Non-malleable Digital Lockers for Efficiently Sampleable Distributions

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Abstract

An obfuscated program reveals nothing about its design other than its input/output behavior. A digital locker is an obfuscated program that outputs a stored cryptographic key if and only if a user enters a previously stored password. A digital locker is private if it provides an adversary with no information with high probability. An ideal digital locker would also prevent an adversary from mauling an obfuscation on one password and key into a new program that obfuscates a related password or key. There are no known constructions of non-malleable digital lockers (in the standard model).

Komargodski and Yogev (Eurocrypt, 2018) constructed a simpler primitive: a non-malleable keyless digital locker. For this functionality, a user can only confirm if their point is correct. This primitive is known as non-malleable point obfuscation. Their construction prevents an adversary from transforming an obfuscation into an obfuscation on a related password.

This work first describes two nonmalleable digital lockers for short keys, one for a single bit key and a second for a logarithmic length keys. These constructs can be safely composed with the same input password. We then show how to extend to an arbitrarily polynomial length key and provide non-malleability over the stored password and key. Our full design combines a digital locker for short keys, non-malleable codes, and seed-dependent condensers. Seed-dependent condensers inherently require the distribution of passwords to be efficient sampleable.

Keywords: Digital Lockers; Point obfuscation; Virtual black-box obfuscation; Non-malleable codes; Seed-dependent condensers

1 Introduction

Obfuscation hides the implementation of a program from all users of the program. This work is concerned with *virtual black-box obfuscation*, where an obfuscator creates a program that reveals nothing about the program other than its input and output behavior [BGI+01, BGI+12]. (We do not consider indistinguishability obfuscation in this work [GGH+13, GGH+16, SW14, PST14, GLSW15, AJ15] [BR17]). Barak et al. showed that a virtual black-box obfuscator cannot exist for all polynomial time circuits [BGI+01]. However, this leaves open the possibility of virtual black-box obfuscators for interesting classes of programs [CD08, BC10] [CRV10, WZ17, BR17].

Our focus is on *digital lockers*. A digital locker obfuscator inputs a value, val, and key, key. The output is a program $\mathsf{unlock}_{\mathsf{val},\mathsf{key}}(\cdot)$ which outputs key if and only if the input is val. Privacy says $\mathsf{unlock}_{\mathsf{val},\mathsf{key}}$ should reveal nothing about val or key if the adversary cannot guess val. A digital locker is also known as

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a multi-bit point obfuscation [CD08, BC10]. Digital lockers have applications in password [Can97] and biometric authentication [CFP+16].

A simpler object to construct is a *point function* $\mathsf{unlockPoint}_{\mathsf{val}}$ which stores val, outputting 1 if and only if the input is val. An obfuscated point function only needs to hide val [Can97]. It is possible to compose point functions to build a digital locker (if the point function retains security when composed) [CD08]. The construction is straightforward, for each bit of the key, either a random point or val is obfuscated producing $\mathsf{unlockPoint}_{y_i}$. When running the program, the user tries to open each $\mathsf{unlockPoint}_{y_i}$, the point functions that output 1 represent 1s in key. The point function functions that output 0 correspond to a 0 in key. We call this digital locker the *real-or-random* construction.

Nonmalleability A desirable property of an obfuscated program is non-malleability. A *non-malleable* obfuscator prevents any tampering of the obfuscation (for some family of functions $f \in \mathcal{F}$) to obtain a related obfuscation [CV09]. For example, it is desirable to prevent $\mathsf{unlockPoint}_{\mathsf{val}}$ from being mauled to $\mathsf{unlockPoint}_{f(\mathsf{val})}$. In the random oracle model, designing non-malleable digital lockers and point functions is easy: for random oracle h one outputs the program $h(\mathsf{val}) \oplus (\mathsf{key}||h(\mathsf{key}))$, where $h(\mathsf{key})$ is truncated. Furthermore, in the common reference string model, one can achieve nonmalleability using appropriate zero-knowledge proofs of knowledge [BCFW09].

Recently, Komargodski and Yogev showed how to build a non-malleable obfuscated point function in the standard model [KY18]. Their construction is as follows, let g be a fixed group generator, to obfuscate the point val, the obfuscator computes a random r and outputs $\mathcal{O}(x) = (r, r^{g^{h(\mathsf{val})}})$. Here $h(x) = x^4 + x^3 + x^2 + x$. As we explain below, the function h is designed specifically to prevent mauling.

Security of the construction relies on two variants of the Decisional Diffie-Hellman (DDH) assumption [DH76] - the strong DDH assumption (discussed in [BC10]) and the power DDH assumption (introduced in [GJM02]). For an obfuscated point val, the construction prevents the adversary from changing val to $f(\mathsf{val})$ for a polynomial f (of degree at most t).

In this work, we construct the first non-malleable digital lockers in the standard model. One could try and compose Komargodski and Yogev's construction for each bit of the key using the *real-or-random* construction. However, that approach does not ensure nonmalleability over the bits of key. It is easy to permute bits of key by reordering group elements and to set individual bits of key to 0 by replacing a group element by a random group element. Consider a case when a user encrypts their files with key. If the adversary can create $\mathsf{lock}_{\mathsf{val},\mathsf{key}'}(\cdot)$, the user may use some cryptographic key that is known to the adversary or susceptible to cryptanalytic attack [BCM11].

1.1 Our Contribution

We present the first construction of a non-malleable digital locker in the standard model. As mentioned, the *real-or-random* digital locker does not prevent mauling of key. To prevent mauling of key, a natural idea is to use some integrity primitive Tag and include $\mathsf{Tag}(\mathsf{key})$ in the real-or-random construction.

A natural keyed primitive is a one-time message authentication code (MAC) or their non-malleable extensions [DW09]. The only other source of private randomness is in val which may be highly nonuniform and is highly correlated to the value being locked. Any approach that uses val as part of a MAC needs to ensure tampering on key and the MAC must be done "together." We use a keyless primitive to ensure the entire that key and any authentication are tampered "together."

Before describing the authentication in detail, we make a small but important change to the cryptographic primitive that is being composed. Instead of composing point functions, we compose digital

lockers that natively store a small number of key bits. We present two constructions of digital lockers that support short keys: one that supports a single bit and the second for a logarithmic number of bits.

The single-bit digital locker Point obfuscators output 1 on the correct input and 0 everywhere else. Our *single-bit digital locker* has three possible outputs: 1, 0, and $\perp$ (to indicate any incorrect point). To specify the output, the obfuscator now takes an additional bit b as input. Our construction of a single-bit digital locker is:

$$\mathcal{O}(\mathsf{val}, b) = (r, r^{g^{h(2*\mathsf{val}+b)}}) = (r, r^{g^{h(\mathsf{val}||b)}}).$$

Here $h(x) = x^4 + x^3 + x^2 + x$ as in Komargodski and Yogev's point function obfuscator. With a known value val' the user attempts to unlock with both values of b. Since the underlying function h is one-to-one, the construction remains correct. The intuitive argument for nonmallability of a single obfuscation is simple, if an adversary could change either x or b that would correspond to a break of the original system.

However, to create a digital locker requires composition of several obfuscations. This small modification makes it more difficult to show nonmalleability (even just for functions on val). This is because the adversary now gets access to obfuscations of two distinct points $2\mathsf{val}$ and $2\mathsf{val}+1$. Now the adversary may be able to compute a function f' that takes $h(2\mathsf{val}), h(2\mathsf{val}+1)$ and 1, yielding $h(2\mathsf{val}'+b')$. Despite this additional information provided to the adversary, Theorem 3.2 states it is still difficult to maul using a bounded degree polynomial.

In Theorem 3.2, we have to directly reduce to the strong power and strong vector Diffie-Hellman assumptions (defined in Section 2.1). Previous constructions of multi-bit point obfuscators [CD08, BC10] could be constructed generically from any point obfuscator (assuming the underlying point obfuscator is "secure" when composed).

Extending to nonbinary alphabet Our second construction builds a digital locker that encodes a logarithmic number of bits in a pair of group elements. Suppose that we wish to encode τ bit symbols in each group element. The most natural idea is to extend our first construction by computing $h(\tau*\mathsf{val}+\mathsf{key}_i)$. As stated above, in the single bit case, nonmalleability required showing that given $h(2\mathsf{val})$ and $h(2\mathsf{val}+1)$ it was difficult to compute $h(2\mathsf{val}'+b')$. Under the power DDH assumption (see Section 2.1) this requires one to show that $h(2\mathsf{val}'+b)$ is linearly independent of $h(2\mathsf{val})$ and $h(2\mathsf{val}+1)$. Since $h(x) = x^4+x^3+x^2+x$, the space spanned by differed $h(x)$ is only dimension 4. Naturally, as the adversary is given more linearly independent values $h(\tau * \mathsf{val} + \mathsf{key}_i)$ these span a larger dimensional space and it becomes impossible to show that fresh values are linearly independent. To address this problem we use a new hash function for symbols $y \in \{0,1\}^\tau$:

$$h(x, y) = \left(\sum_{i=0}^{4} (x+2)^{4+\tau}\right) + \sum_{j=0}^{\tau-1} y_i \cdot (x+2)^j.$$

This new hash function ensures that all τ different values of y create a $\log \tau$ dimensional subspace and cannot be used to predict the value of the hash function for any $x' \neq x$. There are a few subtle notes about this new construction:

1. Reconstruction proceeds by exhaustive checking of possible y values so running time is proportional to 2^τ making this construction efficient when $\tau = O(\log(\lambda))$.

2. Our single bit construction used a polynomial that included four nonzero powers. Interestingly, if we include 4 powers that are not multiplied by a bit of key there is a problem. For three values

of τ there are values of val' in the linear span. This is because the choice of τ introduced a new degree of freedom to the linear system. The three values depend on field arithmetic (solutions to $\tau^3 - 15\tau^2 - 52\tau - 12 \equiv 0 \mod p$). It would be possible to avoid these τ by checking when τ is a solution for a particular p. Instead, our construction adds another power of x, which retains a single bad τ which is $\tau \equiv p - 5$. This value of τ never occurs for logarithmic τ.

3. To ensure the construction is correct and one-to-one, we restrict $x \in \{0,1\}^{\lambda-1}$ and compute powers of $x + 2$. We can think of the construction as taking a subset of powers of x, we need to manually exclude $x = \{0, 1\}$ to ensure the function is one-to-one.

4. The group operations need to be in a group of size $(6+\tau)\lambda$ to ensure that operations do not overflow. In the first construction this size is only 5λ.

We note that neither of these constructions make any attempt to prevent tampering of key.

1.2 Authenticating the key

We now describe how we authenticate a potential value $\bar{c}$ resulting from a (potentially tampered) sequence of single bit digital lockers. Our authenticator uses seed-dependent condensers as a MAC [DRV12] and non-malleable codes [DPW10] to bind together the MAC and key. Our goal is construct a string s that is decodable to key and difficult to tamper. The construction proceeds as follows:

- First, we use the output of a seed-dependent condenser [DRV12], namely $\mathtt{cond}(\mathsf{val}, \mathsf{seed})$ as a MAC on key. That is we set $c = \mathsf{key}||\mathtt{cond}(\mathsf{val}; \mathsf{seed})$. A seed dependent condenser is an object that has a high entropy output even if the distribution over val is adversarially dependent on the randomness seed. Seed-dependent condensers for efficiently sampleable distributions are instantiable using collision-resistant hash functions [DRV12, Theorem 4.1]. Since there is no requirement on the distribution of val being independent of seed, the same seed can be used universally and be a system parameter. It is possible to check tampering over $\mathtt{cond}(\mathsf{val}; \mathsf{seed})$ since this value has high entropy.

- To ensure that any tampering on d (and thus key) results in tampering on $\mathtt{cond}(\mathsf{val}; \mathsf{seed})$ we use a nonmalleable code. Let $\mathcal{F}$ be some function class. A non-malleable code is a pair Enc and Dec where for functions $f \in \mathcal{F}$ the value $\tilde{s} = \mathsf{Dec}(f(\mathsf{Enc}(s)))$ is independent of s. That is, we set $s = \mathsf{Enc}(c) = \mathsf{Enc}(\mathsf{key}||\mathtt{cond}(\mathsf{val}; \mathsf{seed}))$.

  However, there are problems with using a non-malleable code for this application:

  1. In a non-malleable code, the adversary specifies the tampering function before seeing any information about $\mathsf{Enc}(\mathsf{key})$. In our setting, the adversary sees obfuscations correlated to $\mathsf{Enc}(\mathsf{key})$ before deciding how to tamper. We show that nonmalleable codes can be used a nonstandard way where the tampering function is chosen after seeing the obfuscated values (assuming a pseudorandomness condition, see Theorem 4.3).
  2. Nonmalleable codes allow tampering to an independent value $\tilde{s}$. Having to match the output of the condenser guarantees such an attack is likely to be detected.

Nonmalleable extractors [DW09, CRS14] and non-malleable one-way hashes [BCFW09, BFS11, CQZ+16] also hold promise for creating this binding. These objects both guarantee that the adversary tampers to an "independent" value like nonmalleable codes. We leave exploration of these objects as future work.

