Efficient Inversion In (Pseudo-)Mersenne Prime Order Fields

Kaushik Nath and Palash Sarkar

Applied Statistics Unit
Indian Statistical Institute
203, B. T. Road
Kolkata
India 700108
email:{kaushikn_r, palash}@isical.ac.in

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Abstract

Efficient scalar multiplication algorithms require a single finite field inversion at the end to convert from projective to affine coordinates. This inversion consumes a significant proportion of the total time. The present work makes a comprehensive study of inversion over Mersenne and pseudo-Mersenne prime order fields. Inversion algorithms for such primes are based on exponentiation which in turn requires efficient algorithms for multiplication, squaring and modulo reduction. From a theoretical point of view, we present a number of algorithms for multiplication/squaring and reduction leading to a number of different inversion algorithms which are appropriate for different settings. Our algorithms collect together and generalise ideas which are scattered across various papers and codes. At the same time, they also introduce new ideas to improve upon existing works. A key theoretical feature of our work, which is not present in previous works, is that we provide formal statements and detailed proofs of correctness of the different reduction algorithms that we describe. On the implementation aspect, a total of twenty primes are considered, covering all previously proposed cryptographically relevant (pseudo-)Mersenne prime order fields at various security levels. For each of these fields, we provide 64-bit assembly implementations of all the relevant inversion algorithms for a wide range of Intel processors. We were able to find previous 64-bit implementations of inversion for six of the twenty primes considered in this work. On the Haswell, Skylake and Kabylake processors of Intel, for all the six primes where previous implementations are available, our implementations outperform such previous implementations. The assembly codes that we have developed are publicly available and can be used as a plug-in to replace the inversion routines in existing softwares for scalar multiplication.

Keywords: finite field, inversion, multiplication, reduction, elliptic curve cryptography, scalar multiplication.

1 Introduction

Elliptic curve cryptography was independently introduced by Koblitz [17] and Miller [20], and later cryptography based on hyper-elliptic curves was introduced by Koblitz [18]. Over the last three decades, there has been a tremendous amount of research on various aspects of secure and efficient curve based cryptography.

Presently, there are several different approaches for choosing a suitable curve. The basic task for all such approaches is to choose the underlying finite field. This field can either have composite or prime order. Further, for a prime order field, the prime may be a Mersenne or a pseudo-Mersenne prime. Some well known examples are the Mersenne prime $2^{127} - 1$ used in [13, 4, 7, 10], and the pseudo-Mersenne primes $2^{255} - 19$ and $2^{256} - 2^{32} - 977$ [25] used in Curve25519 [2] and the Bitcoin [21] protocol respectively. The NIST proposals [11], on the other hand, provide examples of more unstructured primes and efficient implementation of arithmetic using such primes have been studied [15]. In this work, we will focus only on Mersenne and pseudo-Mersenne prime order fields.

Two of the basic applications of curve-based cryptography are key agreement and signature schemes. Both of these schemes require scalar multiplications. The scalar multiplications are computed using projective coordinates and at the end a single inversion over the underlying finite field is required to convert to affine coordinates. For example, on the Skylake processor of Intel, the existing sandy2x [9] software for Curve25519 [2] requires a total of 141374 cycles, out of which the inversion requires 13223 cycles; the existing software for the signing algorithm of Ed25519 [5] requires a total of 50224 cycles for signing a 64-byte message, out of which the inversion requires 13901 cycles. In the former case, inversion requires about 9% of the time while in the later case, inversion requires about 27% of the time. These figures illustrate the importance of finite field inversion in curve based cryptography.

Our Contributions

The goal of this work is to carry out a comprehensive study of inversion over fields whose order is either a Mersenne or a pseudo-Mersenne prime. There are two aspects of our contributions.

Survey contribution: The ideas behind efficient inversion are scattered across a number of papers. In some cases, the ideas are present in the code accompanying a paper rather than the paper itself. Examples are the papers [2, 6, 5, 8, 9, 12, 22] and associated codes devoted to the pseudo-Mersenne prime $2^{255} - 19$; and in papers [4, 10, 7] and associated codes devoted to the Mersenne prime $2^{127} - 1$. The present work can be viewed as describing previous ideas in a single unified framework. The survey value of the work arises from providing researchers with a single point of reference on algorithms for inversion over (pseudo-)Mersenne prime order fields.

While some of the algorithms that we present generalise existing ideas, we also introduce new ideas leading to new algorithms which are not available in the literature. So, the importance of the work goes beyond that of a survey.

Research contribution: Apart from the survey contribution, this paper makes a number of research contributions of both theoretical and practical nature. To describe these contributions, we first provide some context.

An inversion algorithm for a cryptographically relevant (pseudo-)Mersenne prime p essentially computes the map $x \mapsto x^{p-2} \bmod p$. The implementation of this computation requires multiplication and squaring over the field $\mathbb{F}_p$. Such field multiplication and squaring have two broad phases, namely, a multiplication phase and a reduction phase. Of the two, the reduction phase is relatively more complex.

Let the prime $p = 2^m - \delta$. Elements of $\mathbb{F}_p$ fit within an m-bit string. Such an m-bit string is formatted into κ binary strings where the first $\kappa - 1$ strings are each η bits long and the last string is ν bits long with $0 < \nu \leq \eta$. Following the usual convention, we call each of the individual κ binary strings to be limbs. For 64-bit arithmetic, each limb fits into a 64-bit word. Two kinds of representations have been considered in the literature. In the first kind of representation, $\eta = 64$, and so the limbs (except possibly for the last one) are packed tightly into 64-bit words. In the second kind of representation, $\eta < 64$, and so the 64-bit words containing the limbs have some free or redundant bits. We call the first kind of representation to be *saturated limb representation* and the second kind to be *unsaturated limb representation*.

The rationale for using the unsaturated limb representation is that the redundancies make it possible to avoid overflow during computations. This approach has been extensively used in the context of Curve25519 implementations [2, 6, 5, 9]. It is possible to handle overflows even with the saturated limb representation, but, this can become costly. New generations of Intel processors (from the Broadwell processor onwards) provide a set of instructions which make it possible to very efficiently handle overflow issues arising in the saturated limb representation. Two white papers [24, 23] describe integer multiplication and squaring algorithms for 512-bit integers.

Theoretical contributions: We provide various algorithms for multiplication/squaring and reduction using both the saturated and the unsaturated limb representations. A brief summary of these contributions is as follows.

Multiplication/squaring for saturated limb representation: We describe two sets of algorithms with each set consisting of an algorithm for multiplication and one for squaring. The first set of algorithms (which we call mulSLDCC/sqrSLDCC) generalises the multiplication/squaring algorithms in the Intel white papers [24, 23] to work for $64i$-bit integers for any $i \geq 2$. These algorithms use two independent carry chains and can be implemented in the newer generation of processors. The second set of algorithms (which we call mulSLa/sqrSLa) do not use double carry chains and can be implemented across all generation of processors. These algorithms combine an initial step of the reduction with the multiplication. The idea behind mulSLa/sqrSLa have not appeared earlier in the literature.

Multiplication/squaring for unsaturated limb representation: Again, we describe two sets of algorithms. The first set of algorithms (which we call mulUSL/sqrUSL) generalise the ideas used in [5] for the prime $2^{255} - 19$. These algorithms, however, lead to overflow for certain primes such as the prime $2^{256} - 2^{32} - 977$. To handle such overflow issues, we describe a second set of algorithms (which we call mulUSLa/sqrUSLa) which have not appeared earlier in the literature.

Reduction for saturated limb representation: We describe four reduction algorithms, namely, reduceSLMP, reduceSLPMP, reduceSLPMPa and reduceSL. Algorithms reduceSLMP, reduceSLPMP and reduceSLPMPa reduce the outputs of mulSLDCC/sqrSLDCC. Specifically, reduceSLMP works for all Mersenne primes and is a generalisation of the ideas used in [4] for the prime $2^{127} - 1$. Algorithm reduceSLPMP works for a large class of pseudo-Mersenne primes and has not appeared earlier. Algorithm reduceSLPMPa works for a large class of pseudo-Mersenne primes and is a generalisation of the ideas for 4-limb representation used in [5] for the prime $2^{255} - 19$. Algorithm reduceSL reduces the output mulSLa/sqrSLa and has not appeared earlier in the literature.

Reduction for unsaturated limb representation: Again we describe three reduction algorithms, namely, reduceUSL, reduceUSLA and reduceUSLB. Algorithm reduceUSL works for a large class of pseudo-Mersenne primes and is a generalisation of the ideas for 5-limb representation used in [9][1] for the prime $2^{255} - 19$. For certain primes, reduceUSLA is more efficient than reduceUSL and generalises ideas used in [5] for the prime $2^{255} - 19$. For certain other primes, reduceUSLB is more efficient than both reduceUSL and reduceUSLA. The idea behind reduceUSLB has not appeared earlier in the literature.

In Table 1, for each algorithm presented in this work, we state whether it is new or, the earlier work that it generalises.

There are two key theoretical features of our work. First, while previous works have developed code for a single prime, we describe the algorithms in their full generality. Second, for each reduction

[1]See also https://github.com/floodyberry/supercop/tree/master/crypto_scalarmult/curve25519/amd64-51.

algorithm, we state precise theorems about their correctness and provide detailed proofs of correctness. Such formal treatment of correctness of reduction algorithms do not appear earlier in the literature.

Practical contributions: The second aspect of our research contribution is in the implementation of the various inversion algorithms. The various algorithms for multiplication/squaring and reduction have been combined to obtain a number of algorithms for inversion. All the algorithms described in this paper, have been implemented in assembly for Intel processors. The implementations are divided into two groups, namely maa and maax. For implementations in the maa group, the only arithmetic instructions used are `mul, imul, add` and `adc`, while for implementations in the maax group, the arithmetic instructions `mulx, adcx` and `adox` are also used. These second set of instructions are available from the Broadwell processor onwards.

algorithm	feature
mulSLDCC/sqrSLDCC	generalises [24, 23]
mulSLa/sqrSLa	new
mulUSL/sqrUSL	generalises [5]
mulUSLa/sqrUSLa	new
reduceSLMP	generalises [4]
reduceSLPMP	new
reduceSLPMPa	generalises [5, 4-limb]
reduceSL	new
reduceUSL	generalises [9, 5-limb]
reduceUSLA	generalises [5, 5-limb]
reduceUSLB	new

Table 1: The various algorithms for multiplication/squaring and reduction described in this paper.

In this work we have considered a total of twenty primes which include all the cryptographically relevant (pseudo-)Mersenne primes at various security levels. These primes are shown in Table 2. For the prime $2^{255} - 19$, we have found earlier implementations of both maa and maax types and for five of the other primes, we have found implementations of maa type. So, for fourteen of the twenty primes, we provide the first maa type implementation and for nineteen of the twenty primes, we provide the first maax type implementations.

Timings for the new implementations and the existing implementations have been measured on the Haswell, Skylake and Kabylake processors. For each prime where a previous implementation is available, our implementation improves upon such previous implementations. This holds for both maa and maax type implementations. We highlight the improvements obtained for three well known primes.

1. For the prime $2^{127} - 1$, previous implementation of only maa type is available. The speed-up percentage is about 10% on all three processors.

2. For the prime $2^{256} - 2^{32} - 977$, previous implementation of only maa type is available. The speed-up percentage is about 36% on Haswell and about 28% on the Skylake and Kabylake processors.

3. For the prime $2^{255} - 19$, compared to previous maa type implementation, the speed-up percentage is about 4% on all three processors. Compared to previous maax type implementation, the speed-up percentage is about 23% on the Skylake and the Kabylake processors.

Source codes for all our implementations are publicly available at the following link.

https://github.com/kn-cs/pmp-inv.

Structure of the Paper

We start by providing a brief summary of the relevant Intel instructions in Section 2. The background on representation of elements of $\mathbb{F}_p$ and the exponentiation based inversion algorithm are covered in Section 3. An overview of the various algorithms presented in this paper is provided in Section 4. Algorithms for integer multiplication and squaring using the saturated limb representation and using double independent carry chains are given in Section 5. Corresponding reduction algorithms along with formal statements of correctness and proofs of security are presented in Section 6. Multiplication using unsaturated limb representation is considered in Section 7 and the corresponding reduction algorithms are described in Section 8. Multiplication and reduction using saturated limb representation without using double carry chains are described in Section 9. Detailed timing results and their consequences are presented in Section 10. Finally, Section 11 concludes the paper.

prime	curve(s)
$2^{127} - 1$	Kummer_2 [4], Four$\mathbb{Q}$ [10]
$2^{221} - 3$	M-221 [1]
$2^{222} - 117$	E-222 [1]
$2^{251} - 9$	Curve1174 [1], KL2519(81,20) [16]
$2^{255} - 19$	Curve25519 [2], KL25519(82,77) [16]
$2^{256} - 2^{32} - 977$	secp256k1 [25]
$2^{266} - 3$	KL2663(260,139) [16]
$2^{382} - 105$	E-382 [1]
$2^{383} - 187$	M-383 [1]
$2^{414} - 17$	Curve41417 [3]
$2^{511} - 187$	M-511 [1]
$2^{512} - 569$	-
$2^{521} - 1$	P-521 [1], E-521 [1]
$2^{607} - 1$	-
$2^{751} - 165$	-
$2^{832} - 143$	-
$2^{896} - 213$	-
$2^{960} - 167$	-
$2^{1024} - 105$	-
$2^{1088} - 89$	-

Table 2: The primes considered in this work.

2 A Brief Summary of Relevant Intel Instructions

The 64-bit architecture of the Intel x86 processors has sixteen 64-bit registers, namely rax, rbx, rcx, rdx, rsi, rdi, rsp, rbp, r8, r9, r10, r11, r12, r13, r14, r15. Except rsp (which is the stack pointer), all other registers can be used for storing data and operating on them. There is a register named FLAGS, which consists of various available flags. We note two of these flags. Bit 0 of FLAGS is the carry flag CF and bit 11 of FLAGS is the overflow flag OF. Integer addition and multiplication affect the states of these two flags and are relevant to our work.

The basic 64-bit arithmetic operations in the x86 processors are `mul`, `imul`, `add` and `adc`. From the Broadwell processor onwards, Intel also provides another set of arithmetic instructions, namely, `mulx`, `adcx` and `adox`. The structure of multiplication and addition instructions and their operations are as follows.

`mul` src_2;	$\mathsf{rdx} : \mathsf{rax} \leftarrow \mathsf{src}_2 \cdot \mathsf{rax}$.
`imul` src_1, src_2, dst;	$\mathsf{dst} \leftarrow \mathsf{lsb}_{64}(\mathsf{src}_1 \cdot \mathsf{src}_2)$.
`add` src, dst;	$\mathsf{dst} \leftarrow \mathsf{src} + \mathsf{dst}$.
`adc` src, dst;	$\mathsf{dst} \leftarrow \mathsf{src} + \mathsf{dst} + \mathsf{CF}$.
`mulx` src_1, dst_ℓ, dst_h;	$\mathsf{dst}_h : \mathsf{dst}_\ell \leftarrow \mathsf{src}_1 \cdot \mathsf{rax}$.
`adcx` src, dst;	$\mathsf{dst} \leftarrow \mathsf{src} + \mathsf{dst} + \mathsf{CF}$.
`adox` src, dst;	$\mathsf{dst} \leftarrow \mathsf{src} + \mathsf{dst} + \mathsf{OF}$.

The operation `mulx` is available from the Haswell processor onwards; `adcx` and `adox` are available from the Broadwell processor onwards. Processors previous to Haswell had only `mul`, `imul`, `add` and `adc`.

The effect on the carry and the overflow flags for the above mentioned arithmetic operations are the following.

- `mul, imul, add` and `adc` affect *both* CF and OF;
- `mulx` affects neither CF nor OF;
- `adcx` affects only CF but, not OF;
- `adox` affects only OF but, not CF.

Suppose there is an interleaved sequence of multiplications and additions to be performed. The additions generate carries which need to be taken into consideration for subsequent additions. The `mul` and `imul` instructions affect the carry flag and so the carry out of the previous addition gets lost. On the other hand, a sequence of `mulx` and `adc` instructions can efficiently perform such an interleaved sequence of multiplications and additions. The `mulx` instruction does not affect the carry flag and so the sequence of `adc` instructions can carry out the instructions using a *single carry chain.*

The combination of `mulx`, `adcx` and `adox` provides a more powerful tool. As mentioned above, the `mulx` instruction does not affect either CF or OF. A sequence of `adcx` instructions proceeds by using a carry chain using only CF, while a sequence of `adox` instructions proceeds by using a carry chain using only OF. So, in effect, it is possible to use two *independent* carry chains which we call *double carry chain.* This greatly facilitates arithmetic computations as we will see later.

Remark: Let x be an ℓ-bit number and $\eta \leq \ell$. The operation $x \bmod 2^\eta$ returns $\mathsf{lsb}_\eta(x)$, i.e., the η least significant bits of x, whereas the operation $\lfloor x/2^\eta \rfloor$ returns the $\ell - \eta$ most significant bits of x. It will be helpful to keep this simple observation in mind while going through the various algorithms given later.

3 Representation of Elements and Inversion in $\mathbb{F}_p$

Let η be a positive integer and $\theta = 2^\eta$. Consider a polynomial

$$h(\theta) \quad = \quad h_0 + h_1\theta + \cdots + h_{k-1}\theta^{k-1} \tag{1}$$

where $h_0, h_1, \ldots, h_{k-1}$ are non-negative integers. The polynomial $h(\theta)$ is given by the vector of coefficients $(h_0, h_1, \ldots, h_{k-1})$. We will call these coefficients to be the limbs of the polynomial. Note that we do not insist that the coefficients are less than 2^η; in fact, at intermediate steps, the coefficients will not necessarily be less than 2^η.

Given positive integers m and η, let κ and ν be positive integers such that

$$m \quad = \quad \eta(\kappa - 1) + \nu \quad \text{with } 0 < \nu \leq \eta. \tag{2}$$

Given m and η, the values of κ and ν are uniquely determined. An integer in the range $[0, \ldots, 2^m - 1]$ can be represented by an m-bit string. From (2), an m-bit string can be considered to be the concatenation of κ strings, where the first $\kappa - 1$ strings are each η bits long while the last string is ν bits long. So, an m-bit integer can be represented as $h(\theta) = h_0 + h_1\theta + \cdots + h_{\kappa-1}\theta^{\kappa-1}$ where $0 \leq h_0, h_1, \ldots, h_{\kappa-1} < 2^\eta$ and $0 \leq h_{\kappa-1} < 2^\nu$. As mentioned above, each of the κ individual strings will be referred to as a limb. *Given an m-bit integer, by a (κ, η, ν)-representation we will mean a κ-limb representation, where the first $\kappa - 1$ limbs are η bits long, the last limb is ν bits long and $m = \eta(\kappa - 1) + \nu$.*

Proposition 1. *Let x and y be two m-bit integers both having a (κ, η, ν)-representation and let $z = x \cdot y$. Then z has a (κ', η, ν')-representation where*

$$\kappa' = 2\kappa - 1,\ \nu' = 2\nu \text{ if } 0 < \nu \leq \eta/2; \quad \text{and} \quad \kappa' = 2\kappa,\ \nu' = 2\nu - \eta \text{ if } \eta/2 < \nu \leq \eta.$$

Proof. We have $m = \eta(\kappa - 1) + \nu$. The number of bits in z is at most $2m$ and we may write $2m = \eta(2\kappa - 2) + 2\nu$. If $0 < \nu \leq \eta/2$, then z has a $(2\kappa - 1)$-limb representation where the first $2\kappa - 2$ limbs are each η bits long and the last limb is $\nu' = 2\nu$ bits long. On the other hand, if $\eta/2 < \nu \leq \eta$, then we may write $2m = \eta(2\kappa - 1) + 2\nu - \eta$ and so z has a 2κ-limb representation where the first $2\kappa - 1$ limbs are each η bits long and the last limb is $2\nu - \eta$ bits long. (Note that $\eta/2 < \nu \leq \eta$ implies $0 < 2\nu - \eta \leq \eta$.) ☐

Consider a (κ, η, ν)-representation of an m-bit integer w. Suppose that ω-bit arithmetic will be used for implementation. Using ω-bit arithmetic, w will be represented by κ ω-bit words $w_0, \ldots, w_{\kappa-1}$ such that the binary representation of w is given by

$$\mathsf{lsb}_\nu(w_{\kappa-1}) || \mathsf{lsb}_\eta(w_{\kappa-2}) || \cdots || \mathsf{lsb}_\eta(w_0). \tag{3}$$

Here $\mathsf{lsb}_i(x)$ denotes the i least significant bits of the binary string x and $0 < \nu \leq \eta \leq \omega$. Depending on the value of η, we identify two kinds of representation.

Saturated limb representation: In this case $\eta = \omega$. So, each of the ω-bit words $w_0, w_1, \ldots, w_{\kappa-2}$ are "saturated" in the sense that there are no leading redundant bits in these words. The ω-bit word $w_{\kappa-1}$ is saturated or unsaturated depending on whether $\nu = \eta$ or $\nu < \eta$ respectively.

Unsaturated limb representation: In this case $\eta < \omega$. So, each of the ω-bit words $w_0, w_1, \ldots, w_{\kappa-1}$ are "unsaturated" in the sense that they contain some leading redundant bits. The word $w_{\kappa-1}$ contains the same or more leading redundant bits according as whether $\nu = \eta$ or $\nu < \eta$ respectively.

Remark: *In this work, we will consider 64-bit arithmetic and so $\omega = 64$.* The general ideas of the algorithms apply to arbitrary values of ω. The actual value of $\omega = 64$ is used at some places in the (non-)overflow analysis of the correctness proofs.

The primes p that we consider are of the form

$$p \quad = \quad 2^m - \delta, \tag{4}$$

where δ is sufficiently small. Given (κ, η, ν)-representation of m-bit integers, we have

$$2^{\eta(\kappa-1)+\nu} \quad = \quad 2^m \equiv \delta \bmod p. \tag{5}$$

For future reference, we define

$$c_p \quad = \quad 2^{\eta-\nu}\delta. \tag{6}$$

The values of m, δ, κ, η and ν for the various primes p considered in this work are given in Table 3. For each prime, two sets of values of κ, η and ν are provided, one using saturated limb representation and the other using unsaturated limb representation. For the last six primes in Table 3, we do not report the unsaturated limb representations since, we have not implemented the relevant algorithms for these primes.

