to simulate $\mathsf{aux}, \{\mathsf{ct}_j\}_{j\neq i}$ and h_{13}, h_2 to simulate sk. See Lemma 23 for details.

Sub-Game$_{i,2}$. We modify the distribution of sk_j for all $j \neq i$ (while keeping sk_i unchanged):

$$\mathsf{sk}_j\ (j \neq i)\ : h_1^{r_j} \cdot h_2^{r_j} \cdot \boxed{h_3^{r_j}},\ h_1^{r_j(w_0+j\cdot w_1)} \cdot h_2^{r_j u_j} \cdot \boxed{h_3^{r_j u_j}}$$

We claim that Sub-Game$_{i,1} \approx_c$ Sub-Game$_{i,2}$. This follows from the $\mathrm{DDH}_{p_3}^{H_N}$ assumption, which tells us that

$$\{h_3^{r_j}, h_3^{r_j w_1}\}_{j\neq i} \approx_c \{h_3^{r_j}, \boxed{h_3^{r_j u'_j}}\}_{j \neq i} \text{ given } g_1, g_2, g_3, h_1, h_2, h_3.$$

where $u'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. In the reduction, we will program $w_0 := \tilde{w}_0 - i \cdot w_1 \bmod p_3$ with $\tilde{w}_0 \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ so that we can simulate $g_3^{s_i(w_0+i\cdot w_1)}$ in ct_i, and then implicitly set $u_j = \tilde{w}_0 + (j-i)\cdot u'_j \bmod p_3$ for all $j \neq i$. See Lemma 24 for details.

Sub-Game$_{i,3}$. We modify the distribution of ct_i and sk_i simultaneously:

$$\begin{aligned}\mathsf{ct}_i &: g_1^{s_i(w_0+i\cdot w_1)} \cdot \boxed{g_3^{s_i u_i}},\ g_1^{s_i} \cdot g_3^{s_i}\\ \mathsf{sk}_i &: h_1^{r_i} \cdot h_2^{r_i} \cdot h_3^{r_i},\ h_1^{r_i(w_0+i\cdot w_1)} \cdot h_2^{r_i u_i} \cdot \boxed{h_3^{r_i u_i}}\end{aligned}$$

We claim that Sub-Game$_{i,2} \equiv$ Sub-Game$_{i,3}$. This follows from the fact that for all $j \neq i$, the quantity $w_0 + j \cdot w_1 \bmod p_3$ leaked in sk_j is masked by u_j and therefore $\{w_0 + i \cdot w_1 \bmod p_3\} \equiv \{u_i \bmod p_3\}$. See Lemma 25 for details.

Sub-Game$_{i,4}$. We modify the distribution of ct_i as follows:

$$\mathsf{ct}_i : g_1^{s_i(w_0+i\cdot w_1)} \cdot \boxed{g_2^{s_i u_i}} \cdot g_3^{s_i u_i},\ g_1^{s_i} \cdot \boxed{g_2^{s_i}} \cdot g_3^{s_i}$$

We claim that Sub-Game$_{i,3} \approx_c$ Sub-Game$_{i,4}$. This follows from the $\mathrm{SD}^{G_N}_{p_3 \mapsto p_3 p_2}$ assumption, which tells us that

$$g_3^{s_i} \approx_c \boxed{g_2^{s_i}} \cdot g_3^{s_i} \ \text{ given }\ g_1, g_2, h_1, h_{23}.$$

In the reduction, we will sample $w_0, w_1, u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ and use g_1, g_2 to simulate $\mathsf{aux}, \{\mathsf{ct}_j\}_{j\neq i}$. In addition, we will use generator h_{23} to sample $\{h_2^{r_j} \cdot h_3^{r_j}, h_2^{r_j u_j} \cdot h_3^{r_j u_j}\}_{j\in[n]}$ in sk. See Lemma 26 for details.

Sub-Game$_{i,5}$. We modify the distribution of ct_i and sk_i:

$$\begin{aligned}\mathsf{ct}_i &: g_1^{s_i(w_0+i\cdot w_1)} \cdot g_2^{s_i u_i} \cdot \boxed{g_3^{s_i(w_0+i\cdot w_1)}},\ g_1^{s_i} \cdot g_2^{s_i} \cdot g_3^{s_i}\\ \mathsf{sk}_i &: h_1^{r_i} \cdot h_2^{r_i} \cdot h_3^{r_i},\ h_1^{r_i(w_0+i\cdot w_1)} \cdot h_2^{r_i u_i} \cdot \boxed{h_3^{r_i(w_0+i\cdot w_1)}}\end{aligned}$$

We claim that Sub-Game$_{i,4} \equiv$ Sub-Game$_{i,5}$. The proof is completely analogous to that of Sub-Game$_{i,2} \equiv$ Sub-Game$_{i,3}$. See Lemma 27 for details.

Sub-Game$_{i,6}$. We modify the distribution of sk_j for all $j \neq i$:

$$\mathsf{sk}_j\ (j \neq i)\ : h_1^{r_j} \cdot h_2^{r_j} \cdot \boxed{h_3^{r_j}},\ h_1^{r_j(w_0+j\cdot w_1)} \cdot h_2^{r_j u_j} \cdot \boxed{h_3^{r_j(w_0+j\cdot w_1)}}$$

We claim that Sub-Game$_{i,5} \approx_c$ Sub-Game$_{i,6}$. The proof is completely analogous to that of Sub-Game$_{i,1} \approx_c$ Sub-Game$_{i,2}$. See Lemma 28 for details.

Sub-Game$_{i,7}$. We modify the distribution of ct_i:

$$\mathsf{ct}_i : g_1^{s_i(w_0+i\cdot w_1)} \cdot g_2^{s_i u_i} \cdot \cancel{g_3^{s_i(w_0+i\cdot w_1)}},\ g_1^{s_i} \cdot g_2^{s_i} \cdot \cancel{g_3^{s_i}}$$

We claim that Sub-Game$_{i,6} \approx_c$ Sub-Game$_{i,7}$. The proof is completely analogous to that of Game$_i \approx_c$ Sub-Game$_{i,1}$. See Lemma 29 for details. Furthermore, observe that Sub-Game$_{i,7}$ is actually identical to Game$_{i+1}$.

<u>Game$_{n+1}$.</u> In Game$_{n+1}$, we have:

$$\left\{ \begin{array}{l} \mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1}, g_2 \\ \quad \mathsf{ct}: \{g_1^{s_j(w_0+j\cdot w_1)} \cdot \boxed{g_2^{s_j u_j}}, \, g_1^{s_j} \cdot \boxed{g_2^{s_j}}\}_{j\in[n]} \\ \quad \mathsf{sk}: \{h_{13}^{r_j} \cdot \boxed{h_2^{r_j}}, \, h_{13}^{r_j(w_0+j\cdot w_1)} \cdot \boxed{h_2^{r_j u_j}}\}_{j\in[n]} \end{array} \right\}.$$

This is exactly the right distribution of Lemma 3.

Game	CT $j<i$	CT i	CT $j>i$	SK $j \neq i$	SK i	Remark & Aux
		$g_1^{s_j} \cdot ?$		$(h_1^{r_j} \cdot h_2^{r_j} \cdot h_3^{r_j})$		
		$g_1^{s_j(w_0+j\cdot w_1)} \cdot ?$		$h_1^{r_j(w_0+j w_1)} \cdot ?$		
0		1				Left distribution
		1		$h_2^{r_j(w_0+j\cdot w_1)} \cdot h_3^{r_j(w_0+j\cdot w_1)}$		
$0'$		1				$\mathrm{DDH}_{p_2}^{H_N}: \{h_2^{r_j}, h_2^{r_j w_0}\}_{j\in[n]} \approx_c \{h_2^{r_j}, h_2^{r_j u'_j}\}_{j\in[n]}$
		1		$\boxed{h_2^{r_j u_j}} \cdot h_3^{r_j(w_0+j\cdot w_1)}$		define u_j from u'_j
i	$\boxed{g_2^{s_j}}$	1	1			$\mathsf{Game}_1 = \mathsf{Game}_{0'}$
	$\boxed{g_2^{s_j u_j}}$	1	1	$h_2^{r_j u_j} \cdot h_3^{r_j(w_0+j\cdot w_1)}$		$\mathsf{Game}_{i-1,7} = \mathsf{Game}_i$
$i,1$	$g_2^{s_j}$	$\boxed{g_3^{s_i}}$	1			$\mathrm{SD}_{p_1 \mapsto p_1 p_3}^{G_N}: g_1^{s_i} \approx_c g_1^{s_i} \cdot g_3^{s_i}$
	$g_2^{s_j u_j}$	$\boxed{g_3^{s_i(w_0+i\cdot w_1)}}$	1	$h_2^{r_j u_j} \cdot h_3^{r_j(w_0+j\cdot w_1)}$		given g_1, g_2, h_{13}, h_2
$i,2$	$g_2^{s_j}$	$g_3^{s_i}$	1			$\mathrm{DDH}_{p_3}^{H_N}: \{h_3^{r_j}, h_3^{r_j w_1}\}_{j\neq i} \approx_c \{h_3^{r_j}, h_3^{r_j u'_j}\}_{j\neq i}$
	$g_2^{s_j u_j}$	$g_3^{s_i(w_0+i\cdot w_1)}$	1	$h_2^{r_j u_j} \cdot \boxed{h_3^{r_j u_j}}$	$h_2^{r_i u_i} \cdot h_3^{r_i(w_0+i\cdot w_1)}$	given $w_0 + i \cdot w_1$, define u_j from u'_j
$i,3$	$g_2^{s_j}$	$g_3^{s_i}$	1			statistical argument: for fixed $\{\mathsf{sk}_j, \mathsf{ct}_j\}_{j\neq i}$
	$g_2^{s_j u_j}$	$\boxed{g_3^{s_i u_i}}$	1	$h_2^{r_j u_j} \cdot h_3^{r_j u_j}$	$h_2^{r_i u_i} \cdot \boxed{h_3^{r_i u_i}}$	$\{w_0 + i \cdot w_1 \bmod p_3\} \equiv \{u_i \bmod p_3\}$
$i,4$	$g_2^{s_j}$	$\boxed{g_2^{s_i}} \cdot g_3^{s_i}$	1			$\mathrm{SD}_{p_3 \mapsto p_3 p_2}^{G_N}: g_3^{s_i} \approx_c g_2^{s_i} \cdot g_3^{s_i}$
	$g_2^{s_j u_j}$	$\boxed{g_2^{s_i u_i}} \cdot g_3^{s_i u_i}$	1	$h_2^{r_j u_j} \cdot h_3^{r_j u_j}$	$h_2^{r_i u_i} \cdot h_3^{r_i u_i}$	given g_1, g_2, h_1, h_{23}
$i,5$	$g_2^{s_j}$	$g_2^{s_i} \cdot g_3^{s_i}$	1			statistical argument
	$g_2^{s_j u_j}$	$g_2^{s_i u_i} \cdot \boxed{g_3^{s_i(w_0+i\cdot w_1)}}$	1	$h_2^{r_j u_j} \cdot h_3^{r_j u_j}$	$h_2^{r_i u_i} \cdot \boxed{h_3^{r_i(w_0+i\cdot w_1)}}$	analogous to Sub-Game$_{i,3}$
$i,6$	$g_2^{s_j}$	$g_2^{s_i} \cdot g_3^{s_i}$	1			$\mathrm{DDH}_{p_3}^{H_N}$
	$g_2^{s_j u_j}$	$g_2^{s_i u_i} \cdot g_3^{s_i(w_0+i\cdot w_1)}$	1	$h_2^{r_j u_j} \cdot \boxed{h_3^{r_j(w_0+j\cdot w_1)}}$		analogous to Sub-Game$_{i,2}$
$i,7$	$g_2^{s_j}$	$g_2^{s_i} \cdot \cancel{g_3^{s_i}}$	1			$\mathrm{SD}_{p_1 \mapsto p_1 p_3}^{G_N}$
	$g_2^{s_j u_j}$	$g_2^{s_i u_i} \cdot \cancel{g_3^{s_i(w_0+i\cdot w_1)}}$	1	$h_2^{r_j u_j} \cdot h_3^{r_j(w_0+j\cdot w_1)}$		analogous to Sub-Game$_{i,1}$

Fig. 4. Game sequence for our proof of Lemma 3 (Bilinear entropy expansion lemma (step one)).

4 Simulating Composite-Order Groups in Prime-Order Groups

We build upon and extend the previous framework of Chen et al. [6, 11] for simulating composite-order groups in prime-order ones. We provide prime-order analogues of the static assumptions $\mathrm{SD}^{G_N}_{p_1 \mapsto p_1 p_2}, \mathrm{DDH}^{H_N}_{p_1}$ used in the previous sections. Moreover, we show that these prime-order analogues follow from the standard k-Linear assumption (and more generally, the MDDH assumption [9]) in prime-order bilinear groups.

Additional notation. Let $\mathbf{A}$ be a matrix over $\mathbb{Z}_p$. We use $\mathsf{span}(\mathbf{A})$ to denote the column span of $\mathbf{A}$, and we use $\mathsf{span}^\ell(\mathbf{A})$ to denote matrices of width ℓ where each column lies in $\mathsf{span}(\mathbf{A})$; this means $\mathbf{M} \leftarrow_{\mathrm{R}} \mathsf{span}^\ell(\mathbf{A})$ is a random matrix of width ℓ where each column is chosen uniformly from $\mathsf{span}(\mathbf{A})$. We use $\mathsf{basis}(\mathbf{A})$ to denote a basis of $\mathsf{span}(\mathbf{A})$, and we use $(\mathbf{A}_1 \mid \mathbf{A}_2)$ to denote the concatenation of matrices $\mathbf{A}_1, \mathbf{A}_2$. If $\mathbf{A}$ is a m-by-n matrix with $m > n$, we use $\overline{\mathbf{A}}$ to denote the sub-matrix consisting of the first n rows and $\underline{\mathbf{A}}$ the sub-matrix with remaining $m-n$ rows. We let $\mathbf{I}_n$ be the n-by-n identity matrix and $\mathbf{0}$ be a zero matrix whose size will be clear from the context.

4.1 Prime-order groups and matrix Diffie-Hellman assumptions

A generator $\mathcal{G}$ takes as input a security parameter λ and outputs a description $\mathbb{G} := (p, G_1, G_2, G_T, e)$, where p is a prime of $\Theta(\lambda)$ bits, G_1, G_2 and G_T are cyclic groups of order p, and $e : G_1 \times G_2 \to G_T$ is a non-degenerate bilinear map. We require that the group operations in G_1, G_2 and G_T as well the bilinear map e are computable in deterministic polynomial time with respect to λ. Let $g_1 \in G_1$, $g_2 \in G_2$ and $g_T = e(g_1, g_2) \in G_T$ be the respective generators. We employ the *implicit representation* of group elements: for a matrix $\mathbf{M}$ over $\mathbb{Z}_p$, we define $[\mathbf{M}]_1 := g_1^{\mathbf{M}}, [\mathbf{M}]_2 := g_2^{\mathbf{M}}, [\mathbf{M}]_T := g_T^{\mathbf{M}}$, where exponentiation is carried out component-wise. Also, given $[\mathbf{A}]_1, [\mathbf{B}]_2$, we let $e([\mathbf{A}]_1, [\mathbf{B}]_2) = [\mathbf{AB}]_T$.

We define the matrix Diffie-Hellman (MDDH) assumption on G_1 [9]:

Assumption 3 ($\mathrm{MDDH}^m_{k,\ell}$ Assumption) *Let $\ell > k \geq 1$ and $m \geq 1$. We say that the $\mathrm{MDDH}^m_{k,\ell}$ assumption holds if for all PPT adversaries $\mathcal{A}$, the following advantage function is negligible in λ.*

$$\mathsf{Adv}_{\mathcal{A}}^{\mathrm{MDDH}^m_{k,\ell}}(\lambda) := |\Pr[\mathcal{A}(\mathbb{G}, [\mathbf{M}]_1, [\mathbf{MS}]_1) = 1] - \Pr[\mathcal{A}(\mathbb{G}, [\mathbf{M}]_1, [\mathbf{U}]_1) = 1]|$$

where $\mathbf{M} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times k}$, $\mathbf{S} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k \times m}$ and $\mathbf{U} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times m}$.

The MDDH assumption on G_2 can be defined in an analogous way. Escala *et al.* [9] showed that

$$k\text{-Lin} \Rightarrow \mathrm{MDDH}^1_{k,k+1} \Rightarrow \mathrm{MDDH}^m_{k,\ell} \ \forall \ell > k, m \geq 1$$

with a tight security reduction. Henceforth, we will use MDDH_k to denote $\mathrm{MDDH}^1_{k,k+1}$.

4.2 Basis structure

We want to simulate composite-order groups whose order is the product of three primes. Fix parameters $\ell_1, \ell_2, \ell_3, \ell_W \geq 1$. Pick random

$$\mathbf{A}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_1}, \mathbf{A}_2 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_2}, \mathbf{A}_3 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_3}$$

where $\ell := \ell_1 + \ell_2 + \ell_3$. Let $(\mathbf{A}_1^\parallel \mid \mathbf{A}_2^\parallel \mid \mathbf{A}_3^\parallel)^\top$ denote the inverse of $(\mathbf{A}_1 \mid \mathbf{A}_2 \mid \mathbf{A}_3)$, so that $\mathbf{A}_i^\top \mathbf{A}_i^\parallel = \mathbf{I}$ (known as *non-degeneracy*) and $\mathbf{A}_i^\top \mathbf{A}_j^\parallel = \mathbf{0}$ if $i \neq j$ (known as *orthogonality*), as depicted in Fig 5. This generalizes the constructions in [10, 11] where $\ell_1 = \ell_2 = \ell_3 = k$.

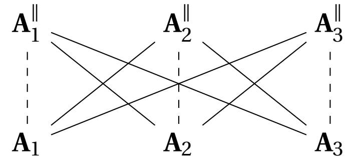

Fig. 5. Basis relations. Solid lines mean orthogonal, dashed lines mean non-degeneracy. Similar relations hold in composite-order groups with (g_1, g_2, g_3) in place of $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$ and (h_1, h_2, h_3) in place of $(\mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|})$.

Correspondence. We have the following correspondence with composite-order groups:

$$g_i \mapsto [\mathbf{A}_i]_1, \qquad g_i^s \mapsto [\mathbf{A}_i \mathbf{s}]_1$$
$$w \in \mathbb{Z}_N \mapsto \mathbf{W} \in \mathbb{Z}_p^{\ell \times \ell_W}, \qquad g_i^w \mapsto [\mathbf{A}_i^\top \mathbf{W}]_1$$

The following statistical lemma is analogous to the Chinese Remainder Theorem, which tells us that w mod p_2 is uniformly random given g_1^w, g_3^w, where $w \leftarrow_{\mathrm{R}} \mathbb{Z}_N$:

Lemma 5 (statistical lemma). *With probability* $1 - 1/p$ *over* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|}$, *the following two distributions are statistically identical.*

$$\{\mathbf{A}_1^\top \mathbf{W}, \mathbf{A}_3^\top \mathbf{W}, \boxed{\mathbf{W}}\} \quad \textit{and} \quad \{\mathbf{A}_1^\top \mathbf{W}, \mathbf{A}_3^\top \mathbf{W}, \boxed{\mathbf{W} + \mathbf{U}^{(2)}}\}$$

where $\mathbf{W} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_W}$ *and* $\mathbf{U}^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|})$.

Proof. For any $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|})$ satisfying the basis relations (shown in Fig 5), we may alternatively pick

$$\mathbf{W} = \widetilde{\mathbf{W}} + \mathbf{W}^{(2)} \text{ where } \widetilde{\mathbf{W}} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_W},\ \mathbf{W}^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|}).$$

This does not change the two distributions, but they now become

$$\{\mathbf{A}_1^\top \widetilde{\mathbf{W}}, \mathbf{A}_3^\top \widetilde{\mathbf{W}}, \widetilde{\mathbf{W}} + \boxed{\mathbf{W}^{(2)}}\} \quad \text{and} \quad \{\mathbf{A}_1^\top \widetilde{\mathbf{W}}, \mathbf{A}_3^\top \widetilde{\mathbf{W}}, \widetilde{\mathbf{W}} + \boxed{\mathbf{W}^{(2)} + \mathbf{U}^{(2)}}\}.$$

Because the boxed terms have the same distribution, these two distributions are statistically identical. □

4.3 Prime-order Subgroup Decision Assumptions

We first describe the *prime-order* $(\mathbf{A}_1 \mapsto \mathbf{A}_1, \mathbf{A}_2)$*-subgroup decision assumption*, denoted by $\mathrm{SD}^{G_1}_{\mathbf{A}_1 \mapsto \mathbf{A}_1, \mathbf{A}_2}$. This is analogous to the subgroup decision assumption in composite-order groups $\mathrm{SD}^{G_N}_{p_1 \mapsto p_1 p_2}$ which asserts that $G_{p_1} \approx_c G_{p_1 p_2}$ given h_1, h_3, h_{12} along with g_1, g_2, g_3. By symmetry, we can permute the indices for $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$.

Lemma 6 ($\mathbf{MDDH}_{\ell_1, \ell_1+\ell_2} \Rightarrow \mathbf{SD}^{G_1}_{\mathbf{A}_1 \mapsto \mathbf{A}_1, \mathbf{A}_2}$). *Under the* $\mathrm{MDDH}_{\ell_1, \ell_1+\ell_2}$ *assumption in* G_1, *there exists an efficient sampler outputting random* $([\mathbf{A}_1]_1, [\mathbf{A}_2]_1, [\mathbf{A}_3]_1)$ *(as described in Section 4.2) along with base* $\mathsf{basis}(\mathbf{A}_1^{\|})$, $\mathsf{basis}(\mathbf{A}_3^{\|})$, $\mathsf{basis}(\mathbf{A}_1^{\|}, \mathbf{A}_2^{\|})$ *(of arbitrary choice) such that the following advantage function is negligible in* λ.

$$\mathsf{Adv}_{\mathcal{A}}^{\mathrm{SD}^{G_1}_{\mathbf{A}_1 \mapsto \mathbf{A}_1, \mathbf{A}_2}}(\lambda) := |\Pr[\mathcal{A}(D, [\mathbf{t}_0]_1) = 1] - \Pr[\mathcal{A}(D, [\mathbf{t}_1]_1) = 1]|$$

where

$$D := (\,[\mathbf{A}_1]_1, [\mathbf{A}_2]_1, [\mathbf{A}_3]_1, \mathsf{basis}(\mathbf{A}_1^{\|}), \mathsf{basis}(\mathbf{A}_3^{\|}), \mathsf{basis}(\mathbf{A}_1^{\|}, \mathbf{A}_2^{\|})\,),$$
$$\mathbf{t}_0 \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1),\ \mathbf{t}_1 \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2).$$

Similar statements were also implicit in [10, 11].

Proof. We prove the lemma from $\mathrm{MDDH}_{\ell_1,\ell_1+\ell_2}$ assumption: for all PPT adversaries $\mathcal{A}$, we construct an algorithm $\mathcal{B}$ such that

$$\mathsf{Adv}_{\mathcal{A}}^{\mathrm{SD}^{G_1}_{\mathbf{A}_1\mapsto\mathbf{A}_1,\mathbf{A}_2}}(\lambda) \leq \mathsf{Adv}_{\mathcal{B}}^{\mathrm{MDDH}_{\ell_1,\ell_1+\ell_2}}(\lambda).$$

The adversary $\mathcal{B}$ gets as input $[\mathbf{M}]_1 \in G_1^{(\ell_1+\ell_2)\times\ell_1}$ and $[\mathbf{u}]_1 \in G_1^{\ell_1+\ell_2}$, either $\mathbf{u} = \mathbf{Ms}$ for $\mathbf{s} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1}$ or $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1+\ell_2}$, and proceeds as follows:

Programming $\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3,\mathbf{A}_1^{\|},\mathbf{A}_2^{\|},\mathbf{A}_3^{\|}$. Pick $\widetilde{\mathbf{A}} \leftarrow_{\mathrm{R}} \mathrm{GL}_\ell(\mathbb{Z}_p)$ and define

$$(\mathbf{A}_1 \mid \mathbf{A}_2 \mid \mathbf{A}_3) := \widetilde{\mathbf{A}} \begin{pmatrix} \overline{\mathbf{M}} & & \\ \underline{\mathbf{M}} & \mathbf{I}_{\ell_2} & \\ & & \mathbf{I}_{\ell_3} \end{pmatrix} \quad \text{and} \quad (\mathbf{A}_1^{\|} \mid \mathbf{A}_2^{\|} \mid \mathbf{A}_3^{\|}) := (\widetilde{\mathbf{A}}^{-1})^\top \begin{pmatrix} (\overline{\mathbf{M}}^{-1})^\top & -(\overline{\mathbf{M}}^{-1})^\top\underline{\mathbf{M}}^\top & \\ & \mathbf{I}_{\ell_2} & \\ & & \mathbf{I}_{\ell_3} \end{pmatrix}.$$

Observe that $\mathcal{B}$ can simulate $[\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3]_1$ since it knows $\widetilde{\mathbf{A}}$ and $[\mathbf{M}]_1$.

Simulating $\mathsf{basis}(\mathbf{A}_1^{\|}), \mathsf{basis}(\mathbf{A}_3^{\|}), \mathsf{basis}(\mathbf{A}_1^{\|},\mathbf{A}_2^{\|})$. $\mathcal{B}$ can readily compute

$$\mathsf{basis}(\mathbf{A}_3^{\|}) := (\widetilde{\mathbf{A}}^{-1})^\top \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_{\ell_3} \end{pmatrix}$$

from $\widetilde{\mathbf{A}}$. Observe that we can define

$$\mathsf{basis}(\mathbf{A}_1^{\|}) := \mathbf{A}_1^{\|}\,\overline{\mathbf{M}}^\top = (\widetilde{\mathbf{A}}^{-1})^\top \begin{pmatrix} \mathbf{I}_{\ell_1} \\ \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathsf{basis}(\mathbf{A}_1^{\|},\mathbf{A}_2^{\|}) := (\mathbf{A}_1^{\|} \mid \mathbf{A}_2^{\|}) \begin{pmatrix} \overline{\mathbf{M}}^\top & \underline{\mathbf{M}}^\top \\ & \mathbf{I}_{\ell_2} \end{pmatrix} = (\widetilde{\mathbf{A}}^{-1})^\top \begin{pmatrix} \mathbf{I}_{\ell_1+\ell_2} \\ \mathbf{0} \end{pmatrix}$$

since $\overline{\mathbf{M}}$ is full-rank with overwhelming probability, and $\mathcal{B}$ can also compute them from $\widetilde{\mathbf{A}}$.

Simulating the challenge. $\mathcal{B}$ outputs the challenge as

$$\left[\widetilde{\mathbf{A}}\begin{pmatrix} \mathbf{u} \\ \mathbf{0} \end{pmatrix}\right]_1.$$

Observe that if $\mathbf{u} = \mathbf{Ms}$ for $\mathbf{s} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1}$, then the output challenge will be uniformly distributed over $[\mathsf{span}(\mathbf{A}_1)]_1$; and if $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1+\ell_2}$, then the output challenge will be a random element from $[\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)]_1$.

The lemma then follows readily. □

4.4 Prime-order Subgroup Diffie-Hellman assumptions

We then formalize the *prime-order* $\mathbf{A}_1$*-subgroup Diffie-Hellman assumption,* denoted by $\mathrm{DDH}^{G_2}_{\mathbf{A}_1}$. This is analogous to the subgroup Diffie-Hellman assumption in the composite-order group $\mathrm{DDH}^{H_N}_{p_1}$ which ensures that $\{h_1^{r_j w}, h_1^{r_j}\}_{j\in[Q]} \approx_c \{h_1^{r_j w}\cdot h_1^{u_j}, h_1^{r_j}\}_{j\in[Q]}$ given g_1,g_2,g_3,h_1,h_2,h_3 for $Q = \mathrm{poly}(\lambda)$. One can permute the indices for $\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3$.

Lemma 7 ($\mathbf{MDDH}^{\ell_1}_{\ell_W,Q} \Rightarrow \mathbf{DDH}^{G_2}_{\mathbf{A}_1}$). *Fix $Q = poly(\lambda)$ with $Q > \ell_W \geq 1$. Under the $MDDH^{\ell_1}_{\ell_W,Q}$ assumption in G_2, the following advantage function is negligible in λ*

$$\mathsf{Adv}_{\mathcal{A}}^{\mathrm{DDH}^{G_2}_{\mathbf{A}_1}}(\lambda) := |\Pr[\mathcal{A}(D,T_0)=1] - \Pr[\mathcal{A}(D,T_1)=1]|$$

where

$$D := (\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|};\ \mathbf{A}_2^{\top}\mathbf{W}, \mathbf{A}_3^{\top}\mathbf{W}), \qquad \mathbf{W} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_W};$$
$$T_0 := ([\mathbf{W}\mathbf{D}]_2, [\mathbf{D}]_2),\ T_1 := ([\mathbf{W}\mathbf{D} + \mathbf{R}^{(1)}]_2, [\mathbf{D}]_2), \quad \mathbf{D} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W \times Q}, \mathbf{R}^{(1)} \leftarrow_{\mathrm{R}} \mathsf{span}^Q(\mathbf{A}_1^{\|}).$$

Proof. We prove the lemma from $\mathrm{MDDH}_{\ell_W,Q}^{\ell_1}$ assumption which says that

$$([\mathbf{D}]_2, [\mathbf{S}\mathbf{D}]_2) \approx_c ([\mathbf{D}]_2, [\mathbf{S}\mathbf{D} + \mathbf{U}]_2)$$

where $\mathbf{D} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W \times Q}$, $\mathbf{S} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1 \times \ell_W}$ and $\mathbf{U} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1 \times Q}$. Concretely, for all PPT adversaries $\mathcal{A}$, we construct an algorithm $\mathcal{B}$ such that

$$\mathsf{Adv}_{\mathcal{A}}^{\mathrm{DDH}_{\mathbf{A}_1}^{G_2}}(\lambda) \le \mathsf{Adv}_{\mathcal{B}}^{\mathrm{MDDH}_{\ell_W,Q}^{\ell_1}}(\lambda).$$

On input $([\mathbf{D}]_2, [\mathbf{T}]_2)$, algorithm $\mathcal{B}$ samples $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|}$, pick $\widetilde{\mathbf{W}} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_W}$ and implicitly set $\mathbf{W} := \widetilde{\mathbf{W}} + \mathbf{A}_1^{\|}\mathbf{S}$. Output

$$\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|};\ \mathbf{A}_2^{\top}\widetilde{\mathbf{W}}, \mathbf{A}_3^{\top}\widetilde{\mathbf{W}},$$
$$[\widetilde{\mathbf{W}}\mathbf{D} + \mathbf{A}_1^{\|}\mathbf{T}]_2, [\mathbf{D}]_2.$$

Observe that when $\mathbf{T} = \mathbf{S}\mathbf{D}$, the output is identical to (D, T_0); and when $\mathbf{T} = \mathbf{S}\mathbf{D} + \mathbf{U}$ and we set $\mathbf{R}^{(1)} := \mathbf{A}_1^{\|}\mathbf{U}$, the output is identical to (D, T_1). This readily proves the lemma. □

5 Bilinear Entropy Expansion in Prime-Order Groups

In this section, we present our (bilinear) entropy expansion lemma in prime-order groups.

Entropy expansion lemma. We start by sampling $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|}$ as in Section 4.2. Our prime-order entropy expansion lemma is as follows:

Lemma 8 (prime-order entropy expansion lemma). *Suppose $\ell_1, \ell_3, \ell_W \ge k$. Then, under the MDDH$_k$ assumption, we have*

$$\left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^{\top}]_1, [\mathbf{A}_1^{\top}\mathbf{W}]_1, [\mathbf{A}_1^{\top}\mathbf{W}_0]_1, [\mathbf{A}_1^{\top}\mathbf{W}_1]_1 \\ \mathsf{ct}: [\mathbf{c}^{\top}]_1,\ \{[\mathbf{c}^{\top}\mathbf{W} + \mathbf{c}_j^{\top}(\mathbf{W}_0 + j \cdot \mathbf{W}_1)]_1,\ [\mathbf{c}_j^{\top}]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[\mathbf{W}\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{D}_j]_2\}_{j\in[n]} \end{array}\right\}$$
$$\approx_c \left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^{\top}]_1, [\mathbf{A}_1^{\top}\mathbf{W}]_1, [\mathbf{A}_1^{\top}\mathbf{W}_0]_1, [\mathbf{A}_1^{\top}\mathbf{W}_1]_1 \\ \mathsf{ct}: [\boxed{\mathbf{c}}^{\top}]_1,\ \{[\boxed{\mathbf{c}}^{\top}(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}}) + \boxed{\mathbf{c}_j}^{\top}(\mathbf{W}_0 + j \cdot \mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1,\ [\boxed{\mathbf{c}_j}^{\top}]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j \cdot \mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2\}_{j\in[n]} \end{array}\right\}$$

where $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_W}, \mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|}), \mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W \times \ell_W}$*, and* $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ *in the left distribution while* $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)$ *in the right distribution. Concretely, the distinguishing advantage* $\mathsf{Adv}_{\mathcal{A}}^{\mathrm{ExpLem}}(\lambda)$ *is at most*

$$\mathsf{Adv}_{\mathcal{B}}^{\mathrm{SD}_{\mathbf{A}_1 \mapsto \mathbf{A}_1,\mathbf{A}_2}^{G_1}}(\lambda) + \mathsf{Adv}_{\mathcal{B}_0}^{\mathrm{DDH}_{\mathbf{A}_2}^{G_2}}(\lambda)$$
$$+\ n \cdot \big(\mathsf{Adv}_{\mathcal{B}_1}^{\mathrm{SD}_{\mathbf{A}_1 \mapsto \mathbf{A}_1,\mathbf{A}_3}^{G_1}}(\lambda) + \mathsf{Adv}_{\mathcal{B}_2}^{\mathrm{DDH}_{\mathbf{A}_3}^{G_2}}(\lambda) + \mathsf{Adv}_{\mathcal{B}_4}^{\mathrm{SD}_{\mathbf{A}_3 \mapsto \mathbf{A}_3,\mathbf{A}_2}^{G_1}}(\lambda) + \mathsf{Adv}_{\mathcal{B}_6}^{\mathrm{DDH}_{\mathbf{A}_3}^{G_2}}(\lambda) + \mathsf{Adv}_{\mathcal{B}_7}^{\mathrm{SD}_{\mathbf{A}_1 \mapsto \mathbf{A}_1,\mathbf{A}_3}^{G_1}}(\lambda)\big) + \mathsf{Adv}_{\mathcal{B}_8}^{\mathrm{DDH}_{\mathbf{A}_2}^{G_2}}(\lambda)$$

where $\mathsf{Time}(\mathcal{B}), \mathsf{Time}(\mathcal{B}_0), \mathsf{Time}(\mathcal{B}_1), \mathsf{Time}(\mathcal{B}_2), \mathsf{Time}(\mathcal{B}_4), \mathsf{Time}(\mathcal{B}_6), \mathsf{Time}(\mathcal{B}_7), \mathsf{Time}(\mathcal{B}_8) \approx \mathsf{Time}(\mathcal{A})$.