Choosing a non-malleable code There are non-malleable codes that prevent the adversary from permuting the bit vector, setting individual bits, and arbitrary functions that are applied separately to different parts of the encoded value. This class of of adversary is called a "split-state" adversary. More recently, Chattopadhyay and Li described a construction that prevents tampering in the class ACO [CL17]. We recommend using a non-malleable code that prevents at least setting and flipping of individual bits and permutations [AGM+15b, AGM+15a]. One concern about nonmalleable codes is that the adversary is *necessarily* restricted to low complexity classes, not including the code's encoding and decoding functions. Note, we are encoding and decoding the code "in the clear" while the adversary is tampering "in the exponent."

Non-malleable codes with manipulation detection Recently, Kiayias et al. [KLT18] introduced *non-malleable codes with manipulation detection*. Here, the adversary has low probability of producing any codeword $\tilde{c}$ that successfully decodes. (Clearly, the class of tampering functions cannot contain constant functions.) Kiayias et al. constructed a non-malleable code with manipulation detection but their construction requires each symbol of the code to come from a polynomial size alphabet or equivalently for each symbol to have logarithmic length. Thus, necessitating the multi-bit construction. With this strengthened object our construction does not need the seed-dependent condenser in c. We do note that Kiayias et al.'s construction does not include permutations which are efficiently computable by the adversary in our construction. To the best of our knowledge, this is not an inherent restriction of the definition.

Open Questions We present two main open questions resulting from this work. The first is whether our construction can be modified to use other nonmalleable primitives such as extractors or one-way hashes. The second open question in our multi-bit construction a different polynomial would allow for a more compact group and thus more efficient operations. It seems necessary for the group size to scale with τ if the linear independence argument is used. It may be possible to compress group size by allowing imperfect correctness.

Organization The rest of this work is organized as follows Section 2 reviews definitions and computational assumptions, Section 3 introduces the single bit digital locker and shows security under composition for val, Section 4 describes the full construction with nonmallebility for both val and key using non-malleable codes, lastly Section 5 shows how to support multiple bits in each pair of group elements, enabling support for logarithmic size symbols.

2 Definitions and Background

For a random variables X_i over some alphabet $\mathcal{Z}$ we denote by $X = X_1, ..., X_n$ the tuple $(X_1, \ldots, X_n)$. For a set of indices J, X_J is the restriction of X to the indices in J. The *minentropy* of X is $\mathrm{H}_\infty(X) = -\log(\max_x \Pr[X = x])$, and the *average (conditional)* minentropy [DORS08, Section 2.4] of X given Y is

$$\tilde{\mathrm{H}}_\infty(X|Y) = -\log\left(\mathop{\mathbb{E}}_{y\in Y} \max_x \Pr[X = x|Y = y]\right).$$

The *statistical distance* between random variables X and Y with the same domain is

$$\Delta(X, Y) = \frac{1}{2}\sum_x |\Pr[X = x] - \Pr[Y = x]|.$$

For a distinguisher D, the *computational distance* between X and Y is $\delta^D(X,Y) = |\mathbb{E}[D(X)] - \mathbb{E}[D(Y)]|$ (we extend it to a class of distinguishers $\mathcal{D}$ by taking the maximum over all distinguishers $D \in \mathcal{D}$). We denote by $\mathcal{D}_s$ the class of randomized circuits which output a single bit and have size at most s. Logarithms are base 2. Usually, we use capitalized letters for random variables and corresponding lowercase letters for their samples.

Single Bit Digital Locker A point function is a function I_{val}: $\{0,1\}^n \mapsto \{0,1\}$ outputs 1 on input x and 0 elsewhere. The goal of an obfuscator is to preserve functionality while hiding the point val if val is not provided as input to the program. We will build a function that outputs a single bit when val is provided: $I_{\mathsf{val},b}$: $\{0,1\}^n \mapsto \{\perp, 0, 1\}$. Here $I_{\mathsf{val},b}(\mathsf{val}) = b$ and $I_{\mathsf{val},b}(\mathsf{val}') = \perp$ for all other points $\mathsf{val}' \neq \mathsf{val}$. We call this primitive a *single-bit digital locker*. We do not include condition of *polynomial slowdown* in our definitions, but note the running time of our constructions throughout. We define a single-bit digital locker by adapting Komargodski and Yogev's definition [KY18]:

Definition 2.1. *For security parameter $\lambda \in \mathbb{N}$ a **single bit digital locker** lock is a probabilistic polynomial-time algorithm that inputs a point $\mathsf{val} \in \{0,1\}^{\lambda-1}$ and a bit $b \in \{0,1\}$, and outputs a circuit unlock such that the following two conditions are met with error γ:*

1. ***Completeness**: For all $\lambda \in \mathbb{N}$, all $x \in \{0,1\}^{\lambda-1}$, and either $b \in \{0,1\}$, it holds that $\Pr[\mathsf{unlock}(\cdot) \equiv I_{x,b}(\cdot) | \mathsf{unlock} \leftarrow \mathsf{lock}(x,b)] = 1$, where the probability is over the randomness of lock.*
2. ***Soundness**: For every probabilistic polynomial-time algorithm $\mathcal{A}$ and any polynomial function p, there exists a (possibly inefficient) simulator $\mathcal{S}$ and a polynomial $q(\lambda)$ such that, for all large enough $\lambda \in \mathbb{N}$, all $\mathsf{val} \in \{0,1\}^{\lambda-1}$, any single bit b, and for any $\mathcal{P} : \{0,1\}^\lambda \mapsto \{0,1\}$,*

$$\left| \Pr[\mathcal{A}(\mathsf{lock}(\mathsf{val}, b)) = \mathcal{P}(\mathsf{val}, b)] - \Pr[\mathcal{S}^{I_{\mathsf{val},b}}(1^\lambda) = \mathcal{P}(\mathsf{val}, b)] \right| \leq \frac{1}{p(\lambda)},$$

where S is allowed $q(\lambda)$ oracle queries to $I_{\mathsf{val},b}$ and the probabilities are over the internal randomness of $\mathcal{A}$ and lock, and of $\mathcal{S}$, respectively. Here $I_{x,b}$ is an oracle that returns b when provided input x otherwise $I_{x,b}$ returns $\perp$.

We note the above definition is *virtual grey-box obfuscation* as the simulator is allowed unbounded time but a limited number of queries. The definition of a point function is analagous with the removal of b and a suitable change to the ideal oracle I.

We directly adapt the definition of nonmalleability from Komargodski and Yogev. We do note that in their definition the adversary that is performing the mauling must output the mauling function f. See Komargodski and Yogev for definitional considerations [KY18]. Their construction also depends on an obfuscation being easy to recognize. Our constructions consist of pairs of group elements and are easy to recognize.

Definition 2.2. *A PPT algorithm $\mathcal{V}$ for a digital locker, lock, for $\mathsf{val} \in \{0,1\}^{\lambda-1}$, $b \in \{0,1\}$ is called a verifier if for all $\lambda \in \mathbb{N}$ and all $\mathsf{val} \in \{0,1\}^{\lambda-1}, b \in \{0,1\}$, it holds that $\Pr[\mathcal{V}(\mathsf{lock}(x,b)) = 1] = 1$, (prob. over the randomness of $\mathcal{V}$ and lock).*

With this definition, we can define nonmalleability. Our definition includes the bit b, does not assume a distribution over b, and does not assume hardness of the adversary tampering with b.

Definition 2.3. *Let* lock *be a single bit digital locker for* $\mathsf{val} \in \{0,1\}^{\lambda-1}$ *with an associated verifier* $\mathcal{V}$*. Let* $\mathcal{F} : \{0,1\}^{\lambda-1} \to \{0,1\}^{\lambda-1}$ *be a family of functions and let* $\mathcal{X}$ *be a family of distributions over* $\{0,1\}^{\lambda-1}$*. A single bit digital locker* lock *is* non-malleable *for* $\mathcal{F}$ *and* $\mathcal{X}$ *if for any polynomial-time adversary* $\mathcal{A}$*, there exists a negligible function* ϵ*, such that for all* $b \in \{0,1\}$ *it holds that:*

$$\Pr_{\mathsf{val} \leftarrow \mathcal{X}} \left[\mathcal{V}(C) = 1, f \in \mathcal{F}, (I_{f(\mathsf{val}),0} \equiv C \vee I_{f(\mathsf{val}),1} \equiv C) | (C, f) \leftarrow \mathcal{A}(\mathsf{lock}(\mathsf{val})) \right] \leq \epsilon.$$

Digital Locker Our full construction is a non-malleable digital locker. In this section we present two definitions, the first ensures nonmalleability over just the encoded point val. The second provides nonmalleability over both the encoded point val as well as key. These definitions are used in Section 3 and Section 4 respectively.

Definition 2.4. *The algorithm* lock *with security parameter* λ *is secure digital locker if the following hold:*

- ***Correctness*** *For every* key *and* val*,*

$$\Pr[\mathsf{unlock}(\cdot) \equiv I_{\mathsf{val},\mathsf{key}}(\cdot) | \mathsf{unlock} \leftarrow \mathsf{lock}(\mathsf{val}, \mathsf{key})] = 1.$$

- ***Security*** *For every PPT adversary* $\mathcal{A}$ *and every positive polynomial* p*, there exists a simulator* S *and a polynomial* $q(\lambda)$ *such that for any sufficiently large* λ*, any polynomially-long sequence of values* $(\mathsf{val}_i, \mathsf{key}_i)$ *for* $i = 1, \ldots, \ell$*, and for any predicate* $\mathcal{P}$*,*

$$\left| \Pr\left[\mathcal{A}\left(\mathsf{lock}(\mathsf{val}, \mathsf{key})\right) = \mathcal{P}(\mathsf{val}, \mathsf{key})\right] - \Pr\left[\mathcal{S}^{I_{\mathsf{val},\mathsf{key}}}(1^\lambda) = \mathcal{P}(\mathsf{val}, \mathsf{key})\right] \right| \leq \frac{1}{p(\lambda)}$$

where $\mathcal{S}$ *is allowed* $q(\lambda)$ *oracle queries to the oracle* $I_{\mathsf{val},\mathsf{key}}$*.*

Definition 2.5. *Let* $\mathcal{F} : \{0,1\}^\lambda \to \{0,1\}^\lambda$ *be a family of functions and let* $\mathcal{X}$ *be a family of distributions over* $\{0,1\}^\lambda$*. A digital locker,* lock*, with security parameter* λ *is* point *non-malleable for* $\mathcal{F}$ *if for any PPT* $\mathcal{A}$ *there exists a negligible function* ϵ *such that for all* $\mathsf{key} \in \{0,1\}^k$ *it holds that:*

$$\Pr_{x \leftarrow \mathcal{X}} \left[\begin{array}{c} (\mathsf{unlock}', f) \leftarrow \mathcal{A}(\mathsf{lock}(\mathsf{val}, \mathsf{key})) \\ \mathcal{V}(\mathsf{unlock}') = 1, f \in \mathcal{F}, \exists \mathsf{key}' \in \{0,1\}^k \wedge (I_{f(\mathsf{val}),\mathsf{key}'} \equiv \mathsf{unlock}') \end{array} \right] \leq \epsilon.$$

Definition 2.6. *Let* $\mathcal{F} : \{0,1\}^\lambda \to \{0,1\}^\lambda, \mathcal{G} : \{0,1\}^n \to \{0,1\}^n$ *be families of functions and let* $\mathcal{X}$ *be a family of distributions over* $\{0,1\}^\lambda$*. A digital locker,* $(\mathsf{lock}, \mathsf{unlock})$*, with security parameter* λ *is* point *non-malleable for* $\mathcal{F}$ *and* key *non-malleable for* $\mathcal{G}$ *if for any PPT* $\mathcal{A}$ *there exists a negligible function* ϵ *such that for all* $\mathsf{key} \in \{0,1\}^k$ *it holds that for any* $\mathsf{key}' \{0,1\}^k$*:*

$$\Pr_{x \leftarrow X} \left[\begin{array}{c} (\mathsf{unlock}', f, f') \leftarrow \mathcal{A}(\mathsf{lock}(\mathsf{val}, \mathsf{key})) \\ \mathcal{V}(\mathsf{unlock}') = 1, f \in \mathcal{F}, f' \in \mathcal{G} \wedge (I_{(f(\mathsf{val}),f'(\mathsf{key}))} \equiv \mathsf{unlock}') \end{array} \right] \leq \epsilon.$$

2.1 Hardness Assumptions

Our constructions will rely on multiple decisional assumptions in suitable groups. The most well known assumption is the decisional Diffie-Hellman (DDH) assumption which says that for a prime p for a generator g of $\mathbb{Z}_p^*$ the tuple (g, g^x, g^y, g^{xy}) is computationally indistinguishable from (g, g^x, g^y, g^u) where x, y, u

are uniform elements in $\mathbb{Z}_p^*$. We consider an ensemble of groups with efficient operations where $\mathbb{G}_\lambda$ is a group of prime order $p \in (2^\lambda, 2^{\lambda+1})$.