The main goal of obtaining a representation of m of the form (2) is to minimise the value of κ. As the value of κ grows, so does the complexity of multiplication. Clearly the value of κ is minimised for the saturated limb representations. An issue with the saturated limb representations is that each limb has the maximum possible size and so handling overfull issues can be complex. It is for this reason that one considers the unsaturated limb representations. For the unsaturated limb representations, in Table 3 there is a column labelled "type". This column refers to the particular type of reduction algorithm which applies best to the corresponding prime and will be explained in details later.

prime	m	δ	unsaturated limb				saturated limb		
			κ	η	ν	type	κ	η	ν
$2^{127}-1$	127	1	3	43	41	A	2	64	63
$2^{221}-3$	221	3	4	56	53	A	4	64	29
$2^{222}-117$	222	117	4	56	54	A	4	64	30
$2^{251}-9$	251	9	5	51	47	A	4	64	59
$2^{255}-19$	255	19	5	51	51	A	4	64	63
$2^{256}-2^{32}-977$	256	$2^{32}+977$	5	52	48	A	4	64	64
$2^{266}-3$	266	3	5	54	50	A	5	64	10
$2^{382}-105$	382	105	7	55	52	B	6	64	62
$2^{383}-187$	383	187	7	55	53	B	6	64	63
$2^{414}-17$	414	17	8	52	50	A	7	64	30
$2^{511}-187$	511	187	9	57	55	G	8	64	63
$2^{512}-569$	512	569	9	57	56	G	8	64	64
$2^{521}-1$	521	1	9	58	57	A	9	64	9
$2^{607}-1$	607	1	10	61	58	G	10	64	31
$2^{751}-165$	751	165	-	-	-	-	12	64	47
$2^{832}-143$	832	143	-	-	-	-	13	64	64
$2^{896}-213$	896	213	-	-	-	-	14	64	64
$2^{960}-167$	960	167	-	-	-	-	15	64	64
$2^{1024}-105$	1024	105	-	-	-	-	16	64	64
$2^{1088}-89$	1088	89	-	-	-	-	17	64	64

Table 3: The primes considered in this work and their saturated and unsaturated limb representations.

Suppose m-bit integers have a (κ, η, ν)-representation and $\theta = 2^\eta$. The prime $p = 2^m - \delta$ is represented as the polynomial $\mathfrak{p}(\theta)$, defined as

$$\mathfrak{p}(\theta) \quad = \quad p_0 + p_1\theta + \ldots + p_{\kappa-1}\theta^{\kappa-1}, \tag{7}$$

where $p_0 = 2^\eta - \delta$, $p_1, p_2, \ldots, p_{\kappa-2} = 2^\eta - 1$, and $p_{\kappa-1} = 2^\nu - 1$.

An element in $\mathbb{F}_p$ is represented as a polynomial $f(\theta)$ of degree at most $\kappa - 1$, defined as

$$f(\theta) \quad = \quad f_0 + f_1\theta + \cdots + f_{\kappa-1}\theta^{\kappa-1}, \tag{8}$$

where $0 \le f_0, f_1, \ldots, f_{\kappa-1} < 2^\eta$ and $0 \le f_{\kappa-1} < 2^\nu$.

It should be noted that the polynomials $f(\theta)$ are in one-one correspondence with the integers $0, 1, \ldots, 2^m - 1$. This leads to a non-unique representation of δ elements in $\mathbb{F}_p$, i.e., the elements $0, 1, \ldots, \delta - 1$ are also represented as $2^m - \delta, 2^m - \delta + 1, \ldots, 2^m - 1$. The non-unique representation does not affect the correctness of the computations. At the end of the computation, the final result is converted to a unique representation using a simple algorithm. This procedure is shown in Algorithm 1.

Algorithm 1 Converts to unique representation in $\mathbb{F}_p$, where $p = 2^m - \delta$.

```
1: function makeUnique($h^{(0)}(\theta)$)
2: input: $h^{(0)}(\theta) = h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1}$, where $0 \leq h_0^{(0)}, h_1^{(0)}, \ldots, h_{\kappa-2}^{(0)} < 2^\eta$ and $0 \leq h_{\kappa-1}^{(0)} < 2^\nu$.
3: output: $h^{(1)}(\theta) = h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1}$, where $0 \leq h_0^{(1)} < 2^\eta - \delta$, $0 \leq h_1^{(1)}, h_2^{(1)}, \ldots, h_{\kappa-2}^{(1)} < 2^\eta$, $0 \leq h_{\kappa-1}^{(0)} < 2^\nu$ and $h^{(1)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.
4:     $u \leftarrow h_0^{(0)} \geq p_0$
5:     for $i \leftarrow 1$ to $\kappa - 2$ do
6:         $u \leftarrow u \ \& \ (h_i^{(0)} = 2^\eta - 1)$
7:     end for
8:     $u \leftarrow u \ \& \ (h_{\kappa-1}^{(0)} = p_{\kappa-1})$
9:     $v \leftarrow -u$; $u \leftarrow \neg v$
10:    $h_0^{(1)} \leftarrow (h_0^{(0)} \ \& \ u) \mid ((h_0^{(0)} - p_0) \ \& \ v)$
11:    for $i \leftarrow 1$ to $\kappa - 1$ do
12:        $h_i^{(1)} \leftarrow (h_i^{(0)} \ \& \ u) \mid (0 \ \& \ v)$
13:    end for
14:    return $h^{(1)}(\theta) = h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1}$
15: end function.
```

3.1 Inversion

Fermat's little theorem states that for a prime p and any non-zero $a \in \mathbb{F}_p$, $a^{p-1} \equiv 1 \bmod p$. So, a^{p-2} is the inverse of a in $\mathbb{F}_p$. Thus, the computation of inverse of any non-zero element in $\mathbb{F}_p$ reduces to the problem of exponentiating a to the power $p-2$. The standard way to compute this exponentiation is to use the square-and-multiply algorithm. Since, the value $p-2$ is fixed, the numbers of squarings and multiplications are fixed and do not depend on the value of a. So, if squaring and multiplication in $\mathbb{F}_p$ are constant time algorithms, then the exponentiation based inversion is also a constant time algorithm. This is the method of choice for inversion for almost all practical cryptographically relevant field sizes.

Thus, the problem of efficient constant time inversion reduces to the problem of obtaining efficient constant time multiplication and squaring algorithms in $\mathbb{F}_p$. A multiplication/squaring in $\mathbb{F}_p$ consists of two phases. In the first phase, an integer multiplication/squaring is performed and in the second phase, the result is reduced modulo the prime p. We have identified two different representations, namely saturated and unsaturated, of the elements of the field $\mathbb{F}_p$. For both of these representations, we provide the corresponding integer multiplication/squaring and the reduction algorithms.

Remark: In the rest of the paper, we will focus entirely on field multiplication and squaring. In comparison, field addition, negation and subtraction are much faster. We note one important difference in these operations which arises from the representation of the elements. For saturated limb representations, field addition/negation/subtraction can be implemented using `add/adc/sub/sbb`. On the other hand, for unsaturated limb representations, implementation of these operations also require shift operations.

4 Overview of the Algorithms

All algorithms in this work are described keeping 64-bit arithmetic in mind.

Meanings of various abbreviations:

SL	: saturated limb;
USL	: unsaturated limb;
SCC	: single carry chain;
DCC	: double (independent) carry chains;
MP	: Mersenne prime;
PMP	: pseudo-Mersenne prime;
maa	: algorithms implemented using only `mul`, `imul`, `add` and `adc`;
maax	: algorithms which also use `mulx`, `adcx` and `adox`.

Brief descriptions of the tasks of the different algorithms that we consider are given below.

Algorithms for the saturated limb representation:

mulSCC: Multiply a word whose value is less than 2^{64} to an integer given by a saturated limb representation using a single carry chain.

mulSLDCC: Multiply two integers given in saturated limb representations using double (independent) carry chains.

sqrSLDCC: Square an integer given in saturated limb representation using double carry chains.

reduceSLMP: Reduction algorithm to be applied to the outputs of mulSLDCC or sqrSLDCC when the underlying prime is a Mersenne prime.

reduceSLPMP: Reduction algorithm to be applied to the outputs of mulSLDCC or sqrSLDCC when the underlying prime is a pseudo-Mersenne prime.

reduceSLPMPa: A partial reduction algorithm to be applied to the outputs of mulSLDCC or sqrSLDCC when the underlying prime is a pseudo-Mersenne prime.

mulSLa: Multiply two integers given in saturated limb representations and perform an initial step of the reduction.

sqrSLa: Square an integer given in saturated limb representation and perform an initial step of the reduction.

reduceSL: A generic reduction algorithm to be applied to the outputs of mulSLa/sqrSLa.

invSLa: Computes a field inverse by calling mulSLa, sqrSLa and reduceSL.

invSLMP: Computes a field inverse by calling mulSL, sqrSL and reduceSLMP. See the remark below for mulSL and sqrSL.

invxSLMP: Computes a field inverse by calling mulSLDCC, sqrSLDCC and reduceSLMP.

invxSLPMP: Computes a field inverse by calling mulSLDCC, sqrSLDCC and reduceSLPMP.

Remark: The output of mulSLDCC is the product of the two integers and the output of sqrSLDCC is the square of an integer. Algorithms mulSLDCC/sqrSLDCC utilise double carry chains to perform the computations. The product of two integers in the saturated limb representation can also be performed without using double carry chains and similarly, the square of an integer in the saturated limb representation can be performed without using double carry chains. For the prime $2^{255} - 19$, the 4-limb algorithms in [5] perform such computations. The 4-limb algorithms in [5] can be extended to work for arbitrary limb representations. We will denote the resulting multiplication and squaring algorithms by mulSL and sqrSL. Note that mulSL/sqrSL are different from mulSLa/sqrSLa since mulSLa/sqrSLa also perform an initial step of reduction while this is not done by mulSL/sqrSL.

Algorithms for the unsaturated limb representation:

mulUSL: Multiply two integers given in unsaturated limb representations and perform an initial step of the reduction.

sqrUSL: Square an integer given in unsaturated limb representation and perform an initial step of the reduction.

mulUSLa: Multiply two integers given in unsaturated limb representations and perform an initial step of the reduction. This is a variant of mulUSL which is to be used when mulUSL leads to overflows.

sqrUSLa: Square an integer given in unsaturated limb representation and perform an initial step of the reduction. This is a variant of sqrUSL which is to be used when sqrUSL leads to overflows.

reduceUSL: A generic reduction algorithm to be applied to the outputs of mulUSL/sqrUSL or mulUSLa/sqrUSLa.

reduceUSLA: An algorithm to be applied to the outputs of mulUSL/sqrUSL or mulUSLa/sqrUSLa when the prime is of Type A. For such primes, reduceUSLA is more efficient than reduceUSL.

reduceUSLB: An algorithm to be applied to the outputs of mulUSL/sqrUSL or mulUSLa/sqrUSLa when the prime is of Type B. For such primes, reduceUSLB is more efficient than reduceUSL or reduceUSLA.

invUSL: Computes a field inverse by calling mulUSL, sqrUSL and reduceUSL.

invUSLA: Computes a field inverse by calling mulUSL, sqrUSL and reduceUSLA.

invUSLB: Computes a field inverse by calling mulUSL, sqrUSL and reduceUSLB.

invUSLa: Computes a field inverse by calling mulUSLa, sqrUSLa and reduceUSLA.

The implementations of the various inversion algorithms are divided into two groups.

Algorithms in the maa setting: The algorithms invSLa, invUSL, invUSLA, invUSLB and invUSLa have been implemented in assembly using only the instructions `mul`, `imul`, `add` and `adc` to do arithmetic. These implementations are downward compatible with previous generations of Intel processors.

Algorithms in the maax setting: The implementatinos of the algorithms invxSLMP and invxSLPMP also use the instructions `mulx`, `adcx` and `adox` for doing arithmetic. These implementations work on the Broadwell and later generation processors.

Descriptions of the algorithms. We describe a number of algorithms. The descriptions of the algorithms are at a fairly high level. They are provided in a form which make it easy to understand the algorithms and present the proofs of correctness. For the various reduction algorithms, the input is considered to be a polynomial $h^{(0)}(\theta)$, with $\theta = 2^\eta$, and the output is $h^{(k)}(\theta)$ for some $k \geq 1$, such that

$$h^{(0)}(\theta) \equiv h^{(1)}(\theta) \equiv \cdots \equiv h^{(k)}(\theta) \bmod p.$$

Conceptually, the algorithm proceeds in stages where the i-th stage computes $h^{(i)}(\theta)$ from $h^{(i-1)}(\theta)$ for $i = 1, 2, \ldots, k$. The proofs of correctness show that $h^{(i)}(\theta) \equiv h^{(i-1)}(\theta) \bmod p$ and also provide precise bounds on the coefficients of $h^{(i)}(\theta)$. In order to define the polynomials $h^{(i)}(\theta)$, the algorithms use certain statements which simply copy some of the coefficients of $h^{(i-1)}(\theta)$ to $h^{(i)}(\theta)$. Also, for ease of reference in the proofs, certain temporary variables are indexed by the loop counter creating the impression that a number of such variables are required, whereas in actual implementation one variable is sufficient.

For actual assembly implementation, it is desirable to use the registers as much as possible and also to avoid using load/store instructions to the extent possible. As such, the strict distinction between the various stages of the algorithm is not maintained so that some of the copy statements become redundant and are not implemented. Also, the use of temporary variables are minimised as much as possible and such variables are reused whenever feasible. Modulo such routine simplifications, the implementations follow the general flow of the algorithms. For each of the algorithms, we provide efficient assembly implementations for a number of primes. Studying the code together with the algorithm descriptions will make the associations between them clear and lead to a better understanding of the code.

5 Integer Multiplication/Squaring for Saturated Limb Representation Using Independent Carry Chains

Let c be an η-bit constant, $\theta = 2^\eta$ and $f(\theta)$ be a polynomial in θ of degree at most $d-1$ whose coefficients are from $\mathbb{Z}_\theta$. A basic step in the multiplication and squaring algorithms is the computation $c \cdot f(\theta)$. The result is a polynomial $h(\theta)$ of degree at most d and whose coefficients are from $\mathbb{Z}_\theta$. Function mulSCC given in Algorithm 2 performs this computation.

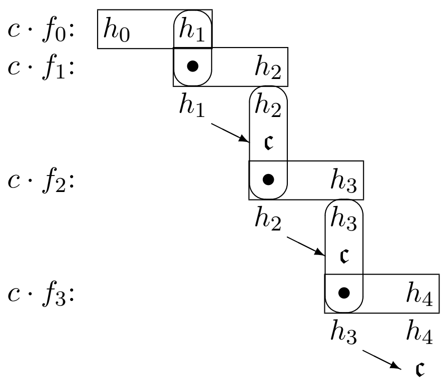
Algorithm 2 Multiply $f(\theta)$ with an η-bit constant c; $\theta = 2^\eta, \eta = 64$.

```
1:  function mulSCC($f(\theta), c$)
2:  input: $f(\theta) = f_0 + f_1\theta + \cdots + f_{d-1}\theta^{d-1}$, $c$, where $0 \leq c, f_0, f_1, \ldots, f_{d-1} < 2^\eta$ and $d \geq 1$.
3:  output: $h(\theta) = h_0 + h_1\theta + \cdots + h_d\theta^d = c \cdot f$, where $0 \leq h_0, h_1, \ldots, h_d < 2^\eta$.
4:      $t \leftarrow c \cdot f_0$; $h_0 \leftarrow t \bmod 2^\eta$; $h_1 \leftarrow \lfloor t/2^\eta \rfloor$
5:      $\mathfrak{c} \leftarrow 0$
6:      for $i \leftarrow 1$ to $d-1$ do
7:          $t \leftarrow c \cdot f_i$; $h_{i+1} \leftarrow \lfloor t/2^\eta \rfloor$
8:          $t \leftarrow h_i + t \bmod 2^\eta + \mathfrak{c}$; $h_i \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
9:      end for
10:     $h_d \leftarrow h_d + \mathfrak{c}$
11:     return $h(\theta) = h_0 + h_1\theta + \cdots + h_d\theta^d$
12: end function.
```

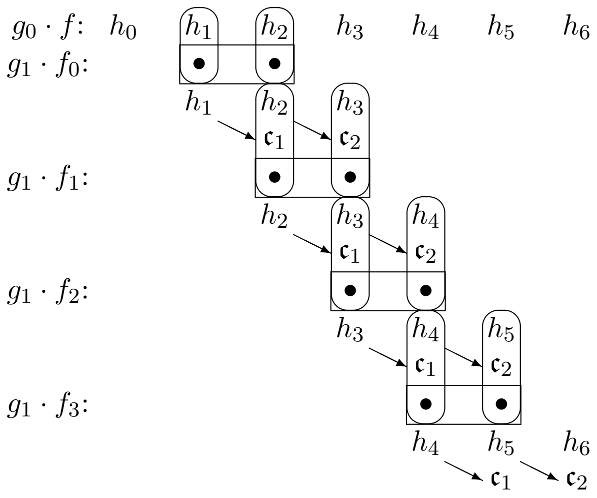
The multiplication in Step 4 of mulSCC can be completed using a single `mulx` operation. The **for loop** in Steps 6 to 9 uses an interleaved sequence of multiplications and additions. The additions involve a carry propagation through the variable $\mathfrak{c}$. Step 4 can be completed using a single `mulx` instruction while Step 5 can be completed using an `adc` instruction. The single bit value of the carry variable $\mathfrak{c}$ is carried through CF. Note that `mulx` does not affect CF and so it is possible to use `adc` instructions to implement the carry chain. Since the `mul` instruction affects CF, using `mul` instead of `mulx` would not have allowed an efficient implementation of the carry chain using `adc` instructions.

The single carry chain of mulSCC is pictorially depicted in Figure 1a. The horizontal rectangular boxes denote the two η-bit quantities arising out of the multiplication shown at the left end of the corresponding row. The vertical oval shape encapsulates the quantities that are added using the `adc` instruction. These consist of two η-bit quantities and the carry $\mathfrak{c}$ whose value is available in the CF flag.

In general, it is required to multiply two integers written as polynomials $f(\theta)$ and $g(\theta)$ having degrees d and e respectively. This is performed using Function mulSLDCC given in Algorithm 3. The algorithm is written in a manner so that there are two independent carry chains in action. This is illustrated in Figure 1b.



(a) Single carry chain for mulSCC.



(b) Two independent carry chains for mulSLDCC.

Figure 1: Illustration of carry chains.

Algorithm 3 Multiply $f(\theta)$ and $g(\theta)$; $\theta = 2^\eta, \eta = 64$.

```
1: function mulSLDCC($f(\theta), g(\theta)$)
2: input: $f(\theta) = f_0 + f_1\theta + \cdots + f_{d-1}\theta^{d-1}$ and $g(\theta) = g_0 + g_1\theta + \cdots + g_{e-1}\theta^{e-1}$, where $0 \le f_0, f_1, \ldots, f_{d-1}, g_0, g_1, \ldots, g_{e-1} < 2^\eta$, and $d \ge e \ge 2$.
3: output: $h(\theta) = h_0 + h_1\theta + \cdots + h_{d+e-1}\theta^{d+e-1} = f \cdot g$, where $0 \le h_0, h_1, \ldots, h_{d+e-1} < 2^\eta$.
4:     $h_0 + h_1\theta + \cdots + h_d\theta^d \leftarrow$ mulSCC($f(\theta), g_0$)
5:     for $i \leftarrow 1$ to $e-1$ do
6:         $\mathfrak{c}_1 \leftarrow 0$; $\mathfrak{c}_2 \leftarrow 0$
7:         for $j \leftarrow 0$ to $d-1$ do
8:             $t \leftarrow g_i \cdot f_j$
9:             $r \leftarrow h_{i+j} + (t \bmod 2^\eta) + \mathfrak{c}_1$
10:            $s \leftarrow h_{i+j+1} + \lfloor t/2^\eta \rfloor + \mathfrak{c}_2$
11:            $h_{i+j} \leftarrow r \bmod 2^\eta$;    $\mathfrak{c}_1 \leftarrow \lfloor r/2^\eta \rfloor$
12:            $h_{i+j+1} \leftarrow s \bmod 2^\eta$; $\mathfrak{c}_2 \leftarrow \lfloor s/2^\eta \rfloor$
13:        end for
14:        $h_{i+j+1} \leftarrow h_{i+j+1} + \mathfrak{c}_1$
15:    end for
16:    return $h(\theta) = h_0 + h_1\theta + \cdots + h_{d+e-1}\theta^{d+e-1}$
17: end function.
```

The multiplications in mulSLDCC are independent and can be performed simultaneously. The two additions are also independent and can be performed simultaneously. The additions, however, depend on the result of the previous multiplication. The `mulx` instruction is used to perform the multiplications. This instruction does not affect either CF or OF. The two independent carry chains arising in Function mulSLDCC (and as illustrated in Figure 1b) are implemented using a sequence of `adcx` and `adox` instructions. The `adcx` instruction uses CF to propagate the carry while the `adox` instruction uses OF to propagate the carry.

