How to Record Quantum Queries, and Applications to Quantum Indifferentiability

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Abstract

The quantum random oracle model (QROM) has become the standard model in which to prove the post-quantum security of random-oracle-based constructions. Unfortunately, none of the known proof techniques allow the reduction to record information about the adversary's queries, a crucial feature of many classical ROM proofs, including all proofs of indifferentiability for hash function domain extension.

In this work, we give a new QROM proof technique that overcomes this "recording barrier". Our central observation is that when viewing the adversary's query *and the oracle itself* in the Fourier domain, an oracle query switches from writing to the adversary's space to writing to the oracle itself. This allows a reduction to simulate the oracle by simply recording information about the adversary's query in the Fourier domain.

We then use this new technique to show the indifferentiability of the Merkle-Damgard domain extender for hash functions. Given the threat posed by quantum computers and the push toward quantum-resistant cryptosystems, our work represents an important tool for efficient post-quantum cryptosystems.

1 Introduction

The random oracle model [BR93] has proven to be a powerful tool for heuristically proving the security of schemes that otherwise lacked a security proof. In the random oracle model (ROM), a hash function H is modeled as a truly random function that can only be evaluated by querying an oracle for H. A scheme is secure in the ROM if it can be proven secure in this setting. Of course, random oracles cannot be efficiently realized; in practice, the random oracle is replaced with a concrete efficient hash function. The hope is that the ROM proof will indicate security in the real world, provided there are no structural weaknesses in the concrete hash function.

Meanwhile, given the looming threat of quantum computers [IBM17], there has been considerable interest in analyzing schemes for "post-quantum" security — security against quantum attack[NIS17, Son14, ATTU16, CBH+17, YAJ+17, CDG+17, CDG+15]. Many of the proposed scheme are random oracle schemes; Boneh et al. [BDF+11] argue that the right way of modeling the random oracle in the quantum setting is to use the quantum random oracle model, or QROM. Such a model allows a quantum attacker to query the random oracle on a quantum superposition of inputs. The idea is that a real-world quantum attacker, who knows the code for the concrete hash function, can evaluate the hash function on superposition in order to perform tasks such as Grover search [Gro96] or collision finding [BHT98]. In order to accurately capture such real-world attacks, it is crucial to model the random oracle to allow for such superposition queries. The quantum random oracle

model has been used in a variety of subsequent works to prove the post-quantum security of cryptosystems [BDF+11, Zha12b, Zha15, TU16, Eat17].

The Recording Barrier. Unfortunately, proving security in the random oracle model can be extremely difficult. Indeed, in the classical random oracle model, one can copy down the adversary's queries as a means to learning what points the adversary is interested in. Many classically security proofs crucially use this information in order to construct a new adversary which solves some hard underlying problem, reaching a contradiction. In the quantum setting, such recording is impossible, by the quantum no-cloning theorem. One can try to record some information about the query, but this amounts to a measurement of the adversary's query state which can be detected by the adversary. A mischievous adversary may refuse to continue if it detects such a measurement, rendering the adversary useless for solving the underlying hard problem. Because of the difficulty in reading an adversary's query, it also becomes hard to adaptively program the random oracle, another very common proof technique.

This difficulty has led authors to develop new quantum-sound proof techniques to replace classical techniques, such as Zhandry's small-range distributions [Zha12a]. These proof techniques choose the oracle from a careful distribution that allows for proofs to go through. However, every such proof technique always chooses a classical oracle at the beginning of the experiment, and never changes it afterward. This seems to limit what can be proved using such non-recording techniques. For example, Dagdelen, Fischlin, and Gagliardoni [DFG13] show that such natural proof techniques are likely incapable of proving the security of Fiat-Shamir[1]. This leads to a natural question:

Is it possible to record information about an adversary's query without the adversary detecting

Enter Indifferentiability. The random oracle model (quantum or otherwise) assumes the adversary treats the hash function as a monolithic object. Unfortunately, hash functions in practice are usually built from smaller building blocks, called compression functions. If one is not careful, hash functions built in this way are vulnerable to attacks such as length-extension attacks. Coron et al. [CDMP05] show that a hash function built from a compression function can be eas good as a monolithic oracle in many settings if it satisfies a notion of *indifferentiability*, due to Maurer, Renner, and Holenstein [MRH04]. Roughly, in indifferentiability, an adversary A has oracle access to both h and H. In one case, h is a random function, and H is built from h according to the hash function construction. In the other case, H is a random function, and h is simulated so as to be consistent with H. A hash function is indifferentiable from a random oracle if no efficient adversary can distinguish the two cases.

Coron et al.'s proof of indifferentiability for Merkle-Damgard requires the simulator to remember the queries that the adversary has made. This is actually inherent for any domain extender, by a simple counting argument discussed below. In the quantum setting, such recording presents a serious issue, as recording a query is equivalent (from the adversary's point of view) to measuring the query. As any measurement will disturb the quantum system, such measurement may be detectable to the adversary. Note that in the case where A is interacting with a truly random h, there is no measurement happening. Therefore, if such a measurement can be detected, the adversary can distinguish the two cases, breaking indifferentiability.

[1] We note that if the underlying building blocks are strengthened, Fiat-Shamir was proven secure by Unruh [Unr16]

Example. To illustrate what might go wrong, we will use the simple example from Coron et al. [CDMP05]. Here, we will actually assume access to two independent compression functions $h_0, h_1 : \{0,1\}^{2n} \to \{0,1\}^n$. We will define $H : \{0,1\}^{3n} \to \{0,1\}^n$ as $H(x,y) = h_1(h_0(x), y)$, where $x \in \{0,1\}^2 n, y \in \{0,1\}^n$.

To argue that H is indifferentiable from a random oracle, Coron et al. use the following simulator S, which has access to H, and tries to implement the oracles h_0, h_1. S works as follows:

- S keeps databases D_0, D_1, which will contain tuples (x,y). D_b containing (x,y) means that S has set the $h_b(x) = y$.
- h_0 is implemented on the fly: every query on x looks up $(x,y) \in D_0$, and returns y if it is found; if no such pair is found, a random y is chosen and returns, and (x,y) is added to D_0.
- h_1 is more interesting. By default, h_1 is answered randomly on the fly as in h_0. However, it needs to make sure that $h_1(h_0(x), y)$ always evaluates to $H(x,y)$, else it is trivial to distinguish the two words. Therefore, on a query (z,y), h_1 will check if there is a pair (x,z) in D_0. If so, it will reasonably guess that the adversary is trying to evaluate $H(x,y)$, and respond by making a query to $H(x,y)$. Otherwise it will resort to the default simulation.

Note that by defining the simulator in this way, if the adversary ever tries to evaluate H on (x,z) by first making a query x to h_0 to get y, and then making a query (y,z) to h_1, the simulator will correctly set the output of h_1 to $H(x,z)$, so that the adversary will get a result that is consistent with H. However, note that it is crucial that S wrote down the queries made to h_0, or else it will not know which point to query H when simulating h_1.