The first assumption that we will use is the strong ℓ-vector assumption introduced by Bitansky and Canetti [BC10]. This strengthens the DDH assumption in two ways. First, the values x are not drawn from uniform powers but rather from a well spread distribution. Second, the adversary is given multiple samples from correlated distributions $\mathcal{X}_1, ..., \mathcal{X}_\ell$. The only guarantee is on the marginal distribution of each $\mathcal{X}_i$. Nothing is guaranteed about the joint distribution. In particular, each distribution could be identically distributed. Here we introduce a new variant of this assumption where each distribution has average min-entropy.

Definition 2.7. *An ensemble of joint distributions $(\mathcal{X}, \mathcal{Y}) = \{X_\lambda, Y_\lambda\}_{\lambda\in\mathbb{N}}$, where $\mathcal{X}_\lambda$ is over $\{0,1\}^\lambda$, is* average case well-spread *if*

1. *It is efficiently and uniformly samplable. That is, there exists a PPT algorithm given 1^λ as input whose output is identically distributed as (X_λ, Y_λ).*
2. *For all large enough $\lambda \in \mathbb{N}$, it has super-logarithmic min-entropy. Namely, $\tilde{\mathrm{H}}_\infty(X_\lambda|Y_\lambda) = \omega(\log \lambda)$.*

Assumption 2.1 (t-Strong Average Vector DDH)**.** *Let $\ell = \mathtt{poly}(\lambda)$ be a parameter and let $\mathbb{G}_\lambda$ be an ensemble of groups with efficient representation and operations. We say that the* t-strong average vector decision Diffie-Hellman assumption *holds if for any vector $\mathcal{X}, \mathcal{Y}$ where $\mathcal{X}$ is a vector in $\mathbb{Z}_p^*)^t$ that is average case well-spread it holds that for every $s_{sec} = poly(\lambda)$ there exists some $\epsilon = \mathtt{ngl}(\lambda)$ such that :*

$$\delta^{s_{sec}}\left((g_1, g_1^{x_1}, y_1, ..., g_t, g_t^{x_t}, y_t), (g_1, g_1^{u_1}, y_1, ..., g_t, g_t^{u_t}, y_t)\right) \leq \epsilon.$$

Where $((x_1, y_1), ..., (x_t, y_t) \leftarrow (\mathcal{X}, \mathcal{Y})$ and u_i is sampled uniformly from $\mathbb{Z}_p^*$.

We use this variant as it is conceptually cleaner but random variables with super logarithmic average min-entropy have worst-case super-logarithmic min-entropy with overwhelming probability. Thus, there is not a qualitative difference between the average case assumption and a worst case formulation.

The second assumption we will consider is a variant of the power DDH assumption. The t-power DDH assumption (introduced in [GJM02]) says that increasing powers of a single element x are indistinguishable from uniformly random. That is, $(g, g^x, g^{x^2}, ..., g^{x^t})$ is pseudorandom for a uniformly random x. One can naturally extend the power DDH assumption assumption to the strong setting:

Assumption 2.2 (t-Strong Average Power DDH)**.** *The* t-strong average DDH assumption *is said to hold for an ensemble of groups $\mathbb{G}_\lambda$ with associated generator g if for any average-case well-spread distribution ensemble $\mathcal{X}, \mathcal{Y}$, the following holds for any $s_{sec} = \mathtt{poly}(\lambda)$ there exists $\epsilon = \mathtt{ngl}(\lambda)$ such that*

$$\delta^{s_{sec}}((g, g^{X_\lambda}, g^{X_\lambda^2}, ..., g^{X_\lambda^t}, Y_\lambda), (g, g^{r_1}, g^{r_2}, ..., g^{r_t}, Y_\lambda)) \leq \epsilon.$$

where the distribution (X_λ, Y_λ) are jointly sampled. Here the elements $r_1, ..., r_t$ are sampled uniformly at random from $\mathbb{Z}_p^*$.

3 Non-malleability for the locked value

Construction 3.1. *Let $\lambda \in \mathbb{N}$ be a security parameter and let $\{0,1\}^\lambda$ be the domain. Let $\mathcal{F}_{t,\mathtt{poly}} \stackrel{def}{=} \{f : \{0,1\}^\lambda \to \{0,1\}^\lambda\}_{\lambda\in\mathbb{N}}$ the ensemble of all functions that can be computed by polynomials of degree*

at most t except constant and identity functions. Let $\mathcal{G} = \{\mathbb{G}_\lambda\}_{\lambda\in\mathbb{N}}$ be a group ensemble with efficient representation and operations where each $\mathbb{G}_\lambda$ is a group of prime order $q \in (2^\lambda, 2^{\lambda+1})$. We assume that for every $\lambda \in \mathbb{N}$ there is a canonical and efficient mapping between the elements of $\{0,1\}^\lambda$ to $\mathbb{G}_\lambda$. Let g be a generator of a group $\mathbb{G}_{5\lambda}$. For known generator $g \in \mathbb{G}_{5\lambda}$, Our single bit digital locker gets an element $x \in \{0,1\}^{\lambda-1}, b \in \{0,1\}$ and randomness $r \in \mathbb{G}_{5\lambda}$. and outputs:

$$\mathsf{lock}(x, b; r) \stackrel{def}{=} \left(r, r^{g^{(2x+b)^4+(2x+b)^3+(2x+b)^2+(2x+b)}}\right).$$

Given a program unlock *consisting of two group elements g_1, g_2 for test password x' and obfuscated program* unlock*, the user runs both* $\mathsf{unlock}(x', 0)$ *and* $\mathsf{unlock}(x', 1)$*. That is, the user computes:*

$$\mathsf{unlock}(x, 0) = \left(r_1^{g^{(2x)^4+(2x)^3+(2x)^2+(2x)}} \stackrel{?}{=} g_2.\right)$$
$$\mathsf{unlock}(x, 1) = \left(r_1^{g^{(2x+1)^4+(2x+1)^3+(2x+1)^2+(2x+1)}} \stackrel{?}{=} g_2.\right)$$

If $\mathsf{unlock}(x, b)$ *outputs* 1 *then the user outputs b. Otherwise, $\perp$ is the output.*

We will first show this construction is secure when the adversary receives a single instance and then discuss nonmalleability when composed. Throughout our discussion we use $h(y) = y^4 + y^3 + y^2 + y$ as shorthand for the polynomial being computed in the exponent. In order to prove security of this construction for a single obfuscation we reduce to the construction of Komgrodski and Yogev which computes $\mathsf{lockPoint}(x; r) = (r, r^{g^{h(x)}})$ for input $x \in \{0,1\}^\lambda$. The proof is deferred to Appendix A.

Theorem 3.1. *Suppose that* $\mathsf{lockPoint}(x; r) = (r, r^{g^{h(x)}})$ *is a non-malleable point function obfuscator for points $x \in \{0,1\}^\lambda$ for all distributions $X = \mathcal{X}_\lambda$ such that $H_\infty(X) = \omega(\log\lambda)$. Then* $\mathsf{lock}(x, b) = \mathsf{lockPoint}(2x + b)$ *is a single bit digital locker for inputs $x \in \{0,1\}^{\lambda-1}, b \in \{0,1\}$ for all distributions X such that $H_\infty(X) = \omega(\log\lambda)$.*

3.1 Composing the Single Bit Construction

We now show this construction can be composed to built digital locker. This requires showing that soundness, completeness, and nonmalleability are preserved when the adversary is provided with single bit digital lockers that are correlated. There are two things that could go wrong when an adversary receives single bit digital lockers on correlated points: 1) the inclusion of correlated points may allow the adversary to maul and 2) having multiple samples of points under different randomness may break privacy. To prevent against tampering, we show security against an adversary that obtains $g^{h(2x)}$ and $g^{h(2x+1)}$. That is, we don't rely on the generators r_i in providing any protection against tampering. To show that privacy is preserved we rely on the t-strong vector DDH assumption. Our construction is a simple concatenation of the single bit digital locker but the proof is substantially more involved.

Construction 3.2. *Let all variables be as in Construction 3.1 and let* $\mathsf{key} \in \{0,1\}^n$ *be some arbitrary value. Then define* $\mathsf{lock}(\mathsf{val}, \mathsf{key})$ *as follows (initialize* $\mathtt{Out} = \perp$*): for $i = 1$ to n compute:*

1. *Sample $r_i \leftarrow \mathbb{G}_{5\lambda}$.*
2. *Append* $\mathtt{Out} = \mathtt{Out}||(r_i, (r_i)^{g^{(2\mathsf{val}+\mathsf{key}_i)^4+(2\mathsf{val}+\mathsf{key}_i)^3+(2\mathsf{val}+\mathsf{key}_i)^2+(2\mathsf{val}+\mathsf{key}_i)}})$.

Define $\mathsf{unlock}(\mathsf{val})$ *as follows for input* $\{r_i, y_i\}_{i=1}^n$*:*
For $i = 1$ *to* n *compute:*

$$\gamma_{i,0} = (2\mathsf{val})^4 + (2\mathsf{val})^3 + (2\mathsf{val})^2 + (2\mathsf{val})$$
$$\gamma_{i,1} = (2\mathsf{val}+1)^4 + (2\mathsf{val}+1)^3 + (2\mathsf{val}+1)^2 + (2\mathsf{val}+1)$$
$$P(x,0,i) = \left(r_i^{g^{\gamma_{i,0}}} \stackrel{?}{=} y_i.\right)$$
$$P(x,1,i) = \left(r_i^{g^{\gamma_{i,1}}} \stackrel{?}{=} y_i.\right)$$

If $P(x,b,i)$ *outputs* 1 *then the user sets* $\mathsf{key}_i = b$*. Otherwise output* $\perp$*.*

Theorem 3.2. *Suppose that*

1. *The* n*-strong vector DDH assumption holds,*
2. *The* $4t$*-strong power DDH assumption holds,*
3. *The selected prime* $p \notin \{2,3,5,7,11\}$*. (As* $\mathbb{G}_\lambda$ *increases these primes will never be selected. We include this condition as the proof does not apply for the listed* p*.)*
4. X *is a distribution over* $\{0,1\}^{\lambda-1}$ *such that* $\mathrm{H}_\infty(X) = \omega(\log\lambda)$*.*

Then Construction 3.2 is point non-malleable for $\mathcal{F} = \{f | \deg(f) \leq t\}$ *(excluding constant polynomials and the identity polynomial).*

Proof. We separately consider correctness, soundness, and nonmalleability. Correctness is a direct extension of correctness for the single bit digital locker whose correctness is shown in Theorem 3.1. (The crucial fact is that $g^{h(x,b)}$ is a one-to-one function.)

Privacy Define the random variables $\vec{X} \stackrel{def}{=} \{X_i = (X||\mathsf{key}_i)\}_{i=1}^n$. Since X is distribution where $\mathrm{H}_\infty(X) = \omega(\log\lambda)$, $\vec{X}$ is a average-case well spread distribution (according to Definition 2.7). Since the function

$$f(x,b) = g^{(2x+b)^4+(2x+b)^3+(2x+b)^2+(2x+b)}$$

is one-to-one it is also true that $g^{h(\vec{X})}$ is average-case well spread. Komargodski and Yogev showed that a one-to-one function can be applied before obfuscation without effecting privacy [KY18, Claim 3.1]. This proof directly carries over to the single bit digital locker setting. Under the strong vector DDH assumption, $\{r_i, r_i^{X_i}\}_{i=1}^n \approx \{r_i, u_i\}_{i=1}^n$ for uniform group elements u_i. This means the construction satisfies a weaker notion called distributional indistinguishability [BC10, Definition 5.3], which says no adversary can tell between obfuscations of related points and independent uniform points. Bitanski and Canetti [BC10] show that this definition implies composition for virtual-grey box obfuscation. (Their proof is for point obfuscators but can be modified for this setting.) Overall virtual grey box security then follows using arguments from [CD08].

Nonmalleability We first recall that we use $h(x)$ to denote $x^4 + x^3 + x^2 + x$. In order to prove nonmalleability is preserved, we will show how, given an adversary that can maul our obfuscation given two distinct obfuscations with the same point x, we can create an algorithm that can break the String Power DDH assumption. We assume that key is a value known to both the reduction and the adversary.