Algorithm 4 Square $f(\theta)$; $\theta = 2^\eta, \eta = 64$.

```
1:  function sqrSLDCC($f(\theta)$)
2:  input: $f(\theta) = f_0 + f_1\theta + \cdots + f_{d-1}\theta^{d-1}$ such that $0 \leq f_0, f_1, \ldots, f_{d-1} < 2^\eta$
3:  output: $h = h_0 + h_1\theta + \ldots + h_{2d-1}\theta^{2d-1} = f^2$ such that $0 \leq h_0, h_1, \ldots, h_{2d-1} < 2^\eta$.
4:      $h_1 + h_2\theta + \cdots + h_d\theta^d \leftarrow$ mulSCC($f_1 + f_2\theta + \cdots + f_{d-1}\theta^{d-2}, f_0$)
5:      for $i \leftarrow 1$ to $d-3$ do  $h_{d+i} \leftarrow 0$ end for
6:      for $i \leftarrow 1$ to $d-3$ do
7:          $\mathfrak{c}_1 \leftarrow 0$;  $\mathfrak{c}_2 \leftarrow 0$
8:          for $j \leftarrow i+1$ to $d-1$ do
9:              $t \leftarrow f_i \cdot f_j$
10:             $r \leftarrow h_{i+j} + (t \bmod 2^\eta) + \mathfrak{c}_1$; $s \leftarrow h_{i+j+1} + \lfloor t/2^\eta \rfloor + \mathfrak{c}_2$
11:             $h_{i+j} \leftarrow r \bmod 2^\eta$;     $\mathfrak{c}_1 \leftarrow \lfloor r/2^\eta \rfloor$
12:             $h_{i+j+1} \leftarrow s \bmod 2^\eta$;  $\mathfrak{c}_2 \leftarrow \lfloor s/2^\eta \rfloor$
13:         end for
14:         $h_{i+j+1} \leftarrow h_{i+j+1} + \mathfrak{c}_1$
15:     end for
16:     $t \leftarrow f_{d-1} \cdot f_{d-2}$
17:     $r \leftarrow h_{2d-3} + (t \bmod 2^\eta)$; $h_{2d-3} \leftarrow r \bmod 2^\eta$
18:     $\mathfrak{c} \leftarrow \lfloor r/2^\eta \rfloor$; $h_{2d-2} \leftarrow \lfloor t/2^\eta \rfloor + \mathfrak{c}$
19:     $h_{2d-1} \leftarrow \lfloor h_{2d-2}/2^{\eta-1} \rfloor$
20:     for $i \leftarrow 2d-1$ down to 2 do $h_i \leftarrow (2h_i \bmod 2^\eta) + \lfloor h_{i-1}/2^{\eta-1} \rfloor$ end for
21:     $h_1 \leftarrow 2h_1 \bmod 2^\eta$
22:     $t \leftarrow f_0 \cdot f_0$
23:     $h_0 \leftarrow t \bmod 2^\eta$; $t \leftarrow h_1 + \lfloor t/2^\eta \rfloor$
24:     $h_1 \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
25:     for $i \leftarrow 1$ to $d-1$ do
26:         $t \leftarrow f_i \cdot f_i$
27:         $r \leftarrow h_{2i} + (t \bmod 2^\eta) + \mathfrak{c}$; $h_{2i} \leftarrow r \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor r/2^\eta \rfloor$
28:         $r \leftarrow h_{2i+1} + \lfloor t/2^\eta \rfloor + \mathfrak{c}$; $h_{2i+1} \leftarrow r \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor r/2^\eta \rfloor$
29:     end for
30:     return $h(\theta) = h_0 + h_1\theta + \ldots + h_{2d-1}\theta^{2d-1}$
31: end function.
```

Intel processors have multiple ALUs. So, the independent additions can be simultaneously executed on two separate ALUs. Further, subject to availability, the independent multiplications can be scheduled on separate ALUs and the multiplications and additions can be scheduled in a pipelined manner on separate ALUs such that the time for addition does not cause any delay in the overall computation.

Squaring an integer of the form $f(\theta)$ can be performed by setting both inputs in mulSLDCC to be equal to $f(\theta)$. On the other hand, it is possible to reduce the number of multiplications. Function sqrSLDCC given in Algorithm 4 squares $f(\theta)$. It consists of three phases. In the first phase, the cross product terms are computed; in the second phase, these are multiplied by 2 (which is a doubling operation); and in the third phase, the squares of the coefficients of $f(\theta)$ are computed. Multiplications are performed in the first and the third phase. In the first phase, two independent carry chains arise in a manner similar to that of mulSLDCC. These two chains are implemented using the instructions `mulx`, `adcx` and `adox`. In the third phase, there is a single carry chain which is implemented using the instructions `mulx` and `adc` in a manner similar to that used in mulSCC.

6 Reduction in $\mathbb{F}_p$ Using Saturated Limb Representation

For $p = 2^m - \delta$, elements of $\mathbb{F}_p$ are m-bit integers and have a (κ, η, ν)-representation. In this section, we consider saturated limb representation and so $\eta = 64$. As mentioned earlier, a multiplication/squaring in $\mathbb{F}_p$ consists of an integer multiplication/squaring followed by a reduction. The integer multiplication and squaring operations are respectively performed by the functions mulSLDCC and sqrSLDCC described in Section 5. In both cases, two m-bit integers having (κ, η, ν)-representations are multiplied and the product is a $2m$-bit integer having (κ', η, ν')-representation where the values of κ' and ν' are given by Proposition 1. The task of the reduction is to reduce the product modulo p to an m-bit integer which again has a (κ, η, ν)-representation.

Algorithm 5 Reduction for saturated limb representation. Performs reduction modulo p, where $p = 2^m - 1$ is a Mersenne prime; $\theta = 2^\eta$.

```
1: function reduceSLMP($h^{(0)}(\theta)$)
2: Input: $h^{(0)}(\theta)$.
3: Output: $h^{(3)}(\theta)$.
4:     for $i \leftarrow 2\kappa - 1$ down to $\kappa$ do
5:         $h_i^{(1)} \leftarrow (2^{\eta-\nu} h_i^{(0)}) \bmod 2^\eta + \lfloor h_{i-1}^{(0)}/2^\nu \rfloor$; $h_{i-\kappa-1}^{(1)} \leftarrow h_{i-\kappa-1}^{(0)}$
6:     end for
7:     $h_{\kappa-1}^{(1)} \leftarrow h_{\kappa-1}^{(0)} \bmod 2^\nu$; $h_0^{(1)} \leftarrow h_0^{(0)}$
8:     $t \leftarrow h_0^{(1)} + h_\kappa^{(1)}$; $h_0^{(2)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
9:     for $i \leftarrow 1$ to $\kappa - 2$ do
10:        $t \leftarrow h_i^{(1)} + h_{\kappa+i}^{(1)} + \mathfrak{c}$; $h_i^{(2)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
11:    end for
12:    $h_{\kappa-1}^{(2)} \leftarrow h_{\kappa-1}^{(1)} + h_{2\kappa-1}^{(1)} + \mathfrak{c}$
13:    $t \leftarrow h_0^{(2)} + \lfloor h_{\kappa-1}^{(2)}/2^\nu \rfloor$; $h_0^{(3)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
14:    for $i \leftarrow 1$ to $\kappa - 2$ do
15:        $t \leftarrow h_i^{(2)} + \mathfrak{c}$; $h_i^{(3)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
16:    end for
17:    $h_{\kappa-1}^{(3)} \leftarrow h_{\kappa-1}^{(2)} \bmod 2^\nu + \mathfrak{c}$
18:    FULL REDUCTION: return $h^{(3)}(\theta) = h_0^{(3)} + h_1^{(3)}\theta + \cdots + h_{\kappa-1}^{(3)}\theta^{\kappa-1}$
19: end function.
```

We provide two reduction algorithms using the saturated limb representation, namely reduceSLMP which works for Mersenne primes and reduceSLPMP which works for pseudo-Mersenne primes. A Mersenne prime is also a pseudo-Mersenne prime and so reduceSLPMP also works for Mersenne primes, but, for such primes it will be slower than reduceSLMP. On the other hand, reduceSLMP does not work for pseudo-Mersenne primes.

6.1 Mersenne Primes

Let $p = 2^m - 1$ and suppose m-bit integers have a (κ, η, ν)-representation. Function reduceSLMP given in Algorithm 5 takes as input the output of either mulSLDCC or sqrSLDCC and outputs an m-bit integer in an (κ, η, ν)-representation, which is congruent to the input modulo p.

The following result states the correctness of $\mathsf{reduceSLMP}$.

Theorem 2. *Let $p = 2^m - 1$ be a Mersenne prime and let $\kappa \geq 2$, η and ν be such that, m-bit integers have a (κ, η, ν)-representation. Suppose that the input $h^{(0)}(\theta)$ to* $\mathsf{reduceSLMP}$ *is the output of either* $\mathsf{mulSLDCC}(f(\theta), g(\theta))$ *or* $\mathsf{sqrSLDCC}(f(\theta))$ *where $f(\theta)$ and $g(\theta)$ represent m-bit integers having (κ, η, ν)-representations. Then the output $h^{(3)}(\theta)$ of* $\mathsf{reduceSLMP}$ *has a (κ, η, ν)-representation and $h^{(3)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*

Proof. Since m-bit integers have a (κ, η, ν)-representation, we have $m = \eta(\kappa - 1) + \nu$ with $0 < \nu \leq \eta$. If $\nu = \eta$, then $m = \kappa\eta$ and so $p = 2^m - 1 = 2^{\kappa\eta} - 1 = (2^\kappa)^\eta - 1$, which has a factor $2^\kappa - 1$ contradicting that p is a prime. So, if p is a Mersenne prime, then it necessarily follows that $\nu < \eta$.

The input $h^{(0)}(\theta)$ to $\mathsf{reduceSLMP}$ is the product of two m-bit integers each having a (κ, η, ν)-representation. From Proposition 1, the $2m$-bit integer $h^{(0)}(\theta)$ has a (κ', η, ν')-representation where the values of κ' and ν' are given by Proposition 1. Using these values, we have the following bounds on the coefficients of $h^{(0)}(\theta)$.

$$\begin{array}{llll} 0 \leq h^{(0)}_0, h^{(0)}_1, \ldots, h^{(0)}_{2\kappa-3} < 2^\eta; & & \text{and} & \\ 0 \leq h^{(0)}_{2\kappa-2} < 2^{2\nu}, & h^{(0)}_{2\kappa-1} = 0 & \text{if} & 0 < \nu \leq \eta/2; \\ 0 \leq h^{(0)}_{2\kappa-2} < 2^{\eta}, & 0 \leq h^{(0)}_{2\kappa-1} < 2^{2\nu-\eta} & \text{if} & \eta/2 < \nu < \eta. \end{array} \tag{9}$$

The input $h^{(0)}(\theta)$ can be written as

$$\begin{aligned} h^{(0)}(\theta) &= h^{(0)}_0 + h^{(0)}_1\theta + \cdots + h^{(0)}_{\kappa-1}\theta^{\kappa-1} + h^{(0)}_{\kappa}\theta^{\kappa} + h^{(0)}_{\kappa+1}\theta^{(\kappa+1)} + \cdots + h^{(0)}_{2\kappa-1}\theta^{(2\kappa-1)}, \\ &= (h^{(0)}_0 + h^{(0)}_1\theta + \cdots + h^{(0)}_{\kappa-1}\theta^{\kappa-1}) + (h^{(0)}_{\kappa} + h^{(0)}_{\kappa+1}\theta + \cdots + h^{(0)}_{2\kappa-1}\theta^{\kappa-1})\theta^{\kappa}, \\ &\equiv (h^{(0)}_0 + h^{(0)}_1\theta + \cdots + h^{(0)}_{\kappa-1}\theta^{\kappa-1}) + (h^{(0)}_{\kappa} + h^{(0)}_{\kappa+1}\theta + \cdots + h^{(0)}_{2\kappa-1}\theta^{\kappa-1})2^{\eta-\nu} \bmod p, \end{aligned} \tag{10}$$

since using (5) and $\delta = 1$, we have $\theta^\kappa = 2^{\kappa\eta} = 2^{(\kappa-1)\eta+\nu} \cdot 2^{\eta-\nu} = 2^m \cdot 2^{\eta-\nu} \equiv 2^{\eta-\nu} \bmod p$. For $j = \kappa-1, \kappa, \ldots, 2\kappa-2$, define

$$h^{(0)}_j = h^{(0)}_{j,0} + h^{(0)}_{j,1}2^\nu, \text{ where } h^{(0)}_{j,0} = h^{(0)}_j \bmod 2^\nu, \text{ and } h^{(0)}_{j,1} = \lfloor h^{(0)}_j / 2^\nu \rfloor. \tag{11}$$

Using (9), we have the following bounds on $h^{(0)}_{j,0}$ and $h^{(0)}_{j,1}$.

Claim 3. *$0 \leq h^{(0)}_{j,0} < 2^\nu$ for $j = \kappa-1, \kappa, \ldots, 2\kappa-2$; $0 \leq h^{(0)}_{j,1} < 2^{\eta-\nu}$ for $j = \kappa-1, \kappa, \ldots, 2\kappa-3$; and $0 \leq h^{(0)}_{2\kappa-2,1} < 2^\nu$ if $0 < \nu \leq \eta/2$; $0 \leq h^{(0)}_{2\kappa-2,1} < 2^{\eta-\nu}$ if $\eta/2 < \nu < \eta$.*

Substituting (11) into (10) we obtain

$$\begin{aligned} & h^{(0)}(\theta) \\ \equiv\; & (h^{(0)}_0 + h^{(0)}_1\theta + \cdots + h^{(0)}_{\kappa-2}\theta^{\kappa-2} + (h^{(0)}_{\kappa-1,0} + h^{(0)}_{\kappa-1,1}2^\nu)\theta^{\kappa-1}) \;+ \\ & ((h^{(0)}_{\kappa,0} + h^{(0)}_{\kappa,1}2^\nu) + (h^{(0)}_{\kappa+1,0} + h^{(0)}_{\kappa+1,1}2^\nu)\theta + \cdots + (h^{(0)}_{2\kappa-2,0} + h^{(0)}_{2\kappa-2,1}2^\nu)\theta^{\kappa-2} + h^{(0)}_{2\kappa-1}\theta^{\kappa-1})2^{\eta-\nu} \bmod p \\ =\; & (h^{(0)}_0 + h^{(0)}_1\theta + \cdots + h^{(0)}_{\kappa-2}\theta^{\kappa-2} + h^{(0)}_{\kappa-1,0}\theta^{\kappa-1}) \;+ \\ & (h^{(0)}_{\kappa-1,1}2^{((\kappa-1)\eta+\nu)} + h^{(0)}_{\kappa,0}2^{\eta-\nu} + h^{(0)}_{\kappa,1}2^{\eta} + h^{(0)}_{\kappa+1,0}2^{2\eta-\nu} + h^{(0)}_{\kappa+1,1}2^{2\eta} + \cdots + \qquad (12) \\ & h^{(0)}_{2\kappa-2,0}2^{(\kappa-1)\eta-\nu} + h^{(0)}_{2\kappa-2,1}2^{(\kappa-1)\eta} + h^{(0)}_{2\kappa-1}2^{\kappa\eta-\nu}) \text{ [using } \theta = 2^\eta] \qquad (13) \\ \equiv\; & (h^{(0)}_0 + h^{(0)}_1\theta + \cdots + h^{(0)}_{\kappa-2}\theta^{\kappa-2} + h^{(0)}_{\kappa-1,0}\theta^{\kappa-1}) \;+ \\ & (h^{(0)}_{\kappa-1,1} + 2^{\eta-\nu}h^{(0)}_{\kappa,0}) + (h^{(0)}_{\kappa,1} + 2^{\eta-\nu}h^{(0)}_{\kappa+1,0})\theta + (h^{(0)}_{\kappa+1,1} + 2^{\eta-\nu}h^{(0)}_{\kappa+2,0})\theta^2 + \cdots + \\ & (h^{(0)}_{2\kappa-3,1} + 2^{\eta-\nu}h^{(0)}_{2\kappa-2,0})\theta^{\kappa-2} + (h^{(0)}_{2\kappa-2,1} + 2^{\eta-\nu}h^{(0)}_{2\kappa-1})\theta^{\kappa-1} \bmod p \text{ [using (5) and } \delta = 1]. \qquad (14) \end{aligned}$$

In reduceSLMP, Steps 4 to 7 perform the computations in (14), giving us

$$h^{(0)}(\theta) \equiv \underbrace{(h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1}) + (h_\kappa^{(1)} + h_{\kappa+1}^{(1)}\theta + \cdots + h_{2\kappa-1}^{(1)}\theta^{\kappa-1})}_{\text{through Steps 4 to 7}} = h^{(1)}(\theta), \tag{15}$$

where $h_j^{(1)} = h_j^{(0)}$ for $j = 1, 2, \ldots, \kappa-2$, $h_{\kappa-1}^{(1)} = h_{\kappa-1,0}^{(0)}$,

$$h_j^{(1)} = 2^{\eta-\nu}h_{j,0}^{(0)} + h_{j-1,1}^{(0)} \text{ for } j = \kappa, \kappa+1, \ldots, 2\kappa-2, \text{ and } h_{2\kappa-1}^{(1)} = 2^{\eta-\nu}h_{2\kappa-1}^{(0)} + h_{2\kappa-2,1}^{(0)}.$$

In (15), it directly follows that $0 \le h_0^{(1)}, h_1^{(1)}, \ldots, h_{\kappa-2}^{(1)} < 2^\eta$ and $h_{\kappa-1}^{(1)} < 2^\nu$. The bounds on $h_\kappa^{(1)}, h_{\kappa+1}^{(1)}, \ldots, h_{2\kappa-1}^{(1)}$ are given in the following result.

Claim 4. *$0 \le h_\kappa^{(1)}, h_{\kappa+1}^{(1)}, \ldots, h_{2\kappa-2}^{(1)} < 2^\eta$ and $0 \le h_{2\kappa-1}^{(1)} < 2^\nu$.*

Proof. Using Claim 3, for $j = \kappa, \kappa+1, \ldots, 2\kappa-2$ we have

$$0 \le h_j^{(1)} = \lfloor h_{j-1}^{(0)}/2^\nu \rfloor + (2^{\eta-\nu}h_j^{(0)}) \bmod 2^\eta = h_{j-1,1}^{(0)} + (2^{\eta-\nu}h_{j,0}^{(0)} + 2^\eta h_{j,1}^{(0)}) \bmod 2^\eta = h_{j-1,1}^{(0)} + 2^{\eta-\nu}h_{j,0}^{(0)},$$

which implies $0 \le h_j^{(1)} < 2^{\eta-\nu} + 2^{\eta-\nu}(2^\nu - 1) = 2^\eta$. The argument for the bounds on $h_{2\kappa-1}^{(1)}$ is in two cases.

Case 1: $0 < \nu \le \eta/2$. From (9) and Claim 3, $h_{2\kappa-1}^{(1)} = 2^{\eta-\nu}h_{2\kappa-1}^{(0)} + h_{2\kappa-2,1}^{(0)} < 2^\nu$.

Case 2: $\eta/2 < \nu < \eta$. From (9) and Claim 3, $h_{2\kappa-1}^{(1)} = 2^{\eta-\nu}h_{2\kappa-1}^{(0)} + h_{2\kappa-2,1}^{(0)} < 2^{\eta-\nu}(2^{2\nu-\eta} - 1) + 2^{\eta-\nu} = 2^\nu$. □

In Steps 8 to 12, we pairwise add the coefficients of $\theta^0, \theta^1, \ldots, \theta^{\kappa-1}$ sequentianlly in (15) by forwarding the 1-bit carry to the subsequent pair to get an intermediate κ-limb polynomial $h^{(2)}$ as

$$\begin{aligned} h^{(1)}(\theta) &= (h_0^{(1)} + h_\kappa^{(1)}) + (h_1^{(1)} + h_{\kappa+1}^{(1)})\theta + \cdots + (h_{\kappa-1}^{(1)} + h_{2\kappa-1}^{(1)})\theta^{\kappa-1}, \\ &= \underbrace{h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1}}_{\text{through Steps 8 to 12}} = h^{(2)}(\theta). \end{aligned} \tag{16}$$

From the computation done in the Steps 8 to 11 it follows that

$$0 \le h_0^{(2)}, h_1^{(2)}, \ldots, h_{\kappa-2}^{(2)} < 2^\eta. \tag{17}$$

Also, since $0 \le h_{\kappa-1}^{(1)}, h_{2\kappa-1}^{(1)} \le 2^\nu - 1$ and $0 \le \mathfrak{c} \le 1$, we have

$$0 \le h_{\kappa-1}^{(2)} = h_{\kappa-1}^{(1)} + h_{2\kappa-1}^{(1)} + \mathfrak{c} \quad \le \quad 2^{\nu+1} - 1 \tag{18}$$

$$\le \quad 2^\eta - 1 \text{ (using } \nu < \eta). \tag{19}$$

From (15), (16), (17) and (18), we have $h^{(2)}(\theta) \equiv h^{(0)}(\theta) \bmod p$ and $h^{(2)}(\theta)$ has a $(\kappa, \eta, \nu+1)$-representation.

Equation (19) proves that Step 12 does not lead to an overflow. Define

$$h_{\kappa-1}^{(2)} = h_{\kappa-1,0}^{(2)} + h_{\kappa-1,1}^{(2)}2^\nu, \text{ where } h_{\kappa-1,0}^{(2)} = h_{\kappa-1}^{(2)} \bmod 2^\nu \text{ and } h_{\kappa-1,1}^{(2)} = \lfloor h_{\kappa-1}^{(2)}/2^\nu \rfloor. \tag{20}$$

From (18), it follows that that $0 \le h^{(2)}_{\kappa-1,0} < 2^\nu$ and $0 \le h^{(2)}_{\kappa-1,1} \le 1$. We write

$$\begin{aligned} h^{(2)}(\theta) &= h^{(2)}_0 + h^{(2)}_1\theta + \cdots + h^{(2)}_{\kappa-2}\theta^{\kappa-2} + (h^{(2)}_{\kappa-1,0} + h^{(2)}_{\kappa-1,1}2^\nu)\theta^{\kappa-1} \\ &= h^{(2)}_0 + h_1\theta + \cdots + h^{(2)}_{\kappa-2}\theta^{\kappa-2} + h^{(2)}_{\kappa-1,0}\theta^{\kappa-1} + h^{(2)}_{\kappa-1,1}2^{(\kappa-1)\eta+\nu} \\ &\equiv (h^{(2)}_0 + h^{(2)}_{\kappa-1,1}) + h^{(2)}_1\theta + \cdots + h^{(2)}_{\kappa-2}\theta^{\kappa-2} + h^{(2)}_{\kappa-1,0}\theta^{\kappa-1} \bmod p \text{ [using (5) and } \delta = 1]. \end{aligned} \tag{21}$$

The following result is crucial in arguing that the carry will be absorbed at some point in the computation.

