Reproducible Codes and Cryptographic Applications

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Abstract. In this paper we study structured linear block codes, starting from well known examples and generalizing them to a wide class of codes that we call *reproducible codes*. These codes have the property that can be entirely generated from a small number of signature vectors, and consequently admit matrices that can be described in a very compact way. We then show some cryptographic applications of this class of codes and explain why the general framework we introduce may pave the way for future developments of code-based cryptography based on structured codes.

Keywords: Linear block codes, code-based cryptography, post-quantum cryptography, reproducible codes

1 Introduction

Defining linear block codes that possess a certain inner structure and verify some regularity properties is a natural process in coding theory. Arguably, the most relevant example is represented by the class of *cyclic codes*, which includes several families of codes that proved to be important throughout the history of communications, such as BCH and Hamming codes, as well as the binary Golay codes, Reed-Solomon codes and many others. This class is defined by the property of having codewords that are invariant under the action of a specific permutation, namely the cyclic (circular) shift, i.e. the rotation of the vector to the right (equivalently, to the left). Other examples which are well-known in literature include *constacyclic* codes, *negacyclic* codes, *quasi-cyclic* codes etc.

Recently, this research direction has been investigated further: Misoczki and Barreto in 2009 introduced the family of *quasi-dyadic* codes [1], which contain codewords that are invariant under a different type of permutation. The work was motivated by its applications in the framework of the McEliece cryptosystem [2], and in particular by the necessity of having a family of codes which possess generator and parity-check matrices that can be represented in a compact way. Another family of codes that have been considered for application in this context is that of quasi-cyclic (QC) codes [3] and, more recently, of quasi-cyclic low-density parity-check (QC-LDPC) [4] and quasi-cyclic moderate-density parity-check (QC-MDPC) codes [5]. These codes have characteristic matrices formed

by circulant blocks, which can be described by one of their rows, thus yielding very compact representations.

All these efforts are motivated by the fact that, in code-based cryptography, the public key of an encryption (or signature) scheme usually consists precisely of a generator or parity-check matrix of a linear block code. With the size of the codes used in code-based cryptography (code length is in the order of 10^3 to 10^4), describing a whole matrix results in a public key of several kilobytes, and this size increases quadratically in the code length. This has historically prevented the use of the original McEliece cryptosystem [2], exploiting random-like public codes, in many applications. On the other hand, structured codes admit a generator and parity-check matrix which can be entirely described by one or few rows; this allows for a very important reduction in public-key size, and it is arguably a fundamental step towards making code-based cryptography truly practical.

The importance of code-based cryptography has risen dramatically in recent years due to the work of Peter Shor [6], which shows how it will be possible to effectively break cryptography based on "classical" number theory problems by introducing polynomial-time algorithms for factoring and computing discrete logarithms on a quantum computer. This means that the cryptographic community has to devise primitives which rely on different hard problems, which will not be affected once quantum computers of an appropriate size will be available. Code-based cryptography is one of the most important solutions in this sense, and ever since McEliece's seminal work in 1978, has shown no vulnerabilities against quantum attackers.

Our Contribution In this paper we analyze in details the construction of structured codes. We introduce the notion of *reproducible* codes, which captures the generic idea of a code which admits matrices that can be entirely described by a subset of rows. We show that all the existing constructions are in fact but a simple, special case of our general formulation. We then propose a framework for constructing reproducible codes of any kind, and present concrete instantiations of non-trivial reproducible codes which have not been proposed in literature before.

2 Preliminaries

We denote with $\mathbb{F}_q$ the finite field with q elements, where q is a prime power. For two sets X and Y we denote by X^Y the set of all functions from X to Y. For a set S we then denote by 2^S its power set, exploiting the well-known bijection between the power set of S and the set of functions from S to $\{0,1\}$.

2.1 Coding Theory

A linear code $\mathcal{C}$ is a k-dimensional subspace of the n-dimensional vector space over the finite field $\mathbb{F}_q$. The parameters n (*length*) and k (*dimension*) are positive integers with $k \leq n$. The value $r = n - k$ is known as *codimension* of the code.

Definition 1 (Hamming metric). *The Hamming weight* $\mathsf{wt}(x)$ *of a vector* $x \in \mathbb{F}_q^n$ *is the number of its non-zero entries. The Hamming distance* $\mathsf{d}(x, y)$ *between two vectors* $x, y \in \mathbb{F}_q^n$ *is defined as the weight of their difference, i.e.* $\mathsf{d}(x, y) = \mathsf{wt}(x - y)$*. The minimum distance* d *of a code* $\mathcal{C}$ *is defined as the minimum distance between any two different codewords of* $\mathcal{C}$*, or equivalently as the minimum weight over all non-zero codewords.*

A linear code of length n, dimension k, and minimum distance d is called an $[n, k, d]$-code.

The error-correcting capability of a linear code is connected to its minimum distance, and in particular it corresponds to $\lfloor (d-1)/2 \rfloor$.