(That is, we do not rely on any uncertainty with respect to key.) That is, we assume that there exists some key and a PPT $\mathcal{A}$ such that for all negligible functions ϵ:

$$\Pr_{x \leftarrow X}\left[\begin{array}{c}(\mathsf{unlock}', f) \leftarrow \mathcal{A}(\mathsf{lock}(\mathsf{val}, \mathsf{key})) \\ \mathcal{V}(\mathsf{unlock}') = 1, f \in \mathcal{F}, \exists \mathsf{key}' \{0,1\}^n \wedge (I_{(f(\mathsf{val}),\mathsf{key}')} \equiv C)\end{array}\right] > \epsilon.$$

We show how to construct $\mathcal{A}'$ that breaks the $\tau = 4t$ strong power DDH assumption (Assumption 2.2). Suppose we receive a sequence $\{g, g^{z_1}, g^{z_2}, ..., g^{z_\tau}\}$ where each z_i either equals x^i (sampled $x \leftarrow X$ where $X \in \{0,1\}^{\lambda-1}$) or a random group element r_i. We first compute two values:

$$y_0 = g^{16z_4+8z_3+4z_2+2z_1} = g^{16*z_4} \cdot g^{8*z_3} \cdot g^{4z_2} \cdot g^{2z_1},$$
$$y_1 = g^{16z_4+40z_3+40z_2+20z_1+4}.$$

$\mathcal{A}'$ then computes a vector based on these values:

$$\{r_i \xleftarrow{\$} \mathbb{G}, r_i^{y_{\mathsf{key}_i}}\}_{i=1}^n.$$

Then $\mathcal{A}$ is initialized based on these values and outputs a function f and a $2n$ vector of group elements $\{r_{\mathcal{A},i}, w_{\mathcal{A},i}\}_{i=1}^n$. We assume f is specified by coefficients (if not, these coefficients can be interpolated using points from the distribution X, see [KY18]). We can then use the f provided by $\mathcal{A}$ to check if each point in the vector is a valid single bit digital locker of $f(x)$ and a bit 0 or 1. Details for this check are in Algorithm 1. We proceed to analyze the success of this algorithm in both the real case $z_i = x^i$ and the random case $z_i = r_i$.

The real case In the real case the adversary $\mathcal{A}$ sees pairs $\left(r_i, r_i^{g^{h(2x+\mathsf{key}_i)}}\right)$. This is exactly the distribution expected by $\mathcal{A}$. Furthermore, $\mathcal{A}'$ outputs 1 when the mauled obfuscation is a valid obfuscation of x on some key'. Thus, given the real distribution $\mathcal{A}'$ outputs 1 with probability at least ϵ.

The random case We now assume that each z_i is a uniform and randomly distributed s_i for $1 \leq i \leq \tau$. We assume that the adversary is computationally unbounded and is provided with two points $c_0 = 16s_4 + 8s_3 + 4s_2 + 2s_1$ and $c_1 = 16s_4 + 40s_3 + 40s_2 + 20s_1 + 4$. (We can provide the adversary also with the values r_i.) That is, we give the adversary direct access to the value in the exponent. Its clear if no adversary can win in this game, then no adversary can win in the original game.

In order for $\mathcal{A}$ to succeed she needs to compute $c_\alpha = \sum_{i=0}^{\tau} \alpha_i s_i$ or $c_\beta = \sum_{i=0}^{\tau} \beta_i s_i$ (the vectors $\vec{\alpha}$ and $\vec{\beta}$ are defined in Algorithm 1). If the degree of the polynomial f is greater than 1 this requires computing a linear combination with some s_i where $i > 4$ that is independent of the adversary's view. In this case, both the distribution of both random variables C_α and C_β has entropy $\log |G_\lambda| = \lambda$ conditioned on the adversary's view [KY18, Claim 4.4]. By a union bound the probability of matching either C_α or C_β for some i is at most $\Pr[\text{success}] \leq \frac{2}{2^{-\lambda}} = \frac{1}{2^{\lambda-1}}$.

We now move to the case where the polynomial f is of degree 1. That is, $f(x) = \mu x + \nu$. If the function f is a linear one, then we will show how an accurate obfuscation of $h(2f(x') + b)$ cannot be formed from the inputs. To do this, we will look at what information about the points that the adversary receives. The adversary receives a multiple linear combinations of s_1, s_2, s_3, s_4.

$$\begin{bmatrix} 16 & 8 & 4 & 2 & 0 \\ 16 & 40 & 40 & 20 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_4 \\ s_3 \\ s_2 \\ s_1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ 1 \end{bmatrix}.$$

input : $g, g^{z_1}, ..., g^{\tau}$
output: $\mathcal{P}(x)$

1. Sample $\mathsf{key} \leftarrow \{0,1\}^n$.
2. Compute $y_0 = g^{16z_4+8z_3+4z_2+2z_1}$ and $y_1 = g^{16z_4+40z_3+40z_2+20z_1+4}$.
3. Compute $\{r_i \xleftarrow{\$} \mathbb{G}, r_i^{y_{\mathsf{key}_i}}\}_{i=1}^n$..
4. Run $(f, \{r_{\mathcal{A},i}, w_{\mathcal{A},i}\}_{i=1}^n) \leftarrow \mathcal{A}(\{r_i, r_i^{y_{\mathsf{key}_i}}\}_{i=1}^n)$. If the output is not a function followed by $2n$ group elements output 0.
5. Compute coefficients α_i of $h(2f(x))$ and β_i of $h(2 * f(x) + 1)$.
6. For $i = 1$ to n
   (a) Check if $w_{\mathcal{A},i} \stackrel{?}{=} r_{\mathcal{A},i}^{(\sum_{i=0}^{\tau} \alpha_i z_i)}$ or $w_{\mathcal{A},i} \stackrel{?}{=} r_{\mathcal{A},i}^{(\sum_{i=0}^{\tau} \beta_i z_i)}$.
   (b) If neither check is true, output 0.
7. Output 1.

Algorithm 1: Construction of $\mathcal{A}'$ from $\mathcal{A}$

The first row of the above matrix corresponds to the linear combination used when $\mathsf{key}_i = 1$, the second row when $\mathsf{key}_i = 0$ and the last row is the constant group element. As stated, the function f can be rewritten as $f(x) = \mu * x' + \nu$. By substituting and simplifying, f can finally be rewritten as:

$$2f(x) + b' = 2 * (\mu x + \nu) + b' = 2\mu x + 2\nu + b' = 2ax + b$$

for some $b \in \{0,1\}$ and a is a field element. Note we consider this as an existential argument so a so $a = (\mu x + \nu)x^{-1} = \mu + \nu x^{-1}$ is a valid assignment for a. We can write the desired linear combination as follows:

$$\begin{bmatrix} 16a^4 \\ 32a^3b + 8a^3 \\ 20a^2b^2 + 16a^2b + 4a^2 \\ 6ab^3 + 6ab^2 + 6ab + 2a \\ b^4 + b^3 + b^2 + b \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} s_4 \\ s_3 \\ s_2 \\ s_1 \\ 1 \end{bmatrix}.$$

We now show that even for an unbounded $\mathcal{A}$, this value is information theoretically hidden (given $c_0, c_1, 1$).

Lemma 3.1. *Let S_1, S_2, S_3, S_4 be uniformly distributed in $\mathbb{G}_{5\lambda}$ then define $C_0 = 16S_1 + 8S_2 + 4S_3 + 2S_4$ mod $G_{5\lambda}$ and $C_1 = 16S_1 + 40S_2 + 40S_3 + 20S_4 + 4 \mod G_{5\lambda}$ Define for $a \in \mathbb{G}_{5\lambda}, b \in \{0,1\}$,*

$$C^*_{a,b} = 16a^4S_4 + (32a^3b + 8a^3)S_3 + (20a^2b^2 + 16a^2b + 4a^2)S_2 + (6ab^3 + 6ab^2 + 6ab + 2a)S_1 + (b^4 + b^3 + b^2 + b).$$

Then the value $\mathrm{H}_\infty(C^*_{a,b}|C_0, C_1) \geq \lambda - 1$ if $a \neq 0, 1$ and $p \notin \{2, 3, 5, 11, 17\}$.

Proof of Lemma 3.1. We first show that when $a \neq 0, 1$, the value $c_{a,b}$ is linearly independent of the values $c_0, c_1, 1$. That is, we show the following system has no solutions $\alpha_0 c_0 + \alpha_1 c_1 + \alpha_2 c_2 = c_{a,b}$. To show linear independence we consider the following system of equations: Since $\mathcal{A}$ only has access to linear combinations of these variables in order for $\mathcal{A}$ to properly output some correct mauled obfuscation, they must find a solution to the following:

$$\begin{bmatrix} 16 & 8 & 4 & 2 & 0 \\ 16 & 40 & 40 & 20 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 16a^4 \\ 32a^3b + 8a^3 \\ 20a^2b^2 + 16a^2b + 4a^2 \\ 6ab^3 + 6ab^2 + 6ab + 2a \\ b^4 + b^3 + b^2 + b \end{bmatrix}$$

where $\alpha_0, \alpha_1, \alpha_2$ are field elements. Because b is a single bit (i.e. either 0 or 1), we will examine the existence of solutions under these two possibilities. In both cases we assume that the adversary can exactly solve the last equation using α_2 as a free variable and consider the following reduced system:

$$\begin{bmatrix} 16 & 16 \\ 40 & 8 \\ 40 & 4 \\ 20 & 2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 16a^4 \\ 32a^3b + 8a^3 \\ 20a^2b^2 + 16a^2b + 4a^2 \\ 6ab^3 + 6ab^2 + 6ab + 2a \end{bmatrix}$$

Case 1: $b = 0$ Considering the two by two system formed by the second and third equation yields that:

$$\begin{aligned} 40\alpha_0 &= 8a^2 - 8a^3, \\ 4\alpha_1 &= 8a^3 - 4a^2. \end{aligned}$$

Substituting these values into the fourth equation yields a quadratic for a:

$$4a^2 - 2a = 0$$

This has solutions of $a = 0, 2^{-1}$. Substituting the solutions for α_0, α_1 into the constraint from the first equation yields the quartic:

$$16a^4 - (32 + 16 * 5^{-1})a^3 + (16 - 16 * 5^{-1})a^2 = 0$$

This equation is consistent with the solutions where $a = 0$. However, when $a = 2^{-1}$, the first equation is only satisfied when $3 \equiv 0 \mod p$. Thus, it suffices for $p \neq 3$. Since the solution when $a = 0$ is considered trivial, nontrivial solutions exist in this case only when $p = 3$.

Case 2: $b = 1$ Again starting with the second and third linear constraints we have that:

$$\begin{aligned} \alpha_0 &= 2a^2 - a^3, \\ \alpha_1 &= 10a^3 - 10a^2. \end{aligned}$$

Substituting these values into the fourth equation yields $180a^3 - 160a^2 - 20a = 0$. Which has the trivial solutions of $a = 0, 1$ and nontrivial solution $a = -1 * 9^{-1}$. However, when $a = -1 * 9^{-1}$, the first equation is only satisfied when $11968 \equiv 0 \mod p$. Thus, it suffices for $p \notin \{2, 11, 17\}$. Since the solution when $a = 0$ is considered trivial, nontrivial solutions exist only when $p \in \{2, 11, 17\}$.

Putting things together So, regardless of choice of b, if a is nontrivial, the value $c_{a,b}$ is linearly independent of c_0, c_1 and 1. Consider the following system of equations:

$$\begin{bmatrix} 16 & 16 & 0 & 16a^4 \\ 8 & 40 & 0 & 32a^3b + 8a^3 \\ 1 & 40 & 0 & 20a^2b^2 + 16a^2b + 4a^2 \\ 2 & 20 & 0 & 6ab^3 + 6ab^2 + 6ab + 2a \\ 0 & 4 & 1 & b^4 + b^3 + b^2 + b \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} s_4 \\ s_3 \\ s_2 \\ s_1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ 1 \\ c_{a,b} \end{bmatrix}$$

This system of equations is rank 4 (in the case when a is nontrivial). This means for any particular c_0, c_1 there are at least 2^λ tuples of S_1, S_2, S_3, S_4 that produce any particular value of $c_{a,b}$. In particular, the probability that $\Pr[C_{a,b} = c | C_0, C_1] = \frac{1}{2^\lambda}$. Since the adversary has to match one of two values by union bound they succeed with probability at most $2^{-\lambda+1}$. This completes the proof of Lemma 3.1. □

Thus, in both random cases the probability of mauling is at most $2^{-(\lambda-1)}$. This allows us to state the distinguishing capability of $\mathcal{A}$:

$$\Pr[\mathcal{A}(\{g^{x^i}\}_{i=1}^{\tau}) = 1] - \Pr[\mathcal{A}(\{g^{r_i}\}_{i=1}^{\tau}) = 1] \geq \epsilon - \frac{1}{2^{\lambda-1}}.$$

This is a contradiction and completes the proof of Theorem 3.2. □

4 Non-malleability for the key

In this section, we extend our construction to prevent tampering over both val and key. We need several new tools including non-malleable codes and seed-dependent condensers. We first introduce these in turn.

Non-malleable codes We first need to introduce the notion of non-malleable codes introduced by Dziembowski, Pietrzak, and Wichs [DPW10].

Definition 4.1. *A pair of algorithms* $(\mathsf{Enc}, \mathsf{Dec})$ *is called a* coding scheme *if for* $\mathsf{Enc} : \{0,1\}^k \to \{0,1\}^n$ *is randomized and* $\mathsf{Dec} : \{0,1\}^n \to \{0,1\}^k \cup \perp$ *is deterministic and for each* $s \in \{0,1\}^k$ *it holds that* $\Pr[\mathsf{Dec}(\mathsf{Enc}(s)) = s] = 1$.