Remark 1 (Differences from overview in Section 1.3). We stated our prime-order expansion lemma for general ℓ_1, ℓ_2, ℓ_3; for our KP-ABE, it suffices to set $(\ell_1, \ell_2, \ell_3) = (k, 1, k)$. Compared to the informal statement (8) in Section 1.3, we use $\mathbf{A}_2 \in \mathbb{Z}_p^{2k+1}$ instead of $\mathbf{A}_2 \in \mathbb{Z}_p^{(2k+1)\times k}$, and we introduced extra $\mathbf{A}_2$-components corresponding to $\mathbf{A}_2^\top \mathbf{W}, \mathbf{A}_2^\top(\mathbf{W}_0 + j \cdot \mathbf{W}_1)$ in ct on the RHS. We have $\mathbf{D}_j$ in place of $\mathbf{Br}_j$ in the above statement, though we will introduce $\mathbf{B}$ later on in Lemma 12. We also picked $\mathbf{D}_j$ to be square matrices to enable random self-reducibility of the sk-terms. Finally, $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)}$ correspond to $\mathbf{V}_j, \mathbf{U}_j$ in the informal statement, and in particular, we have $\mathbf{A}_1^\top \mathbf{V}_j^{(2)} = \mathbf{A}_1^\top \mathbf{U}_j^{(2)} = \mathbf{0}$.

Analogous to the composite-order setting, we prove the lemma in two steps via the following two lemmas.

Lemma 9 (prime-order entropy expansion lemma (step one)). *Suppose $\ell_1, \ell_3, \ell_W \geq k$. Then, under the MDDH$_k$ assumption, we have*

$$\left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_2^\top]_1 \\ \mathsf{ct}: \{[\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1,\ [\mathbf{c}_j^\top]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot \mathbf{W}_1)\mathbf{D}_j]_2\}_{j\in[n]} \end{array}\right\} \approx_c \left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_2^\top]_1 \\ \mathsf{ct}: \{[\boxed{\mathbf{c}_j}^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1,\ [\boxed{\mathbf{c}_j}^\top]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot \mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2\}_{j\in[n]} \end{array}\right\}$$

where $\mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}, \mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^\parallel), \mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$*, and* $\mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ *in the left distribution while* $\mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)$ *in the right distribution. Concretely, the distinguishing advantage* $\mathsf{Adv}_{\mathcal{A}}^{\mathrm{STEP1}}(\lambda)$ *is at most*

$$\mathsf{Adv}_{\mathcal{B}_0}^{\mathrm{DDH}_{\mathbf{A}_2}^{G_2}}(\lambda) + n\cdot\left(\mathsf{Adv}_{\mathcal{B}_1}^{\mathrm{SD}_{\mathbf{A}_1\mapsto\mathbf{A}_1,\mathbf{A}_3}^{G_1}}(\lambda) + \mathsf{Adv}_{\mathcal{B}_2}^{\mathrm{DDH}_{\mathbf{A}_3}^{G_2}}(\lambda) + \mathsf{Adv}_{\mathcal{B}_4}^{\mathrm{SD}_{\mathbf{A}_3\mapsto\mathbf{A}_3,\mathbf{A}_2}^{G_1}}(\lambda) + \mathsf{Adv}_{\mathcal{B}_6}^{\mathrm{DDH}_{\mathbf{A}_3}^{G_2}}(\lambda) + \mathsf{Adv}_{\mathcal{B}_7}^{\mathrm{SD}_{\mathbf{A}_1\mapsto\mathbf{A}_1,\mathbf{A}_3}^{G_1}}(\lambda)\right)$$

where $\mathsf{Time}(\mathcal{B}_0), \mathsf{Time}(\mathcal{B}_1), \mathsf{Time}(\mathcal{B}_2), \mathsf{Time}(\mathcal{B}_4), \mathsf{Time}(\mathcal{B}_6), \mathsf{Time}(\mathcal{B}_7) \approx \mathsf{Time}(\mathcal{A})$.

Lemma 10 (prime-order entropy expansion lemma (step two)). *Suppose $\ell_W \geq k$. Then, under the MDDH$_k$ assumption, we have*

$$\left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1 \\ \mathsf{ct}: [\mathbf{c}^{(2)\top}]_1, \{[\mathbf{c}^{(2)\top}\mathbf{W} + \mathbf{c}_j^{(2)\top}\mathbf{U}_j^{(2)}]_1,\ [\mathbf{c}_j^{(2)\top}]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[\mathbf{W}\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2,\ [\mathbf{U}_j^{(2)}\mathbf{D}_j]_2\}_{j\in[n]} \end{array}\right\} \approx_c \left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1 \\ \mathsf{ct}: [\mathbf{c}^{(2)\top}]_1, \{[\mathbf{c}^{(2)\top}(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}}) + \mathbf{c}_j^{(2)\top}\mathbf{U}_j^{(2)}]_1,\ [\mathbf{c}_j^{(2)\top}]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2,\ [\mathbf{U}_j^{(2)}\mathbf{D}_j]_2\}_{j\in[n]} \end{array}\right\}$$

where $\mathbf{W} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$*,* $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^\parallel)$*,* $\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$*, and* $\mathbf{c}^{(2)}, \mathbf{c}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_2)$*. Concretely, the distinguishing advantage* $\mathsf{Adv}_{\mathcal{A}}^{\mathrm{STEP2}}(\lambda)$ *is bounded by* $\mathsf{Adv}_{\mathcal{B}_8}^{\mathrm{DDH}_{\mathbf{A}_2}^{G_2}}(\lambda)$ *where* $\mathsf{Time}(\mathcal{B}_8) \approx \mathsf{Time}(\mathcal{A})$.

Proof. This follows from the $\mathrm{DDH}_{\mathbf{A}_2}^{G_2}$ assumption (with $Q = n\cdot\ell_W$), which asserts

$$\{\,[\mathbf{W}\mathbf{D}_j]_2, [\mathbf{D}_j]_2\,\}_{j\in[n]} \approx_c \{\,[\mathbf{W}\mathbf{D}_j + \mathbf{R}_j^{(2)}]_2, [\mathbf{D}_j]_2\,\}_{j\in[n]} \qquad \text{where } \mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}, \mathbf{R}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^\parallel).$$

Formally, the adversary $\mathcal{B}_8$ gets as input $\{[\mathbf{T}_j]_2, [\mathbf{D}_j]_2\}_{j\in[n]}$ along with $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_2^\parallel, \mathbf{A}_1^\top\mathbf{W}$. It samples $\mathbf{s} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_2}$ and $\Delta_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_2\times\ell_2}$ for all $j\in[n]$, and define $\mathbf{s}_j^\top = \mathbf{s}^\top\Delta_j$. Pick $\widetilde{\mathbf{U}}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_2\times\ell_W}$ for all $j\in[n]$ and implicitly program

$$\mathbf{U}_j^{(2)} := \mathbf{A}_2^\parallel(\widetilde{\mathbf{U}}_j - \Delta_j^{-1}\mathbf{A}_2^\top\mathbf{W}),$$

and runs $\mathcal{A}$ on input

$$\left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1 \\ \mathsf{ct}: [\mathbf{s}^\top\mathbf{A}_2^\top]_1, \{[\mathbf{s}^\top\Delta_j\widetilde{\mathbf{U}}_j]_1,\ [\mathbf{s}^\top\Delta_j\mathbf{A}_2^\top]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[\mathbf{T}_j]_2,\ [\mathbf{D}_j]_2,\ [\mathbf{A}_2^\parallel\widetilde{\mathbf{U}}_j\mathbf{D}_j - \mathbf{A}_2^\parallel\Delta_j^{-1}\mathbf{A}_2^\top\mathbf{T}_j]_2\}_{j\in[n]} \end{array}\right\}.$$

Observe that

- When $\mathbf{T}_j = \mathbf{W}\mathbf{D}_j$ and if we write $\mathbf{U}_j^{(2)}\mathbf{D}_j = \mathbf{A}_2^{\|}\widetilde{\mathbf{U}}_j\mathbf{D}_j - \mathbf{A}_2^{\|}\Delta_j^{-1}\mathbf{A}_2^{\top}\mathbf{W}\mathbf{D}_j$, then $\mathbf{s}^{\top}\Delta_j\widetilde{\mathbf{U}}_j = (\mathbf{s}^{\top}\Delta_j\mathbf{A}_2^{\top})(\mathbf{A}^{\|}\widetilde{\mathbf{U}}_j) = \mathbf{s}^{\top}\mathbf{A}_2^{\top}\mathbf{W} + (\mathbf{s}^{\top}\Delta_j)\mathbf{A}_2^{\top}\mathbf{U}_j^{(2)}$ and the distribution we feed to $\mathcal{A}$ is identical to the left distribution.
- When $\mathbf{T}_j = \mathbf{W}\mathbf{D}_j + \mathbf{R}_j^{(2)}$ and if we write $\mathbf{U}_j^{(2)}\mathbf{D}_j = \mathbf{A}_2^{\|}\widetilde{\mathbf{U}}_j\mathbf{D}_j - \mathbf{A}_2^{\|}\Delta_j^{-1}\mathbf{A}_2^{\top}(\mathbf{W}\mathbf{D}_j + \mathbf{R}_j^{(2)})$, then $\mathbf{s}^{\top}\Delta_j\widetilde{\mathbf{U}}_j = \mathbf{s}^{\top}\mathbf{A}_2^{\top}(\mathbf{W} + \mathbf{R}_j^{(2)}\mathbf{D}_j^{-1}) + (\mathbf{s}^{\top}\Delta_j)\mathbf{A}_2^{\top}\mathbf{U}_j^{(2)}$ and the distribution we feed to $\mathcal{A}$ is identical to the right distribution when we set $\mathbf{V}_j^{(2)} := \mathbf{R}_j^{(2)}\mathbf{D}_j^{-1}$.

This completes the proof. □

We may now complete the proof of Lemma 8 using Lemmas 9 and 10. The proof of Lemmas 9 is deferred to Section 5.1.

Proof (of Lemma 8). The statement follows readily from the following hybrid argument:

$$\left\{\begin{array}{l}\mathsf{aux}: [\mathbf{A}_1^{\top}]_1, [\mathbf{A}_1^{\top}\mathbf{W}]_1, [\mathbf{A}_1^{\top}\mathbf{W}_0]_1, [\mathbf{A}_1^{\top}\mathbf{W}_1]_1 \\ \mathsf{ct}: [\mathbf{c}^{\top}]_1,\ \{[\mathbf{c}^{\top}\mathbf{W} + \mathbf{c}_j^{\top}(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1,\ [\mathbf{c}_j^{\top}]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[\mathbf{W}\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{D}_j]_2\}_{j\in[n]}\end{array}\right\} \text{ LHS in Lemma 8}$$

$$\overset{\mathbf{A}_1\mapsto(\mathbf{A}_1,\mathbf{A}_2)}{\approx_c}\left\{\begin{array}{l}\mathsf{aux}: [\mathbf{A}_1^{\top}]_1, [\mathbf{A}_1^{\top}\mathbf{W}]_1, [\mathbf{A}_1^{\top}\mathbf{W}_0]_1, [\mathbf{A}_1^{\top}\mathbf{W}_1]_1 \\ \mathsf{ct}: [\boxed{\mathbf{c}}^{\top}]_1,\ \{[\boxed{\mathbf{c}}^{\top}\mathbf{W} + \mathbf{c}_j^{\top}(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1,\ [\mathbf{c}_j^{\top}]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[\mathbf{W}\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{D}_j]_2\}_{j\in[n]}\end{array}\right\} \text{ where } \boxed{\mathbf{c} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)}$$

$$\overset{\text{Lemma 9}}{\approx_c}\left\{\begin{array}{l}\mathsf{aux}: [\mathbf{A}_1^{\top}]_1, [\mathbf{A}_1^{\top}\mathbf{W}]_1, [\mathbf{A}_1^{\top}\mathbf{W}_0]_1, [\mathbf{A}_1^{\top}\mathbf{W}_1]_1 \\ \mathsf{ct}: [\mathbf{c}^{\top}]_1,\ \{[\mathbf{c}^{\top}\mathbf{W} + \boxed{\mathbf{c}_j}^{\top}(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1,\ [\boxed{\mathbf{c}_j}^{\top}]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[\mathbf{W}\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2\}_{j\in[n]}\end{array}\right\} \text{ where } \boxed{\mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)}$$

$$\overset{\text{Lemma 10}}{\approx_c}\left\{\begin{array}{l}\mathsf{aux}: [\mathbf{A}_1^{\top}]_1, [\mathbf{A}_1^{\top}\mathbf{W}]_1, [\mathbf{A}_1^{\top}\mathbf{W}_0]_1, [\mathbf{A}_1^{\top}\mathbf{W}_1]_1 \\ \mathsf{ct}: [\mathbf{c}^{\top}]_1,\ \{[\mathbf{c}^{\top}(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}}) + \mathbf{c}_j^{\top}(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})]_1,\ [\mathbf{c}_j^{\top}]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})\mathbf{D}_j]_2\}_{j\in[n]}\end{array}\right\} \text{ RHS in Lemma 8}$$

It is easy to check every transitions as follows:

- The first transition follows from the $\mathrm{SD}^{G_1}_{\mathbf{A}_1\mapsto\mathbf{A}_1,\mathbf{A}_2}$ assumption asserting that

$$[\mathbf{c} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)]_1 \approx_c [\mathbf{c} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)]_1 \text{ given } [\mathbf{A}_1]_1, [\mathbf{A}_2]_1.$$

  In the reduction, we sample $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$ and generate aux and sk honestly.
- The second transition follows from Lemma 9. In the reduction, we sample $\mathbf{W} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$ and simulate $[\mathbf{A}_1^{\top}\mathbf{W}]_1$ in aux honestly. By $[\mathbf{A}_1^{\top}]_1$ and $[\mathbf{A}_2^{\top}]_1$, we can simulate $([\mathbf{c}^{\top}]_1, [\mathbf{c}^{\top}\mathbf{W}]_1)$ honestly where $\mathbf{c} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)$. The simulation of $\{[\mathbf{W}\mathbf{D}_j]_2\}_{j\in[n]}$ in sk is direct.
- The third transition follows from Lemma 10. In the reduction, we sample $\mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$. Then we can simulate $[\mathbf{A}_1^{\top}\mathbf{W}_0]_1$ and $[\mathbf{A}_1^{\top}\mathbf{W}_1]_1$ in aux, and $\{[(\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{D}_j]_2\}_{j\in[n]}$ in sk from $\{[\mathbf{D}_j]_2\}_{j\in[n]}$. As for ct, we simulate $\{[\mathbf{c}^{(1)\top}]_1, [\mathbf{c}^{(1)\top}\mathbf{W} + \mathbf{c}_j^{(1)\top}(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1, [\mathbf{c}_j^{(1)\top}]_1\}_{j\in[n]}$ where $\mathbf{c}^{(1)}, \mathbf{c}_j^{(1)} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ using $([\mathbf{A}_1^{\top}]_1, [\mathbf{A}_1^{\top}\mathbf{W}]_1)$. □

5.1 Entropy epansion lemma: Step one

We prove Lemma 9 via the following game sequence summarized in Fig 6. By ct_j (resp. sk_j), we denote the j'th tuple of ct (resp. sk).

Game$_0$. The adversary $\mathcal{A}$ is given the left distribution in Lemma 9.

$$\left\{\begin{array}{r}\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_2^\top]_1 \\ \mathsf{ct}: \left\{[\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1,\ [\mathbf{c}_j^\top]_1\right\}_{j\in[n]} \\ \mathsf{sk}: \left\{[\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{D}_j]_2\right\}_{j\in[n]}\end{array}\right\}.$$

Game$_{0'}$. We modify the distribution of sk as follows:

$$\mathsf{sk}: \left\{[\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2\right\}_{j\in[n]}$$

where $\mathbf{U}_1^{(2)},\ldots,\mathbf{U}_n^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^\parallel)$. We claim that Game$_0 \approx_c$ Game$_{0'}$. This follows from $\mathrm{DDH}_{\mathbf{A}_2}^{G_2}$ assumption which tells us that

$$\{[\mathbf{D}_j]_2, [\mathbf{W}_0\mathbf{D}_j]_2\}_{j\in[n]} \approx_c \{[\mathbf{D}_j]_2, [(\mathbf{W}_0 + \mathbf{U}_j^{(2)})\mathbf{D}_j]_2\}_{j\in[n]}$$

given $\mathbf{A}_1^\top, \mathbf{A}_1^\top\mathbf{W}_0$. See Lemma 30 for details.

Game$_i$ $(i = 1,\ldots,n+1)$. We change the distribution of ct:

$$\mathsf{ct}: \left\{[\boxed{\mathbf{c}_j}^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1,\ [\boxed{\mathbf{c}_j}^\top]_1\right\}_{j<i}$$
$$\left\{[\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1,\ [\mathbf{c}_j^\top]_1\right\}_{j\geq i}$$

where $\boxed{\mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)}$ for $j < i$ and $\mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ for all remaining $j \geq i$. It is easy to see that Game$_{0'} \equiv$ Game$_1$. To show that Game$_i \approx_c$ Game$_{i+1}$, we will require another sequence of sub-games.

Sub-Game$_{i,1}$. Identical to Game$_i$ except that we modify ct$_i$ as follows:

$$\mathsf{ct}_i: [\boxed{\mathbf{c}_i}^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1)]_1,\ [\boxed{\mathbf{c}_i}^\top]_1$$

where $\boxed{\mathbf{c}_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_3)}$. We claim that Game$_i \approx_c$ Sub-Game$_{i,1}$. This follows from $\mathrm{SD}_{\mathbf{A}_1\mapsto\mathbf{A}_1,\mathbf{A}_3}^{G_1}$ assumption, which tells us that

$$[\mathsf{span}(\mathbf{A}_1)]_1 \approx_c [\mathsf{span}(\mathbf{A}_1,\mathbf{A}_3)]_1 \text{ given } [\mathbf{A}_1,\mathbf{A}_2]_1, \mathsf{basis}(\mathbf{A}_2^\parallel).$$

In the reduction, we will sample $\mathbf{W}_0,\mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$ and $\mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^\parallel)$ using $\mathsf{basis}(\mathbf{A}_2^\parallel)$, and simulate aux, $\{\mathsf{ct}_j\}_{j\neq i}$, sk honestly. See Lemma 31 for details.

Sub-Game$_{i,2}$. We modify the distributions of all sk$_j$ with $j \neq i$ (while keeping sk$_i$ unchanged):

$$\mathsf{sk}_j(j\neq i):\ [\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)} + \boxed{\mathbf{U}_j^{(3)}})\mathbf{D}_j]_2$$

where $\mathbf{U}_j^{(3)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_3^\parallel)$. We claim that Sub-Game$_{i,1} \approx_c$ Sub-Game$_{i,2}$. This follows from $\mathrm{DDH}_{\mathbf{A}_3}^{G_2}$, which tells us that

$$\{[\mathbf{D}_j]_2, [\mathbf{W}_1\mathbf{D}_j]_2\}_{j\neq i} \approx_c \{[\mathbf{D}_j]_2, [(\mathbf{W}_1 + \widetilde{\mathbf{U}}_j^{(3)})\mathbf{D}_j]_2\}_{j\neq i} \text{ given } \mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3,\mathbf{A}_2^\parallel,\mathbf{A}_1^\top\mathbf{W}_1,\mathbf{A}_2^\top\mathbf{W}_1$$

where $\widetilde{\mathbf{U}}_j^{(3)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_3^\parallel)$. In the reduction, we will program $\mathbf{W}_0 := \widetilde{\mathbf{W}}_0 - i\cdot\mathbf{W}_1$ with $\widetilde{\mathbf{W}}_0 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$ so that we can simulate $\mathbf{W}_0 + i\cdot\mathbf{W}_1$ in ct$_i$, and then implicitly set $\mathbf{U}_j^{(3)} = (j-i)\cdot\widetilde{\mathbf{U}}_j^{(3)}$ for all $j\neq i$. See Lemma 32 for details.

Sub-Game$_{i,3}$. We modify the distributions of ct_i and sk_i:

$$\mathsf{ct}_i : [\mathbf{c}_i^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_i^{(2)} + \mathbf{U}_i^{(3)}})]_1,\ [\mathbf{c}_i^\top]_1$$
$$\mathsf{sk}_i : [\mathbf{D}_i]_2,\ [(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)} + \boxed{\mathbf{U}_i^{(3)}})\mathbf{D}_i]_2$$

where $\mathbf{U}_i^{(3)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_3^{\|})$. We claim that Sub-Game$_{i,2} \equiv$ Sub-Game$_{i,3}$. This follows from the following statistical argument:

$$(\mathbf{A}_1^\top\mathbf{W}_0, \mathbf{A}_2^\top\mathbf{W}_0, \mathbf{W}_0, \{\mathbf{W}_0 + \mathbf{U}_j^{(3)}\}_{j\neq i}) \equiv (\mathbf{A}_1^\top\mathbf{W}_0, \mathbf{A}_2^\top\mathbf{W}_0, \mathbf{W}_0 + \boxed{\mathbf{U}_i^{(3)}}, \{\mathbf{W}_0 + \mathbf{U}_j^{(3)}\}_{j\neq i}).$$

Furthermore, $\mathbf{c}_i^\top\mathbf{U}_i^{(2)} = \mathbf{0}$ since $\mathbf{c}_i \in \mathsf{span}(\mathbf{A}_1, \mathbf{A}_3)$ and $\mathbf{A}_1^\top\mathbf{A}_2^{\|} = \mathbf{A}_3^\top\mathbf{A}_2^{\|} = \mathbf{0}$, so the term $\mathbf{U}_i^{(2)}$ in ct_i is introduced "for free". See Lemma 33 for details.

Sub-Game$_{i,4}$. We modify the distribution of ct_i:

$$\mathsf{ct}_i : [\boxed{\mathbf{c}_i}^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)} + \mathbf{U}_i^{(3)})]_1,\ [\boxed{\mathbf{c}_i}^\top]_1 \ \text{ where } \boxed{\mathbf{c}_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)}.$$

We claim that Sub-Game$_{i,3} \approx_c$ Sub-Game$_{i,4}$. This follows from $\mathrm{SD}^{G_1}_{\mathbf{A}_3 \mapsto \mathbf{A}_3, \mathbf{A}_2}$ which tells us that

$$[\mathsf{span}(\mathbf{A}_3)]_1 \approx_c [\mathsf{span}(\mathbf{A}_3, \mathbf{A}_2)]_1 \ \text{ given } \ [\mathbf{A}_2, \mathbf{A}_1]_1, \mathsf{basis}(\mathbf{A}_2^{\|}, \mathbf{A}_3^{\|}).$$

In the reduction, we sample $\mathbf{W}_0, \mathbf{W}_1$ and $(\mathbf{U}_j^{(2)} + \mathbf{U}_j^{(3)}) \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|}, \mathbf{A}_3^{\|})$ using $\mathsf{basis}(\mathbf{A}_2^{\|}, \mathbf{A}_3^{\|})$ for all $j \in [n]$. Then aux, sk and $\{\mathsf{ct}_j\}_{j\neq i}$ can be simulated honestly. See Lemma 34 for details.

Sub-Game$_{i,5}$. We change the distributions of ct_i and sk_i:

$$\mathsf{ct}_i : [\mathbf{c}_i^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)} + \cancel{\mathbf{U}_i^{(3)}})]_1,\ [\mathbf{c}_i^\top]_1$$
$$\mathsf{sk}_i : [\mathbf{D}_i]_2,\ [(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)} + \cancel{\mathbf{U}_i^{(3)}})\mathbf{D}_i]_2$$

We claim that Sub-Game$_{i,4} \equiv$ Sub-Game$_{i,5}$. The proof is completely analogous to that of Sub-Game$_{i,2} \equiv$ Sub-Game$_{i,3}$. See Lemma 35 for details.

Sub-Game$_{i,6}$. We change the distributions of all sk_j with $j \neq i$:

$$\mathsf{sk}_j (j\neq i) : [\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)} + \cancel{\mathbf{U}_j^{(3)}})\mathbf{D}_j]_2$$

We claim that Sub-Game$_{i,5} \approx_c$ Sub-Game$_{i,6}$. The proof is completely analogous to that of Sub-Game$_{i,1} \equiv$ Sub-Game$_{i,2}$. See Lemma 36 for details.

Sub-Game$_{i,7}$. We change the distribution of ct_i:

$$\mathsf{ct}_i : [\boxed{\mathbf{c}_i}^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)})]_1,\ [\boxed{\mathbf{c}_i}^\top]_1,$$

where $\boxed{\mathbf{c}_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)}$. We claim that Sub-Game$_{i,6} \approx_c$ Sub-Game$_{i,7}$. The proof is completely analogous to that of Game$_i \equiv$ Sub-Game$_{i,1}$. See Lemma 37 for details. Furthermore, observe that Sub-Game$_{i,7}$ is actually identical to Game$_{i+1}$.

Game$_{n+1}$. In Game$_{n+1}$, we have:

$$\left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_2^\top]_1 \\ \mathsf{ct}: \left\{[\boxed{\mathbf{c}_j}^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1,\ [\boxed{\mathbf{c}_j}^\top]_1\right\}_{j\in[n]} \\ \mathsf{sk}: \left\{[\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2\right\}_{j\in[n]} \end{array}\right\}$$

where $\mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)$, $\mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\parallel})$ for all $j\in[n]$. This is exactly the right distribution of Lemma 9.

Game	CT $j<i$	CT i	CT $j>i$	SK $j\neq i$	SK i	Remark & aux
		$\mathbf{c}_j \leftarrow_{\mathrm{R}}$? $\mathbf{W}_0 + j\cdot\mathbf{W}_1 + ?$		$\mathbf{W}_0 + j\cdot\mathbf{W}_1 + ?$		
0		$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$		$\mathbf{0}$		
$0'$		$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$		$\boxed{\mathbf{U}_j^{(2)}}$		$\mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\parallel})$; $\mathrm{DDH}^{G_2}_{\mathbf{A}_2}$: $\{[\mathbf{D}_j]_2,[\mathbf{W}_0\mathbf{D}_j]_2\}_{j\in[n]} \approx_c \{[\mathbf{D}_j]_2,[(\mathbf{W}_0+\mathbf{U}_j^{(2)})\mathbf{D}_j]_2\}_{j\in[n]}$
i	$\boxed{\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)}$ $\boxed{\mathbf{U}_j^{(2)}}$	$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$	$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$	$\mathbf{U}_j^{(2)}$	$\mathbf{U}_i^{(2)}$	$\mathsf{Game}_1 = \mathsf{Game}_{0''}$, $\mathsf{Game}_i = \mathsf{Game}_{i-1,7}$
$i,1$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)$ $\mathbf{U}_j^{(2)}$	$\boxed{\mathsf{span}(\mathbf{A}_1,\mathbf{A}_3)}$ $\mathbf{0}$	$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$	$\mathbf{U}_j^{(2)}$	$\mathbf{U}_i^{(2)}$	$\mathrm{SD}^{G_1}_{\mathbf{A}_1\mapsto\mathbf{A}_1,\mathbf{A}_3}$: $[\mathsf{span}(\mathbf{A}_1)]_1 \approx_c [\mathsf{span}(\mathbf{A}_1,\mathbf{A}_3)]_1$ given $[\mathbf{A}_1,\mathbf{A}_2]_1, \mathsf{basis}(\mathbf{A}_2^{\parallel})$
$i,2$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)$ $\mathbf{U}_j^{(2)}$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_3)$ $\mathbf{0}$	$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$	$\mathbf{U}_j^{(2)} + \boxed{\mathbf{U}_j^{(3)}}$	$\mathbf{U}_i^{(2)}$	$\mathbf{U}_j^{(3)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_3^{\parallel})$; $\mathrm{DDH}^{G_2}_{\mathbf{A}_3}$: Given $\mathbf{W}_0 + i\cdot\mathbf{W}_1$, $\{[\mathbf{D}_j]_2,[\mathbf{W}_1\mathbf{D}_j]_2\}_{j\neq i} \approx_c \{[\mathbf{D}_j]_2,[(\mathbf{W}_1+\tilde{\mathbf{U}}_j^{(3)})\mathbf{D}_j]_2\}_{j\neq i}$ with $\mathbf{U}_j^{(3)} = (j-i)\cdot\tilde{\mathbf{U}}_j^{(3)}$
$i,3$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)$ $\mathbf{U}_j^{(2)}$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_3)$ $\boxed{\mathbf{U}_i^{(2)} + \mathbf{U}_i^{(3)}}$	$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$	$\mathbf{U}_j^{(2)} + \mathbf{U}_j^{(3)}$	$\mathbf{U}_i^{(2)} + \boxed{\mathbf{U}_i^{(3)}}$	statistical lemma: $(\mathbf{W}_0, \{\mathbf{W}_0 + \mathbf{U}_j^{(3)}\}_{j\neq i}) \equiv (\mathbf{W}_0 + \mathbf{U}_i^{(3)}, \{\mathbf{W}_0 + \mathbf{U}_j^{(3)}\}_{j\neq i})$
$i,4$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)$ $\mathbf{U}_j^{(2)}$	$\boxed{\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)}$ $\mathbf{U}_i^{(2)} + \mathbf{U}_i^{(3)}$	$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$	$\mathbf{U}_j^{(2)} + \mathbf{U}_j^{(3)}$	$\mathbf{U}_i^{(2)} + \mathbf{U}_i^{(3)}$	$\mathrm{SD}^{G_1}_{\mathbf{A}_3\mapsto\mathbf{A}_3,\mathbf{A}_2}$: $[\mathsf{span}(\mathbf{A}_3)]_1 \approx_c [\mathsf{span}(\mathbf{A}_2,\mathbf{A}_3)]_1$ given $[\mathbf{A}_2,\mathbf{A}_1]_1, \mathsf{basis}(\mathbf{A}_2^{\parallel},\mathbf{A}_3^{\parallel})$
$i,5$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)$ $\mathbf{U}_j^{(2)}$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)$ $\mathbf{U}_i^{(2)} + \cancel{\mathbf{U}_i^{(3)}}$	$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$	$\mathbf{U}_j^{(2)} + \mathbf{U}_j^{(3)}$	$\mathbf{U}_i^{(2)} + \cancel{\mathbf{U}_i^{(3)}}$	statistical argument analogous to Sub-Game$_{i,3}$
$i,6$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)$ $\mathbf{U}_j^{(2)}$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)$ $\mathbf{U}_i^{(2)}$	$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$	$\mathbf{U}_j^{(2)} + \cancel{\mathbf{U}_j^{(3)}}$	$\mathbf{U}_i^{(2)}$	$\mathrm{DDH}^{G_2}_{\mathbf{A}_3}$ analogous to Sub-Game$_{i,2}$
$i,7$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)$ $\mathbf{U}_j^{(2)}$	$\mathsf{span}(\mathbf{A}_1,\mathbf{A}_2,\cancel{\mathbf{A}_3})$ $\mathbf{U}_i^{(2)}$	$\mathsf{span}(\mathbf{A}_1)$ $\mathbf{0}$	$\mathbf{U}_j^{(2)}$	$\mathbf{U}_i^{(2)}$	$\mathrm{SD}^{G_1}_{\mathbf{A}_1\mapsto\mathbf{A}_1,\mathbf{A}_3}$ analogous to Sub-Game$_{i,1}$

Fig. 6. Game sequence for our proof of Lemma 9 (Bilinear entropy expansion lemma (step one)). There are two rows for each game: The 1st one indicates where $\mathbf{c}_j$ are sampled from; and the 2nd one indicates whether the hiding matrix $\mathbf{U}_j^2$ (and $\mathbf{U}_j^3$) will appear.

6 KP-ABE for Monotone Span Programs in Composite-Order Groups

In this section, we present our adaptively secure, unbounded KP-ABE for monotone span programs based on static assumptions in composite-order groups (cf. Section 3.1).

6.1 Construction

$\mathsf{Setup}(1^\lambda, 1^n)$: On input $(1^\lambda, 1^n)$, sample $\mathbb{G} := (N = p_1p_2p_3, G_N, H_N, G_T, e) \leftarrow \mathcal{G}(1^\lambda)$ and select random generators g_1, h_1 and h_{123} of G_{p_1}, H_{p_1} and H_N, respectively. Pick

$$w, w_0, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ \alpha \leftarrow_{\mathrm{R}} \mathbb{Z}_N,$$

a pairwise independent hash function $\mathsf{H} : G_T \to \{0,1\}^\lambda$, and output the master public and secret key pair

$$\mathsf{mpk} := \big(\,(N, G_N, H_N, G_T, e);\ g_1,\ g_1^{w},\ g_1^{w_0},\ g_1^{w_1},\ e(g_1, h_{123})^\alpha;\ \mathsf{H}\,\big)$$

and

$$\mathsf{msk} := (h_{123}, h_1, \alpha, w, w_0, w_1\,).$$

$\mathsf{Enc}(\mathsf{mpk}, \mathbf{x}, m)$: On input an attribute vector $\mathbf{x} := (x_1, \ldots, x_n) \in \{0,1\}^n$ and $m \in \{0,1\}^\lambda$, pick $s, s_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$ and output

$$\mathsf{ct}_{\mathbf{x}} := \big(\,C_0 := g_1^s,\ \{\,C_{1,j} := g_1^{sw + s_j(w_0 + j\cdot w_1)},\ C_{2,j} := g_1^{s_j}\,\}_{j:x_j=1},\ C := \mathsf{H}(e(g_1, h_{123})^{\alpha s}) \cdot m\,\big) \in G_N^{2n+1} \times \{0,1\}^\lambda.$$

$\mathsf{KeyGen}(\mathsf{mpk}, \mathsf{msk}, \mathbf{M})$: On input a monotone span program $\mathbf{M} \in \mathbb{Z}_N^{n\times\ell'}$, pick $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_N^{\ell'-1}$ and $r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$, and output

$$\mathsf{sk}_{\mathbf{M}} := \big(\,\{\,K_{0,j} := h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_1^{r_j w},\ K_{1,j} := h_1^{r_j},\ K_{2,j} := h_1^{r_j(w_0 + j\cdot w_1)}\,\}_{j\in[n]}\,\big) \in H_N^{3n}.$$

$\mathsf{Dec}(\mathsf{mpk}, \mathsf{sk}_{\mathbf{M}}, \mathsf{ct}_{\mathbf{x}})$: If $\mathbf{x}$ satisfies $\mathbf{M}$, compute $\omega_1, \ldots, \omega_n \in \mathbb{Z}_p$ such that

$$\sum_{j:x_j=1} \omega_j \mathbf{M}_j = \mathbf{1}.$$

Then, compute

$$K \leftarrow \prod_{j:x_j=1} \big(e(C_0, K_{0,j}) \cdot e(C_{1,j}, K_{1,j})^{-1} \cdot e(C_{2,j}, K_{2,j})\big)^{\omega_j},$$

and recover the message as $m \leftarrow C / \mathsf{H}(K) \in \{0,1\}^\lambda$.