Definition 2 (Generator and Parity Check Matrices). *Let* $\mathcal{C}$ *be a linear code over* $\mathbb{F}_q$*. We call* Generator Matrix *of* $\mathcal{C}$ *a matrix* G *whose rows form a basis for the vector space defined by* $\mathcal{C}$*, i.e.:*

$$\mathcal{C} = \{xG : x \in \mathbb{F}_q^k\}.$$

For any matrix H *and any vector* x*, the vector* Hx^T *is called* syndrome *of* x*. We then call* Parity-Check Matrix *of* $\mathcal{C}$ *a matrix* H *such that every codeword has syndrome 0 with respect to* H*, i.e.*

$$\mathcal{C} = \{x \in \mathbb{F}_q^n : Hx^T = 0\}.$$

Note that the parity-check matrix of a code $\mathcal{C}$ is also a generator matrix for the *dual* code $\mathcal{C}^\perp$, i.e. the linear code formed by all the words of $\mathbb{F}_q^n$ that are orthogonal to $\mathcal{C}$. It follows that for any generator matrix G and parity check matrix H of a code, we have $GH^T = 0$.

For an $[n, k, d]$-code, the generator matrix G has size $k \times n$, and the parity check matrix has size $(n-k) \times n = r \times n$. Both matrices always have to have full rank. Moreover, notice that, clearly, neither matrix is unique: for instance, given a generator matrix G it is always possible to obtain another generator matrix for the same code by a linear transformation, that is, the multiplication on the left by an invertible $k \times k$ matrix S, so that $G' = SG$. This corresponds simply to a change of basis for the vector space. A similar property is verified by the parity-check matrix. Finally, two generator matrices generate *equivalent codes* if one is obtained from the other by a permutation of columns. These two facts are at the basis of the McEliece cryptosystem.

Joining these two properties, we can write any generator matrix G in *systematic form* as $G = [I_k|A]$, where I_k is the identity matrix of size k and $|$ denotes

concatenation. If $\mathcal{C}$ is generated by $G = [I_k|A]$, then a parity check matrix for $\mathcal{C}$ is $H = [-A^T|I_{n-k}]$ (up to permutation, H can be transformed so that the identity submatrix is on the left hand side).

2.2 The McEliece Cryptosystem

The McEliece public-key encryption scheme [2] was introduced by R. J. McEliece in 1978. The original scheme uses binary Goppa codes, with which it remains unbroken (with a proper choice of parameters), but the scheme can be used with any class of codes for which an efficient decoding algorithm is known.

Key Generation Let G be a generator matrix for a linear $[n, k, d]$-code over $\mathbb{F}_q$ with an efficient decoding algorithm $\mathcal{D}$ which can correct up to $t = \lfloor (d-1)/2 \rfloor$ errors. Let S be an invertible $k \times k$ matrix and P be a random $n \times n$ permutation matrix over $\mathbb{F}_q$. The private key is (S, G, P) and the public key is $G' := SGP$.

Encryption To be able to encrypt a plaintext, it has to be represented as a vector m of length k over $\mathbb{F}_q$. The encryption algorithm chooses a random error vector e of weight t in $\mathbb{F}_q^n$, and computes the ciphertext $c = mG' + e$.

Decryption The decryption algorithm first computes $\hat{c} = cP^{-1} = mSG + eP^{-1}$. As P is a permutation matrix, eP^{-1} has the same weight as e. Therefore, $\mathcal{D}$ can be used to decode the errors: $\hat{m} = mS = \mathcal{D}(\hat{c})$. Finally, the plaintext is retrieved as $m = \hat{m}S^{-1}$.

In successive papers, the original McEliece cryptosystem was refined and tweaked many times; for example it is now common practice to replace the scrambling method given by S and P with the computation of the systematic form, i.e. G' is the systematic form of G. This is possible when the McEliece cryptosystem is embedded into a larger framework to convert it into an IND-CCA2 secure PKE or KEM, and has the additional advantage (beyond the obvious simper formulation) of a smaller public key (since only the non-identity submatrix needs to be stored).

The (one-way) security of McEliece is largely based on the following hard problem.

Problem 1 (Syndrome Decoding Problem). Given an $(n-k) \times n$ full-rank matrix H and a vector s, both with entries in $\mathbb{F}_q$, and a non-negative integer t; find a vector $e \in \mathbb{F}_q^n$ of weight t such that $He^T = s^T$.

The Syndrome Decoding Problem (SDP) is a well-known problem in complexity theory, and it has been shown to be NP complete [7]. Note that, since

the McEliece cryptosystem uses an $[n, k, d]$ code, the number of error vectors is $\binom{n}{t}(q-1)^t$, while the number of possible syndromes is q^r. Therefore, if

$$\binom{n}{t}(q-1)^t < q^r,$$

there is at most one solution to the problem, which guarantees the decoding process has a unique solution.

3 Sparse-Matrix Codes

One of the most delicate points about the McEliece cryptosystem is that, in order for the security to reduce to SDP, it is assumed that the matrix produced as the public key is indistinguishable from a uniformly random matrix of the same size. This is, as we just mentioned, an assumption, and while plausible, it has been shown to be false in several cases. For many variants of McEliece (e.g. [8], in fact, this opened up avenues of attack which simply ruled out the variant altogether. Even the long-standing binary Goppa codes have been shown to be distinguishable [9] when the code rate is chosen carelessly (too high). This is arguably one of the main reasons that pushed researchers away from algebraic codes, and towards codes of a different nature.