Definition 4.2. *A coding scheme,* $(\mathsf{Enc}, \mathsf{Dec})$ *is called* $(\epsilon_{nmc}, s_{nmc}, \mathcal{F})$*-non-malleable if for each* $f \in \mathcal{F}$ *and each* $s \in \{0,1\}^k$ *there exists a distribution* $D_f()$ *over* $\{\{0,1\}^k, \mathtt{same}\}$ *that is efficiently samplable given oracle access to* f *such that the following holds:*

$$\delta^{s_{nmc}}(\{c \leftarrow \mathsf{Enc}(s); \overline{c} \leftarrow f(c), \overline{s} = \mathsf{Dec}(\overline{c}) : \textit{Output } \overline{s}\},$$
$$\{\tilde{s} \leftarrow D_f, \ \textit{Output } s \textit{ if } \tilde{s} = \mathtt{same} \textit{ else } \tilde{s}\}) \leq \epsilon_{nmc}.$$

Seed Dependent Condensers Seed dependent condensers were introduced by Dodis, Ristenpart, and Vadhan [DRV12]. The goal of a condenser is similar to a traditional randomness extractor except that rather than considering a uniform output, the output only has to be statistical close to a distribution with min-entropy. Importantly, it is possible to construct condensers where the adversary is allowed to output the chosen distribution after seeing the seed (called *seed-dependent*).

Definition 4.3. *Let* $\mathtt{cond} : \{0,1\}^\lambda \times \{0,1\}^d \to \{0,1\}^\alpha$ *is a* (k, k', s, ϵ) seed-dependent condenser *if for all probabilistic adversaries of size at most* s *who take a random seed* $\mathsf{seed} \leftarrow U_d$ *and output a distribution* $X_{\mathsf{seed}} \leftarrow \mathcal{A}(\mathsf{seed})$ *of entropy* $\mathrm{H}_\infty(X|\mathsf{seed}) \geq k$. *Define the joint distribution* (X, U_d) *as the joint distribution over* X_{seed} *arising from a random* $\mathsf{seed} \leftarrow U_d$. *Then there exists a distribution* Y *such that* $\tilde{\mathrm{H}}_\infty(Y|U_d)$ *such that*

$$\Delta((Y, U_d), (\mathtt{cond}(X; U_d), U_d)) \leq \epsilon.$$

Dodis, Ristenpart, and Vadhan showed that seed-dependent condensers can be constructed using the standard tool of collision resistant hash functions:

Definition 4.4. *A family of hash functions* $\mathcal{H} = \{h : \{0,1\}^\lambda \to \{0,1\}^\alpha$ *is* (t, δ)-collision resistant *if for any circuit* $\mathcal{A}$ *of size at most* t,

$$\Pr_{h \leftarrow \mathcal{H} \wedge (x_1, x_2) \leftarrow \mathcal{A}(h)}[h(x_1) = h(x_2) \wedge x_1 \neq x_2] \leq \delta.$$

The following theorem states that collision-resistance translates into a seed-dependent condenser:

Theorem 4.1. *[DRV12, Theorem 4.1] Let* $\mathcal{H}$ *be a* $(2s, \delta)$*-collision-resistant hash function family, then* $\mathtt{cond}(x; h) = h(x)$ *for* $h \leftarrow \mathcal{H}$ *is a* $(-\log(\delta), \frac{-\log(\delta)-1}{2}, s, 0)$*-seed-dependent condenser.*

4.1 The Construction

Intuitively, we can combine non-malleable codes and seed-dependent condensers to check if the adversary tampers over the key value. We use the locked point val as input to a seed dependent condenser as part of the value encoded in the nonmalleable code. If the adversary tampers to an *independent value* there are unlikely to match the output of the condenser on the real val.

Construction 4.1. *Let* $\lambda \in \mathbb{N}$ *be a security parameter and let* $\{0,1\}^\lambda$ *be the domain.*

1. *Let* $\mathcal{F}_{t,\mathtt{poly}}$ *be as above.*
2. *Let* $(\mathsf{Enc}, \mathsf{Dec})$ *be a coding scheme where* $\mathsf{Enc} : \{0,1\}^{k+\beta} \to \{0,1\}^n$.
3. *Let* $\mathcal{G} = \{\mathbb{G}_\lambda\}_{\lambda \in \mathbb{N}}$ *be a group ensemble with efficient representation and operations where each* $\mathbb{G}_{5\lambda}$ *is a group of prime order* $q \in (2^{5\lambda}, 2^{5\lambda+1})$.
4. X *is a distribution such that* $\mathrm{H}_\infty(X) \geq \mu = \omega(\log \lambda)$. *There is no requirement that* X *is independent of system parameters (in particular* seed*) as long as it is efficiently samplable.*
5. *Suppose for any* $s = \mathtt{poly}(\lambda)$ *there exists* $\beta = \omega(\log \lambda)$ *such* $\mathtt{cond} : \{0,1\}^\lambda \times \{0,1\}^d \to \{0,1\}^\alpha$ *is a* $(\mu, \beta, s, 0)$*-seed-dependent condenser (instantiable using Theorem 4.1).*
6. *Let a description of* $\mathbb{G}_{5\lambda}$*, a generator* g *for* $\mathbb{G}_{5\lambda}$ *and* $\mathsf{seed} \leftarrow \{0,1\}^d$ *be system parameters.*

Define the algorithms $(\mathsf{lock}, \mathsf{unlock})$ *as in Figure 1.*

Theorem 4.2. *Suppose that*

1. *The* n*-strong average vector DDH assumption holds,*
2. *The* $4t$*-strong average power DDH assumption holds,*
3. *The selected prime* $p \notin \{2, 3, 5, 7, 11\}$.

$\mathsf{lock}(\mathsf{val}, \mathsf{key})$, input in $\{0,1\}^{\lambda+k}$:

1. Compute $y = \mathtt{cond}(\mathsf{val}, \mathsf{seed})$.
2. Compute $z = \mathsf{Enc}(\mathsf{key}||y)$.
3. Initialize $\mathtt{Out} = \perp$.
4. For $i = 1$ to n compute:
   (a) Sample random generator $r_i \leftarrow \mathbb{G}_{5\lambda}$.
   (b) Compute
   $$\gamma_i = (2\mathsf{val} + z_i)^4 + (2\mathsf{val} + z_i)^3 + (2\mathsf{val} + z_i)^2 + (2\mathsf{val} + z_i).$$
   (c) Append $\mathtt{Out} = \mathtt{Out}||\ (r_i, (r_i)^{g^{\gamma_i}})$.
5. Output $\mathtt{Out}$.

$\mathsf{unlock}(\mathsf{val})$, input in $\{0,1\}^{\lambda}$:

1. Compute $y = \mathtt{cond}(\mathsf{val}, \mathsf{seed})$.
2. For $i = 1$ to n, input r_i, y_i compute:
   $$\gamma_{i,0} = (2\mathsf{val})^4 + (2\mathsf{val})^3 + (2\mathsf{val})^2 + (2\mathsf{val})$$
   $$\gamma_{i,1} = (2\mathsf{val}+1)^4 + (2\mathsf{val}+1)^3 + (2\mathsf{val}+1)^2 + (2\mathsf{val}+1)$$
   $$P(x,0,i) = \left(r_i^{g^{\gamma_{i,0}}} \stackrel{?}{=} y_i.\right), P(x,1,i) = \left(r_i^{g^{\gamma_{i,1}}} \stackrel{?}{=} y_i.\right)$$
   (a) If $P(x,b,i)$ outputs 1 then set $z_i = b$. Otherwise output $\perp$.
3. Run decode $\mathsf{key}' = \mathsf{Dec}(z)$.
4. If $\mathsf{key}'_{k\ldots k+n} \neq y$ output $\perp$. Else output $\mathsf{key}'_{0\ldots k-1}$.

Figure 1: Non-malleable digital locker preventing tampering over both val and key. A group generator g and a seed of a seed-dependent condenser are global system parameters.

4. *Suppose that* $\mu - \beta = \omega(\log \lambda)$.
5. *The code* $(\mathsf{Enc}, \mathsf{Dec})$ *is an* $(\epsilon_{nmc}, s_{nmc}, \mathcal{F}_{nmc})$ *non-malleable code.*

Then $(\mathsf{lock}, \mathsf{unlock})$ *in Construction 4.1 is point non-malleable for* $\mathcal{F}_{\mathtt{poly},t}$ *and key nonmallable* $\mathcal{F}_{nmc}$.

Proof. We note a couple of key differences between the above theorem and Theorem 3.2 that provided non-malleable only over the point val. First, the condition on the distribution X is changed. We now require that $X|(\mathtt{cond}(X;S),S)$ has min-entropy (recalling that X can be chosen dependent on S). Note that if the output length β is a constant fraction of $\mathrm{H}_\infty(X)$ this condition is implied by the entropy of X using standard entropy arguments [DORS08, Lemma 2.2b].

Correctness, privacy, and point nonmalleability follow using the same arguments as Theorem 3.2 under the strengthened average case vector and power DDH assumptions. The core of our proof is a theorem (which may be of independent interest) that allows us to use non-malleable codes in a nontraditional way where the adversary is provided with pseudorandom information that is correlated to the encoded codeword before choosing which function $f \in \mathcal{F}$ to tamper with. We first define an adaptive tampering experiment as follows for arbitrary distributions X, Y, Z and binary predicate $\mathtt{Test}$:

Experiment $\mathbf{Exp}^{\mathrm{ad-nmc}}_{\mathcal{F}_{nmc},X,Y,Z,\mathcal{A},\mathtt{Test}}$:
Sample $(x,y,z) \leftarrow (X,Y,Z)$
Sample $f \leftarrow \mathcal{A}(x)$.
If $f \notin \mathcal{F}_{nmc}$ output 0.
If $f(y) = y$ output 0.
If $\mathtt{Test}(f(y), z)$ output 1.
Else output 0.

Theorem 4.3. *Let $Z \in \{0,1\}^\alpha$ be a distribution such that $\mathrm{H}_\infty(Z) \geq \beta$. Let $(\mathsf{Enc}, \mathsf{Dec})$ be a $(\epsilon_{nmc}, s_{nmc}, \mathcal{F})$ non-malleable code and $\mathsf{Enc} : \{0,1\}^k \to \{0,1\}^n$ where $k = \alpha + \gamma$. Let $\mathsf{key} \in \{0,1\}^\gamma$, define the distribution Y_{key} by sampling $z \leftarrow Z$ and computing $\mathsf{Enc}(\mathsf{key}, z)$. For inputs y and z, define $\mathtt{Test}(y,z) = 1$ if and only if $\mathsf{Dec}(y)_{\gamma \ldots (\gamma+\alpha-1)} = z$. Suppose that*

1. *For all $f \in \mathcal{F}$ it is possible to compute f in a circuit of size at most $s_{\mathcal{F},\mathtt{eval}}$.*
2. *It is possible to evaluate $\mathtt{Test}$ using a circuit of size at most $|\mathtt{Test}|$ and $s_{nmc} > |\mathtt{Test}|$.*
3. *For a function f it is possible to check if $f \in \mathcal{F}$ in size at most $s_{\mathcal{F},\mathtt{check}}$. Furthermore, this check is correct with probability 1.*
4. *X be an arbitrary distribution over $\mathcal{M}$ such that*

$$\delta^{\mathcal{D}_{s_{pr}}}((X, Y_{\mathsf{key}}, Z), (U_{\mathcal{M}}, Y_{\mathsf{key}}, Z)) \leq \epsilon_{pr}.$$

Then for all $\mathcal{A}$ of size s it holds that

$$\Pr\left[\mathbf{Exp}^{\mathrm{ad-nmc}}_{\mathcal{F},X,Y,Z,\mathcal{A}} = 1\right] \leq 2^{-\beta} + \epsilon_{nmc} + \epsilon_{pr}.$$

Here $s = \min\{s_{pr} - |\mathtt{Test}| - s_{\mathcal{F},\mathtt{eval}} - s_{\mathcal{F},\mathtt{check}}, s_{nmc}\}$.

The above lemma says that provided an adversary with some information X that may be correlated to the encoded codeword Y is not harmful as long as X is pseudorandom in the presence of Y. Crucially, it must be possible to test if the adversary tampers to an independent codeword. This necessitates the use of the auxiliary distribution Z that is part of the value encoded in Y. (In our construction Z is the output of a seed-dependent condenser applied to val.)

Proof. We begin by defining a standard non-malleable code experiment with a simulator for a function f defined by a distribution $D_f(\cdot)$:

Experiment $\mathbf{Exp}^{\mathrm{sim}}_{f,Z,D_f}$:
Sample $\tilde{s} \leftarrow D_f(\cdot), z \leftarrow Z$
If $\tilde{s} = \mathtt{same}$ output 0.
If $\mathtt{Test}(\tilde{s}, z) = 1$ output 1.
Else output 0.

