ISO/IEC 9797-1 Revisited: Beyond Birthday Bound

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Abstract. The international standard ISO/IEC 9797-1:2011 specifies six versions of MACs, called MAC Algorithm 1-6, and many of these MACs enjoy widespread use in practical applications. However, security guarantees of these MACs are all capped at birthday bound since they all use single CBC-MAC computations. It is recommended in this standard to improve the security level by concatenating outputs of two MACs with independent keys rather than XORing them.

In this paper, we show such claim is wrong by giving birthday forgery attacks on concatenations of two MACs with independent keys in this standard. Furthermore, we revisit the impact of XORing of two MACs on ISO/IEC 9797-1:2011 and show this operation can only lift up the security level. We give the first two provable-security bounds for XORing of two MAC Algorithm 1 (XMAC1) in ISO/IEC 9797-1:2011 with either padding scheme 3 or 2. We prove that XMAC1 with padding scheme 3 is secure beyond birthday bound with $O(\sigma q^2 \ell / 2^{2n})$. Note that our result implies that this is the first CBC-type MAC that provably goes beyond birthday barrier with only two secret keys. When instantiated with padding scheme 2, we prove that XMAC1 is secure with birthday bound $O(\sigma^2 / 2^n)$. Illustrated with Joux *et al*'s attack, this bound is tight up to a constant factor. We also prove that XORing of two MAC Algorithm 5 (XMAC5) is secure with a bound $O(\sigma q^2 \ell / 2^{2n})$.

Finally, together with previous results, we give a summary of the impact of XORing of two MACs on ISO/IEC 9797-1:2011 and conclude that such operation can only lift up the security bound.

Keywords: ISO/IEC 9797-1, birthday forgery attack, XMAC1, XMAC5, beyond birthday bound

1 Introduction

A Message Authentication Code (MAC) is a fundamental symmetric-key primitive to provide integrity and authenticity of messages between two parties. There are several ways to realize a MAC including by using a universal hash [24], using a compression function [1] or using a block cipher [2]. Block cipher-based MACs are a large family of MACs that use block ciphers to construct secure PRFs (Pseudo-Random Functions) under the assumption that underlying block

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ciphers are secure PRPs (Pseudo-Random Permutations). A MAC is said to be secure if it is a good PRF under all adversaries. In information-theoretic proofs of MACs, the underlying block cipher are usually replaced with a random permutation at first step and the transformed construction would be given a rigorous security proof. The advantage of PRF-security is often measured by n the block size, q the total number of queries, ℓ the length of the longest message and σ the total number of blocks of all queries. Many block cipher-based MACs can achieve the so-called birthday security, generally with a bound like $O(q^2/2^n)$.

It is not enough to reach birthday security for block cipher-based MACs, especially when the block size is small. For lightweight block ciphers (e.g. PRESENT [7] and PRINCE [8]) as well as legacy block ciphers such as 3DES, the block size is usually $n = 64$, in which the birthday bound becomes 2^{32} and vulnerable in many practical applications. For instance, Bhargavan and Leurent [6] have demonstrated two practical attacks called Sweet32 that exploit collision on short block ciphers. Hence, performing MACs with beyond birthday security in practical devices is of great importance.

ISO/IEC 9797-1 is an international standard that defines MACs using a block cipher. ISO/IEC 9797-1:2011[13] specifies six different mechanisms of CBC MACs, called MAC Algorithm 1-6, where each MAC is defined by specifying the final iteration and output transformation. Many of these MACs enjoy widespread use in practical devices and thus are of great importance. Since each of these MACs uses single CBC-MAC computations, they all suffer from birthday forgery attacks as explained in [21,22]. Hence security guarantees of these MACs are all capped at birthday bound which is less satisfying. It is suggested to improve the security level by concatenating outputs of two MACs in Annex C, C.2 Rationale, ISO/IEC 9797-1:2011:

> if a MAC algorithm with a higher security level is needed, it is recommended to perform two MAC calculations with independent keys and concatenate the results (rather than XORing them).

However, this claim is wrong and will be shown later that such concatenation can not enhance the security guarantee except doubling communication complexity among users and indeed XORing outputs of two MACs with independent keys can lift up the security to beyond birthday bound for all kinds of MACs specified in this standard.

Our contributions. Firstly, we notice a wrong claim in ISO/IEC 9797-1:2011, that is, concatenation of two MACs can not improve the security level but only double communication complexity among users. We argue this by giving birthday forgery attacks on these concatenations, and these attacks show that security guarantees of concatenations of any two MACs in ISO/IEC 9797-1:2011 are all still capped at birthday bound.

Secondly, we revisit the impact of XORing of two MACs on ISO/IEC 9797-1:2011 and show that this operation can lift up the security level to beyond birthday bound. Due to Joux *et al*'s double collision attack [16], the security

Table 1: Comparison of XMAC1, XMAC5 and other CBC-type MACs with beyond birthday security. XMAC1 resp. XMAC5 denotes the XORing of two MAC Algorithm 1 resp. 5 in ISO/IEC 9797-1:2011.

Algorithm	#keys	Security	Ref.
SUM-ECBC	4	$O(q^3\ell^3/2^{2n})$	[25]
3kf9	3	$O(q^3\ell^3/2^{2n} + q\ell/2^n)$	[27]
XMAC1 with pad3	2	$O(\sigma q^2\ell/2^{2n})$	Sect. 5
XMAC1 with pad2	2	$O(\sigma^2/2^n)$	App. A
XMAC5	2	$O(\sigma q^2\ell/2^{2n})$	Sect. 6

guarantee of XORing of two MAC Algorithm 1 (XMAC1[1]) with padding scheme 2 is stopped at birthday barrier. On the other hand, the provable-security bounds of XMAC1 with either padding scheme 2 or 3 still remain open which has been pointed out by Rogaway in [23]. In 2010 [25], Yasuda proved that XORing of two MAC Algorithm 2 (termed SUM-ECBC in his paper) is secure beyond birthday bound with either padding scheme 2 or 3, and his proof can also apply to XORing of two MAC Algorithm 4[2] and obtain similar security bounds. However, provable-security bounds of XORing of the four rest standardized MACs (XMAC1, XMAC3, XMAC5, XMAC6) remain absent until now. In particular, provable-security bounds of XMAC1 and XMAC5 are related to the open problem of XORing of single-key CBC-MACs, which has been mentioned by Yasuda [25] and Zhang et al [27].

In this paper, we give the first two security bounds for XMAC1 either instantiated with padding scheme 3 or 2. When instantiated with padding scheme 3, we prove that XMAC1 is secure beyond birthday barrier with a bound $O(\sigma q^2\ell)$. Compared with other two CBC-like MACs with beyond birthday bound security SUM-ECBC [25] and 3kf9 [27], requiring four and three secret keys respectively, XMAC1 only need two keys. Thus our result implies that XMAC1 is the first CBC-type MAC that provably goes beyond birthday bound with only two secret keys. When instantiated with padding scheme 2, we prove that XMAC1 is secure with birthday bound $O(\sigma^2/2^n)$. Together with Joux *et al.*'s attack [16], this bound is tight up to a constant factor. The reason behind these two different bounds for XMAC1 is that padding scheme 3 is a prefix-free encoding while padding scheme 2 is not. Furthermore, we prove that XMAC5 can also achieve the $O(2^{2n/3})$ security. XMAC5 is more efficient in a sense that it is not necessary to compute the length of messages before beginning the process of authentication. On the other hand, XMAC3 and XMAC6 are similar to XMAC2 and can be proved by Yasuda's proof [25].

We finally give a summary of the impact of XORing of two MACs on ISO/IEC 9797-1:2011 and show that the resulted MAC can only be lifted to a higher

[1] this is also the MAC Algorithm 5 specified in ISO/IEC 9797-1:1999 [12]
[2] this is also the MAC Algorithm 6 specified in ISO/IEC 9797-1:1999

security level. Thus, according to our results, one can simply XORing two CBC-like MACs at hand and achieve better security guarantees.

Organization. In Sect. 2, we give essential notations and definitions. We give birthday forgery attacks on concatenations of two MACs of ISO/IEC 9797-1:2011 in Sect. 3. We present our main results on XMAC1, XMAC5 in Sect. 4. Then we give proofs of XMAC1 with padding scheme 3, XMAC1 with padding scheme 2 and XMAC5 in Sect. 5, Appendix A, Sect. 6 respectively. Finally, we discuss the impact of XORing of two MACs on ISO/IEC 9797-1:2011 in Sect. 7 and conclude this paper in Sect. 8.

2 Preliminaries

2.1 Notation

If $\mathcal{X}$ is a set, then $X \xleftarrow{\$} \mathcal{X}$ denotes the operation of drawing X from $\mathcal{X}$ uniformly at random. $\{0,1\}^*$ denotes all bit strings including the empty string. The bit length of a string X is written by $|X|$. Concatenation of strings X and Y is written as either $X\|Y$ or simply XY. We denote $X \oplus Y$ the bitwise exclusive-or of two equal-length strings. For a string $X \in \{0,1\}^{n\ell}$ with $\ell \geq 1$, we divide X into n-bit blocks as $X = X[1]\| \ldots \|X[\ell]$ where $|X[1]| = \cdots = |X[\ell]| = n$. If ℓ is a non-negative integer such that $\ell < 2^n$, we write $\mathsf{bin}_n(\ell)$ for the n-bit binary representation of ℓ.

We write $\mathrm{Perm}(n)$ for the set of all permutations over $\{0,1\}^n$, and $\mathrm{Rand}(n)$ for the set of all functions mapping $\{0,1\}^*$ to $\{0,1\}^n$. We often perform lazy sampling for specifying a random permutation $P \xleftarrow{\$} \mathrm{Perm}(n)$. We denote $\mathrm{Dom}(P)$ and $\mathrm{Ran}(P)$ the sets of already-defined domain points and range points of P respectively, and $\overline{\mathrm{Dom}(P)}$ and $\overline{\mathrm{Ran}(P)}$ for the complementary sets. A block cipher E is a family of permutations $\{E_K : K \in \mathcal{K}\}$, where $E_K(\cdot) = E(K, \cdot)$ is a permutation over $\{0,1\}^n$ specified by a key K. $\mathcal{K}$ is the key space and n is the block length.