Claim 5. *If $h^{(2)}_{\kappa-1,1} = 1$, then it is impossible to simultaneously have $h^{(2)}_0 = h^{(2)}_1 = \cdots = h^{(2)}_{\kappa-2} = 2^\eta - 1$ and $h^{(2)}_{\kappa-1,0} = 2^\nu - 1$.*

Proof. Suppose $h^{(2)}_{\kappa-1,1} = 1$ and let if possible, $h^{(2)}_0 = h^{(2)}_1 = \cdots = h^{(2)}_{\kappa-2} = 2^\eta - 1$, $h^{(2)}_{\kappa-1,0} = 2^\nu - 1$. So, from (20), we have $h^{(2)}_{\kappa-1} = h^{(2)}_{\kappa-1,0} + h^{(2)}_{\kappa-1,1}2^\nu = 2^{\nu+1} - 1$. In this case, the polynomial $h^{(2)}(\theta)$ is given as follows.

$$\begin{aligned} h^{(2)}(\theta) &= h^{(2)}_0 + h^{(2)}_1\theta + \cdots + h^{(2)}_{\kappa-1}\theta^{\kappa-1}, \\ &= (2^\eta - 1) + (2^\eta - 1)2^\eta + \cdots + (2^{\nu+1} - 1)2^{(\kappa-1)\eta}, \\ &= 2^{(\kappa-1)\eta+\nu+1} - 1, \\ &= 2^{m+1} - 1 \text{ [using (5)]}. \end{aligned} \tag{22}$$

From (16), $h^{(2)}(\theta)$ is obtained by adding the polynomials $(h^{(1)}_0 + h^{(1)}_1\theta + \cdots + h^{(1)}_{\kappa-1}\theta^{\kappa-1})$ and $(h^{(1)}_\kappa + h^{(1)}_{\kappa+1}\theta + \cdots + h^{(1)}_{2\kappa-1}\theta^{\kappa-1})$, where $0 \le h^{(1)}_0, h^{(1)}_1, \ldots, h^{(1)}_{\kappa-2}, h^{(1)}_\kappa, \ldots, h^{(1)}_{2\kappa-2} < 2^\eta$ and $0 \le h^{(1)}_{\kappa-1}, h^{(1)}_{2\kappa-1} < 2^\nu$. So, the maximum possible value of each of the polynomials is $2^m - 1$ and hence the bounds of $h^{(2)}(\theta)$ should be $0 \le h^{(2)}(\theta) < 2^{m+1} - 1$, which contradicts what is obtained in (22). Hence the result. □

The computation $h^{(3)}_0 = (h^{(2)}_0 + \lfloor h^{(2)}_{\kappa-1}/2^\nu \rfloor) \bmod 2^\eta$ in (21) is performed in Step 13, and the 1-bit $\mathfrak{c}$ is forwarded to the subsequent terms for addition, which are performed in Steps 14 to 17 producing the values of $h^{(3)}_1, h^{(3)}_2, \ldots, h^{(3)}_{\kappa-1}$. Hence, (21) can be written as

$$h^{(2)}(\theta) \equiv \underbrace{h^{(3)}_0 + h^{(3)}_1\theta + \cdots + h^{(3)}_{\kappa-1}\theta^{\kappa-1}}_{\text{through Steps 14 to 17}} \bmod p = h^{(3)}(\theta). \tag{23}$$

We now argue that either the $\mathfrak{c}$ out of Step 13 is 0 or it is absorbed in one of the subsequent additions in Steps 15 or in the addition in Step 17. If $h^{(2)}_{\kappa-1,1} = 0$, then the $\mathfrak{c}$ out of Step 13 itself is 0. So, suppose that $h^{(2)}_{\kappa-1,1} = 1$. From Claim 5, it follows that either there is a $j \in \{0, 1, \ldots, \kappa-2\}$ such that $h^{(2)}_j < 2^\eta - 1$ or $h^{(2)}_{\kappa-1,0} < 2^\nu - 1$. In the former case, the $\mathfrak{c}$ is absorbed by one of the additions in Step 15; if this does not happen, then the later case arises and the carry is absorbed by the addition in Step 17.

This shows that the algorithm terminates without any overflow and at the end of the algorithm we have $0 \le h^{(3)}_0, h^{(3)}_1, \ldots, h^{(3)}_{\kappa-2} < 2^\eta$ and $0 \le h^{(3)}_{\kappa-1} < 2^\nu$ and so $h^{(3)}(\theta)$ has a (κ, η, ν)-representation. Further, By combining (15), (16), (21) and (23) we have $h^{(3)}(\theta) \equiv h^{(0)}(\theta) \bmod p$, which proves the statement of the theorem on full reduction. □

6.2 Pseudo-Mersenne Primes

Let $p = 2^m - \delta$ and suppose m-bit integers have a (κ, η, ν)-representation. Function reduceSLPMP given in Algorithm 6 takes as input the output of either mulSLDCC or sqrSLDCC and outputs an m-bit integer in an (κ, η, ν)-representation which is congruent to the input modulo p.

As in the case of reduceSLMP, for the correctness of reduceSLPMP, it is not required to have $\eta = 64$. The value of $\eta = 64$ is used for 64-bit implementation and the algorithm can equally well be used with η-bit arithmetic for any value of η (say $\eta = 32$ or $\eta = 128$).

Algorithm 6 Reduction for saturated limb representation. Performs reduction modulo p, where $p = 2^m - \delta$ is a pseudo-Mersenne prime; $c_p = 2^{\eta-\nu}\delta$, $2^{\alpha-1} \le \delta < 2^\alpha$, $\nu' = 2(1 - \lfloor \nu/\eta \rfloor)$ and $\theta = 2^\eta$.

```
1: **function** reduceSLPMP($h_0^{(0)}(\theta)$)
2: **input**: $h^{(0)}(\theta)$.
3: **output**: $h^{(3)}(\theta)$ or $h^{(4)}(\theta)$.
4:     **for** $i \leftarrow 0$ to $\kappa - 1$ **do** $h_i^{(1)} \leftarrow h_i^{(0)}$ **end for**
5:     $h_\kappa^{(1)} + h_{\kappa+1}^{(1)}\theta + \cdots + h_{2\kappa}^{(1)}\theta^\kappa \leftarrow \mathsf{mulSCC}(h_\kappa^{(0)} + h_{\kappa+1}^{(0)}\theta + \cdots + h_{2\kappa-1}^{(0)}\theta^{\kappa-1}, c_p)$
6:     $t \leftarrow h_0^{(1)} + h_\kappa^{(1)}$; $h_0^{(2)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
7:     **for** $i \leftarrow 1$ to $\kappa - 1$ **do**
8:         $t \leftarrow h_i^{(1)} + h_{\kappa+i}^{(1)} + \mathfrak{c}$; $h_i^{(2)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
9:     **end for**
10:    $h_\kappa^{(2)} \leftarrow h_{2\kappa}^{(1)} + \mathfrak{c}$
11:    $r \leftarrow 2^{\eta-\nu}h_\kappa^{(2)} + \lfloor h_{\kappa-1}^{(2)}/2^\nu \rfloor$; $h_{\kappa-1}^{(2)} \leftarrow h_{\kappa-1}^{(2)} \bmod 2^\nu$
12:    $u \leftarrow h_0^{(2)} + \delta r$; $h_0^{(3)} \leftarrow u \bmod 2^\eta$; $q \leftarrow \lfloor u/2^\eta \rfloor$
13:    $v \leftarrow h_1^{(2)} + q$; $h_1^{(3)} \leftarrow v \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor v/2^\eta \rfloor$
14:    **for** $i \leftarrow 2$ to $\kappa - 2$ **do**
15:        $t \leftarrow h_i^{(2)} + \mathfrak{c}$; $h_i^{(3)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
16:    **end for**
17:    $h_{\kappa-1}^{(3)} \leftarrow h_{\kappa-1}^{(2)} + \mathfrak{c}$
18:    $\boxed{\text{PARTIAL REDUCTION FOR } \nu < \eta\text{: } \textbf{return } h^{(3)}(\theta) = h_0^{(3)} + h_1^{(3)}\theta + \cdots + h_{\kappa-1}^{(3)}\theta^{\kappa-1}}$
19:    $t \leftarrow h_{\kappa-1}^{(3)}$; $h_{\kappa-1}^{(3)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$; $h_\kappa^{(3)} \leftarrow \mathfrak{c}$
20:    $s \leftarrow 2^{\eta-\nu}h_\kappa^{(3)} + \lfloor h_{\kappa-1}^{(3)}/2^\nu \rfloor$; $h_{\kappa-1}^{(3)} \leftarrow h_{\kappa-1}^{(3)} \bmod 2^\nu$
21:    $z = h_0^{(3)} + \delta s$
22:    **if** $\max(2^{\eta-\nu+\alpha}, 2^{2\alpha+\nu'}) + 2^{\eta-\nu+\alpha} - 2^\alpha \le 2^{\eta-1}$ **then**
23:        $h_0^{(4)} \leftarrow z$; $h_1^{(4)} \leftarrow h_1^{(3)}$
24:    **else**
25:        $h_0^{(4)} \leftarrow z \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor z/2^\eta \rfloor$; $h_1^{(4)} = h_1^{(3)} + \mathfrak{c}$
26:    **end if**
27:    **for** $i \leftarrow 2$ to $\kappa - 1$ **do** $h_i^{(4)} \leftarrow h_i^{(3)}$ **end for**
28:    $\boxed{\text{FULL REDUCTION: } \textbf{return } h^{(4)}(\theta) = h_0^{(4)} + h_1^{(4)}\theta + \cdots + h_{\kappa-1}^{(4)}\theta^{\kappa-1}}$
29: **end function**.
```

We note that reduceSLMP does not work if $\delta > 1$. This may not be immediately obvious from the description of reduceSLMP. To see that reduceSLMP does not work when $\delta > 1$, one needs to consider the proof of correctness of the algorithm. In the proof of Theorem 2, the step from (13) to (14) uses $2^{(\kappa-1)\eta+\nu} = 2^m \equiv \delta \bmod p$. In the case of Mersenne primes, $\delta = 1$ and so the step from (13) to (14) works; for $\delta > 1$, this step does not work. Instead, we consider a multiplication of the upper half of the input by $c_p = 2^{\eta-\nu}\delta$ at the very beginning and then the resulting polynomial is reduced in several steps. Due to this multiplication, the number of iterations required to obtain the complete reduction in reduceSLPMP is one more than that required in reduceSLMP. Also, the termination argument (that after a certain stage there is no carry) is more complicated.

Remark: The boolean condition in Step 22 of reduceSLPMP does not depend on the input $h^{(0)}(\theta)$ and is determined entirely by η, ν and α. So, once the prime and the values of η and ν are fixed, either the 'then' part of the 'if' statement will be required or, the 'else' part of the 'if' statement will be required. Among the primes considered in Table 3, the 'else' part is required only for the prime $2^{256} - 2^{32} - 977$.

We state a simple result which will be useful in arguing about the termination of reduceSLPMP.

Lemma 6. *Let x, y_1 and y_2 be two integers such that $0 \leq x < 2^\eta$ and $0 \leq y_1, y_2 \leq 2^{\eta-1}$. Then either $x + y_1 < 2^\eta$ or $y_2 + (x + y_1 \bmod 2^\eta) < 2^\eta$.*

Proof. If $0 \leq x < 2^\eta - y_1$, then $x + y_1 < 2^\eta$ and so the result holds. Otherwise, assume that $2^\eta - y_1 \leq x < 2^\eta$. In this case, $2^\eta \leq x + y_1 < 2^\eta + y_1$. So, $0 \leq x + y_1 \bmod 2^\eta < y_1 \leq 2^{\eta-1}$. Consequently, $y_2 \leq y_2 + (x + y_1 \bmod 2^\eta) < y_2 + 2^{\eta-1} \leq 2^\eta$, which proves the result. □

The following result states the correctness of reduceSLPMP.

Theorem 7. *Let $p = 2^m - \delta$ be a prime and let $\kappa \geq 2$, η and ν be such that, m-bit integers have a (κ, η, ν)-representation. Let α be such that $2^{\alpha-1} \leq \delta < 2^\alpha$ and $\alpha < \min(\nu + 1, \eta - 2(1 - \lfloor \nu/\eta \rfloor))$. Suppose that the input $h^{(0)}(\theta)$ to* reduceSLPMP *is the output of either* mulSLDCC$(f(\theta), g(\theta))$ *or* sqrSLDCC$(f(\theta))$ *where*

- *$f(\theta)$ and $g(\theta)$ are m-bit integers having (κ, η, ν)-representations, if $\nu = \eta$;*
- *$f(\theta)$ and $g(\theta)$ are $(m + 1)$-bit integers having $(\kappa, \eta, \nu + 1)$-representations, if $\nu < \eta$.*

Then the following holds.

1. *In the case of partial reduction for $\nu < \eta$, the output $h^{(3)}(\theta)$ of* reduceSLPMP *has a $(\kappa, \eta, \nu + 1)$-representation and $h^{(3)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*
2. *In the case of full reduction, the output $h^{(4)}(\theta)$ of* reduceSLPMP *has a (κ, η, ν)-representation and $h^{(4)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*

Proof. Note that since $p = 2^m - \delta$ is a prime, for $\delta > 1$, δ cannot be a power of 2. Let $\nu' = 2(1 - \lfloor \nu/\eta \rfloor)$ and so $\nu' = 0$ if $\nu = \eta$ and $\nu' = 2$ for $0 < \nu < \eta$. From $\alpha < \min(\nu + 1, \eta - \nu')$, we have

$$\alpha \leq \alpha + \nu' \leq \eta - 1. \tag{24}$$

Also, since $\delta < 2^\alpha$ and $\alpha \leq \nu$, we have $\delta < 2^\nu$. Using $2^{\alpha-1} \leq \delta < 2^\alpha$,

$$\begin{aligned} 2^{\eta-\nu+\alpha-1} \leq c_p = 2^{\eta-\nu}\delta &< 2^{\eta-\nu+\alpha} \\ &\leq 2^\eta \text{ (since } \alpha \leq \nu). \end{aligned} \tag{25}$$

So, $c_p < 2^\eta$ and hence can be considered to be an η-bit word.

The input $h^{(0)}(\theta)$ is the product of two m-bit integers each having a (κ, η, ν)-representation. As in the proof of Theorem 2, using Proposition 1 we have the following bounds on the coefficients of $h^{(0)}(\theta)$.

Case 1: $\nu < \eta$.

$$\begin{array}{lll} 0 \le h_0^{(0)}, h_1^{(0)}, \ldots, h_{2\kappa-3}^{(0)} < 2^\eta; & & \text{and} \\ 0 \le h_{2\kappa-2}^{(0)} < 2^{2(\nu+1)}, & h_{2\kappa-1}^{(0)} = 0 & \text{if} \quad 1 < \nu+1 \le \eta/2; \\ 0 \le h_{2\kappa-2}^{(0)} < 2^\eta, & 0 \le h_{2\kappa-1}^{(0)} < 2^{2(\nu+1)-\eta} & \text{if} \quad \eta/2 < \nu+1 \le \eta. \end{array}$$

Case 2: $\nu = \eta$.

$$0 \le h_0^{(0)}, h_1^{(0)}, \ldots, h_{2\kappa-1}^{(0)} < 2^\eta.$$

For the case $0 < \nu + 1 \le \eta/2$, we have $0 \le h_{2\kappa-2}^{(0)} < 2^{2\nu+2} \le 2^\eta$. The above cases can be merged and the following bounds can be stated for all $0 < \nu \le \eta$.

$$0 \le h_0^{(0)}, h_1^{(0)}, \ldots, h_{2\kappa-3}^{(0)}, h_{2\kappa-2}^{(0)} < 2^\eta \quad \text{and} \quad 0 \le h_{2\kappa-1}^{(0)} < \max(1, 2^{2\nu-\eta+\nu'}). \tag{26}$$

Using $\theta = 2^\eta$ and $p = 2^m - \delta$ we have

$$\theta^\kappa = 2^{\kappa\eta} = 2^{(\kappa-1)\eta+\nu} \cdot 2^{\eta-\nu} = 2^m \cdot 2^{\eta-\nu} \equiv 2^{\eta-\nu}\delta \bmod p = c_p. \tag{27}$$

The input $h^{(0)}$ to reduceSLPMP can be written as

$$\begin{aligned} h^{(0)}(\theta) &= h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} + h_\kappa^{(0)}\theta^\kappa + h_{\kappa+1}^{(0)}\theta^{(\kappa+1)} + \cdots + h_{2\kappa-1}^{(0)}\theta^{(2\kappa-1)} \\ &= (h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1}) + (h_\kappa^{(0)} + h_{\kappa+1}^{(0)}\theta + \cdots + h_{2\kappa-1}^{(0)}\theta^{\kappa-1})\theta^\kappa \\ &\equiv (h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1}) + (h_\kappa^{(0)} + h_{\kappa+1}^{(0)}\theta + \cdots + h_{2\kappa-1}^{(0)}\theta^{\kappa-1})c_p. \end{aligned} \tag{28}$$

Step 5 computes the product $(h_\kappa^{(0)} + h_{\kappa+1}^{(0)}\theta + \cdots + h_{2\kappa-1}^{(0)}\theta^{\kappa-1})c_p$ of (28) using mulSCC, obtaining the output as $(h_\kappa^{(1)} + h_{\kappa+1}^{(1)}\theta + \cdots + h_{2\kappa}^{(1)}\theta^\kappa)$. Step 4 simply copies the values of $h_i^{(0)}$ to $h_i^{(1)}$, for $i = 0, 1, \ldots, \kappa-1$. This defines the polynomial $h^{(1)}(\theta)$ and from (28) we have

$$h^{(0)}(\theta) \equiv \underbrace{(h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1})}_{\text{through Step 4}} + \underbrace{(h_\kappa^{(1)} + h_{\kappa+1}^{(1)}\theta + \cdots + h_{2\kappa}^{(1)}\theta^\kappa)}_{\text{through Step 5}} \bmod p = h^{(1)}(\theta). \tag{29}$$

Limb bounds of $h^{(1)}(\theta)$: The bounds on $h_j^{(0)}$ are given in (26). So, by Step 4, $0 \le h_j^{(1)} < 2^\eta, j = 0, 1, \ldots, \kappa-1$. Let $X(\theta) = h_\kappa^{(0)} + h_{\kappa+1}^{(0)}\theta + \cdots + h_{2\kappa-1}^{(0)}\theta^{\kappa-1}$ and $Y(\theta) = h_\kappa^{(1)} + h_{\kappa+1}^{(1)}\theta + \cdots + h_{2\kappa}^{(1)}\theta^\kappa$. Using (26), the size of the integer represented by $X(\theta)$ is at most $(\kappa-1)\eta + 2\nu - \eta + \nu'$ bits. The integer represented by $Y(\theta)$ is obtained by multiplying $X(\theta)$ by the constant c_p. From (25), the size of c_p is at most $(\eta - \nu + \alpha)$ bits. Hence, the number of bits in the integer represented by $Y(\theta)$ is at most

$$(\kappa-1)\eta + 2\nu - \eta + \nu' + (\eta - \nu + \alpha) \quad = \quad (\kappa-1)\eta + \nu + \alpha + \nu'.$$

If $\nu + \alpha + \nu' \le \eta$, then $Y(\theta)$ has a $(\kappa, \eta, \nu + \alpha + \nu')$-representation. Suppose that $\nu + \alpha + \nu' > \eta$. Since $0 < \nu \le \eta$ and from (24), $\alpha + \nu' < \eta$ we have $\alpha + \nu' < \nu + \alpha + \nu' < 2\eta$. Wrting $(\kappa-1)\eta + (\nu + \alpha + \nu') = \kappa\eta + (\nu + \alpha + \nu' - \eta)$, in this case, $Y(\theta)$ has a $(\kappa+1, \eta, \nu + \alpha + \nu' - \eta)$-representation. Combining the two cases, the limb bounds for $Y(\theta) = (h_\kappa^{(1)} + h_{\kappa+1}^{(1)}\theta + \cdots + h_{2\kappa}^{(1)}\theta^\kappa)$ are $0 \le h_j^{(1)} < 2^\eta, j = \kappa, \kappa+1, \ldots 2\kappa-1$, $0 \le h_{2\kappa}^{(1)} < \max(1, 2^{\nu+\alpha+\nu'-\eta})$. Hence, the limb bounds of $h^{(1)}(\theta)$ can be stated as

$$0 \le h_j^{(1)} < 2^\eta \text{ for } j = 0, 1, \ldots, 2\kappa-1, \text{ and } 0 \le h_{2\kappa}^{(1)} < \max(1, 2^{\nu+\alpha+\nu'-\eta}). \tag{30}$$

Through Steps 6 to 10, we pairwise add the coefficients of $\theta^0, \theta^1, \ldots, \theta^{\kappa-1}$ given in (29) sequentially by forwarding the 1-bit carry, and in Step 10 we add the last carry to $h_{2\kappa}^{(1)}$ producing the $(\kappa+1)$-limb polynomial $h^{(2)}(\theta)$. Hence, from (29) we have

$$\begin{aligned} h^{(1)}(\theta) &= (h_0^{(1)} + h_\kappa^{(1)}) + (h_1^{(1)} + h_{\kappa+1}^{(1)})\theta + \cdots + (h_{\kappa-1}^{(1)} + h_{2\kappa-1}^{(1)})\theta^{\kappa-1} + h_{2\kappa}^{(1)}\theta^\kappa, \\ &= \underbrace{h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1} + h_\kappa^{(2)}\theta^\kappa}_{\text{through Steps 6 to 10}} = h^{(2)}(\theta). \end{aligned} \tag{31}$$

Limb bounds of $h^{(2)}(\theta)$: Using the bounds in (30), the bounds on the limbs of $h^{(2)}(\theta)$ defined in (31) are given by

$$0 \le h_j^{(2)} < 2^\eta \text{ for } j = 0, 1, \ldots, \kappa-1, \text{ and } 0 \le h_\kappa^{(2)} \le \max(1, 2^{\nu+\alpha+\nu'-\eta}). \tag{32}$$

In Step 11, $r = 2^{\eta-\nu}h_\kappa^{(2)} + \lfloor h_{\kappa-1}^{(2)}/2^\nu \rfloor$ is computed and the product δr is used in Step 12. The bounds on r and δr are obtained as follows.