Lemma 4.1. *Suppose that $\mathrm{H}_\infty(Z) \geq \beta$, for any f, $\Pr[\mathbf{Exp}^{\mathrm{sim}}_{f,Z,D_f}(k) = 1] \leq 2^{-\beta}$.*

Proof of Lemma 4.1. We note that whenever $\tilde{s} = \mathtt{same}$ the output of the experiment is 0. Thus, we can restrict our attention to cases when $\tilde{s} \neq \mathtt{same}$. Then $\Pr[Z = \tilde{s}_{k-\alpha \ldots k}] \leq 2^{-\mathrm{H}_\infty(Z)} = 2^{-\beta}$. This completes the proof of Lemma 4.1. □

We will now argue that the adversary in the adaptive adversary does not perform substantially better than in the simulated experiment. We use a hybrid argument with two intermediate games, $\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}}$ and $\mathbf{Exp}^2_{\mathcal{F},X,Y,Z,\mathcal{A}}$. In moving from $\mathbf{Exp}^{\mathrm{ad-nmc}}_{\mathcal{F},X,Y,Z,\mathcal{A}}$ to $\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}}$ we will replace the distribution X with a random distribution that is uncorrelated to Y, Z. In $\mathbf{Exp}^2_{f,Y,Z}$ we will eliminate the uniform distribution as input and move from the adversary picking a function to defining the experiment for a particular function f. Finally in moving to $\mathbf{Exp}^{\mathrm{sim}}_{f,Z,D_f}(k)$ we will rely on the hardness of non-malleable codes. The two experiments are described formally below.

Experiment $\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}}$:	**Experiment** $\mathbf{Exp}^2_{f,Y,Z}$:
Sample $(y,z) \leftarrow (Y,Z)$	Sample $(y,z) \leftarrow (Y,Z)$
Sample $u \leftarrow \mathcal{M}$.	If $f(y) = y$ output 0.
Sample $f \leftarrow \mathcal{A}(u)$.	Output $\mathtt{Test}(f(y), z)$.
If $f \notin \mathcal{F}$ output 0.	
If $f(y) = y$ output 0.	
Output $\mathtt{Test}(f(y), z)$.	

We now show each of these games are computationally close.

Lemma 4.2. *Suppose that*

1. *For each $f \in \mathcal{F}$, the function f is computable in size at most $s_{\mathcal{F}}$.*
2. *For f it is possible to correctly check $f \in \mathcal{F}$ in size $s_{\mathcal{F},\mathtt{check}}$.*
3. *That $\delta^{\mathcal{D}_{s_{pr}}}((X, Y_{\mathsf{key}}, Z), (U_{\mathcal{M}}, Y_{\mathsf{key}}, Z)) \leq \epsilon_{pr}$.*

Then for $\mathcal{A}$ of size at most $s_{pr} - |\mathtt{Test}| - s_{\mathcal{F},\mathtt{eval}} - s_{\mathcal{F},\mathtt{check}}$,

$$\left|\Pr\left[\mathbf{Exp}^{\mathrm{ad-nmc}}_{\mathcal{F},X,Y,Z,\mathcal{A}}(k) = 1\right] - \Pr[\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}}(k) = 1]\right| \leq \epsilon_{pr}.$$

Proof of Lemma 4.2. Suppose not. That is, suppose that there exists an $\mathcal{A}$ of size at most $s_{pr} - |\mathtt{Test}| - s_{\mathcal{F},\mathtt{eval}} - s_{\mathcal{F},\mathtt{check}}$ such that

$$|\Pr[\mathbf{Exp}^{\mathrm{ad-nmc}}_{\mathcal{F},X,Y,Z,\mathcal{A}} = 1] - \Pr[\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}} = 1]| > \epsilon_{pr}.$$

Then the following program D (of size at most s_{pr}) is a distinguisher for $((X,Y,Z)$ and $(U_{\mathcal{M}},Y,Z))$:

1. On input x, y, z.
2. Run $f \leftarrow \mathcal{A}(x)$.
3. If $f \notin \mathcal{F}$ or $f(y) = y$ output 0.
4. Else output $\mathtt{Test}(f(y), z)$.

That is,

$$\begin{aligned}&|\Pr[D(X,Y,Z) = 1] - \Pr[D(U_{\mathcal{M}},Y,Z) = 1]| \\ &= \left|\Pr[\mathbf{Exp}^{\mathrm{ad-nmc}}_{\mathcal{F},X,Y,Z,\mathcal{A}} = 1] - \Pr[\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}} = 1]\right| > \epsilon_{pr}.\end{aligned}$$

This contradicts the pseudorandomness of $X|(Y,Z)$ and completes the proof of Lemma 4.2. □

Lemma 4.3. *There exists some $f \in \mathcal{F}$ such that for any $\mathcal{A}$ (here $\mathcal{A}$ need not be computationally bounded):*

$$\Pr[\mathbf{Exp}^2_{f,Y,Z}(k) = 1] \geq \Pr[\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}}(k) = 1].$$

Proof of Lemma 4.3. First we consider the circuits $\mathcal{A}$ that always output $f \in \mathcal{F}$. Given any $\mathcal{A}$ that outputs an $f \notin \mathcal{F}$ we can design another $\mathcal{A}'$ that runs $f \leftarrow \mathcal{F}$ and simply outputs a fixed $f' \in \mathcal{F}$ whenever $f \notin \mathcal{F}$. This $\mathcal{A}'$ does not perform worse in $\mathbf{Exp}^1$ than $\mathcal{A}$.

Now consider some $\mathcal{A}$ that always outputs functions $f \in \mathcal{F}$. There is a distribution $D_{\mathcal{A}}$ that outputs exactly the distribution that is output by $\mathcal{A}$. Note that this distribution is independent of y. Note that

$$\Pr[\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}} = 1] = \sum_{f \in \mathcal{F}} \Pr\left[D_{\mathcal{A}} = f\right] \Pr_{(y,z) \leftarrow (Y,Z)}\left[f(y) \neq y \wedge \mathtt{Test}(y,z) = 1\right].$$

Now suppose that for all $f \in \mathcal{F}$,

$$\Pr[\mathbf{Exp}^2_{f,Y,Z}(k) = 1] < \Pr[\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}}(k) = 1].$$

Then one has

$$\begin{aligned}
\Pr[\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}} = 1] &= \sum_{f \in \mathcal{F}} \Pr\left[D_{\mathcal{A}} = f\right] \Pr_{(y,z) \leftarrow (Y,Z)}\left[f(y) \neq y \wedge \mathtt{Test}(y,z) = 1\right] \\
&= \sum_{f \in \mathcal{F}} \Pr\left[D_{\mathcal{A}} = f\right] \Pr[\mathbf{Exp}^2_{f,Y,Z} = 1] \\
&< \sum_{f \in \mathcal{F}} \Pr\left[D_{\mathcal{A}} = f\right] \Pr[\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}} = 1] \\
&= \Pr[\mathbf{Exp}^1_{\mathcal{F},Y,Z,\mathcal{A}} = 1]
\end{aligned}$$

This is a contradiction and completes the proof of Lemma 4.3. □

Before showing distinguishability of the last two games we consider the definition of non-malleable codes where the encoded secret is drawn from a distribution instead of considering a single point:

Lemma 4.4. *Let* $(\mathsf{Enc}, \mathsf{Dec})$ *be a* $(\epsilon_{nmc}, s_{nmc})$*-non-malleable code for functions in* $\mathcal{F}$*. Then for any distribution* Z *over points in* $\{0,1\}^k$ *it holds that*

$$\begin{aligned}
\delta^{\mathcal{D}_{s_{nmc}}}(&(\{c \leftarrow \mathsf{Enc}(Z); \overline{c} \leftarrow f(c), \overline{s} = \mathsf{Dec}(\overline{c}) : \textit{Output } \overline{s}\}, Z), \\
&(\{\tilde{s} \leftarrow D_f, \ \textit{Output } Z \textit{ if } \tilde{s} = \mathtt{same} \textit{ else } \tilde{s}\}, Z)) \\
&\leq \epsilon_{nmc}.
\end{aligned}$$

Proof of Lemma 4.4. Suppose not, that is there exists some Z for which the statement is not true. In particular, there must be some $z \in Z$ where $\Pr[Z = z] > 0$ such that there exists $\mathcal{D}_{s_{nmc}}$,

$$\begin{aligned}
\delta^{\mathcal{D}_{s_{nmc}}}(&(\{c \leftarrow \mathsf{Enc}(z); \overline{c} \leftarrow f(c), \overline{s} = \mathsf{Dec}(\overline{c}) : \text{Output } \overline{s}\}, z), \\
&(\{\tilde{s} \leftarrow D_f, \ \text{Output } z \text{ if } \tilde{s} = \mathtt{same} \text{ else } \tilde{s}\}, z) \\
&> \epsilon_{nmc}.
\end{aligned}$$

This contradicts security of the non-malleable code. □

With this distributional version of security for non-malleable codes we can turn to indistinguishability of the last two games.

Lemma 4.5. *For every $f \in \mathcal{F}$, if $s_{nmc} \geq |Test|$ then*

$$\left|\Pr[\mathbf{Exp}^{\text{sim}}_{f,Z,D_f} = 1] - \Pr[\mathbf{Exp}^{2}_{f,Y,Z} = 1]\right| \leq \epsilon_{nmc}.$$

Proof of Lemma 4.5. Suppose not, that is suppose that there exists some $f \in \mathcal{F}$ such that

$$\left|\Pr[\mathbf{Exp}^{\text{sim}}_{f,Z,D_f}(k) = 1] - \Pr[\mathbf{Exp}^{2}_{f,Y,Z}(k) = 1]\right| > \epsilon_{nmc}.$$

Then we have a distinguisher D (of size $|\texttt{Test}|$) for the distributional version of non-malleable code security guarantee 1) input $\tilde{s}, z$ and 2) Compute $\texttt{Test}(\tilde{s}, z)$. This completes the proof of Lemma 4.5. □

Combining Lemmas 4.1, 4.2, 4.3 and 4.5 completes the proof of Theorem 4.3. □

Application of Theorem 4.3 yields Theorem 4.2. □

5 Encoding multiple key bits in each digital locker

In this section, we transform Construction 3.2 to support encoding multiple bits in each group element. The reason for this extension is to support non-malleable codes where $\mathcal{F}_{nmc}$ allows tampering over nonbinary symbols. An example of this type of non-malleable code is the recent work by Kiayias et al. [KLT18]. We show how to support τ bits in each obfuscation at the cost of running time proportional to 2^τ. This increase in running time is due to exhaustively checking each possible value of the symbol. In addition, it allows a weaker vector DDH assumption at the cost of a stronger power DDH assumption. It thus allows a tradeoff between these two parameters.

Construction 5.1. *Let all variables be as in Construction 3.1, let* $\mathsf{key} \in \{0,1\}^n$ *be some arbitrary value, and let* $\tau = O(\log \lambda)$*. Then define* $\mathsf{lock}(\mathsf{val}, \mathsf{key})$ *as follows: for* $i = 1$ *to* n/τ *compute:*

1. *Sample* $r_i \leftarrow \mathbb{G}_{(6+\tau)\lambda}$.
2. *Set* $z = x + 2$.
3. *Output* $(r_i, (r_i)^{g^{z^{4+\tau}+z^{3+\tau}+z^{2+\tau}+z^{1+\tau}+z^{\tau}+\sum_{i=0}^{\tau-1} \mathsf{key}_{(i \mod \tau)} z^i}})$.

Define $\mathsf{unlock}(\mathsf{val})$ *as follows: for* $i = 1$ *to* n/τ*, input* r_i, y_i *for each* $v_j \in [0, 2^\tau)$ *compute:*

$$P(x, v_j, i) = \left(r_i^{g^{(x+2)^{4+\tau}+(x+2)^{3+\tau}+(x+2)^{2+\tau}+(x+2)^{1+\tau}+(x+2)^{\tau}+\sum_{j=0}^{\tau-1} v_j (x+2)^j}} \stackrel{?}{=} y_i\right).$$

If $P(x, v_j, i)$ *outputs* 1 *then the user sets* $\mathsf{key}_i = v_j$*. Otherwise output* $\perp$.

Nonmalleability over keys This construction can be augmented using a seed-dependent condenser and a non-malleable code in the same method as in Construction 4.1. In theorem below we only consider nonmalleability over the locked val and assume no distribution over key.

Theorem 5.1. *Suppose that*

1. *The $\frac{n}{\tau}$-strong average vector DDH assumption holds,*
2. *The $(5+\tau)t$-strong average power DDH assumption holds,*
3. *The selected prime $p \notin \{2, 31, 41, 73\}$. (As $\mathbb{G}_\lambda$ increases these primes will never be selected. We include this condition as the proof does not apply if $p \in \{2, 31, 41, 73\}$.)*
4. *X is a distribution over $\{0,1\}^{\lambda-1}$ such that $\mathrm{H}_\infty(X) = \omega(\log \lambda)$.*

Then the $(\mathsf{lock}, \mathsf{unlock})$ *in Construction 5.1 is point non-malleable for $\mathcal{F} = \{f | \deg(f) \leq t\}$ (excluding constant polynomials and the identity polynomial).*

Proof of Theorem 5.1. As before we separately consider correctness, soundness, and nonmalleability.