A MAC is an algorithm that takes two inputs a key K and a message M then outputs a fixed-length tag T. K, M and T are all binary strings. The CBC MAC is built from cipher block chaining some underlying block cipher. Let $M_i = M_i[1]\|M_i[2]\| \cdots \|M_i[m_i]$ be a message, where $|M_i[1]| = |M_i[2]| = \cdots = |M_i[m_i]| = n$ and m_i is the block length. Then $\mathrm{CBC}[E_K](M_i)$, the CBC MAC of M, is defined as $y^i_{m_i}$, where

$$y^i_j = E_K(M_i[j] \oplus y^i_{j-1})$$

for $j = 1, \ldots, m_i$ and $y^i_0 = 0^n$.

There are total four padding schemes specified in ISO/IEC 9797-1:2011. As padding scheme 1 allows a trivial forgery, we only consider padding scheme 2, 3 and 4 in this paper, and padding scheme 4 is only used in MAC Algorithm 5:

pad2 the message M is always right-padded with a single '1' bit then right-padded with i bits '0' where i is the least non-negative integer such that $|M| + i + 1$ is a positive multiple of n.

pad3 the message M is mapped to $\mathsf{bin}_n(|M|)\|M0^i$ where i is the least non-negative integer such that $|M|+i$ is a positive multiple of n.

pad4 if the message has length that is positive multiple of n, then no padding shall be applied. Otherwise, the message shall be right-padded with a single '1' bit then right-padded with i bits '0' where i is the least non-negative integer such that $|M|+i+1$ is a positive multiple of n.

Note that after padded via padding scheme 3, the messages list would become prefix-free, meaning that it is any such pair from this list where neither string is a prefix of the other. We simply denote $\mathbf{pad2}(M)$, $\mathbf{pad3}(M)$ and $\mathbf{pad4}(M)$ the operations of mapping M to a sequence of n-bit blocks with padding scheme 2, 3 and 4 respectively.

2.2 Security Notions

An adversary $\mathcal{A}$ is an algorithm that always outputs a bit. We write $\mathcal{A}^{O(\cdot)} \Rightarrow 1$ to denote the event that $\mathcal{A}$ outputs 1 after interacting with oracle $O(\cdot)$. We focus on the information-theoretic setting, namely, all keyed block ciphers are replaced with random permutations. Throughout this paper, an adversary $\mathcal{A}$ is allowed to unbounded computational power and assumed to be deterministic without loss of generality. Its complexity is measured by the number of queries, the maximum block length of messages, the total number of blocks of messages. Recalling that any pseudo-random function (PRF) is a secure MAC [2], our goal is to prove $F[P]$ is a secure PRF, where $F[P]$ is an interested function based on random permutations. We say that $F[P]$ is a secure PRF if it is indistinguishable from a random function $\mathcal{R} \xleftarrow{\$} \mathrm{Rand}(n)$. Formally, we define

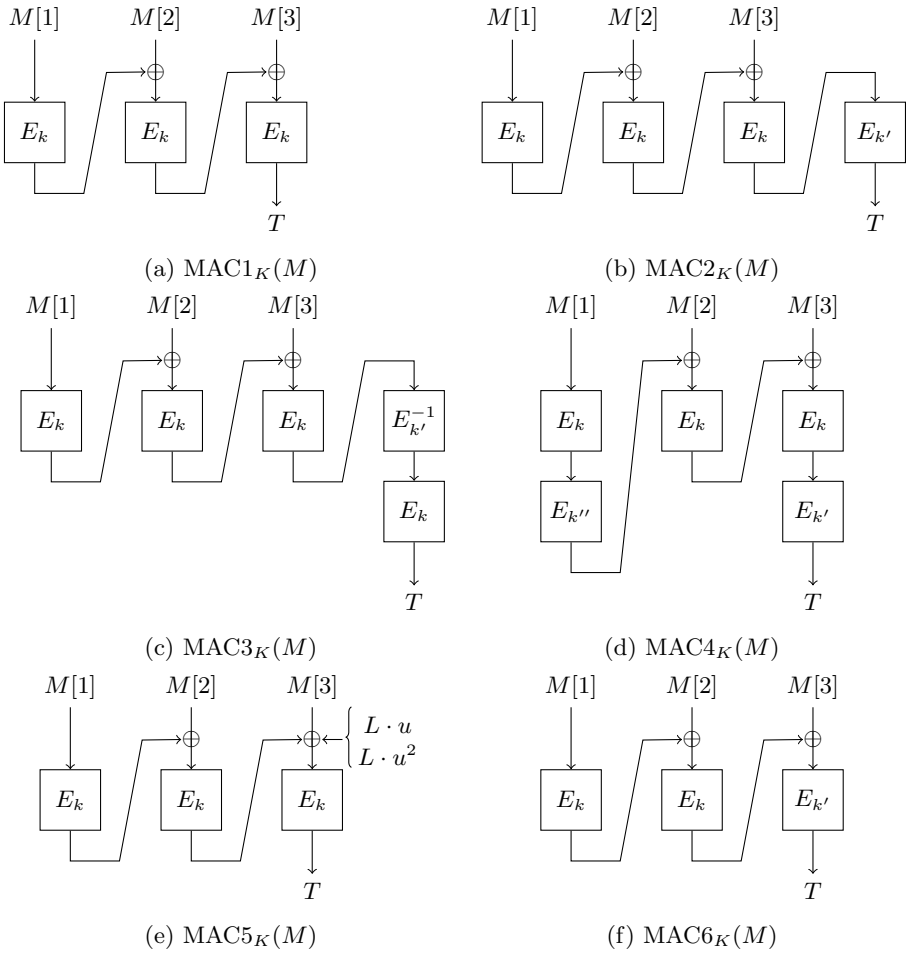
$$\mathbf{Adv}^{\mathrm{prf}}_{F[P]}(\mathcal{A}) \stackrel{\mathrm{def}}{=} \Pr[P \xleftarrow{\$} \mathrm{Perm}(n) : \mathcal{A}^{F[P](\cdot)} \Rightarrow 1] - \Pr[\mathcal{R} \xleftarrow{\$} \mathrm{Rand}(n) : \mathcal{A}^{\mathcal{R}(\cdot)} \Rightarrow 1].$$

Note that the probabilities are taken over $P, \mathcal{R}$, and $\mathcal{A}'s$ coins.

3 Forgery Attacks on The Concatenation of Two MACs in ISO/IEC 9797-1:2011

In this section, we present birthday forgery attacks on the concatenation of six ISO/IEC 9797-1:2011 MACs, which are depicted in Fig. 1. We only present the attack for the concatenation of a MAC Algorithm with two independent keys here and the attacks for the concatenation of two different MAC Algorithms is the same. We denote $\mathrm{MAC}i_K(M)$ the MAC for a message M computed using the MAC Algorithm i specified in ISO/IEC 9797-1:2011 for $1 \leq i \leq 6$. We denote $\mathrm{MAC}i_{K_1}(M)\|\mathrm{MAC}i_{K_2}(M)$ the concatenation of two MACs with independent keys K_1, K_2. Since padding scheme 1 allows trivial forgery attack, we only consider padding scheme 2, 3 and 4 here. The attack procedures are detailed below.

At first, we consider messages after padded with scheme 2. For $\mathrm{MAC}1_{K_1}(M)\|$ $\mathrm{MAC}1_{K_2}(M)$, i.e., the concatenation of two MAC Algorithm 1, we first query



(a) $\mathrm{MAC1}_K(M)$ (b) $\mathrm{MAC2}_K(M)$

(c) $\mathrm{MAC3}_K(M)$ (d) $\mathrm{MAC4}_K(M)$

(e) $\mathrm{MAC5}_K(M)$ (f) $\mathrm{MAC6}_K(M)$

Fig. 1: Illustration of the six ISO/IEC MACs. The message M has been padded with specific padding scheme. For $\mathrm{MAC1}_K(M)$ and $\mathrm{MAC5}_K(M)$, the underlying key is $K = k$. For $\mathrm{MAC2}_K(M)$, MAC3 and $\mathrm{MAC6}_K(M)$, the underlying key is $K = (k, k')$. For $\mathrm{MAC1}_K(M)$, the underlying key is $K = (k, k', k'')$.

a single block message M_1 and obtain $\mathrm{MAC1}_{K_1}(M_1) \| \mathrm{MAC1}_{K_2}(M_1)$. Then we query two strings $M_1 \oplus \mathrm{MAC1}_{K_1}(M_1)$ and $M_1 \oplus \mathrm{MAC1}_{K_2}(M_1)$, and receive $\mathrm{MAC1}_{K_2}(M_1 \oplus \mathrm{MAC1}_{K_1}(M_1))$ (the right half of concatenation) and $\mathrm{MAC1}_{K_1}(M_1 \oplus \mathrm{MAC1}_{K_2}(M_1))$ (the left half of concatenation) respectively. In this stage, we can forge a MAC of the message $M_1 \| M_1 \oplus \mathrm{MAC1}_{K_1}(M_1) \oplus \mathrm{MAC1}_{K_2}(M_1)$ without querying this message since $\mathrm{MAC1}_{K_1}(M_1 \| M_1 \oplus \mathrm{MAC1}_{K_1}(M_1) \oplus \mathrm{MAC1}_{K_2}(M_1)) = \mathrm{MAC1}_{K_1}(M_1 \oplus \mathrm{MAC1}_{K_2}(M_1))$ and $\mathrm{MAC1}_{K_2}(M_1 \| M_1 \oplus \mathrm{MAC1}_{K_1}(M_1) \oplus \mathrm{MAC1}_{K_2}(M_1)) = \mathrm{MAC1}_{K_2}(M_1 \oplus \mathrm{MAC1}_{K_1}(M_1))$. This attack only requires 3 queries.