Now consider a quantum adversary. A quantum query to, say, h_0 has the form

$$\sum_{x \in \{0,1\}^{2n}, u \in \{0,1\}^n} \alpha_{x,u} |x, u\rangle$$

In response, the oracle for h_0 will perform the map $|x,u\rangle \mapsto |x, u \oplus h_0(x)\rangle$. This will transform the query state to

$$\sum_{x \in \{0,1\}^{2n}, u \in \{0,1\}^n} \alpha_{x,u} |x, u \oplus h_0(x)\rangle$$

Now, imagine our simulator trying to answer queries to h_0 in superposition. For simplicity, suppose this is the first query to h_0, so D_0 is empty. The natural approach is to just have S store it's database D_0 in superposition, performing a map that may look like $|x,u\rangle \mapsto |x, u \oplus y\rangle \otimes |x, y\rangle$, where y is chosen randomly, and everything to the left of the $\otimes$ is the simulators state.

But now consider the following query by an adversary. It sets up the uniform superposition $\sum_{x,u} |x,u\rangle$ and queries. In the case where h_0 is a classical function, then this state becomes

$$\sum_{x,u} |x, u \oplus h_0(x)\rangle = \sum_{x,u} |x,u\rangle$$

Namely, the state is unaffected by making the query. In contrast, the simulated query would result in

$$\sum_{x,u} |x, u \oplus y\rangle \otimes |x, y\rangle$$

Here, the adversary's state is now entangled with the simulator's. It is straightforward to detect this entanglement by applying the Quantum Fourier Transform (QFT) to the adversary's x registers,

and then measuring the result. In the case where the adversary is interacting with a random h_0, the QFT will result in a 0. In the simulated case, the QFT will result in a random string. These two cases are therefore easily distinguishable.

One natural way to try to remedy the issue is to not store a database for h_0 or h_1. This might be achieved by setting $h_0(x)$ to be a quantum pseudorandom function [Zha12a] evaluated at x. Such a function will be indisitnguishable from a truly random $h_0(x)$ [2]. This will certainly fix the issue above, but introduce new problems. Now when the adversary makes a query to h_1, the simulator needs to decide if the query represents an attempt at evaluating H, and if so, it must program the output of h_1 accordingly. However, without knowing what inputs the adversary has queried to h_0, it seems impossible for the simulator to determine which point the adversary is interested in. For example, if the adversary queries h_1 on (y, z), there will be roughly 2^n possible x that gave rise to this y (since h_0 is compressing). Therefore, the simulator must choose from one of 2^n inputs of the form (x, z) on which to query H.

To make matters even more complicated, an adversary can submit the uniform superposition $\sum_x |x, 0\rangle$, resulting in the state $\sum_x |x, h_0(x)\rangle$, which causes it to "learn" $h_0(x)$. At this point, the simulator should be ready to respond to an h_1 query using x, meaning the simulator must be entangled with x. Then, at some later time, the adversary can query again on the state $\sum_x |x, h_0(x)\rangle$, resulting in the original state $\sum_x |x, 0\rangle$ again. The adversary can test that it received the correct state using the quantum Fourier transform. Therefore, after this later query, the simulator must be un-entangled with x. Even more complex strategies are possible, where the adversary can compute and un-compute h_0 in stages, so as to try to hide what it is doing from any potential simulator.

These issues are apparent in every construction of a hash function from a compression function. Indeed, it is easy to show the following: suppose there is a hash function $H : \{0,1\}^M \to \{0,1\}^N$ built from a compression function $h : \{0,1\}^m \to \{0,1\}^n$, and suppose it holds that $M + \log_2 N > m + \log_2 n$. Then it must be that any simulator for classical indifferentiability cannot answer the queries with the same exact function each time. This follows from the simple fact that in the simulated world, there are $2^{N \times 2^M}$ choices for H, while in the real world there are only $2^{n \times 2^m}$. If the simulator simulates using a fixed function S^H, then we have that $C^{S^H} = H$ in all but a negligible fraction of positions. However, since S^H is a function, and so there are only $2^{n \times 2^m}$ possibilities. And yet it must match a random function on most of its inputs, which is clearly impossible. Therefore, any simulator must actually answer queries dependent on previous queries — in other words, it must keep some state.

We therefore ask:

Is it possible to build large hash functions from small compression functions such that the hash functions are indifferentiable from a quantum-accessible random oracle?

1.1 This Work

In this work, perhaps surprisingly, we answer the question above in the affirmative. Namely, we give a new *compressed oracle technique*, which allows for recording the adversary's queries in a way that the adversary can never detect. The intuition is surprisingly simple: an adversary interacting with a random oracle can be thought of as being entangled with a uniform superposition of oracles. As entanglement is symmetric, if the adversary every has any information about the oracle, the

[2] Zhandry [Zha12b] shows that actually, a t-wise independent function will suffice, for an appropriate choice of t

oracle must also have information about the adversary. Therefore a simulator can always get away with recording *some* information about the adversary.

We then use the technique to prove the indifferentiability of the Merkle-Damqård construction. We believe our new technique will be of independent interest; for example our technique also gives very simple proofs of several quantum query lower bounds.

The Compressed Oracle Technique. In order to prove indifferentiability, we devise a new way of analyzing quantum query algorithms

Consider an adversary interacting with an oracle $h : \{0,1\}^m \to \{0,1\}^n$. It is well established that the usual quantum oracle mapping $|x, y\rangle \mapsto |x, y \oplus h(x)\rangle$ is equivalent to the "phase" oracle, which maps $|x, u\rangle \mapsto (-1)^{u \cdot h(x)}|x, u\rangle$ (we discuss this equivalence in Section 3). For simplicity, in this paper we will focus on the phase oracle, which is without loss of generality.

Next, we note that the oracle h being chosen at random is equivalent (from the adversary's point of view) to h being in uniform superposition $\sum_h |h\rangle$. This is because the superposition can be reduced to a random h by measuring, and measuring the h registers (which it outside of A's view) is undetectable to A.

Therefore, we will imagine the h oracle as actually containing $\sum_h |h\rangle$. When A makes a query on $\sum_{x,u} \alpha_{x,u}|x, u\rangle$, the joint system of the adversary and oracle are

$$\sum_{x,u} \alpha_{x,u}|x, u\rangle \otimes \sum_h |h\rangle$$

The query introduces a phase term $(-1)^{u \cdot h(x)}$, so the joint system becomes

$$\sum_{x,u} \alpha_{x,u}|x, u\rangle \otimes \sum_h |h\rangle(-1)^{u \cdot h(x)}$$

We normally think of the phase as being returned to the adversary, but the phase really affects the entire system.

Now, we will think of h as a vector of length $2^m \times n$ by simply writing down all of the outputs of h as a list. We will think of each x, u pair as a a point function $P_{x,u}$ which outputs y on x and 0 elsewhere. Using our encoding of functions as vectors, we can write $u \cdot h(x)$ as $P_{x,y} \cdot h$.

We can therefore write the post-query state as

$$\sum_{x,u} \alpha_{x,u}|x, u\rangle \otimes \sum_h |h\rangle(-1)^{h \cdot P_{x,u}}$$

In general, the state after making q queries can be written as

$$\sum_{x_1,\ldots,x_q,u_1,\ldots,u_q} \alpha_{x_1,\ldots,x_q,u_1,\ldots,u_q}|\psi_{x_1,\ldots,x_q,u_1,\ldots,u_q}\rangle \otimes \sum_h |h\rangle(-1)^{h \cdot (P_{x_1,u_1}+\cdots+P_{x_q,u_q})}$$

Another view of the above is that we can view the oracle as a purification of the adversary's state, and the purification can be updated through the adversary's queries by following simple update rules.