Low-Density Parity-Check (LDPC) codes are defined by matrices whose only requirement is to be very sparse, with a very low, constant row weight. These codes are easy to generate, and moreover admit a variety of choices for the decoding algorithm $\mathcal{D}$, like the Bit Flipping (BF) decoder of Gallager [10], which is very efficient in practice. For these reasons, this class of codes is a natural candidates for the McEliece cryptosystem. In such a framework, the secret code $\mathcal{C}$ is represented through its parity check matrix H; the public key corresponds to a generator matrix G for $\mathcal{C}$. It is important to note that, from the knowledge of G, the opponent can compute several parity-check matrices H' for $\mathcal{C}$, but they will not lead to an efficient decoding, unless they are sparse. As explained in section 2.2, typically having G in systematic form is enough to guarantee such property. Indeed, we can always write $H = [H_0|H_1]$, where H_0 and H_1 have dimensions, respectively, $r \times k$ and $r \times r$. Then, the corresponding generator matrix in systematic form is obtained as $G = [I_k|H_0^T H_1^{-T}]$. Typically (unless for specific choices of H) the inverse of a sparse matrix is dense, and so H_1^{-T} is dense: in such a case, the multiplication of H_0^T by H_1^{-T} is enough to hide the structure of H into the one of G.

It is important to note that, due to their probabilistic nature, decoding algorithms for LDPC codes are characterized by a non-trivial decoding failure rate (DFR). This means that, in the case of a decoding failure, Bob must ask Alice for a retransmission of the plaintext, encrypted with a different error vector. In order to avoid frequent retransmissions, which would obviously increase the latency of the system, the DFR must be kept sufficiently low; typically, values

are in the range of 10^{-6} to 10^{-9}. As we will discuss later, this fact represents a crucial difference, with respect to the case of algebraic codes, since it leads to a new family of attacks, aimed at recovering the secret key.

3.1 Security

The advantage of using LDPC codes is that the indistinguishability issue boils down to recovering low-weight words, and specifically low-weight codewords in the dual code, which is again a decoding problem. In particular, let us denote by $\mathcal{C}^\perp$ the dual code of $\mathcal{C}$, which admits H as generator matrix. Since the rows of H are sparse, with maximum weight $w \ll n$, with overwhelming probability they represent minimum-weight codewords in $\mathcal{C}^\perp$, and so can be searched with a generic algorithm for finding low-weight words.

At the current state of the art, the best procedure for this task is the information set decoding (ISD) algorithm, which was first introduced by Prange in 1962 [11], and has received many improvements during the years [12–15]. However, ISD and all its variants are characterized by an exponential complexity: the search for a weight-w codeword has asymptotic complexity equal to $2^{\alpha w}$, where the value of the constant α depends on the particular algorithm we are analyzing. Even in a quantum setting, ISD algorithms are still characterized by exponential complexity: indeed, the only known application of a quantum algorithm to an ISD algorithm, which consists in using Grover's algorithm [16] to speed up the search, leads to a reduction in the complexity, with respect to the classical case, which cannot be larger than the square root of the exponent α [17]. This means that the adoption of Low-density parity-check (LDPC) and moderate-density parity-check (MDPC) codes does not reduce the security of the McEliece cryptosystem, since attacks deriving from the structure of the secret code can be easily avoided by fixing the minimum weight of the rows of H.

Since the main threat to the use of LDPC codes is represented by a search for low-weight words, it makes sense to relax the notion of "Low-Density": the authors in [18] introduce the notion of "Moderate-Density" by increasing the allowed row weight in the parity-check matrix from $O(1)$ to $O(\sqrt{n})$. It is still possible to decode such codes (called MDPC by analogy) with the previously-mentioned algorithms; the error-correction capacity gets obviously worse, but the gain in security makes this tradeoff worth it.

3.2 Structures Codes

Using generic LDPC and MDPC is not a practical choice, since the resulting public-key sizes are significantly larger than the ones we can obtain with other families of codes, like Goppa codes. Indeed, even if the secret parity-check matrix can be compactly represented just by storing the positions of the ones (and so, each row can be stored just with $w \log_2 n$ bits), applying this technique to the

public key is not possible, since a sparse G might compromise the security of the system. One way to avoid this issue is to add some structure to the code family. This idea was first introduced in the context of algebraic codes [3, 19], and was therefore extended to sparse codes [5,20]. In all cases, the authors propose to use QC codes to reduce key size. A QC code is simply a code which admits parity-check and generator matrices made of *circulant* blocks. A circulant matrix is a matrix in which every row is obtained as the cyclic shift of the previous one; an example of a circulant matrix is depicted below.

$$\begin{bmatrix} a_0 & a_1 & \dots & a_{p-1} \\ a_1 & a_2 & \dots & a_0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{p-1} & a_0 & \dots & a_{p-2} \end{bmatrix}$$

This means that, in the McEliece cryptosystem, we can describe the public key completely using just the first row of each circulant block; it is clear that this results in a significant reduction in the public key size. However, this additional structure presents some drawbacks, since it exposes the system to further weaknesses. In particular, the QC structure summed to the algebraic structure of the underlying codes, provides a lot of information to the attacker, and opens up the possibility of structural attacks aimed at recovering the private code. The most famous structural attack of this type is known as FOPT [21], and works by solving a multivariate algebraic system with Gröbner bases techniques together with the QC property which greatly reduces the number of unknowns of the system. As a result, it seems very hard to provide secure schemes which involve QC algebraic codes (Goppa, GRS etc.), while still obtaining an effective key reduction: the recent NIST proposal BIG QUAKE [**?**] shows a reduction of about $1/4$ of the key size which would be obtained in a "classical" McEliece using unstructured binary Goppa codes.