Bounds on r and δr: From (32), we have $0 \le h_{\kappa-1}^{(2)} < 2^\eta$ and so $\lfloor h_{\kappa-1}^{(2)}/2^\nu \rfloor < 2^{\eta-\nu}$ i.e., $\lfloor h_{\kappa-1}^{(2)}/2^\nu \rfloor \le 2^{\eta-\nu} - 1$. Also, from (32), we have $0 \le h_\kappa^{(2)} \le \max(1, 2^{\nu+\alpha+\nu'-\eta})$. From the definition of r, we obtain

$$\begin{aligned} 0 &\le r \le \max(2^{\eta-\nu}, 2^{\alpha+\nu'}) + 2^{\eta-\nu} - 1 \\ \Rightarrow 0 &\le \delta r < \max(2^{\eta-\nu+\alpha}, 2^{2\alpha+\nu'}) + 2^{\eta-\nu+\alpha} - 2^\alpha \text{ [since } \delta < 2^\alpha]. \end{aligned} \tag{33}$$

If the boolean condition in Step 22 holds, then

$$0 \le \delta r < 2^{\eta-1}. \tag{34}$$

Otherwise, a bound on δr is obtained by continuing the computation of (33) as follows.

$$\begin{aligned} \Rightarrow 0 &\le \delta r < \max(2^{2\eta-\nu-1}, 2^{2\eta-2}) + 2^{2\eta-\nu-1} - 2^{\eta-1} \text{ [from (24) } \alpha, \alpha+\nu' \le \eta - 1] \\ \Rightarrow 0 &\le \delta r < 2^{2\eta-2} + 2^{2\eta-\nu-1} - 2^{\eta-1} \text{ [since } \nu \ge 1] \\ \Rightarrow 0 &\le \delta r < 2^{2\eta-1} - 2^{\eta-1} \text{ [again since } \nu \ge 1]. \end{aligned} \tag{35}$$

So, (35) holds irrespective of whether the boolean condition in Step 22 holds or not. The variable u is defined in Step 12. From (32) and (35) an upper bound on u is as follows.

$$u = h_0^{(2)} + \delta r < 2^\eta - 1 + 2^{2\eta-1} - 2^{\eta-1}. \tag{36}$$

Write $h_{\kappa-1}^{(2)} = h_{\kappa-1,0}^{(2)} + h_{\kappa-1,1}^{(2)}2^\nu$ where $h_{\kappa-1,0}^{(2)} = h_{\kappa-1}^{(2)} \bmod 2^\nu$ and $h_{\kappa-1,1}^{(2)} = \lfloor h_{\kappa-1}^{(2)}/2^\nu \rfloor$. Then

$$\begin{aligned} h^{(2)}(\theta) &= h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1} + h_\kappa^{(2)}\theta^\kappa \\ &= h_0^{(2)} + h_1^{(2)}\theta + \cdots + (h_{\kappa-1,0}^{(2)} + h_{\kappa-1,1}^{(2)}2^\nu)\theta^{\kappa-1} + h_\kappa^{(2)}\theta^\kappa \\ &= h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1,0}^{(2)}\theta^{\kappa-1} + h_{\kappa-1,1}^{(2)}2^{\eta(\kappa-1)+\nu} + h_\kappa^{(2)}\theta^\kappa \\ &= h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1,0}^{(2)}\theta^{\kappa-1} + h_{\kappa-1,1}^{(2)}2^m + h_\kappa^{(2)}\theta^\kappa \\ &\equiv h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1,0}^{(2)}\theta^{\kappa-1} + h_{\kappa-1,1}^{(2)}\delta + h_\kappa^{(2)}c_p \text{ [using } 2^m \equiv \delta \bmod p \text{ and (25)]} \\ &= h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1,0}^{(2)}\theta^{\kappa-1} + (h_{\kappa-1,1}^{(2)} + h_\kappa^{(2)}2^{\eta-\nu})\delta \text{ [using (25)]} \\ &= (r\delta + h_0^{(2)}) + h_1^{(2)}\theta + \cdots + h_{\kappa-1,0}^{(2)}\theta^{\kappa-1} \text{ [from Step 11]} \\ &= u + h_1^{(2)}\theta + \cdots + h_{\kappa-1,0}^{(2)}\theta^{\kappa-1} \text{ [from Step 12]} \qquad (37) \\ &\equiv \underbrace{h_0^{(3)} + h_1^{(3)}\theta + \cdots + h_\kappa^{(3)}\theta^\kappa}_{\text{through Steps 12 to 19}} \bmod p = h^{(3)}(\theta). \qquad (38) \end{aligned}$$

The analysis of the rest of the algorithm, i.e., Steps 12 to 27 is divided into two cases depending on whether $u < 2^\eta$ or $u \geq 2^\eta$.

Case 1: $u < 2^\eta$. In this case, $q = \lfloor u/2^\eta \rfloor = 0$, and so from Steps 13 to 16 we have $0 \leq h_j^{(3)} = h_j^{(2)} < 2^\eta$ for $j = 1, 2, \ldots, \kappa - 2$. Also, $0 \leq h_{\kappa-1}^{(3)} = h_{\kappa-1}^{(2)} < 2^\nu$, because in Step 11 we have already updated $h_{\kappa-1}^{(2)}$ by $h_{\kappa-1}^{(2)} \bmod 2^\nu$. By Step 12, we have $h_0^{(3)} = u \bmod 2^\eta < 2^\eta$, and Step 19 gives $h_\kappa^{(3)} = 0$. So, in this case, $h^{(3)}(\theta)$ returned at Step 18 has a (κ, η, ν)-representation irrespective of whether $\nu < \eta$ or $\nu = \eta$. By (29), (31) and (38) we have $h^{(3)}(\theta) \equiv h^{(0)}(\theta) \bmod p$. This proves the statement of the theorem on partial reduction for **Case 1**.

Further, we have $s = 0$ by Step 20, and so by the remaining steps of the algorithm we have $0 \leq h_j^{(4)} = h_j^{(3)} < 2^\eta$ for $j = 0, 1, \ldots, \kappa - 2$ and $0 \leq h_{\kappa-1}^{(4)} = h_{\kappa-1}^{(3)} < 2^\nu$, i.e., $h^{(4)}(\theta)$ has a (κ, η, ν)-representation. It follows that $h^{(4)}(\theta) = h^{(3)}(\theta)$ and using (29), (31) and (38), we have $h^{(4)}(\theta) \equiv h^{(0)}(\theta) \bmod p$ which proves the statement of the theorem on full reduction for **Case 1**.

Case 2: $u \geq 2^\eta$. Step 12 defines q to be $q = \lfloor u/2^\eta \rfloor$. Since in this case $u \geq 2^\eta$, the bounds on q are the following.

$$\begin{aligned}
1 &\leq q \leq \left\lfloor \frac{2^\eta - 1 + 2^{2\eta-1} - 2^{\eta-1}}{2^\eta} \right\rfloor \quad \text{[using (36)]} \\
\Rightarrow\ 1 &\leq q \leq \left\lfloor 1 - \frac{1}{2^\eta} + 2^{\eta-1} - \frac{1}{2} \right\rfloor \\
\Rightarrow\ 1 &\leq q \leq 2^{\eta-1} + \left\lfloor \frac{1}{2} - \frac{1}{2^\eta} \right\rfloor \\
\Rightarrow\ 1 &\leq q \leq 2^{\eta-1} < 2^\eta - 1.
\end{aligned} \tag{39}$$

In Step 13, the algorithm computes $v = h_1^{(2)} + q$. There are two sub cases to consider depending on whether $v < 2^\eta$ or $v \geq 2^\eta$.

Subcase 2a: $v < 2^\eta$. Step 13 defines $\mathfrak{c} = \lfloor v/2^\eta \rfloor$ and so $\mathfrak{c} = 0$ at this step. This simplifies the analysis and the rest of the proof is similar to that of **Case 1**.

Subcase 2b: $v \geq 2^\eta$. This is the non-trivial case and it is required to argue that there are no overflows. In this case, using (32) and (39), we have

$$2^\eta \leq v = h_1^{(2)} + q < 2^\eta + 2^\eta - 1 = 2^{\eta+1} - 1. \tag{40}$$

So, $2^\eta \leq v \leq 2^{\eta+1} - 2 = 2^\eta + 2^\eta - 2$ implying $v \bmod 2^\eta \leq 2^\eta - 2 < 2^\eta - 1$. After Step 13,

$$h_1^{(3)} = v \bmod 2^\eta < 2^\eta - 1 \text{ and } 0 \leq \mathfrak{c} \leq 1.$$

Consider $h^{(3)}(\theta)$ as given in (38).

- By Step 12, we have $0 \leq h_0^{(3)} < 2^\eta$.
- Since $\mathfrak{c} \leq 1$, considering Step 15 for $i = 2, 3, \ldots, \kappa - 2$, shows $0 \leq h_j^{(3)} < 2^\eta$ for $j = 2, 3, \ldots, \kappa - 2$.
- Recall that $h_{\kappa-1,0}^{(2)} = h_{\kappa-1}^{(2)} \bmod 2^\nu \leq 2^\nu - 1$ and consider Step 15 for $i = \kappa - 1$. Since $\mathfrak{c} \leq 1$, we have $h_{\kappa-1}^{(3)} \leq 2^\nu$ and so $h_{\kappa-1}^{(3)}$ is a $(\nu + 1)$-bit integer.

Consequently, if $\nu < \eta$, then $h^{(3)}(\theta)$ has a $(\kappa, \eta, \nu+1)$-representation and by (29), (31) and (38) we have $h^{(3)}(\theta) \equiv h^{(0)}(\theta) \bmod p$. This proves the statement of the theorem on partial reduction for **Subcase 2b**.

On the other hand, if $\nu = \eta$, then $h^{(3)}_{\kappa-1}$ will be an $(\eta+1)$-bit string (equivalently, $h^{(3)}(\theta)$ will be an $(m+1)$-bit integer) and further reduction is required to ensure that the number of limbs in the final result is κ. So, for $\nu = \eta$, partial reduction is not useful. The general analysis for obtaining the final reduction irrespective of whether $\nu < \eta$ or $\nu = \eta$ is given below.

In Steps 20 and 21, the algorithm computes $s = 2^{\eta-\nu}h^{(3)}_{\kappa} + \lfloor h^{(3)}_{\kappa-1}/2^{\nu} \rfloor$ and $z = h^{(3)}_0 + \delta s$ respectively. Write $h^{(3)}_{\kappa-1} = h^{(3)}_{\kappa-1,0} + h^{(3)}_{\kappa-1,1}2^{\nu}$ where $h^{(3)}_{\kappa-1,0} = h^{(3)}_{\kappa-1} \bmod 2^{\nu}$ and $h^{(3)}_{\kappa-1,1} = \lfloor h^{(3)}_{\kappa-1}/2^{\nu} \rfloor$. Then

$$\begin{aligned}
h^{(3)}(\theta) &= h^{(3)}_0 + h^{(3)}_1\theta + \cdots + h^{(3)}_{\kappa-1}\theta^{\kappa-1} + h^{(3)}_{\kappa}\theta^{\kappa} \\
&= h^{(3)}_0 + h^{(3)}_1\theta + \cdots + (h^{(3)}_{\kappa-1,0} + h^{(3)}_{\kappa-1,1}2^{\nu})\theta^{\kappa-1} + h^{(3)}_{\kappa}\theta^{\kappa} \\
&= h^{(3)}_0 + h^{(3)}_1\theta + \cdots + h^{(3)}_{\kappa-1,0}\theta^{\kappa-1} + h^{(3)}_{\kappa-1,1}2^{\eta(\kappa-1)+\nu} + h^{(3)}_{\kappa}\theta^{\kappa} \\
&= h^{(3)}_0 + h^{(3)}_1\theta + \cdots + h^{(3)}_{\kappa-1,0}\theta^{\kappa-1} + h^{(3)}_{\kappa-1,1}2^{m} + h^{(3)}_{\kappa}\theta^{\kappa} \\
&\equiv h^{(3)}_0 + h^{(3)}_1\theta + \cdots + h^{(3)}_{\kappa-1,0}\theta^{\kappa-1} + h^{(3)}_{\kappa-1,1}\delta + h^{(3)}_{\kappa}c_p \bmod p \text{ [using (5) and (25)]} \\
&= h^{(3)}_0 + h^{(3)}_1\theta + \cdots + h^{(3)}_{\kappa-1,0}\theta^{\kappa-1} + (h^{(3)}_{\kappa-1,1} + h^{(3)}_{\kappa}2^{\eta-\nu})\delta \text{ [using (27)]} \\
&= (s\delta + h^{(3)}_0) + h^{(3)}_1\theta + \cdots + h^{(3)}_{\kappa-1,0}\theta^{\kappa-1} \text{ [from Step 20]} \\
&= z + h^{(3)}_1\theta + \cdots + h^{(3)}_{\kappa-1,0}\theta^{\kappa-1} \text{ [from Step 21]}. \qquad (41)
\end{aligned}$$

Claim 8. *The value of s computed in Step 20 is at most 1.*

Proof. The value of s is $2^{\eta-\nu}h^{(3)}_{\kappa} + \lfloor h^{(3)}_{\kappa-1}/2^{\nu} \rfloor$.

In Step 11, $h^{(2)}_{\kappa-1}$ is set to $h^{(2)}_{\kappa-1} \bmod 2^{\nu}$ and so after this step $0 \le h^{(2)}_{\kappa-1} < 2^{\nu}$. Consider Steps 17 and 19. We have $0 \le h^{(2)}_{\kappa-1} < 2^{\nu}$ and so the value of t at Step 19 is at most 2^{ν}.

- The value of $h^{(3)}_{\kappa-1}$ is set to be equal to $t \bmod 2^{\eta}$. So, if $\nu < \eta$ then $0 \le h^{(3)}_{\kappa-1} \le 2^{\nu}$, while if $\nu = \eta$ then $0 \le h^{(3)}_{\kappa-1} < 2^{\eta}$.
- The updated value of $\mathfrak{c}$ is $\lfloor t/2^{\eta} \rfloor$ and this can be equal to 1 only if $\nu = \eta$. This value of $\mathfrak{c}$ is assigned to $h^{(3)}_{\kappa}$ in Step 19. So, $h^{(3)}_{\kappa} = 1$ only if $\nu = \eta$.

If $\nu < \eta$, then $h^{(3)}_{\kappa} = 0$ and $\lfloor h^{(3)}_{\kappa-1}/2^{\nu} \rfloor \le 1$ implying that $s \le 1$. On the other hand, if $\nu = \eta$, then $2^{\eta-\nu}h^{(3)}_{\kappa} = h^{(3)}_{\kappa} \le 1$ and $\lfloor h^{(3)}_{\kappa-1}/2^{\nu} \rfloor = \lfloor h^{(3)}_{\kappa-1}/2^{\eta} \rfloor = 0$ again implying that $s \le 1$. □

If $s = 0$, then $z = h^{(3)}_0$, implying $h^{(4)}_0 = h^{(3)}_0$ and $\mathfrak{c}$ at Step 25 is 0. So, $h^{(4)}_0 = h^{(3)}_0$ and $h^{(4)}_1 = h^{(3)}_1$ hold for both branches of the 'if' statement in Step 22. From Step 27 it follows that $h^{(4)}(\theta) = h^{(3)}(\theta)$.

If $s = 1$, then $z = h^{(3)}_0 + \delta$. The termination arguments for the two branches of the 'if' statement at Step 22 are different.

First suppose that the boolean condition of the 'if' statement evaluates to true. We apply Lemma 6 with $x = h^{(2)}_0$, $y_1 = \delta r$ and $y_2 = \delta$. From (34), we have $0 \le \delta r < 2^{\eta-1}$ which also implies $0 < \delta < 2^{\eta-1}$. In Step 12, u is computed as $u = h^{(2)}_0 + \delta r = x + y_1$ and $h^{(3)}_0 = u \bmod 2^{\eta} = x + y_1 \bmod 2^{\eta}$. In Step 23, $h^{(0)}_4 = z = h^{(3)}_0 + \delta = y_2 + (x + y_1 \bmod 2^{\eta})$. In **Case 2**, $u \ge 2^{\eta}$, i.e., $x + y_1 \ge 2^{\eta}$. Then from Lemma 6, we have $h^{(0)}_4 = y_2 + (x + y_1 \bmod 2^{\eta}) < 2^{\eta}$. So, the procedure terminates.

Now consider the case that the boolean condition of the 'if' statement evaluates to false. By Step 20 we have $0 \le h^{(3)}_{\kappa-1} < 2^{\nu}$ and $0 \le h^{(4)}_0 < 2^{\eta}$ by Step 25. The value of $\mathfrak{c}$ in Step 25 can be at most 1, and since the bound of $h^{(3)}_1$ is $0 \le h^{(3)}_1 < 2^{\eta} - 1$, hence after Step 25 we have $0 \le h^{(4)}_1 < 2^{\eta}$.

So, after both branches of the 'if' statement in Step 22, the limb bounds of $h^{(4)}(\theta)$ are $0 \leq h_j^{(4)} < 2^\eta$ for $j = 0, 1, \ldots, \kappa - 2$, and $0 \leq h_{\kappa-1}^{(4)} < 2^\nu$. From (41) we can write

$$h^{(3)}(\theta) \quad \equiv \quad \underbrace{h_0^{(4)} + h_1^{(4)}\theta + \cdots + h_{\kappa-1}^{(4)}\theta^\kappa}_{\text{through Steps 20 to 27}} \bmod p = h^{(4)}(\theta). \tag{42}$$

Combining (29), (31), (38) and (42) we have $h^{(4)}(\theta) \equiv h^{(0)}(\theta) \bmod p$, which proves the statement of the theorem on full reduction for **Subcase 2b**. □

Usefulness of partial reduction: The statement of Theorem 7 identifies two cases. If $\nu < \eta$, then the input to reduceSLPMP is considered to be a $(2m+2)$-bit integer, whereas if $\nu = \eta$, then the input to reduceSLPMP is considered to be a $2m$-bit integer. This is the consequence of whether partial reduction is used or not. In the case of $\nu < \eta$, a partial reduction strategy is used whereas for $\nu = \eta$, such a strategy is not used. For the partial reduction strategy, the output $h^{(3)}(\theta)$ returned by reduceSLPMP is an $(m + 1)$-bit integer. So, if partial reduction strategy is used throughout, then the inputs to mulSLDCC and sqrSLDCC will also be $(m + 1)$-bit integers and so their outputs will be $(2m + 2)$-bit integers. Subsequent applications of reduceSLPMP will have to handle $(2m + 2)$-bit integers. This is the reason why the statement of Theorem 7 specifies the input to reduceSLPMP to be a $(2m + 2)$-bit integer for the case $\nu < \eta$. On the other hand, for $\nu = \eta$, partial reduction is not used and so the output $h^{(4)}(\theta)$ of reduceSLPMP is an m-bit integer and consequently, the outputs of mulSLDCC and sqrSLDCC will be $2m$-bit integers.

Partial reduction is useful since it avoids the computation required to reduce $h^{(3)}(\theta)$ to $h^{(4)}(\theta)$. All intermediate computations are performed using partially reduced results and the full reduction is invoked only once at the end. This strategy leads to substantial savings in the number of operations and hence on the consequent speed of computation.

There does not seem to be an efficient way in which the partial reduction strategy can be made to work for Mersenne primes. A possible partially reduced result in reduceSLMP would be $h^{(2)}(\theta)$. It can be shown that when the input to reduceSLMP is the product of two m-bit integers each having (κ, η, ν)-representation, then $h^{(2)}(\theta)$ has a $(\kappa, \eta, \nu + 1)$-representation, i.e., it is an $(m + 1)$-bit integer. So, mulSLDCC and sqrSLDCC will produce as output $(2m + 2)$-bit integers. Feeding such an integer as the input of reduceSLMP results in $h^{(2)}(\theta)$ having a $(\kappa, \eta, \nu + 3)$-representation. In other words, the size of the last limb grows. This can be brought down, but, doing this requires additional computation and results in the partial reduction being less efficient than the full reduction that we have described. In the case of pseudo-Mersenne primes, the partial result returned by reduceSLPMP avoids such growth of the last limb.

Remark: Step 5 of reduceSLPMP performs a mulSCC call. As described in Section 5, this call can be implemented using a single carry chain by using the `mulx` and `adc` instructions. In reduceSLPMP, Steps 6 to 9 add the output of mulSCC to the initial κ limbs of the input $h^{(0)}$. It is possible to consider a strategy whereby the multiplications and the additions within mulSCC are done simultaneously with the additions in Steps 6 to 9. It is possible to organise the code such that two independent carry chains arise so that one of the addition chains is implemented using `adcx` and the other using `adox`. We have implemented this strategy, but, the gain in speed is not significant and so we do not describe the details.

A variant of reduceSLPMP: Bernstein et al. [5] have used an algorithm for partial reduction using a 4-limb representation of $2^{255} - 19$. Function reduceSLPMPa in Algorithm 7 provides a generalisation of this algorithm which works for a large class of pseudo-Mersenne primes.

Algorithm 7 Partial reduction for saturated limb representation. Performs reduction modulo p, where $p = 2^m - \delta$ is a pseudo-Mersenne prime; $c_p = 2^{\eta-\nu}\delta$ and $\theta = 2^\eta$.

```
 1: function reduceSLPMPa($h_0^{(0)}(\theta)$)
 2: input: $h^{(0)}(\theta)$.
 3: output: $h^{(4)}(\theta)$.
 4:     for $i \leftarrow 0$ to $\kappa - 1$ do $h_i^{(1)} \leftarrow h_i^{(0)}$ end for
 5:     $h_\kappa^{(1)} + h_{\kappa+1}^{(1)}\theta + \cdots + h_{2\kappa}^{(1)}\theta^\kappa \leftarrow \mathsf{mulSCC}(h_\kappa^{(0)} + h_{\kappa+1}^{(0)}\theta + \cdots + h_{2\kappa-1}^{(0)}\theta^{\kappa-1}, c_p)$
 6:     $t \leftarrow h_0^{(1)} + h_\kappa^{(1)}$; $h_0^{(2)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
 7:     for $i \leftarrow 1$ to $\kappa - 1$ do
 8:         $t \leftarrow h_i^{(1)} + h_{\kappa+i}^{(1)} + \mathfrak{c}$; $h_i^{(2)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
 9:     end for
10:     $h_\kappa^{(2)} \leftarrow h_{2\kappa}^{(1)} + \mathfrak{c}$
11:     $r \leftarrow h_\kappa^{(2)} \cdot c_p$
12:     $u \leftarrow h_0^{(2)} + r$; $h_0^{(3)} \leftarrow u \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor u/2^\eta \rfloor$
13:     for $i \leftarrow 1$ to $\kappa - 1$ do
14:         $t \leftarrow h_i^{(2)} + \mathfrak{c}$; $h_i^{(3)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
15:     end for
16:     $h_\kappa^{(3)} \leftarrow \mathfrak{c}$
17:     $s \leftarrow h_\kappa^{(3)} \cdot c_p$
18:     $z = h_0^{(3)} + s$;
19:     if $\max(2^{\eta-\nu+\alpha}, 2^{2\alpha+\nu'}) \leq 2^{\eta-1}$ then
20:         $h_0^{(4)} \leftarrow z$; $h_1^{(4)} \leftarrow h_1^{(3)}$
21:     else
22:         $h_0^{(4)} \leftarrow z \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor z/2^\eta \rfloor$; $h_1^{(4)} = h_1^{(3)} + \mathfrak{c}$
23:     end if
24:     for $i \leftarrow 2$ to $\kappa - 1$ do $h_i^{(4)} \leftarrow h_i^{(3)}$ end for
25:     PARTIAL REDUCTION: return $h^{(4)}(\theta) = h_0^{(4)} + h_1^{(4)}\theta + \cdots + h_{\kappa-1}^{(4)}\theta^{\kappa-1}$
26: end function.
```

Similar to reduceSLPMP, the boolean condition in Step 19 of reduceSLPMPa does not depend on the input $h^{(0)}(\theta)$ and is determined entirely by η, ν and α. So, either the 'then' part of the 'if' statement will be required or, the 'else' part of the 'if' statement will be required. Among the primes considered in Table 3, the 'else' part is required only for the prime $2^{256} - 2^{32} - 977$.

The following result states the correctness of reduceSLPMPa.