Next, notice that by applying the Quantum Fourier transform to h, the h registers will now contain $(P_{x_1,u_1} + \cdots + P_{x_q,u_q})$ mod 2. Working in the Fourier domain, we see that each query simply adds $P_{x,u}$ (modulo 2) to the result. In the Fourier domain, the initial state is 0.

Therefore, from A's point of view, it is indistinguishable whether the oracle for h is implemented as above, or it is implemented as follows:

- The oracle keeps as state a vector $D \in \{0,1\}^{n \times 2^m}$, initially set to 0.
- On any oracle query, the oracle performs the map $|x,u\rangle \otimes |D\rangle \mapsto |x,u\rangle \otimes |D \oplus v(x,u)\rangle$

In other words, the oracle can actually be implemented by recording and updating phase information about the queries being in made.

We can now take this a couple steps further. Notice that after q queries, D is non-zero on at most q inputs. Therefore, we can store the database in an extremely compact form, namely the list of (x,y) pairs where $y = D(x)$ and $y \neq 0$. Notice that this allows us to efficiently simulate a random oracle, without an a priori bound on the number of queries. Previously, simulating random oracles exactly and efficiently required $2q$-wise independent functions [Zha12b] and hence required knowing q up front. We therefore believe this simulation will have independent applications for the efficient simulation of quantum oracles. We will call this the compressed Fourier oracle.

We can then take our compressed Fourier oracle, and convert it back into a primal-domain oracle. Namely, for each (x,y) pair, we perform the QFT on the y registers. The result is a superposition of databases of (x,w) pairs, where w represents $h(x)$. For any pair not in the database, $h(x)$ is implicitly a uniform superposition of inputs. We call this the compressed standard oracle. It represents what the adversary knows about the function h: if (x,y) is in the database then the adversary "knows" $h(x) = y$, and otherwise, the adversary "knows" nothing about $h(x)$.

Applying Compressed Oracles to Indifferentiability. The compressed standard oracle offers a simple way to keep track of the queries the adversary has made. In particular, it tracks exactly the kind of information needed in the classical indifferentiability proof above, namely whether or not a particular value has been queries by the adversary, and what the value of the query at that point is. We use this to give a quantum indifferentiability proof for Merkle-Damgård construction using prefix-free encodings [CDMP05].

To illustrate our ideas, consider our simple example above with h_0, h_1 and H. Our simulator will simulate h_0 as in the compressed standard oracle, keeping a list D_0 of (x,y) pairs. Next, our simulator must handle h_1 queries. When given a phase query $|y,z\rangle$, the simulator does the following. If first looks for a pair (x,y') in D_0 with $y' = y$. If one is found, it reasonably guesses that the adversary is interested in computing $H(x,z)$, and so it makes a query on x,z to H. Otherwise, it is reasonable to guess that the adversary is not trying to compute H on any input, since the adversary does not "know" any inputs to h_0 that would result in a query to h_1 on y,z.

While the above intuition appears to work, we need to make sure the simulator does not disturb the compressed oracle. Unfortunately, some disturbance is inherent. This is because testing whether an element is contained in D_0 amounts to a measurement in the Fourier domain; meanwhile, determining the value of $h_0(x)$ is a measurement in the primal. These two measurements do not commute, so by the uncertainty principle it is impossible to perform both measurements perfectly.

Nonetheless, we show that the errors are small. Intuitively, we implement the simulator where it only performs a polynomial number of measurements of the form "is $h_0(x) = y$". Indeed, the classical simulator can easily be implemented using such tests, and we just use the same implementation. Meanwhile, testing whether an element is contained in D_0 is simply a Fourier-domain test of the form "is $h_0(x) = 0$". These primal and Fourier measurements still do not commute, but it is straightforward to show that they "almost" commute. Thus, the simulator can perform these measurements without perturbing the state significantly.

This shows that h_0 queries are correctly simulated; we also need to show that h_1 queries are correctly simulated and consistent with H. The intuition above suggests that h_1 should be consistent with H, and indeed we show this using a careful sequence of hybrids.

The Power of Forgetting. Surprisingly, our simulator ends up strongly resembling the classical simulator. It is natural to ask, therefore, how the simulator gets around the difficulties outlined above.

First, notice that the query $\sum_{x,u} |x,u\rangle$ in our example, when implemented as a phase query, simply becomes $\sum_x |x,0\rangle$. Since $u = 0$, this query has no effect on the oracle's state. This means the oracle remains un-entangled with the adversary, as desired.

Second, a query $\sum_x |x,0\rangle$ becomes $\sum_{x,u} |x,u\rangle$ for a phase query. After applying the query, the joint quantum system of the adversary and simulator becomes

$$\sum_x |x,u\rangle \sum_y |\{(x,y)\}\rangle (-1)^{y\cdot u}$$

Thus, the simulator can clearly tell that the adversary has queried on x. Later, when the adversary queries on the same state a second time, the query will erase the phase term. Then the simulator will actually remove the (x,y) pair from it's database. This is because in the Fourier domain, the compressed Fourier oracle now contains $(x,0)$, which gets removed. Thus, after this later query, the database contains no information about x. Hence, the adversary is un-entangled with x, and so it's tests will output the correct value.

Ultimately then, the key difference between our simulator and the natural quantum analog of the classical simulator is that our simulator must be ready to *forget* some of the oracle points it simulated previously. By implementing h_0 as a compressed oracle, it will forget *exactly* when it needs to so that the adversary can never detect that it is interacting with a simulated oracle.

Simple Quantum Query Complexity Lower Bounds. We also show that our compressed oracles can be used to give very simple quantum query complexity lower bounds for problems for *random functions*, such as pre-image search, collision finding, k-SUM, and k-collision.

Our proof strategy is roughly as follows. First, since intuitively the adversary has no knowledge of values of h outside of D, except with very small probability any successful algorithm will output points in D. Therefore it suffices to bound the number of queries required to get D to contain a pre-image/collision/k-sum/k-collision.

For pre-image search, we re-prove the optimal lower bound of $\Omega(2^{n/2})$ queries of [BBBV97], but for random functions; note that pre-image search for random functions and worst-case functions is equivalent using simple reductions. The proof appears superficially similar to [BBBV97]: we show that each query can increase the "amplitude" on "good" databases by a small $O(2^{-n/2})$ amount. After q queries, this amplitude becomes $O(q/2^{n/2})$, which we then square to get the probability of a "good" database. The proof is only slightly over a page once the compressed oracle formalism has been given.

We then re-prove the optimal collision lower bound of $\Omega(2^{n/3})$ queries for random functions, matching the worst case bound [AS04] and the more recent average case bound [Zha15]. Remarkably, our proof involves only a few lines of modification to the pre-image lower bound. We show that the amplitude on "good" databases increases by $O(\sqrt{q} \times 2^{n/2})$ for each query, where the extra $\sqrt{q}$

intuitively comes from the fact that the database has size at most q, giving q opportunities for a collision every time a new entry is added to the database[3].

In contrast to our very simple proof, the prior bounds involved very different techniques and were much more complicated. Also note that prior works could not prove directly that finding collisions were hard. Instead, they show that distinguishing a function with many collisions from an injective function was hard. This then only works directly for expanding functions, which are of little interest to cryptographers. Zhandry [Zha15] shows for random functions a reduction from expanding functions to compressing functions, giving the desired lower bound for compressing functions. Our proof, in contrast, works directly with functions of arbitrary domain and range. These features suggests that our proof technique is fundamentally different than those of prior works.