Correctness. For all $\mathbf{M}$ and $\mathbf{x}$ such that $\mathbf{x}$ satisfies $\mathbf{M}$, we have

$$\begin{aligned} &\prod_{j:x_j=1} \big(e(C_0, K_{0,j}) \cdot e(C_{1,j}, K_{1,j})^{-1} \cdot e(C_{2,j}, K_{2,j})\big)^{\omega_j} \\ =&\prod_{j:x_j=1} \Big(e(g_1^s,\ h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_1^{r_j w}) \cdot e(g_1^{sw + s_j(w_0+j\cdot w_1)},\ h_1^{r_j})^{-1} \cdot e(g_1^{s_j},\ h_1^{r_j(w_0+j\cdot w_1)})\Big)^{\omega_j} \\ =&e(g_1, h_{123})^{s\sum_{j:x_j=1}\omega_j \mathbf{M}_j\binom{\alpha}{\mathbf{u}}} = e(g_1, h_{123})^{s\alpha}. \end{aligned}$$

This readily proves the correctness.

6.2 Proof of Security

We prove the following theorem:

Theorem 1. *Under the subgroup decision assumptions and the subgroup Diffie-Hellman assumptions (cf. Section 3.1), the unbounded KP-ABE scheme described in this section (cf. Section 6.1) is adaptively secure (cf. Section 2.2).*

Main technical lemma. We prove the following technical lemma. Our proof consists of two steps. We first apply the entropy expansion lemma (see Lemma 2) and obtain a copy of the LOSTW KP-ABE (variant thereof) in the p_2-subgroup. We may then carry out the classic dual system methodology used for establishing adaptive security of the LOSTW KP-ABE in the p_2-subgroup with the p_3-subgroup as the semi-functional space.

Lemma 11. *For any adversary $\mathcal{A}$ that makes at most Q key queries against the unbounded KP-ABE scheme, there exist adversaries $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_2$ such that:*

$$\mathsf{Adv}^{\mathrm{ABE}}_{\mathcal{A}}(\lambda) \leq \mathsf{Adv}^{\textsc{ExpLem}}_{\mathcal{B}_0}(\lambda) + \mathsf{Adv}^{\mathrm{SD}^{G_N}_{p_2 \mapsto p_2p_3}}_{\mathcal{B}_1}(\lambda) + Q \cdot \mathsf{Adv}^{\mathrm{SD}^{H_N}_{p_2 \mapsto p_2p_3}}_{\mathcal{B}_2}(\lambda) + Q \cdot \mathsf{Adv}^{\mathrm{SD}^{H_N}_{p_2 \mapsto p_2p_3}}_{\mathcal{B}_3}(\lambda)$$

where $\mathsf{Time}(\mathcal{B}_0), \mathsf{Time}(\mathcal{B}_1), \mathsf{Time}(\mathcal{B}_2), \mathsf{Time}(\mathcal{B}_3) \approx \mathsf{Time}(\mathcal{A})$. *In particular, we achieve security loss $O(n + Q)$ based on the $SD^{H_N}_{p_1 \mapsto p_1p_2}$, $SD^{H_N}_{p_1 \mapsto p_1p_3}$, $SD^{G_N}_{p_1 \mapsto p_1p_2}$, $DDH^{H_N}_{p_2}$, $SD^{G_N}_{p_1 \mapsto p_1p_3}$, $DDH^{H_N}_{p_3}$, $SD^{G_N}_{p_3 \mapsto p_3p_2}$, $SD^{G_N}_{p_2 \mapsto p_2p_3}$, $SD^{H_N}_{p_2 \mapsto p_2p_3}$ assumptions.*

The proof follows a series of games based on the dual system methodology, and outlined in Fig. 7. We first define the auxiliary distributions, upon which we can describe the games.

Game	CT	SK $\kappa < i$	SK $\kappa = i$	SK $\kappa > i$	Justification
0	Normal	Normal			real game
0'	E-normal	E-normal			entropy expansion lemma, Lemma 2
i	SF	SF	E-normal	E-normal	$\mathrm{SD}^{G_N}_{p_2 \mapsto p_2p_3}$, $\mathsf{Game}_i = \mathsf{Game}_{i-1,3}$
$i,1$	—	—	P-normal	—	$\mathrm{SD}^{H_N}_{p_2 \mapsto p_2p_3}$
$i,2$	—	—	P-SF	—	statistical lemma, Lemma 1
$i,3$	—	—	SF	—	$\mathrm{SD}^{H_N}_{p_2 \mapsto p_2p_3}$
Final	random m	SF			statistical hiding

Fig. 7. Game sequence for proving the adaptive security of our composite-order unbounded KP-ABE.

Auxiliary distributions. We define various forms of a ciphertext (of message m under attribute vector $\mathbf{x}$):

- Normal: Generated by Enc.
- E-normal: Same as a normal ciphertext except that a copy of normal ciphertext is created in G_{p_2} and then we use the substitution:

$$w \mapsto v_j \bmod p_2 \text{ in } j\text{'th component} \quad \text{and} \quad w_0 + j \cdot w_1 \mapsto u_j \bmod p_2 \tag{10}$$

  where $v_j, u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. Concretely, an E-normal ciphertext is of the form

$$\mathsf{ct}_{\mathbf{x}} := \big(\, g_1^s \cdot \boxed{g_2^s},\ \{\, g_1^{sw+s_j(w_0+j\cdot w_1)} \cdot \boxed{g_2^{sv_j+s_ju_j}},\ g_1^{s_j} \cdot \boxed{g_2^{s_j}} \,\}_{j:x_j=1},\ \mathsf{H}(e(g_1^s \cdot \boxed{g_2^s}, h_{123}^{\alpha})) \cdot m \,\big)$$

where g_2 is a random generator of G_{p_2} and $s, s_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$.

– SF: Same as E-normal ciphertext except that we copy all entropy from G_{p_2} to G_{p_3}. Concretely, an SF ciphertext is of the form

$$\mathsf{ct}_{\mathbf{x}} := \big(g_1^s \cdot g_2^s \cdot \boxed{g_3^s}, \{ g_1^{sw+s_j(w_0+j\cdot w_1)} \cdot g_2^{sv_j+s_ju_j} \cdot \boxed{g_3^{sv_j+s_ju_j}}, g_1^{s_j} \cdot g_2^{s_j} \cdot \boxed{g_3^{s_j}} \}_{j:x_j=1}, \mathsf{H}(e(g_1^s \cdot g_2^s \cdot \boxed{g_3^s}, h_{123}^{\alpha})) \cdot m \big)$$

where g_3 is a random generator of G_{p_3} and $s, s_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$.

Then we pick $\hat{\alpha} \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ and define various forms of a key (for span program $\mathbf{M}$):

– Normal: Generated by KeyGen.

– E-normal: Same as a normal key except that we make a copy of $\{h_1^{r_jw}, h_1^{r_j}, h_1^{r_j(w_0+j\cdot w_1)}\}_{j\in[n]}$ in H_{p_2} and use the same substitution as in (10). Concretely, an E-normal key is of the form

$$\mathsf{sk}_{\mathbf{M}} := \big(\{ h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_1^{r_jw} \cdot \boxed{h_2^{r_jv_j}}, h_1^{r_j} \cdot \boxed{h_2^{r_j}}, h_1^{r_j(w_0+j\cdot w_1)} \cdot \boxed{h_2^{r_ju_j}} \}_{j\in[n]} \big)$$

where h_{123}, h_1 and h_2 are respective random generators of H_N, H_{p_1} and H_{p_2}, $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_N^{\ell'-1}$ and $r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$.

– P-normal: Same as E-normal key except that we make a copy of $\{h_2^{r_jv_j}, h_2^{r_j}, h_2^{r_ju_j}\}_{j\in[n]}$ in H_{p_3}. Concretely, a P-normal key is of the form

$$\mathsf{sk}_{\mathbf{M}} := \big(\{ h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_1^{r_jw} \cdot h_2^{r_jv_j} \cdot \boxed{h_3^{r_jv_j}}, h_1^{r_j} \cdot h_2^{r_j} \cdot \boxed{h_3^{r_j}}, h_1^{r_j(w_0+j\cdot w_1)} \cdot h_2^{r_ju_j} \cdot \boxed{h_3^{r_ju_j}} \}_{j\in[n]} \big)$$

where h_3 is a random generator of H_{p_3}, $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_N^{\ell'-1}$ and $r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$.

– P-SF: Same as P-normal key except that $\hat{\alpha}$ is introduced in H_{p_3}. Concretely, a P-SF key is of the form

$$\mathsf{sk}_{\mathbf{M}} := \big(\{ h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot \boxed{h_3^{\mathbf{M}_j\binom{\hat{\alpha}}{\mathbf{0}}}} \cdot h_1^{r_jw} \cdot h_2^{r_jv_j} \cdot h_3^{r_jv_j}, h_1^{r_j} \cdot h_2^{r_j} \cdot h_3^{r_j}, h_1^{r_j(w_0+j\cdot w_1)} \cdot h_2^{r_ju_j} \cdot h_3^{r_ju_j} \}_{j\in[n]} \big)$$

where $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_N^{\ell'-1}$ and $r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$.

– SF: Same as P-SF key except that $\{h_3^{r_jv_j}, h_3^{r_j}, h_3^{r_ju_j}\}_{j\in[n]}$ is removed. Concretely, a SF key is of the form

$$\mathsf{sk}_{\mathbf{M}} := \big(\{ h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_3^{\mathbf{M}_j\binom{\hat{\alpha}}{\mathbf{0}}} \cdot h_1^{r_jw} \cdot h_2^{r_jv_j} \cdot \cancel{h_3^{r_jv_j}}, h_1^{r_j} \cdot h_2^{r_j} \cdot \cancel{h_3^{r_j}}, h_1^{r_j(w_0+j\cdot w_1)} \cdot h_2^{r_ju_j} \cdot \cancel{h_3^{r_ju_j}} \}_{j\in[n]} \big)$$

where $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_N^{\ell'-1}$ and $r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$.

Here E, P, SF means “expanded”, “pesudo”, “semi-functional”, respectively.

Games. We describe the game sequence in detail.

<u>Game_0.</u> The real security game (cf. Section 2.2) where all keys and ciphertext are normal.

<u>Game_0'.</u> Identical to Game_0 except that all keys and the challenge ciphertext are E-normal. We claim that $\mathsf{Game}_0 \approx_c \mathsf{Game}_{0'}$. This follows from the entropy expansion lemma (see Lemma 2). In the reduction, on input

$$\left\{ \begin{array}{l} \mathsf{aux}: g_1, g_1^w, g_1^{w_0}, g_1^{w_1} \\ \mathsf{ct}: C_0, \{C_{1,j}, C_{2,j}\}_{j\in[n]} \\ \mathsf{sk}: \{K_{0,j}, K_{1,j}, K_{2,j}\}_{j\in[n]} \end{array} \right\},$$

we select a random generator h_{123} of H_N, sample $\alpha \leftarrow_R \mathbb{Z}_N$, $\mathbf{u}_\kappa \leftarrow_R \mathbb{Z}_N^{\ell'-1}$, $\tilde{r}_{j,\kappa} \leftarrow_R \mathbb{Z}_N$ for $j \in [n]$ and $\kappa \in [Q]$, and simulate the game with

$$\left\{ \begin{array}{l} \mathsf{mpk} : \mathsf{aux},\ e(g_1, h_{123})^\alpha \\ \mathsf{ct}_{\mathbf{x}^*} : \{C_0,\ C_{1,j},\ C_{2,j}\}_{j:x_j^*=1}, e(C_0, h_{123}^\alpha) \cdot m_b \\ \mathsf{sk}_{\mathbf{M}}^\kappa : \{\ h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}_\kappa}} \cdot K_{0,j}^{\tilde{r}_{j,\kappa}}, K_{1,j}^{\tilde{r}_{j,\kappa}}, K_{2,j}^{\tilde{r}_{j,\kappa}} \}_{j\in[n]} \end{array} \right\}.$$

See Lemma 38 for details.

<u>Game_i.</u> Identical to $\mathsf{Game}_{0'}$ except that the first $i-1$ keys and the challenge ciphertext is SF. We claim that $\mathsf{Game}_{0'} \approx_c \mathsf{Game}_1$. This follows from the $\mathsf{SD}^{G_N}_{p_2 \mapsto p_2p_3}$ assumption, which asserts that

$$(\,g_2^s,\ \{g_2^{s_j}\}_{j\in[n]}\,) \approx_c (\,g_2^s \cdot \boxed{g_3^s}, \{g_2^{s_j} \cdot \boxed{g_3^{s_j}}\}_{j\in[n]}\,) \text{ given } g_1, h_1, h_2.$$

In the reduction, we sample $w, w_0, w_1, v_j, u_j \leftarrow_R \mathbb{Z}_N, h_{123} \leftarrow_R H_N$, $\alpha \leftarrow_R \mathbb{Z}_N$ and simulate $\mathsf{mpk}, \mathsf{sk}_{\mathbf{M}}^\kappa$ honestly. See Lemma 39 for details. To show that $\mathsf{Game}_i \approx_c \mathsf{Game}_{i+1}$, we will require another sequence of sub-games.

<u>$\mathsf{Game}_{i,1}$.</u> Identical to Game_i except that the i'th key is P-normal. We claim that $\mathsf{Game}_i \approx_c \mathsf{Game}_{i,1}$. This follows from $\mathsf{SD}^{H_N}_{p_2 \mapsto p_2p_3}$ assumption which asserts that

$$\{h_2^{r_j}\}_{j\in[n]} \approx_c \{h_2^{r_j} \cdot h_3^{r_j}\}_{j\in[n]} \text{ given } g_1, g_{23}, h_1, h_2, h_3$$

In the reduction, we sample $w, w_0, w_1, v_j, u_j, \alpha, \hat{\alpha} \leftarrow_R \mathbb{Z}_N$ and select a random generator h_{123} of H_N, and simulate $\mathsf{mpk}, \mathsf{ct}, \{\mathsf{sk}_{\mathbf{M}}^\kappa\}_{\kappa \neq i}$ honestly. See Lemma 40 for details.

<u>$\mathsf{Game}_{i,2}$.</u> Identical to Game_i except that the i'th key is P-SF. We claim that $\mathsf{Game}_{i,1} \equiv \mathsf{Game}_{i,2}$. This follows from Lemma 1 in Section 2 which ensures that for any $\mathbf{x}$ that does not satisfy $\mathbf{M}$,

$$\begin{aligned} &(\,\overbrace{h_2, \{h_2^{v_j}\}_{j\in[n]}, \alpha, \hat{\alpha}}^{\kappa\text{'th sk}, \kappa \neq i};\ \overbrace{\{g_2, g_2^{v_j}, g_3, g_3^{v_j}\}_{j:x_j=1}}^{\text{SF ct}};\ \overbrace{\{h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_3^{r_j v_j}, h_3^{r_j}\}_{j\in[n]}}^{\text{P-normal } i\text{'th sk}}\,) \\ \equiv\ &(\,h_2, \{h_2^{v_j}\}_{j\in[n]}, \alpha, \hat{\alpha};\ \{g_2, g_2^{v_j}, g_3, g_3^{v_j}\}_{j:x_j=1};\ \underbrace{\{h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot \boxed{h_3^{\mathbf{M}_j\binom{\hat{\alpha}}{\mathbf{0}}}} \cdot h_3^{r_j v_j}, h_3^{r_j}\}_{j\in[n]}}_{\text{P-SF } i\text{'th sk}}\,) \end{aligned}$$

where $v_j \leftarrow_R \mathbb{Z}_N$ and $\mathbf{u} \leftarrow_R \mathbb{Z}_N^{\ell'-1}$, and for all α, $\hat{\alpha}$, and $r_j \neq 0 \bmod p_3$. It is straight-forward to compute the remaining terms in mpk, the challenge ciphertext and the Q secret keys by sampling $g_1, w, w_0, w_1, u_j, s, s_j$ ourselves. See Lemma 41 for details.

<u>$\mathsf{Game}_{i,3}$.</u> Identical to Game_i except that the i'th key is SF. We claim that $\mathsf{Game}_{i,2} \approx_c \mathsf{Game}_{i,3}$. The proof is completely analogous to that of $\mathsf{Game}_i \approx_c \mathsf{Game}_{i,1}$. See Lemma 42 for details. Furthermore, observe that $\mathsf{Game}_{i,3}$ is actually identical to Game_{i+1}.

<u>$\mathsf{Game}_{\mathsf{Final}}$.</u> Identical to Game_{Q+1} except that the challenge ciphertext is a SF one for a random message in G_T. We claim that $\mathsf{Game}_{Q+1} \equiv \mathsf{Game}_{\mathsf{Final}}$. This follows from the fact that

$$(\,\overbrace{e(g_1, h_{123}^\alpha)}^{\text{mpk}},\ \overbrace{h_{123}^\alpha \cdot h_3^{\hat{\alpha}}}^{\text{SF sk}},\ \overbrace{e(g_{123}^s, h_{123}^\alpha)}^{\text{SF ct}}\,) \equiv (\,e(g_1, h_{123}^\alpha),\ h_{123}^\alpha,\ e(g_{123}^s, h_{123}^\alpha \cdot \boxed{h_3^{\hat{\alpha}}})\,)$$

where g_{123}, h_{123} and h_3 are respective random generators of G_N, H_N and H_{p_3}, $\alpha, \hat{\alpha} \leftarrow_R \mathbb{Z}_N$. The message m_b is statistically hidden by $e(g_{123}^s, h_3^{\hat{\alpha}})$. See Lemma 43 for details. In $\mathsf{Game}_{\mathsf{Final}}$, the view of the adversary is statistically independent of the challenge bit b. Hence, $\mathsf{Adv}_{\mathsf{Final}} = 0$.

7 KP-ABE for Monotone Span Programs in Prime-Order Groups

In this section, we present our adaptively secure, unbounded KP-ABE for monotone span programs programs based on the k-Lin assumption in prime-order groups.

7.1 Construction

$\mathsf{Setup}(1^\lambda, 1^n)$: On input $(1^\lambda, 1^n)$, sample

$$\mathbf{A}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times k}, \mathbf{B} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(k+1)\times k}, \mathbf{W}, \mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(k+1)}, \mathbf{k} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{2k+1}$$

and output the master public and secret key pair

$$\mathsf{mpk} := \big([\mathbf{A}_1^\top, \mathbf{A}_1^\top \mathbf{W}, \mathbf{A}_1^\top \mathbf{W}_0, \mathbf{A}_1^\top \mathbf{W}_1]_1,\ e([\mathbf{A}_1^\top]_1, [\mathbf{k}]_2) \big) \in G_1^{k\times(2k+1)} \times (G_1^{k\times(k+1)})^3 \times G_T^k$$

and

$$\mathsf{msk} := (\,\mathbf{k},\ \mathbf{B},\ \mathbf{W},\ \mathbf{W}_0,\ \mathbf{W}_1\,).$$

$\mathsf{Enc}(\mathsf{mpk}, \mathbf{x}, m)$: On input an attribute vector $\mathbf{x} := (x_1, \ldots, x_n) \in \{0,1\}^n$ and $m \in G_T$, pick $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ for all $j \in [n]$ and output

$$\mathsf{ct}_{\mathbf{x}} := \big(C_0 := [\mathbf{c}^\top]_1,\ \{\, C_{1,j} := [\mathbf{c}^\top \mathbf{W} + \mathbf{c}_j^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1)]_1, C_{2,j} := [\mathbf{c}_j^\top]_1 \,\}_{j:x_j=1},\ C := e([\mathbf{c}^\top]_1, [\mathbf{k}]_2) \cdot m \big)$$
$$\in G_1^{2k+1} \times (G_1^{k+1} \times G_1^{2k+1})^n \times G_T.$$

$\mathsf{KeyGen}(\mathsf{mpk}, \mathsf{msk}, \mathbf{M})$: On input a monotone span program $\mathbf{M} \in \mathbb{Z}_p^{n\times \ell'}$, pick $\mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}$, $\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B})$ for all $j \in [n]$, and output

$$\mathsf{sk}_{\mathbf{M}} := \big(\{\, K_{0,j} := [(\mathbf{k} \| \mathbf{K}')\mathbf{M}_j^\top + \mathbf{W}\mathbf{d}_j]_2,\ K_{1,j} := [\mathbf{d}_j]_2,\ K_{2,j} := [(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{d}_j]_2 \,\}_{j\in[n]} \big)$$
$$\in (G_2^{2k+1} \times G_2^{k+1} \times G_2^{2k+1})^n.$$

$\mathsf{Dec}(\mathsf{mpk}, \mathsf{sk}_{\mathbf{M}}, \mathsf{ct}_{\mathbf{x}})$: If $\mathbf{x}$ satisfies $\mathbf{M}$, compute $\omega_1, \ldots, \omega_n \in \mathbb{Z}_p$ such that

$$\sum_{j:x_j=1} \omega_j \mathbf{M}_j = \mathbf{1}.$$

Then, compute

$$K \leftarrow \prod_{j:x_j=1} \big(e(C_0, K_{0,j}) \cdot e(C_{1,j}, K_{1,j})^{-1} \cdot e(C_{2,j}, K_{2,j}) \big)^{\omega_j},$$

and recover the message as $m \leftarrow C/K \in G_T$.

Correctness. For all $\mathbf{M}$ and $\mathbf{x}$ such that $\mathbf{x}$ satisfies $\mathbf{M}$, we have

$$\begin{aligned}
& e(C_0, K_{0,j}) \cdot e(C_{1,j}, K_{1,j})^{-1} \cdot e(C_{2,j}, K_{2,j}) \\
=& e([\mathbf{c}^\top]_1, [(\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^\top + \mathbf{W}\mathbf{d}_j]_2) \cdot e([\mathbf{c}^\top \mathbf{W} + \mathbf{c}_j^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1)]_1, [\mathbf{d}_j]_2)^{-1} \cdot e([\mathbf{c}_j^\top]_1, [(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{d}_j]_2) \\
=& [\mathbf{c}^\top (\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^\top + \mathbf{c}^\top \mathbf{W}\mathbf{d}_j - \mathbf{c}^\top \mathbf{W}\mathbf{d}_j - \mathbf{c}_j^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{d}_j + \mathbf{c}_j^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{d}_j]_T \\
=& [\mathbf{c}^\top (\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^\top]_T
\end{aligned}$$

for each $j \in [n]$ and

$$K = \prod_{j:x_j=1} \big(e(C_0, K_{0,j}) \cdot e(C_{1,j}, K_{1,j})^{-1} \cdot e(C_{2,j}, K_{2,j}) \big)^{\omega_j}$$

$$= \prod_{j:x_j=1} [\omega_j \mathbf{c}^\top (\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^\top]_T = [\mathbf{c}^\top (\mathbf{k}\|\mathbf{K}') \cdot \sum_{j:x_j=1} \omega_j \mathbf{M}_j^\top]_T = [\mathbf{c}^\top (\mathbf{k}\|\mathbf{K}')\mathbf{1}^\top]_T = [\mathbf{c}^\top \mathbf{k}]_T = e([\mathbf{c}^\top]_1, [\mathbf{k}]_2).$$

This readily proves correctness.

7.2 Bilinear entropy expansion lemma, revisited

With the additional basis $\mathbf{B} \in \mathbb{Z}_p^{(k+1)\times k}$, we need a variant of the entropy expansion lemma in Lemma 8 with $(\ell_1, \ell_2, \ell_3, \ell_W) = (k, 1, k, k+1)$ where the columns of $\mathbf{D}_j$ are drawn from $\mathsf{span}(\mathbf{B})$ instead of $\mathbb{Z}_p^{k+1}$ (see Lemma 12), which we may deduce readily from Lemma 8, thanks to the MDDH_k assumption.

Lemma 12 (prime-order entropy expansion lemma, revisited). *Pick basis* $(\mathbf{A}_1, \mathbf{a}_2, \mathbf{A}_3) \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(k+1)} \times \mathbb{Z}_p^{2k+1} \times \mathbb{Z}_p^{(2k+1)\times(k+1)}$ *and define its dual* $(\mathbf{A}_1^\|, \mathbf{a}_2^\|, \mathbf{A}_3^\|)$ *as in Section 4.2. With* $\mathbf{B} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(k+1)\times k}$*, we have*

$$\left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1 \\ \mathsf{ct}: [\mathbf{c}^\top]_1, \left\{[\mathbf{c}^\top \mathbf{W} + \mathbf{c}_j^\top (\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1, [\mathbf{c}_j^\top]_1\right\}_{j\in[n]} \\ \mathsf{sk}: \left\{[\mathbf{W}\mathbf{D}_j]_2, [\mathbf{D}_j]_2, [(\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{D}_j]_2\right\}_{j\in[n]} \end{array}\right\}$$
$$\approx_c \left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1 \\ \mathsf{ct}: [\boxed{\mathbf{c}}^\top]_1, \left\{[\boxed{\mathbf{c}}^\top (\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}}) + \boxed{\mathbf{c}_j}^\top (\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1, [\boxed{\mathbf{c}_j}^\top]_1\right\}_{j\in[n]} \\ \mathsf{sk}: \left\{[(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})\mathbf{D}_j]_2, [\mathbf{D}_j]_2, [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2\right\}_{j\in[n]} \end{array}\right\}$$

where $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(k+1)}, \mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{a}_2^\|), \mathbf{D}_j \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{B})$*, and* $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ *in the left distribution while* $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{a}_2)$ *in the right distribution. We let* $\mathsf{Adv}_{\mathcal{A}}^{\text{ExpLemRev}}(\lambda)$ *denote the distinguishing advantage.*

We claim that the lemma follows from the basic entropy expansion lemma (Lemma 8) and the MDDH_k assumption, which tells us that

$$\{[\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(k+1)\times(k+1)}]_2\}_{j\in[n]} \approx_c \{[\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{B})]_2\}_{j\in[n]}.$$

Concretely, for all $\mathcal{A}$, we can construct $\mathcal{B}_0$ and $\mathcal{B}_1$ with $\mathsf{Time}(\mathcal{B}_0), \mathsf{Time}(\mathcal{B}_1) \approx \mathsf{Time}(\mathcal{A})$ such that

$$\mathsf{Adv}_{\mathcal{A}}^{\text{ExpLemRev}}(\lambda) \le \mathsf{Adv}_{\mathcal{B}_0}^{\text{ExpLem}}(\lambda) + 2\cdot \mathsf{Adv}_{\mathcal{B}_1}^{\text{MDDH}_{k,k+1}^{n(k+1)}}(\lambda).$$

The proof is straight-forward by demonstrating that the left (resp. right) distributions in Lemma 8 and Lemma 12 are indistinguishable under the MDDH_k assumption and then applying Lemma 8. In the reduction, we sample $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(k+1)}$ (and $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{a}_2^\|)$ for the right distributions) and simulate $\mathsf{aux}, \mathsf{ct}$ honestly.

7.3 Proof of Security

We prove the following theorem:

Theorem 2. *Under the* MDDH_k *assumption in prime-order groups (cf. Section 4.1), the unbounded KP-ABE scheme for monotone span programs described in this Section (cf. Section 7.1) is adaptively secure (cf. Section 2.2).*

Main technical lemma. We prove the following technical lemma. As with the composite-order scheme in Section 6, we first apply the new entropy expansion lemma in Lemma 12 and obtain a copy of the CGW KP-ABE (variant-thereof) in the $\mathbf{a}_2$-subspace. We may then carry out the classic dual system methodology used for establishing adaptive security of the CGW KP-ABE.

Lemma 13. *For any adversary $\mathcal{A}$ that makes at most Q key queries against the unbounded KP-ABE scheme, there exist adversaries $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2$ such that:*

$$\mathsf{Adv}^{\mathrm{ABE}}_{\mathcal{A}}(\lambda) \le \mathsf{Adv}^{\mathrm{ExpLemRev}}_{\mathcal{B}_0}(\lambda) + Q \cdot \mathsf{Adv}^{\mathrm{MDDH}^n_{k,k+1}}_{\mathcal{B}_1}(\lambda) + Q \cdot \mathsf{Adv}^{\mathrm{MDDH}^n_{k,k+1}}_{\mathcal{B}_2}(\lambda) + O(1/p).$$

where $\mathsf{Time}(\mathcal{B}_0), \mathsf{Time}(\mathcal{B}_1), \mathsf{Time}(\mathcal{B}_2) \approx \mathsf{Time}(\mathcal{A})$. *In particular, we achieve security loss $O(n+Q)$ based on the MDDH$_k$ assumption.*

Game	CT	SK $\kappa < i$	SK $\kappa = i$	SK $\kappa > i$	Justification
0	Normal	Normal			real game
0'	E-normal	E-normal			entropy expansion lemma (revisited), Lemma 12
i	—	SF	E-normal	E-normal	$\mathsf{Game}_i = \mathsf{Game}_{i-1,3}$
$i,1$	—	—	P-normal	—	MDDH$_k$
$i,2$	—	—	P-SF	—	statistical lemma, Lemma 1
$i,3$	—	—	SF	—	MDDH$_k$
Final	random m	SF			statistical hidding

Fig. 8. Game sequence for proving the adaptive security of our prime-order unbounded KP-ABE.

The proof follows a series of games based on the dual system methodology, and outlined in Fig. 8. We first define the auxiliary distributions, from which we can describe the games. A notable difference from the composite-order setting in Section 6 is that we do not define SF ciphertexts.

Auxiliary distributions. We define various forms of ciphertext (of message m under attribute vector $\mathbf{x}$):

- Normal: Generated by Enc; in particular, $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$.
- E-normal: Same as a normal ciphertext except that $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{a}_2)$ and we use the substitution:

$$\mathbf{W} \mapsto \widehat{\mathbf{V}}_j := \mathbf{W} + \mathbf{V}^{(2)}_j \text{ in } j\text{'th component} \quad \text{and} \quad \mathbf{W}_0 + j \cdot \mathbf{W}_1 \mapsto \widehat{\mathbf{U}}_j := \mathbf{W}_0 + j \cdot \mathbf{W}_1 + \mathbf{U}^{(2)}_j \tag{11}$$

where $\mathbf{U}^{(2)}_j, \mathbf{V}^{(2)}_j \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{a}^{\parallel}_2)$. Concretely, an E-normal ciphertext is of the form

$$\mathsf{ct}_{\mathbf{x}} := \Big([\mathbf{c}^\top]_1, \{ [\mathbf{c}^\top \boxed{\widehat{\mathbf{V}}_j} + \mathbf{c}_j^\top \boxed{\widehat{\mathbf{U}}_j}]_1, [\mathbf{c}_j^\top]_1 \}_{j: x_j = 1}, e([\mathbf{c}^\top]_1, [\mathbf{k}]_2) \cdot m \Big) \text{ where } \boxed{\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{a}_2)}.$$

Then we pick $\alpha \leftarrow_{\mathrm{R}} \mathbb{Z}_p$ and define various forms of key (for span program $\mathbf{M}$):

- Normal: Generated by KeyGen.

– E-normal: Same as a normal key except that we use the same substitution as in (11). Concretely, an E-normal key is of the form

$$\mathsf{sk}_{\mathbf{M}} := \left(\{ [(\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^\top + \boxed{\widehat{\mathbf{V}}_j}\mathbf{d}_j]_2,\ [\mathbf{d}_j]_2,\ [\boxed{\widehat{\mathbf{U}}_j}\mathbf{d}_j]_2 \}_{j\in[n]} \right) \quad \text{where } \mathbf{d}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B}), \mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}.$$

– P-normal: Sample $\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k+1}$ in an E-normal key. Concretely, a P-normal key is of the form

$$\mathsf{sk}_{\mathbf{M}} := \left(\{ [(\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^\top + \widehat{\mathbf{V}}_j\mathbf{d}_j]_2,\ [\mathbf{d}_j]_2,\ [\widehat{\mathbf{U}}_j\mathbf{d}_j]_2 \}_{j\in[n]} \right) \quad \text{where } \boxed{\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k+1}}, \mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}.$$

– P-SF: Replace $\mathbf{k}$ with $\mathbf{k}+\alpha\mathbf{a}_2^\|$ in a P-normal key. Concretely, a P-SF key is of the form

$$\mathsf{sk}_{\mathbf{M}} := \left(\{ [(\mathbf{k}+\boxed{\alpha\mathbf{a}_2^\|}\|\mathbf{K}')\mathbf{M}_j^\top + \widehat{\mathbf{V}}_j\mathbf{d}_j]_2,\ [\mathbf{d}_j]_2,\ [\widehat{\mathbf{U}}_j\mathbf{d}_j]_2 \}_{j\in[n]} \right) \quad \text{where } \mathbf{d}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k+1}, \mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}.$$

– SF: Sample $\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B})$ in a P-SF key. Concretely, a SF key is of the form

$$\mathsf{sk}_{\mathbf{M}} := \left(\{ [(\mathbf{k}+\alpha\mathbf{a}_2^\|\|\mathbf{K}')\mathbf{M}_j^\top + \widehat{\mathbf{V}}_j\mathbf{d}_j]_2,\ [\mathbf{d}_j]_2,\ [\widehat{\mathbf{U}}_j\mathbf{d}_j]_2 \}_{j\in[n]} \right) \quad \text{where } \boxed{\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B})}, \mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}.$$

Here E, P, SF means "expanded", "pesudo", "semi-functional", respectively.

Games. We describe the game sequence in detail.

Game$_0$. The real security game (c.f. Section 2.2) where all keys and ciphertext are normal.

Game$_{0'}$. Identical to Game$_0$ except that all keys and the challenge ciphertext are E-normal. We claim that Game$_0 \approx_c$ Game$_{0'}$. This follows from our new prime-order entropy expansion lemma (see Lemma 12). In the reduction, on input

$$\left\{ \begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1 \\ \mathsf{ct}: [\mathbf{C}_0]_1,\ \{[\mathbf{C}_{1,j}]_1,\ [\mathbf{C}_{2,j}]_1\}_{j\in[n]} \\ \mathsf{sk}: \{[\mathbf{K}_{0,j}]_2,\ [\mathbf{K}_{1,j}]_2,\ [\mathbf{K}_{2,j}]_2\}_{j\in[n]} \end{array} \right\},$$

we sample $\mathbf{k} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{2k+1}$, $\mathbf{K}'_\kappa \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}$, $\tilde{\mathbf{d}}_{j,\kappa} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k+1}$ for $j \in [n]$ and $\kappa \in [Q]$, and simulate the game with

$$\left\{ \begin{array}{l} \mathsf{mpk}: \mathsf{aux},\ e([\mathbf{A}_1^\top]_1, [\mathbf{k}]_2) \\ \mathsf{ct}_{\mathbf{x}^*}: [\mathbf{C}_0]_1,\ \{[\mathbf{C}_{1,j}]_1,\ [\mathbf{C}_{2,j}]_1\}_{j:x_j^*=1}, e([\mathbf{C}_0]_1, [\mathbf{k}]_2)\cdot m_\beta \\ \mathsf{sk}_{\mathbf{M}}^\kappa: \{ [(\mathbf{k}\|\mathbf{K}'_\kappa)\mathbf{M}_j^\top + \mathbf{K}_{0,j}\tilde{\mathbf{d}}_{j,\kappa}]_2,\ [\mathbf{K}_{1,j}\tilde{\mathbf{d}}_{j,\kappa}]_2,\ [\mathbf{K}_{2,j}\tilde{\mathbf{d}}_{j,\kappa}]_2 \}_{j\in[n]} \end{array} \right\}.$$

In both cases, we set $\mathbf{d}_{j,\kappa} := \mathbf{D}_j\tilde{\mathbf{d}}_{j,\kappa}$ where $\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{B})$ as defined in the new entropy expansion lemma (Lemma 12). Therefore all $\mathbf{d}_{j,\kappa}$ are uniformly distributed over $\mathsf{span}(\mathbf{B})$ with high probability. See Lemma 44 for details.