Therefore, once again, it seems safer to deploy code-based schemes using sparse codes, since in this case there is no additional algebraic structure, and the QC property alone is not enough to provide a structural attack. However, some care is still necessary when using such systems. In particular, we can consider two main aspects which need to be addressed:

– ISD algorithms might exploit the QC structure, and thus obtain a speed up; this technique, consisting of analyzing multiple samples at once, is known as Decoding One Out of Many (DOOM) [22], and results in a reduction in the complexity of these attacks. Note that this speedup is valid not only for attacks aiming at recovering the secret key, but also for attacks aiming at decoding intercepted ciphertexts (one-way security). While the speedup is not dramatic (it is in the order of the circulant size), it still has an important impact, since it leads to an increase in the row weight of H and in the number of errors applied during encryption, which in turn results in an increase

in the key length.

– It has been recently shown that the probability of a decoding failure depends on the number of overlapping ones between the error vector and rows of H. In addition, in a circulant matrix, all the rows are characterized by the same set of cyclic distances between set symbols (given two ones at positions i and j, the corresponding cyclic distance is computed as $\min\{\pm(i-j) \mod p\}$, with p being the circulant size). As shown in [23], an adversary can mount a Chosen Ciphertext Attack (CCA) attack, in which he impersonates an honest user, producing ciphertexts and requesting their decryption. The adversary can then exploit the events of decoding failures, which are of public knowledge, in order to gather information about distances among ones. The set of all such distances is called *distance spectrum*, and can be used to reconstruct the secret parity-check matrix. This problem can be related to a graph problem, in which a row of H corresponds to a clique with maximum size. For a sparse QC matrix, such graph is sparse as well, and so it is typically characterized by a small number of cliques. This means that, once the distance spectrum is known, recovering the corresponding parity-check matrix is, in most of the cases, not a hard task.

Among the cited attacks, reaction attacks represent the main threat, since they can be fully avoided only with significant trade-offs. In fact, currently existing solutions are based on the use of ephemeral keys [24, 25], or on the use of particular families of codes which make the reconstruction of the secret key unfeasible [26]. However, as we mentioned, both these solutions come with a significant price to pay, since we must generate a new key-pair for each encryption (in the first case) or increase the size of the public key. Another solution could be that of reducing the DFR to a negligible value, in order to increase the number of ciphertexts that the opponent must produce to recover the secret distance spectrum [27].

As we will see in the rest of this paper, the idea of using some structure to reduce the public key size can be strongly generalized. In particular, we will show that existing solutions are just very special cases of a wider framework, characterized by several different aspects. This generalization comes with no increase in the public key, while it might allow avoiding the attack of DOOM and/or reaction attacks, or at the very least reduce their efficiency.

4 Reproducible and Quasi-Reproducible Codes

We are now ready to introduce the fundamental notions of this paper.

Definition 3. *Consider a matrix $M \in \mathbb{F}_q^{k\times n}$. Let $\mathcal{R}$ be the set of the rows of M and let $2^{\mathcal{R}}$ be its power set. We say that M is* reproducible *if M can be entirely described as $\mathcal{F}(V)$, where $\mathcal{F}$ is a family of linear transformations from $\mathbb{F}_q^n$ to $\mathbb{F}_q^n$ and V is an element of $2^{\mathcal{R}}$ called the* signature set.

Definition 4. *Let $\mathcal{C}$ be a linear code over $\mathbb{F}_q$, described by a generator matrix $G \in \mathbb{F}_q^{k\times n}$; if G is reproducible, then we say that $\mathcal{C}$ is in* reproducible form.

Basically, a reproducible matrix is described just by its signature set and by the corresponding family of linear functions. Consequently, the reproducible form of the generator matrix leads to a compact representation of the code. Actually, the condition on the reproducibility of a matrix can be relaxed, in order to take into account also other structures that allow a compact representation.

Definition 5. *Let us consider some reproducible matrices $G_{i,j} \in \mathbb{F}_q^{k_{i,j}\times n_{i,j}}$, each one of them with dimensions $k_{i,j} \times n_{i,j}$, signature set $V_{i,j} \in \mathbb{F}_q^{m_{i,j}\times n_{i,j}}$ and family of linear functions $\mathcal{F}_{i,j}$. Let G be a matrix obtained using as building blocks the matrices $G_{i,j}$; then, we say that G is* quasi-reproducible. *If G is the generator matrix of a code $\mathcal{C}$, then we say that $\mathcal{C}$ is in* quasi-reproducible form.

It is clear that, in order to describe a quasi-reproducible matrix, we just need the ensemble of the signature sets of its building blocks, together with the corresponding families of linear functions. One common case of quasi-reproducible codes is the one in which the blocks $G_{i,j}$ are square matrices, are defined by the same family $\mathcal{F}$ and form a group.

Definition 6. *Let us consider a family of linear functions $\mathcal{F} = \left\{\sigma_0, \sigma_1, \cdots, \sigma_{\frac{p}{m}-1}\right\}$, where each σ_i is a $p \times p$ matrix. We denote as $\mathcal{M}_{\mathcal{F},m}$ the ensemble of all reproducible matrices with signature of dimensions $m \times p$ and family of linear transformations $\mathcal{F}$. Then, if $\mathcal{M}_{\mathcal{F},m}$ is a group, which means*

$$A + B, AB \in \mathcal{M}_{\mathcal{F},m}, \forall \;\; A, B \in \mathcal{M}_{\mathcal{F},m},$$

we say that $\mathcal{F}$ induces a reproducible group *over $\mathbb{F}_q$.*

4.1 Families of transformations inducing reproducible groups

In this section we derive the properties that a family of transformations $\mathcal{F}$ must have, in order to let it induce a reproducible group over $\mathbb{F}_q$.