By generalizing our collision bound slightly, we can obtain an $\Omega(2^{n/(k+1)})$ lower bound for finding a set of distinct points $x_1, \ldots, x_k$ such that $\sum_i H(x_i) = 0$. This bound is tight by adapting the collision-finding algorithm of [BHT98] to this problem. Again, our proof is obtained by modifying just a few lines of the pre-image search proof.

This problem is usually called the k-SUM problem. However, k-SUM is typically described as a decisional problem, deciding whether a set of such points exists. This decisional problem is usually stated to have a query complexity of $\Theta(2^{m\frac{k}{(k+1)}})$ [BS13], where m is the number of input bits. By setting $n = mk$ — which is the setting where a random function will have approximately a single k-SUM solution — we coincide with the existing lower bounds. The prior proof [BS13] actually only works in the setting where $n \gtrsim km$ and does not apply to settings where there are many k-SUM solutions. In contrast, our lower bound works for *any* setting of m and n.

With simple modifications, we can also get new bounds for the quantum query complexity of finding k-collisions — that is, k distinct inputs that all map to the same output. Intuitively, we observe that if adding a point to a database caused it to contain a k-collision, then it must already have a $(k-1)$-collision. By inductively using the lower bound for $(k-1)$-collisions, we can get stronger bounds for the amplitude increase of each query. Ultimately, we obtain a $\Omega(2^{n\times\frac{k-1}{2k-1}})$ query lower bound for finding k-collisions. This is tight for $k = 2$ (that is, for regular collisions), though we do not know of any matching upper bound for $k \geq 3$.

1.2 Related Works

Ristenpart, Shacham, and Shrimpton [RSS11] shows that indifferentiability is insufficient for replacing a concrete hash function with a random oracle in the setting of multi-stage games. Nonetheless, Mittelbach [Mit14] shows that indifferentiability can still be useful in these settings. Exploring the quantum analogs of these results is an interesting direction for future research.

1.3 Independent Work

Independently and concurrently, Carstens et al. [CETU18] also discuss the difficulties of quantum indifferentiability. They present a very similar argument for the difficulty of proving indifferentiability as we do. They then conclude that "quantum indifferentiability is probably impossible to achieve in many situations", and state a formal conjecture regarding such an impossibility. Finally, they prove concretely an impossibility for *perfect* simulation for certain protocols.

[3] and the square root comes from the fact that the norm of the sum of q unit vectors of disjoint support is $\sqrt{q}$

Our work demonstrates that Carstens et al.'s conjecture is false, as their conjecture would apply to Merkle-Damgård. Note also that our simulator is *im*perfect as it perturbs the state slightly with every query, and therefore the formal impossibility does not apply.

2 Preliminaries

2.1 Quantum

A quantum system Q is defined over a finite set B of classical states. We will generally consider $B = \{0,1\}^n$. A **pure** state over Q is an L_2-normalized vector in $\mathbb{C}^{|B|}$, which assigns a (complex) weight to each element in B. Thus the set of pure states forms a complex Hilbert space. A **qubit** is a quantum system defined over $B = \{0,1\}$. Given a quantum system Q_0 over B_0 and a quantum system Q_1 over B_1, we can define the product system $Q = Q_0 \times Q_1$ over $B = B_0 \times B_1 = \{(b_0, b_1) : b_0 \in B_0, b_1 \in B_1\}$. Given a state $v_0 \in Q_0$ and $v_1 \in Q_1$, we define the product state $v_0 \otimes v_1$ in the natural way. An n-qubit system is then $Q = Q_0^{\oplus n}$ where Q_0 is a single qubit.

Bra-ket notation. We will think of pure states as column vectors. The pure state that assigns weight 1 to x and weight 0 to each $y \neq x$ is denoted $|x\rangle$. The set $\{|x\rangle\}$ therefore gives an orthonormal basis for the Hilbert space of pure states. We will call this basis the "computational basis." If a state $|\phi\rangle$ is a linear combination of several $|x\rangle$, we say that $|\phi\rangle$ is in "superposition." For a pure state $|\phi\rangle$, we will denote the conjugate transpose as the row vector $\langle\phi|$.

Entanglement. In general, a pure state $|\phi\rangle$ over $Q_0 \times Q_1$ cannot be expressed as a product state $|\phi_0\rangle \otimes |\phi_1\rangle$ where $|\phi_b\rangle \in Q_b$. If $|\phi\rangle$ is not a product state, we say that the systems Q_0, Q_1 are **entangled**. If $|\phi\rangle$ is a product state, we say the systems are **un-entangled**.

Evolution of quantum systems. A pure state $|\phi\rangle$ can be manipulated by performing a unitary transformation U to the state $|\phi\rangle$. We will denote the resulting state as $|\phi'\rangle = U|\phi\rangle$.

Basic Measurements. A pure state $|\phi\rangle$ can be measured; the measurement outputs the value x with probability $|\langle x|\phi\rangle|^2$. The normalization of $|\phi\rangle$ ensures that the distribution over x is indeed a probability distribution. After measurement, the state "collapses" to the state $|x\rangle$. Notice that subsequent measurements will always output x, and the state will always stay $|x\rangle$.

If $Q = Q_0 \times Q_1$, we can perform a **partial measurement** in the system Q_0 or Q_1. If $|\phi\rangle = \sum_{x \in B_0, y \in B_1} \alpha_{x,y}|x,y\rangle$, partially measuring in Q_0 will give x with probability $p_x = \sum_{y \in B_1} |\alpha_{x,y}|^2$. $|\phi\rangle$ will then collapse to the state $\sum_{y \in B_1} \frac{\alpha_{x,y}}{\sqrt{p_x}}|x,y\rangle$. In other words, the new state has support only on pairs of the form (x,y) where x was the output of the measurement, and the weight on each pair is proportional to the original weight in $|\phi\rangle$. Notice that subsequent partial measurements over Q_0 will always output x, and will leave the state unchanged.

The above corresponds to measurement in the computational basis. Measurements in other bases are possible to, and defined analogously. We will generally only consider measurements in the computational basis; measurements in other bases can be implemented by composing unitary operations with measurements in the computational basis.

Efficient Computation. A quantum computer will be able to perform a fixed, finite set G of unitary transformations, which we will call **gates**. For concreteness, we will use so-called Hadamard, phase, CNOT and $\pi/8$ gates, but the precise choice is not important for this work, so long as the gate set is "universal" for quantum computing.

Let Q be a quantum system on n qubits. Each gate costs unit time to apply, and each partial measurement also costs unit time. Therefore, an efficient quantum algorithm will be able to make a polynomial-length sequence of operations, where each operation is either a gate from G or a partial measurement in the computational basis. Here, "polynomial" will generally mean polynomial in n.

Examples of Quantum Computations.

- **Quantum Fourier Transform.** Let Q_0 be a quantum system over $B = \mathbb{Z}_q$ for some integer q. Let $Q = Q_0^{\otimes n}$. The Quantum Fourier Transform (QFT) performs the following operation efficiently:

$$\mathsf{QFT}|x\rangle = \frac{1}{\sqrt{q^n}}\omega_q^{x\cdot y}\sum_{y\in\{0,1\}^n}|y\rangle$$

  where $\omega_q = e^{2\pi i/q}$.

  In this paper, we will always consider $q = 2$, so that $\omega_q = (-1)$.