For $i \in \{2,3,4,6\}$, we adopt the idea of multicollisions attack in iterated hash functions proposed by Joux [15]. For any $i \in \{2,3,4,6\}$, we first focus on collisions of the left half of concatenation, i.e., MAC$i_{K_1}(M)$. We search two two-block messages $a_{1,1}\|r_{1,1}$ and $a_{2,1}\|r_{2,1}$ such that MAC$i_{K_1}(a_{1,1}\|r_{1,1}) =$ MAC$i_{K_1}(a_{2,1}\|r_{2,1})$. This requires about $2^{n/2}$ MAC computations due to birthday paradox. Fixing $a_{1,1}\|r_{1,1}$ and $a_{2,1}\|r_{2,1}$, we then search $a_{1,2}\|r_{1,2}$ and $a_{2,2}\|r_{2,2}$ such that MAC$i_{K_1}(a_{1,1}\|r_{1,1}\|a_{1,2}\|r_{1,2}) =$ MAC$i_{K_1}(a_{2,1}\|r_{2,1}\|a_{2,2}\|r_{2,2})$. This also requires about $2^{n/2}$ MAC computations. We do this until find two $2t$-block ($t \geq n/2$) messages $a_{1,1}\|r_{1,1}\|\ldots\|a_{1,t}\|r_{1,t}$ and $a_{2,1}\|r_{2,1}\|\ldots\|a_{2,t}\|r_{2,t}$ such that MAC$i_{K_1}(a_{1,1}\|\ldots\|r_{1,t}) =$ MAC$i_{K_1}(a_{2,1}\|\ldots\|r_{2,t})$. This yields 2^t different messages $a_{i_1,1}\|r_{i_1,1}\|\ldots\|a_{i_t,t}\|r_{i_t,t}$ for $i_1,\ldots,i_t \in \{1,2\}$ with the same MAC value on the left half of concatenation. Assume $t \geq n/2$, then with high probability there exists a collision among these 2^t elements such that the MAC value on the right half are also equal. Assume the collided messages are M_1 and M_2. Then the two MAC values for $M_1\|A$ and $M_2\|A$ are also a collision for any n-bit block A. This attack requires total about $(1+n)\cdot 2^{n/2}$ MAC computations.

Secondly, we consider messages after padded with scheme 3 for $i \in \{1,2,3,4,6\}$ and padded with scheme 4 for $i = 5$. For any $i \in \{1,2,3,4,5,6\}$, we first focus on collisions of the left half of concatenation, i.e., MAC$i_{K_1}(M)$. Let $t \geq n/2$ and each message has same block-length $2t$. We search two $2t$-block messages $a_{1,1}\|r_{1,1}\|0^n\|\ldots\|0^n$ and $a_{2,1}\|r_{2,1}\|0^n\|\ldots\|0^n$ (the last $2t-2$ blocks are all zero) such that MAC$i_{K_1}(\mathsf{bin}_n(2t)\|a_{1,1}\|r_{1,1}\|0^n\|\ldots\|0^n) =$ MAC$i_{K_1}(\mathsf{bin}_n(2t)\|a_{2,1}\|r_{2,1}$ $\|0^n\|\ldots\|0^n)$. This requires about $2^{n/2}$ MAC computations due to birthday paradox. Fixing $a_{1,1}\|r_{1,1}$ and $a_{2,1}\|r_{2,1}$, we then search $a_{1,2}\|r_{1,2}$ and $a_{2,2}\|r_{2,2}$ such that MAC$i_{K_1}(\mathsf{bin}_n(2t)\|a_{1,1}\|r_{1,1}\|a_{1,2}\|r_{1,2}\|0^n\|\ldots\|0^n) =$ MAC$i_{K_1}(\mathsf{bin}_n(2t)\|a_{2,1}$ $\|r_{2,1}\|a_{2,2}\|r_{2,2}\|0^n\|\ldots\|0^n)$. This also requires about $2^{n/2}$ MAC computations. We do this until find two $2t$-block ($t \geq n/2$) messages $a_{1,1}\|r_{1,1}\ldots\|a_{1,t}\|r_{1,t}$ and $a_{2,1}\|r_{2,1}\ldots\|a_{2,t}\|r_{2,t}$ such that MAC$i_{K_1}(\mathsf{bin}_n(2t)\|a_{1,1}\|\ldots\|r_{1,t}) =$ MAC$i_{K_1}(\mathsf{bin}_n$ $(2t)\|a_{2,1}\|\ldots\|r_{2,t})$. This yields 2^t different messages $a_{i_1,1}\|r_{i_1,1}\|\ldots\|a_{i_t,t}\|r_{i_t,t}$ for $i_1,\ldots,i_t \in \{1,2\}$ with the same MAC value on the left half of concatenation. Assume $t \geq n/2$, then with high probability there exists a collision among these 2^t elements such that the MAC value on the right half are also equal. Assume the collided messages are $a_{i_1,1}\|r_{i_1,1}\ldots\|a_{i_t,t}\|r_{i_t,t}$ and $a_{j_1,1}\|r_{j_1,1}\ldots\|a_{j_t,t}\|r_{j_t,t}$, then the two MAC values for $a_{i_1,1}\|r_{i_1,1}\ldots\|a_{i_t,t}\|r_{i_t,t}\oplus A$ and $a_{j_1,1}\|r_{j_1,1}\ldots\|a_{j_t,t}\|r_{j_t,t}\oplus$ A are also a collision for any n-bit block A. This attack requires total about $(1+n)\cdot 2^{n/2}$ MAC computations.

4 Main Results on XMAC1 and XMAC5

In the information-theoretic setting, we simply denote XMAC1[P] the XORing of two MAC Algorithm 1 based on random permutations. Similarly, we denote XMAC5[P] the XORing of two MAC Algorithm 5 based on random permutations. We consider an adversary $\mathcal{A}$ that makes at most q queries to its oracle, each query being at most ℓ blocks, and the total number of blocks of all queries being at most σ.

As for XMAC1, we have the following two results.

Theorem 1. *With padding scheme 3, if $\ell \leq 2^{n/3}$, one has*

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[P]}(\mathcal{A}) \leq \frac{844\sigma q^2 \ell}{2^{2n}}.$$

Theorem 2. *With padding scheme 2, one has*

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[P]}(\mathcal{A}) \leq \frac{2\sigma^2}{2^n} + \frac{2\sigma q}{2^n} + \frac{0.5q^2}{2^n}.$$

As for XMAC5, we have

Theorem 3. *For $\ell \leq 2^{n/3}$, one has*

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC5}[P]}(\mathcal{A}) \leq \frac{4}{2^n} + \frac{58\sigma^2 q}{2^{2n}} + \frac{841\sigma q^2 \ell}{2^{2n}}.$$

The proof of Theorem 1 is given in Sec. 5, the proof of Theorem 2 is given in Appendix A, and the proof of Theorem 3 is given in Sec. 6.

5 Security of XMAC1 with Padding Scheme 3

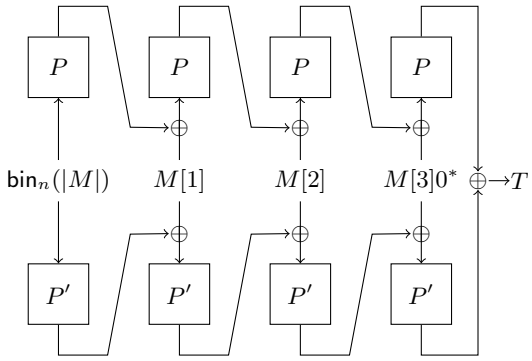


Fig. 2: Illustration of XMAC1[P] with padding scheme 3 for $M = M[1]\|M[2]\|M[3]$, where $|M[1]| = |M[2]| = n$ and $1 \leq |M[3]| \leq n$.

In this section, we adopt the framework used in proofs for SUM-ECBC [25] and PMAC_Plus [26], and prove that XMAC1[P] instantiated with padding scheme 3 (described in Fig. 2) is an $O(2^{2n/3})$-secure PRF. Note that in the rest of this section, we always consider the messages list after padded with padding scheme 3, i.e., $M_i = \mathbf{pad3}(M_i)$ and denote by $m_i = |\mathbf{pad3}(M_i)|$ the length of message for $1 \leq i \leq q$. This messages list is easily seen to be prefix-free.

5.1 Main Ideas

We focus on the last input of random permutations P and P' at each query, denoted by $x^i_{m_i}$ and $u^i_{m_i}$ for $1 \le i \le q$. We consider an adversary $\mathcal{A}$ that aims at distinguishing XMAC1$[P]$ from a random function $\mathcal{R} : \{0,1\}^* \to \{0,1\}^n$. $\mathcal{A}$ is allowed to unlimited computational power but can make at most q queries to its oracle, each query being at most ℓ blocks, and the total number of blocks of all queries being at most σ. Without loss of generality, $\mathcal{A}$ is assumed to be deterministic and never to repeat a query. The main game is presented in Fig. 3 and codes of the four cases are given in Fig. 4, Fig. 5 and Fig. 6 respectively. Depending on the behavior after bad events, this game can simulate either XMAC1$[P]$ or a random function $\mathcal{R}$. These two games are identical until bad events occur, so by the fundamental lemma of game-playing [5] we have

$$\Pr[\mathcal{A}^{\mathrm{XMAC1}[P](\cdot)} \Rightarrow 1] - \Pr[\mathcal{A}^{\mathcal{R}(\cdot)} \Rightarrow 1] \le \Pr[\mathcal{A}^{\mathcal{R}(\cdot)} \text{ sets } \mathbf{bad}].$$

Note that in the game simulating random function $\mathcal{R}$, the respond returning to the adversary is always a random n-bit string, unrelated to adversary's query or the setting of **bad**. Thus even if $\mathcal{A}$ prepares all of its queries $M_1, \ldots, M_q$ in advance, the probability that $\mathcal{A}$ sets a **bad** flag is not made smaller, therewith the interaction being vacuous in this game. We write bad events in more detail:

$$\begin{aligned}
&\Pr[\mathcal{A} \text{ sets } \mathbf{bad}] \\
=&\sum_{i=1}^{q} \big(\Pr[x^i_{m_i} \notin \mathrm{Dom}(P) \wedge u^i_{m_i} \notin \mathrm{Dom}(P')] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case A}] \\
&+ \Pr[x^i_{m_i} \in \mathrm{Dom}(P) \wedge u^i_{m_i} \notin \mathrm{Dom}(P')] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case B}] \\
&+ \Pr[x^i_{m_i} \notin \mathrm{Dom}(P) \wedge u^i_{m_i} \in \mathrm{Dom}(P')] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case C}] \\
&+ \Pr[x^i_{m_i} \in \mathrm{Dom}(P) \wedge u^i_{m_i} \in \mathrm{Dom}(P')] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case D}] \big) \\
\le&\sum_{i=1}^{q} \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case A}] + \sum_{i=1}^{q} \Pr[x^i_{m_i} \in \mathrm{Dom}(P)] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case B}] \\
&+ \sum_{i=1}^{q} \Pr[u^i_{m_i} \in \mathrm{Dom}(P')] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case C}] \\
&+ \sum_{i=1}^{q} \Pr[x^i_{m_i} \in \mathrm{Dom}(P) \wedge u^i_{m_i} \in \mathrm{Dom}(P')].
\end{aligned}$$

These four terms are relevant to four cases and will be bounded in following subsections.