Theorem 1. *Let $\mathcal{F} = \left\{\sigma_0 = id, \sigma_1, \cdots, \sigma_{\frac{p}{m}-1}\right\}$ be a family of linear transformations inducing a reproducible group $\mathcal{M}_{\mathcal{F},m}$ over $\mathbb{F}_q$. Then, for every matrix $B \in \mathcal{M}_{\mathcal{F},m}$, it must be*

$$\sigma_i B = B\sigma_i, \;\; \forall i \in \mathbb{N}, 0 \leq i \leq \frac{p}{m} - 1.$$

Proof. Let A and B be two matrices belonging to $\mathcal{M}_{\mathcal{F},m}$, with respective signatures a_0, b_0, that is

$$A = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{\frac{p}{m}-1} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_0\sigma_1 \\ \vdots \\ a_0\sigma_{\frac{p}{m}-1} \end{bmatrix}, \;\; B = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{\frac{p}{m}-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_0\sigma_1 \\ \vdots \\ b_0\sigma_{\frac{p}{m}-1} \end{bmatrix}.$$

It is straightforward to show that $C = A + B$ is again a reproducible matrix, defined by the family $\mathcal{F}$ and signature $a_0 + b_0$.

For the product $C = AB$, we have

$$C = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{\frac{p}{m}-1} \end{bmatrix} = AB = \begin{bmatrix} a_0 B \\ a_1 B \\ \vdots \\ a_{\frac{p}{m}-1} B \end{bmatrix} = \begin{bmatrix} a_0 B \\ a_0 \sigma_1 B \\ \vdots \\ a_0 \sigma_{\frac{p}{m}-1} B \end{bmatrix}. \tag{1}$$

Since we want C to be reproducible and defined by $\mathcal{F}$, its signature must be $c_0 = a_0 B$, and

$$a_0 \sigma_i B = c_i = c_0 \sigma_i = a_0 B \sigma_i, \tag{2}$$

for each integer $i \leq \frac{p}{m} - 1$. Since eq. (2) must be satisfied for every $a_0 \in \mathbb{F}_q^{m \times p}$, then every transformation σ_i must commute with every matrix $B \in \mathcal{M}_{\mathcal{F},m}$, that is $\sigma_i B = B \sigma_i$. □

In the particular case of the functions σ_i being permutations, the previous theorem leads to a further result, which is described in theorem 2. Since a permutation is a matrix in which every row and column has weight equal to 1, it can equivalently described as a bijection over $[0; \frac{p}{m} - 1] \subset \mathbb{N}$. With some abuse of notation, we say that $\sigma_i(v) = z$ if the permutation σ_i is such that it places the v-th element in position z. Then, $\sigma_i(v) = z$ means that the element in position (v, z) in σ_i is equal to 1. In addition, we use $\sigma_i \circ \sigma_j$ to denote the bijection defined by the application of σ_i after σ_j. The actual meaning of σ_i (permutation or bijection) should be clear from the context.

Theorem 2. *Let $\mathcal{F} = \left\{\sigma_0 = id, \sigma_1, \cdots, \sigma_{\frac{p}{m}-1}\right\}$ be a family of linear transformations, with each σ_i being a permutation, and let us suppose that $\mathcal{F}$ induces a reproducible group $\mathcal{M}_{\mathcal{F},m}$ over $\mathbb{F}_q$. Then, the following relation must be satisfied*

$$\sigma_i \sigma_j = \sigma_{\sigma_j(i)}, \quad \forall i, j \in \mathbb{N}, \quad 0 \leq i \leq \frac{p}{m} - 1, \quad 0 \leq j \leq \frac{p}{m} - 1.$$

Proof. Because of theorem 1, for every matrix $B \in \mathcal{M}_{\mathcal{F},m}$ and every function σ_i it must be $\sigma_i B = B \sigma_i$. In particular, the left-side multiplication of B by σ_i corresponds to a row-permutation, such that

$$\sigma_i B = \begin{bmatrix} b_{\sigma_i(0)} \\ b_{\sigma_i(1)} \\ \vdots \\ b_{\sigma_i(\frac{p}{m}-1)} = \end{bmatrix} = \begin{bmatrix} b_0 \sigma_{\sigma_i(0)} \\ b_0 \sigma_{\sigma_i(1)} \\ \vdots \\ b_0 \sigma_{\sigma_i(\frac{p}{m}-1)} \end{bmatrix}. \tag{3}$$

The product $B\sigma_i$ defines, instead, a column permutation of the elements in B, and can be expressed as

$$B\sigma_i = \begin{bmatrix} b_0 \sigma_0 \\ b_0 \sigma_1 \\ \vdots \\ b_0 \sigma_{\frac{p}{m}-1} \end{bmatrix} \sigma_i = \begin{bmatrix} b_0 \sigma_0 \sigma_i \\ b_0 \sigma_1 \sigma_i \\ \vdots \\ b_0 \sigma_{\frac{p}{m}-1} \sigma_i \end{bmatrix}. \tag{4}$$

Then, we can put together eqs. (3) and (4), and thus obtain

$$\sigma_i \sigma_j = \sigma_{\sigma_j(i)}, \tag{5}$$

which must be satisfied for every pair of indexes (i, j). □

From theorem 2 we can derive some other properties about that $\mathcal{F}$ must satisfy.