Game$_i$. Identical to Game$_{0'}$ except that the first $i-1$ keys are SF. It is easy to see that Game$_{0'} \equiv$ Game$_1$. To show that Game$_i \approx_c$ Game$_{i+1}$, we will require another sequence of sub-games.

Game$_{i,1}$. Identical to Game$_i$ except that the i'th key is P-normal. We claim that Game$_i \approx_c$ Game$_{i,1}$. This follows from the MDDH$^n_{k,k+1}$ assumption asserting

$$\{[\mathbf{d}_{j,i} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B})]_2\}_{j\in[n]} \approx_c \{[\mathbf{d}_{j,i} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k+1}]_2\}_{j\in[n]}.$$

In the reduction, on input $[\mathbf{B}]_2, \{[\mathbf{t}_j]_2\}_{j\in[n]}$, we sample $\mathbf{A}_1, \mathbf{a}_2, \mathbf{W}, \mathbf{W}_0, \mathbf{W}_1, \mathbf{k}$ and $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)}, \alpha$, and honestly simulate $\mathsf{mpk}, \mathsf{ct}$ and $\{\mathsf{sk}_{\mathbf{M}}^\kappa\}_{\kappa\neq i}$. The i'th key is created using $\{[\mathbf{t}_j]_2\}_{j\in[n]}$. See Lemma 45 for details.

Game$_{i,2}$. Identical to Game$_i$ except that the i'th key is P-SF. We claim that Game$_{i,1}$ ≡ Game$_{i,2}$. This follows from Lemma 1 in Section 2 which ensures that for any $\mathbf{x}$ that does not satisfy $\mathbf{M}$,

$$\Big(\overbrace{\mathbf{k},\alpha,\mathbf{B},\mathbf{V}_j^{(2)}\mathbf{B}}^{\kappa\text{'th sk}, \kappa\neq i};\ \overbrace{\{\mathbf{V}_j^{(2)}\}_{j:x_j=1}}^{\text{E-normal ct}};\ \overbrace{\{(\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^\top+\mathbf{V}_j^{(2)}\mathbf{d}_j,\mathbf{d}_j\}_{j\in[n]}}^{\text{P-normal } i\text{'th sk}}\Big)$$
$$\equiv \Big(\mathbf{k},\alpha,\mathbf{B},\mathbf{V}_j^{(2)}\mathbf{B};\ \{\mathbf{V}_j^{(2)}\}_{j:x_j=1};\ \underbrace{\{(\mathbf{k}+\boxed{\alpha\mathbf{a}_2^\|}\|\mathbf{K}')\mathbf{M}_j^\top+\mathbf{V}_j^{(2)}\mathbf{d}_j,\mathbf{d}_j\}_{j\in[n]}}_{\text{P-SF } i\text{'th sk}}\Big)$$

where $\mathbf{K}' \leftarrow_R \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}, \mathbf{V}_j^{(2)} \leftarrow_R \mathsf{span}^{k+1}(\mathbf{a}_2^\|)$, and for any $\mathbf{k},\alpha,\mathbf{B}$, and $\mathbf{d}_j \notin \mathsf{span}(\mathbf{B})$. It is straight-forward to compute the remaining terms in mpk, the challenge ciphertext and the Q secret keys by sampling $\mathbf{A}_1,\mathbf{W},\mathbf{W}_0,\mathbf{W}_1,\mathbf{U}_j^{(2)},\mathbf{c},\mathbf{c}_j,\mathbf{d}_j$ ourselves. See Lemma 46 for details.

Game$_{i,3}$. Identical to Game$_i$ except that the i'th key is SF. We claim that Game$_{i,2}$ $\approx_c$ Game$_{i,3}$. The proof is completely analogous to that of Game$_i$ $\approx_c$ Game$_{i,1}$. See Lemma 47 for details. Furthermore, observe that Game$_{i,3}$ is actually identical to Game$_{i+1}$.

Game$_{\mathsf{Final}}$. Identical to Game$_{Q+1}$, except that the challenge ciphertext is a SF one for a random message in $\mathbb{G}_T$. We claim that Game$_{Q+1}$ ≡ Game$_{\mathsf{Final}}$. This follows from the fact that

$$(e(\overbrace{[\mathbf{A}_1]_1,[\mathbf{k}]_2}^{\mathsf{mpk}}),\ \overbrace{\mathbf{k}+\alpha\mathbf{a}_2^\|}^{\mathsf{SF\ sk}},\ \overbrace{e([\mathbf{c}^\top]_1,[\mathbf{k}]_2)\cdot m_b}^{\mathsf{SF\ ct}}) = (e([\mathbf{A}_1]_1,[\mathbf{k}]_2),\ \mathbf{k},\ e([\mathbf{c}^\top]_1,[\mathbf{k}+\boxed{\alpha\mathbf{a}_2^\|}]_2)),$$

where $\mathbf{k} \leftarrow_R \mathbb{Z}_p^{2k+1}$ and $\alpha \leftarrow_R \mathbb{Z}_p$. The message m_b is statistically hidden by $e([\mathbf{c}^\top]_1,[\mathbf{a}_2^\|]_2)^\alpha$ since $\mathbf{c} \in \mathsf{span}(\mathbf{A}_1,\mathbf{a}_2)$. See Lemma 48 for details. In Game$_{\mathsf{Final}}$, the view of the adversary is statistically independent of the challenge bit b. Hence, $\mathsf{Adv}_{\mathsf{Final}} = 0$.

8 CP-ABE for Monotone Span Programs

In this section, we present our adaptively secure, unbounded CP-ABE for monotone span programs based on the k-Lin assumption in prime-order groups.

8.1 Warm-up: a composite-order scheme

As before, we begin with a scheme in composite-order groups:

$$\begin{aligned}
\mathsf{mpk} &:= (g_1, g_1^{u_0}, g_1^{w}, g_1^{w_0}, g_1^{w_1}, e(g_1,h_{123})^\alpha) \\
\mathsf{ct}_{\mathbf{M}} &:= (g_1^s, \{g_1^{s(u_0,\mathbf{u})\mathbf{M}_j^\top+s_jw},\ g_1^{s_j},\ g_1^{s_j(w_0+j\cdot w_1)}\}_{j\in[n]},\ e(g_1,h_{123})^{\alpha s}\cdot m) \\
\mathsf{sk}_{\mathbf{x}} &:= (h_{123}^\alpha\cdot h_1^{u_0r},\ h_1^r,\ \{h_1^{rw+r_j(w_0+j\cdot w_1)},\ h_1^{r_j}\}_{j:x_j=1})
\end{aligned} \tag{12}$$

Decryption proceeds by first computing $\{e(g_1,h_1)^{s(u_0,\mathbf{u})\mathbf{M}_j^\top r}\}_{j:x_j=1}$ and then $e(g_1,h_1)^{su_0r}$ and $e(g_1,h_{123})^{\alpha s}$.

Bilinear entropy expansion. To analyze this scheme, we would require the following variant of our entropy expansion lemma where we basically swap ct and sk (on the LHS, instead of having g_1^s, g_1^{sw} in ct, we have h_1^r, h_1^{rw} in sk):

$$\left\{\begin{array}{l}\mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}\\ \mathsf{ct}: g_1^s, \{g_1^{s_jw},\ g_1^{s_j},\ g_1^{s_j(w_0+j\cdot w_1)}\}_{j\in[n]}\\ \mathsf{sk}: h_1^r, \{h_1^{rw+r_j(w_0+j\cdot w_1)},\ h_1^{r_j}\}_{j\in[n]}\end{array}\right\} \approx_c \left\{\begin{array}{l}\mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}\\ \mathsf{ct}: g_1^s\cdot\boxed{g_2^s}, \{g_1^{s_jw}\cdot\boxed{g_2^{s_jv_j}},\ g_1^{s_j}\cdot\boxed{g_2^{s_j}},\ g_1^{s_j(w_0+j\cdot w_1)}\cdot\boxed{g_2^{s_ju_j}}\}_{j\in[n]}\\ \mathsf{sk}: h_1^r\cdot\boxed{h_2^r}, \{h_1^{rw+r_j(w_0+j\cdot w_1)}\cdot\boxed{h_2^{rv_j+r_ju_j}},\ h_1^{r_j}\cdot\boxed{h_2^{r_j}}\}_{j\in[n]}\end{array}\right\} \tag{13}$$

where

$$w, w_0, w_1 \leftarrow_R \mathbb{Z}_N,\ v_j, u_j \leftarrow_R \mathbb{Z}_N,\ s, s_j \leftarrow_R \mathbb{Z}_N,\ r, r_j \leftarrow_R \mathbb{Z}_N.$$

This would in turn require the following change to step two (whereas step one remains intact).

$$\left\{\begin{array}{l}\mathsf{aux}: g_1, g_2, g_1^w, h_1, h_1^w \\ \mathsf{ct}: \{g_2^{s_j w},\ g_2^{s_j},\ g_2^{s_j u_j}\}_{j\in[n]} \\ \mathsf{sk}: h_2^r,\ \{\, h_2^{rw}\cdot h_2^{r_j u_j},\ h_2^{r_j}\,\}_{j\in[n]}\end{array}\right\} \approx_c \left\{\begin{array}{l}\mathsf{aux}: g_1, g_2, g_1^w, h_1, h_1^w \\ \mathsf{ct}: \{\boxed{g_2^{s_j v_j}},\ g_2^{s_j},\ g_2^{s_j u_j}\}_{j\in[n]} \\ \mathsf{sk}: h_2^r,\ \{\,\boxed{h_2^{r v_j}}\cdot h_2^{r_j u_j},\ h_2^{r_j}\,\}_{j\in[n]}\end{array}\right\} \quad (14)$$

where

$$w \leftarrow_R \mathbb{Z}_N,\ v_j, u_j \leftarrow_R \mathbb{Z}_N,\ s_j \leftarrow_R \mathbb{Z}_N,\ r, r_j \leftarrow_R \mathbb{Z}_N.$$

We can justify (13) via a hybrid argument analogous to that shown in Section 3.2. See Appendix E for details.

8.2 Our Prime-order Scheme

Our prime-order construction is presented as follows:

$\mathsf{Setup}(1^\lambda, 1^n)$: On input $(1^\lambda, 1^n)$, sample

$$\mathbf{A}_1 \leftarrow_R \mathbb{Z}_p^{3k\times k}, \mathbf{B} \leftarrow_R \mathbb{Z}_p^{(k+1)\times k},\ \mathbf{W}, \mathbf{W}_0, \mathbf{W}_1, \mathbf{U}_0 \leftarrow_R \mathbb{Z}_p^{3k\times(k+1)},\ \mathbf{k} \leftarrow_R \mathbb{Z}_p^{3k}$$

and output the master public and secret key pair

$$\mathsf{mpk} := \big(\, [\mathbf{A}_1^\top, \mathbf{A}_1^\top\mathbf{W}, \mathbf{A}_1^\top\mathbf{W}_0, \mathbf{A}_1^\top\mathbf{W}_1, \mathbf{A}_1^\top\mathbf{U}_0]_1,\ e([\mathbf{A}_1^\top]_1, [\mathbf{k}]_2)\,\big) \in G_1^{k\times 3k} \times (G_1^{k\times(k+1)})^4 \times G_T^k$$

and

$$\mathsf{msk} := (\, \mathbf{k},\ \mathbf{B},\ \mathbf{W},\ \mathbf{W}_0,\ \mathbf{W}_1,\ \mathbf{U}_0\,).$$

$\mathsf{Enc}(\mathsf{mpk}, \mathbf{M}, m)$: On input a monotone span program $\mathbf{M} \in \mathbb{Z}_p^{n\times\ell'}$ and $m \in G_T$, pick $\mathbf{c}, \mathbf{c}_j \leftarrow_R \mathsf{span}(\mathbf{A}_1)$ for all $j \in [n]$, sample $\mathbf{U} \leftarrow_R \mathbb{Z}_p^{(\ell'-1)\times(k+1)}$ and output

$$\mathsf{ct}_{\mathbf{M}} := \Big(\, C_0 := [\mathbf{c}^\top]_1,\ \{\, C_{1,j} := [\mathbf{M}_j \begin{pmatrix}\mathbf{c}^\top\mathbf{U}_0 \\ \mathbf{U}\end{pmatrix} + \mathbf{c}_j^\top\mathbf{W}]_1,\ C_{2,j} := [\mathbf{c}_j^\top]_1,\ C_{3,j} := [\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1\,\}_{j\in[n]},\ C := e([\mathbf{c}^\top]_1, [\mathbf{k}]_2)\cdot m\,\Big)$$
$$\in G_1^{3k} \times (G_1^{k+1} \times G_1^{3k} \times G_1^{k+1})^n \times G_T.$$

$\mathsf{KeyGen}(\mathsf{mpk}, \mathsf{msk}, \mathbf{x})$: On input an attribute vector $\mathbf{x} := (x_1, \ldots, x_n) \in \{0,1\}^n$, pick $\mathbf{d}, \mathbf{d}_j \leftarrow_R \mathsf{span}(\mathbf{B})$ for all $j \in [n]$, and output

$$\mathsf{sk}_{\mathbf{x}} := \big(\, K_0 := [\mathbf{k} + \mathbf{U}_0\mathbf{d}]_2,\ K_1 := [\mathbf{d}]_2,\ \{\, K_{2,j} := [\mathbf{W}\mathbf{d} + (\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{d}_j]_2\ K_{3,j} := [\mathbf{d}_j]_2\,\}_{j:x_j=1}\,\big)$$
$$\in G_2^{3k} \times G_2^{k+1} \times (G_2^{3k} \times G_2^{k+1})^n.$$

$\mathsf{Dec}(\mathsf{mpk}, \mathsf{sk}_{\mathbf{x}}, \mathsf{ct}_{\mathbf{M}})$: If $\mathbf{x}$ satisfies $\mathbf{M}$, compute $\omega_1, \ldots, \omega_n \in \mathbb{Z}_p$ such that

$$\sum_{j:x_j=1} \omega_j \mathbf{M}_j = \mathbf{1}.$$

Then, compute

$$K \leftarrow e(C_0, K_0) / \prod_{j:x_j=1} \big(e(C_{1,j}, K_1)\cdot e(C_{2,j}, K_{2,j})^{-1}\cdot e(C_{3,j}, K_{3,j})\big)^{\omega_j},$$

and recover the message as $n \leftarrow C/K \in G_T$.

Correctness. For all $\mathbf{M}$ and $\mathbf{x}$ such that $\mathbf{x}$ satisfies $\mathbf{M}$, we have

$$
\begin{aligned}
& e(C_{1,j}, K_1)\cdot e(C_{2,j}, K_{2,j})^{-1}\cdot e(C_{3,j}, K_{3,j})\\
=& e([\mathbf{M}_j \begin{pmatrix} \mathbf{c}^\top \mathbf{U}_0 \\ \mathbf{U}\end{pmatrix} + \mathbf{c}_j^\top \mathbf{W}]_1, [\mathbf{d}]_2)\cdot e([\mathbf{c}_j^\top]_1, [\mathbf{W}\mathbf{d} + (\mathbf{W}_0 + j\cdot \mathbf{W}_1)\mathbf{d}_j]_2)^{-1}\cdot e([\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1, [\mathbf{d}_j]_2)\\
=& [\mathbf{M}_j \begin{pmatrix} \mathbf{c}^\top \mathbf{U}_0\mathbf{d} \\ \mathbf{U}\mathbf{d}\end{pmatrix}]_T
\end{aligned}
$$

for all $j \in [n]$. Then we have

$$
\begin{aligned}
e(C_0, K_0)/\prod_{j:x_j=1}[\mathbf{M}_j \begin{pmatrix} \mathbf{c}^\top \mathbf{U}_0\mathbf{d} \\ \mathbf{U}\mathbf{d}\end{pmatrix}]_T^{\omega_j} &= [\mathbf{c}^\top\mathbf{k}]_T\cdot[\mathbf{c}^\top\mathbf{U}_0\mathbf{d}]_T/[\sum_{j:x_j=1}\omega_j\mathbf{M}_j\begin{pmatrix} \mathbf{c}^\top \mathbf{U}_0\mathbf{d} \\ \mathbf{U}\mathbf{d}\end{pmatrix}]_T\\
&= [\mathbf{c}^\top\mathbf{k}]_T\cdot[\mathbf{c}^\top\mathbf{U}_0\mathbf{d}]_T/[\mathbf{c}^\top\mathbf{U}_0\mathbf{d}]_T = [\mathbf{c}^\top\mathbf{k}]_T.
\end{aligned}
$$

This readily proves correctness.

8.3 Prime-order Bilinear Entropy Expansion for CP-ABE (for Monotone Span Program)

We prove the adaptive security using the following entropy expansion lemma. Let $\mathbf{A}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_1}, \mathbf{A}_2 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_2}, \mathbf{A}_3 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_3}$ and $\mathbf{A}_1^\| \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_1}, \mathbf{A}_2^\| \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_2}, \mathbf{A}_3^\| \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_3}$ with the constraints described in Section 4.2. Let $\ell_W \in \mathbb{N}$.

Lemma 14 (prime-order entropy expansion lemma for CP-ABE). *Suppose $\ell_1, \ell_2, \ell_3 \geq k$. Then, under the MDDH$_k$ assumption, we have*

$$
\left\{
\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1\\
\mathsf{ct}: [\mathbf{c}^\top]_1, \left\{[\mathbf{c}_j^\top\mathbf{W}]_1,\ [\mathbf{c}_j]_1,\ [\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1\right\}_{j\in[n]}\\
\mathsf{sk}: \left\{[\mathbf{D}]_2,\ [\mathbf{W}\mathbf{D} + (\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2\right\}_{j\in[n]}
\end{array}
\right\}
$$
$$
\approx_c \left\{
\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1\\
\mathsf{ct}: [\boxed{\mathbf{c}}^\top]_1, \left\{[\boxed{\mathbf{c}_j}^\top(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})]_1,\ [\boxed{\mathbf{c}_j}]_1,\ [\boxed{\mathbf{c}_j}^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1\right\}_{j\in[n]}\\
\mathsf{sk}: \left\{[\mathbf{D}]_2,\ [(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})\mathbf{D} + (\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2\right\}_{j\in[n]}
\end{array}
\right\}
$$

where $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$, $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^\|)$, $\mathbf{D}, \mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$, *and* $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ *in the left distribution while* $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)$ *in the right distribution.*

The lemma is proved in two steps via Lemma 9 and Lemma 15 described below. The new lemma for the second step (i.e., Lemma 15) is adapted from Lemma 10 in an analogous way to the composite case. To establish the lemma from MDDH$_k$ assumption, we additionally require that $\ell_2 \geq k$.

Lemma 15 (prime-order entropy expansion lemma for CP-ABE (step two)). *Suppose $\ell_2 \geq k$. Then, under the MDDH$_k$ assumption, we have*

$$
\left\{
\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_2^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1\\
\mathsf{ct}: \left\{[\mathbf{c}_j^{(2)\top}\mathbf{W}]_1, [\mathbf{c}_j^{(2)\top}]_1, [\mathbf{c}_j^{(2)\top}\mathbf{U}_j^{(2)}]_1\right\}_{j\in[n]}\\
\mathsf{sk}: \left\{[\mathbf{D}]_2, [\mathbf{W}\mathbf{D} + \mathbf{U}_j^{(2)}\mathbf{D}_j]_2\ [\mathbf{D}_j]_2,\right\}_{j\in[n]}
\end{array}
\right\}
\approx_c
\left\{
\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_2^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1\\
\mathsf{ct}: \left\{[\mathbf{c}_j^{(2)\top}(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})]_1, [\mathbf{c}_j^{(2)\top}]_1, [\mathbf{c}_j^{(2)\top}\mathbf{U}_j^{(2)}]_1\right\}_{j\in[n]}\\
\mathsf{sk}: \left\{[\mathbf{D}]_2, [(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})\mathbf{D} + \mathbf{U}_j^{(2)}\mathbf{D}_j]_2\ [\mathbf{D}_j]_2,\right\}_{j\in[n]}
\end{array}
\right\}
$$

where $\mathbf{W} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$, $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^\|)$, $\mathbf{D}, \mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$, *and* $\mathbf{c}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_2)$.

Proof. This follows from the $\mathrm{MDDH}^{\ell_W}_{\ell_2,n}$ assumption, which tells us that

$$\{[\mathbf{s}_j^\top]_1, [\mathbf{s}_j^\top \mathbf{W}_{\{2\}}]_1\}_{j\in[n]} \approx_c \{[\mathbf{s}_j^\top]_1, [\mathbf{s}_j^\top(\mathbf{W}_{\{2\}}+\mathbf{V}_j)]_1\}_{j\in[n]}$$

where $\mathbf{s}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_2}$, $\mathbf{W}_{\{2\}}, \mathbf{V}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_2\times\ell_W}$. On input $\{[\mathbf{s}_j^\top],[\mathbf{t}_j^\top]\}_{j\in[n]}$, we select $\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3,\mathbf{A}_1^\parallel,\mathbf{A}_2^\parallel,\mathbf{A}_3^\parallel$, sample $\mathbf{W}_{\{1\}} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1\times\ell_W}$, $\mathbf{W}_{\{3\}} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_3\times\ell_W}$, and implicitly define $\mathbf{W} := \mathbf{A}_1^\parallel\mathbf{W}_{\{1\}} + \mathbf{A}_2^\parallel\mathbf{W}_{\{2\}} + \mathbf{A}_3^\parallel\mathbf{W}_{\{3\}}$. We then sample $\mathbf{U}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_2\times\ell_W}$ and $\mathbf{D},\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$ for all $j\in[n]$, and output

$$\left\{\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_2^\top]_1, [\mathbf{W}_{\{1\}}]_1 \\
\ \ \mathsf{ct}: \left\{[\mathbf{t}_j]_1, [\mathbf{s}_j^\top\mathbf{A}_2^\top]_1, [\mathbf{s}_j^\top\mathbf{U}_j - \mathbf{t}_j^\top\mathbf{D}\mathbf{D}_j^{-1}]_1\right\}_{j\in[n]} \\
\ \ \mathsf{sk}: \left\{[\mathbf{D}]_2, [(\mathbf{A}_1^\parallel\mathbf{W}_{\{1\}} + \mathbf{A}_3^\parallel\mathbf{W}_{\{3\}})\mathbf{D} + \mathbf{A}_2^\parallel\mathbf{U}_j\mathbf{D}_j]_2\ [\mathbf{D}_j]_2,\right\}_{j\in[n]}
\end{array}\right\}.$$

Observe that

- when $\mathbf{t}_j^\top = \mathbf{s}_j^\top\mathbf{W}_{\{2\}}$ and let $\mathbf{c}_j^{(2)} := \mathbf{A}_2\mathbf{s}_j$ and $\mathbf{A}_2^\parallel\mathbf{U}_j = \mathbf{U}_j^{(2)} + \mathbf{A}_2^\parallel\mathbf{W}_{\{2\}}\mathbf{D}\mathbf{D}_j^{-1}$, then $\mathbf{s}_j^\top\mathbf{U}_j - \mathbf{t}_j^\top\mathbf{D}\mathbf{D}_j^{-1} = \mathbf{s}_j^\top\mathbf{A}_2^\top\mathbf{U}_j^{(2)}$ and the output is identical to the left distribution;
- when $\mathbf{t}_j^\top = \mathbf{s}_j^\top(\mathbf{W}_{\{2\}}+\mathbf{V}_j)$ and let $\mathbf{c}_j^{(2)} := \mathbf{A}_2\mathbf{s}_j$ and $\mathbf{A}_2^\parallel\mathbf{U}_j = \mathbf{U}_j^{(2)} + \mathbf{A}_2^\parallel(\mathbf{W}_{\{2\}}+\mathbf{V}_j)\mathbf{D}\mathbf{D}_j^{-1}$, then $\mathbf{s}_j^\top\mathbf{U}_j - \mathbf{t}_j^\top\mathbf{D}\mathbf{D}_j^{-1} = \mathbf{s}_j^\top\mathbf{A}_2^\top\mathbf{U}_j^{(2)}$ and the output is identical to the right distribution where we define $\mathbf{V}_j^{(2)} := \mathbf{A}_2^\parallel\mathbf{V}_j$.

This readily proves the lemma. □

It is easy to check that Lemma 14 is implied by Lemma 9 and Lemma 15 by the following hybrid argument.

$$\begin{array}{rll}
& \left\{\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1 \\
\ \ \mathsf{ct}: [\mathbf{c}^\top]_1,\ \left\{[\mathbf{c}_j^\top\mathbf{W}]_1,\ [\mathbf{c}_j^\top]_1,\ [\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1\right\}_{j\in[n]} \\
\ \ \mathsf{sk}: \left\{[\mathbf{D}]_2,\ [\mathbf{W}\mathbf{D} + (\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2\right\}_{j\in[n]}
\end{array}\right\} & \text{LHS in Lemma 14} \\
\overset{\mathbf{A}_1\mapsto(\mathbf{A}_1,\mathbf{A}_2)}{\approx_c} & \left\{\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1 \\
\ \ \mathsf{ct}: [\boxed{\mathbf{c}}^\top]_1,\ \left\{[\mathbf{c}_j^\top\mathbf{W}]_1,\ [\mathbf{c}_j^\top]_1,\ [\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1\right\}_{j\in[n]} \\
\ \ \mathsf{sk}: \left\{[\mathbf{D}]_2,\ [\mathbf{W}\mathbf{D} + (\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2\right\}_{j\in[n]}
\end{array}\right\} & \text{where } \boxed{\mathbf{c} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1) \\
\overset{\text{Lemma 9}}{\approx_c} & \left\{\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1 \\
\ \ \mathsf{ct}: [\mathbf{c}^\top]_1,\ \left\{[\boxed{\mathbf{c}_j}^\top\mathbf{W}]_1,\ [\boxed{\mathbf{c}_j}^\top]_1,\ [\boxed{\mathbf{c}_j}^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1\right\}_{j\in[n]} \\
\ \ \mathsf{sk}: \left\{[\mathbf{D}]_2,\ [\mathbf{W}\mathbf{D} + (\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2\right\}_{j\in[n]}
\end{array}\right\} & \text{where } \boxed{\mathbf{c},\mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_2)} \\
\overset{\text{Lemma 15}}{\approx_c} & \left\{\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1 \\
\ \ \mathsf{ct}: [\mathbf{c}^\top],\ \left\{[\mathbf{c}_j^\top(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})]_1,\ [\mathbf{c}_j^\top]_1,\ [\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})]_1\right\}_{j\in[n]} \\
\ \ \mathsf{sk}: \left\{[\mathbf{D}]_2,\ [(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})\mathbf{D} + (\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})\mathbf{D}_j]_2,\ [\mathbf{D}_j]_2\right\}_{j\in[n]}
\end{array}\right\} & \text{RHS in Lemma 14}
\end{array}$$

8.4 Proof of Security

Theorem 3. *Under the MDDH_k assumption in prime-order groups (cf. Section 4.1), the unbounded CP-ABE scheme described in this section (cf. Section 8.2) is adaptively secure (cf. Section 2.2).*

The proof is completely analogous to that of Theorem 2, using instead the entropy expansion lemma (Lemma 14) in this section, and the following statistical lemma:

Lemma 16 (statistical lemma [6, Appendix A.5]). *For any* $\mathbf{x}$ *that does not satisfy* $\mathbf{M}$, *the distributions*

$$(\{s, \mathbf{M}_j \binom{su_0}{\mathbf{u}} + s_j v_j, s_j\}_{j\in[n]},\ \{\alpha + r u_0, r, r v_j\}_{j:x_j=1})$$

perfectly hide α, *where the randomness is taken over* $u_0, v_i, \leftarrow_R \mathbb{Z}_p, \mathbf{u} \leftarrow_R \mathbb{Z}_p^{\ell'-1}$, *and for any fixed* $s, s_j, r \neq 0$.

The proof follows the same series of games as described in Fig. 8, and we simply state the auxiliary distributions here.

Auxiliary distributions and game sequence. We define various forms of ciphertext (for span program $\mathbf{M}$ and message m):

- Normal: Generated by Enc; in particular, $\mathbf{c}, \mathbf{c}_j \leftarrow_R \mathsf{span}(\mathbf{A}_1)$.
- E-normal: Same as a normal ciphertext except that $\mathbf{c}, \mathbf{c}_j \leftarrow_R \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)$ and we use the substitution:

$$\mathbf{W} \mapsto \widehat{\mathbf{V}}_j := \mathbf{W} + \mathbf{V}_j^{(2)} \text{ in } j\text{'th component} \quad \text{and} \quad \mathbf{W}_0 + j\cdot \mathbf{W}_1 \mapsto \widehat{\mathbf{U}}_j := \mathbf{W}_0 + j \cdot \mathbf{W}_1 + \mathbf{U}_j^{(2)} \tag{15}$$

  where $\mathbf{U}_j^{(2)}, \mathbf{V}_j^{(2)} \leftarrow_R \mathsf{span}^{k+1}(\mathbf{A}_2^{\parallel})$. Concretely, an E-normal ciphertext is of the form

$$\mathsf{ct}_{\mathbf{M}} := \Big([\mathbf{c}^\top]_1, \{ [\mathbf{M}_j \begin{pmatrix} \mathbf{c}^\top \mathbf{U}_0 \\ \mathbf{U} \end{pmatrix} + \mathbf{c}_j^\top \boxed{\widehat{\mathbf{V}}_j}\,]_1, [\mathbf{c}_j^\top]_1, [\mathbf{c}_j^\top \boxed{\widehat{\mathbf{U}}_j}\,]_1 \}_{j\in[n]}, e([\mathbf{c}^\top]_1, [\mathbf{k}]_2) \cdot m \Big)$$

  where $\mathbf{U} \leftarrow_R \mathbb{Z}_p^{(\ell'-1)\times(k+1)}$ and $\boxed{\mathbf{c}, \mathbf{c}_j \leftarrow_R \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)}$.

Then we pick $\mathbf{k}^{(2)} \leftarrow_R \mathsf{span}(\mathbf{A}_2^{\parallel})$ and define various forms of key (for attribute vector $\mathbf{x}$):

- Normal: Generated by KeyGen.
- E-normal: Same as a Normal key except that we use the same substitution as in (15). Concretely, an E-normal key is of the form

$$\mathsf{sk}_{\mathbf{x}} := \Big([\mathbf{k} + \mathbf{U}_0 \mathbf{d}]_2, [\mathbf{d}]_2, \{ [\boxed{\widehat{\mathbf{V}}_j}\,\mathbf{d} + \boxed{\widehat{\mathbf{U}}_j}\,\mathbf{d}_j]_2\ [\mathbf{d}_j]_2 \}_{j:x_j=1} \Big) \quad \text{where } \mathbf{d}, \mathbf{d}_j \leftarrow_R \mathsf{span}(\mathbf{B}).$$

- P-normal: Sample $\mathbf{d}, \mathbf{d}_j \leftarrow_R \mathbb{Z}_p^{k+1}$ in an E-normal key. Concretely, a P-normal key is of the form

$$\mathsf{sk}_{\mathbf{x}} := \Big([\mathbf{k} + \mathbf{U}_0 \mathbf{d}]_2, [\mathbf{d}]_2, \{ [\widehat{\mathbf{V}}_j \mathbf{d} + \widehat{\mathbf{U}}_j \mathbf{d}_j]_2\ [\mathbf{d}_j]_2 \}_{j:x_j=1} \Big) \quad \text{where } \boxed{\mathbf{d}, \mathbf{d}_j \leftarrow_R \mathbb{Z}_p^{k+1}}.$$

- P-SF: Replace $\mathbf{k}$ with $\mathbf{k} + \hat{\mathbf{k}}^{(2)}$ in a P-normal key. Concretely, a P-SF key is of the form

$$\mathsf{sk}_{\mathbf{x}} := \Big([\mathbf{k} + \boxed{\mathbf{k}^{(2)}} + \mathbf{U}_0 \mathbf{d}]_2, [\mathbf{d}]_2, \{ [\widehat{\mathbf{V}}_j \mathbf{d} + \widehat{\mathbf{U}}_j \mathbf{d}_j]_2\ [\mathbf{d}_j]_2 \}_{j:x_j=1} \Big) \quad \text{where } \mathbf{d}, \mathbf{d}_j \leftarrow_R \mathbb{Z}_p^{k+1}.$$

- SF: Sample $\mathbf{d}, \mathbf{d}_j \leftarrow_R \mathsf{span}(\mathbf{B})$ in a P-SF key. Concretely, a SF key is of the form

$$\mathsf{sk}_{\mathbf{x}} := \Big([\mathbf{k} + \mathbf{k}^{(2)} + \mathbf{U}_0 \mathbf{d}]_2, [\mathbf{d}]_2, \{ [\widehat{\mathbf{V}}_j \mathbf{d} + \widehat{\mathbf{U}}_j \mathbf{d}_j]_2\ [\mathbf{d}_j]_2 \}_{j:x_j=1} \Big) \quad \text{where } \boxed{\mathbf{d}, \mathbf{d}_j \leftarrow_R \mathsf{span}(\mathbf{B})}.$$

The game sequence and claims follow Section 7.3. We simply provide the proofs of the claims that $\mathsf{Game}_0 \approx \mathsf{Game}_{0'}$ and $\mathsf{Game}_{i,1} \equiv \mathsf{Game}_{i,2}$: The first one follows from a revisited version of Lemma 14 (entropy expansion lemma for CP-ABE), which will change the distributions of $\mathbf{D}_j$ (analogous to Lemma 12) and employ parameter setting $(\ell_1, \ell_2, \ell_3, \ell_W) = (k, k, k, k+1)$. This can be established from the MDDH_k assumption and Lemma 14; The second claim follows from Lemma 16 which ensures that for any $\mathbf{x}$ that does not satisfy $\mathbf{M}$,

$$\Big(\overbrace{\mathbf{A}_1^\top, \mathbf{A}_1^\top \mathbf{U}_0}^{\text{mpk}};\ \overbrace{\mathbf{k}, \mathbf{k}^{(2)}, \mathbf{B}, \mathbf{U}_0 \mathbf{B}, \mathbf{V}_j^{(2)} \mathbf{B}}^{\kappa\text{'th sk}, \kappa \neq i};\ \overbrace{\{\mathbf{c}^\top, \mathbf{M}_j \begin{pmatrix} \mathbf{c}^\top \mathbf{U}_0 \\ \mathbf{U} \end{pmatrix} + \mathbf{c}_j^\top \mathbf{V}_j^{(2)}, \mathbf{c}_j^\top\}_{j\in[n]}}^{\text{E-normal ct}};\ \overbrace{\mathbf{k} + \mathbf{U}_0 \mathbf{d}, \mathbf{d}, \{\mathbf{V}_j^{(2)} \mathbf{d}\}_{j:x_j=1}}^{\text{P-normal } i\text{'th sk}}\Big)$$

$$\equiv \Big(\mathbf{A}_1^\top, \mathbf{A}_1^\top \mathbf{U}_0;\ \mathbf{k}, \mathbf{k}^{(2)}, \mathbf{B}, \mathbf{U}_0 \mathbf{B}, \mathbf{V}_j^{(2)} \mathbf{B};\ \{\mathbf{c}^\top, \mathbf{M}_j \begin{pmatrix} \mathbf{c}^\top \mathbf{U}_0 \\ \mathbf{U} \end{pmatrix} + \mathbf{c}_j^\top \mathbf{V}_j^{(2)}, \mathbf{c}_j^\top\}_{j\in[n]};\ \underbrace{\mathbf{k} + \boxed{\mathbf{k}^{(2)}} + \mathbf{U}_0 \mathbf{d}, \mathbf{d}, \{\mathbf{V}_j^{(2)} \mathbf{d}\}_{j:x_j=1}}_{\text{P-SF } i\text{'th sk}}\Big)$$

where $\mathbf{U}_0 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{3k\times(k+1)}$, $\mathbf{V}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{A}_2^{\|})$, $\mathbf{U} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(\ell'-1)\times(k+1)}$, and for any $\mathbf{A}_1, \mathbf{k}, \mathbf{c}, \mathbf{c}_j, \mathbf{k}^{(2)}$, and $\mathbf{d} \notin \mathsf{span}(\mathbf{B})$. It is straight-forward to compute the remaining terms in mpk, the challenge ciphertext and the Q secret keys by sampling $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1, \mathbf{U}_j^{(2)}, \mathbf{d}_j$ ourselves.