Theorem 9. *Let $p = 2^m - \delta$ be a prime and let $\kappa \geq 2$, η and ν be such that, m-bit integers have a (κ, η, ν)-representation. Let α be such that $2^{\alpha-1} \leq \delta < 2^\alpha$. Suppose that the input $h^{(0)}(\theta)$ to* reduceSLPMP *is the output of either* mulSLDCC$(f(\theta), g(\theta))$ *or* sqrSLDCC$(f(\theta))$ *where $f(\theta)$ and $g(\theta)$ are $\kappa\eta$-bit integers having (κ, η, η)-representations. Then the output $h^{(4)}(\theta)$ of* reduceSLPMPa *has a (κ, η, η)-representation and $h^{(4)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*

Proof. The proof is similar to the proof of Theorem 7. The inputs to reduceSLPMP and reduceSLPMPa are of different sizes. On the other hand, Steps 4 to 10 of reduceSLPMPa is exactly the same as that of reduceSLPMP. So, the bounds on the limbs of $h^{(2)}(\theta)$ can be derived in a manner similar to the bounds obtained in (32) and are as follows: $0 \leq h_j^{(2)} < 2^\eta$ for $j = 0, 1, \ldots, \kappa - 1$, and $0 \leq h_\kappa^{(2)} \leq \max(1, 2^{\nu+\alpha-\eta})$. Since r is computed as $r = h_\kappa^{(2)} \cdot c_p = h_\kappa^{(2)} \cdot 2^{\eta-\nu}\delta < 2^{\eta-\nu+\alpha}$, we have $0 \leq r \leq \max(2^{\eta-\nu+\alpha}, 2^{2\alpha})$. The upper bound on r gives rise to the boolean condition in Step 19. The rest of the argument proceeds along the same lines as that of Theorem 7 and is in fact a bit simpler. The 'if' statement in Step 19 determines whether the last addition takes place at limb number 0 (which is the 'then' part), or, whether it takes place at limb number 1 (which is the 'else' part). The 'else' part is required only if the addition to limb number 0 can produce a carry. This part of the argument is similar to the argument for termination corresponding to $s = 1$ in the proof of Theorem 7. □

Comparison of reduceSLPMP and reduceSLPMPa: We note the following points.

1. Full reduction can be obtained using reduceSLPMP, but, reduceSLPMPa always performs partial reduction. So, if reduceSLPMPa is used, then the last reduction is to be done by reduceSLPMP, or, the final output of reduceSLPMPa is to be further reduced using some other method. On the other hand, if $\nu < \eta$ and reduceSLPMP is used, then partial reduction will be done for all but the last invocation, and the last invocation will perform full reduction. No other code is required to ensure full reduction.

2. The computation of r in reduceSLPMP is slightly more expensive than the computation of r in reduceSLPMPa. Using partial reduction for reduceSLPMP avoids generating $h^{(4)}(\theta)$ from $h^{(3)}(\theta)$ saving a few instructions. Compared to reduceSLPMPa, saving these instructions more or less balances the extra cost of generating r.

3. We have implemented both reduceSLPMP and reduceSLPMPa as part of the various inversion algorithms. There does not appear to be any significant difference in the timings.

Remark: Theorems 2, 7 and 9 also hold if the algorithms mulSL and sqrSL are used instead of mulSLDCC and sqrSLDCC respectively.

7 Multiplication Using Unsaturated Limb Representation

The unsaturated limb representation has been very effectively used in the various implementations of Curve25519 [2, 6, 5, 9].

In the case of saturated limb representation, the tasks of integer multiplication/squaring and reduction are completely separate, i.e., the integer multiplication step simply multiplies two integers while the integer squaring step simply squares an integer without any reference to the prime which will be used to perform the reduction step. In the case of unsaturated limb representation, the multiplication/squaring step is not simply an integer multiplication/squaring. It uses the underlying prime to return an intermediate reduced result which is then provided as input to the reduction algorithm. Two strategies are described below. The first strategy is a generalisation to arbitrary pseudo-Mersenne primes of a strategy used for Curve25519 [5]. For some primes, however, this strategy leads to overflow in the intermediate result. To handle such cases, we describe a modified strategy which works for a larger class of primes. To the best of our knowledge, this modified strategy has not appeared earlier in the literature either in its general form or, for any particular prime.

As in Section 3, let $p = 2^m - \delta$, $\theta = 2^\eta$, and $c_p = 2^{\eta-\nu}\delta$. Since we are working with the unsaturated limb representation, $\eta < 64$. Let $f(\theta)$ and $g(\theta)$ be two elements of $\mathbb{F}_p$ written as

$$\begin{aligned} f(\theta) &= f_0 + f_1\theta + \cdots + f_{\kappa-1}\theta^{\kappa-1}, \\ g(\theta) &= g_0 + g_1\theta + \cdots + g_{\kappa-1}\theta^{\kappa-1}, \end{aligned}$$

where $0 \leq f_i, g_i < 2^\eta$ for $i = 0, 1, \ldots, \kappa - 2$, and $0 \leq f_{\kappa-1},\ g_{\kappa-1} < 2^\nu$. The product of $f(\theta)$ and $g(\theta)$

modulo p can be written as the polynomial $h(\theta) = h_0 + h_1\theta + \cdots + h_{\kappa-1}\theta^{\kappa-1}$ where

$$\begin{array}{rcl} h_0 &=& f_0g_0 + c_p(f_1g_{\kappa-1} + f_2g_{\kappa-2} + \cdots + f_{\kappa-2}g_2 + f_{\kappa-1}g_1), \\ h_1 &=& f_0g_1 + f_1g_0 + c_p(f_2g_{\kappa-1} + \cdots + c_pf_{\kappa-2}g_3 + f_{\kappa-1}g_2), \\ \cdots & \cdots & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ h_{\kappa-2} &=& f_0g_{\kappa-2} + f_1g_{\kappa-3} + f_2g_{\kappa-4} + \cdots + f_{\kappa-2}g_0 + c_pf_{\kappa-1}g_{\kappa-1}, \\ h_{\kappa-1} &=& f_0g_{\kappa-1} + f_1g_{\kappa-2} + f_2g_{\kappa-3} + \cdots + f_{\kappa-2}g_1 + f_{\kappa-1}g_0. \end{array} \tag{43}$$

Substituting $g = f$, we get similar equations for squaring and during the squaring computation, each cross-product term is computed only once. We have

$$\mathfrak{h}_{\max} = \max(h_0, \ldots, h_{\kappa-1}). \tag{44}$$

If $\mathfrak{h}_{\max} < 2^{128}$, then each of the coefficients h_i, $i = 0, \ldots, \kappa - 1$ fit in two 64-bit words. In such cases, the above strategy for multiplication/squaring is feasible. We denote the resulting algorithm for multiplication (resp. squaring) as mulUSL (resp. sqrUSL). We note that for many primes, $\mathfrak{h}_{\max}$ is significantly below 2^{128} and this plays a role in the efficient implementation of the subsequent reduction algorithm.

7.1 Modified Multiplication Strategy

In the case where $\mathfrak{h}_{\max} \geq 2^{128}$, the coefficients in (43) do not fit within two 64-bit words. This happens when the value of $c_p = 2^{\eta-\nu}\delta$ is a bit large. One example of such a prime is $2^{256} - 2^{32} - 977$ for which $\delta = 2^{32} + 977$, $\kappa = 5$, $\eta = 52$ and $\nu = 48$ so that $c_p = 16(2^{32} + 977)$ is a 37-bit integer. To handle such primes, we describe a simple modification of the previous strategy. Define

$$\begin{array}{rcl} u_0 &=& f_1g_{\kappa-1} + f_2g_{\kappa-2} + \cdots + f_{\kappa-2}g_2 + f_{\kappa-1}g_1, \\ u_1 &=& f_2g_{\kappa-1} + \cdots + f_{\kappa-2}g_3 + f_{\kappa-1}g_2, \\ \cdots & \cdots & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ u_{\kappa-2} &=& f_{\kappa-1}g_{\kappa-1}, \end{array} \tag{45}$$

where $\max(u_0, u_1, \ldots, u_{\kappa-2}) = u_0 \leq \mathfrak{u}_{\max}$ with $\mathfrak{u}_{\max} = (2^\eta - 1)^2(\kappa - 1)$. For $i = 0, 1, \ldots, \kappa - 2$, define

$$u_{i,0} = u_i \bmod 2^\eta,\ u_{j,1} = \lfloor u_j/2^\eta \rfloor \quad \text{so that} \quad u_j = u_{j,0} + 2^\eta u_{j,1} = u_{j,0} + u_{j,1}\theta. \tag{46}$$

Then for $f(\theta) \cdot g(\theta) = h(\theta) = h_0 + h_1\theta + \cdots + h_{\kappa-1}\theta^{\kappa-1}$ such that the coefficients $h_0, \ldots, h_{\kappa-1}$ are given by (43), we have $h(\theta) = h'(\theta) = h'_0 + h'_1\theta + \cdots + h'_{\kappa-1}\theta$ where

$$\begin{array}{rcl} h'_0 &=& f_0g_0 + c_pu_{0,0}, \\ h'_1 &=& f_0g_1 + f_1g_0 + c_p(u_{1,0} + u_{0,1}), \\ \cdots & \cdots & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ h'_{\kappa-2} &=& f_0g_{\kappa-2} + f_1g_{\kappa-3} + \cdots + f_{\kappa-2}g_0 + c_p(u_{\kappa-2,0} + u_{\kappa-1,1}), \\ h'_{\kappa-1} &=& f_0g_{\kappa-1} + f_1g_{\kappa-2} + \cdots + f_{\kappa-1}g_0 + c_pu_{\kappa-2,1}. \end{array} \tag{47}$$

Let

$$\mathfrak{h}'_{\max} = \max(h'_0, \ldots, h'_{\kappa-1}). \tag{48}$$

If $\mathfrak{u}_{\max} < 2^{128}$ and $\mathfrak{h}'_{\max} < 2^{128}$, then each of the coefficients u_i, $i = 0, 1, \ldots, \kappa - 2$ and also each of the coefficients h_j, $j = 0, 1, \ldots, \kappa - 1$ fit in two 64-bit words. So, even if some coefficient of $h(\theta)$ is greater than or equal to 2^{128}, it is still feasible to compute $h'(\theta)$ using 64-bit arithmetic without any overflow. We denote the resulting multiplication and squaring algorithms by mulUSLa and sqrUSLa respectively.

Remarks:

1. The rationale for obtaining $h'_0, h'_1, \ldots, h'_{\kappa-1}$ is that $h_0, h_1, \ldots, h_{\kappa-1}$ are not 128-bit quantities. For certain primes, it may happen that there is an $i \in \{0, 1, \ldots, \kappa-1\}$ such that $h_0, h_1, \ldots, h_i$ are greater than $2^{128}-1$ while $h_{i+1}, h_{i+2} \ldots, h_{\kappa-1}$ are each at most $2^{128}-1$. In such cases, it would be sufficient to use $h'_0, h'_1 \ldots, h'_i, h_{i+1}, \ldots, h_{\kappa-1}$.

2. In [16], a different strategy was used to tackle the situation when $\mathfrak{h}_{\max} \geq 2^{128}$. This strategy consists of 'expanding' $u_0, u_1, \ldots, u_{\kappa-2}$ to κ, η-bit quantities $\mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_{\kappa-2}$ and then adding $c_p u_0$ to $f_0 g_0$; $c_p u_1$ to $f_0 g_1 + f_1 g_0$ and so on. In the present case, this strategy turns out to be less efficient than the strategy used to obtain $h'_0, h'_1, \ldots, h'_{\kappa-1}$.

7.2 Dovetailing with Reduction Algorithms

The outputs of the multiplication/squaring algorithms are fed as inputs into the reduction algorithms and the outputs of the reduction algorithms are fed as inputs to the multiplication/squaring algorithms. In the (κ, η, ν)-representation of an m-bit integer, each of the first $\kappa-1$ limbs is η bits long and the last limb is ν bits long. So, one may consider the goal of the reduction algorithms to ensure that the output indeed has a (κ, η, ν)-representation. It is, however, more efficient to obtain a partial reduction, where some of the coefficients of the output of the reduction algorithms may have one extra bit. Such a strategy is feasible, if the multiplication/squaring algorithms applied to such inputs do not lead to any overflow. Based on such criterion, we describe three reduction algorithms.

Let $f(\theta) = f_0 + f_1\theta + \cdots + f_{\kappa-1}\theta^{\kappa-1}$ and $g(\theta) = g_0 + g_1\theta + \cdots + g_{\kappa-1}\theta^{\kappa-1}$.

General reduction algorithm: Let $\mathfrak{f}(\theta) = \mathfrak{f}_0 + \mathfrak{f}_1\theta + \cdots + \mathfrak{f}_{\kappa-1}\theta^{\kappa-1}$. Define a predicate $\mathsf{genCond}(\mathfrak{f})$ to be true if and only if

$$\begin{array}{rclcl} 0 & \leq & \mathfrak{f}_0, \mathfrak{f}_2, \ldots, \mathfrak{f}_{\kappa-2} & < & 2^{\eta}; \\ 0 & \leq & \mathfrak{f}_1 & < & 2^{\eta+1}; \\ 0 & \leq & \mathfrak{f}_{\kappa-1} & < & 2^{\nu}. \end{array} \tag{49}$$

Consider the following conditions.

1. The inputs $f(\theta)$ and $g(\theta)$ to $\mathsf{mulUSL}/\mathsf{sqrUSL}$ or $\mathsf{mulUSLa}/\mathsf{sqrUSLa}$ satisfy $\mathsf{genCond}(f)$ and $\mathsf{genCond}(g)$.

2. Let the output of $\mathsf{mulUSL}/\mathsf{sqrUSL}$ or $\mathsf{mulUSLa}/\mathsf{sqrUSLa}$ on such $f(\theta)$ and $g(\theta)$ be $h(\theta)$ or $h'(\theta)$ respectively. Suppose that $\mathfrak{h}_{\max} < 2^{127}$ or $\mathfrak{h}'_{\max} < 2^{127}$ as the case may be.

3. $\kappa \geq 3$.

If the above conditions hold, then we describe the reduction algorithm $\mathsf{reduceUSL}$ which takes as input either $h(\theta)$ or $h'(\theta)$ as the case may be and produces an output for which (49) holds.

Reduction algorithm for primes of Type A: Let $\mathfrak{f}(\theta) = \mathfrak{f}_0 + \mathfrak{f}_1\theta + \cdots + \mathfrak{f}_{\kappa-1}\theta^{\kappa-1}$. Define a predicate $\mathsf{condA}(\mathfrak{f})$ to be true if and only if

$$\begin{array}{rclcl} 0 & \leq & \mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{\kappa-2} & < & 2^{\eta}; \\ 0 & \leq & \mathfrak{f}_0 & < & 2^{\eta+1}; \\ 0 & \leq & \mathfrak{f}_{\kappa-1} & < & 2^{\nu}. \end{array} \tag{50}$$

Consider the following conditions.

1. The inputs $f(\theta)$ and $g(\theta)$ to $\mathsf{mulUSL}/\mathsf{sqrUSL}$ or $\mathsf{mulUSLa}/\mathsf{sqrUSLa}$ satisfy $\mathsf{condA}(f)$ and $\mathsf{condA}(g)$.

2. Let the output of mulUSL/sqrUSL or mulUSLa/sqrUSLa on such $f(\theta)$ and $g(\theta)$ be $h(\theta)$ or $h'(\theta)$ respectively. Suppose that $\mathfrak{h}_{\max} < 2^{\ell}$ or $\mathfrak{h}'_{\max} < 2^{\ell}$ as the case may be and $\ell < 63 + \nu$.

3. $\kappa \geq 3$.

If the above three conditions hold, then we describe the reduction algorithm reduceUSLA which takes as input either $h(\theta)$ or $h'(\theta)$ as the case may be and produces an output for which (50) holds. *The primes for which* reduceUSLA *applies have been identified as Type A in Table 3.*

Reduction algorithm for primes of Type B: Let $\mathfrak{f}(\theta) = \mathfrak{f}_0 + \mathfrak{f}_1\theta + \cdots + \mathfrak{f}_{\kappa-1}\theta^{\kappa-1}$. Define a predicate condB($\mathfrak{f}$) to be true if and only if

$$\begin{array}{ccccc} 0 & \leq & \mathfrak{f}_0, \mathfrak{f}_1, \ldots, \mathfrak{f}_{\kappa-2} & < & 2^{\eta+1}; \\ 0 & \leq & \mathfrak{f}_{\kappa-1} & < & 2^{\nu+1}. \end{array} \tag{51}$$

Consider the following conditions.

1. The inputs $f(\theta)$ and $g(\theta)$ to mulUSL/sqrUSL or mulUSLa/sqrUSLa satisfy condA(f) and condA(g).

2. Let the output of mulUSL/sqrUSL or mulUSLa/sqrUSLa on such $f(\theta)$ and $g(\theta)$ be $h(\theta)$ or $h'(\theta)$ respectively. Suppose that $\mathfrak{h}_{\max} < 2^{\ell}$ or $\mathfrak{h}'_{\max} < 2^{\ell}$ as the case may be and $64 + \nu < \ell \leq 128$.

3. $\kappa \geq 3$.

If the above three conditions hold, then we describe the reduction algorithm reduceUSLB which takes as input either $h(\theta)$ or $h'(\theta)$ as the case may be and produces an output for which (51) holds. *The primes for which* reduceUSLB *applies have been identified as Type B in Table 3.* There are two such primes. Note that the condition $\ell < 63 + \nu$ used to identify Type A primes and the condition $64 + \nu < \ell \leq 128$ used to identify Type B primes are non-exhaustive. They do not cover the values of $\ell = 63 + \eta$ and $\ell = 64 + \eta$. None of the primes in Table 3 correspond to such values of ℓ and so this is not an issue.

For the three primes identified as Type G in Table 3, the conditions required to apply either reduceUSLA or reduceUSLB do not hold. It is possible to consider further conditions to develop an algorithm for the Type G primes and we have indeed implemented such an algorithm. This algorithm, however, turns out to be slower than the generic reduceUSL and so we do not describe the details of it.

8 Reduction in $\mathbb{F}_p$ Using Unsaturated Limb Representation

In this section, we describe several reduction algorithms which works with the unsaturated limb representation. The first of this is Function reduceUSL and is shown in Algorithm 8.

Theorem 10 below states the correctness of reduceUSL. The correctness is based on two assumptions both of which are valid for all the primes considered in this work.

Theorem 10. *Let $p = 2^m - \delta$ and m be such that m-bit integers have (κ, η, ν)-representation with $\kappa \geq 3$ and $\delta < 2^{2\eta+\nu-129}$. Suppose the input $h^{(0)}(\theta) = h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1}$ to* reduceUSL *is such that $0 \leq h_i^{(0)} < 2^{128} - 2^{128-\eta}$ for $i = 0, 1, \ldots, \kappa - 1$.*

1. *For partial reduction, the output of* reduceUSL *is $h^{(1)}(\theta) = h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1}$, where $0 \leq h_1^{(1)} < 2^{\eta+1}$, $0 \leq h_0^{(1)}, h_2^{(1)}, \ldots, h_{\kappa-2}^{(1)} < 2^{\eta}$ and $0 \leq h_{\kappa-1}^{(1)} < 2^{\nu}$ satisfying $h^{(1)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*

2. *For full reduction, the output $h^{(2)}(\theta)$ of* reduceUSL *has a (κ, η, ν)-representation and $h^{(2)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*

Algorithm 8 Reduction for unsaturated limb representation. Performs reduction modulo $p = 2^m - \delta$; m-bit integers have a (κ, η, ν)-representation with $\eta < 64$; $\theta = 2^\eta$.

```
1: function reduceUSL($h^{(0)}(\theta)$)
2: input: $h^{(0)}(\theta)$.
3: output: $h^{(1)}(\theta)$ or $h^{(2)}(\theta)$.
4:     $u \leftarrow h_0^{(0)} \bmod 2^\eta$; $r_0 \leftarrow \lfloor h_0^{(0)}/2^\eta \rfloor$
5:     $t_1 \leftarrow h_1^{(0)} + r_0$; $v \leftarrow t_1 \bmod 2^\eta$; $r_1 \leftarrow \lfloor t_1/2^\eta \rfloor$
6:     for $i \leftarrow 2$ to $\kappa - 2$ do
7:         $t_i \leftarrow h_i^{(0)} + r_{i-1}$; $h_i^{(1)} \leftarrow t_i \bmod 2^\eta$; $r_i \leftarrow \lfloor t_i/2^\eta \rfloor$
8:     end for
9:     $t_{\kappa-1} \leftarrow h_{\kappa-1}^{(0)} + r_{\kappa-2}$; $h_{\kappa-1}^{(1)} \leftarrow t_{\kappa-1} \bmod 2^\nu$; $r_{\kappa-1} \leftarrow \lfloor t_{\kappa-1}/2^\nu \rfloor$
10:    $t \leftarrow u + \delta r_{\kappa-1}$; $h_0^{(1)} \leftarrow t \bmod 2^\eta$; $r_0 \leftarrow \lfloor t/2^\eta \rfloor$
11:    $h_1^{(1)} \leftarrow v + r_0$
12:    $\boxed{\text{PARTIAL REDUCTION: } \textbf{return } h^{(1)}(\theta) = h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1}}$
13:    $w \leftarrow h_1^{(1)} \bmod 2^\eta$; $\mathfrak{c}_1 \leftarrow \lfloor h_1^{(1)}/2^\eta \rfloor$
14:    for $i \leftarrow 2$ to $\kappa - 2$ do
15:        $t \leftarrow h_i^{(1)} + \mathfrak{c}_{i-1}$; $h_i^{(2)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c}_i \leftarrow \lfloor t/2^\eta \rfloor$
16:    end for
17:    $t \leftarrow h_{\kappa-1}^{(1)} + \mathfrak{c}_{\kappa-2}$; $h_{\kappa-1}^{(2)} \leftarrow t \bmod 2^\nu$; $\mathfrak{c}_{\kappa-1} \leftarrow \lfloor t/2^\nu \rfloor$
18:    $t \leftarrow h_0^{(1)} + \delta\mathfrak{c}_{\kappa-1}$; $h_0^{(2)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c}_0 \leftarrow \lfloor t/2^\eta \rfloor$
19:    $t \leftarrow w + \mathfrak{c}_0$; $h_1^{(2)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
20:    $h_2^{(2)} \leftarrow h_2^{(2)} + \mathfrak{c}$
21:    $\boxed{\text{FULL REDUCTION: } \textbf{return } h^{(2)}(\theta) = h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1}}$
22: end function.
```