Corollary 1. *Let $\mathcal{F}$ be a family of permutations satisfying theorem 2. Then, $\mathcal{F}$ has the following properties*

(a) $\sigma_i(0) = i, \ \forall i$;
(b) $\forall i \ \exists j \ \ s.t. \ \sigma_i \circ \sigma_j = id$.

Proof. From theorem 2, we have

$$\sigma_j = \sigma_0 \sigma_j = \sigma_{\sigma_j(0)}, \tag{6}$$

from which it is clear that it must be $\sigma_j(0) = j$, and this proves property (a). Since each σ_i is a bijection of the integers in $[0; \frac{p}{m} - 1]$, we know that

$$\forall j \ \ \exists i \ \text{ s.t. } \ \sigma_j(i) = 0. \tag{7}$$

Since we have defined $\sigma_0 = id$, then

$$\forall j \ \ \exists i \ \text{ s.t. } \ \sigma_j \sigma_i = id, \tag{8}$$

which proves property (b). □

4.2 Some examples of reproducible groups

In the previous section we have derived some properties that a family of linear transformations $\mathcal{F}$ must satisfy, in order to guarantee that it induces a reproducible group over $\mathbb{F}_q$. Well known cases of such groups, with common use in cryptography, are the ones of circulant matrices and dyadic matrices. It is quite easy to show that these families of matrices are compliant with theorem 2 and corollary 1.

Circulant Matrices As we have seen before, a circulant matrix is a $p \times p$ matrix for which each row is obtained as the cyclic shift of the previous one. In particular, a circulant matrix can be seen as a square reproducible matrix, whose signature corresponds to the first row and the functions σ_i defining $\mathcal{F}$ correspond to π^i, where the elements of π are defined as

$$\pi_{l,j} = \begin{cases} 1 & \text{if } l + 1 \equiv j \mod p \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

Basically, the bijection representing π is defined as

$$\pi(v) = v + 1 \mod p. \tag{10}$$

It can be easily shown that

$$\sigma_i(v) = \pi^i(v) = \underbrace{\pi \circ \pi \cdots \circ \pi}_{i \text{ times}}(v) = v + i \mod p, \tag{11}$$

which leads to $\pi^p = I_p$ and $\pi^i \pi^j = \pi^{i+j \mod p}$. Since permutation matrices are orthogonal, their inverse correspond to their transpose, and thus we have $(\pi^i)^T = \pi^{p-i}$. With these properties, we have

$$\begin{aligned} \sigma_i \circ \sigma_j(v) &= (v + j) + i \mod p = \\ &= v + (i + j) \mod p = \\ &= \sigma_{i+j \mod p}(v) = \\ &= \sigma_{\sigma_j(i)}(v), \end{aligned} \tag{12}$$

which corresponds to the thesis of theorem 2.

Dyadic Matrices A *dyadic* matrix is a $p \times p$ matrix, with p being a power of 2, whose signature corresponds to its first row. The rows of a dyadic matrix are obtained by permuting the elements of the signature, such that the element in position (i, j) is the one in the signature in position $i \oplus j$, where $\oplus$ denotes the bitwise XOR between i and j. Then, a dyadic can be described in terms of reproducible matrices, for which each function σ_i is the dyadic matrix whose signature have all null entries, except for the one in position i. This means that σ_i can be described as

$$\sigma_i(v) = v \oplus i \mod p. \tag{13}$$

If we combine two transformations, we obtain

$$\begin{aligned} \sigma_i \circ \sigma_j(v) &= (v \oplus j) \oplus i = \\ &= v \oplus (i \oplus j) = \\ &= \sigma_{i \oplus j}(v) = \\ &= \sigma_{\sigma_j(i)}(v), \end{aligned}$$

which again is compliant with the thesis of theorem 2. It straightforwardly follows that $\sigma_i \circ \sigma_i = id$.

Circulant and dyadic matrices are just two particular cases of reproducible groups, and can obviously be further generalized by considering signatures that are composed by more than one row. In addition, several more constructions can be obtained. For example, for every orthogonal matrix ψ (i.e., $\psi\psi^T = I_p$) and every reproducible group $\mathcal{M}_{\mathcal{F},m}$, we can obtain a new group as $\left\{ \psi M \psi^T \middle| M \in \mathcal{M}_{\mathcal{F},m} \right\}$.

Indeed, for any two matrices $A = \psi M_A \psi^T$ and $B = \psi M_B \psi^T$, with $M_A, M_B \in \mathcal{M}_{\mathcal{F},m}$, we have

$$A + B = \psi M_A \psi^T + \psi M_B \psi^T = \psi(M_A + M_B)\psi^T, \tag{14}$$

$$AB = \psi M_A \psi^T \psi M_B \psi^T = \psi M_A M_B \psi^T, \tag{15}$$

which return matrices belonging to the defined group, since $M_A + M_B, M_A M_B \in \mathcal{M}_{\mathcal{F},m}$.

The two examples we have just presented illustrate how the structures that have been proposed for the McEliece cryptosystem (i.e, circulant and dyadics) are just special cases of a wider framework. As we show in the next section, such framework can be further generalized.