- **Efficient Classical Computations.** Any function that can be computed efficiently classically can be computed efficiently on a quantum computer. More specifically, if f is computable by a polynomial-sized circuit, then there is a efficiently computable unitary U_f on the quantum system $Q = Q_{in} \otimes Q_{out} \otimes Q_{work}$ with the property that: $U_f|x, y, 0\rangle = |x, y + f(x), 0\rangle$.

  Here, Q_{in} is a quantum system over the set of possible inputs, Q_{out} is a quantum system over the set of possible outputs, and Q_{work} is another quantum system that is just used for workspace, and is reset after use.

2.2 Almost Commuting Measurements

Lemma 2.1. *Consider a quantum state $|\psi\rangle$ on n qubits. Let P be the measurement that checks if $|\psi\rangle$ is zero in the computational basis. Let Q be the measurement that checks if $|\psi\rangle$ is zero in the Fourier basis. Let $\rho_{P,Q}$.....*

2.3 Indifferentiability

Let $h : \{0,1\}^m \to \{0,1\}^n$ be a random oracle, and let $C^h : \{0,1\}^M \to \{0,1\}^N$ be a polynomial-sized circuit that makes oracle queries to h.

Definition 2.2. C^h is quantum indifferentiable from a random oracle $H : \{0,1\}^M \to \{0,1\}^N$ if, for any polynomial-time distinguisher D, there exists a polynomial-time simulator S such that S makes queries to H and:

$$|\Pr[D^{h,C^h}() = 1] - \Pr[D^{S^H,H}() = 1]| < \mathsf{negl}$$

Here, D is given quantum oracle access to h and H. On the left hand size, $H = C^h$ and h is a random oracle, while on the right-hand side $h = S^H$ and H is a random oracle.

It is also straightforward to adapt this definition to handle the case of many random compression functions $h_1, \ldots, h_n$.

3 Oracle Variations

Here, we describe several oracle variations.

Standard Oracle. Here, the oracle $H : \{0,1\}^m \to \{0,1\}^n$ is represented as its truth table: a vector of size 2^m where each component is an n-bit string.

The oracle takes as input a tuple $|x, y\rangle \otimes |H\rangle$. Here, x, y is the query, and H is the truth-table of the function, as above. It performs the map

$$|x, y\rangle \otimes |H\rangle \mapsto |x, y \oplus H(x)\rangle \otimes |H\rangle$$

We will call this oracle StO.

Phase Oracle. This oracle takes as input a tuple $|x, z\rangle \otimes |H\rangle$. Here, x, z is the query, and H is the truth-table of the function. It performs the map

$$|x, z\rangle \otimes |H\rangle \mapsto (-1)^{y \cdot H(x)} |x, z\rangle \otimes |H\rangle$$

We will call this oracle PhO. Notice that PhO and StO are equivalent by applying the Fourier transform to the y registers. That is,

$$\mathsf{PhO} = (\mathsf{Id} \otimes \mathsf{H}^{\otimes n} \otimes \mathsf{Id}) \cdot \mathsf{StO} \cdot (\mathsf{Id} \otimes \mathsf{H}^{\otimes n} \otimes \mathsf{Id})$$

Fourier Oracle. This oracle takes as input a tuple $|x, z\rangle \otimes |D\rangle$, and D is the truth-table of the a function. It performs the map:

$$|x, z\rangle \otimes |D\rangle \mapsto |x, z\rangle \otimes |D \oplus P_{x,z}\rangle$$

Here, $P_{x,z}$ is the point function $P_{x,z}(x') = \begin{cases} z & \text{if } x' = x \\ 0 & \text{if } x' \neq n \end{cases}$, and $D \oplus P_{x,z}$ is the function $(D \oplus P_{x,z})(x') = D(x') \oplus P_{x,z}(x')$.

We will call this oracle $\mathsf{FourierO}$. Notice that $\mathsf{FourierO}$ and PhO are equivalent by applying the Fourier transform to the H registers. That is,

$$\mathsf{FourierO} = (\mathsf{Id} \otimes \mathsf{Id} \otimes \mathsf{H}^{\otimes n \times 2^m}) \cdot \mathsf{PhO} \cdot (\mathsf{Id} \otimes \mathsf{Id} \otimes \mathsf{H}^{\otimes n \times 2^m})$$

Notice that whether the Fourier or Phase oracle is simply a change of basis on the oracle side. This means it is completely inconsequential to the adversary which oracle is used.

Whether the oracle is implemented in the computational of Fourier domains is orthogonal to whether queries are made in the computational or phase domains. Therefore, we actually get four different oracle types: $\mathsf{StO}, \mathsf{PhO}, \mathsf{FourierStO}, \mathsf{FourierPhsO}$, where $\mathsf{FourierPhsO}$ is the PhO oracle described above, and $\mathsf{FourierStO}$ is the standard oracle, except that the oracle is represented in the Fourier domain.

Compressed Fourier Oracle. Typically, the oracle H will be chosen at random. We can simulate this by initially setting the oracle registers to be the uniform superposition over all truth tables. In the Fourier domain, this corresponds to the all-zeros function. Therefore, when we implement the Fourier Oracle, the oracle will typically start off containing just $D = 0^{n2^m}$.

Notice furthermore that after q queries, D will be the sum of q point functions — in particular, it will be zero in all but q locations. Therefore, we can actually compress D into an unordered list of (x, z) pairs such that $z \neq 0$.

Using this encoding, we get the compressed Fourier oracle $\mathsf{CFourierO}$. It performs the map:

$$|x, z\rangle \otimes |D\rangle \mapsto |x, z\rangle \otimes |D \oplus (x, z)\rangle$$

where $D \oplus (x, z)$ is the procedure that does the following:

- If $z = 0$ it outputs D.
- If there is a pair $(x, z') \in D$ for some z', it does the following:
  - If $z = z'$, it removes (x, z) from D, and outputs the new D
  - if $z \neq z'$, it replaces (x, z') with $(x, z \oplus z')$ and outputs the new D.
- Finally, if there is no such pair $(x, z') \in D$, it adds the pair (x, z) to D and outputs the new D.

Note that as described above, the compressed Fourier oracle takes implements a phase query to the adversary. We can also imagine it implementing a standard query. Thus we can actually get two compressed Fourier oracles $\mathsf{CFourierStO}$ and $\mathsf{CFourierPhsO}$. We will typically only use $\mathsf{CFourierO} = \mathsf{CFourierPhsO}$.

Compressed Standard and Phase Oracles. Finally, we can also compress the oracle in the computational basis, though more care is required. The compressed standard oracle will work by taking the compressed Fourier oracle, and simply performing the Fourier transform on all of the z terms. Thus, the state of the Compressed Standard Oracle will be an unordered list D of pairs (x, y). Note here that y can potentially be zero, but we still require that y is non-zero in the Fourier domain.

As always, this representation is independent of the adversary's view, and we can choose to give the adversary either a phase or standard oracle by appropriate change of basis on query registers. We will label these two oracles as $\mathsf{CStO}, \mathsf{CPhsO}$.

Note that in the compressed standard or phase oracle, each (x, y) in D pair corresponds to two statements: (1) x has been queries by the adversary, and (2) the value of $H(x) = y$. Unlike the standard oracle in which we can measure $H(x)$ to get y, here we cannot measure the y part of x, as this will destroy the invariant that the y registers must be non-zero in the Fourier domain. In particular, if we perform such a measurement, we will not be able to correctly remove (x, y) from D in future queries if needed.