Proof. Since $0 \le h_0^{(0)} < 2^{128} - 2^{128-\eta}$, after Step 4, the bounds on u and r_0 are $0 \le u < 2^\eta$ and $0 \le r_0 < 2^{128-\eta}$ respectively. In Step 5, t_1 is set to $h_1^{(0)} + r_0$ implying $0 \le t_1 < 2^{128}$. Consequently, $0 \le v < 2^\eta$ and $0 \le r_1 < 2^{128-\eta}$ respectively. After Steps 4-5, $h^{(0)}(\theta)$ can be written as

$$\begin{aligned}
h^{(0)}(\theta) &= h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\
&= (u + r_0\theta) + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\
&= u + (h_1^{(0)} + r_0)\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\
&= u + t_1\theta + h_2^{(0)}\theta^2 + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\
&= u + (v + r_1\theta)\theta + h_2^{(0)}\theta^2 + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\
&= u + v\theta + (h_2^{(0)} + r_1)\theta^2 + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1}. \qquad (52)
\end{aligned}$$

The coefficients $h_2^{(1)}, h_3^{(1)} \ldots, h_{\kappa-1}^{(1)}$ are computed in Steps 6 to 9 as follows.

$$\begin{aligned}
h_j^{(1)} &= (h_j^{(0)} + r_{j-1}) \bmod 2^\eta, & r_j &= \lfloor (h_j^{(0)} + r_{j-1})/2^\eta \rfloor, \quad j = 2, 3, \ldots, \kappa - 2, \\
h_{\kappa-1}^{(1)} &= (h_{\kappa-1}^{(0)} + r_{\kappa-2}) \bmod 2^\nu, & r_{\kappa-1} &= \lfloor (h_{\kappa-1}^{(0)} + r_{\kappa-2})/2^\nu \rfloor
\end{aligned} \qquad (53)$$

where $0 \le r_j < 2^{128-\eta}$ and $0 \le r_{\kappa-1} < 2^{128-\nu}$. Starting from (52) and using (53), the effect of Steps 6 to 9 can be written in the following manner.

$$\begin{aligned}
h^{(0)}(\theta) &= u + v\theta + (h_2^{(0)} + r_1)\theta^2 + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\
&= u + v\theta + t_1\theta^2 + h_3^{(0)}\theta^3 + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\
&= u + v\theta + (h_2^{(1)} + r_2\theta)\theta^2 + h_3^{(0)}\theta^3 + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\
&= u + v\theta + h_2^{(1)}\theta^2 + (h_3^{(0)} + r_2)\theta^3 + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\
\cdots \quad & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
&= u + v\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-2}^{(1)}\theta^{\kappa-2} + (h_{\kappa-1}^{(0)} + r_{\kappa-2})\theta^{\kappa-1} \\
&= u + v\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-2}^{(1)}\theta^{\kappa-2} + t_{\kappa-1}\theta^{\kappa-1} \\
&= u + v\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-2}^{(1)}\theta^{\kappa-2} + (h_{\kappa-1}^{(1)} + r_{\kappa-1}2^{\nu})\theta^{\kappa-1} \\
&= u + v\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} + r_{\kappa-1}2^{(\kappa-1)\eta+\nu} \text{ [since } \theta = 2^{\eta}] \\
&\equiv u + v\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} + r_{\kappa-1}\delta \bmod\ p \text{ [using (5)]} \qquad (54)
\end{aligned}$$

where $0 \leq u, v, h_2^{(1)}, h_3^{(1)}, \ldots, h_{\kappa-2}^{(1)} < 2^{\eta}$ and $0 \leq h_{\kappa-1}^{(1)} < 2^{\nu}$. The bounds on $\delta r_{\kappa-1}$ are $0 \leq \delta r_{\kappa-1} < 2^{2\eta+\nu-129} \cdot 2^{128-\nu} = 2^{2\eta-1}$. In Step 10, t is assigned to $u + \delta r_{\kappa-1}$ and so $0 \leq t < 2^{2\eta}$. By the remaining two instructions of Step 10, we get $0 \leq h_0^{(1)} < 2^{\eta}$ and $0 \leq r_0 < 2^{\eta}$. By Step 11 we have, $0 \leq h_1^{(1)} < 2^{\eta+1}$. Hence, from (54), through Steps 10 and 11 we obtain

$$\begin{aligned}
h^{(0)}(\theta) &\equiv (u + \delta r_{\kappa-1}) + v\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \bmod p \\
&= t + v\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \\
&= (t \bmod 2^{\eta} + \lfloor t/2^{\eta} \rfloor\theta) + v\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \text{ [since } \theta = 2^{\eta}] \\
&= h_0^{(1)} + (v + r_0)\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \\
&= h_0^{(1)} + h_1^{(1)}\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} = h^{(1)}(\theta),
\end{aligned}$$

where $0 \leq h_1^{(1)} < 2^{\eta+1}$, $0 \leq h_0^{(1)}, h_2^{(1)}, \ldots, h_{\kappa-2}^{(1)} < 2^{\eta}$, and $h_{\kappa-1}^{(1)} < 2^{\nu}$. From (55) we have $h^{(1)}(\theta) \equiv h^{(0)}(\theta) \bmod p$ which proves the statement on partial reduction.

To ensure full reduction another pass over the limbs is required. This is performed in Steps 13 to 20. First assume that $h_2^{(2)}$ computed in Step 20 satisfies $0 \leq h_2^{(2)} < 2^{\eta}$. Then, it is routine to argue in a manner similar to above that the final computed $h^{(2)}(\theta)$ is such that $h^{(2)}(\theta) \equiv h^{(0)}(\theta) \bmod p$ and $0 \leq h_i^{(2)} < 2^{\eta}$ for $i = 0, 1, \ldots, \kappa - 2$ and $0 \leq h_{\kappa-1}^{(2)} < 2^{\nu}$.

We now show that $h_2^{(2)}$ computed in Step 20 satisfies $0 \leq h_2^{(2)} < 2^{\eta}$. Since $h_1^{(1)} < 2^{\eta+1}$, this implies $\mathfrak{c}_1 \leq 1$. In the first iteration of the loop in Steps 14 to 17, $\mathfrak{c}_1$ is added to $h_2^{(1)}$ to obtain t. From this t, $h_2^{(2)}$ is obtained as $t \bmod 2^{\eta}$ and $\mathfrak{c}_2$ is obtained as $\mathfrak{c} = \lfloor t/2^{\eta} \rfloor$. So, $\mathfrak{c}_2 \leq 1$. Since $h_2^{(1)} < 2^{\eta}$, $\mathfrak{c}_2 = 1$ if and only if $h_2^{(1)} = 2^{\eta} - 1$ and in this case, $h_2^{(2)} = 0$. The following observations can be noted.

1. $\mathfrak{c}_i, \mathfrak{c} \leq 1$ for $i = 0, 1, \ldots, \kappa - 1$.
2. If $\mathfrak{c}_2 = 0$, then $\mathfrak{c}_i = 0$ for $i = 3, 4, \ldots, \kappa - 1$ and $\mathfrak{c}_0 = \mathfrak{c} = 0$.

So, if $\mathfrak{c}_2 = 0$, then $\mathfrak{c} = 0$ and so the value of $h_2^{(2)}$ computed in Step 20 is equal to the value of $h_2^{(2)}$ computed in Step 15. Since the value of $h_2^{(2)}$ computed in this step satisfies $0 \leq h_2^{(2)} < 2^{\eta}$ so does the value of $h_2^{(2)}$ computed in Step 20. On the other hand, if $\mathfrak{c}_2 = 1$, then the value of $h_2^{(2)}$ computed in Step 15 is 0 and so, the value of $h_2^{(2)}$ computed in Step 20 is equal to $\mathfrak{c} \leq 1$. So, in both cases, the bounds $0 \leq h_2^{(2)} < 2^{\eta}$ hold. □

An important implementation issue: In Steps 4, 5, 7, 9 and 10, the operations $w \bmod 2^\tau$ and $\lfloor w/2^\tau \rfloor$ are performed on a 128-bit quantity w where τ is either η or ν. The operation $\lfloor w/2^\tau \rfloor$ heavily influences the overall performance of the algorithm. We describe the implementation of the operations $w \bmod 2^\tau$ and $\lfloor w/2^\tau \rfloor$ in more details. The 128-bit quantity w is stored in two 64-bit words w_0 and w_1 such that $w = w_0 + w_1 2^{64}$. There are two cases to consider.

Case 1: $0 \leq w < 2^{128-\tau}$. In this case, $0 \leq w_1 < 2^{64-\tau}$, i.e., w_1 is at most a $(64-\tau)$-bit word. So, it is possible to left shift w_1 by τ bits and at the same time move in the τ most significant bits of w_0 into the τ least significant bits of w_1. The assembly level Intel instruction for doing this is `shld` and the two operations $w \bmod 2^\tau$ and $\lfloor w/2^\tau \rfloor$ are executed as follows.

```
shld 64 − τ, w0, w1
and  2^τ − 1, w0.
```

After executing these two steps, w_1 stores $\lfloor w/2^\tau \rfloor$ and w_0 stores $w \bmod 2^\tau$.

Case 2: $w \geq 2^{128-\tau}$. In this case, $0 \leq w_1 < 2^{64-\tau}$ and the length of w_1 in bits is more than $64-\tau$ bits. So, left shifting w_1 by τ bits will result in loss of information and the strategy of Case 1 does not work. Further, the result of $\lfloor w/2^\tau \rfloor$ is more than 64 bits in length and requires two 64-bit words to be stored. The strategy in this case is the following. First copy w_0 to another 64-bit location x_0. Right shift τ bits of w_0 while moving in τ least significant bits of w_1 into the most significant bits of w_0. (The Intel instruction for doing this is `shld`.) Then, right shift w_1 by τ bits. The two operations $w \bmod 2^\tau$ and $\lfloor w/2^\tau \rfloor$ are executed as follows.

```
mov  w0, x0
and  2^τ − 1, x0
shrd τ, w1, w0
shr  τ, w1.
```

After executing the above steps, x_0 stores $w \bmod 2^\tau$; w_0 stores the 64 least significant bits of $\lfloor w/2^\tau \rfloor$ and the $(64-\tau)$ least significant bits of w_1 stores the $(64-\tau)$ most significant bits of $\lfloor w/2^\tau \rfloor$.

Clearly Case 2 is more time consuming than Case 1. The applicability of Case 1 and Case 2 to the primes that we have considered are as follows.

1. For primes identified as Type A in Table 3, Case 1 can be applied, except for the prime $2^{222} - 117$ where Case 2 needs to be applied only for Step 4 of reduceUSL.

2. For primes identified as Type B in Table 3, Case 2 needs to be applied.

A computational bottleneck: The various computations $\lfloor \cdot/2^\tau \rfloor$ in reduceUSL are strictly sequential. Correspondingly, the operations `shld` or `shrd` as the case may be, are not independent and have to be executed in sequence. These are relatively high latency operations and so the strict sequential execution of these operations have a negative impact on the overall performance of the algorithm.

We next describe two other reduction algorithms. The main motivation of these algorithms is to try and ensure that the operations `shld` or `shrd` are independent. Achieving such independence comes at the cost of increasing the total number of operations. Even then, for certain primes, the independence of these operations result in an overall faster algorithm.

Remark: Steps 13 to 19 also use the operation $\lfloor \cdot/2^\tau \rfloor$, but, these are on 64-bit quantities and can be efficiently implemented using the `shr` instruction.

8.1 Improved Reduction for Type A Primes

Function reduceUSLA in Algorithm 9 describes a reduction algorithm which improves upon reduceUSL for primes identified as Type A in Table 3.

Algorithm 9 Improved reduction algorithm for primes identified as Type A in Table 3 using unsaturated limb representation. Performs reduction modulo $p = 2^m - \delta$ and m-bit integers have a (κ, η, ν)-representation with $\eta < 64$; $\theta = 2^\eta$.

```
1: function reduceUSLA($h^{(0)}(\theta)$)
2: input: $h^{(0)}(\theta)$.
3: output: $h^{(2)}(\theta)$ or $h^{(3)}(\theta)$.
4:     $r \leftarrow h_0^{(0)} \bmod 2^\eta$
5:     for $i \leftarrow 1$ to $\kappa - 2$ do
6:         $h_i^{(1)} \leftarrow h_i^{(0)} \bmod 2^\eta + \lfloor h_{i-1}^{(0)}/2^\eta \rfloor$
7:     end for
8:     $h_{\kappa-1}^{(1)} \leftarrow h_{\kappa-1}^{(0)} \bmod 2^\nu + \lfloor h_{\kappa-2}^{(0)}/2^\eta \rfloor$
9:     $s \leftarrow \lfloor h_{\kappa-1}^{(0)}/2^\nu \rfloor$; $h_0^{(1)} \leftarrow r + \delta s$
10:    $u \leftarrow h_0^{(1)} \bmod 2^\eta$; $r_0 \leftarrow \lfloor h_0^{(1)}/2^\eta \rfloor$
11:    for $i \leftarrow 1$ to $\kappa - 2$ do
12:        $t_i \leftarrow h_i^{(1)} + r_{i-1}$; $h_i^{(2)} \leftarrow t_i \bmod 2^\eta$; $r_i \leftarrow \lfloor t_i/2^\eta \rfloor$
13:    end for
14:    $t_{\kappa-1} \leftarrow h_{\kappa-1}^{(1)} + r_{\kappa-2}$; $h_{\kappa-1}^{(2)} \leftarrow t_{\kappa-1} \bmod 2^\nu$; $r_{\kappa-1} \leftarrow \lfloor t_{\kappa-1}/2^\nu \rfloor$
15:    $h_0^{(2)} \leftarrow u + \delta r_{\kappa-1}$
16:    $\boxed{\text{PARTIAL REDUCTION: } \textbf{return } h^{(2)}(\theta) = h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1}}$
17:    $v \leftarrow h_0^{(2)} \bmod 2^\eta$; $\mathfrak{c}_0 \leftarrow \lfloor h_0^{(2)}/2^\eta \rfloor$
18:    for $i \leftarrow 1$ to $\kappa - 2$ do
19:        $t \leftarrow h_i^{(2)} + \mathfrak{c}_{i-1}$; $h_i^{(3)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c}_i \leftarrow \lfloor t/2^\eta \rfloor$
20:    end for
21:    $t \leftarrow h_{\kappa-1}^{(2)} + \mathfrak{c}_{\kappa-2}$; $h_{\kappa-1}^{(3)} \leftarrow t \bmod 2^\nu$; $\mathfrak{c}_{\kappa-1} \leftarrow \lfloor t/2^\nu \rfloor$
22:    $t \leftarrow v + \delta \mathfrak{c}_{\kappa-1}$; $h_0^{(3)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
23:    $h_1^{(3)} \leftarrow h_1^{(3)} + \mathfrak{c}$
24:    $\boxed{\text{FULL REDUCTION: } \textbf{return } h^{(3)}(\theta) = h_0^{(3)} + h_1^{(3)}\theta + \cdots + h_{\kappa-1}^{(3)}\theta^{\kappa-1}}$
25: end function.
```

The following result states the correctness of reduceUSLA.

Theorem 11. *Let $p = 2^m - \delta$ be a Type A prime as identified in Table 3; m be such that m-bit integers have (κ, η, ν)-representation; and $\delta < 2^{2\eta+\nu-130}$. Suppose the input $h^{(0)}(\theta) = h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1}$ to* reduceUSLA *is such that $0 \le h_i^{(0)} < 2^\ell$ for $i = 0, 1, \ldots, \kappa - 1$ where $\ell < 63 + \nu$.*

1. *For partial reduction, the output of* reduceUSLA *is $h^{(2)}(\theta) = h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1}$, where $0 \le h_0^{(2)} < 2^{\eta+1}$, $0 \le h_1^{(2)}, \ldots, h_{\kappa-2}^{(2)} < 2^\eta$ and $0 \le h_{\kappa-1}^{(2)} < 2^\nu$ satisfying $h^{(2)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*
2. *For full reduction, the output $h^{(3)}(\theta)$ of* reduceUSLA *has a (κ, η, ν)-representation and $h^{(3)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*

Proof. Note that for all the primes identified as Type A in Table 3, we have $\eta < 62$, Steps 4 to 9 convert the $h^{(0)}(\theta)$ to $h^{(1)}(\theta)$ ensuring $h^{(1)}(\theta) \equiv h^{(0)}(\theta) \bmod p$ and Steps 10 to 15 convert $h^{(1)}(\theta)$ to $h^{(2)}(\theta)$

ensuring $h^{(2)}(\theta) \equiv h^{(1)}(\theta) \bmod p$. Write

$$\begin{array}{rcll} h_j^{(0)} & = & h_{j,0}^{(0)} + h_{j,1}^{(0)} 2^\eta & \text{where } h_{j,0}^{(0)} = h_j^{(0)} \bmod 2^\eta,\ h_{j,1}^{(0)} = \lfloor h_j^{(0)}/2^\eta \rfloor,\ j = 0,1,\ldots,\kappa-2; \\ h_{\kappa-1}^{(0)} & = & h_{\kappa-1,0}^{(0)} + h_{\kappa-1,1}^{(0)} 2^\nu & \text{where } h_{\kappa-1,0}^{(0)} = h_{\kappa-1}^{(0)} \bmod 2^\nu,\ h_{\kappa-1,1}^{(0)} = \lfloor h_{\kappa-1}^{(0)}/2^\nu \rfloor. \end{array} \tag{55}$$

Clearly, $0 \le h_{j,0}^{(0)} < 2^\eta < 2^{62}$ and $0 \le h_{j,1}^{(0)} < 2^{\ell-\eta}$ for $j = 0, 1 \ldots, \kappa - 2$; $0 \le h_{\kappa-1,0}^{(0)} < 2^\nu$ and $0 \le h_{\kappa-1,1}^{(0)} < 2^{\ell-\nu}$. Using $\ell < 63 + \nu$ and $\eta \ge \nu$, we have $0 \le h_{j,1}^{(0)} < 2^{62}$ for $j = 0, 1, \ldots, \kappa - 1$.

In Step 4, r is assigned the value $h_{0,0}^{(0)}$; for $i = 1, 2, \ldots, \kappa - 2$, the i-th iteration of the loop in Steps 5 to 7 assigns the value $(h_{i,0}^{(0)} + h_{i-1,1}^{(0)})$ to $h_i^{(1)}$; Step 8 assigns the value $(h_{\kappa-1,0}^{(0)} + h_{\kappa-2,1}^{(0)})$ to $h_{\kappa-1}^{(1)}$; Step 9 assigns the value $h_{\kappa-1,1}^{(0)}$ to s. So, $0 \le r < 2^\eta$, $0 \le s < 2^{\ell-\nu}$. Note that $2^{(\kappa-1)\eta+\nu} = 2^m \equiv \delta \bmod p$ and so $0 \le \delta s < 2^{2\eta+\ell-130} < 2^{2\eta-2}$ since $\ell < 128$. Step 9 assigns the value $r + \delta s$ to $h_0^{(1)}$. The bounds on $h_i^{(1)}$ are

$$0 \le h_0^{(1)} < 2^{2\eta-1} \quad \text{and} \quad 0 \le h_i^{(1)} < 2^{63} \text{ for } i = 1, 2, \ldots, \kappa - 1. \tag{56}$$

Using $\theta = 2^\eta$, we can write $h^{(0)}(\theta)$ as

$$\begin{aligned} h^{(0)}(\theta) &= h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\ &= (h_{0,0}^{(0)} + h_{0,1}^{(0)}\theta) + (h_{1,0}^{(0)} + h_{1,1}^{(0)}\theta)\theta + \cdots + (h_{\kappa-1,0}^{(0)} + h_{\kappa-1,1}^{(0)}2^\nu)\theta^{\kappa-1} \\ &= h_{0,0}^{(0)} + (h_{0,1}^{(0)} + h_{1,0}^{(0)})\theta + \cdots + (h_{\kappa-2,1}^{(0)} + h_{\kappa-1,0}^{(0)})\theta^{\kappa-1} + h_{\kappa-1,1}^{(0)}2^{(\kappa-1)\eta+\nu} \\ &\equiv (r + \delta s) + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)})\theta^{\kappa-1} \bmod p \text{ [using (5)]} \\ &= h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} = h^{(1)}(\theta). \end{aligned} \tag{57}$$

The argument that Steps 10-14 compute $h^{(2)}(\theta)$ such that $h^{(2)}(\theta) \equiv h^{(1)}(\theta) \bmod p$ and the limbs of $h^{(2)}(\theta)$ satisfy the stated bounds for partial reduction is similar to the proof of Theorem 10. The points to be noted are the following.

1. Since $0 \le h_0^{(1)} < 2^{2\eta-1}$, the values of u and r_0 computed in Step 10 satisfy $0 \le u < 2^\eta$ and $0 \le r_0 < 2^{\eta-1}$ respectively.

2. Since $0 \le h_i^{(1)} < 2^{63}$ for $i = 1, 2, \ldots, \kappa - 2$, and $\eta < 64$, in Step 12 we have $0 \le t_i < 2^{64}$, $0 \le h_i^{(2)} < 2^\eta$ and $0 \le r_i < 2^{64-\eta}$ for $i = 1, 2, \ldots, \kappa - 2$.

3. Since $0 \le h_{\kappa-1}^{(1)} < 2^{63}$, in Step 14 we have $0 \le t_{\kappa-1} < 2^{64}$, $0 \le h_{\kappa-1}^{(2)} < 2^\nu$ and $0 \le r_{\kappa-1} < 2^{64-\nu}$.

4. Since $0 \le \delta < 2^{2\eta+\nu-130}$ and $0 \le r_{\kappa-1}^{(1)} < 2^{64-\nu}$, in Step 15 we have $0 \le \delta r_{\kappa-1} < 2^{2\eta-66} < 2^\eta$ for all the primes identified as Type A in Table 3. This, along with $0 \le u < 2^\eta$ imply $0 \le h_0^{(2)} < 2^{\eta+1}$ in Step 15.