Correctness For correctness first note that

$$(x+1)^{4+\tau} + (x+1)^{3+\tau} + (x+1)^{2+\tau} + (x+1)^{1+\tau} + (x+1)^{\tau} > \sum_{i=0}^{4+\tau} x^i$$

In particular, the binomial expansion of $(x+1)^{4+\tau}$ has nonzero coefficients on every power $i \leq 4+\tau$. This shows that for any $\mathsf{key}, \mathsf{key}' \in \{0,1\}^\tau$ and any $x > x'$ it is true that

$$g^{(x+2)^{4+\tau}+(x+2)^{3+\tau}+(x+2)^{2+\tau}+(x+2)^{1+\tau}+(x+2)^{\tau}+\sum_{i=0}^{\tau-1} \mathsf{key}_i (x+2)^i} \neq$$
$$g^{(x'+2)^{4+\tau}+(x'+2)^{3+\tau}+(x'+2)^{2+\tau}+(x'+2)^{1+\tau}+(x'+2)^{\tau}+\sum_{i=0}^{\tau-1} \mathsf{key}'_i (x'+2)^i}.$$

That is, the value of key cannot cause the obfuscation to unlock on a different point. Furthermore, the function is one-to-one as long as $x \notin \{0,1\}$. These cases are avoiding by computing $x+2$ before computing the polynomial. Note that since $x \in \{0,1\}^{\lambda-1}$ it always holds that $(x+2) \in \{0,1\}^\lambda$.

Privacy The privacy argument for this construction is exactly the same as in Theorem 3.2 since the function

$$f(x, \mathsf{key}_i) = g^{(x+2)^{4+\tau}+x^{3+\tau}+(x+2)^{2+\tau}+(x+2)^{1+\tau}+(x+2)^{\tau}+\sum_{j=0}^{\tau-1} \mathsf{key}_{i,j}(x+2)x^j}$$

is a one-to-one function.

Nonmalleability The analysis for the random case when the mauling function has degree greater than 1 are exactly the same as in Theorem 3.2. We focus on showing that the adversary cannot find a function linear f using linear combinations of the known values. We can think of the adversary being given values of following type:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \mathsf{key}_{0,0} & \dots & \mathsf{key}_{0,\tau-1} \\ 1 & 1 & 1 & 1 & 1 & \mathsf{key}_{0,0} & \dots & \mathsf{key}_{0,\tau-1} \\ \dots & & & & & & & \\ 1 & 1 & 1 & 1 & 1 & \mathsf{key}_{n/\tau,0} & \dots & \mathsf{key}_{n/\tau,\tau-1} \end{bmatrix} \begin{bmatrix} r_{4+\tau} \\ r_{3+\tau} \\ r_{2+\tau} \\ r_{1+\tau} \\ r_\tau \\ \dots \\ r_1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_2 \\ \dots \\ c_{n/tau} \end{bmatrix}$$

Without loss of generality, we assume that an adversary can perfectly set/predict the powers x^j where $j < \tau$. However, to change the obfuscated point they will also need to change the higher order powers. We can think of the adversary having to find α, β, γ such that

$$\sum_{i=0}^{4} (\alpha x + \beta)^{i+\tau} = \gamma \sum_{i=0}^{4} x^{i+\tau}.$$

We can write the desired linear combination as follows:

$$\begin{bmatrix} \alpha^{4+\tau} \\ \alpha^{3+\tau}\left(\binom{\tau+4}{1}\beta + \binom{\tau+3}{0}\right) \\ \alpha^{2+\tau}\left(\binom{\tau+4}{2}\beta^2 + \binom{\tau+3}{1}\beta + \binom{\tau+2}{0}\right) \\ \alpha^{1+\tau}\left(\sum_{i=0}^{3}\left(\binom{\tau+4-i}{3-i}\beta^{3-i}\right)\right) \\ \alpha^{\tau}\left(\sum_{i=0}^{4}\left(\binom{\tau+4-i}{4-i}\beta^{4-i}\right)\right) \end{bmatrix}^{\intercal} \begin{bmatrix} r_{4+\tau} & 0 & 0 & 0 & 0 \\ 0 & r_{3+\tau} & 0 & 0 & 0 \\ 0 & 0 & r_{2+\tau} & 0 & 0 \\ 0 & 0 & 0 & r_{1+\tau} & 0 \\ 0 & 0 & 0 & 0 & r_{\tau} \end{bmatrix} = \gamma \begin{bmatrix} r_{4+\tau} \\ r_{3+\tau} \\ r_{2+\tau} \\ r_{1+\tau} \\ r_{\tau} \end{bmatrix}$$

Substituting one has that

1. If $\beta = 0$ then this implies $\alpha^{\tau+4} = \alpha^{\tau+3} = \alpha^{\tau+2} = \alpha^{\tau+1} = \alpha^{\tau}$ which only has solutions if $\alpha = 0$ or $\alpha = 1$. These are both considered trivial solutions.
2. $\gamma = \alpha^{\tau+4}$ (using first equation),
3. $(\tau + 4)\beta + 1 = \alpha$ (using second equation),
4. $(\tau + 4)\beta = 2$ (using third equation, and relying on $\beta \neq 0$).
5. $\alpha = 3$ (substitution of third constraint into second equation)
6. $\gamma = 81$ (substitution of α in first equation)
7. $\tau = -5$ or $\tau = -114 * 31^{-1}$ (solving fourth equation using prior constraints).

Thus, using the first four equations we are able rule out all solutions unless $\tau = -5$ or $\tau = -114 * 31^{-1}$. Using the last equation we can almost always rule out this second possibility for τ. Namely, the fifth equation (recalling that $\beta = 2/(\tau + 4)$)

$$\binom{\tau+4}{4}\left(\frac{2}{\tau+4}\right)^4 + \binom{\tau+3}{3}\left(\frac{2}{\tau+4}\right)^3 + \binom{\tau+2}{2}\left(\frac{2}{\tau+4}\right)^2 + (\tau+1)\frac{2}{\tau+4} + 1 = 81$$

This solution is only satisfied for $\tau = -114 * 31^{-1}$ if $191552 \equiv 0 \mod p$. Thus, it suffices to choose $p \notin \{2, 31, 41, 73\}$. This allows us to conclude that the adversary's value in the random case is linearly independent, which again leads to this value having entropy λ. Since the adversary only has to match a single value for each index by union bound their overall probability may be as high as $\tau/2^{\lambda}$. Thus, in both random cases the probability of mauling is at most $\chi/2^{(\lambda)}$. We note that since $\tau = O(\log \lambda)$ for large enough λ one can be sure that $\tau \not\equiv -5 \mod \mathbb{G}_{\lambda}$. This allows us to state the distinguishing capability of $\mathcal{A}$:

$$\Pr[\mathcal{A}(\{g^{x^i}\}_{i=1}^{4t}) = 1] - \Pr[\mathcal{A}(\{g^{r_i}\}_{i=1}^{4t}) = 1] \geq \epsilon - \frac{\tau}{2^{\lambda}}.$$

In particular, this breaks the $(5+\tau)t$-strong average power DDH assumption. This is a contradiction and completes the proof of Theorem 5.1. □

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References

[AGM+15a] Shashank Agrawal, Divya Gupta, Hemanta K Maji, Omkant Pandey, and Manoj Prabhakaran. Explicit non-malleable codes against bit-wise tampering and permutations. In *Advances in Cryptology – CRYPTO*, pages 538–557. Springer, 2015.

[AGM+15b] Shashank Agrawal, Divya Gupta, Hemanta K Maji, Omkant Pandey, and Manoj Prabhakaran. A rate-optimizing compiler for non-malleable codes against bit-wise tampering and permutations. In *Theory of Cryptography Conference*, pages 375–397. Springer, 2015.

[AJ15] Prabhanjan Ananth and Abhishek Jain. Indistinguishability obfuscation from compact functional encryption. In *Advances in Cryptology – CRYPTO*, pages 308–326. Springer, 2015.

[BC10] Nir Bitansky and Ran Canetti. On strong simulation and composable point obfuscation. In *Advances in Cryptology–CRYPTO 2010*, pages 520–537. Springer, 2010.

[BCFW09] Alexandra Boldyreva, David Cash, Marc Fischlin, and Bogdan Warinschi. Foundations of non-malleable hash and one-way functions. In *International Conference on the Theory and Application of Cryptology and Information Security*, pages 524–541. Springer, 2009.

[BCM11] Mihir Bellare, David Cash, and Rachel Miller. Cryptography secure against related-key attacks and tampering. In *Advances in Cryptology – ASIACRYPT*, pages 486–503. Springer, 2011.

[BFS11] Paul Baecher, Marc Fischlin, and Dominique Schröder. Expedient non-malleability notions for hash functions. In *Cryptographers Track at the RSA Conference*, pages 268–283. Springer, 2011.

[BGI+01] Boaz Barak, Oded Goldreich, Rusell Impagliazzo, Steven Rudich, Amit Sahai, Salil Vadhan, and Ke Yang. On the (im) possibility of obfuscating programs. In *Advances in Cryptology–CRYPTO*, pages 1–18. Springer, 2001.

[BGI+12] Boaz Barak, Oded Goldreich, Russell Impagliazzo, Steven Rudich, Amit Sahai, Salil Vadhan, and Ke Yang. On the (im) possibility of obfuscating programs. *Journal of the ACM (JACM)*, 59(2):6, 2012.

[BR17] Zvika Brakerski and Guy N Rothblum. Obfuscating conjunctions. *Journal of Cryptology*, 30(1):289–320, 2017.

[Can97] Ran Canetti. Towards realizing random oracles: Hash functions that hide all partial information. In *Advances in Cryptology-CRYPTO'97*, pages 455–469. Springer, 1997.

[CD08] Ran Canetti and Ronny Ramzi Dakdouk. Obfuscating point functions with multibit output. In *Advances in Cryptology–EUROCRYPT 2008*, pages 489–508. Springer, 2008.

[CFP+16] Ran Canetti, Benjamin Fuller, Omer Paneth, Leonid Reyzin, and Adam Smith. Reusable fuzzy extractors for low-entropy distributions. In *Advances in Cryptology–Eurocrypt 2016*, pages 117–146. Springer, 2016.

[CL17] Eshan Chattopadhyay and Xin Li. Non-malleable codes and extractors for small-depth circuits, and affine functions. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1171–1184. ACM, 2017.

[CQZ+16] Yu Chen, Baodong Qin, Jiang Zhang, Yi Deng, and Sherman SM Chow. Non-malleable functions and their applications. In *IACR International Workshop on Public Key Cryptography*, pages 386–416. Springer, 2016.

[CRS14] Gil Cohen, Ran Raz, and Gil Segev. Nonmalleable extractors with short seeds and applications to privacy amplification. *SIAM Journal on Computing*, 43(2):450–476, 2014.

[CRV10] Ran Canetti, Guy N Rothblum, and Mayank Varia. Obfuscation of hyperplane membership. In *Theory of Cryptography Conference*, pages 72–89. Springer, 2010.

[CV09] Ran Canetti and Mayank Varia. Non-malleable obfuscation. In Omer Reingold, editor, *Theory of Cryptography*, pages 73–90, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg.

[DH76] Whitfield Diffie and Martin Hellman. New directions in cryptography. *IEEE transactions on Information Theory*, 22(6):644–654, 1976.

[DORS08] Yevgeniy Dodis, Rafail Ostrovsky, Leonid Reyzin, and Adam Smith. Fuzzy extractors: How to generate strong keys from biometrics and other noisy data. *SIAM Journal on Computing*, 38(1):97–139, 2008.

[DPW10] Stefan Dziembowski, Krzysztof Pietrzak, and Daniel Wichs. Non-malleable codes. In *ICS*, volume 2010, page 1st. Citeseer, 2010.

[DRV12] Yevgeniy Dodis, Thomas Ristenpart, and Salil Vadhan. Randomness condensers for efficiently samplable, seed-dependent sources. In *Theory of Cryptography Conference*, pages 618–635. Springer, 2012.

[DW09] Yevgeniy Dodis and Daniel Wichs. Non-malleable extractors and symmetric key cryptography from weak secrets. In *Proceedings of the forty-first annual ACM symposium on Theory of computing*, pages 601–610. ACM, 2009.

[GGH+13] Sanjam Garg, Craig Gentry, Shai Halevi, Mariana Raykova, Amit Sahai, and Brent Waters. Candidate indistinguishability obfuscation and functional encryption for all circuits. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 40–49. IEEE, 2013.

[GGH+16] Sanjam Garg, Craig Gentry, Shai Halevi, Mariana Raykova, Amit Sahai, and Brent Waters. Candidate indistinguishability obfuscation and functional encryption for all circuits. *SIAM Journal on Computing*, 45(3):882–929, 2016.

[GJM02] Philippe Golle, Stanislaw Jarecki, and Ilya Mironov. Cryptographic primitives enforcing communication and storage complexity. In *International Conference on Financial Cryptography*, pages 120–135. Springer, 2002.