9 KP-ABE for Arithmetic Span Programs

In this section, we present our adaptively secure, unbounded KP-ABE for arithmetic span programs (ASPs) based on the k-Lin assumption in prime-order groups. Arithmetic span programs capture both boolean as well as arithmetic formula and branching programs [15].

9.1 Definitions

We define arithmetic span programs:

Definition 2 (arithmetic span program [15]). *An arithmetic span program $(\mathcal{V}, \rho)$ is a collection of row vectors $\mathcal{V} = \{(\mathbf{y}_j, \mathbf{z}_j) : j \in [\ell]\}$ in $\mathbb{Z}_p^{\ell'}$ and $\rho : [\ell] \to [n]$. We say that*

$$\mathbf{x} \in \mathbb{Z}_p^n \text{ satisfies } (\mathcal{V}, \rho) \text{ iff } \mathbf{1} \in \mathsf{span}\langle \mathbf{y}_j + x_{\rho(j)}\mathbf{z}_j \rangle,$$

where $\mathbf{1} := (1, 0, \ldots, 0)^\top \in \mathbb{Z}_p^{\ell'}$ and span refers to linear span of a collection of row vectors.

That is, $\mathbf{x}$ satisfies $(\mathcal{V}, \rho)$ iff there exists constants $\omega_1, \ldots, \omega_\ell \in \mathbb{Z}_p$ such that

$$\sum_{j=1}^{\ell} \omega_j (\mathbf{y}_j + x_{\rho(j)}\mathbf{z}_j) = \mathbf{1}. \tag{16}$$

Like in prior works [6], we need to impose a one-use restriction, that is, ρ is a permutation and $\ell = n$. By re-ordering the coordinates in $\mathcal{V}$, we may assume WLOG that ρ is the identity map, which we omit in the rest of this section.

9.2 Warm-Up: a Composite-Order Scheme

As a warm-up, we begin with a composite-order candidate. We build upon the "bounded" KP-ABE scheme for arithmetic span programs in [15, 6]. Roughly speaking, we start with our unbounded KP-ABE for monotone span programs in Section 6 and perform the following substitutions "in the exponent":

$$\begin{aligned}
\mathsf{mpk}: &\quad w, w_0, w_1 \mapsto (w, w_0, w_1),\ (w', w_0', w_1') \\
\mathsf{ct}: &\quad x_j s w \mapsto s(w + x_j w') \\
&\quad s_j,\ x_j s_j (w_0 + j\cdot w_1) \mapsto s_j,\ s_j',\ s_j(w_0 + j\cdot w_1) + x_j s_j'(w_0' + j\cdot w_1') \\
\mathsf{sk}: &\quad \mathbf{M}_j \binom{\alpha}{\mathbf{u}} + r_j w \mapsto \mathbf{y}_j \binom{\alpha}{\mathbf{u}} + r_j w,\ \mathbf{z}_j \binom{\alpha}{\mathbf{u}} + r_j w'
\end{aligned}$$

This yields the following scheme:

$$\begin{aligned}
\mathsf{mpk} &:= (g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}, g_1^{w'}, g_1^{w_0'}, g_1^{w_1'}, e(g_1, h_{123})^{\alpha}) \qquad (17) \\
\mathsf{ct}_{\mathbf{x}} &:= (g_1^s, \{g_1^{s(w+x_j w') + s_j(w_0 + j w_1) + x_j s_j'(w_0' + j w_1')}, g_1^{s_j}, g_1^{s_j'}\}_{j\in[n]}, e(g_1, h_{123})^{\alpha s}\cdot m) \\
\mathsf{sk}_{\mathcal{V}} &:= (\{h_{123}^{\mathbf{y}_j\binom{\alpha}{\mathbf{u}}} \cdot h_1^{r_j w}, h_{123}^{\mathbf{z}_j\binom{\alpha}{\mathbf{u}}} \cdot h_1^{r_j w'}, h_1^{r_j}, h_1^{r_j(w_0 + j w_1)}, h_1^{r_j(w_0' + j w_1')}\}_{j\in[n]})
\end{aligned}$$

To analyze this scheme, we would require the following extension to our basic entropy expansion lemma, Lemma 2 (cf. Section 3.2), which essentially involves two parallel instances in Lemma 2, with respect to parameters (w, w_0, w_1) and (w', w'_0, w'_1) respectively, bound together via common random coins s, r_j:

$$\begin{Bmatrix} \mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}, g_1^{w'}, g_1^{w'_0}, g_1^{w'_1} \\ \mathsf{ct}: g_1^{s},\ \{g_1^{sw+s_j(w_0+j\cdot w_1)},\ g_1^{s_j},\ g_1^{sw'+s'_j(w'_0+j\cdot w'_1)},\ g_1^{s'_j}\}_{j\in[n]} \\ \mathsf{sk}: \{h_1^{r_j w},\ h_1^{r_j w'},\ h_1^{r_j},\ h_1^{r_j(w_0+j\cdot w_1)},\ h_1^{r_j(w'_0+j\cdot w'_1)}\}_{j\in[n]} \end{Bmatrix}$$
$$\approx_c \begin{Bmatrix} \mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}, g_1^{w'}, g_1^{w'_0}, g_1^{w'_1} \\ \mathsf{ct}: g_1^{s}\cdot \boxed{g_2^{s}},\ \{g_1^{sw+s_j(w_0+j\cdot w_1)}\cdot \boxed{g_2^{sv_j+s_ju_j}},\ g_1^{s_j}\cdot\boxed{g_2^{s_j}},\ g_1^{sw'+s'_j(w'_0+j\cdot w'_1)}\cdot\boxed{g_2^{sv'_j+s'_ju'_j}},\ g_1^{s'_j}\cdot\boxed{g_2^{s'_j}}\}_{j\in[n]} \\ \mathsf{sk}: \{h_1^{r_j w}\cdot\boxed{h_2^{r_jv_j}},\ h_1^{r_jw'}\cdot\boxed{h_2^{r_jv'_j}},\ h_1^{r_j}\cdot\boxed{h_2^{r_j}},\ h_1^{r_j(w_0+j\cdot w_1)}\cdot\boxed{h_2^{r_ju_j}},\ h_1^{r_j(w'_0+j\cdot w'_1)}\cdot\boxed{h_2^{r_ju'_j}}\}_{j\in[n]} \end{Bmatrix} \quad (18)$$

where

$$w, w_0, w_1, w', w'_0, w'_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ v_j, u_j, v'_j, u'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ s, s_j, s'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N.$$

We can justify (18) following the hybrid arguement in Section 3.2. See Appendix F for details. However, we will rely on a parallel version of Lemma 3 described as follows for step one:

$$\begin{Bmatrix} \mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1}, g_1^{w'_0}, g_1^{w'_1}, g_2 \\ \mathsf{ct}: \{g_1^{s_j(w_0+j\cdot w_1)},\ g_1^{s_j},\ g_1^{s'_j(w'_0+j\cdot w'_1)},\ g_1^{s'_j}\}_{j\in[n]} \\ \mathsf{sk}: \{h_{123}^{r_j},\ h_{123}^{r_j(w_0+j\cdot w_1)},\ h_{123}^{r_j(w'_0+j\cdot w'_1)}\}_{j\in[n]} \end{Bmatrix}$$
$$\approx_c \begin{Bmatrix} \mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1}, g_1^{w'_0}, g_1^{w'_1}, g_2 \\ \mathsf{ct}: \{g_1^{s_j(w_0+j\cdot w_1)}\cdot\boxed{g_2^{s_ju_j}},\ g_1^{s_j}\cdot\boxed{g_2^{s_j}},\ g_1^{s'_j(w'_0+j\cdot w'_1)}\cdot\boxed{g_2^{s'_ju'_j}},\ g_1^{s'_j}\cdot\boxed{g_2^{s'_j}}\}_{j\in[n]} \\ \mathsf{sk}: \{h_{12}^{r_j}\cdot\boxed{h_2^{r_j}},\ h_{12}^{r_j(w_0+j\cdot w_1)}\cdot\boxed{h_2^{r_ju_j}},\ h_{12}^{r_j(w'_0+j\cdot w'_1)}\cdot\boxed{h_2^{r_ju'_j}}\}_{j\in[n]} \end{Bmatrix} \quad (19)$$

where

$$w_0, w_1, w'_0, w'_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ u_j, u'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ s_j, s'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,$$

and a similar extension of Lemma 4 described as follows for step two:

$$\begin{Bmatrix} \mathsf{aux}: g_1, g_1^{w}, g_1^{w'}, h_1, h_1^{w}, h_1^{w'} \\ \mathsf{ct}: g_2^{s},\ \{g_2^{sw}\cdot g_2^{s_ju_j},\ g_2^{s_j},\ g_2^{sw'}\cdot g_2^{s'_ju'_j},\ g_2^{s'_j}\}_{j\in[n]} \\ \mathsf{sk}: \{h_2^{r_jw},\ h_2^{r_jw'},\ h_2^{r_j},\ h_2^{r_ju_j},\ h_2^{r_ju'_j}\}_{j\in[n]} \end{Bmatrix} \approx_c \begin{Bmatrix} \mathsf{aux}: g_1, g_1^{w}, g_1^{w'}, h_1, h_1^{w}, h_1^{w'} \\ \mathsf{ct}: g_2^{s},\ \{\boxed{g_2^{sv_j}}\cdot g_2^{s_ju_j},\ g_2^{s_j},\ \boxed{g_2^{sv'_j}}\cdot g_2^{s'_ju'_j},\ g_2^{s'_j}\}_{j\in[n]} \\ \mathsf{sk}: \{\boxed{h_2^{r_jv_j}},\ \boxed{h_2^{r_jv'_j}},\ h_2^{r_j},\ h_2^{r_ju_j},\ h_2^{r_ju'_j}\}_{j\in[n]} \end{Bmatrix} \quad (20)$$

where

$$w, w' \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ v_j, u_j, v'_j, u'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ s, s_j, s'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N,\ r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N.$$

9.3 Our Prime-Order Construction

We present our scheme in prime-order groups:

$\mathsf{Setup}(1^\lambda, 1^n)$: On input $(1^\lambda, 1^n)$, sample

$$\mathbf{A}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times k}, \mathbf{B} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(k+1)\times k},\ \mathbf{W}, \mathbf{W}_0, \mathbf{W}_1, \mathbf{W}', \mathbf{W}'_0, \mathbf{W}'_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(k+1)},\ \mathbf{k} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{2k+1}$$

and output the master public and secret key pair

$$\mathsf{mpk} := \big(\, [\mathbf{A}_1^\top, \mathbf{A}_1^\top\mathbf{W}, \mathbf{A}_1^\top\mathbf{W}_0, \mathbf{A}_1^\top\mathbf{W}_1, \mathbf{A}_1^\top\mathbf{W}', \mathbf{A}_1^\top\mathbf{W}'_0, \mathbf{A}_1^\top\mathbf{W}'_1]_1,\ e([\mathbf{A}_1^\top]_1, [\mathbf{k}]_2)\, \big) \in G_1^{k\times(2k+1)} \times (G_1^{k\times(k+1)})^6 \times G_T^k$$

and

$$\mathsf{msk} := \big(\, \mathbf{k},\, \mathbf{B},\, \mathbf{W},\, \mathbf{W}_0,\, \mathbf{W}_1,\, \mathbf{W}',\, \mathbf{W}'_0,\, \mathbf{W}'_1 \,\big).$$

$\mathsf{Enc}(\mathsf{mpk}, \mathbf{x}, m)$: On input an attribute vector $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ and $m \in G_T$, pick $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ for all $j \in [n]$ and output

$$\mathsf{ct}_{\mathbf{x}} := \left(\begin{array}{c} C_0 := [\mathbf{c}^\top]_1, \left\{ C_{1,j} := [\mathbf{c}^\top(\mathbf{W} + x_j \cdot \mathbf{W}') + \mathbf{c}_j^\top(\mathbf{W}_0 + j \cdot \mathbf{W}_1) + x_j \cdot {\mathbf{c}'_j}^\top(\mathbf{W}'_0 + j \cdot \mathbf{W}'_1)]_1, \begin{array}{c} C_{2,j} := [\mathbf{c}_j^\top]_1, \\ C'_{2,j} := [{\mathbf{c}'_j}^\top]_1 \end{array} \right\}_{j \in [n]} \\ C := e([\mathbf{c}^\top]_1, [\mathbf{k}]_2) \cdot m \end{array} \right)$$

$$\in G_1^{2k+1} \times (G_1^{k+1} \times (G_1^{2k+1})^2)^n \times G_T.$$

$\mathsf{KeyGen}(\mathsf{mpk}, \mathsf{msk}, \mathcal{V})$: On input an arithmetic span program $\mathcal{V} = \{(\mathbf{y}_j, \mathbf{z}_j)\}_{j \in [n]}$, pick $\mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}$, $\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B})$ for all $j \in [n]$ and output

$$\mathsf{sk}_{\mathcal{V}} := \left(\left\{ \begin{array}{l} K_{0,j} := [(\mathbf{k}\|\mathbf{K}')\mathbf{y}_j^\top + \mathbf{W}\mathbf{d}_j]_2, \\ K'_{0,j} := [(\mathbf{k}\|\mathbf{K}')\mathbf{z}_j^\top + \mathbf{W}'\mathbf{d}_j]_2, \end{array} K_{1,j} := [\mathbf{d}_j]_2, \begin{array}{l} K_{2,j} := [(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{d}_j]_2 \\ K'_{2,j} := [(\mathbf{W}'_0 + j \cdot \mathbf{W}'_1)\mathbf{d}_j]_2 \end{array} \right\}_{j \in [n]} \right)$$

$$\in ((G_2^{2k+1})^2 \times G_2^{k+1} \times (G_2^{2k+1})^2)^n.$$

$\mathsf{Dec}(\mathsf{mpk}, \mathsf{sk}_{\mathcal{V}}, \mathsf{ct}_{\mathbf{x}})$: If $\mathbf{x}$ satisfies $\mathcal{V}$, compute $\omega_1, \ldots, \omega_n \in \mathbb{Z}_p$ such that

$$\sum_{j \in [n]} \omega_j (\mathbf{y}_j + x_j \cdot \mathbf{z}_j) = \mathbf{1}.$$

Then, compute

$$K \leftarrow \prod_{j \in [n]} \Big(e(C_0, K_{0,j} \cdot (K'_{0,j})^{x_j}) \cdot e(C_{1,j}, K_{1,j})^{-1} \cdot e(C_{2,j}, K_{2,j}) \cdot e(C'_{2,j}, (K'_{2,j})^{x_j}) \Big)^{\omega_j},$$

and recover the message as $m \leftarrow C/K \in G_T$.

Correctness. For all $\mathcal{V}$ and $\mathbf{x}$ such that $\mathbf{x}$ satisfies $\mathcal{V}$, we have

$$\begin{aligned} & e(C_0, K_{0,j} \cdot (K'_{0,j})^{x_j}) \cdot e(C_{1,j}, K_{1,j})^{-1} \cdot e(C_{2,j}, K_{2,j}) \cdot e(C'_{2,j}, (K'_{2,j})^{x_j}) \\ =& e([\mathbf{c}^\top]_1, [(\mathbf{k}\|\mathbf{K}')(\mathbf{y}_j^\top + x_j \cdot \mathbf{z}_j^\top) + (\mathbf{W} + x_j \cdot \mathbf{W}')\mathbf{d}_j]_2) \\ & \cdot e([\mathbf{c}^\top(\mathbf{W} + x_j \cdot \mathbf{W}') + \mathbf{c}_j^\top(\mathbf{W}_0 + j \cdot \mathbf{W}_1) + x_j \cdot {\mathbf{c}'_j}^\top(\mathbf{W}'_0 + j \cdot \mathbf{W}'_1)]_1, [\mathbf{d}_j]_2)^{-1} \\ & \cdot e([\mathbf{c}_j^\top]_1, [(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{d}_j]_2) \cdot e([{\mathbf{c}'_j}^\top]_1, [x_j \cdot (\mathbf{W}'_0 + j \cdot \mathbf{W}'_1)\mathbf{d}_j]_2) \\ =& \left[\begin{array}{l} \mathbf{c}^\top(\mathbf{k}\|\mathbf{K}')(\mathbf{y}_j^\top + x_j \cdot \mathbf{z}_j^\top) + \mathbf{c}^\top(\mathbf{W} + x_j \cdot \mathbf{W}')\mathbf{d}_j \\ -\mathbf{c}^\top(\mathbf{W} + x_j \cdot \mathbf{W}')\mathbf{d}_j - \mathbf{c}_j^\top(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{d}_j - x_j \cdot {\mathbf{c}'_j}^\top(\mathbf{W}'_0 + j \cdot \mathbf{W}'_1)\mathbf{d}_j \\ \mathbf{c}_j^\top(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{d}_j + x_j \cdot {\mathbf{c}'_j}^\top(\mathbf{W}'_0 + j \cdot \mathbf{W}'_1)\mathbf{d}_j \end{array} \right]_T = [\mathbf{c}^\top(\mathbf{k}\|\mathbf{K}')(\mathbf{y}_j^\top + x_j \cdot \mathbf{z}_j^\top)]_T \end{aligned}$$

for each $j \in [n]$ and

$$K = \prod_{j \in [n]} \Big(e(C_0, K_{0,j} \cdot (K'_{0,j})^{x_j}) \cdot e(C_{1,j}, K_{1,j})^{-1} \cdot e(C_{2,j}, K_{2,j}) \cdot e(C'_{2,j}, (K'_{2,j})^{x_j}) \Big)^{\omega_j}$$

$$
\begin{aligned}
&= \prod_{j \in [n]} [\omega_j \mathbf{c}^\top (\mathbf{k} \| \mathbf{K}')(\mathbf{y}_j^\top + x_j \cdot \mathbf{z}_j^\top)]_T \\
&= [\mathbf{c}^\top (\mathbf{k} \| \mathbf{K}') \sum_{j \in [n]} \omega_j (\mathbf{y}_j^\top + x_j \cdot \mathbf{z}_j^\top)]_T \\
&= [\mathbf{c}^\top (\mathbf{k} \| \mathbf{K}') \mathbf{1}^\top]_T = [\mathbf{c}^\top \mathbf{k}]_T = e([\mathbf{c}^\top]_1, [\mathbf{k}]_2).
\end{aligned}
$$

This readily proves correctness.

9.4 Bilinear Entropy Expansion Lemma

To prove the adaptive security of our unbounded KP-ABE scheme for arithmetic span programs, we require the following variant of Lemma 8, the prime-order entropy expansion lemma for our unbounded KP-ABE for boolean span program (cf. Section 5).

Lemma 17 (entropy expansion lemma for ASPs). *Define bases $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^\|, \mathbf{A}_2^\|, \mathbf{A}_3^\|$ as in Section 4.2. Suppose $\ell_1, \ell_3, \ell_W \geq k$. Then, under the MDDH$_k$ assumption, we have*

$$
\left\{
\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_1^\top \mathbf{W}']_1, [\mathbf{A}_1^\top \mathbf{W}_0']_1, [\mathbf{A}_1^\top \mathbf{W}_1']_1 \\
\mathsf{ct}: [\mathbf{c}^\top]_1, \left\{ \begin{array}{ll} [\mathbf{c}^\top \mathbf{W} + \mathbf{c}_j^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1)]_1, & [\mathbf{c}_j^\top]_1 \\ [\mathbf{c}^\top \mathbf{W}' + \mathbf{c}_j'^\top (\mathbf{W}_0' + j \cdot \mathbf{W}_1')]_1, & [\mathbf{c}_j'^\top]_1 \end{array} \right\}_{j \in [n]} \\
\mathsf{sk}: \left\{ \begin{array}{lll} [\mathbf{W}\mathbf{D}_j]_2, & & [(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{D}_j]_2 \\ & [\mathbf{D}_j]_2, & \\ [\mathbf{W}'\mathbf{D}_j]_2, & & [(\mathbf{W}_0' + j \cdot \mathbf{W}_1')\mathbf{D}_j]_2 \end{array} \right\}_{j \in [n]}
\end{array}
\right\}
$$

$$
\approx_c
\left\{
\begin{array}{l}
\mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_1^\top \mathbf{W}']_1, [\mathbf{A}_1^\top \mathbf{W}_0']_1, [\mathbf{A}_1^\top \mathbf{W}_1']_1 \\
\mathsf{ct}: [\boxed{\mathbf{c}}^\top]_1, \left\{ \begin{array}{ll} [\boxed{\mathbf{c}}^\top (\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}}) + \boxed{\mathbf{c}_j}^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1, & [\boxed{\mathbf{c}_j}^\top]_1 \\ [\boxed{\mathbf{c}}^\top (\mathbf{W}' + \boxed{\mathbf{V}_j'^{(2)}}) + \boxed{\mathbf{c}_j'}^\top (\mathbf{W}_0' + j \cdot \mathbf{W}_1' + \boxed{\mathbf{U}_j'^{(2)}})]_1, & [\boxed{\mathbf{c}_j'}^\top]_1 \end{array} \right\}_{j \in [n]} \\
\mathsf{sk}: \left\{ \begin{array}{lll} [(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})\mathbf{D}_j]_2, & & [(\mathbf{W}_0 + j \cdot \mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2 \\ & [\mathbf{D}_j]_2, & \\ [(\mathbf{W}' + \boxed{\mathbf{V}_j'^{(2)}})\mathbf{D}_j]_2, & & [(\mathbf{W}_0' + j \cdot \mathbf{W}_1' + \boxed{\mathbf{U}_j'^{(2)}})\mathbf{D}_j]_2 \end{array} \right\}_{j \in [n]}
\end{array}
\right\}
$$

where $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1, \mathbf{W}', \mathbf{W}_0', \mathbf{W}_1' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_W}$, $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)}, \mathbf{V}_j'^{(2)}, \mathbf{U}_j'^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^\|)$, $\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W \times \ell_W}$, and $\mathbf{c}, \mathbf{c}_j, \mathbf{c}_j' \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ in the left distribution while $\mathbf{c}, \mathbf{c}_j, \mathbf{c}_j' \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)$ in the right distribution.

One can prove Lemma 17 via a hybrid argument analogous to that in Section 5: the support of $\mathbf{c}$ is changed from $\mathsf{span}(\mathbf{A}_1)$ to $\mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)$, and then Lemma 18 (for step one) and Lemma 19 (for step two) described as follows are applied successively.

Lemma 18 (entropy expansion lemma for ASPs (step one)). *Suppose $\ell_1, \ell_3 \geq k$. Then, under the MDDH$_k$ assumption, we have*

$$\left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_1^\top \mathbf{W}_0']_1, [\mathbf{A}_1^\top \mathbf{W}_1']_1, [\mathbf{A}_2^\top] \\ \mathsf{ct}: \left\{\begin{array}{ll} [\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1, & [\mathbf{c}_j^\top]_1 \\ [\mathbf{c}_j'^\top(\mathbf{W}_0' + j\cdot\mathbf{W}_1')]_1, & [\mathbf{c}_j'^\top]_1 \end{array}\right\}_{j\in[n]} \\ \mathsf{sk}: \left\{ [\mathbf{D}_j]_2, \begin{array}{l} [(\mathbf{W}_0 + j\cdot\mathbf{W}_1)\mathbf{D}_j]_2 \\ [(\mathbf{W}_0' + j\cdot\mathbf{W}_1')\mathbf{D}_j]_2 \end{array}\right\}_{j\in[n]} \end{array}\right\}$$

$$\approx_c \left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_1^\top \mathbf{W}_0']_1, [\mathbf{A}_1^\top \mathbf{W}_1']_1, [\mathbf{A}_2^\top] \\ \mathsf{ct}: \left\{\begin{array}{ll} [\boxed{\mathbf{c}_j}^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1, & [\boxed{\mathbf{c}_j}^\top]_1 \\ [\boxed{\mathbf{c}_j'}^\top(\mathbf{W}_0' + j\cdot\mathbf{W}_1' + \boxed{\mathbf{U}_j'^{(2)}})]_1, & [\boxed{\mathbf{c}_j'}^\top]_1 \end{array}\right\}_{j\in[n]} \\ \mathsf{sk}: \left\{ [\mathbf{D}_j]_2, \begin{array}{l} [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2 \\ [(\mathbf{W}_0' + j\cdot\mathbf{W}_1' + \boxed{\mathbf{U}_j'^{(2)}})\mathbf{D}_j]_2 \end{array}\right\}_{j\in[n]} \end{array}\right\}$$

where $\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_0', \mathbf{W}_1' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$, $\mathbf{U}_j^{(2)}, \mathbf{U}_j'^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|})$, $\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$, *and* $\mathbf{c}_j, \mathbf{c}_j' \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ *in the left distribution while* $\mathbf{c}_j, \mathbf{c}_j' \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)$ *in the right distribution.*

Lemma 19 (entropy expansion lemma for ASPs (step two)). *Suppose $\ell_W \geq k$. Then, under the MDDH$_k$ assumption, we have*

$$\left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}']_1 \\ \mathsf{ct}: [\mathbf{c}^{(2)\top}]_1, \left\{\begin{array}{ll} [\mathbf{c}^{(2)\top}\mathbf{W} + \mathbf{c}_j^{(2)\top}\mathbf{U}_j^{(2)}]_1, & [\mathbf{c}_j^{(2)\top}]_1 \\ [\mathbf{c}^{(2)\top}\mathbf{W}' + \mathbf{c}_j'^{(2)\top}\mathbf{U}_j'^{(2)}]_1, & [\mathbf{c}_j'^{(2)\top}]_1 \end{array}\right\}_{j\in[n]} \\ \mathsf{sk}: \left\{\begin{array}{lll} [\mathbf{W}\mathbf{D}_j]_2, & [\mathbf{D}_j]_2, & [\mathbf{U}_j^{(2)}\mathbf{D}_j]_2 \\ [\mathbf{W}'\mathbf{D}_j]_2, & & [\mathbf{U}_j'^{(2)}\mathbf{D}_j]_2 \end{array}\right\}_{j\in[n]} \end{array}\right\}$$

$$\approx_c \left\{\begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}']_1 \\ \mathsf{ct}: [\mathbf{c}^{(2)\top}]_1, \left\{\begin{array}{ll} [\mathbf{c}^{(2)\top}(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}}) + \mathbf{c}_j^{(2)\top}\mathbf{U}_j^{(2)}]_1, & [\mathbf{c}_j^{(2)\top}]_1 \\ [\mathbf{c}^{(2)\top}(\mathbf{W}' + \boxed{\mathbf{V}_j'^{(2)}}) + \mathbf{c}_j'^{(2)\top}\mathbf{U}_j'^{(2)}]_1, & [\mathbf{c}_j'^{(2)\top}]_1 \end{array}\right\}_{j\in[n]} \\ \mathsf{sk}: \left\{\begin{array}{lll} [(\mathbf{W} + \boxed{\mathbf{V}_j^{(2)}})\mathbf{D}_j]_2, & [\mathbf{D}_j]_2, & [\mathbf{U}_j^{(2)}\mathbf{D}_j]_2 \\ [(\mathbf{W}' + \boxed{\mathbf{V}_j'^{(2)}})\mathbf{D}_j]_2, & & [\mathbf{U}_j'^{(2)}\mathbf{D}_j]_2 \end{array}\right\}_{j\in[n]} \end{array}\right\}$$

where $\mathbf{W}, \mathbf{W}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$, $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)}, \mathbf{V}_j'^{(2)}, \mathbf{U}_j'^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|})$, $\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$, *and* $\mathbf{c}^{(2)}, \mathbf{c}_j^{(2)}, \mathbf{c}_j'^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_2)$.

Proving Lemma 18 and Lemma 19. Before we proceed, we develop a parallel variant of the $\mathrm{DDH}_{\mathbf{A}_1}^{G_2}$ assumption (cf. Section 4), which is denoted by $\mathrm{pDDH}_{\mathbf{A}_1}^{G_2}$.

Lemma 20 ($\mathbf{MDDH}^{2\ell_1}_{\ell_W,Q} \Rightarrow \mathbf{pDDH}^{G_2}_{\mathbf{A}_1}$)**.** *Fix $Q = poly(\lambda)$ with $Q > \ell_W \geq 1$. Under the MDDH$^{2\ell_1}_{\ell_W,Q}$ assumption in G_2, the following advantage function is negligible in λ*

$$\mathsf{Adv}_{\mathcal{A}}^{\mathrm{DDH}^{G_2}_{\mathbf{A}_1}}(\lambda) := |\Pr[\mathcal{A}(D, T_0 = 1] - \Pr[\mathcal{A}(D, T_1) = 1]|$$

where

$$D := (\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|};\ \mathbf{A}_2^\top \mathbf{W}, \mathbf{A}_3^\top \mathbf{W}, \mathbf{A}_2^\top \mathbf{W}', \mathbf{A}_3^\top \mathbf{W}'), \qquad \mathbf{W}, \mathbf{W}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_W};$$

$$T_0 := ([\mathbf{WD}]_2, [\mathbf{W}'\mathbf{D}]_2[\mathbf{D}]_2),\ T_1 := ([\mathbf{WD} + \mathbf{R}^{(1)}]_2, [\mathbf{W}'\mathbf{D} + \mathbf{R}'^{(1)}]_2, [\mathbf{D}]_2), \quad \mathbf{D} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W \times Q},\ \mathbf{R}^{(1)}, \mathbf{R}'^{(1)} \leftarrow_{\mathrm{R}} \mathsf{span}^Q(\mathbf{A}_1^{\|}).$$

Proof. We prove the lemma from

$$([\mathbf{D}]_2, [\mathbf{SD}]_2, [\mathbf{S}'\mathbf{D}]_2) \approx_c ([\mathbf{D}]_2, [\mathbf{SD} + \mathbf{U}]_2, [\mathbf{S}'\mathbf{D} + \mathbf{U}']_2)$$

where $\mathbf{D} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W \times Q}$, $\mathbf{S}, \mathbf{S}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1 \times \ell_W}$ and $\mathbf{U}, \mathbf{U}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1 \times Q}$, which is implied by the MDDH$^{2\ell_1}_{\ell_W,Q}$ assumption. On input $([\mathbf{D}]_2, [\mathbf{T}]_2, [\mathbf{T}']_2)$, algorithm $\mathcal{B}$ samples $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|}$, pick $\widetilde{\mathbf{W}}, \widetilde{\mathbf{W}}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell \times \ell_W}$, and implicitly set $\mathbf{W} := \widetilde{\mathbf{W}} + \mathbf{A}_1^{\|}\mathbf{S}$, $\mathbf{W}' := \widetilde{\mathbf{W}}' + \mathbf{A}_1^{\|}\mathbf{S}'$. Output

$$\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|};\ \mathbf{A}_2^\top \widetilde{\mathbf{W}}, \mathbf{A}_3^\top \widetilde{\mathbf{W}}, \mathbf{A}_2^\top \widetilde{\mathbf{W}}', \mathbf{A}_3^\top \widetilde{\mathbf{W}}',$$
$$[\widetilde{\mathbf{W}}\mathbf{D} + \mathbf{A}_1^{\|}\mathbf{T}]_2, [\widetilde{\mathbf{W}}'\mathbf{D} + \mathbf{A}_1^{\|}\mathbf{T}']_2, [\mathbf{D}]_2.$$

Observe that when $\mathbf{T} = \mathbf{SD}$ and $\mathbf{T}' = \mathbf{S}'\mathbf{D}$, the output is identical to (D, T_0); and when $\mathbf{T} = \mathbf{SD} + \mathbf{U}$ and $\mathbf{T}' = \mathbf{S}'\mathbf{D} + \mathbf{U}'$, the output is identical to (D, T_1) if we set $\mathbf{R}^{(1)} := \mathbf{A}_1^{\|}\mathbf{U}$ and $\mathbf{R}'^{(1)} := \mathbf{A}_1^{\|}\mathbf{U}'$. This readily proves the lemma. □

It is not hard to see that the proof of Lemma 19 is completely analogous to that of Lemma 10 using the pDDH$^{G_2}_{\mathbf{A}_2}$ assumption instead of the basic DDH$^{G_2}_{\mathbf{A}_2}$ assumption, while Lemma 18 can be proved using the following game sequence, which is analogous to that shown in Section 5.1 for proving Lemma 9.