5.2 Analysis of Case A

We handle this case via the technique of fair sets developed by Lucks [17], which has also been used in proofs of SUM-ECBC [25] and PMAC_Plus [26].

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1: for i = 1 to q do
2:    x^i_{m_i} ← P(x^i_{m_i−1}) ⊕ M_i[m_i]
3:    u^i_{m_i} ← P'(u^i_{m_i−1}) ⊕ M_i[m_i]
4:    if x^i_{m_i} ∉ Dom(P) and u^i_{m_i} ∉ Dom(P') then
5:       go to Case A
6:    end if
7:    if x^i_{m_i} ∈ Dom(P) and u^i_{m_i} ∉ Dom(P') then
8:       go to Case B
9:    end if
10:   if x^i_{m_i} ∉ Dom(P) and u^i_{m_i} ∈ Dom(P') then
11:      go to Case C
12:   end if
13:   if x^i_{m_i} ∈ Dom(P) and u^i_{m_i} ∈ Dom(P') then
14:      go to Case D
15:   end if
16: end for
```

Fig. 3: We omit the internal computations and present the computation on the last input of permutation P and P' at each query. $x^i_{m_i}$ and $u^i_{m_i}$ respectively denote the last input to P and P' at i-th query. $x^i_{m_i-1}$ and $u^i_{m_i-1}$ respectively denote (m_i-1)-th input to P and P' at i-th query.

Lemma 1. *In Case A, we have*

$$\sum_{i=1}^{q} \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case A}] \leq \frac{4\sigma^2 q}{2^{2n}},$$

for $\sigma \leq 2^{n-1}$.