[GLSW15] Craig Gentry, Allison Bishop Lewko, Amit Sahai, and Brent Waters. Indistinguishability obfuscation from the multilinear subgroup elimination assumption. In *2015 IEEE 56th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 151–170. IEEE, 2015.

[KLT18] Aggelos Kiayias, Feng-Hao Liu, and Yiannis Tselekounis. Non-malleable codes for partial functions with manipulation detection. In *Advances in Cryptology – CRYPTO*, 2018.

[KY18] Ilan Komargodski and Eylon Yogev. Another step towards realizing random oracles: Non-malleable point obfuscation. In *Advances in Cryptology – EUROCRYPT*, pages 259–279. Springer, 2018.

[PST14] Rafael Pass, Karn Seth, and Sidharth Telang. Indistinguishability obfuscation from semantically-secure multilinear encodings. In *Advances in Cryptology – CRYPTO*, pages 500–517. Springer, 2014.

[SW14] Amit Sahai and Brent Waters. How to use indistinguishability obfuscation: deniable encryption, and more. In *Proceedings of the forty-sixth annual ACM symposium on Theory of computing*, pages 475–484. ACM, 2014.

[WZ17] Daniel Wichs and Giorgos Zirdelis. Obfuscating compute-and-compare programs under lwe. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 600–611. IEEE, 2017.

A Analysis of One Time Security

Proof of Theorem 3.1. We separately argue completeness, soundness, and nonmalleability.

Completeness For completeness, it suffices to show that the pair $(r, r^{g^{h(2x+b)}})$ is unique for each choice of x and b is unique for each pair x, b. First note that $2x + b$ is injective in both variables. Then $h(x) = x^4 + x^3 + x^2 + x$ is injective, thus the value $h(2x + b)$ will also be unique for each x, b. Let p be the prime corresponding to the chosen group $\mathbb{G}_{5\lambda}$. Then, it holds that $h(2x + b) \leq 2^{5\lambda}$ and thus, the value of $g^{h(2x+b)}$ is one-to-one. This ensures the pair $(r, r^{g^{h(2x+b)}})$ is unique for unique x, b. Completeness immediately follows.

Soundness Note no distribution is assumed on the bit b. Fix some $b \in \{0, 1\}$. Let $\mathcal{Z}_\lambda$ be the set of distributions Z over $\{0, 1\}^\lambda$ where $\mathrm{H}_\infty(Z) = \omega(\log \lambda)$ and similarly define the set of distributions $\mathcal{X}_{\lambda-1}$ as the set of distributions X over $\{0, 1\}^{\lambda-1}$ where $\mathrm{H}_\infty(X) = \omega(\log \lambda)$. Lastly, define the set of distributions $\mathcal{Y} = \{2X + b | X \in \mathcal{X}_{\lambda-1}\}$ where we understand $2X + b$ to be a distribution created by sampling $x \leftarrow X$ and computing $2x + b$. Then $\mathcal{Y} \subseteq \mathcal{Z}_\lambda$. That is, for each distribution that lock is intended to secure it is contained in the set of distributions that $\mathsf{lockPoint}'$ is intended to secure. Similarly, let $Z \in \mathcal{Z}_\lambda$ and define the distribution X using the probability density function $\Pr[X = x] = \Pr[Z = (2x+0)] + \Pr[Z = (2x+1)]$. Then $\mathrm{H}_\infty(X) \geq \omega(\log \lambda)$, thus $X \in \mathcal{X}_{\lambda-1}$.

We show soundness by contradiction. First assume that the construction described $\mathsf{lockPoint}(x) = (r, r^{g^{h(x)}})$ is sound. That is, for all $c \in \mathbb{Z}^+$ there exists some $\lambda_{p,c}$ such that for $\lambda > \lambda_{p,c}$ for all PPT $\mathcal{A}_p$, there exists a simulator $\mathcal{S}_p$ such that:

$$\left| \Pr_{x \in \{0,1\}^\lambda} [\mathcal{A}_p(\mathsf{lockPoint}(x)) = 1] - \Pr_{x \in \{0,1\}^\lambda} [\mathcal{S}_p^{I_x}(1^\lambda) = 1] \right| \leq \frac{1}{\lambda^c}.$$

input : Oracle access to $I_{x,b}$
output: $\mathcal{P}(x,b)$

Initialize $\mathcal{S}_p$.
Receive query y from $\mathcal{S}_p$, send $x_{0,\ldots\lambda-2}$ to $I_{x,b}$.
If response b, check if $x_{\lambda-1} = b$, if so return 1.
Otherwise **return** $\perp$ to S'.
Output S_{dl}s output.

Algorithm 2: Construction of $\mathcal{S}_{dl}$ from $\mathcal{S}_p$

In addition, suppose our construction is insecure, that is, there exists some $c \in \mathbb{Z}^+$ such that for all $\lambda_{c,dl}$ there exists a $\lambda > \lambda_{c,dl}$ such that there exists a PPT $\mathcal{A}_{dl}$ where for all S_{dl} using a polynomial number of queries

$$\left| \Pr_{x\in\{0,1\}^{\lambda^*}}[\mathcal{A}_{dl}(\mathsf{lock}(x,b)) = 1] - \Pr_{x\in\{0,1\}^{\lambda}}[\mathcal{S}_{dl}^{I_{x,b}}(1^{\lambda^*}) = 1] \right| > \frac{1}{\lambda^c}.$$

For a fixed $c \in \mathbb{Z}^+$, denote by $\lambda_{\max,c} = \max\{\lambda_{p,c}, \lambda_{dl,c}\}$. There must exist some $\lambda > \lambda_{c,max}$ where the distance between $\mathcal{A}_p$ and S_p is less than $1/\lambda^c$ while there exists some $\mathcal{A}_{dl}$ such that for all $\mathcal{S}_{dl}$ the distance between $\mathcal{A}_{dl}$ and $\mathcal{S}_{dl}$ is greater than $1/\lambda^c$. Denote by $\mathcal{A}_{dl}^*$ one such adversary, that is for $\mathcal{A}_{dl}^*$ and all S_{dl} making a polynomial number of queries their statistical distance is at least $1/\lambda^c$. With the goal of arriving at a contradiction we use $\mathcal{A}_{dl}^*$ to construct an adversary $\mathcal{A}_p^*$ for lock that cannot be effectively simulated.

This adversary $\mathcal{A}_p^*$ works as follows, on input $\mathsf{lockPoint}(x)$ where $x \in \{0,1\}^\lambda$, $\mathcal{A}_p^*$ initializes $\mathcal{A}_{dl}^*$ with input $\mathsf{lockPoint}(x)$. Note this is equivalent to initializing $\mathcal{A}_{dl}^*$ with input $\mathsf{lock}(x'||b)$ for $x = x'||b$. Any predicate $\mathcal{P}(x'||b) = \mathcal{P}(x)$ output by $\mathcal{A}_{dl}^*$ is a valid predicate on on x. So, this predicate can be immediately output.

Thus, $\mathcal{A}_{dl}^*$ implies the existence of an $\mathcal{A}_p^*$ that outputs 1 with the same probability, and in particular they succeed with the same probability because we have exactly produced the probability distribution that $\mathcal{A}_{dl}^*$ is expecting. That is,

$$\Pr[\mathcal{A}_p^*[(\mathsf{lockPoint}(x)) = 1] = \Pr[\mathcal{A}_{dl}^*(\mathsf{lock}(x',b)) = 1].$$

By assumption for each $\mathcal{A}_p$ there exists $\mathcal{S}_p$ such that for $\lambda > \lambda_{\max,c}$,

$$\left| \Pr_{x\in\mathcal{Z}^\lambda}[\mathcal{A}_p(\mathsf{lockPoint}(x)) = \mathcal{P}(x)] - \Pr_{x\in\mathcal{Z}^\lambda}[\mathcal{S}_p^{I_x(\cdot)}(1^\lambda) = \mathcal{P}(x)] \right| \leq \frac{1}{\lambda^c}.$$

This $\mathcal{S}_p$ implies the existence of an $\mathcal{S}_{dl}$ that computes any predicate $\mathcal{P}$ on (x',b). $\mathcal{S}_{dl}$ initializes S_p and for every query from S_p it drops the last bit and asks its oracle the query. It only returns a match to S_p if the returned bit matches the last bit of S_p's query. A formal description is in Algorithm 2.

So, the existence of $\mathcal{S}_p$ directly leads to the simulator $\mathcal{S}_{dl}$ that computes a 1 with the same probability. That is,

$$\Pr[\mathcal{S}_p^{I_x}(1^\lambda) = 1] = \Pr[\mathcal{S}_{dl}^{I_{x,b}}(1^\lambda) = 1].$$

These three equations allow us to state:

$$\left| \Pr_{x \leftarrow X}[\mathcal{A}^*_{dl}(\mathsf{lock}(x,b)) = 1] - \Pr_{x \leftarrow X}[\mathcal{S}^{I_{x,b}}_{dl}(1^{\lambda^*}) = 1] \right| \leq \frac{1}{\lambda^c}.$$

This is a contradiction and completes the argument of soundness.

Nonmalleability We proceed by contradiction assuming that there exists some ensemble X'_λ where $\mathrm{H}_\infty(X_\lambda) = \omega(\log \lambda)$, some $\mathcal{A}_{dl}$, and some bit $b \in \{0,1\}$ such that there exists some c where

$$\Pr_{x' \leftarrow X'_\lambda} \left[\begin{array}{c} (\mathsf{lock}', f) \leftarrow \mathcal{A}_{dl}(\mathsf{lock}(x', b)) \\ \mathcal{V}(\mathsf{lock}') = 1, f \in \mathcal{F}, (I_{f(x'),0} \equiv \mathsf{lock}' \vee I_{f(x'),1} \equiv \mathsf{lock}')| \end{array} \right] > 1/\lambda^c$$

We build an mauling adversary for the original obfuscation $\mathsf{lockPoint}$. We consider the ensemble of distributions $Z_\lambda = X_\lambda || b$, that is the distribution of X_λ with the bit b appended. Note that this is a valid ensemble for the obfuscator $\mathsf{lockPoint}$ In showing non-malleability, we are able to inherit the non-malleability guarantees of the underlying obfuscation, with a slight adjustment for bounds. This follows both for polynomial functions of x as well as for bit flipping on b. We design $\mathcal{A}_p$ as follows:

1. Receive $\mathsf{lockPoint}(x)$ as input (equivalently receive $\mathsf{lock}(2x' + b)$).
2. Initialize $\mathcal{A}_{dl}$ with $\mathsf{lockPoint}(x)$.
3. Receive lock', f.
4. Set $f_0(x) = 2^{-1}(x - b)$ and $f_2(x) = 2 * (x) + b$. Compute $f_3 = f_2 \circ f \circ f_0$.
5. Flip $r \xleftarrow{\$} \{0,1\}$. If $r = 0$ set $f' = f_3 - b$. Else set $f' = f_3 - b + 1$.
6. Output lock', f'.

There are two pieces to how $\mathcal{A}_p$ is using $\mathcal{A}_{dl}$. First, $\mathcal{A}_p$ is trying to produce a tampering function on $x \in \{0,1\}^\lambda$, while $\mathcal{A}^*$ is tampering on x' contained in $x'||b$. Consider the example when $f(x) = 3x + 1$ is output by the adversary. This is akin to the adversary producing a valid obfuscation of $6x + 2 + 1$ or $6x + 2$. The two functions f_0 and f_2 are for this difference. The second main difference is that the $\mathcal{A}_{dl}$ does not know if the last bit of x will be 0 or 1 so they output each possibility with probability $1/2$. Together these facts allow us to conclude:

$$\begin{aligned}
&\Pr_{x \leftarrow Z_\lambda} \left[\begin{array}{c} (\mathsf{lockPoint}', f) \leftarrow \mathcal{A}_p(\mathsf{lockPoint}(x)) \\ \mathcal{V}(\mathsf{lockPoint}') = 1, f \in \mathcal{F}, (I_{f(x)} \equiv \mathsf{lockPoint}') \end{array} \right] \\
&\quad = \Pr[R = r] \Pr_{x' \leftarrow X_\lambda} \left[\begin{array}{c} (\mathsf{lockPoint}', f) \leftarrow \mathcal{A}_p(\mathsf{lockPoint}(x')) \\ \mathcal{V}(\mathsf{lockPoint}') = 1, f \in \mathcal{F}, (I_{f(x'),r} \equiv \mathsf{lockPoint}) \end{array} \right] \\
&\quad = \frac{1}{2} \Pr_{x' \leftarrow X_\lambda} \left[\begin{array}{c} (\mathsf{lock}', f) \leftarrow \mathcal{A}_{dl}(\mathsf{lock}(x', b) \\ \mathcal{V}(\mathsf{lock}') = 1, f \in \mathcal{F}, (I_{f(x'),0} \equiv \mathsf{lock}' \vee I_{f(x'),1} \equiv \mathsf{lock}') \end{array} \right] \\
&\quad > \frac{1}{2\lambda^c}.
\end{aligned}$$

This contradicts the nonmalleability of $\mathsf{lockPoint}$ and completes the proof of nonmalleability. This completes the proof of Theorem 3.1. □