<u>Game$_0$.</u> The adversary $\mathcal{A}$ is given the left distribution in Lemma 18.

$$\left\{ \begin{array}{l} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_1^\top \mathbf{W}_0']_1, [\mathbf{A}_1^\top \mathbf{W}_1']_1, [\mathbf{A}_2^\top] \\ \mathsf{ct}: \left\{ \begin{array}{ll} [\mathbf{c}_j^\top (\mathbf{W}_0 + j \cdot \mathbf{W}_1)]_1, & [\mathbf{c}_j^\top]_1 \\ [\mathbf{c}_j'^\top (\mathbf{W}_0' + j \cdot \mathbf{W}_1')]_1, & [\mathbf{c}_j'^\top]_1 \end{array} \right\}_{j \in [n]} \\ \mathsf{sk}: \left\{ [\mathbf{D}_j]_2, \begin{array}{l} [(\mathbf{W}_0 + j \cdot \mathbf{W}_1)\mathbf{D}_j]_2 \\ [(\mathbf{W}_0' + j \cdot \mathbf{W}_1')\mathbf{D}_j]_2 \end{array} \right\}_{j \in [n]} \end{array} \right\}.$$

<u>Game$_{0'}$.</u> We modify the distribution of sk as follows:

$$\mathsf{sk}: \left\{ [\mathbf{D}_j]_2, \begin{array}{l} [(\mathbf{W}_0 + j \cdot \mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2 \\ [(\mathbf{W}_0' + j \cdot \mathbf{W}_1' + \boxed{\mathbf{U}_j'^{(2)}})\mathbf{D}_j]_2 \end{array} \right\}_{j \in [n]}$$

where $\mathbf{U}_1^{(2)}, \ldots, \mathbf{U}_n^{(2)}, \mathbf{U}_1'^{(2)}, \ldots, \mathbf{U}_n'^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|})$. We claim that $\mathsf{Game}_0 \approx_c \mathsf{Game}_{0'}$. This follows from the pDDH$^{G_2}_{\mathbf{A}_2}$ assumption which tells us that

$$\left\{ [\mathbf{D}_j]_2,\ [\mathbf{W}_0\mathbf{D}_j]_2,\ [\mathbf{W}_0'\mathbf{D}_j]_2 \right\}_{j \in [n]} \approx_c \left\{ [\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + \mathbf{U}_j^{(2)})\mathbf{D}_j]_2,\ [(\mathbf{W}_0' + \mathbf{U}_j'^{(2)})\mathbf{D}_j]_2 \right\}_{j \in [n]}$$

given $\mathbf{A}_1^\top, \mathbf{A}_1^\top \mathbf{W}_0, \mathbf{A}_1^\top \mathbf{W}_0'$. The proof is analogous to that of Lemma 30.

Game$_i$ $(i = 1,\dots,n+1)$. We change the distribution of ct:

$$\mathsf{ct}: \left\{ \left\{ \begin{array}{l} [\boxed{\mathbf{c}_j}^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1,\ [\boxed{\mathbf{c}_j}^\top]_1 \\ [\boxed{\mathbf{c}'_j}^\top(\mathbf{W}'_0 + j\cdot\mathbf{W}'_1 + \boxed{\mathbf{U}_j'^{(2)}})]_1,\ [\boxed{\mathbf{c}'_j}^\top]_1 \end{array} \right\}_{j<i}, \left\{ \begin{array}{l} [\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1,\ [\mathbf{c}_j^\top]_1 \\ [\mathbf{c}_j'^\top(\mathbf{W}'_0 + j\cdot\mathbf{W}'_1)]_1,\ [\mathbf{c}_j'^\top]_1 \end{array} \right\}_{j\geq i} \right\}$$

where $\boxed{\mathbf{c}_j, \mathbf{c}'_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)}$ for $j < i$ and $\mathbf{c}_j, \mathbf{c}'_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ for all remaining $j \geq i$. It is easy to see that Game$_{0'}$ $\equiv$ Game$_1$. To show that Game$_i$ $\approx_c$ Game$_{i+1}$, we will require another sequence of sub-games.

Sub-Game$_{i,1}$. Identical to Game$_i$ except that we modify ct$_i$ as follows:

$$\mathsf{ct}_i: \begin{array}{l} [\boxed{\mathbf{c}_i}^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1)]_1,\ [\boxed{\mathbf{c}_i}^\top]_1 \\ [\boxed{\mathbf{c}'_i}^\top(\mathbf{W}'_0 + i\cdot\mathbf{W}'_1)]_1,\ [\boxed{\mathbf{c}'_i}^\top]_1 \end{array}$$

where $\boxed{\mathbf{c}_i, \mathbf{c}'_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_3)}$. We claim that Game$_i$ $\approx_c$ Sub-Game$_{i,1}$. This follows from

$$[\mathbf{c}_i, \mathbf{c}'_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)]_1 \approx_c [\mathbf{c}_i, \mathbf{c}'_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_3)]_1 \text{ given } [\mathbf{A}_1, \mathbf{A}_2]_1, \mathsf{basis}(\mathbf{A}_2^{\|})$$

which is ensured by the $\mathsf{SD}^{G_1}_{\mathbf{A}_1 \mapsto \mathbf{A}_1, \mathbf{A}_3}$ assumption. The proof is analogous to that of Lemma 31.

Sub-Game$_{i,2}$. We modify the distributions of all sk$_j$ with $j \neq i$ (while keeping sk$_i$ unchanged):

$$\mathsf{sk}_j (j \neq i): [\mathbf{D}_j]_2, \begin{array}{l} [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)} + \boxed{\mathbf{U}_j^{(3)}})\mathbf{D}_j]_2 \\ [(\mathbf{W}'_0 + j\cdot\mathbf{W}'_1 + \mathbf{U}_j'^{(2)} + \boxed{\mathbf{U}_j'^{(3)}})\mathbf{D}_j]_2 \end{array}$$

where $\mathbf{U}_j^{(3)}, \mathbf{U}_j'^{(3)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_3^{\|})$. We claim that Sub-Game$_{i,1}$ $\approx_c$ Sub-Game$_{i,2}$. This follows from $\mathsf{pDDH}^{G_2}_{\mathbf{A}_3}$, which tells us that

$$\left\{ [\mathbf{D}_j]_2, [\mathbf{W}_1\mathbf{D}_j]_2, [\mathbf{W}'_1\mathbf{D}_j]_2 \right\}_{j\neq i} \approx_c \left\{ [\mathbf{D}_j]_2, [(\mathbf{W}_1 + \widetilde{\mathbf{U}}_j^{(3)})\mathbf{D}_j]_2, [(\mathbf{W}'_1 + \widetilde{\mathbf{U}}_j'^{(3)})\mathbf{D}_j]_2 \right\}_{j\neq i}$$

given $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_2^{\|}, \mathbf{A}_1^\top\mathbf{W}_1, \mathbf{A}_2^\top\mathbf{W}_1, \mathbf{A}_1^\top\mathbf{W}'_1, \mathbf{A}_2^\top\mathbf{W}'_1$, where $\widetilde{\mathbf{U}}_j^{(3)}, \widetilde{\mathbf{U}}_j'^{(3)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_3^{\|})$. The proof is analogous to that of Lemma 32.

Sub-Game$_{i,3}$. We modify the distributions of ct$_i$ and sk$_i$:

$$\mathsf{ct}_i: \left\{ \begin{array}{l} [\mathbf{c}_i^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_i^{(2)} + \mathbf{U}_i^{(3)}})]_1,\quad [\mathbf{c}_i^\top]_1 \\ [\mathbf{c}_i'^\top(\mathbf{W}'_0 + i\cdot\mathbf{W}'_1 + \boxed{\mathbf{U}_i'^{(2)} + \mathbf{U}_i'^{(3)}})]_1,\ [\mathbf{c}_i'^\top]_1 \end{array} \right\}$$

$$\mathsf{sk}_i: \left\{ [\mathbf{D}_i]_2, \begin{array}{l} [(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)} + \boxed{\mathbf{U}_i^{(3)}})\mathbf{D}_i]_2 \\ [(\mathbf{W}'_0 + i\cdot\mathbf{W}'_1 + \mathbf{U}_i'^{(2)} + \boxed{\mathbf{U}_i'^{(3)}})\mathbf{D}_i]_2 \end{array} \right\}$$

where $\mathbf{U}_i^{(3)}, \mathbf{U}_i'^{(3)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_3^{\|})$. We claim that Sub-Game$_{i,2}$ $\equiv$ Sub-Game$_{i,3}$. This follows from the following statistical argument:

$$\begin{pmatrix} \mathbf{A}_1^\top\mathbf{W}_0, \mathbf{A}_2^\top\mathbf{W}_0\ \mathbf{W}_0, \left\{\mathbf{W}_0 + \mathbf{U}_j^{(3)}\right\}_{j\neq i} \\ \mathbf{A}_1^\top\mathbf{W}'_0, \mathbf{A}_2^\top\mathbf{W}'_0\ \mathbf{W}'_0, \left\{\mathbf{W}'_0 + \mathbf{U}_j'^{(3)}\right\}_{j\neq i} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{A}_1^\top\mathbf{W}_0, \mathbf{A}_2^\top\mathbf{W}_0\ \mathbf{W}_0 + \boxed{\mathbf{U}_i^{(3)}}, \left\{\mathbf{W}_0 + \mathbf{U}_j^{(3)}\right\}_{j\neq i} \\ \mathbf{A}_1^\top\mathbf{W}'_0, \mathbf{A}_2^\top\mathbf{W}'_0\ \mathbf{W}'_0 + \boxed{\mathbf{U}_i'^{(3)}}, \left\{\mathbf{W}'_0 + \mathbf{U}_j'^{(3)}\right\}_{j\neq i} \end{pmatrix}$$

and $\mathbf{U}_i^{(2)}$, $\mathbf{U}_i'^{(2)}$ in ct$_i$ are introduced "for free". The proof is analogous to that of Lemma 33.

Sub-Game$_{i,4}$. We modify the distribution of ct_i:

$$\mathsf{ct}_i : \begin{array}{ll} [\boxed{\mathbf{c}_i}^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)} + \mathbf{U}_i^{(3)})]_1, & [\boxed{\mathbf{c}_i}^\top]_1 \\ [\boxed{\mathbf{c}'_i}^\top(\mathbf{W}'_0 + i\cdot\mathbf{W}'_1 + \mathbf{U}_i'^{(2)} + \mathbf{U}_i'^{(3)})]_1, & [\boxed{\mathbf{c}'_i}^\top]_1 \end{array} \quad \text{where } \boxed{\mathbf{c}_i, \mathbf{c}'_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)}.$$

We claim that Sub-Game$_{i,3} \approx_c$ Sub-Game$_{i,4}$. This follows from $\mathrm{SD}^{G_1}_{\mathbf{A}_3 \mapsto \mathbf{A}_3, \mathbf{A}_2}$ which tells us that

$$[\mathbf{c}_i, \mathbf{c}'_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_3)]_1 \approx_c [\mathbf{c}_i, \mathbf{c}'_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_3, \mathbf{A}_2)]_1 \text{ given } [\mathbf{A}_2, \mathbf{A}_1]_1, \mathsf{basis}(\mathbf{A}_2^{\parallel}, \mathbf{A}_3^{\parallel}).$$

The proof is analogous to that of Lemma 34, where we independently sample $(\mathbf{U}_j^{(2)} + \mathbf{U}_j^{(3)})$ and $(\mathbf{U}_j'^{(2)} + \mathbf{U}_j'^{(3)})$ using $\mathsf{basis}(\mathbf{A}_2^{\parallel}, \mathbf{A}_3^{\parallel})$ for all $j \in [n]$.

Sub-Game$_{i,5}$. We change the distributions of ct_i and sk_i:

$$\mathsf{ct}_i : \left\{ \begin{array}{ll} [\mathbf{c}_i^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)} + \cancel{\mathbf{U}_i^{(3)}})]_1, & [\mathbf{c}_i^\top]_1 \\ [\mathbf{c}_i'^\top(\mathbf{W}'_0 + i\cdot\mathbf{W}'_1 + \mathbf{U}_i'^{(2)} + \cancel{\mathbf{U}_i'^{(3)}})]_1, & [\mathbf{c}_i'^\top]_1 \end{array} \right\}$$
$$\mathsf{sk}_i : \left\{ [\mathbf{D}_i]_2, \begin{array}{l} [(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)} + \cancel{\mathbf{U}_i^{(3)}})\mathbf{D}_i]_2 \\ [(\mathbf{W}'_0 + i\cdot\mathbf{W}'_1 + \mathbf{U}_i'^{(2)} + \cancel{\mathbf{U}_i'^{(3)}})\mathbf{D}_i]_2 \end{array} \right\}$$

We claim that Sub-Game$_{i,4} \equiv$ Sub-Game$_{i,5}$. The proof is completely analogous to that of Sub-Game$_{i,2} \equiv$ Sub-Game$_{i,3}$.

Sub-Game$_{i,6}$. We change the distributions of all sk_j with $j \neq i$:

$$\mathsf{sk}_j (j \neq i) : [\mathbf{D}_j]_2, \begin{array}{l} [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)} + \cancel{\mathbf{U}_j^{(3)}})\mathbf{D}_j]_2 \\ [(\mathbf{W}'_0 + j\cdot\mathbf{W}'_1 + \mathbf{U}_j'^{(2)} + \cancel{\mathbf{U}_j'^{(3)}})\mathbf{D}_j]_2 \end{array}$$

We claim that Sub-Game$_{i,5} \approx_c$ Sub-Game$_{i,6}$. The proof is completely analogous to that of Sub-Game$_{i,1} \equiv$ Sub-Game$_{i,2}$.

Sub-Game$_{i,7}$. We change the distribution of ct_i:

$$\mathsf{ct}_i : \begin{array}{l} [\boxed{\mathbf{c}_i}^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(2)})]_1, \ [\boxed{\mathbf{c}_i}^\top]_1 \\ [\boxed{\mathbf{c}'_i}^\top(\mathbf{W}'_0 + i\cdot\mathbf{W}'_1 + \mathbf{U}_i'^{(2)})]_1, \ [\boxed{\mathbf{c}'_i}^\top]_1 \end{array}$$

where $\boxed{\mathbf{c}_i, \mathbf{c}'_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)}$. We claim that Sub-Game$_{i,6} \approx_c$ Sub-Game$_{i,7}$. The proof is completely analogous to that of Game$_i \equiv$ Sub-Game$_{i,1}$.

Game$_{n+1}$. In Game$_{n+1}$, we have:

$$\left\{ \begin{array}{l} \mathsf{aux} : [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1, [\mathbf{A}_1^\top\mathbf{W}'_0]_1, [\mathbf{A}_1^\top\mathbf{W}'_1]_1, [\mathbf{A}_2^\top] \\ \mathsf{ct} : \left\{ \begin{array}{ll} [\boxed{\mathbf{c}_j}^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})]_1, & [\boxed{\mathbf{c}_j}^\top]_1 \\ [\boxed{\mathbf{c}'_j}^\top(\mathbf{W}'_0 + j\cdot\mathbf{W}'_1 + \boxed{\mathbf{U}_j'^{(2)}})]_1, & [\boxed{\mathbf{c}'_j}^\top]_1 \end{array} \right\}_{j\in[n]} \\ \mathsf{sk} : \left\{ [\mathbf{D}_j]_2, \begin{array}{l} [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \boxed{\mathbf{U}_j^{(2)}})\mathbf{D}_j]_2 \\ [(\mathbf{W}'_0 + j\cdot\mathbf{W}'_1 + \boxed{\mathbf{U}_j'^{(2)}})\mathbf{D}_j]_2 \end{array} \right\}_{j\in[n]} \end{array} \right\}$$

where $\mathbf{c}_j, \mathbf{c}'_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2)$ and $\mathbf{U}_j^{(2)}, \mathbf{U}_j'^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2)$ for all $j \in [n]$. This is exactly the right distribution of Lemma 18.

9.5 Proof of Security

Theorem 4. *Under the MDDH_k assumption in prime-order groups (cf. Section 4.1), the unbounded KP-ABE scheme for arithmetic span programs described in this section (cf. Section 9.3) is adaptively secure (cf. Section 2.2).*

The proof is completely analogous to that of Theorem 2, using instead the entropy expansion lemma (Lemma 17) in this section, and the following statistical lemma:

Lemma 21 (statistical lemma [6, Appendix A.6]). *For any $\mathbf{x}$ that does not satisfy $\mathcal{V} = \{(\mathbf{y}_j, \mathbf{z}_j)\}_{j\in[n]}$, the distributions*

$$\Big(\big\{v_j + x_j v'_j\big\}_{j\in[n]},\ \big\{\mathbf{y}_j \left(\begin{smallmatrix}\alpha\\ \mathbf{u}\end{smallmatrix}\right) + r_j v_j,\ \mathbf{z}_j \left(\begin{smallmatrix}\alpha\\ \mathbf{u}\end{smallmatrix}\right) + r_j v'_j,\ r_j\big\}_{j\in[n]}\Big)$$

perfectly hide α, where the randomness is taken over $v_j, v'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p, \mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell'-1}$, and for any fixed $r_j \neq 0$.

The proof follows the same series of games as described in Fig. 8, and we simply state the auxiliary distributions here.

Auxiliary distributions. We define various forms of ciphertext (of attribute vector $\mathbf{x}$ and message m):

- Normal: Generated by Enc; in particular, $\mathbf{c}, \mathbf{c}_j, \mathbf{c}'_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$.
- E-normal: Same as a normal ciphertext except that $\mathbf{c}, \mathbf{c}_j, \mathbf{c}'_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{a}_2)$ and we use the substitution:

$$\begin{array}{l} \mathbf{W} \mapsto \widehat{\mathbf{V}}_j := \mathbf{W} + \mathbf{V}_j^{(2)} \\ \mathbf{W}' \mapsto \widehat{\mathbf{V}}'_j := \mathbf{W}' + \mathbf{V}_j'^{(2)} \end{array} \text{ in } j\text{'th component and } \begin{array}{l} \mathbf{W}_0 + j\cdot\mathbf{W}_1 \mapsto \widehat{\mathbf{U}}_j := \mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)} \\ \mathbf{W}'_0 + j\cdot\mathbf{W}'_1 \mapsto \widehat{\mathbf{U}}'_j := \mathbf{W}'_0 + j\cdot\mathbf{W}'_1 + \mathbf{U}_j'^{(2)} \end{array} \quad (21)$$

  where $\mathbf{U}_j^{(2)}, \mathbf{V}_j^{(2)}, \mathbf{U}_j'^{(2)}, \mathbf{V}_j'^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{a}_2^{\parallel})$. Concretely, an E-normal ciphertext is of the form

$$\mathsf{ct}_{\mathbf{x}} := \Big([\mathbf{c}^\top]_1,\ \big\{[\mathbf{c}^\top \boxed{\widehat{\mathbf{V}}_j} + x_j\cdot\mathbf{c}^\top \boxed{\widehat{\mathbf{V}}'_j} + \mathbf{c}_j^\top \boxed{\widehat{\mathbf{U}}_j} + x_j\cdot{\mathbf{c}'_j}^\top \boxed{\widehat{\mathbf{U}}'_j}]_1,\ [\mathbf{c}_j^\top]_1, [{\mathbf{c}'_j}^\top]_1\big\}_{j\in[n]},\ e([\mathbf{c}^\top]_1, [\mathbf{k}]_2)\cdot m\Big)$$

  where $\boxed{\mathbf{c}, \mathbf{c}_j, \mathbf{c}'_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{a}_2)}$.

Then we pick $\alpha \leftarrow_{\mathrm{R}} \mathbb{Z}_p$ and define various forms of key (for span program $\mathbf{M}$):

- Normal: Generated by KeyGen.
- E-normal: Same as a normal key except that we use the same substitution as in (21). Concretely, a E-normal key is of the form

$$\mathsf{sk}_{\mathcal{V}} := \left(\left\{\begin{array}{ll} [(\mathbf{k}\|\mathbf{K}')\mathbf{y}_j^\top + \boxed{\widehat{\mathbf{V}}_j}\mathbf{d}_j]_2, & [\boxed{\widehat{\mathbf{U}}_j}\mathbf{d}_j]_2 \\ & [\mathbf{d}_j]_2, \\ [(\mathbf{k}\|\mathbf{K}')\mathbf{z}_j^\top + \boxed{\widehat{\mathbf{V}}'_j}\mathbf{d}_j]_2, & [\boxed{\widehat{\mathbf{U}}'_j}\mathbf{d}_j]_2 \end{array}\right\}_{j\in[n]}\right) \quad \text{where } \mathbf{d}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B}),\ \mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}.$$

- P-normal: Sample $\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k+1}$ in an E-normal key. Concretely, a P-normal key is of the form

$$\mathsf{sk}_{\mathcal{V}} := \left(\left\{\begin{array}{ll} [(\mathbf{k}\|\mathbf{K}')\mathbf{y}_j^\top + \widehat{\mathbf{V}}_j\mathbf{d}_j]_2, & [\widehat{\mathbf{U}}_j\mathbf{d}_j]_2 \\ & [\mathbf{d}_j]_2, \\ [(\mathbf{k}\|\mathbf{K}')\mathbf{z}_j^\top + \widehat{\mathbf{V}}'_j\mathbf{d}_j]_2, & [\widehat{\mathbf{U}}'_j\mathbf{d}_j]_2 \end{array}\right\}_{j\in[n]}\right) \quad \text{where } \boxed{\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k+1}},\ \mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}.$$

– P-SF: Replace $\mathbf{k}$ with $\mathbf{k}+\alpha\mathbf{a}_2^{\|}$ in a P-normal key. Concretely, a P-SF key is of the form

$$\mathsf{sk}_{\mathcal{Y}} := \left(\left\{ \begin{array}{ll} [(\mathbf{k}+\boxed{\alpha\mathbf{a}_2^{\|}}\|\mathbf{K}')\mathbf{y}_j^\top + \widehat{\mathbf{V}}_j\mathbf{d}_j]_2, & [\widehat{\mathbf{U}}_j\mathbf{d}_j]_2 \\ & [\mathbf{d}_j]_2, \\ [(\mathbf{k}+\boxed{\alpha\mathbf{a}_2^{\|}}\|\mathbf{K}')\mathbf{z}_j^\top + \widehat{\mathbf{V}}'_j\mathbf{d}_j]_2, & [\widehat{\mathbf{U}}'_j\mathbf{d}_j]_2 \end{array} \right\}_{j\in[n]} \right) \quad \text{where } \mathbf{d}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k+1},\ \mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}.$$

– SF: Sample $\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B})$ in a P-SF key. Concretely, a SF key is of the form

$$\mathsf{sk}_{\mathcal{Y}} := \left(\left\{ \begin{array}{ll} [(\mathbf{k}+\alpha\mathbf{a}_2^{\|}\|\mathbf{K}')\mathbf{y}_j^\top + \widehat{\mathbf{V}}_j\mathbf{d}_j]_2, & [\widehat{\mathbf{U}}_j\mathbf{d}_j]_2 \\ & [\mathbf{d}_j]_2, \\ [(\mathbf{k}+\alpha\mathbf{a}_2^{\|}\|\mathbf{K}')\mathbf{z}_j^\top + \widehat{\mathbf{V}}'_j\mathbf{d}_j]_2, & [\widehat{\mathbf{U}}'_j\mathbf{d}_j]_2 \end{array} \right\}_{j\in[n]} \right) \text{where } \boxed{\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B})},\ \mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}.$$

The game sequence and claims follow Section 7.3. We simply provide the proofs of the claims that $\mathsf{Game}_0 \approx \mathsf{Game}_{0'}$ and $\mathsf{Game}_{i,1} \equiv \mathsf{Game}_{i,2}$: The first one follows from a revisited version of Lemma 17 (entropy expansion lemma for KP-ABE for ASPs), which will change the distributions of $\mathbf{D}_j$ (analogous to Lemma 12) and employ parameter setting $(\ell_1,\ell_2,\ell_3,\ell_W) = (k,k,k,k+1)$. This can be established from the MDDH_k assumption and Lemma 17; The second claim follows from Lemma 21 which ensures that for any $\mathbf{x}$ that does not satisfy $\mathbf{M}$,

$$\begin{aligned} &\Big(\overbrace{\mathbf{k},\alpha,\mathbf{B},\mathbf{V}_j^{(2)}\mathbf{B},\mathbf{V}_j'^{(2)}\mathbf{B}}^{\kappa\text{'th sk},\ \kappa\neq i};\ \overbrace{\{\mathbf{V}_j^{(2)}+x_j\cdot\mathbf{V}_j'^{(2)}\}_{j\in[n]}}^{\text{E-normal ct}},\ \overbrace{\{(\mathbf{k}\|\mathbf{K}')\mathbf{y}_j^\top+\mathbf{V}_j^{(2)}\mathbf{d}_j,(\mathbf{k}\|\mathbf{K}')\mathbf{z}_j^\top+\mathbf{V}_j'^{(2)}\mathbf{d}_j,\mathbf{d}_j\}_{j\in[n]}}^{\text{P-normal } i\text{'th sk}}\Big) \\ \equiv\ &\Big(\mathbf{k},\alpha,\mathbf{B},\mathbf{V}_j^{(2)}\mathbf{B},\mathbf{V}_j'^{(2)}\mathbf{B};\ \{\mathbf{V}_j^{(2)}+x_j\cdot\mathbf{V}_j'^{(2)}\}_{j\in[n]},\ \underbrace{\{(\mathbf{k}+\boxed{\alpha\mathbf{a}_2^{\|}}\|\mathbf{K}')\mathbf{y}_j^\top+\mathbf{V}_j^{(2)}\mathbf{d}_j,(\mathbf{k}+\boxed{\alpha\mathbf{a}_2^{\|}}\|\mathbf{K}')\mathbf{z}_j^\top+\mathbf{V}_j'^{(2)}\mathbf{d}_j,\mathbf{d}_j\}_{j\in[n]}}_{\text{P-SF } i\text{'th sk}}\Big) \end{aligned}$$

where $\mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}, \mathbf{V}_j^{(2)}, \mathbf{V}_j'^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{a}_2^{\|})$, and for any $\mathbf{k},\alpha,\mathbf{B}$ and $\mathbf{d}_j \notin \mathsf{span}(\mathbf{B})$. It is straightforward to compute the remaining terms in mpk, the challenge ciphertext and the Q secret keys by sampling $\mathbf{A}_1, \mathbf{W}, \mathbf{W}_0, \mathbf{W}_1, \mathbf{W}', \mathbf{W}'_0, \mathbf{W}'_1, \mathbf{U}_j^{(2)}, \mathbf{U}_j'^{(2)}, \mathbf{c}, \mathbf{c}_j, \mathbf{c}'_j, \mathbf{d}_j$ ourselves.

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Appendix

A Additional claims for Section 3.3

Fixing an adversary $\mathcal{A}$, we use Adv_{xx} to denote the advantage of $\mathcal{A}$ in Game_{xx}.

Lemma 22 ($\mathsf{Game}_0 \approx_c \mathsf{Game}_{0'}$)**.** *There exists* $\mathcal{B}_0$ *with* $\mathsf{Time}(\mathcal{A}) \approx \mathsf{Time}(\mathcal{B}_0)$ *such that*

$$|\mathsf{Adv}_0 - \mathsf{Adv}_{0'}| \le \mathsf{Adv}_{\mathcal{B}_0}^{\mathrm{DDH}_{p_2}^{H_N}}(\lambda).$$

Proof. This follows from

$$\{h_2^{r_j}, h_2^{r_j w_0}\}_{j\in[n]} \approx_c \{h_2^{r_j}, \boxed{h_2^{r_j u_j'}}\}_{j\in[n]} \text{ given } g_1, g_2, h_{13}$$

where $u_j' \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$ and h_{13} is a random generator of $H_{p_1p_3}$, which is implied by the $\mathrm{DDH}_{p_2}^{H_N}$ assumption. On input $\{h_2^{r_j}, T_j\}_{j\in[n]}$ along with g_1, g_2, h_{13}, algorithm $\mathcal{B}_0$ samples $\tilde{w}_0, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ as well as $s_j, \tilde{r}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$, and outputs

$$\left\{\begin{array}{l} \mathsf{aux}: g_1, g_1^{\tilde{w}_0}, g_1^{w_1}, g_2 \\ \quad \mathsf{ct}: \{g_1^{s_j(\tilde{w}_0 + j\cdot w_1)},\ g_1^{s_j}\}_{j\in[n]} \\ \quad \mathsf{sk}: \{h_{13}^{\tilde{r}_j}\cdot h_2^{r_j},\ h_{13}^{\tilde{r}_j(\tilde{w}_0+j\cdot w_1)}\cdot T_j \cdot (h_2^{r_j})^{j\cdot w_1}\}_{j\in[n]} \end{array}\right\}.$$

By the Chinese Remainder Theorem, we have that

$$(g_1^{w_0}, \{h_{13}^{r_j}, h_{13}^{r_j w_0}\}_{j\in[n]}, \{h_2^{r_j}, h_2^{r_j w_0}\}_{j\in[n]}) \equiv (g_1^{\tilde{w}_0}, \{h_{13}^{\tilde{r}_j}, h_{13}^{\tilde{r}_j \tilde{w}_0}\}_{j\in[n]}, \{h_2^{r_j}, h_2^{r_j w_0}\}_{j\in[n]})$$

where $r_j, w_0, \tilde{r}_j, \tilde{w}_0 \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. Observe that when $T_j = h_2^{r_j w_0}$ the output is identical to Game_0; and when $T_j = h_2^{r_j u_j'}$ and let $u_j = u_j' + j \cdot w_1$, the output is identical to $\mathsf{Game}_{0'}$. This readily proves the lemma. □

Lemma 23 ($\mathsf{Game}_i \approx_c \mathsf{Sub\text{-}Game}_{i,1}$)**.** *There exists* $\mathcal{B}_1$ *with* $\mathsf{Time}(\mathcal{A}) \approx \mathsf{Time}(\mathcal{B}_1)$ *such that*

$$|\mathsf{Adv}_i - \mathsf{Adv}_{i,1}| \le \mathsf{Adv}_{\mathcal{B}_1}^{\mathrm{SD}_{p_1\mapsto p_1p_3}^{G_N}}(\lambda).$$

Proof. Recall that the $\mathrm{SD}_{p_1\mapsto p_1p_3}^{G_N}$ assumption asserts that

$$g_1^{s_i} \approx_c g_1^{s_i}\cdot \boxed{g_3^{s_i}} \text{ given } g_1, g_2, h_{13}, h_2.$$

On input T along with g_1, g_2, h_{13}, h_2, algorithm $\mathcal{B}_1$ samples $w_0, w_1, u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$ as well as $s_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for $j \in [n] \setminus \{i\}$, and outputs

$$\left\{\begin{array}{l} \mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1}, g_2 \\ \quad \mathsf{ct}: \{g_1^{s_j(w_0+j\cdot w_1)}\cdot g_2^{s_j u_j},\ g_1^{s_j}\cdot g_2^{s_j}\}_{j<i} \\ \qquad\quad \{T^{w_0+i\cdot w_1},\ T\} \\ \qquad\quad \{g_1^{s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j>i} \\ \quad \mathsf{sk}: \{h_{13}^{r_j}\cdot h_2^{r_j},\ h_{13}^{r_j(w_0+j\cdot w_1)}\cdot h_2^{r_j u_j}\}_{j\in[n]} \end{array}\right\}.$$

Observe that when $T = g_1^{s_i}$ the output is identical to Game_i; and when $T = g_1^{s_i}\cdot g_3^{s_i}$, the output is identical to $\mathsf{Sub\text{-}Game}_{i,1}$. This readily proves the lemma. □

Lemma 24 (Sub-Game$_{i,1} \approx_c$ Sub-Game$_{i,2}$). *There exists* $\mathcal{B}_2$ *with* $\mathsf{Time}(\mathcal{A}) \approx \mathsf{Time}(\mathcal{B}_2)$ *such that*

$$|\mathsf{Adv}_{i,1} - \mathsf{Adv}_{i,2}| \le \mathsf{Adv}_{\mathcal{B}_2}^{\mathrm{DDH}_{p_3}^{H_N}}(\lambda).$$

Proof. This follows from

$$\{h_3^{r_j}, h_3^{r_j w_1}\}_{j\neq i} \approx_c \{h_3^{r_j}, \boxed{h_3^{r_j u'_j}}\}_{j\neq i} \text{ given } g_1, g_2, g_3, h_1, h_2, h_3,$$

where $u'_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$, which is implied by the $\mathrm{DDH}_{p_3}^{H_N}$ assumption. On input $\{h_3^{r_j}, T_j\}_{j\neq i}$, algorithm $\mathcal{B}_1$ samples $\tilde{w}_0, \tilde{w}_1, u_j, s_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$ and programs

$$w_0 := \tilde{w}_0 \bmod p_1 p_2,\ w_0 := \tilde{w}_0 - i \cdot w_1 \bmod p_3,$$

and outputs

$$\left\{\begin{array}{l} \mathsf{aux}: g_1, g_1^{\tilde{w}_0}, g_1^{\tilde{w}_1}, g_2 \\ \quad \mathsf{ct}: \{g_1^{s_j(\tilde{w}_0 + j\cdot\tilde{w}_1)} \cdot g_2^{s_j u_j},\ g_1^{s_j} \cdot g_2^{s_j}\}_{j<i} \\ \qquad \{g_1^{s_i(\tilde{w}_0 + i\cdot\tilde{w}_1)} \cdot g_3^{s_i \tilde{w}_0},\ g_1^{s_i} \cdot g_3^{s_i}\} \\ \qquad \{g_1^{s_j(\tilde{w}_0 + j\cdot\tilde{w}_1)},\ g_1^{s_j}\}_{j>i} \\ \quad \mathsf{sk}: \{h_1^{r_j} \cdot h_2^{r_j} \cdot h_3^{r_j},\ h_1^{r_j(\tilde{w}_0 + j\cdot\tilde{w}_1)} \cdot h_2^{r_j u_j} \cdot h_3^{r_j \tilde{w}_0} \cdot T_j^{j-i}\}_{j\neq i} \\ \qquad \{h_1^{r_i} \cdot h_2^{r_i} \cdot h_3^{r_i},\ h_1^{r_i(\tilde{w}_0 + i\cdot\tilde{w}_1)} \cdot h_2^{r_i u_i} \cdot h_3^{r_i \tilde{w}_0}\} \end{array}\right\}.$$

By the Chinese Remainder Theorem, we have $(g_1^{w_1}, h_1^{w_1}, h_3^{w_1}) \equiv (g_1^{\tilde{w}_1}, h_1^{\tilde{w}_1}, h_3^{w_1})$ where $\tilde{w}_1, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. Observe that when $T_j = g_3^{r_j w_1}$ the output is identical to Sub-Game$_{i,1}$; and when $T_j = g_3^{r_j u'_j}$ and if we re-define $u_j = \tilde{w}_0 + (j-i)\cdot u'_j \bmod p_3$ for $j \neq i$, the output is identical Sub-Game$_{i,2}$. This readily proves the lemma. □

Lemma 25 (Sub-Game$_{i,2} \equiv$ Sub-Game$_{i,3}$). $\mathsf{Adv}_{i,2} \equiv \mathsf{Adv}_{i,3}$.