```
1: Choose a fair set U ⊂ Ran(P) × Ran(P')                    XMAC1[P]/[R]
2: (y^i_{m_i}, w^i_{m_i}) ←$ Ran(P) × Ran(P')
3: if (y^i_{m_i}, w^i_{m_i}) ∉ U then
4:    bad←true [(y^i_{m_i}, w^i_{m_i}) ←$ U]
5: end if
6: T_i ← y^i_{m_i} ⊕ w^i_{m_i}
7: return T_i
```

Fig. 4: Case A

Proof. We consider the game as described in Fig. 4. The code without the boxed statement faithfully simulates $P(x^i_{m_i}) \oplus P'(u^i_{m_i})$ for $1 \leq i \leq q$, while the code with boxed statement always returns a n-bit random string T_i. We choose a fair set U as follows. Enumerate $\mathrm{Ran}(P)$ as $\{y_1, \ldots, y_\alpha\}$ and $\mathrm{Ran}(P')$ as

$\{w_1, \ldots, w_\beta\}$. For each $y_i \in \{y_1, \ldots, y_\alpha\}$ and $w_j \in \{w_1, \ldots, w_\beta\}$, we choose arbitrarily representatives $(y'_i, w'_j) \in \overline{\mathrm{Ran}(P)} \times \overline{\mathrm{Ran}(P')}$ such that $y'_i \oplus w'_j = y_i \oplus w_j$ for $1 \le i \le \alpha$ and $1 \le j \le \beta$. We remove these $\alpha\beta$ pairs (y'_i, w'_j) from $\overline{\mathrm{Ran}(P)} \times \overline{\mathrm{Ran}(P')}$ and obtain U. For each value $T \in \{0,1\}^n$, we have

$$|\{(y, w) \in U \mid y \oplus w = T\}| = 2^n - \alpha - \beta,$$

i.e., the chance to induce T from U is equal. Let α_i and β_i respectively denote the number of new defined domain points of P and P' at i-th query. Then

$$\begin{aligned}
&\sum_{i=1}^{q} \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case A}] \\
\le &\sum_{i=1}^{q} \frac{|(\overline{\mathrm{Ran}(P)} \times \overline{\mathrm{Ran}(P')}) \setminus U|}{|\overline{\mathrm{Ran}(P)} \times \overline{\mathrm{Ran}(P')}|} \\
= &\sum_{i=1}^{q} \frac{(\alpha_1 + \cdots + \alpha_i - 1)(\beta_1 + \cdots + \beta_i - 1)}{(2^n - \alpha_1 - \cdots - \alpha_i + 1)(2^n - \beta_1 - \cdots - \beta_i + 1)} \\
\le &\sum_{i=1}^{q} \frac{\sigma^2}{(2^n - \sigma)^2} \le \frac{4\sigma^2 q}{2^{2n}}
\end{aligned}$$

under the condition $\sigma \le 2^{n-1}$ and concludes the proof of Lemma 1.

5.3 Analysis of Case B

In this case, $x^i_{m_i}$ collides with previous inputs to P. The output string is $T_i = y^i_{m_i} \oplus w^i_{m_i}$. That is, either $y^i_{m_i}$ or $w^i_{m_i}$ being random may make T_i a random string. Our goal is to bound the probability that $x^i_{m_i}$ collides with previous inputs of P and subsequently $w^i_{m_i}$ deviates from a random n-bit string.

We first use the following *full collision probability* lemma proved in [3,4,14] to bound the probability of $x^i_{m_i}$ colliding with previous inputs of P. For any two prefix-free messages M_i and M_j, the full collision probability $\mathbf{FCP}_n(M_i, M_j)$ is the probability of the event $x^j_{m_j} \in \{x^i_1, \ldots, x^i_{m_i},$
$x^j_1, \ldots, x^j_{m_j - 1}\}$ where for each $b \in \{i, j\}$, we have $x^b_k = P(x^b_{k-1}) \oplus M_b[k]$ for $2 \le k \le m_b$ and $x^b_1 = M_b[1]$.

Lemma 2 (Full Collision Probability). *For any two prefix-free messages $M_i \in \{0,1\}^{m_i n}$ and $M_j \in \{0,1\}^{m_j n}$, we have*

$$\mathbf{FCP}_n(M_i, M_j) \le \frac{3(m_i + m_j)}{2^n - m_i - m_j} + \frac{(m_i + m_j)^4}{2^{2n}}.$$

Remark on This Lemma. The full collision probability lemma is first proved by Bellare *et al.* in [3] and then refined in its full version [4]. Recently Jha and Nandi [14] pointed out a flaw in the previous proof.

We denote by $\mathbf{FCP}_n(\emptyset, M_1)$ the probability of the special case $x^1_{m_1} \in \{x^1_1, \ldots, x^1_{m_1-1}\}$ and it is easily seen that $\mathbf{FCP}_n(\emptyset, M_1) \leq \frac{3m_1}{2^n - m_1} + \frac{m_1^4}{2^{2n}}$. Then at i-th query, we have

$$\Pr[x^i_{m_i} \in \mathrm{Dom}(P)] \leq \sum_{j=1}^{i-1} \mathbf{FCP}_n(M_j, M_i).$$

We next utilize game-playing techniques to examine the randomness of string $w^i_{m_i}$. Note that if we pick $w^i_{m_i}$ as $w^i_{m_i} \xleftarrow{\$} \{0,1\}^n$, then the distribution of $T_i = y^i_{m_i} \oplus w^i_{m_i}$ would be uniformly random. We consider the game presented in Fig. 5. The code with the boxed statement is the simulation of $P(x^i_{m_i}) \oplus P'(u^i_{m_i})$ while the code without boxed statement corresponds to a random function. Without a bad event occurring, the responses that $\mathcal{A}$ receives from the oracle are uniform and independent binary strings. We see that the bad event occurs with a probability of $|\mathrm{Ran}(P')|/2^n$ for each sampling operation, which is at most $\sigma/2^n$.

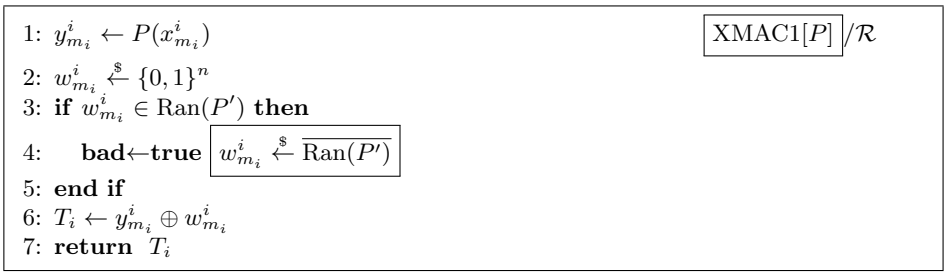

Fig. 5: Case B

Let $M_1, M_2, \ldots, M_q$ be a sequence of messages, then

$$\begin{aligned}
&\sum_{i=1}^{q} \Pr[x^i_{m_i} \in \mathrm{Dom}(P)] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case B}] \\
\leq &\sum_{i=1}^{q} \Pr[x^i_{m_i} \in \mathrm{Dom}(P)] \cdot \frac{|\mathrm{Ran}(P')|}{2^n} \\
\leq &\frac{\sigma}{2^n} \cdot \left(\mathbf{FCP}_n(\emptyset, M_1) + \sum_{i=2}^{q} \sum_{j=1}^{i-1} \mathbf{FCP}_n(M_j, M_i) \right) \\
\leq &\frac{\sigma}{2^n} \cdot \left(\frac{6m_1}{2^n} + \frac{m_1^4}{2^{2n}} + \sum_{i=2}^{q} \sum_{j=1}^{i-1} \left(\frac{6(m_i + m_j)}{2^n} + \frac{(m_i + m_j)^4}{2^{2n}} \right) \right) \\
\leq &\frac{\sigma}{2^n} \cdot \left(\frac{12\sigma q}{2^n} + \frac{16\sigma q \ell^3}{2^{2n}} \right) \leq \frac{28\sigma^2 q}{2^{2n}},
\end{aligned}$$

if $\ell \leq 2^{n/3}$.

5.4 Analysis of Case C

The analysis of Case C is identical to Case B since P and P' are two independent random permutations. We obtain the same upper bound in this case:

$$\sum_{i=1}^{q} \Pr[u^i_{m_i} \in \mathrm{Dom}(P')] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case C}] \leq \frac{28\sigma^2 q}{2^{2n}},$$

if $\ell \leq 2^{n/3}$.

5.5 Analysis of Case D

In this case, $y^i_{m_i}$ and $w^i_{m_i}$ both have appeared before and T_i is not a random string anymore. As shown in Fig.6, we always set **bad** flag in this case. The code with boxed statements simulates $P(x^i_{m_i}) \oplus P'(u^i_{m_i})$ for $1 \leq i \leq q$ while the code without boxed statements is the simulation of a random function.

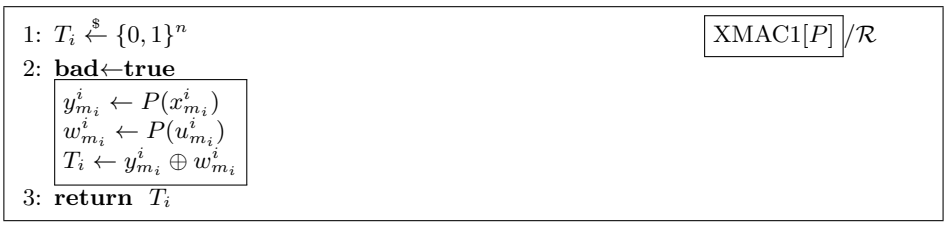

Fig. 6: Case D

Let $M_1, \ldots, M_q$ be a sequence of messages, then by using Lemma 2, we have

$$\begin{aligned}
&\sum_{i=1}^{q} \Pr[x^i_{m_i} \in \mathrm{Dom}(P) \wedge u^i_{m_i} \in \mathrm{Dom}(P')] \\
\leq &\mathbf{FCP}_n(\emptyset, M_1)^2 + \sum_{i=2}^{q}\sum_{j=1}^{i-1}\sum_{k=1}^{i-1} \mathbf{FCP}_n(M_j, M_i) \cdot \mathbf{FCP}_n(M_k, M_i) \\
\leq &\left(\frac{6m_1}{2^n} + \frac{m_1^4}{2^{2n}}\right)^2 + \sum_{i=2}^{q}\sum_{j=1}^{i-1}\sum_{k=1}^{i-1} \left(\frac{6(m_i+m_j)}{2^n} + \frac{(m_i+m_j)^4}{2^{2n}}\right) \cdot \left(\frac{6(m_i+m_k)}{2^n} + \frac{(m_i+m_k)^4}{2^{2n}}\right) \\
\leq &\frac{144\sigma q^2 \ell}{2^{2n}} + \frac{384\sigma q^2 \ell^4}{2^{3n}} + \frac{256\sigma q^2 \ell^7}{2^{4n}} \leq \frac{784\sigma q^2 \ell}{2^{2n}},
\end{aligned}$$

if $\ell \leq 2^{n/3}$.

5.6 Summation

Finally we sum up the probabilities over above four cases and obtain

$$\begin{aligned}&\Pr[\mathcal{A}^{\mathcal{R}(\cdot)} \text{ sets } \mathbf{bad}]\\ \leq&\frac{4\sigma^2 q}{2^{2n}} + \frac{28\sigma^2 q}{2^{2n}} + \frac{28\sigma^2 q}{2^{2n}} + \frac{784\sigma q^2 \ell}{2^{2n}}\\ \leq&\frac{844\sigma q^2 \ell}{2^{2n}}\end{aligned}$$

under the condition $\ell \leq 2^{n/3}$, which completes the proof of Theorem 1.

6 Security of XMAC5

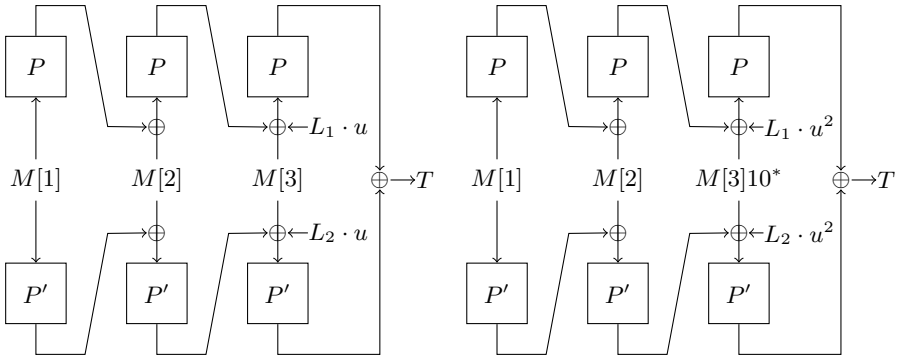

Fig. 7: Illustration of XMAC5[P]. The left is the case where the message length is a positive multiple of n while the right is the case where the message length is not a positive multiple of n. $L_1 = P(0^n)$ and $L_2 = P'(0^n)$, u is some non-zero constant, and $\cdot$ is field multiplication.

The proof of XMAC5 is similar to the proof of XMAC1 with padding scheme 3 and we only outline their main differences here. We use exactly the same main game except that we need to set **bad** flag in following additional events:

- $L_1 \cdot u = 0^n$, $L_2 \cdot u = 0^n$, $L_1 \cdot u^2 = 0^n$, $L_2 \cdot u^2 = 0^n$, i.e., $L_1 = 0^n$ or $L_2 = 0^n$. $\Pr[L_1 = 0^n \vee L_2 = 0^n] = \Pr[P, P' \xleftarrow{\$} \mathrm{Perm}(n) : P(0^n) = 0^n \vee P'(0^n) = 0^n] = \frac{2}{2^n}$;
- $L_1 \cdot u = L_1 \cdot u^2$, $L_2 \cdot u = L_2 \cdot u_2$, i.e., $L_1 = \mathrm{constant}_1$, $L_2 = \mathrm{constant}_2$. $\Pr[L_1 = \mathrm{constant}_1 \vee L_2 = \mathrm{constant}_2] = \Pr[P, P' \xleftarrow{\$} \mathrm{Perm}(n) : P(0^n) = \mathrm{constant}_1 \vee P'(0^n) = \mathrm{constant}_2] = \frac{2}{2^n}$.