For completeness, we specify the Compressed Standard Oracle CStO. Initially, the database D is empty. To answer a query on $|x, y\rangle$, do the following:

- Look for a tuple $(x, y') \in D$. If one is found, respond with $|x, y \oplus y'\rangle$

- If no tuple is found, create new registers initialized to the state $\frac{1}{\sqrt{2^n}}\sum_{y'}|y'\rangle$. At the registers (x, y') to D. Then respond with $|x, y \oplus y'\rangle$.
- Finally, regardless of whether the tuple was found or added, there is now a tuple (x, y') in D, which may have to be removed. To do so, measure whether the registers containing y' contain 0 in the Fourier basis. If so, remove the tuple from D. Otherwise, leave the tuple in D.

4 Quantum Query Lower Bounds Using Compressed Oracles

In this section, we re-prove several known query complexity lower bounds, as well as provide some new bounds. All these bounds follow from simple applications of our compressed oracles

4.1 Optimality of Grover Search

Here, we re-prove that the quadratic speed-up of Grover search is optimal. Specifically, we prove that for a random function $H : \{0,1\}^m \to \{0,1\}^n$, any q query algorithm has a success probability of at most $O(q^2/2^n)$ for finding a pre-image of 0^n. We will actually prove a stronger statement, namely that:

Theorem 4.1. *After q queries, if the compressed (standard or phase) oracle for H is measured, the resulting database will contain a pair of the form $(x, 0)$ with probability at most $O(q^2/2^n)$*

Since the adversary only has information about the points in the compressed oracle, the only way for it to achieve a non-trivial success probability is to output an element in the compressed oracle's database. Theorem 4.1 bounds the probability such points can be a pre-image of zero.

Proof. We will prove the theorem for the compressed phase oracle, the compressed standard oracle following from the fact that they are equivalent by applying a unitary to the adversary's registers.

Clearly, at the beginning, the compressed oracle's database is empty, so the probability the database contains an $(x, 0)$ is 0. Let $0 \in D$ mean that D contains a pair of the form $(x, 0)$. Let $D(x)$ be the function that outputs y if there is a pair (x, y) in D, and outputs $\perp$ if there is no such pair.Now, we will show that the probability cannot rise too much with each query. Consider the joint state of the adversary and oracle just before the qth query:

$$|\psi\rangle = \sum_{x,y,z,D} \alpha_{x,y,z,D}|x, y, z\rangle \otimes |D\rangle$$

Where D represents the compressed phase oracle, x, y as the query registers, and z as the adversary's private storage. We will write $|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle$ for three orthogonal unnormalized states:

- $|\psi_0\rangle$ is the state where all vectors have $y = 0$
- $|\psi_1\rangle$ is the state where $y \neq 0$ and $D(x) = \perp$. This means under the query, D will grow by a pair of the form (x, w)
- $|\psi_2\rangle$ is the state such that $y \neq 0$ and, after applying the query, $D(x) = \perp$. This means that $D(x) \neq \perp$ originally. In other words, it is the pre-image of the query over the state $|\psi_2'\rangle$, which contains only registers with $y \neq 0$ and $D(x) = \perp$.

- $|\psi_3\rangle$ is the state such that $y \neq 0$, $D(x) \neq \perp$, and $D(x)$ remains not equal to $\perp$ after the query.

Let $|\psi_i'\rangle$ be the image of $|\psi_i\rangle$ under the query. Let P be the projection onto the space spanned by states containing D with $0 \in D$.

We notice the following:

- $|\psi_0'\rangle = |\psi_0\rangle$.
- For a basis state $|x, y, z\rangle \otimes |D\rangle$ in the support of $|\psi_1\rangle$, its image under the query will by $|x, y, z\rangle \otimes \frac{1}{\sqrt{2^n}} \sum_w (-1)^{y \cdot w} |D \cup (x, w)\rangle$. Therefore, if we project onto P, we have $|x, y, z\rangle \otimes \frac{1}{\sqrt{2^n}} \sum_w (-1)^{y \cdot w} |D \cup (x, w)\rangle$ if $0 \in D$, and $\frac{1}{\sqrt{2^n}} |x, y, z\rangle \otimes |D \cup (x, 0)\rangle$ if $0 \notin D$.

  This means that $P|\psi_1'\rangle = UP|\psi_1\rangle + \frac{1}{\sqrt{2^n}} V(I - P)|\psi_1\rangle$, where U is the unitary that maps $|x, y, z\rangle \otimes |D\rangle \mapsto |x, y, z\rangle \otimes \frac{1}{\sqrt{2^n}} \sum_w (-1)^{y \cdot w} |D \cup (x, w)\rangle$, $I - P$ is the projection onto the space spanned by D with $0 \notin D$, and V is the unitary that maps $|x, y, z\rangle \otimes |D\rangle \mapsto |x, y, z\rangle \otimes |D \cup (x, 0)\rangle$.
- By symmetry, we have that $P|\psi_2\rangle = UP|\psi_2'\rangle + \frac{1}{\sqrt{2^n}} V(I - P)|\psi_2'\rangle$.
- Finally, for $|\psi_3\rangle$, since the query cannot change $D(x)$ except by changing $\perp$ to something or something to $\perp$, all D in $|\psi_3\rangle$ remain unchanged.

  Therefore, $P|\psi_3'\rangle = P|\psi_3\rangle$.

Putting this all toegher, we have that $\|P|\psi'\rangle\| = \|P|\psi\rangle\| + \frac{1}{\sqrt{2^n}} \|V(I - P)(|\phi_1\rangle - |\phi_2'\rangle)\| \leq \|P|\psi\rangle\| + \frac{2}{\sqrt{2^n}}$.

Therefore, after q queries, we have that $\|P|\psi\rangle\| \leq \frac{2q}{\sqrt{2^n}}$. Now, the probability the database in $|\psi\rangle$ contains a 0 is $\|P|\psi\rangle\|^2 \leq \frac{4q^2}{2^n}$.

□

4.2 Collision Lower Bound

We can also adapt the above proof for the Collision lower bound. If we define P to project onto databases D containing a collision, we have that if we project $|x, y, z\rangle \otimes \frac{1}{\sqrt{2^n}} \sum_w (-1)^{y \cdot w} |D \cup (x, w)\rangle$ in $|\phi_1'\rangle$ onto P, we will get $|z, y, z\rangle \otimes \frac{1}{\sqrt{2^n}} \sum_{w \in D} |D \cup (x, w)\rangle\rangle$

This allows us to write $P|\psi_1'\rangle = UP|\psi_1\rangle + \frac{1}{\sqrt{2^n}} |\phi\rangle$ where

$$|\phi\rangle = \sum_{x, y \neq 0, z, D : D(x) = \perp, w \in D} \alpha_{x,y,z,D} |x, y, z\rangle \otimes |D \cup (x, w)\rangle$$

Notice that $|\phi\rangle$ is the sum of $|\phi_i\rangle$ for $i \in q$, where w is set to be the ith image point in D. Notice that after q queries, the total size of D is at most q. Each of the $|\phi_i\rangle$ have norm at most 1, and have disjoint support, so $\||\phi\rangle\| \leq \sqrt{q}$.