5 Codes in reproducible form

In the previous section we have described the properties that a family of functions $\mathcal{F}$ must have, in order to let it induce a reproducible group over $\mathbb{F}_q$. Our analysis has shown that there is a wide range of possibilities for obtaining a code with a compact representation. Basically, it is enough to choose a family of functions $\mathcal{F}$ that induces a reproducible group, and use elements from such group to construct a parity-check matrix. Then, a quasi-reproducible generator matrix can be easily obtained (e.g., by using the systematic form); because of the group structure, the so obtained matrix is completely described by the signatures of the blocks. It is clear that the use of reproducible groups allows obtaining codes that can be efficiently described. We remember that the use of such codes has a crucial meaning for the McEliece cryptosystem, in which we are interested in obtaining public keys that can be compactly represented. In this section we describe how this whole construction can be further generalized. Indeed, we can obtain codes that are described by a generator matrix that is not made of reproducible square blocks. This fact offers a new wide range of possible constructions. Basically, what we do in this section is removing the condition on reproducible groups and just consider the case of codes that can be described by a reproducible generator matrix. In addition, we provide a simple method that allows obtaining random codes in reproducible form, starting from their parity-check matrix.

The following theorem states some properties about the parity-check matrix that are sufficient conditions (but not necessary) for having a code in reproducible form.

Theorem 3. *Let $\mathcal{C}$ over $\mathbb{F}_q$ be a code with length n, dimension k and codimension r. Let $m \in \mathbb{N}$ be a factor of k, and consider a family of linear transformations $\mathcal{F} = \left\{\sigma_0, \sigma_1, \cdots, \sigma_{\frac{k}{m}-1}\right\}$, with $\sigma_0 = I_n$. Let $H \in \mathbb{F}_q^{r \times n}$ be a parity-check matrix for $\mathcal{C}$, and $s \in \mathbb{N}$ be a factor of r. Let h_i denote the subset of rows of H in positions $\{is, is+1, \cdots, (i+1)s-1\}$. Let $g_0 \in \mathbb{F}_q^{m \times n}$ be a matrix such that $g_0 H^T = 0_{m \times r}$. If we can define a function $f : [0; \frac{k}{m} - 1] \times [0; \frac{r}{s} - 1] \subset \mathbb{N}^2 \to [0; \frac{r}{s} - 1] \subset \mathbb{N}$ with the following properties:*

(a) $h_j \sigma_i^T = h_{f(i,j)}$;
(b) *for any three integers* $i \in [0; \frac{k}{m} - 1]$ *and* $j_0, j_1 \in [0; \frac{r}{s} - 1]$ *it must be* $f(i, j_0) \neq f(i, j_1)$;
(c) *for any three integers* $i_0, i_1 \in [0; \frac{k}{m} - 1]$ *and* $j \in [0; \frac{r}{s} - 1]$ *it must be* $f(i_0, j) \neq f(i_1, j)$;

then $\mathcal{C}$ *admits a generator matrix in reproducible form which is defined by the family* $\mathcal{F}$ *and by the signature* g_0.

Proof. Since the generator matrix G is reproducible, with signature g_0, we have

$$G = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{\frac{k}{m}-1} \end{bmatrix} = \begin{bmatrix} g_0 \\ g_0\sigma_1 \\ \vdots \\ g_0\sigma_{\frac{k}{m}-1} \end{bmatrix}, \tag{16}$$

while for the parity-check matrix H we can write

$$H = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{\frac{r}{s}-1} \end{bmatrix}. \tag{17}$$

Because $GH^T = 0_{k \times r}$, it must be

$$g_i h_j^T = g_0 \sigma_i h_j^T = 0_{m \times s}, \quad \forall i, j \in \mathbb{N} \text{ s.t. } 0 \leq i \leq \frac{k}{m} - 1, \quad 0 \leq j \leq \frac{r}{s} - 1. \tag{18}$$

From the hypothesis, we dispose of an $m \times n$ matrix g_0 such that $g_0 H^T = 0_{m \times r}$, which means

$$g_0 h_j^T = 0_{m \times s}, \quad \forall j \in \mathbb{N} \text{ s.t. } 0 \leq j \leq \frac{r}{s} - 1. \tag{19}$$

Let us now consider the product $g_i h_j^T = g_0 \sigma_i h_j^T$, for $i \geq 1$. If we can define a function $f : [0; \frac{k}{m} - 1] \times [0; \frac{r}{s} - 1] \subset \mathbb{N}^2 \to [0; \frac{r}{s} - 1] \subset \mathbb{N}$, such that

$$\sigma_i h_j^T = h_{f(i,j)}^T, \tag{20}$$

then eq. (18) is surely satisfied, since

$$g_i h_j^T = g_0 \sigma_i h_j^T = g_0 h_{f(i,j)}^T = 0_{m \times s}, \tag{21}$$

where $g_0 h_{f(i,j)}^T$ because of (19). In particular, by transposing both sides of eq. (20), we obtain $h_j \sigma_i^T = h_{f(i,j)}$, which proves property *a*).

For the other two properties, we must consider that we want H and G to have rank, respectively, equal to r and k. One necessary condition for having H with full rank (but obviously not a sufficient one) is that, for a fixed i, the function $f(i, j)$ spans over all the integers in $[0; \frac{r}{s} - 1]$ for different input values

j. Indeed, if for some integers i and $j_0 \neq j_1$ we have $f(i, j_0) = f(i, j_1)$, then this means that $h_{j_0}\sigma_i^T = h_{j_1}\sigma_i^T$, which implies $h_{j_0} = h_{j_1}$. In analogous way, there cannot exist three integers $i_0 \neq i_1$ and j such that $h_j\sigma_{i_0}^T = h_j\sigma_{i_1}^T$, otherwise it must be $\sigma_{i_0} = \sigma_{i_1}$, which results in G having some identical rows. □

Theorem 3 allows obtaining a code in reproducible form in a very simple way. Suppose that, given a family of transformations $\mathcal{F}$, we have found a matrix H, with the characteristics required by theorem 3. Then, for the code $\mathcal{C}$ having H as parity-check matrix we can obtain a variety of reproducible generator matrices. Indeed, let G be a generator matrix for $\mathcal{C}$: by definition, since $GH^T = 0_{k\times r}$, we know that whichever subset g_0 formed by m rows of G is such that $g_0H^T = 0_{m\times r}$. Then, g_0 is a valid signature for our reproducible generator matrix, defined by the family $\mathcal{F}$.