These account to a term $\frac{4}{2^n}$. The definitions of four cases (A,B,C,D) are the same as in Section 5. Note that $P(0^n)$ and $P'(0^n)$ have been defined at the beginning

and thus $P(0^n) \in \mathrm{Ran}(P)$ and $P'(0^n) \in \mathrm{Ran}(P')$. We always consider a padded sequence of messages $M_1, \ldots, M_q$, that is, each message has been padded with 10^* when the length is not a positive multiple of n and further the last block of each message has been bitwise exclusive-or with either $L_i \cdot u$ or $L_i \cdot u^2$ depending on the length, and $i = 1$ when analyzing P, $i = 2$ when analyzing P'.

– In the Case A, we have

$$\begin{aligned}
&\sum_{i=1}^{q} \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case A}] \\
\le &\sum_{i=1}^{q} \frac{|(\overline{\mathrm{Ran}(P)} \times \overline{\mathrm{Ran}(P')}) \setminus U|}{|\overline{\mathrm{Ran}(P)} \times \overline{\mathrm{Ran}(P')}|} \\
= &\sum_{i=1}^{q} \frac{(\alpha_1 + \cdots + \alpha_i)(\beta_1 + \cdots + \beta_i)}{(2^n - \alpha_1 - \cdots - \alpha_i)(2^n - \beta_1 - \cdots - \beta_i)} \\
\le &\sum_{i=1}^{q} \frac{\sigma^2}{(2^n - \sigma)^2} \le \frac{4\sigma^2 q}{2^{2n}}
\end{aligned}$$

under the condition $\sigma \le 2^{n-1}$.

– In the Case B, the probability that $x^i_{m_i}$ collides with the values in $\mathrm{Dom}(P)$ would be slightly enlarged as M_i may be a prefix of previous messages (we consider the messages list that after padded with 10^* when the length is not a positive multiple of n and further the last block of each message has been bitwise exclusive-or with either $L_1 \cdot u$ or $L_1 \cdot u^2$ depending on the length of message). The probability of M_i being a prefix of previous messages or $M_i = 0^n$ is at most $\frac{i}{2^n}$, as if M_i is a prefix of M_j for $1 \le j \le i-1$ or $M_i = 0^n$:

- $m_i = m_j$, then we have $M_i[m_i] \oplus L_1 \cdot u = M_j[m_j] \oplus L_1 \cdot u^2$, which happens with probability of $\frac{1}{2^n}$;
- if $m_i < m_j$, then we have $M_i[1] \| \ldots \| M_i[m_i - 1] = M_j[1] \| \ldots \| M_j[m_i - 1]$ and $M_i[m_i] \oplus L_1 \cdot u = M_j[m_i]$ (or $M_i[m_i] \oplus L_1 \cdot u^2 = M_j[m_i]$), which happens with probability of $\frac{1}{2^n}$;
- if $m_i > m_j$, then it is impossible;
- if $M_i = 0^n$, then $M_i[1] \oplus L_1 \cdot u = 0^n$ (or $M_i[1] \oplus L_1 \cdot u^2 = 0^n$), which happens with probability of $\frac{1}{2^n}$.

Let $M_0 = 0^n$ and $M_1, \ldots, M_q$ be a sequence of messages, then

$$\begin{aligned}
&\sum_{i=1}^{q} \Pr[x^i_{m_i} \in \mathrm{Dom}(P)] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case B}] \\
\leq &\sum_{i=1}^{q} \Pr[x^i_{m_i} \in \mathrm{Dom}(P)] \cdot \frac{|\mathrm{Ran}(P')|}{2^n} \\
\leq &\frac{\sigma}{2^n} \cdot \left(\sum_{i=2}^{q} \Pr[M_i \text{ is a prefix} \vee M_i = 0^n] + \sum_{i=1}^{q} \sum_{j=0}^{i-1} \mathbf{FCP}_n(M_j, M_i) \right) \\
\leq &\frac{\sigma}{2^n} \cdot \left(\sum_{i=1}^{q} \frac{i}{2^n} + \sum_{i=1}^{q} \sum_{j=0}^{i-1} (\frac{6(m_i + m_j)}{2^n} + \frac{(m_i + m_j)^4}{2^{2n}}) \right) \\
\leq &\frac{\sigma}{2^n} \cdot \left(\frac{q(q+1)}{2^{n+1}} + \frac{12\sigma q}{2^n} + \frac{16\sigma q \ell^3}{2^{2n}} \right) \leq \frac{29\sigma^2 q}{2^{2n}},
\end{aligned}$$

if $\ell \leq 2^{n/3}$.

– In the Case C, the analysis is exactly the same as in the Case B, and thus

$$\sum_{i=1}^{q} \Pr[u^i_{m_i} \in \mathrm{Dom}(P')] \cdot \Pr[\mathcal{A} \text{ sets } \mathbf{bad} \mid \text{Case C}] \leq \frac{29\sigma^2 q}{2^{2n}},$$

if $\ell \leq 2^{n/3}$.

– In the Case D, we have

$$\begin{aligned}
&\sum_{i=1}^{q} \Pr[x^i_{m_i} \in \mathrm{Dom}(P) \wedge u^i_{m_i} \in \mathrm{Dom}(P')] \\
\leq &\sum_{i=1}^{q} \left(\frac{i}{2^n} + \sum_{j=0}^{i-1} \mathbf{FCP}_n(M_j, M_i) \right) \cdot \left(\frac{i}{2^n} + \sum_{k=0}^{i-1} \mathbf{FCP}_n(M_k, M_i) \right) \\
\leq &\sum_{i=1}^{q} \sum_{j=0}^{i-1} \sum_{k=0}^{i-1} \mathbf{FCP}_n(M_j, M_i) \cdot \mathbf{FCP}_n(M_k, M_i) \\
&+ \frac{2q}{2^n} \sum_{i=1}^{q} \sum_{j=0}^{i-1} \mathbf{FCP}_n(M_j, M_i) + \sum_{i=1}^{q} \frac{i^2}{2^{2n}} \\
\leq &\sum_{i=1}^{q} \sum_{j=0}^{i-1} \sum_{k=0}^{i-1} \left(\frac{6(m_i + m_j)}{2^n} + \frac{(m_i + m_j)^4}{2^{2n}} \right) \cdot \left(\frac{6(m_i + m_k)}{2^n} + \frac{(m_i + m_k)^4}{2^{2n}} \right) \\
&+ \frac{2q}{2^n} \sum_{i=1}^{q} \sum_{j=0}^{i-1} (\frac{6(m_i + m_j)}{2^n} + \frac{(m_i + m_j)^4}{2^{2n}}) + \frac{q(q+1)^2}{3 \cdot 2^{2n}} \\
\leq &\frac{784\sigma q^2 \ell}{2^{2n}} + \frac{2q}{2^n} \cdot \left(\frac{12\sigma q}{2^n} + \frac{16\sigma q \ell^3}{2^{2n}} \right) + \frac{q(q+1)^2}{3 \cdot 2^{2n}} \leq \frac{841\sigma q^2 \ell}{2^{2n}}
\end{aligned}$$

Table 2: Impact of XORing of two MACs on ISO/IEC 9797-1:2011

MAC Algorithm	#keys	Padding	Security	XORing Security.
1	1	2	0	$O(\sigma^2/2^n)$ App. A
1	1	3	$O(\ell q^2/2^n)$ [3]	$O(\sigma q^2\ell/2^{2n})$ Sec. 5
2	2	2	$O(q^2/2^n)$[20]	$O(q^3\ell^3/2^{2n})$[25]
2	2	3	$O(q^2/2^n)$[20]	$O(q^3\ell^3/2^{2n})$[25]
3	2	2	$O(q^2/2^n)$[20]	$O(q^3\ell^3/2^{2n})$[25]
3	2	3	$O(q^2/2^n)$[20]	$O(q^3\ell^3/2^{2n})$ [25]
4	3	2	$O(q^2/2^n)$[20]	$O(q^3\ell^3/2^{2n})$[25]
4	3	3	$O(q^2/2^n)$[20]	$O(q^3\ell^3/2^{2n})$[25]
5	1	4	$O(\sigma q/2^n)$[18]	$O(\sigma q^2\ell/2^{2n})$ Sec. 6
6	2	2	$O(q^2/2^n)$[20]	$O(q^3\ell^3/2^{2n})$ [25]
6	2	3	$O(q^2/2^n)$[20]	$O(q^3\ell^3/2^{2n})$ [25]

if $\ell \leq 2^{n/3}$.

Summing up the above possibilities, the PRF-security bound of XMAC5 can be bounded by

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC5}[P]}(q,\ell,\sigma) \leq \frac{4}{2^n} + \frac{58\sigma^2 q}{2^{2n}} + \frac{841\sigma q^2\ell}{2^{2n}}.$$

7 Impact of XORing on ISO/IEC 9797-1:2011

We briefly discuss the impact of XORing of two MACs on ISO/IEC 9797-1:2011 in this section and the results are illustrated in Table 2. One can easily obtain the provable-security bounds of XORing of any two MACs by using the similar proof techniques in this paper and we omit the details here. To be consistent with previous notations, we denote by XMACi the XORing of two MAC Algorithm i. Note that we only consider the XORing of a MAC Algorithm with two different keys here since using two different MAC Algorithms to authenticate a message is relatively inefficient and unpractical.

MAC Algorithm 1 XMAC1 can resist to length-extension attack and achieve birthday bound security with padding scheme 2; with padding scheme 3, the security of XMAC1 is lifted up to beyond birthday bound.

MAC Algorithm 2 The security of XMAC2 is lifted to beyond birthday bound for both padding scheme 2 and 3. The proof can be found in [25].

MAC Algorithm 3 As discussed in [23], by replacing the last two permutation calls $P \circ P'$ with another random permutation P'', this algorithm is same as MAC Algorithm 2 in the information-theoretic setting. Hence this algorithm enjoys the same bounds as MAC Algorithm 2.

MAC Algorithm 4 As discussed in [23], in the information-theoretic setting, the first and second permutation can be reversed and then one makes the first permutation public. In this sense, this algorithm enjoys the same security guarantees as MAC Algorithm 2.

MAC Algorithm 5 The security of XMAC5 will be lifted up to beyond birthday bound.

MAC Algorithm 6 This algorithm is same as EMAC except just saving one block cipher call at the final iteration. The security of XMAC6 will be lifted up to beyond birthday bound and the proof is exactly the same as SUM-ECBC[25].

8 Conclusion

In this paper, we prove that XMAC1 can achieve either beyond birthday bound security or birthday bound security depending on padding methods. We also prove that XMAC5 can achieve $2n/3$-bit security. Our results imply that XORing of any two CBC-like MACs in ISO/IEC 9797-1:2011 can lift up the security bound to a higher level. It seems unlikely that $O(2^{2n/3})$ is a tight bound for either XMAC1 or XMAC5. A future work is to further improve the provable-security bounds of these constructions.

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A Security of XMAC1 with Padding Scheme 2

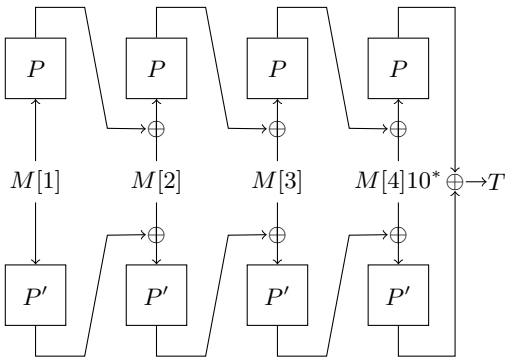

Fig. 8: Illustration of XMAC1[P] with padding scheme 2 for $M = M[1]\|M[2]\|M[3]\|M[4]$, where $|M[1]| = |M[2]| = |M[3]| = n$ and $0 \leq |M[4]| \leq n-1$.

We always consider the messages list after padded with padding scheme 2, i.e., $M_i = \mathbf{pad2}(M_i)$ and denote the message length $m_i = |\mathbf{pad2}(M_i)|$. Note that padding scheme 2 is not a prefix-free encoding, and if the adversary asks a long message $M_1 = M_2\|*$ then asks M_2 a prefix of M_1, the internal values during