Proof. It is sufficient to prove that

$$\begin{aligned} &(\overbrace{\underbrace{g_1^{w_0}, g_1^{w_1}}_{\mathsf{aux}}; \{g_2^{u_j}\}_{j<i}, g_3^{w_0 + i\cdot w_1}}^{\mathsf{ct}}; \overbrace{h_1^{w_0}, h_1^{w_1}, \{h_2^{u_j}\}_{j\in[n]}, \{h_3^{u_j}\}_{j\neq i}, h_3^{w_0+i\cdot w_1}}^{\mathsf{sk}}) \\ \equiv &(g_1^{w_0}, g_1^{w_1}; \{g_2^{u_j}\}_{j<i}, \boxed{g_3^{u_i}}; \quad h_1^{w_0}, h_1^{w_1}, \{h_2^{u_j}\}_{j\in[n]}, \{h_3^{u_j}\}_{j\neq i}, \boxed{h_3^{u_i}}) \end{aligned}$$

where $w_0, w_1, u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. It is direct to see that this is implied by the following statement

$$\begin{aligned} &(w_0 \bmod p_1, w_1 \bmod p_1, \{u_j \bmod p_2\}_{j\in[n]}, \{u_j \bmod p_3\}_{j\neq i}, w_0 + i\cdot w_1 \bmod p_3) \\ \equiv &(w_0 \bmod p_1, w_1 \bmod p_1, \{u_j \bmod p_2\}_{j\in[n]}, \{u_j \bmod p_3\}_{j\neq i}, \boxed{u_i \bmod p_3}) \end{aligned}$$

where $w_0, w_1, u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$, which follows from the Chinese Remainder Theorem and the fact that $\{w_0 + i \cdot w_1 \bmod p_3\} \equiv \{u_i \bmod p_3\}$ when $w_0, w_1, u_i \bmod p_3$ are uniformly distributed over $\mathbb{Z}_{p_3}$. This completes the proof. □

Lemma 26 (Sub-Game$_{i,3} \approx_c$ Sub-Game$_{i,4}$). *There exists* $\mathcal{B}_4$ *with* $\mathsf{Time}(\mathcal{A}) \approx \mathsf{Time}(\mathcal{B}_4)$ *such that*

$$|\mathsf{Adv}_{i,3} - \mathsf{Adv}_{i,4}| \le \mathsf{Adv}_{\mathcal{B}_4}^{\mathrm{SD}_{p_3 \mapsto p_3 p_2}^{G_N}}(\lambda).$$

Proof. Recall that the $\mathrm{SD}_{p_3 \mapsto p_3 p_2}^{G_N}$ assumption asserts that

$$g_3^{s_i} \approx_c \boxed{g_2^{s_i}} \cdot g_3^{s_i} \text{ given } g_1, g_2, h_1, h_{23}.$$

On input T, algorithm $\mathcal{B}_4$ samples $w_0, w_1, u_j, r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$, s_j for all $j \in [n] \setminus \{i\}$ and $\tilde{s}_i \leftarrow_{\mathrm{R}} \mathbb{Z}_N$, and outputs

$$\left\{\begin{array}{l} \mathsf{aux}: g_1, g_1^{w_0}, g_1^{w_1}, g_2 \\ \quad \mathsf{ct}: \{g_1^{s_j(w_0+j\cdot w_1)} \cdot g_2^{s_j u_j},\ g_1^{s_j} \cdot g_2^{s_j}\}_{j<i} \\ \qquad \{g_1^{\tilde{s}_i(w_0+i\cdot w_1)} \cdot T^{u_j},\ g_1^{\tilde{s}_i} \cdot T\} \\ \qquad \{g_1^{s_j(w_0+j\cdot w_1)},\ g_1^{s_j}\}_{j>i} \\ \quad \mathsf{sk}: \{h_1^{r_j} \cdot h_{23}^{r_j},\ h_1^{r_j(w_0+j\cdot w_1)} \cdot h_{23}^{r_j u_j}\}_{j\in[n]} \end{array}\right\}.$$

By the Chinese Remainder Theorem, we have $(g_1^{s_i}, g_2^{s_i}, g_3^{s_i}) \equiv (g_1^{\tilde{s}_i}, g_2^{s_i}, g_3^{s_i})$ where $s_i, \tilde{s}_i \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. Observe that when $T = g_3^{s_i}$ the output is identical to $\mathsf{Sub\text{-}Game}_{i,3}$; and when $T = g_2^{s_i} \cdot g_3^{s_i}$, the output is identical to $\mathsf{Sub\text{-}Game}_{i,4}$. This readily proves the lemma. □

Lemma 27 ($\mathsf{Sub\text{-}Game}_{i,4} \equiv \mathsf{Sub\text{-}Game}_{i,5}$). $\mathsf{Adv}_{i,4} = \mathsf{Adv}_{i,5}$.

Proof. The proof is completely analogous to that of Lemma 25 ($\mathsf{Sub\text{-}Game}_{i,2} \approx_c \mathsf{Sub\text{-}Game}_{i,3}$).

Lemma 28 ($\mathsf{Sub\text{-}Game}_{i,5} \approx_c \mathsf{Sub\text{-}Game}_{i,6}$). *There exists* $\mathcal{B}_6$ *with* $\mathsf{Time}(\mathcal{A}) \approx \mathsf{Time}(\mathcal{B}_6)$ *such that*

$$|\mathsf{Adv}_{i,5} - \mathsf{Adv}_{i,6}| \le \mathsf{Adv}_{\mathcal{B}_6}^{\mathrm{DDH}_{p_3}^{H_N}}(\lambda).$$

Proof. The proof is completely analogous to that of Lemma 24 ($\mathsf{Sub\text{-}Game}_{i,1} \approx_c \mathsf{Sub\text{-}Game}_{i,2}$), except we simulate ct_i as follows

$$\mathsf{ct}_i:\ g_1^{s_i(\tilde{w}_0+i\cdot\tilde{w}_1)} \cdot \boxed{g_2^{s_i u_i}} \cdot g_3^{s_i \tilde{w}_0},\ g_1^{s_i} \cdot \boxed{g_2^{s_i}} \cdot g_3^{s_i}$$

where g_2 and u_i are known to the simulator. □

Lemma 29 ($\mathsf{Sub\text{-}Game}_{i,6} \approx_c \mathsf{Sub\text{-}Game}_{i,7}$). *There exists* $\mathcal{B}_7$ *with* $\mathsf{Time}(\mathcal{A}) \approx \mathsf{Time}(\mathcal{B}_7)$ *such that*

$$|\mathsf{Adv}_{i,6} - \mathsf{Adv}_{i,7}| \le \mathsf{Adv}_{\mathcal{B}_7}^{\mathrm{SD}_{p_1 \mapsto p_1 p_3}^{G_N}}(\lambda).$$

Proof. The proof is completely analogous to that of Lemma 23 ($\mathsf{Game}_i \approx_c \mathsf{Sub\text{-}Game}_{i,1}$), except we generate ct_i as follows

$$\mathsf{ct}_i:\ T^{w_0+i\cdot w_1} \cdot \boxed{g_2^{\tilde{s}_i u_i}},\ T \cdot \boxed{g_2^{\tilde{s}_i}}$$

where g_2 and u_i are known to the simulator and $\tilde{s}_i \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. By the Chinese Remainder Theorem, we have $(g_1^{s_i}, g_2^{s_i}, g_3^{s_i}) \equiv (g_1^{s_i}, g_2^{\tilde{s}_i}, g_3^{s_i})$ where $s_i, \tilde{s}_i \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. □

B Additional claims for Section 5.1

Fix an adversary $\mathcal{A}$, we use Adv_{xx} to denote the advantage of $\mathcal{A}$ in Game_{xx}.

Lemma 30 ($\mathsf{Game}_0 \approx_c \mathsf{Game}_{0'}$). *There exists an adversary* $\mathcal{B}_0$ *with* $\mathsf{Time}(\mathcal{A}) \approx \mathsf{Time}(\mathcal{B}_0)$ *such that*

$$|\mathsf{Adv}_0 - \mathsf{Adv}_{0'}| \le \mathsf{Adv}_{\mathcal{B}_0}^{\mathrm{DDH}_{\mathbf{A}_2}^{G_2}}(\lambda).$$

Proof. This basically follows $\mathrm{DDH}_{\mathbf{A}_2}^{G_2}$ assumption which asserts that

$$\{[\mathbf{W}_0\mathbf{D}_j]_2, [\mathbf{D}_j]_2\}_{j\in[n]} \approx_c \{[(\mathbf{W}_0 + \mathbf{U}_j^{(2)})\mathbf{D}_j]_2, [\mathbf{D}_j]_2\}_{j\in[n]} \text{ given } \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_1^{\top}\mathbf{W}_0$$

where $\mathbf{W}_0 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$, $\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$, $\mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|})$. On input $\{[\mathbf{T}_j]_2, [\mathbf{D}_j]_2\}_{j\in[n]}$ along with $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_1^\top\mathbf{W}_0$, algorithm $\mathcal{B}_0$ samples $\mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$, $\mathbf{s}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1}$ for all $j \in [n]$ and outputs

$$\left\{\begin{array}{r l} \mathsf{aux}: & [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1, [\mathbf{A}_2^\top]_1 \\ \mathsf{ct}: & \left\{[\mathbf{s}_j^\top(\mathbf{A}_1^\top\mathbf{W}_0 + j\cdot\mathbf{A}_1^\top\mathbf{W}_1)]_1,\ [\mathbf{s}_j^\top\mathbf{A}_1^\top]_1\right\}_{j\in[n]} \\ \mathsf{sk}: & \left\{[\mathbf{D}_j]_2,\ [\mathbf{T}_j + j\cdot\mathbf{W}_1\mathbf{D}_j]_2\right\}_{j\in[n]} \end{array}\right\}.$$

Observe that when $\mathbf{T}_j = \mathbf{W}_0\mathbf{D}_j$, the output is identical to Game_0; and when $\mathbf{T}_j = (\mathbf{W}_0 + \mathbf{U}_j^{(2)})\mathbf{D}_j$, the output is identical to $\mathsf{Game}_{0'}$. This readily proves the lemma. □

Lemma 31 ($\mathsf{Game}_i \approx \mathsf{Sub\text{-}Game}_{i,1}$)**.** *For any adversary $\mathcal{A}$, there exists an adversary $\mathcal{B}_1$ with* $\mathsf{Time}(\mathcal{A}) \approx \mathsf{Time}(\mathcal{B}_1)$ *such that*

$$|\mathsf{Adv}_i - \mathsf{Adv}_{i,1}| \le \mathsf{Adv}_{\mathcal{B}_1}^{\mathrm{SD}^{G_1}_{\mathbf{A}_1\mapsto\mathbf{A}_1,\mathbf{A}_3}}(\lambda).$$

Proof. Recall that the $\mathrm{SD}^{G_1}_{\mathbf{A}_1\mapsto\mathbf{A}_1,\mathbf{A}_3}$ assumption asserts

$$[\mathbf{t} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)]_1 \approx_c [\mathbf{t} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_3)]_1 \ \text{ given } \ [\mathbf{A}_1,\mathbf{A}_2]_1, \mathsf{basis}(\mathbf{A}_2^{\|}).$$

On input $[\mathbf{t}]_1$ along with $[\mathbf{A}_1,\mathbf{A}_2]_1, \mathsf{basis}(\mathbf{A}_2^{\|})$, algorithm $\mathcal{B}_1$ samples $\mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$, $\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$ for all $j \in [n]$ and $\mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|})$ for $j \in [n]$ using $\mathsf{basis}(\mathbf{A}_2^{\|})$, and outputs

$$\left\{\begin{array}{r l l} \mathsf{aux}: & [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top\mathbf{W}_0]_1, [\mathbf{A}_1^\top\mathbf{W}_1]_1, [\mathbf{A}_2^\top]_1 & \\ \mathsf{ct}: & \left\{[\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})]_1,\ [\mathbf{c}_j^\top]_1\right\}_{j<i} & \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_2) \\ & \left\{[\mathbf{t}^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1)]_1,\ [\mathbf{t}^\top]_1\right\} & \\ & \left\{[\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1,\ [\mathbf{c}_j^\top]_1\right\}_{j>i} & \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1) \\ \mathsf{sk}: & \left\{[\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})\mathbf{D}_j]_2\right\}_{j\in[n]} & \end{array}\right\}.$$

Observe that when $\mathbf{t} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1)$ the output is identical to that in Game_i; and when $\mathbf{t} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1,\mathbf{A}_3)$, the output is identical to that in $\mathsf{Sub\text{-}Game}_{i,1}$. This readily proves the lemma. □

Lemma 32 ($\mathsf{Sub\text{-}Game}_{i,1} \approx \mathsf{Sub\text{-}Game}_{i,2}$)**.** *There exists an adversary $\mathcal{B}_2$ with* $\mathsf{Time}(\mathcal{A}) \approx \mathsf{Time}(\mathcal{B}_2)$ *such that*

$$|\mathsf{Adv}_{i,1} - \mathsf{Adv}_{i,2}| \le \mathsf{Adv}_{\mathcal{B}_2}^{\mathrm{DDH}^{G_2}_{\mathbf{A}_3}}(\lambda).$$

Proof. This follows the $\mathrm{DDH}^{G_2}_{\mathbf{A}_3}$ assumption asserting that

$$\{\,[\mathbf{D}_j]_2, [\mathbf{W}_1\mathbf{D}_j]_2\,\}_{j\neq i} \approx_c \{\,[\mathbf{D}_j]_2, [(\mathbf{W}_1 + \widetilde{\mathbf{U}}_j^{(3)})\mathbf{D}_j]_2\,\}_{j\neq i} \ \text{ given } \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_2^{\|}, \mathbf{A}_1^\top\mathbf{W}_1, \mathbf{A}_2^\top\mathbf{W}_1$$

where $\widetilde{\mathbf{U}}_j^{(3)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_3^{\|})$. On input $\{[\mathbf{D}_j]_2, [\mathbf{T}_j]_2\}_{j\neq i}$ along with $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_2^{\|}, \mathbf{A}_1^\top\mathbf{W}_1, \mathbf{A}_2^\top\mathbf{W}_1$, algorithm $\mathcal{B}_2$ samples $\widetilde{\mathbf{W}}_0$ and programs $\mathbf{W}_0 := \widetilde{\mathbf{W}}_0 - i\cdot\mathbf{W}_1$, samples $\mathbf{s}_{1,j} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_1}$, $\mathbf{s}_{2,j} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_2}$, $\mathbf{D}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell_W\times\ell_W}$, $\mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}}$

$\mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|})$ for all $j \in [n]$, and outputs

$$\left\{\begin{array}{ll} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \widetilde{\mathbf{W}}_0 - i \cdot \mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_2^\top]_1 & \\ \quad \mathsf{ct}: \left\{[\mathbf{c}_j^\top(\widetilde{\mathbf{W}}_0 + \mathbf{U}_j^{(2)}) + (j-i)\cdot(\mathbf{s}_{1,j}^\top \mathbf{A}_1^\top \mathbf{W}_1 + \mathbf{s}_{2,j}^\top \mathbf{A}_2^\top \mathbf{W}_1)]_1,\ [\mathbf{c}_j^\top]_1\right\}_{j<i} & \mathbf{c}_j := \mathbf{A}_1\mathbf{s}_{1,j} + \mathbf{A}_2\mathbf{s}_{2,j} \\ \qquad \left\{[\mathbf{c}_i^\top \widetilde{\mathbf{W}}_0]_1,\ [\mathbf{c}_i^\top]_1\right\} & \mathbf{c}_i \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_3) \\ \qquad \left\{[\mathbf{s}_{1,j}^\top \mathbf{A}_1^\top \widetilde{\mathbf{W}}_0 + (j-i)\cdot \mathbf{s}_{1,j}^\top \mathbf{A}_1^\top \mathbf{W}_1)]_1,\ [\mathbf{c}_j^\top]_1\right\}_{j>i} & \mathbf{c}_j := \mathbf{A}_1\mathbf{s}_{1,j} \\ \quad \mathsf{sk}: \left\{[\mathbf{D}_j]_2,\ [(\widetilde{\mathbf{W}}_0 + \mathbf{U}_j^{(2)})\mathbf{D}_j + (j-i)\cdot \mathbf{T}_j]_2\right\}_{j\neq i} & \\ \qquad \left\{[\mathbf{D}_i]_2,\ [(\widetilde{\mathbf{W}}_0 + \mathbf{U}_i^{(2)})\mathbf{D}_i]_2\right\} & \end{array}\right\}.$$

Observe that when $\mathbf{T}_j = \mathbf{W}_1\mathbf{D}_j$, the output is identical to $\mathsf{Sub\text{-}Game}_{i,1}$; and when $\mathbf{T}_j = (\mathbf{W}_1 + \widetilde{\mathbf{U}}_j^{(3)})\mathbf{D}_j$ and let $\mathbf{U}_j^{(3)} := (j-i)\cdot\widetilde{\mathbf{U}}_j^{(3)}$, the output is identical to $\mathsf{Sub\text{-}Game}_{i,2}$. This readily proves the lemma. □

Lemma 33 ($\mathsf{Sub\text{-}Game}_{i,2} \equiv \mathsf{Sub\text{-}Game}_{i,3}$)**.** $\mathsf{Adv}_{i,2} = \mathsf{Adv}_{i,3}$.

Proof. It is sufficient to prove that, for $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1^{\|}, \mathbf{A}_2^{\|}, \mathbf{A}_3^{\|}$, we have

$$(\overbrace{\mathbf{A}_1^\top \mathbf{W}_0, \mathbf{A}_2^\top \mathbf{W}_0}^{\mathsf{aux}, \{\mathsf{ct}_j\}_{j\neq i}}, \overbrace{\{\mathbf{W}_0 + \mathbf{U}_j^{(3)}\}_{j\neq i}}^{\{\mathsf{sk}_j\}_{j\neq i}}, \overbrace{\mathbf{W}_0}^{\mathsf{ct}_i, \mathsf{sk}_i}) \equiv (\mathbf{A}_1^\top \mathbf{W}_0, \mathbf{A}_2^\top \mathbf{W}_0, \{\mathbf{W}_0 + \mathbf{U}_j^{(3)}\}_{j\neq i}, \mathbf{W}_0 + \boxed{\mathbf{U}_i^{(3)}})$$

where $\mathbf{W}_0 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$ and $\mathbf{U}_j^{(3)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_3^{\|})$ for all $j \in [n]$. This is further implied by

$$(\mathbf{A}_1^\top \mathbf{W}_0, \mathbf{A}_2^\top \mathbf{W}_0, \mathbf{W}_0) \equiv (\mathbf{A}_1^\top \mathbf{W}_0, \mathbf{A}_2^\top \mathbf{W}_0, \mathbf{W}_0 + \boxed{\mathbf{U}_i^{(3)}})$$

which is described by the statistical lemma, Lemma 5. □

Lemma 34 ($\mathsf{Sub\text{-}Game}_{i,3} \approx_c \mathsf{Sub\text{-}Game}_{i,4}$)**.** *There exists an adversary* $\mathcal{B}_4$ *with* $\mathsf{Time}(\mathcal{B}_4) \approx \mathsf{Time}(\mathcal{A})$ *such that*

$$|\mathsf{Adv}_{i,3} - \mathsf{Adv}_{i,4}| \le \mathsf{Adv}_{\mathcal{B}_4}^{\mathrm{SD}^{G_1}_{\mathbf{A}_3\mapsto\mathbf{A}_3,\mathbf{A}_2}}(\lambda).$$

Proof. Recall that the $\mathrm{SD}^{G_1}_{\mathbf{A}_3\mapsto\mathbf{A}_3,\mathbf{A}_2}$ assumption asserts that

$$[\mathbf{t} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_3)]_1 \approx_c [\mathbf{t} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_3, \mathbf{A}_2)]_1 \ \text{ given } [\mathbf{A}_2, \mathbf{A}_1]_1, \mathsf{basis}(\mathbf{A}_2^{\|}, \mathbf{A}_3^{\|}).$$

On input $[\mathbf{t}]_1$ along with $[\mathbf{A}_2, \mathbf{A}_1]_1, \mathsf{basis}(\mathbf{A}_2^{\|}, \mathbf{A}_3^{\|})$, algorithm $\mathcal{B}_4$ samples $\mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{\ell\times\ell_W}$ and

$$(\mathbf{U}_j^{(2)} + \mathbf{U}_j^{(3)} =)\ \mathbf{U}_j^{(23)} \leftarrow_{\mathrm{R}} \mathsf{span}^{\ell_W}(\mathbf{A}_2^{\|}, \mathbf{A}_3^{\|})$$

for $j \in [n]$ using $\mathsf{basis}(\mathbf{A}_2^{\|}, \mathbf{A}_3^{\|})$, and outputs

$$\left\{\begin{array}{ll} \mathsf{aux}: [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1, [\mathbf{A}_2^\top]_1 & \\ \quad \mathsf{ct}: \left\{[\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(23)})]_1,\ [\mathbf{c}_j^\top]_1\right\}_{j<i} & \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{A}_2) \\ \qquad \left\{[\mathbf{c}_i^\top(\mathbf{W}_0 + i\cdot\mathbf{W}_1 + \mathbf{U}_i^{(23)})]_1,\ [\mathbf{c}_i^\top]_1\right\} & \mathbf{c}_i \leftarrow_{\mathrm{R}} \mathbf{t} + \mathsf{span}(\mathbf{A}_1) \\ \qquad \left\{[\mathbf{c}_j^\top(\mathbf{W}_0 + j\cdot\mathbf{W}_1)]_1,\ [\mathbf{c}_j^\top]_1\right\}_{j>i} & \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1) \\ \quad \mathsf{sk}: \left\{[\mathbf{D}_j]_2,\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(23)})\mathbf{D}_j]_2\right\}_{j\in[n]} & \end{array}\right\}.$$

Observe that when $\mathbf{t} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_3)$ the output is identical to that in $\mathsf{Sub\text{-}Game}_{i,3}$; and when $\mathbf{t} \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_2, \mathbf{A}_3)$, the output is identical to that in $\mathsf{Sub\text{-}Game}_{i,4}$. This readily proves the lemma. □

Lemma 35 ($\text{Sub-Game}_{i,4} \equiv \text{Sub-Game}_{i,5}$)**.** $\text{Adv}_{i,4} = \text{Adv}_{i,5}$.

Proof. The proof is completely analogous to that of Lemma 33 ($\text{Sub-Game}_{i,2} \approx_c \text{Sub-Game}_{i,3}$). □

Lemma 36 ($\text{Sub-Game}_{i,5} \approx_c \text{Sub-Game}_{i,6}$)**.** *There exists an adversary* $\mathcal{B}_6$ *with* $\text{Time}(\mathcal{A}) \approx \text{Time}(\mathcal{B}_6)$ *such that*

$$|\text{Adv}_{i,5} - \text{Adv}_{i,6}| \le \text{Adv}_{\mathcal{B}_6}^{\text{DDH}_{\mathbf{A}_3}^{G_2}}(\lambda).$$

Proof. The proof is completely analogous to that of Lemma 32 ($\text{Sub-Game}_{i,1} \approx_c \text{Sub-Game}_{i,2}$) except that we simulate

$$\text{sk}_i : [\mathbf{c}_i^\top (\widetilde{\mathbf{W}}_0 + \mathbf{U}_i^{(2)})]_1,\ [\mathbf{c}_i^\top]_1$$

where $\mathbf{c}_i \leftarrow_R \mathbb{Z}_p^\ell$ and $\mathbf{U}_i^{(2)}$ are known to the simulator. □

Lemma 37 ($\text{Sub-Game}_{i,6} \approx_c \text{Sub-Game}_{i,7}$)**.** *There exists an adversary* $\mathcal{B}_7$ *with* $\text{Time}(\mathcal{A}) \approx \text{Time}(\mathcal{B}_7)$ *such that*

$$|\text{Adv}_{i,6} - \text{Adv}_{i,7}| \le \text{Adv}_{\mathcal{B}_7}^{\text{SD}_{\mathbf{A}_1 \mapsto \mathbf{A}_1,\mathbf{A}_3}^{G_1}}(\lambda).$$

Proof. The proof is completely analogous to that of Lemma 31 ($\text{Game}_i \approx_c \text{Sub-Game}_{i,1}$) except that we simulate

$$\text{sk}_i : [\mathbf{c}_i^\top (\mathbf{W}_0 + i \cdot \mathbf{W}_1 + \mathbf{U}_i^{(2)})]_1,\ [\mathbf{c}_i^\top]_1$$

where $\mathbf{W}_0, \mathbf{W}_1, \mathbf{U}_i^{(2)}$ are known to the simulator and $\mathbf{c}_i \leftarrow_R \mathbf{t} + \text{span}(\mathbf{A}_2)$. □

C Missing Lemmas for Section 6.2

Fix an adversary $\mathcal{A}$ that makes at most Q key queries, we use Adv_{xx} to denote the advantage of $\mathcal{A}$ in Game_{xx}.

Lemma 38 ($\text{Game}_0 \approx_c \text{Game}_{0'}$)**.** *There exists an adversary* $\mathcal{B}_0$ *with* $\text{Time}(\mathcal{B}_0) \approx \text{Time}(\mathcal{A})$ *such that*

$$|\text{Adv}_0 - \text{Adv}_{0'}| \le \text{Adv}_{\mathcal{B}_0}^{\text{ExpLem}}(\lambda).$$

Proof. This follows from the entropy expansion lemma (see Lemma 2). On input

$$\left\{ \begin{array}{l} \text{aux} : g_1, g_1^w, g_1^{w_0}, g_1^{w_1} \\ \text{ct} : C_0, \{C_{1,j},\ C_{2,j}\}_{j \in [n]} \\ \text{sk} : \{K_{0,j},\ K_{1,j},\ K_{2,j}\}_{j \in [n]} \end{array} \right\},$$

algorithm $\mathcal{B}_0$ proceeds as follows:

Setup. Select a random generator h_{123} of H_N and sample $\alpha \leftarrow_R \mathbb{Z}_N$. Choose a pairwise independent hash function $\mathsf{H} : G_T \to \{0,1\}^\lambda$ and output

$$\text{mpk} := (\,\text{aux},\ e(g_1, h_{123})^\alpha; \mathsf{H}\,).$$

Key Queries. For each query $\mathbf{M}$, sample $\mathbf{u} \leftarrow_R \mathbb{Z}_N^{\ell'-1}$ and $\tilde{r}_j \leftarrow_R \mathbb{Z}_N$ for $j \in [n]$, and output

$$\text{sk}_{\mathbf{M}} := (\,\{\, h_{123}^{\mathbf{M}_j \binom{\alpha}{\mathbf{u}}} \cdot K_{0,j}^{\tilde{r}_j},\ K_{1,j}^{\tilde{r}_j},\ K_{2,j}^{\tilde{r}_j} \,\}_{j \in [n]}\,).$$

Ciphertext. For challenge attribute vector $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ and two equal-length messages m_0, m_1, pick $b \leftarrow_R \{0,1\}$ and output

$$\text{ct}_{\mathbf{x}^*} := (\,C_0, \{\,C_{1,j},\ C_{2,j}\,\}_{j : x_j^* = 1}, e(C_0, h_{123}^\alpha) \cdot m_b\,).$$

Guess. When $\mathcal{A}$ halts with output b', $\mathcal{B}_0$ outputs 1 if $b = b'$ and 0 otherwise.

Observe that when the input is identical to the left distribution in Lemma 2, the output is identical to Game_0; and when the input is identical to the right distribution in Lemma 2, the output is identical to $\mathsf{Game}_{0'}$. This readily proves the lemma. $\square$

Lemma 39 ($\mathsf{Game}_{0'} \approx \mathsf{Game}_1$)**.** *There exists an adversary* $\mathcal{B}_1$ *with* $\mathsf{Time}(\mathcal{B}_1) \approx \mathsf{Time}(\mathcal{A})$ *such that*

$$|\mathsf{Adv}_{0'} - \mathsf{Adv}_1| \le \mathsf{Adv}_{\mathcal{B}_1}^{\mathrm{SD}^{G_N}_{p_2 \mapsto p_2 p_3}}(\lambda).$$

Proof. This follows from

$$(g_2^s, \{g_2^{s_j}\}_{j\in[n]}) \approx_c (g_2^s \cdot \boxed{g_3^s}, \{g_2^{s_j} \cdot \boxed{g_3^{s_j}}\}_{j\in[n]}) \text{ given } g_1, h_1, h_2.$$

which is implied by the $\mathrm{SD}^{G_N}_{p_2\mapsto p_2p_3}$ assumption. On input $(T, \{T_j\}_{j\in[n]})$ along with g_1, h_1, h_2, algorithm $\mathcal{B}_1$ picks $v_j, u_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$ and proceeds as follows:

Setup. Select a random generator h_{123} for H_N and sample $\alpha, w, w_0, w_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. Choose a pairwise independent hash function $\mathsf{H} : G_T \to \{0,1\}^\lambda$ and output

$$\mathsf{mpk} := (\, g_1,\, g_1^w,\, g_1^{w_0},\, g_1^{w_1},\, e(g_1, h_{123})^\alpha;\, \mathsf{H}\,).$$

Key Queries. For each query $\mathbf{M}$, sample $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_N^{\ell'-1}$ and $r_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$ and output

$$\mathsf{sk}_{\mathbf{M}} := (\{\, h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_1^{r_j w} \cdot h_2^{r_j v_j},\, h_1^{r_j} \cdot h_2^{r_j},\, h_1^{r_j(w_0 + j\cdot w_1)} \cdot h_2^{r_j u_j}\,\}_{j\in[n]}).$$

Ciphertext. For challenge attribute vector $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ and two equal-length messages m_0, m_1, pick $b \leftarrow_{\mathrm{R}} \{0,1\}$, $\tilde{s}, \tilde{s}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$ and output

$$\mathsf{ct}_{\mathbf{x}^*} := \big(\, g_1^{\tilde{s}} \cdot T,\, \{\, g_1^{\tilde{s}w + \tilde{s}_j(w_0 + j\cdot w_1)} \cdot T^{v_j} \cdot T_j^{u_j},\, g_1^{\tilde{s}_j} \cdot T_j\,\}_{j: x_j^* = 1},\, \mathsf{H}(e(g_1^{\tilde{s}} \cdot T, h_{123}^\alpha)) \cdot m_b\,\big).$$

Guess. When $\mathcal{A}$ halts with output b', $\mathcal{B}_1$ outputs 1 if $b = b'$ and 0 otherwise.

Observe that when $T = g_2^s$ and $T_j = g_2^{s_j}$, the output is identical to $\mathsf{Game}_{0'}$; and when $T = g_2^s \cdot g_3^s$ and $T_j = g_2^{s_j} \cdot g_3^{s_j}$, the output is identical to Game_1. This readily proves the lemma. $\square$

Lemma 40 ($\mathsf{Game}_i \approx_c \mathsf{Game}_{i,1}$)**.** *There exists an adversary* $\mathcal{B}_2$ *with* $\mathsf{Time}(\mathcal{B}_2) \approx \mathsf{Time}(\mathcal{A})$ *such that*

$$|\mathsf{Adv}_i - \mathsf{Adv}_{i,1}| \le \mathsf{Adv}_{\mathcal{B}_2}^{\mathrm{SD}^{H_N}_{p_2 \mapsto p_2 p_3}}(\lambda).$$

Proof. This follows from

$$\{h_2^{r_j}\}_{j\in[n]} \approx_c \{h_2^{r_j} \cdot h_3^{r_j}\}_{j\in[n]} \text{ given } g_1, g_{23}, h_1, h_2, h_3$$

where g_{23} is a random generator of $G_{p_2p_3}$, which is implied by $\mathrm{SD}^{H_N}_{p_2\mapsto p_2p_3}$ assumption. On input $\{T_j\}_{j\in[n]}$ along with $g_1, g_{23}, h_1, h_2, h_3$, algorithm $\mathcal{B}_2$ samples $v_j, u_j, \hat{\alpha} \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$ and proceeds as follows:

Setup. Select a random generator h_{123} of H_N and sample $\alpha, w, w_0, w_1, \leftarrow_{\mathrm{R}} \mathbb{Z}_N$. Choose a pairwise independent hash function $\mathsf{H} : G_T \to \{0,1\}^\lambda$ and output

$$\mathsf{mpk} := (\, g_1,\, g_1^w,\, g_1^{w_0},\, g_1^{w_1},\, e(g_1, h_{123})^\alpha;\, \mathsf{H}\,).$$

Key Queries. For the κ's query $\mathbf{M}$, sample $\mathbf{u} \leftarrow_{\mathrm{R}} \mathbb{Z}_N^{\ell'-1}$ and $\tilde{r}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_N$ for all $j \in [n]$ and output

$$\mathsf{sk}_{\mathbf{M}} := \begin{cases} (\{\, h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_3^{\mathbf{M}_j\binom{\hat{\alpha}}{\mathbf{0}}} \cdot h_1^{\tilde{r}_j w} \cdot h_2^{\tilde{r}_j v_j},\, h_1^{\tilde{r}_j} \cdot h_2^{\tilde{r}_j},\, h_1^{\tilde{r}_j(w_0 + j\cdot w_1)} \cdot h_2^{\tilde{r}_j u_j}\,\}_{j\in[n]}) & \text{if } \kappa < i \\ (\{\, h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_1^{\tilde{r}_j w} \cdot T_j^{v_j},\, h_1^{\tilde{r}_j} \cdot T_j,\, h_1^{\tilde{r}_j(w_0 + j\cdot w_1)} \cdot T_j^{u_j}\,\}_{j\in[n]}) & \text{if } \kappa = i \\ (\{\, h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_1^{\tilde{r}_j w} \cdot h_2^{\tilde{r}_j v_j},\, h_1^{\tilde{r}_j} \cdot h_2^{\tilde{r}_j},\, h_1^{\tilde{r}_j(w_0 + j\cdot w_1)} \cdot h_2^{\tilde{r}_j u_j}\,\}_{j\in[n]}) & \text{if } \kappa > i \end{cases}$$

Ciphertext. For challenge attribute vector $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ and two equal-length messages m_0, m_1, pick $b \leftarrow_R \{0,1\}$, $s, s_j \leftarrow_R \mathbb{Z}_N$ for all $j \in [n]$ and output

$$\mathsf{ct}_{\mathbf{x}^*} := \big(\, g_1^s \cdot g_{23}^s,\ \{\, g_1^{sw + s_j(w_0 + j\cdot w_1)} \cdot g_{23}^{sv_j + s_j u_j},\ g_1^{s_j} \cdot g_{23}^{s_j} \,\}_{j: x_j^* = 1},\ \mathsf{H}(e(g_1^s \cdot g_{23}^s, h_{123}^{\alpha})) \cdot m_b \,\big).$$

Guess. When $\mathcal{A}$ halts with output b', $\mathcal{B}_2$ outputs 1 if $b = b'$ and 0 otherwise.

Observe that when $T_j = h_2^{r_j}$, the output is identical to Game_i; and when $T_j = h_2^{r_j} \cdot h_3^{r_j}$, the output is identical to $\mathsf{Game}_{i,1}$. This readily proves the lemma. □

Lemma 41 ($\mathsf{Game}_{i,1} \equiv \mathsf{Game}_{i,2}$)**.** $\mathsf{Adv}_{i,1} = \mathsf{Adv}_{i,2}$.