Applying the above arguments, this means $\|P|\psi'\rangle\| \leq \|P|\psi\rangle\| + \frac{2\sqrt{q}}{\sqrt{2^n}}$. Therefore, after q queries, $\|P|\psi\rangle\| \leq \sum_{i=1}^{q} \frac{2\sqrt{q}}{\sqrt{2^n}} \leq \frac{2q^{3/2}}{\sqrt{2^n}}$. Thus, the probability that D contains a collition is at most $4q^3/2^n$.

Then we get the theorem:

Theorem 4.2. *After q queries, if the compressed (standard or phase) oracle for H is measured, the resulting database will contain a collision with probability at most $O(q^3/2^n)$*

4.3 More General Settings

The above proof technique is very general. Suppose we have a relation R such that, for any input x, if R is not satisfied on a database D of size q, then there are at most $k(q)$ possible pairs (x, w) that can be added to D to make R satisfied. Then we have that the total probability R is satisfied on D after q queries is at most $4(\sum_{i=1}^{q} \sqrt{k(q)})^2/2^n$.

This can be used to easily show optimal bounds for the k-sum problem for a random oracle:

Theorem 4.3. *After q queries to a random oracle, D will contain k distinct tuples (x_i, y_i) such that $\sum_i y_i = 0$ with probability at most $O(q^{k+1}/2^n)$, matching the optimal algorithm.*

Furthermore, in many situations, $|\phi\rangle$ can be further bounded. For example, in the setting of 3-collisions, $|\phi\rangle$ must contain terms with a standard 2-collision; using the collision lower bound, we have that the magnitude of $|\phi\rangle$ is at most $O(\sqrt{q^3/2^n})$. This then implies that the probability of finding a 3-collision is at most $O(q^5/2^{2n})$). More generally,

Theorem 4.4. *After q queries to a random oracle, D will contain k distinct tuples (x_i, y_i) such that $y_i = y_j \forall i, j$ with probability at most $O(q^{2k-1}/2^{(k-1)n})$. In other words, the quantum query complexity is at least $\Omega((2^n)^{(k-1)/(2k-1)})$.*

As far as we know, no such lower bounds were previously known other than the trivial collision bound.

5 Quantum Indifferentiability of Merkle-Damgård

In this section, we use our recorded phase technique to prove the indifferentiability of hash functions. Specifically, we will prove the indifferentiability of the Merkle-Damgård construction when using a pre-fix free encoding.

Pre-fix free encoding. A prefix-free code over $\{0,1\}^*$ is a set S such that, for all $x \neq y \in S$, x is not a prefix of y.

Merkle-Damgård. We briefly recall the Merkle-Damgård construction. Let $h : \{0,1\}^{2n} \to \{0,1\}^n$ be a compression function. Let S be a prefix-free code over $(\{0,1\}^n)^*$. Given an input $x \in S$, define $\mathsf{MD}_h(w)$ as follows. First, write w as $(w_1, \ldots, w_\ell)$, where each $w_i \in \{0,1\}^n$. Then:

- Let $z_1 = w_1$.
- For each $i = 2, \ldots, \ell$, let $z_i = h(z_{i-1}, w_i)$.
- Output y_ℓ.

Our Simulator. We now consider an adversary interacting with h, H. In the real world, h is a uniformly random function, and $H = MD_h$. In the ideal world, H is chosen uniformly at random, and we must construct a simulator Sim for h. We must show that these two worlds are indisitnguishable.

Our simulator is defined as follows. The simulator implements h as the compressed standard oracle CStO, but will make occasional exceptions in order to make sure the oracle is "consistent

with" H. Sim maintains an unordered database D of (x, y) pairs, in superposition. D is initially empty.

First, after every query, Sim will check that D contains no collision. That is, it will initialize an auxiliary qubit, and flip the bit if there is a collision. Then it measures the qubit. If the result is 1, Sim immediately aborts. Otherwise, the qubit contains 0, so it can be discarded and Sim continues. This ensures that the state of D never contains collisions.

We describe how Sim operates on basis states $|x, y\rangle$, though Sim will actually operate in superposition. Sim first looks for a "completion of" x in D. A completion is defined in the same was as in the classical case. A completion is a list of entries $((z'_{i-1}, w_i), z_i) \in D$ for $i = 2, \ldots \ell - 1$ such that, if we write $x = (z'_{\ell-1}, w_\ell)$:

- $z'_i = z_i$
- If we let $w_1 = z_1$, then $(w_1, \ldots, w_\ell)$ is in the prefix-free code S.

If Sim finds a single completion, let $w = (w_1, \ldots, w_\ell)$. Then Sim will make the query $|w, y\rangle$ to the oracle H by splicing together the w it just computed with the y registers from the adversary's query. The response will be $|w, y \oplus H(w)\rangle$. Then it uncomputes the completion and w, and finally returns $|x, y \oplus H(w)\rangle$ to the adversary by reconstituting the x and y registers from the original query.

The fact that there are no collisions in D and that S is a prefix-free code implies that there is only a single completion, so we do not need to worry about what happens if Sim finds multiple completions (though for concreteness, imagine that S chooses the first such completion).

If Sim finds no completions, then it proceeds as if it was just the oracle CStO (it does not perform a CStO operation in either of the above two cases).

Security Analysis. We now prove the efficacy of our simulator.

Theorem 5.1. *For any distinguisher A making at most q queries, A cannot distinguish the real from ideal world except with probability $O(q^4 2^{-n/2})$*

Proof. We will prove security using a sequence of hybrids.

Hybrid 0. This hybrid is the real world where h is random and H is implemented as MD_h. Here, we will think of h as implemented using the compressed standard oracle CStO with database D.

Hybrid 1. In this hybrid, we will implement H as MD_h, but we will tweak the implementation of h. After every query, h will look at its database D, and search for a collision: two tuples $(x, y), (x', y')$ such that $x \neq x'$ but $y = y'$. It will initialize an auxiliary bit to 0. If it finds any such collision, it will flip the bit to 1.

Then, it measures the auxiliary bit. If the bit is 0, it discards the bit and continues. Otherwise, if the bit is 1 it will immediately abort.

By the collision bound (Theorem 4.2), the probability that D contains a collision is negligible, so the probability of abort is negligible. This means the measurement negligibly affects the state.

Hybrid 2. In this hybrid, we will change h again, while keeping H as MD_h. Here, instead of storing the state for h as a single database D, we will store the state in "encoded" form as two databases D, E. To process each query, we will decode, apply h as in **Hybrid 1**, and then re-encode.

To encode, start with an empty E and do the following. First, for each tuple $(x, y) \in D$, test if (x, y) has a completion amongst the remaining tuples of D. If a completion is found for (x, y) with string w, then add the tuple (w, y) to E. We need this operation to be reversible, so we actually toggle whether (w, y) is in E. If no completion is found for (x, y), do nothing

Then for each $(w, y) \in E$, test if (w, y) corresponds to some tuple $(x, y) \in D$ that has a completion in D. Concretely, do the following: first set $z_1 = w_1$. Then, for $i = 2, \ldots \ell - 1$, test if there is a tuple $((z_{i-1}, w_i), z_i) \in D$. If not, stop, and report that no completion is found. Otherwise continue. Finally, let $x = (z_{\ell-1}, w_\ell)$, and report that there is a tuple with a completion.

If so, remove (x, y) from D (as before, we want this to be reversible, so we toggle whether (x, y) is in D). Otherwise, do nothing.