computing M_2 have all been defined before. Hence we cannot use the same proof methods in section 5 or section 6. In this section, we resort to the well-known H-coefficient technique [10,19] and prove that XMAC1[P] with padding scheme 3 is a $O(2^{n/2})$-secure PRF.

A.1 Double Collision Attack

For the sake of completeness, we sketch the double collision attack [16] here for XMAC1 with padding scheme 2. First we search for two one-block messages M_1 and M_2 that yield the same tag $T = E_{K_1}(M_1) \oplus E_{K_2}(M_1) = E_{K_1}(M_2) \oplus E_{K_2}(M_2)$. Then we compute the tag values for two one block longer messages of the form $M_1 \| A$ and $M_2 \| B$ for random A, B. If it holds that $A \oplus B = E_{K_1}(M_1) \oplus E_{K_1}(M_2) = E_{K_2}(M_1) \oplus E_{K_2}(M_2)$, then we get collisions for both the upper chain and lower chain and thus a collision on the MAC values of these two extended messages. We can check this kind of double collisions by adding a same block at the end of both extended messages and the resulting messages still collide. This attack requires about $2^{1+n/2}$ MAC computations. However, such a attack does not work out for either XMAC5 or XMAC1 with padding scheme 3, as the padded messages list is always prefix-free except with a negligible probability.

A.2 The H-coefficient Technique

We briefly introduce the H-coefficient technique [10,19] here as the following part of this section adopts this method. A view ν is the query-response tuples that $\mathcal{A}$ receives when interacting with either $F[P]$ (real world) or $\mathcal{R}$ (ideal world). We denote X_{re}, resp. X_{id}, the probability distribution of the ν when $\mathcal{A}$ interacts with $F[P]$, resp. $\mathcal{R}$. We also denote $\Theta = \{\nu \mid \Pr[X_{\mathrm{id}} = \nu] > 0\}$ the set of all attainable views ν while $\mathcal{A}$ interacting with $\mathcal{R}$. The H-coefficient technique evaluates the upper bound of $\mathbf{Adv}^{\mathrm{prf}}_{F[P]}(\mathcal{A})$ by using the following lemma. The proof of this lemma can be found in [9,10].

Lemma 3. *Let Θ_{good} and Θ_{bad} be two disjoint subsets of Θ satisfying $\Theta = \Theta_{\mathrm{good}} \sqcup \Theta_{\mathrm{bad}}$. If there exists ϵ_1 such that $\Pr[X_{\mathrm{id}} \in \Theta_{\mathrm{bad}}] \leq \epsilon_1$ and for each view $\nu \in \Theta_{\mathrm{good}}$, it has*

$$\frac{\Pr[X_{\mathrm{re}} = \nu]}{\Pr[X_{\mathrm{id}} = \nu]} \geq 1 - \epsilon_2.$$

Then $\mathbf{Adv}^{\mathrm{prf}}_{F[P]}(\mathcal{A}) \leq \epsilon_1 + \epsilon_2$.

A.3 Preparations for the H-coefficient Technique

We replace P and P' in XMAC1[P] by random functions F and F', respectively. We write the resulting algorithm as XMAC1[$\mathcal{R}$]. Using the PRP/PRF switching lemma [5], we obtain

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[P]}(q, \ell, \sigma) \leq \frac{\sigma^2}{2^n} + \mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[\mathcal{R}]}(q, \ell, \sigma).$$

We define the following functions from F, F', rand_i and rand'_i, where $\mathsf{rand}_i, \mathsf{rand}'_i \xleftarrow{\$} \{0,1\}^n$ for $1 \le i \le \ell - 1$:

$$\begin{cases} Q_{1,1}(X) = F(X) \oplus \mathsf{rand}_1 \\ Q_{1,i}(X) = F(X \oplus \mathsf{rand}_{i-1}) \oplus \mathsf{rand}_i \text{ for } 2 \le i \le \ell - 1 \\ Q_{1,\ell}(X) = F(X \oplus \mathsf{rand}_{\ell-1}) \\ Q_{2,1}(X) = F'(X) \oplus \mathsf{rand}'_1 \\ Q_{2,i}(X) = F'(X \oplus \mathsf{rand}'_{i-1}) \oplus \mathsf{rand}'_i \text{ for } 2 \le i \le \ell - 1 \\ Q_{2,\ell}(X) = F'(X \oplus \mathsf{rand}'_{\ell-1}) \end{cases}$$

We write $\mathcal{Q}$ for the set of these functions. Let $G_{i,j}$ be 2ℓ independent random functions, for $1 \le i \le 2$ and $1 \le j \le \ell$. We write $\mathcal{G}$ for the set of these functions. For an adversary $\mathcal{B}$, we define

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathcal{Q}}(\mathcal{B}) \stackrel{\mathrm{def}}{=} \Pr[\mathcal{B}^{\mathcal{Q}(\cdot)} \Rightarrow 1] - \Pr[\mathcal{B}^{\mathcal{G}(\cdot)} \Rightarrow 1],$$

where in the right-hand side of the equation, the first probability is taken over $F, F', \mathsf{rand}_i, \mathsf{rand}'_i$ and $\mathcal{B}$'s coin, and the second one is over random functions in $\mathcal{G}$ and $\mathcal{B}$'s coin. $\mathcal{B}$ makes queries of the form $(i, j, X) \in \{1,2\} \times \{1, 2, \cdots, \ell\} \times \{0,1\}^n$, and receives $Q_{i,j}(X)$ or $G_{i,j}(X)$. We prove that $\mathcal{Q}$ is indistinguishable from $\mathcal{G}$ by the following lemma.

Lemma 4. *Let $\mathcal{B}$ be an adversary that makes at most q queries. Then we have $\mathbf{Adv}^{\mathrm{prf}}_{\mathcal{Q}}(\mathcal{B}) \le 0.5q^2/2^n$.*

Proof. When $\mathcal{B}$ interacts with the oracle $\mathcal{Q}$, we define two sets. $\mathcal{I}_1$ is the set of input values of F in $Q_{1,i}$ and $\mathcal{I}_2$ is the set of input values of F' in $Q_{2,i}$, for $1 \le i \le \ell$. We set a **bad** flag if $\mathcal{I}_1$ has a collision or $\mathcal{I}_2$ has a collision. The code with boxed statements simulates $\mathcal{Q}$ while the code without boxed statements simulates $\mathcal{G}$. By the fundamental lemma of game-playing [5], we have

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathcal{Q}}(\mathcal{B}) \le \Pr[\mathcal{B}^{\mathcal{G}(\cdot)} \text{ sets } \mathbf{bad}].$$

Before bad event occurs, $\mathcal{B}$ learns nothing from the values returned by the oracle except a random n-bit string. Hence We only need to consider a fixed sequence of queries made by $\mathcal{B}$. Suppose that $\mathcal{B}$ makes total q_1 queries to $Q_{1,i}$ and makes total q_2 queries to $Q_{2,i}$ for $1 \le i \le \ell$. $\mathcal{I}_1$ has a collision if and only if $X = X' \oplus \mathsf{rand}_i$ or $X \oplus \mathsf{rand}_i = X' \oplus \mathsf{rand}_j$ for $1 \le i, j \le \ell - 1$. Since rand_i is a random string, we have $\Pr[\mathcal{I}_1 \text{ has a collision}] \le 0.5q_1^2/2^n$. Similarly, $\Pr[\mathcal{I}_2$ has a collision$] \le 0.5q_2^2/2^n$. Therefore, we can bound the overall probability of bad event occurring as

$$\Pr[\mathcal{B}^{\mathcal{G}(\cdot)} \text{ sets } \mathbf{bad}] \le \frac{0.5q_1^2}{2^n} + \frac{0.5q_2^2}{2^n} \le \frac{0.5q^2}{2^n},$$

which concludes the proof.

Initialization: **Oracle $\mathcal{G}$/ Oracle $\mathcal{Q}$**

1: **bad**←**false**; $\mathcal{I}_1, \mathcal{I}_2 \leftarrow \emptyset$

Procedure: $\mathcal{O}(i, j, X)$:

2: $T_{i,j} \xleftarrow{\$} \{0,1\}^n$
3: **if** $i = 1$ **then**
4: **if** $j = 1$ **then**
5: **if** $X \in \mathcal{I}_1$ **then**
6: **bad**←**true** $\boxed{,T_{i,j} \leftarrow Q_{1,1}(X)}$
7: **else**
8: $\mathcal{I}_1 \leftarrow \mathcal{I}_1 \cup \{X\}$
9: **end if**
10: **else**
11: **if** $X \oplus \mathsf{rand}_j \in \mathcal{I}_1$ **then**
12: **bad**←**true** $\boxed{,T_{i,j} \leftarrow Q_{1,j}(X)}$
13: **else**
14: $\mathcal{I}_1 \leftarrow \mathcal{I}_1 \cup \{X \oplus \mathsf{rand}_j\}$
15: **end if**
16: **end if**
17: **end if**
18: **if** $i = 2$ **then**
19: **if** $j = 1$ **then**
20: **if** $X \in \mathcal{I}_2$ **then**
21: **bad**←**true** $\boxed{,T_{i,j} \leftarrow Q_{2,1}(X)}$
22: **else**
23: $\mathcal{I}_2 \leftarrow \mathcal{I}_2 \cup \{X\}$
24: **end if**
25: **else**
26: **if** $X \oplus \mathsf{rand}'_j \in \mathcal{I}_2$ **then**
27: **bad**←**true** $\boxed{,T_{i,j} \leftarrow Q_{2,j}(X)}$
28: **else**
29: $\mathcal{I}_2 \leftarrow \mathcal{I}_2 \cup \{X \oplus \mathsf{rand}'_j\}$
30: **end if**
31: **end if**
32: **end if**
33: **return** $T_{i,j}$

Fig. 9: Game used to prove Lemma 4

We consider an algorithm XMAC1[$\mathcal{Q}$], manipulating messages based on $\mathcal{Q}$. Its definition is presented in Fig. 10. We can see that XMAC1[$\mathcal{Q}$] is exactly the same as XMAC1[$\mathcal{R}$] since all the internal values rand_i and rand'_i are canceled during the computation. We next consider another algorithm XMAC1[$\mathcal{G}$] based on $\mathcal{G}$. Its definition is presented in Fig. 11. It is obtained from XMAC1[$\mathcal{Q}$] by replacing $Q_{i,j}$ with $G_{i,j}$, for $i \in \{1,2\}$ and $1 \le j \le \ell$. By using the Lemma 4, we have

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[\mathcal{R}]}(q,\ell,\sigma) = \mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[\mathcal{Q}]}(q,\ell,\sigma) \le \mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[\mathcal{G}]}(q,\ell,\sigma) + \frac{\sigma^2}{2^n}.$$