Proof. Fix generators $g_1, g_2, g_3, h_{123}, h_1, h_2, h_3$, common parameters $w, w_0, w_1, \{u_j\}_{j\in[n]}$, and all random coins except those for the i'th key. It is sufficient to prove that, for all $\alpha, \hat{\alpha}$ and all $r_j \neq 0 \bmod p_3$, we have

$$\big(\, \overbrace{\{h_2^{v_j}\}_{j\in[n]}}^{\kappa\text{'th sk}, \kappa \neq i};\ \overbrace{\{g_2^{v_j}, g_3^{v_j}\}_{j:x_j=1}}^{\text{SF ct}};\ \overbrace{\{h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot h_3^{r_j v_j}, h_3^{r_j}\}_{j\in[n]}}^{\text{P-normal } i\text{'th sk}} \,\big)$$
$$\equiv \big(\, \{h_2^{v_j}\}_{j\in[n]};\ \{g_2^{v_j}, g_3^{v_j}\}_{j:x_j=1};\ \underbrace{\{h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot \boxed{h_3^{\mathbf{M}_j\binom{\hat{\alpha}}{\mathbf{0}}}} \cdot h_3^{r_j v_j}, h_3^{r_j}\}_{j\in[n]}}_{\text{P-SF } i\text{'th sk}} \,\big)$$

where $v_j \leftarrow_R \mathbb{Z}_N$ for all $j \in [n]$ and $\mathbf{u} \leftarrow_R \mathbb{Z}_N^{\ell'-1}$, if $\mathbf{x} = (x_1, \ldots, x_n)$ does not satisfy $\mathbf{M}$. By the Chinese Remainder Theorem, it is implied by

$$\big(\, \{g_3^{v_j}\}_{j:x_j=1},\ \{h_3^{r_j v_j}, h_3^{r_j}\}_{j\in[n]} \,\big) \equiv \big(\, \{g_3^{v_j}\}_{j:x_j=1};\ \{\, \boxed{h_3^{\mathbf{M}_j\binom{\hat{\alpha}}{\mathbf{u}}}} \cdot h_3^{r_j v_j}, h_3^{r_j}\}_{j\in[n]} \,\big)$$

where $v_j \leftarrow_R \mathbb{Z}_N$ for all $j \in [n]$ and $\mathbf{u} \leftarrow_R \mathbb{Z}_N^{\ell'-1}$. This follows from Lemma 1 and readily proves the lemma. □

Lemma 42 ($\mathsf{Game}_{i,2}$ **to** $\mathsf{Game}_{i,3}$)**.** *There exists an adversary* $\mathcal{B}_4$ *with* $\mathsf{Time}(\mathcal{B}_4) \approx \mathsf{Time}(\mathcal{A})$ *such that*

$$|\mathsf{Adv}_{i,2} - \mathsf{Adv}_{i,3}| \leq \mathsf{Adv}_{\mathcal{B}_2}^{\mathrm{SD}^{H_N}_{p_2 \mapsto p_2 p_3}}(\lambda).$$

Proof. The proof is completely analogous to that of Lemma 40 ($\mathsf{Game}_i \approx_c \mathsf{Game}_{i,1}$) except that we generate the κ's key (for $\mathbf{M}$) as

$$\mathsf{sk}_{\mathbf{M}} := \big(\, \{\, h_{123}^{\mathbf{M}_j\binom{\alpha}{\mathbf{u}}} \cdot \boxed{h_3^{\mathbf{M}_j\binom{\hat{\alpha}}{\mathbf{0}}}} \cdot h_1^{\tilde{r}_j w} \cdot T_j^{v_j},\ h_1^{\tilde{r}_j} \cdot T_j,\ h_1^{\tilde{r}_j(w_0 + j\cdot w_1)} \cdot T_j^{u_j} \,\}_{j\in[n]} \,\big)$$

where $\hat{\alpha}$ and h_3 are known to the simulator. □

Lemma 43 ($\mathsf{Game}_{Q+1} \equiv \mathsf{Game}_{\mathbf{Final}}$)**.** $\mathsf{Adv}_{Q+1} = \mathsf{Adv}_{\mathrm{Final}}$.

Proof. Sample w, w_0, w_1, v_j, u_j as usual. Let h_{123} and h_3 be random generators of H_N and H_{p_3}. We sample $\tilde{\alpha}, \hat{\alpha} \leftarrow_R \mathbb{Z}_N$ and define $h_{123}^{\alpha} := h_{123}^{\tilde{\alpha}} / h_3^{\hat{\alpha}}$. Observe that this does not change the distribution of α conditioned on $\hat{\alpha}$. Therefore we can alternatively simulate Game_{Q+1} as follows:

Setup. Choose a pairwise independent hash function $\mathsf{H} : G_T \to \{0,1\}^{\lambda}$ and output

$$\mathsf{mpk} := (\, g_1,\ g_1^{w},\ g_1^{w_0},\ g_1^{w_1},\ e(g_1, h_{123}^{\tilde{\alpha}});\ \mathsf{H} \,).$$

This follows from the fact hat $e(g_1, h_3^{\hat{\alpha}}) = 1$.

Key Queries. For each query $\mathbf{M}$, sample $\mathbf{u} \leftarrow_R \mathbb{Z}_N^{\ell'-1}$ and $r_j \leftarrow_R \mathbb{Z}_N$ for all $j \in [n]$ and output

$$\mathsf{sk}_{\mathbf{M}} := \big(\, \{\, h_{123}^{\mathbf{M}_j\binom{\tilde{\alpha}}{\mathbf{u}}} \cdot h_1^{r_j w} \cdot h_2^{r_j v_j},\ h_1^{r_j} \cdot h_2^{r_j},\ h_1^{r_j(w_0 + j\cdot w_1)} \cdot h_2^{r_j u_j} \,\}_{j\in[n]} \,\big)$$

Ciphertext. For challenge attribute vector $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ and two equal-length messages m_0, m_1, pick $b \leftarrow_R \{0,1\}$, $s, s_j \leftarrow_R \mathbb{Z}_N$ for all $j \in [n]$ and output

$$\mathsf{ct}_{\mathbf{x}^*} := \big(\, g_1^s \cdot g_{23}^s,\ \{\, g_1^{sw + s_j(w_0 + j\cdot w_1)} \cdot g_{23}^{sv_j + s_j u_j},\ g_1^{s_j} \cdot g_{23}^{s_j} \,\}_{j:x_j^*=1},\ \mathsf{H}(e(g_1^s \cdot g_{23}^s, h_{123}^{\tilde{\alpha}}) \cdot \boxed{e(g_3^s, h_3^{-\hat{\alpha}})}\,) \cdot m_b \,\big).$$

Here we let $g_{23} := g_2 \cdot g_3$.

Guess. When $\mathcal{A}$ halts with output b', output 1 if $b = b'$ and 0 otherwise.

Observe that $e(g_3^s, h_3^{-\hat{\alpha}})$ has $\Theta(p_3)$-bit leftover entropy conditioned on mpk and all $\mathsf{sk}_{\mathbf{M}}$. By the leftover hash lemma, the last component of $\mathsf{ct}_{\mathbf{x}^*}$ is uniformly distributed over $\{0,1\}^\lambda$ and the simulation is actually identical $\mathsf{Game}_{\mathsf{Final}}$. This proves the lemma. □

D Missing Lemmas for Section 7.3

Fix an adversary $\mathcal{A}$ that makes at most Q key queries, we use Adv_{xx} to denote the advantage of $\mathcal{A}$ in Game_{xx}.

Lemma 44 ($\mathsf{Game}_0 \approx_c \mathsf{Game}_{0'}$)**.** *There exists an adversary* $\mathcal{B}_0$ *with* $\mathsf{Time}(\mathcal{B}_0) \approx \mathsf{Time}(\mathcal{A})$ *such that*

$$|\mathsf{Adv}_0 - \mathsf{Adv}_{0'}| \le \mathsf{Adv}_{\mathcal{B}_0}^{\textsc{ExpLemRev}}(\lambda).$$

Proof. This follows from Lemma 12 (our new entropy expansion lemma). On input

$$\left\{ \begin{array}{l} \mathsf{aux}:\ [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1 \\ \mathsf{ct}:\ [\mathbf{C}_0]_1,\ \big\{[\mathbf{C}_{1,j}]_1,\ [\mathbf{C}_{2,j}]_1\big\}_{j\in[n]} \\ \mathsf{sk}:\ \big\{[\mathbf{K}_{0,j}]_2,\ [\mathbf{K}_{1,j}]_2,\ [\mathbf{K}_{2,j}]_2\big\}_{j\in[n]} \end{array} \right\},$$

algorithm $\mathcal{B}_0$ proceeds as follows:

Setup. Sample $\mathbf{k} \leftarrow_R \mathbb{Z}_p^{2k+1}$ and output

$$\mathsf{mpk} := (\,\mathsf{aux},\ e([\mathbf{A}_1^\top]_1, [\mathbf{k}]_2)\,).$$

Key Queries. For each query $\mathbf{M}$, sample $\mathbf{K}' \leftarrow_R \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}$ and $\tilde{\mathbf{d}}_j \leftarrow_R \mathbb{Z}_p^{k+1}$ for $j \in [n]$, and output

$$\mathsf{sk}_{\mathbf{M}} := \big(\, \{\, [(\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^\top + \mathbf{K}_{0,j}\tilde{\mathbf{d}}_j]_2,\ [\mathbf{K}_{1,j}\tilde{\mathbf{d}}_j]_2,\ [\mathbf{K}_{2,j}\tilde{\mathbf{d}}_j]_2 \,\}_{j\in[n]} \,\big).$$

Here we implicitly set $\mathbf{d}_j := \mathbf{D}_j \tilde{\mathbf{d}}_j$ which is uniformly distributed over $\mathsf{span}(\mathbf{B})$.

Ciphertext. For challenge attribute vector $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ and two equal-length messages m_0, m_1, pick $b \leftarrow_R \{0,1\}$ and output

$$\mathsf{ct}_{\mathbf{x}^*} := \big([\mathbf{C}_0]_1,\ \big\{[\mathbf{C}_{1,j}]_1,\ [\mathbf{C}_{2,j}]_1\big\}_{j:x_j^*=1}, e([\mathbf{C}_0]_1, [\mathbf{k}]_2) \cdot m_b\big).$$

Guess. When $\mathcal{A}$ halts with output b', output 1 if $b = b'$ and 0 otherwise.

Observe that when $\mathcal{B}_0$'s input is the left distribution in Lemma 12, the output is identical to Game_0; and when it is the right distribution in Lemma 12, the output is identical to $\mathsf{Game}_{0'}$. This proves the lemma. □

Lemma 45 ($\mathsf{Game}_i \approx_c \mathsf{Game}_{i,1}$)**.** *There exists an adversary* $\mathcal{B}_2$ *with* $\mathsf{Time}(\mathcal{B}_2) \approx \mathsf{Time}(\mathcal{A})$ *such that*

$$|\mathsf{Adv}_i - \mathsf{Adv}_{i,1}| \le \mathsf{Adv}_{\mathcal{B}_2}^{\mathrm{MDDH}_{k,k+1}^n}(\lambda).$$

Proof. This follows from the $\mathrm{MDDH}_{k,k+1}^n$ assumption asserting

$$(\,[\mathbf{B}]_2, \{[\mathbf{d}_{j,i} \leftarrow_R \mathsf{span}(\mathbf{B})]_2\}_{j\in[n]}\,) \approx_c (\,[\mathbf{B}]_2, \{[\mathbf{d}_{j,i} \leftarrow_R \mathbb{Z}_p^{k+1}]_2\}_{j\in[n]}\,).$$

On input $[\mathbf{B}]_2, \{[\mathbf{t}_j]_2\}_{j\in[n]}$, algorithm $\mathcal{B}_2$ samples $\mathbf{A}_1, \mathbf{a}_2, \mathbf{a}_2^{\parallel}$ as required and picks $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1 \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(k+1)}$, and prepares $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{a}_2^{\parallel})$ and $\alpha \leftarrow_{\mathrm{R}} \mathbb{Z}_p$, and proceeds as follows.

Setup. Pick $\mathbf{k} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{2k+1}$ and output

$$\mathsf{mpk} := (\,[\mathbf{A}_1^{\top}]_1, [\mathbf{A}_1^{\top}\mathbf{W}]_1, [\mathbf{A}_1^{\top}\mathbf{W}_0]_1, [\mathbf{A}_1^{\top}\mathbf{W}_1]_1,\ e([\mathbf{A}_1^{\top}]_1, [\mathbf{k}]_2)\,).$$

Key Queries. For the κ'th secret key query $\mathbf{M}$, pick $\mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}$ and output

$$\mathsf{sk}_{\mathbf{M}} := \begin{cases} \{\,[(\mathbf{k}+\alpha\mathbf{a}_2^{\parallel}\|\mathbf{K}')\mathbf{M}_j^{\top} + (\mathbf{W}+\mathbf{V}_j^{(2)})\mathbf{d}_j]_2,\ [\mathbf{d}_j]_2\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})\mathbf{d}_j]_2\,\}_{j\in[n]} & \kappa < i \\ \{\,[(\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^{\top} + (\mathbf{W}+\mathbf{V}_j^{(2)})\mathbf{t}_j]_2,\ [\mathbf{t}_j]_2\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})\mathbf{t}_j]_2\,\}_{j\in[n]} & \kappa = i \\ \{\,[(\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^{\top} + (\mathbf{W}+\mathbf{V}_j^{(2)})\mathbf{d}_j]_2,\ [\mathbf{d}_j]_2\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})\mathbf{d}_j]_2\,\}_{j\in[n]} & \kappa > i \end{cases}$$

where $[\mathbf{d}_j]_2 \leftarrow_{\mathrm{R}} [\mathsf{span}(\mathbf{B})]_2$ for all $j \in [n]$.

Ciphertext. For challenge attribute vector $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ and two equal-length messages m_0, m_1, pick $b \leftarrow_{\mathrm{R}} \{0,1\}$ and $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{a}_2)$ for all $j \in [n]$, and output

$$\mathsf{ct}_{\mathbf{x}^*} := (\,[\mathbf{c}^{\top}]_1,\ \{\,[\mathbf{c}^{\top}(\mathbf{W}+\mathbf{V}_j^{(2)}) + \mathbf{c}_j^{\top}(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})]_1,\ [\mathbf{c}_j^{\top}]_1\}_{j:x_j^*=1},\ e([\mathbf{c}^{\top}]_1, [\mathbf{k}]_2)\cdot m_b\,).$$

Guess. When $\mathcal{A}$ halts with output b', output 1 if $b = b'$ and 0 otherwise.

Observe that when $\mathbf{t}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B})$, the output is identical to that in Game_i; and when $\mathbf{t}_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{k+1}$, the output is identical to that in $\mathsf{Game}_{i,1}$. This proves the lemma. □

Lemma 46 ($\mathsf{Game}_{i,1} \equiv \mathsf{Game}_{i,2}$)**.** $\mathsf{Adv}_{i,1} = \mathsf{Adv}_{i,2}$.

Proof. Fix bases $\mathbf{A}_1, \mathbf{a}_2, \mathbf{a}_2^{\parallel}, \mathbf{B}$, common parameters $\mathbf{W}, \mathbf{W}_0, \mathbf{W}_1, \mathbf{k}, \alpha$, and all random coins except those for the i'th key. It is sufficient to prove that, for $\mathbf{x}$ does not satisfy $\mathbf{M}$ and all $\{\mathbf{d}_j \notin \mathsf{span}(\mathbf{B})\}_{j\in[n]}$, we have

$$\Big(\underbrace{\mathbf{V}_j^{(2)}\mathbf{B}}_{\kappa\text{'th sk}, \kappa\neq i};\ \underbrace{\{\mathbf{V}_j^{(2)}\}_{j:x_j=1}}_{\text{E-normal ct}};\ \underbrace{\{(\mathbf{k}\|\mathbf{K}')\mathbf{M}_j^{\top} + \mathbf{V}_j^{(2)}\mathbf{d}_j, \mathbf{d}_j\}_{j\in[n]}}_{\text{P-normal } i\text{'th sk}}\Big) \equiv \Big(\mathbf{V}_j^{(2)}\mathbf{B};\ \{\mathbf{V}_j^{(2)}\}_{j:x_j=1};\ \overbrace{\{(\mathbf{k}+\boxed{\alpha\mathbf{a}_2^{\parallel}}\|\mathbf{K}')\mathbf{M}_j^{\top} + \mathbf{V}_j^{(2)}\mathbf{d}_j, \mathbf{d}_j\}_{j\in[n]}}^{\text{P-SF } i\text{'th sk}}\Big)$$

where $\mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}, \mathbf{V}_j^{(2)} \leftarrow_{\mathrm{R}} \mathsf{span}^{k+1}(\mathbf{a}_2^{\parallel})$.

Let $\mathbf{b}^{\parallel}$ be a (fixed) non-zero vector in $\mathbb{Z}_p^{k+1}$ satisfying $\mathbf{B}^{\top}\mathbf{b}^{\parallel} = \mathbf{0}$. It is direct to see that we can replace $\mathbf{V}_j^{(2)}$ with $\mathbf{V}_j^{(2)} + \mathbf{a}_2^{\parallel} v_j \mathbf{b}^{\parallel\top}$ where $v_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p$ for all $j \in [n]$ in both distributions without changing the distributions. For all $\{\mathbf{V}_j^{(2)}\}_{j\in[n]}$ after substitution, it is sufficient to show that

$$(\,\{\,v_j\,\}_{j:x_j=1}, \{(0\mathbf{a}_2^{\parallel}\|\mathbf{K}')\mathbf{M}_j^{\top} + \mathbf{a}_2^{\parallel} v_j r_j, r_j\}_{j\in[n]}\,) \equiv (\,\{\,v_j\,\}_{j:x_j=1}, \{(\boxed{\alpha\mathbf{a}_2^{\parallel}}\|\mathbf{K}')\mathbf{M}_j^{\top} + \mathbf{a}_2^{\parallel} v_j r_j, r_j\}_{j\in[n]}\,)$$

where $\mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}, v_j \leftarrow_{\mathrm{R}} \mathbb{Z}_p$, and we let $r_j := \mathbf{d}_j^{\top}\mathbf{b}^{\parallel}$. We note that $r_j \neq 0$ since $\mathbf{d}_j \notin \mathsf{span}(\mathbf{B})$ for all $j \in [n]$. This is implied by Lemma 1 and we readily prove the lemma. □

Lemma 47 ($\mathsf{Game}_{i,2} \approx_c \mathsf{Game}_{i,3}$)**.** *There exists an adversary* $\mathcal{B}_3$ *with* $\mathsf{Time}(\mathcal{B}_3) \approx \mathsf{Time}(\mathcal{A})$ *such that*

$$|\mathsf{Adv}_{i,2} - \mathsf{Adv}_{i,3}| \le \mathsf{Adv}_{\mathcal{B}_2}^{\mathrm{MDDH}_{k,k+1}^n}(\lambda).$$

Proof. The proof is completely analogous to that of Lemma 45 ($\mathsf{Game}_i \approx_c \mathsf{Game}_{i,1}$) except that we generate the κ's key as

$$(\,\{\,[(\mathbf{k}+\boxed{\alpha\mathbf{a}_2^{\parallel}}\|\mathbf{K}')\mathbf{M}_j^{\top} + (\mathbf{W}+\mathbf{V}_j^{(2)})\mathbf{t}_j]_2,\ [\mathbf{t}_j]_2\ [(\mathbf{W}_0 + j\cdot\mathbf{W}_1 + \mathbf{U}_j^{(2)})\mathbf{t}_j]_2\,\}_{j\in[n]}\,)$$

where $\mathbf{a}_2^{\parallel}$ and α are known to the simulator. □

Lemma 48 ($\mathsf{Game}_{Q+1} \equiv \mathsf{Game}_{\mathbf{Final}}$). $\mathsf{Adv}_{Q+1} = \mathsf{Adv}_{\mathsf{Final}}$.

Proof. Sample $\mathbf{A}_1, \mathbf{a}_2, \mathbf{a}_2^{\parallel}, \mathbf{B}, \mathbf{W}, \mathbf{W}_0, \mathbf{W}_1$ and $\mathbf{V}_j^{(2)}, \mathbf{U}_j^{(2)}$ as usual. We sample $\tilde{\mathbf{k}} \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{2k+1}$, $\alpha \leftarrow_{\mathrm{R}} \mathbb{Z}_p$, and define $\mathbf{k} := \tilde{\mathbf{k}} - \alpha \mathbf{a}_2^{\parallel}$. The joint distribution of ($\mathbf{k}$, $\mathbf{k} + \alpha \mathbf{a}_2^{\parallel} = \tilde{\mathbf{k}}$) is the same as in Game_{Q+1} and we can simulate the game as follows.

Setup. Output

$$\mathsf{mpk} := (\, [\mathbf{A}_1^\top]_1, [\mathbf{A}_1^\top \mathbf{W}]_1, [\mathbf{A}_1^\top \mathbf{W}_0]_1, [\mathbf{A}_1^\top \mathbf{W}_1]_1,\ e([\mathbf{A}_1^\top]_1, [\tilde{\mathbf{k}}]_2)\,).$$

The simulation is correct here because $e([\mathbf{A}_1^\top]_1, [\mathbf{k}]_2) = e([\mathbf{A}_1^\top]_1, [\tilde{\mathbf{k}}]_2)$.

Key Queries. For each key query $\mathbf{M}$, pick $\mathbf{K}' \leftarrow_{\mathrm{R}} \mathbb{Z}_p^{(2k+1)\times(\ell'-1)}$, $\mathbf{d}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{B})$ for $j \in [n]$ and output

$$\mathsf{sk}_{\mathbf{M}} := (\, \{\, [(\tilde{\mathbf{k}} \| \mathbf{K}')\mathbf{M}_j^\top + (\mathbf{W} + \mathbf{V}_j^{(2)})\mathbf{d}_j]_2,\ [\mathbf{d}_j]_2\ [(\mathbf{W}_0 + j \cdot \mathbf{W}_1 + \mathbf{U}_j^{(2)})\mathbf{d}_j]_2 \,\}_{j \in [n]}\,).$$

Ciphertext. Upon receiving a challenge attribute vector $\mathbf{x}^*$ and two equal-length messages m_0, m_1, pick $b \leftarrow_{\mathrm{R}} \{0,1\}$, $\mathbf{c}, \mathbf{c}_j \leftarrow_{\mathrm{R}} \mathsf{span}(\mathbf{A}_1, \mathbf{a}_2)$ for all $j \in [n]$ and output

$$\mathsf{ct}_{\mathbf{x}^*} := ([\mathbf{c}^\top]_1,\ \{\, [\mathbf{c}^\top(\mathbf{W} + \mathbf{V}_j^{(2)}) + \mathbf{c}_j^\top(\mathbf{W}_0 + j \cdot \mathbf{W}_1 + \mathbf{U}_j^{(2)})]_1,\ [\mathbf{c}_j^\top]_1 \}_{j : x_j^* = 1},\ e([\mathbf{c}^\top]_1, [\tilde{\mathbf{k}}]_2) / \boxed{e([\mathbf{c}^\top]_1, [\alpha \mathbf{a}_2^{\parallel}]_2)} \cdot m_b\,).$$

Guess. When $\mathcal{A}$ halts with output b', $\mathcal{B}_1$ outputs 1 if $b = b'$ and 0 otherwise.

Observe that the boxed term is uniformly distributed over $\mathbb{Z}_p$ with high probability conditioned on other part of the simulation. Hence it does not change the distribution if we replace m_b with a random message m which leads to $\mathsf{Game}_{\mathsf{Final}}$. ☐

E Hybrid Argument for Entropy Expansion in Section 8.1

To prove the entropy expansion statement in (13) for our composite-order unbounded CP-ABE, we employ the following hybrid argument:

$$\left\{ \begin{array}{l} \mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1} \\ \mathsf{ct}: g_1^{s}, \{ g_1^{s_j w},\ g_1^{s_j},\ g_1^{s_j(w_0 + j\cdot w_1)} \}_{j\in[n]} \\ \mathsf{sk}: h_1^{r}, \{ h_1^{rw + r_j(w_0 + j \cdot w_1)},\ h_1^{r_j} \}_{j \in [n]} \end{array} \right\} \quad \text{LHS in (13)}$$

$$\overset{p_1 \mapsto p_1p_2p_3}{\approx_c} \left\{ \begin{array}{l} \mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1} \\ \mathsf{ct}: g_1^{s}, \{ g_1^{s_j w},\ g_1^{s_j},\ g_1^{s_j(w_0 + j\cdot w_1)} \}_{j\in[n]} \\ \mathsf{sk}: \boxed{h_{123}}^{r}, \{ \boxed{h_{123}}^{rw + r_j(w_0 + j \cdot w_1)},\ \boxed{h_{123}}^{r_j} \}_{j \in [n]} \end{array} \right\}$$

$$\overset{p_1 \mapsto p_1p_2}{\approx_c} \left\{ \begin{array}{l} \mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1} \\ \mathsf{ct}: g_1^{s} \cdot \boxed{g_2^{s}}, \{ g_1^{s_j w},\ g_1^{s_j},\ g_1^{s_j(w_0 + j\cdot w_1)} \}_{j\in[n]} \\ \mathsf{sk}: h_{123}^{r}, \{ h_{123}^{rw + r_j(w_0 + j \cdot w_1)},\ h_{123}^{r_j} \}_{j \in [n]} \end{array} \right\}$$

$$\overset{\text{Lemma 3}}{\approx_c} \left\{ \begin{array}{l} \mathsf{aux}: g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1} \\ \mathsf{ct}: g_1^{s} \cdot g_2^{s}, \{ g_1^{s_j w} \cdot \boxed{g_2^{s_j w}},\ g_1^{s_j} \cdot \boxed{g_2^{s_j}},\ g_1^{s_j(w_0 + j\cdot w_1)} \cdot \boxed{g_2^{s_j u_j}} \}_{j\in[n]} \\ \mathsf{sk}: h_{13}^{r} \cdot \boxed{h_2^{r}}, \{ h_{13}^{rw + r_j(w_0 + j \cdot w_1)} \cdot \boxed{h_2^{rw + r_j u_j}},\ h_{13}^{r_j} \cdot \boxed{h_2^{r_j}} \}_{j \in [n]} \end{array} \right\}$$

$$
\stackrel{p_1\mapsto p_1p_3}{\approx_c}
\left\{\begin{aligned}
&\mathsf{aux}:\ g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}\\
&\mathsf{ct}:\ g_1^{s}\cdot g_2^{s},\ \{g_1^{s_j w}\cdot g_2^{s_j w},\ g_1^{s_j}\cdot g_2^{s_j},\ g_1^{s_j(w_0+j\cdot w_1)}\cdot g_2^{s_j u_j}\}_{j\in[n]}\\
&\mathsf{sk}:\ \boxed{h_1}^{r}\cdot h_2^{r},\ \{\boxed{h_1}^{rw+r_j(w_0+j\cdot w_1)}\cdot h_2^{rw+r_j u_j},\ \boxed{h_1}^{r_j}\cdot h_2^{r_j}\}_{j\in[n]}
\end{aligned}\right\}
$$

$$
\stackrel{(14)}{\approx_c}
\left\{\begin{aligned}
&\mathsf{aux}:\ g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}\\
&\mathsf{ct}:\ g_1^{s}\cdot g_2^{s},\ \{g_1^{s_j w}\cdot \boxed{g_2^{s_j v_j}},\ g_1^{s_j}\cdot g_2^{s_j},\ g_1^{s_j(w_0+j\cdot w_1)}\cdot g_2^{s_j u_j}\}_{j\in[n]}\\
&\mathsf{sk}:\ h_1^{r}\cdot h_2^{r},\ \{h_1^{rw+r_j(w_0+j\cdot w_1)}\cdot \boxed{h_2^{r v_j}}\cdot h_2^{r_j u_j},\ h_1^{r_j}\cdot h_2^{r_j}\}_{j\in[n]}
\end{aligned}\right\}\quad \text{RHS in (13)}
$$

F Hybrid Argument for Entropy Expansion in Section 9.2

To prove the entropy expansion statement in (18) for our unbounded KP-ABE for arithmetic span program, we employ the following hybrid argument:

$$
\left\{\begin{aligned}
&\mathsf{aux}:\ g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}, g_1^{w'}, g_1^{w'_0}, g_1^{w'_1}\\
&\mathsf{ct}:\ g_1^{s},\ \{g_1^{sw+s_j(w_0+j\cdot w_1)},\ g_1^{s_j},\ g_1^{sw'+s'_j(w'_0+j\cdot w'_1)},\ g_1^{s'_j}\}_{j\in[n]}\\
&\mathsf{sk}:\ \{h_1^{r_j w},\ h_1^{r_j w'},\ h_1^{r_j},\ h_1^{r_j(w_0+j\cdot w_1)},\ h_1^{r_j(w'_0+j\cdot w'_1)}\}_{j\in[n]}
\end{aligned}\right\}\quad \text{LHS in (18)}
$$

$$
\stackrel{p_1\mapsto p_1p_2p_3}{\approx_c}
\left\{\begin{aligned}
&\mathsf{aux}:\ g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}, g_1^{w'}, g_1^{w'_0}, g_1^{w'_1}\\
&\mathsf{ct}:\ g_1^{s},\ \{g_1^{sw+s_j(w_0+j\cdot w_1)},\ g_1^{s_j},\ g_1^{sw'+s'_j(w'_0+j\cdot w'_1)},\ g_1^{s'_j}\}_{j\in[n]}\\
&\mathsf{sk}:\ \{\boxed{h_{123}}^{r_j w},\ \boxed{h_{123}}^{r_j w'},\ \boxed{h_{123}}^{r_j},\ \boxed{h_{123}}^{r_j(w_0+j\cdot w_1)},\ \boxed{h_{123}}^{r_j(w'_0+j\cdot w'_1)}\}_{j\in[n]}
\end{aligned}\right\}
$$

$$
\stackrel{p_1\mapsto p_1p_2}{\approx_c}
\left\{\begin{aligned}
&\mathsf{aux}:\ g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}, g_1^{w'}, g_1^{w'_0}, g_1^{w'_1}\\
&\mathsf{ct}:\ g_1^{s}\cdot\boxed{g_2^{s}},\ \{g_1^{sw}\cdot\boxed{g_2^{sw}}\cdot g_1^{s_j(w_0+j\cdot w_1)},\ g_1^{s_j},\ g_1^{sw'}\cdot\boxed{g_2^{sw'}}\cdot g_1^{s'_j(w'_0+j\cdot w'_1)},\ g_1^{s'_j}\}_{j\in[n]}\\
&\mathsf{sk}:\ \{h_{123}^{r_j w},\ h_{123}^{r_j w'},\ h_{123}^{r_j},\ h_{123}^{r_j(w_0+j\cdot w_1)},\ h_{123}^{r_j(w'_0+j\cdot w'_1)}\}_{j\in[n]}
\end{aligned}\right\}
$$

$$
\stackrel{(19)}{\approx_c}
\left\{\begin{aligned}
&\mathsf{aux}:\ g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}, g_1^{w'}, g_1^{w'_0}, g_1^{w'_1}\\
&\mathsf{ct}:\ g_1^{s}\cdot g_2^{s},\ \{g_1^{sw}\cdot g_2^{sw}\cdot g_1^{s_j(w_0+j\cdot w_1)}\cdot\boxed{g_2^{s_j u_j}},\ g_1^{s_j}\cdot\boxed{g_2^{s_j}},\ g_1^{sw'}\cdot g_2^{sw'}\cdot g_1^{s'_j(w'_0+j\cdot w'_1)}\cdot\boxed{g_2^{s'_j u'_j}},\ g_1^{s'_j}\cdot\boxed{g_2^{s'_j}}\}_{j\in[n]}\\
&\mathsf{sk}:\ \{h_{13}^{r_j w}\cdot\boxed{h_2^{r_j w}},\ h_{13}^{r_j w'}\cdot\boxed{h_2^{r_j w'}},\ h_{13}^{r_j}\cdot\boxed{h_2^{r_j}},\ h_{13}^{r_j(w_0+j\cdot w_1)}\cdot\boxed{h_2^{r_j u_j}},\ h_{13}^{r_j(w'_0+j\cdot w'_1)}\cdot\boxed{h_2^{r_j u'_j}}\}_{j\in[n]}
\end{aligned}\right\}
$$

$$
\stackrel{p_1\mapsto p_1p_3}{\approx_c}
\left\{\begin{aligned}
&\mathsf{aux}:\ g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}, g_1^{w'}, g_1^{w'_0}, g_1^{w'_1}\\
&\mathsf{ct}:\ g_1^{s}\cdot g_2^{s},\ \{g_1^{sw}\cdot g_2^{sw}\cdot g_1^{s_j(w_0+j\cdot w_1)}\cdot g_2^{s_j u_j},\ g_1^{s_j}\cdot g_2^{s_j},\ g_1^{sw'}\cdot g_2^{sw'}\cdot g_1^{s'_j(w'_0+j\cdot w'_1)}\cdot g_2^{s'_j u'_j},\ g_1^{s'_j}\cdot g_2^{s'_j}\}_{j\in[n]}\\
&\mathsf{sk}:\ \{\boxed{h_1}^{r_j w}\cdot h_2^{r_j w},\ \boxed{h_1}^{r_j w'}\cdot h_2^{r_j w'},\ \boxed{h_1}^{r_j}\cdot h_2^{r_j},\ \boxed{h_1}^{r_j(w_0+j\cdot w_1)}\cdot h_2^{r_j u_j},\ \boxed{h_1}^{r_j(w'_0+j\cdot w'_1)}\cdot h_2^{r_j u'_j}\}_{j\in[n]}
\end{aligned}\right\}
$$

$$
\stackrel{(20)}{\approx_c}
\left\{\begin{aligned}
&\mathsf{aux}:\ g_1, g_1^{w}, g_1^{w_0}, g_1^{w_1}, g_1^{w'}, g_1^{w'_0}, g_1^{w'_1}\\
&\mathsf{ct}:\ g_1^{s}\cdot g_2^{s},\ \{g_1^{sw}\cdot\boxed{g_2^{s v_j}}\cdot g_1^{s_j(w_0+j\cdot w_1)}\cdot g_2^{s_j u_j},\ g_1^{s_j}\cdot g_2^{s_j},\ g_1^{sw'}\cdot\boxed{g_2^{s v'_j}}\cdot g_1^{s'_j(w'_0+j\cdot w'_1)}\cdot g_2^{s'_j u'_j},\ g_1^{s'_j}\cdot g_2^{s'_j}\}_{j\in[n]}\\
&\mathsf{sk}:\ \{h_1^{r_j w}\cdot\boxed{h_2^{r_j v_j}},\ h_1^{r_j w'}\cdot\boxed{h_2^{r_j v'_j}},\ h_1^{r_j}\cdot\boxed{h_2^{r_j}},\ h_1^{r_j(w_0+j\cdot w_1)}\cdot\boxed{h_2^{r_j u_j}},\ h_1^{r_j(w'_0+j\cdot w'_1)}\cdot\boxed{h_2^{r_j u'_j}}\}_{j\in[n]}
\end{aligned}\right\}\quad \text{RHS in (18)}
$$