We note that, since by **Hybrid 1** D has no collisions and since S is a prefix-free code, we have that:

- Any tuple (x, y) has at most one completion
- If (x, y) has a completion, then (x, y) not a part of any other completion
- No two completions correspond to the same w
- This means that it does not matter which completion we use (since there will only be one), or the order in which we process completions
- This also means that whenever we are processing a (w, y), there will be exactly one pair (x, y) corresponding to w that has a completion in D.
- This finally means that encoding is a reversible process. To decode, iterate over all (w, y) in E. Search for the completion corresponding to w (it is guaranteed to exist and be unique). Write the corresponding (x, y) tuple to D. After processing all $(w, y) \in E$, iterate over all (x, y) and see if (x, y) has a completion with string w. If so, remove (w, y) from E.

The initial state of h is D, E are both empty, which corresponds to encoding an empty D.

Hybrid 3. In this hybrid, we will change h again, while keeping H as MD_h. Here, however, we will modify the encoding/decoding procedure above slightly.

During encoding, consider processing a pair $(w, y) \in E$, which resulted in removing some pair $(x = (z_{\ell-1}, w_\ell), y)$ from D. Let $\{((z_{i-1}, w_i), z_i)\}_{i=2,\ldots\ell-1}$ in D be the completion of (x, y). After removing (x, y), we then do the following for $i = \ell - 1, \ldots, 2$:

- Take the pair $((z_{i-1}, w_i), z_i)$, and measure if z_i is 0 in the Fourier domain.
- If z_i is 0 in the Fourier domain, then remove the pair $((z_{i-1}, w_i), z_i)$ from D
- If z_i is non-zero in the Fourier domain, stop and move on to the next (w, y) pair in E

It is important to process the completion in reverse order, since if $((z_{i-1}, w_i), z_i) \in D$, then almost certainly measuring $((z_{i-2}, w_{i-1}), z_{i-1})$ in the Fourier basis will result in non-zero, since z_{i-1} is still in use.

Similarly, during decoding, when processing a $(w, y) \in E$, if we would abort the completion finding process, instead gradually create a completion in D. That is, when testing if there is a tuple $((z_{i-1}, w_i), z_i) \in D$, if we don't find one, we instead do the following: for $j = i, \ldots, \ell-$: create a tuple $((z_{j-1}, w_j), z_j)$ where z_j is the uniform superposition over all possible z_j and add this tuple to D. Finally, let $x = (z_{\ell-1}, w_\ell)$, and report that there is a tuple with a completion.

Notice that if we ever remove a tuple from D during encoding, we will be guaranteed to add it back in during decoding. Therefore, the only difference between **Hybrid 2** and **Hybrid 3** is that introduce some extra measurements (namely, measuring whether a pair is 0 in the Fourier domain). However, we claim that these measurements have negligible effect on the state. Indeed, define $P_{x,y}$ as the measurement that looks up the pair $(x, y') \in D$ and (provided it exists), outputs 1 if and only if $y' = y$. Similarly, define T_x as the measurement that looks up the pair $(x, y') \in D$, and (provided it exists), outputs 1 if and only if y' is 0 in the Fourier domain.

We claim that T_x and $P_{x,y}$ *almost* commute. Indeed, the operator that corresponds to measuring T_x and getting 1 is just $|\psi\rangle\langle\psi|$ where $|\psi\rangle$ is the uniform superposition over all points. Similarly, $P_{x,y}$ giving 1 corresponds to the projection $|y\rangle\langle y|$. If we compute the commutator, we get

$$\frac{1}{2^n}\left(|y\rangle\sum_{y'}\langle y'| - \sum_{y'}|y'\rangle\langle y|\right) = \left(|y\rangle\sum_{y'\neq y}\langle y'| - \sum_{y'}|y'\rangle\langle y|\right) = \frac{\sqrt{2^n-1}}{2^n}\left(|y\rangle\langle\phi'| - |\phi'\rangle\langle y|\right)$$

where $|\phi'\rangle$ is the uniform superposition over all points other than 0.

This is a rank-2 matrix, and since $|\psi'\rangle, |0\rangle$ are unit vectors, the norm of the eigenvalues of this matrix are at most $2 \times \frac{\sqrt{2^n-1}}{2^n}$, which is negligible. Thus, for any state, swapping the order of T_x and $P_{x,y}$ has a negligible affect on the resulting state.

Notice that $T_x, P_{x,y}$ are projections. Moreover, for every point x added to D, the measurement T_x is already applied at least once when x is first added. Finally, it is straightforward to implementing all of the testing for completions as a sequence of polynomially-many applications of $P_{x,y}$. Piecing this all together, we can move every T_x measurements performed in **Hybrid 3** to occur exactly when x is added to D, and then absorb it with the T_x that already exists there. By the near commutativity of T_x and $P_{x,y}$, this incurs only a negligible affect on the final state.

Hybrid 4. Notice that in **Hybrid 3**, if we pretended that T_x and $P_{x,y}$ actually commuted, the ultimate effect of decoding, handling the query $|x, y\rangle$, and then re-encoding is that most of the D, E tuples will be unaffected. The only tuples that may be affected are $(x, y) \in D$, or the tuple $(w, y) \in E$ if w corresponds to a completion of x.

Therefore, in **Hybrid 4**, we do not decode or encode anything except tuples relating to $|x, y\rangle$. By the near commutativity of T_x and $P_{x,y}$, this will only negligibly affect the state.

At this point, an equivalent way of describing **Hybrid 4** is as follows. We implement h as Sim, which makes queries to an oracle H'. The state of Sim will be D, and H' will be implemented using the compressed standard oracle with database E. H will still be MD_h.

Hybrid 5. Finally, we describe the ideal world, which is the same as **Hybrid 4**, except that we set H to be the same as H'. h is still implemented as Sim, which makes oracle queries to $H' = H$.

We now verify that **Hybrid 4** and **Hybrid 5** are identical. Since h is implemented the same way in both hybrids, we only need to verify that a query to MD_h is identical to making a query directly to H'.

Indeed, it is straightforward to verify that after making the initial $\ell - 1$ calls to h, the final query will detect in a completion, resulting in the desired query to H'. Moreover, it is straightforward to verify that any effect on h the queries leading up to the final call have will be uncomputed when MD_h un-computes its intermediate values.

We can make all of the above steps quantitative, using the following lemma of [BBBV97]:

Lemma 5.2 ([**?**])**.** *Let $|\varphi\rangle$ and $|\psi\rangle$ be quantum states with Euclidean distance at most ϵ. Then, performing the same measurement on $|\varphi\rangle$ and $|\psi\rangle$ yields distributions with statistical distance at most 4ϵ.*

The only transitions that cause any change are:

- **Hybrid 0** to **Hybrid 1**. Here, the error caused by each step is a vector of norm $O(\sqrt{q}2^{-n/2})$. The overall error is therefore $O(q^{3/2}2^{-n/2})$
- **Hybrid 2** to **Hybrid 3**. Here, we can implement the completion check using q^2 primal-domain tests. Re-ordering these tests with the Fourier domain tests with the Fourier domain test results in an error vector of norm $O(q^2 2^{-n/2})$. The overall error is $O(q^3 2^{-n/2})$
- **Hybrid 3** to **Hybrid 4**. Here, the only difference is re-ordering up to q tests in the Fourier domain with the up to q^2 primal tests. This results in an error vector of norm $O(q^3 2^{-n/2})$. The overall error is $O(q^4 2^{-n/2})$

Piecing everything together, we find that the total error is at most $O(q^4 2^{-n/2})$.

□

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