$M_i[1] \| \ldots \| M_i[m_i] \xleftarrow{n} \mathbf{pad2}(M_i)$
$y_0^i \leftarrow 0^n$
$w_0^i \leftarrow 0^n$
for $j = 1$ to m_i **do**
  $y_j^i = Q_{1,j}(y_{j-1}^i \oplus M_i[j])$
  $w_j^i = Q_{2,j}(w_{j-1}^i \oplus M_i[j])$
end for
$T_i \leftarrow y_{m_i}^i \oplus w_{m_i}^i$
return T_i

Fig. 10: Definition of XMAC1[$\mathcal{Q}$](M_i)

To upper bound $\mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[\mathcal{G}]}(q, \ell, \sigma)$, we next resort to the H-coefficient technique [10,19], which has been briefly introduced in Sect. A.2.

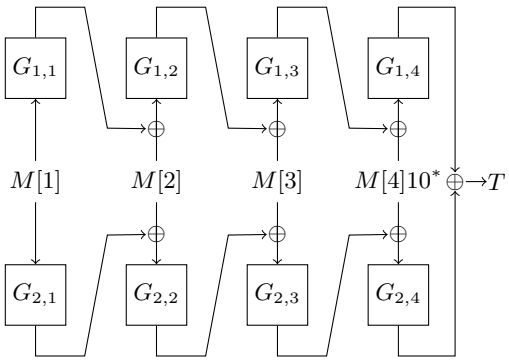

Fig. 11: Definition of XMAC1[$\mathcal{G}$], $G_{i,j}$ are independent random functions for $i \in \{1, 2\}$ and $1 \leq j \leq 4$.

A.4 Analysis of Bad Views

In the real world, the corresponding oracle is XMAC1[$\mathcal{G}$] while in the ideal world it is a random function. We note that the adversary $\mathcal{A}$ can make at most q queries to its oracle, each query being at most ℓ blocks, the total number of blocks of queries being at most σ, and outputs a single bit. Let a view

$$\nu = ((M_1, T_1), \ldots, (M_q, T_q))$$

be a list of queries and corresponding answers. We start by defining bad views and good views.

Definition 1. *A bad view is an attainable view* $\nu = ((M_1, T_1), \ldots, (M_q, T_q))$ *that there exists a collision in* ν *such that*

$$T_i = T_j, \text{ where } 1 \leq i < j \leq q.$$

Otherwise, we call it a good view. We denote Θ_{bad} the set of bad views and Θ_{good} the set of good views.

Now we upper bound the probability to get a bad view in the ideal world.

Lemma 5. *For any integer q, we have*

$$\Pr[X_{\mathrm{id}} \in \Theta_{\mathrm{bad}}] \leq \frac{0.5q^2}{2^n}.$$

Proof. In the ideal world, T_i is simply a random n-bit string and $\Pr[T_i = T_j] = \frac{1}{2^n}$ for any $i \neq j$. Thus,

$$\Pr[X_{\mathrm{id}} \in \Theta_{\mathrm{bad}}] \leq \binom{q}{2}\frac{1}{2^n} \leq \frac{0.5q^2}{2^n}.$$

A.5 Analysis of Good Views

We now analyze good views and prove the following lemma.

Lemma 6. *For any good view ν, we have*

$$\frac{\Pr[X_{\mathrm{re}} = \nu]}{\Pr[X_{\mathrm{id}} = \nu]} \geq 1 - \frac{2\sigma q}{2^n}.$$

Proof. Let $\nu = ((M_1, T_1), \ldots, (M_q, T_q))$ be a good view. Since in the ideal world the oracle is a random function, we simply have

$$\Pr[X_{\mathrm{id}} = \nu] = \frac{1}{2^{qn}}. \tag{1}$$

Now we proceed to lower bound the probability of obtaining ν in the real world. The key point is to count the number of functions that induce ν. From the definition, we have

$$\Pr[X_{\mathrm{re}} = \nu] = \frac{\#\text{functions inducing } \nu}{\#\text{total functions}}.$$

For a message $M_i = M_i[1]\|\cdots\|M_i[m_i]$, we denote x_j^i and y_j^i the input and corresponding output of $G_{1,j}$, u_j^i and w_j^i the input and corresponding output of $G_{2,j}$ in XMAC1[$\mathcal{G}$] for $1 \leq i \leq m_i$. Since our goal is to compute the lower bound of $\Pr[X_{\mathrm{re}} = \nu]$, we can ignore some troublesome functions and merely count the number of ones that induce ν and satisfy the following **condition** to ease the analysis:

$$\textit{if } M_i[1]\|\cdots\|M_i[t] \neq M_j[1]\|\cdots\|M_j[t], \textit{ then } x_t^i \neq x_t^j \textit{ and } u_t^i \neq u_t^j.$$

We denote σ_i the number of messages that have block length at least i. We first compute the probability of $G_{1,1}$ and $G_{2,1}$ satisfying the requirements. We divide these q messages into several groups according to the first block. Messages in each group have the identical first block. After such a classification, we will obtain r

groups. Assume the ith group contains q_i messages and denote $M_j^i (1 \le j \le q_i)$ the jth message in ith group, then we have $q_1 + q_2 + \cdots + q_r = \sigma_1$. By abusing notation, in the ith group, we denote x_1^i and y_1^i the input and output of $G_{1,1}$, u_1^i and w_1^i the input and output of $G_{2,1}$. We note that if there exists a message in the ith group outputting a tag T_i after computation of the first block, then there is a relation between y_1^i and w_1^i as $y_1^i \oplus w_1^i$. Note that each group has at most one relation, otherwise this group has a pair of identical messages. Then we count the number of choices y_1^i and w_1^i in ith group for $1 \le i \le r$ in turn, which will be affected by whether there exists a relation in this group or not:

- For the first group,
  - no relation: there are both 2^n possibilities for y_1^1 and w_1^1, thus total $(2^n)^2$;
  - a relation: there are 2^n possibilities for y_1^1 and once y_1^1 is determined so does w_1^1 as $w_1^1 = T_1 \oplus y_1^1$.
- For the 2nd group, once y_1^1 and w_1^1 are fixed,
  - no relation: due to the additional **condition**, we have $y_1^2 \oplus M_j^2[2] \neq y_1^1 \oplus M_i^1[2]$ and $w_1^2 \oplus M_j^2[2] \neq w_1^1 \oplus M_i^1[2]$ for $1 \le i \le q_1$ and $1 \le j \le q_2$. Therefore, there are both at least $2^n - q_1 q_2$ possibilities for y_1^2 and w_1^2, thus total at lest $(2^n - q_1 q_2)^2$.
  - a relation: there are at least $2^n - 2q_1 q_2$ possibilities for y_1^2 as $y_1^2 \oplus M_j^2[2] \neq y_1^1 \oplus M_i^1[2]$ and $y_1^2 \oplus T_2 \oplus M_j^2[2] = w_1^2 \oplus M_j^2[2] \neq w_1^1 \oplus M_i^1[2]$ for $1 \le i \le q_1$ and $1 \le j \le q_2$.
- ...
- For the k-th group, once $y_1^1, \ldots, y_1^{k-1}$ and $w_1^1, \ldots, w_1^{k-1}$ are fixed,
  - no relation: due to the additional **condition**, we have $y_1^k \oplus M_j^k[2] \neq y_1^t \oplus M_i^t[2]$ and $w_1^k \oplus M_j^k[2] \neq w_1^t \oplus M_i^t[2]$ for $1 \le t \le k-1, 1 \le i \le q_t$ and $1 \le j \le q_k$. Therefore, there are both at least $2^n - q_k(q_1 + q_2 + \cdots + q_{k-1})$ possibilities for y_1^k and w_1^k, thus total at least $(2^n - q_k(q_1 + q_2 + \cdots + q_{k-1}))^2$.
  - a relation: there are at least $2^n - 2q_k(q_1 + q_2 + \cdots + q_{k-1})$ possibilities for y_1^k since $y_1^k \oplus M_j^k[2] \neq y_1^t \oplus M_i^t[2]$ and $y_1^k \oplus T_k \oplus M_j^k[2] = w_1^k \oplus M_j^k[2] \neq w_1^t \oplus M_i^t[2]$ for $1 \le t \le k-1, 1 \le i \le q_t$ and $1 \le j \le q_k$.

Hence, the number of tuples $(y_1^1, w_1^1, \ldots, x_1^r, w_1^r)$ is at least

$$\prod_{i=1}^{r} C^i$$

where for $1 \le i \le r$, either $C^i = (2^n - q_i(q_1 + \cdots + q_{i-1}))^2 \ge (2^n - q_i q)^2 \ge 2^n(2^n - 2q_i q)$ without a relation, or $C^i = 2^n - 2q_i(q_1 + \cdots + q_{i-1}) \ge 2^n - 2q_i q$ with a relation. We denote s the total number of relations among these r groups,

then the probability that $G_{1,1}$ and $G_{2,1}$ meet the requirements is at least

$$\begin{aligned}\frac{\prod_{i=1}^{r} C^i((2^n)^{2^n-r})^2}{((2^n)^{2^n})^2} &= \prod_{i=1}^{r} \frac{C^i}{2^{2n}} \\ &\geq \frac{1}{2^{ns}} \prod_{i=1}^{r} (1 - \frac{2q_i q}{2^n}) \\ &\geq \frac{1}{2^{ns}} (1 - \frac{2\sigma_1 q}{2^n}).\end{aligned}$$

Then we proceed to analyze the remaining blocks and apply the same analysis to the rest of random functions $G_{1,i}$ and $G_{2,i}$ as they are independent from each other for $1 \leq i \leq \ell$. There are at most ℓ pairs of $(G_{1,i}, G_{2,i})$, so the probability of inducing ν in the real world can be bounded by

$$\begin{aligned}\Pr[X_{\mathrm{re}} = \nu] &\geq \prod_{i=1}^{\ell} \frac{1}{2^{ns}} (1 - \frac{2\sigma_i q}{2^n}) \\ &= \frac{1}{2^{nq}} \prod_{i=1}^{\ell} (1 - \frac{2\sigma_i q}{2^n}) \\ &\geq \frac{1}{2^{nq}} (1 - \frac{2\sigma q}{2^n}). \end{aligned} \tag{2}$$

Combining (1) and (2) together, we obtain

$$\frac{\Pr[X_{\mathrm{re}} = \nu]}{\Pr[X_{\mathrm{id}} = \nu]} \geq 1 - \frac{2\sigma q}{2^n}, \tag{3}$$

and this completes the proof of Lemma 6.

Following Lemma 3 and using the results of Lemma 5 and Lemma 6, we have

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[\mathcal{G}]}(q, \ell, \sigma) \leq \frac{2\sigma q}{2^n} + \frac{0.5q^2}{2^n}. \tag{4}$$

Finally we obtain the claimed bound in Theorem 2 as

$$\begin{aligned}\mathbf{Adv}^{\mathrm{prf}}_{\mathrm{XMAC1}[P]}(q, \ell, \sigma) &\leq \frac{\sigma^2}{2^n} + \frac{\sigma^2}{2^n} + \frac{2\sigma q}{2^n} + \frac{0.5q^2}{2^n} \\ &\leq \frac{2\sigma^2}{2^n} + \frac{2\sigma q}{2^n} + \frac{0.5q^2}{2^n}. \end{aligned} \tag{5}$$

We emphasize that this bound is tight up to a constant factor due to the double collision attack as mentioned in Appendix A.1 for XMAC1 with padding scheme 2.