5.1 The relation between reproducible and quasi-reproducible codes

In some cases, a quasi-reproducible code can be seen as a particular case of a reproducible code (and viceversa). Indeed, let us consider a code $\mathcal{C}$ with length $n = n_0p$, dimension $k = p$ and codimension $r = (n_0 - 1)p$, for some integer $n_0 \in \mathbb{N}$. We suppose that $\mathcal{C}$ is described by a generator matrix in quasi-reproducible form: in particular, we suppose that G is obtained as the concatenation of n_0 blocks with dimensions $p \times p$, that is

$$G = [G_0|G_1|\cdots|G_{n_0-1}], \tag{22}$$

where each G_i is an element of the reproducible group $\mathcal{M}_{\mathcal{F}_i,m_i}$, and has signature V_i.

If the signatures have all the same number of rows (that is, $m_i = m$), then such a G can be characterized as a particular reproducible matrix. Indeed, let us denote as $\mathcal{F}_i = \left\{\sigma_0^{(i)}, \sigma_1^{(i)}, \cdots, \sigma_{\frac{p}{m}-1}^{(i)}\right\}$ the i-th family of transformations. It is quite easy to see that a matrix as in (22) can be described as a reproducible matrix, with signature

$$g_0 = \left[g_0^{(0)}\middle|\, g_0^{(1)}\middle|\cdots\middle|g_0^{(n_0-1)}\middle|\right], \tag{23}$$

and described by a unique family of transformations $\mathcal{F} = \left\{\sigma_0, \sigma_1, \cdots, \sigma_{\frac{p}{m}-1}\right\}$, such that

$$\sigma_i = \begin{bmatrix} \sigma_i^{(0)} & 0_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} \\ 0_{p\times p} & \sigma_i^{(1)} & 0_{p\times p} & \cdots & 0_{p\times p} \\ 0_{p\times p} & 0_{p\times p} & \sigma_i^{(2)} & \cdots & 0_{p\times p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{p\times p} & 0_{p\times p} & 0_{p\times p} & \cdots & \sigma_i^{(n_0-1)}. \end{bmatrix} \tag{24}$$

5.2 An efficient technique for obtaining reproducible codes

In this section we present a particular case of codes satisfying theorem 3, and provide an efficient and simple technique to design such codes. Let us here consider a matrix π, such that $\pi^{\frac{r}{s}} = I_n$, and consider the family of transformations $\mathcal{F} = \left\{\sigma_0, \sigma_1, \cdots, \sigma_{\frac{k}{m}-1}\right\}$, with $\sigma_i = \pi^i$. In particular, a matrix H satisfying theorem 3 can be easily obtained just by picking an $s \times n$ matrix h_0 and using it to generate a reproducible matrix of size $r \times n$, with the family of functions $\left\{I_n, (\pi^{\frac{r}{s}-1})^T, (\pi^{\frac{r}{s}-2})^T, \cdots, (\pi^2)^T, \pi^T\right\}$. In other words, we have

$$H = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{\frac{r}{s}-1} \end{bmatrix} = \begin{bmatrix} h_0 \\ h_0(\pi^{\frac{r}{s}-1})^T \\ h_0(\pi^{\frac{r}{s}-2})^T \\ \vdots \\ h_0(\pi^1)^T \end{bmatrix}. \tag{25}$$

It is quite easy to show that such a parity check matrix is compliant with property (a) from theorem 3. Indeed, we have

$$\begin{aligned} h_j \sigma_i^T &= h_0(\pi^{\frac{r}{s}-j})^T(\pi^i)^T = \\ &= h_0\left(\pi^{\frac{r}{s}+i-j}\right)^T, \end{aligned} \tag{26}$$

such that

$$h_0\left(\pi^{\frac{r}{s}+i-j}\right)^T = h_0\left[\pi^{(i-j) \mod \frac{r}{s}}\right]^T = h_{(i-j) \mod \frac{r}{s}}. \tag{27}$$

In such a case the function $f(i,j)$ required by theorem 3 is defined as

$$f(i,j) = i - j \mod \frac{r}{s}. \tag{28}$$

For what concerns property (b), we can consider the following equivalence

$$i - j_0 \equiv i - j_1 \mod \frac{r}{s}, \tag{29}$$

which turns into

$$j_1 - j_0 \equiv 0 \mod \frac{r}{s}. \tag{30}$$

Then, it is clear that it must be $j_0, j_1 < \frac{r}{s}$: however, this condition is quite straightforward, since j denotes the row index of the matrix blocks in H. In the same way, when considering the index of the transformation σ_i, we have

$$i_0 - j \equiv i_1 - j \mod \frac{r}{s}, \tag{31}$$

which turns into

$$i_0 - i_1 \equiv 0 \mod \frac{r}{s}. \tag{32}$$

Again, in order to guarantee that the previous equivalence has no solution, it must be $i_0, i_1 < \frac{r}{s}$. This basically means that we must have $k \leq m\frac{r}{s}$.

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