The effect of Steps 10 to 15 on $h_1^{(1)}(\theta)$ can be written as

$$\begin{aligned}
h^{(1)}(\theta) &= h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \\
&= (u + r_0\theta) + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \\
&= u + (h_1^{(1)} + r_0)\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \\
&= u + t_1\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \\
&= u + (h_1^{(2)} + r_1\theta)\theta + h_2^{(1)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \\
&= u + h_1^{(2)}\theta + (h_2^{(1)} + r_1)\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
&= u + h_1^{(2)}\theta + h_2^{(2)}\theta^2 + \cdots + h_{\kappa-2}^{(1)}\theta^{\kappa-2} + (h_{\kappa-1}^{(0)} + r_{\kappa-2})\theta^{\kappa-1} \\
&= u + h_1^{(2)}\theta + h_2^{(2)}\theta^2 + \cdots + h_{\kappa-2}^{(1)}\theta^{\kappa-2} + t_{\kappa-1}\theta^{\kappa-1} \\
&= u + h_1^{(2)}\theta + h_2^{(2)}\theta^2 + \cdots + h_{\kappa-2}^{(1)}\theta^{\kappa-2} + (h_{\kappa-1}^{(1)} + r_{\kappa-1}2^{\nu})\theta^{\kappa-1} \\
&= u + h_1^{(2)}\theta + h_2^{(2)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} + r_{\kappa-1}2^{(\kappa-1)\eta+\nu} \text{ [since } \theta = 2^{\eta}] \\
&\equiv (u + r_{\kappa-1}\delta) + h_1^{(2)}\theta + h_2^{(2)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \bmod p \text{ [using (5)]} \\
&= h_0^{(2)} + h_1^{(2)}\theta + h_2^{(2)}\theta^2 + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} = h^{(2)}(\theta). \qquad (58)
\end{aligned}$$

Using the points 1-4 mentioned before, we have the desired bounds on the limbs of $h^{(2)}(\theta)$ as $0 \le h_0^{(2)} < 2^{\eta+1}$, $0 \le h_1^{(2)}, h_2^{(2)}, \ldots, h_{\kappa-2}^{(2)} < 2^{\eta}$ and $0 \le h_{\kappa-1}^{(2)} < 2^{\nu}$. Combining (57) and (58) we have $h^{(2)}(\theta) \equiv h^{(0)}(\theta) \bmod p$ which proves the statement of partial reduction.

The statement on full reduction is proved in a manner which is very similar to that of Theorem 10. □

Function reduceUSLA makes two passes over the limbs compared to reduceUSL which makes a single pass over the limbs. So, the total number of operations required by reduceUSLA is more than that of reduceUSL. Even then, for primes of Type A, it turns out that reduceUSLA is faster than reduceUSL. The reason is explained below.

Independent double word shifts: The computations $\lfloor \cdot / 2^{\eta} \rfloor$ in Steps 6 and 8 are on ℓ-bit quantities with $\ell < 63+\eta$. This computation falls under Case 1 discussed after the proof of Theorem 10 and can be completed using a single `shld` instruction. The important difference between reduceUSL and reduceUSLA is that in the later case, the `shld` instructions are *independent*. So, these can be appropriately pipelined and may also be simultaneously scheduled on separate ALUs. It is this feature that leads to the speed up of reduceUSLA over reduceUSL. For example, on the Intel Skylake processor, for $p = 2^{255} - 19$, reduceUSLA takes 25 cycles whereas reduceUSL takes 37 cycles.

Function reduceUSLA has another set of shift operations in Step 14. These operations are on 64-bit words and hence can be computed using the `shr` instruction. This is true for all the primes except for $2^{222} - 117$ for which the first two limbs of $h^{(1)}(\theta)$ have more than 64 bits and hence the `shld` instruction has to be applied to extract the required leading bits of these limbs. The latency of `shr` instruction is much smaller than the latency of `shld` instruction. The independence of the `shld` instructions in reduceUSLA more than compensates for the extra `shr` operations.

It is possible to avoid `shld` instruction and instead implement the desired functionality with the four instructions `shl`, `mov`, `shr` and `or`. We have implemented this strategy to try and speed up reduceUSL, but, the resulting speed is still slower than that of reduceUSLA.

8.2 Improved Reduction for Type B Primes

There are two primes identified as Type B in Table 3. If reduceUSLA is applied to these two primes, then the sizes of all the coefficients of $h^{(1)}(\theta)$ will be more than 64 bits. So, the subsequent steps of reduceUSLA will require application of `shld` instead of `shr`. Further, these `shld` instructions would not be independent. To avoid this situation, it is possible to make an extra pass over the limbs as in Steps 4 to 9 of reduceUSLA. This results in Function reduceUSLB which is given in Algorithm 10. Each limb of the partially reduced output of reduceUSLB has an extra bit. As mentioned in Section 7.2, only those primes are identified as Type B for which this does not lead to an overflow in the multiplication and squaring algorithms.

Algorithm 10 Improved reduction algorithm for primes identified as Type B in Table 3 using unsaturated limb representation. Performs reduction modulo $p = 2^m - \delta$ and m-bit integers have a (κ, η, ν)-representation with $\eta < 64$; $\theta = 2^\eta$.

```
1: function reduceUSLB($h^{(0)}(\theta)$)
2: input: $h^{(0)}(\theta)$.
3: output: $h^{(2)}(\theta)$ or $h^{(4)}(\theta)$.
4:     for $\lambda \leftarrow 0$ to $1$ do
5:         $r \leftarrow h_0^{(\lambda)} \bmod 2^\eta$
6:         for $i \leftarrow 1$ to $\kappa - 2$ do
7:             $h_i^{(\lambda+1)} \leftarrow h_i^{(\lambda)} \bmod 2^\eta + \lfloor h_{i-1}^{(\lambda)}/2^\eta \rfloor$
8:         end for
9:         $h_{\kappa-1}^{(\lambda+1)} \leftarrow h_{\kappa-1}^{(\lambda)} \bmod 2^\nu + \lfloor h_{\kappa-2}^{(\lambda)}/2^\eta \rfloor$
10:        $s \leftarrow \lfloor h_{\kappa-1}^{(\lambda)}/2^\nu \rfloor$; $h_0^{(\lambda+1)} \leftarrow r + \delta s$
11:    end for
12:    $\boxed{\text{PARTIAL REDUCTION: } \textbf{return } h^{(2)}(\theta) = h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1}}$
13:    for $\lambda \leftarrow 2$ to $3$ do
14:        $h_0^{(\lambda+1)} \leftarrow h_0^{(\lambda)} \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor h_0^{(\lambda)}/2^\eta \rfloor$
15:        for $i \leftarrow 1$ to $\kappa - 2$ do
16:            $t \leftarrow h_i^{(\lambda)} + \mathfrak{c}$; $h_i^{(\lambda+1)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
17:        end for
18:        $t \leftarrow h_{\kappa-1}^{(\lambda)} + \mathfrak{c}$; $h_{\kappa-1}^{(\lambda+1)} \leftarrow t \bmod 2^\nu$; $\mathfrak{c} \leftarrow \lfloor t/2^\nu \rfloor$
19:        $h_0^{(\lambda+1)} \leftarrow h_0^{(\lambda+1)} + \delta\mathfrak{c}$
20:    end for
21:    $t \leftarrow h_0^{(4)}$; $h_0^{(4)} \leftarrow t \bmod 2^\eta$; $\mathfrak{c} \leftarrow \lfloor t/2^\eta \rfloor$
22:    $h_1^{(4)} \leftarrow h_1^{(4)} + \mathfrak{c}$
23:    $\boxed{\text{FULL REDUCTION: } \textbf{return } h^{(4)}(\theta) = h_0^{(4)} + h_1^{(4)}\theta + \cdots + h_{\kappa-1}^{(4)}\theta^{\kappa-1}}$
24: end function.
```

The following result states the correctness of reduceUSLB.

Theorem 12. *Let $p = 2^m - \delta$ be a Type B prime as identified in Table 3; m be such that m-bit integers have (κ, η, ν)-representation; and $\delta < 2^{2\eta+\nu-130}$. Suppose the input $h^{(0)}(\theta) = h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1}$ to* reduceUSLA *is such that $0 \le h_i^{(0)} < 2^{128}$ for $i = 0, 1, \ldots, \kappa - 1$.*

1. *For partial reduction, the output of* reduceUSLB *is $h^{(2)}(\theta) = h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1}$, where $0 \le h_0^{(2)}, h_1^{(2)}, \ldots, h_{\kappa-2}^{(2)} < 2^{\eta+1}$ and $0 \le h_{\kappa-1}^{(2)} < 2^{\nu+1}$ satisfying $h^{(2)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*

2. *For full reduction, the output $h^{(4)}(\theta)$ of* reduceUSLB *has a (κ, η, ν)-representation and $h^{(4)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*

Proof. The first iteration of the loop in Steps 4 to 11 converts $h^{(0)}(\theta)$ to $h^{(1)}(\theta)$. The correctness of this conversion can be argued in a manner similar to the first part of the proof of Theorem 11. In particular, we obtain

$$\begin{aligned} h^{(0)}(\theta) &= h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1} \\ &\equiv \underbrace{h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1}}_{\text{through first iteration of Steps 4 to 11}} \bmod p = h^{(1)}(\theta). \end{aligned}$$

Proceeding in a manner similar to the first part of the proof of Theorem 11, it can be shown that $0 \le h_0^{(1)} < 2^{2\eta-1}$ and $0 \le h_1^{(1)}, h_2^{(1)}, \ldots, h_{\kappa-1}^{(1)} < 2^{129-\eta}$.

The second iteration of the loop in Steps 4 to 11 converts $h^{(1)}(\theta)$ to $h^{(2)}(\theta)$. The correctness of this argument is also similar to the first part of the proof of Theorem 11. The only thing required is to argue that the coefficients of $h^{(2)}(\theta)$ satisfy the stated bounds.

Step 5 of the second iteration provides r satisfying $0 \le r < 2^\eta$. We have $\lfloor h_{i-1}^{(1)}/2^\eta \rfloor < 2^{129-2\eta}$, $h_i^{(1)} \bmod 2^\eta < 2^\eta$ for $i = 1, 2, \ldots, \kappa-2$, and $h_{\kappa-1}^{(1)} \bmod 2^\nu < 2^\nu < 2^\eta$. The following three observations hold for both the primes identified as Type B in Table 3. Their consequences are also mentioned.

1. $129 - 2\eta = 129 - 2 \cdot 55 = 129 - 110 = 19 < \nu < \eta$. Consequently, after Steps 6 to 9 of the second iteration are $0 \le h_1^{(2)}, h_2^{(2)}, \ldots, h_{\kappa-2}^{(2)} < 2^{\eta+1}$ and $0 \le h_{\kappa-1}^{(2)} < 2^{\nu+1}$.

2. $\delta < 2^8$. Consequently, after Step 10, $\delta s < 2^8 \cdot 2^{129-2\nu} = 2^{137-2\nu}$.

3. $137 - 2\nu \le 137 - 2 \cdot 52 = 137 - 104 = 33 < \eta$. Consequently, after Step 10, $0 \le h_0^{(2)} < 2^{\eta+1}$.

So, we have

$$\begin{aligned} h^{(1)}(\theta) &= h_0^{(1)} + h_1^{(1)}\theta + \cdots + h_{\kappa-1}^{(1)}\theta^{\kappa-1} \\ &\equiv \underbrace{h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1}}_{\text{through second iteration of Steps 4 to 11}} \bmod p = h^{(2)}(\theta), \end{aligned}$$

where $0 \le h_0^{(2)}, h_1^{(2)}, \ldots, h_{\kappa-2}^{(2)} < 2^{\eta+1}$ and $0 \le h_{\kappa-1}^{(2)} < 2^{\nu+1}$. Combining (59) and (59) we have $h^{(2)}(\theta) \equiv h^{(0)}(\theta) \bmod p$, which proves the statement on partial reduction.

The statement on full reduction is proved routinely. The only point to be noted is that a single pass over the limbs is not sufficient to ensure termination and instead two passes are required. $\square$

9 Saturated Limb Computation Without Double Carry Chains

In Section 5, we have described how the saturated limb representation can be exploited in combination with two independent carry chains to obtain fast squaring and multiplication algorithms. Implementation of these algorithms require the use of the instructions `mulx, adcx` and `adox`. For processors which do not provide these instructions, the algorithms in Section 5 cannot be implemented. In this section, we describe algorithms for saturated limb representation which do not use double carry chains and can be implemented on previous generation processors. In Section 7, we have already described algorithms using the unsaturated limb representation which do not use double carry chains. For the prime $2^{256} - 2^{32} - 977$, it turns out that the implentation using saturated limb representation without double carry chains is more efficient than the implementation using unsaturated limb representation.

As before, let $p = 2^m - \delta$ where m-bit integers have (κ, η, ν)-representation. Since we consider the saturated limb representation, we have $\eta = 64$. As before, $\theta = 2^\eta$ and $c_p = 2^{\eta-\nu}\delta$. Let $f(\theta)$ and $g(\theta)$ be two m-bit integers written as follows.

$$\begin{aligned} f(\theta) &= f_0 + f_1\theta + \cdots + f_{\kappa-1}\theta^{\kappa-1}, \\ g(\theta) &= g_0 + g_1\theta + \cdots + g_{\kappa-1}\theta^{\kappa-1}, \end{aligned}$$

where $0 \le f_i,\ g_i < 2^\eta,\ i = 0, 1, \ldots, \kappa - 2$, and $0 \le f_{\kappa-1},\ g_{\kappa-1} < 2^\nu$. The schoolbook product of $f(\theta)$ and $g(\theta)$ modulo p can be written as $h(\theta) = h_0 + h_1\theta + \cdots + h_{\kappa-1}\theta^{\kappa-1}$ where the coefficients h_i are given by (43). Since we are working with $\eta = 64$, the coefficients h_i are not guaranteed to fit within 128 bits. We show how to tackle this problem. Define

$$f_i \cdot g_j \;=\; u_{i,j} + v_{i,j}2^\eta \;=\; u_{i,j} + v_{i,j}\theta \text{ for } i, j = 0, 1, \ldots, \kappa - 1. \tag{59}$$

Using $\theta^\kappa = 2^{\kappa\eta} \equiv 2^{\eta-\nu}\delta \bmod p = c_p$ and (59) in (43), we have $h(\theta) \equiv z(\theta) \bmod p$ where $z(\theta) = z_0 + z_1\theta + \cdots + z_{\kappa-1}\theta^{\kappa-1}$ and

$$\begin{aligned}
z_0 &= u_{0,0} + c_p(u_{1,\kappa-1} + u_{2,\kappa-2} + \cdots + c_p u_{\kappa-2,2} + u_{\kappa-1,1} \\
&\quad + v_{0,\kappa-1} + v_{1,\kappa-2} + v_{2,\kappa-3} + \cdots + v_{\kappa-2,1} + v_{\kappa-1,0}), \\
z_1 &= u_{0,1} + u_{1,0} + v_{0,0} + c_p(u_{2,\kappa-1} + \cdots + u_{\kappa-2,3} + u_{\kappa-1,2} \\
&\quad + v_{1,\kappa-1} + v_{2,\kappa-2} + \cdots + v_{\kappa-2,2} + v_{\kappa-1,1}), \\
z_2 &= u_{0,2} + u_{1,1} + u_{2,0} + \cdots + v_{0,1} + v_{1,0} \\
&\quad + c_p(u_{\kappa-2,4} + u_{\kappa-1,3} + v_{2,\kappa-1} + \cdots + v_{\kappa-2,3} + v_{\kappa-1,2}), \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
z_{\kappa-3} &= u_{0,\kappa-3} + u_{1,\kappa-4} + u_{2,\kappa-5} + \cdots + v_{0,\kappa-4} + u_{1,\kappa-5} \\
&\quad + c_p(u_{\kappa-2,\kappa-1} + u_{\kappa-1,\kappa-2} + \cdots + v_{\kappa-3,\kappa-1} + v_{\kappa-2,\kappa-2} + v_{\kappa-1,\kappa-3}), \\
z_{\kappa-2} &= u_{0,\kappa-2} + u_{1,\kappa-3} + u_{2,\kappa-4} + \cdots + u_{\kappa-2,0} + v_{0,\kappa-3} \\
&\quad + v_{1,\kappa-4} + v_{2,\kappa-5} + c_p(u_{\kappa-1,\kappa-1} + \cdots + v_{\kappa-2,\kappa-1} + v_{\kappa-1,\kappa-2}), \\
z_{\kappa-1} &= u_{0,\kappa-1} + u_{1,\kappa-2} + u_{2,\kappa-3} + \cdots + u_{\kappa-2,1} + u_{\kappa-1,0} \\
&\quad + v_{0,\kappa-2} + v_{1,\kappa-3} + v_{2,\kappa-4} + \cdots + v_{\kappa-2,0} + c_p v_{\kappa-1,\kappa-1}.
\end{aligned} \tag{60}$$

For all the primes in Table 3, it can be ensured that $0 \le z_0, z_1, \ldots, z_{\kappa-1} < 2^{127}$. Substituting $g = f$, we get similar equations for squaring. Denote the resulting multiplication and squaring algorithms by mulSLa and sqrSLa respectively.

Next we describe how to reduce $z(\theta)$. Function reduceSL in Algorithm 11 performs the required computation. The following results states the correctness of reduceSL. The proof is similar to the proofs of the previous results and hence we skip the proof.

Theorem 13. *Let $p = 2^m - \delta$ be a prime in Table 3 and m be such that m-bit integers have (κ, η, ν)-representation where $\eta = 64$. Suppose the input $h^{(0)}(\theta) = h_0^{(0)} + h_1^{(0)}\theta + \cdots + h_{\kappa-1}^{(0)}\theta^{\kappa-1}$ to* reduceSL *is such that $0 \le h_i^{(0)} < 2^{128}$ for $i = 0, 1, \ldots, \kappa - 1$.*

1. *For partial reduction, the output $h^{(2)}(\theta)$ of* reduceSL *has a $(\kappa, \eta, \nu+1)$-representation and $h^{(2)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*
2. *For full reduction, the output $h^{(3)}(\theta)$ of* reduceSL *has a (κ, η, ν)-representation and $h^{(3)}(\theta) \equiv h^{(0)}(\theta) \bmod p$.*

Remark: The squaring algorithm has some disadvantages using the above strategy. For the primes satisfying $\nu < \eta - 1$, the doubling involved in the squaring operation can take advantage while computing the terms $2f_{\kappa-1} \cdot f_j$ for $j = 0, 1, \ldots, \kappa - 1$, by first computing $2f_{\kappa-1}$, through a shift and then multiplying to f_j. For reduction, we can opt for partial reduction, keeping an extra bit in the last limb, but, then we cannot take the advantage in the doubling operation. For the primes satisfying $\nu = \eta$, we do not get any advantage with the doubling operation in the squaring algorithm and also full reduction is required.

Algorithm 11 Generic reduction algorithm using saturated limb representation for the primes in Table 3. Performs reduction modulo $p = 2^m - \delta$ and m-bit integers have a (κ, η, ν)-representation with $\eta = 64$; $\theta = 2^\eta$.

```
1: function reduceSL($h^{(0)}(\theta)$)
2: input: $h^{(0)}(\theta)$.
3: output: $h^{(2)}(\theta)$ or $h^{(3)}(\theta)$.
4:     $h_0^{(1)} \leftarrow h_0^{(0)} \mod 2^\eta$; $r_0 \leftarrow \lfloor h_0^{(0)}/2^\eta \rfloor$
5:     for $i \leftarrow 1$ to $\kappa - 2$ do
6:         $t_i \leftarrow h_i^{(0)} + r_{i-1}$; $h_i^{(1)} \leftarrow t_i \mod 2^\eta$; $r_i \leftarrow \lfloor t_i/2^\eta \rfloor$
7:     end for
8:     $t_{\kappa-1} \leftarrow h_{\kappa-1}^{(0)} + r_{\kappa-2}$; $h_{\kappa-1}^{(1)} \leftarrow t_{\kappa-1} \mod 2^\nu$; $r_{\kappa-1} \leftarrow \lfloor t_{\kappa-1}/2^\nu \rfloor$
9:     $t \leftarrow h_0^{(1)} + \delta r_{\kappa-1}$; $h_0^{(2)} \leftarrow t \mod 2^\eta$; $\mathfrak{c}_0 \leftarrow \lfloor t/2^\eta \rfloor$
10:    for $i \leftarrow 1$ to $\kappa - 2$ do
11:        $t \leftarrow h_i^{(1)} + \mathfrak{c}_{i-1}$; $h_i^{(2)} \leftarrow t \mod 2^\eta$; $\mathfrak{c}_i \leftarrow \lfloor t/2^\eta \rfloor$
12:    end for
13:    $h_{\kappa-1}^{(2)} \leftarrow h_{\kappa-1}^{(1)} + \mathfrak{c}_{\kappa-2}$
14:    $\boxed{\text{PARTIAL REDUCTION FOR } \nu < \eta\text{: } \textbf{return } h^{(2)}(\theta) = h_0^{(2)} + h_1^{(2)}\theta + \cdots + h_{\kappa-1}^{(2)}\theta^{\kappa-1}}$
15:    $h_{\kappa-1}^{(3)} \leftarrow h_{\kappa-1}^{(2)} \mod 2^\nu$; $\mathfrak{c}_{\kappa-1} \leftarrow \lfloor h_{\kappa-1}^{(2)}/2^\nu \rfloor$
16:    $t \leftarrow h_0^{(2)} + \delta\mathfrak{c}_{\kappa-1}$; $h_0^{(3)} \leftarrow t \mod 2^\eta$; $\mathfrak{c}_0 \leftarrow \lfloor t/2^\eta \rfloor$
17:    $h_1^{(3)} \leftarrow h_1^{(2)} + \mathfrak{c}_0$
18:    for $i \leftarrow 2$ to $\kappa - 2$ do $h_i^{(3)} \leftarrow h_i^{(2)}$ end for
19:    $\boxed{\text{FULL REDUCTION: } \textbf{return } h^{(3)}(\theta) = h_0^{(3)} + h_1^{(3)}\theta + \cdots + h_{\kappa-1}^{(3)}\theta^{\kappa-1}}$
20: end function.
```

10 Implementations and Timings

All the implementations of this work have been developed in the Intel x86 64-bit assembly language. The timing experiments were carried out on single cores of Haswell, Skylake and Kabylake processors. During measurement of the cpu-cycles, turbo-boost and hyper-threading features were turned off. An initial cache warming was done with 25000 iterations and then the median of 100000 iterations was recorded. The time stamp counter TSC was read from the CPU to RAX and RDX registers by RDTSC instruction.

Platform specifications: The details of the hardware and software tools used in our software implementations are as follows.

Haswell: Intel®Core™ i7-4790 4-core CPU 3.60 Ghz. The OS was 64-bit Ubuntu 14.04 LTS and the source code was compiled using GCC version 7.3.0.

Skylake: Intel®Core™ i7-6500U 2-core CPU @ 2.50GHz. The OS was 64-bit Ubuntu 14.04 LTS and the source code was compiled using GCC version 7.3.0.

Kabylake: Intel®Core™ i7-7700U 4-core CPU @ 3.60GHz. The OS was 64-bit Ubuntu 18.04 LTS and the source code was compiled using GCC version 4.8.0.

Recall from Section 4 that maa denotes implementations where arithmetic is performed using only `mul`, `imul`, `add` and `adc`, while maax denotes implementations which also use the instructions `mulx`, adcx and adox. For the meanings of the various inversion algorithms, we again refer to Section 4.

field	maa			
	previous	this work	algorithm	sup
$\mathbb{F}_{2^{127}-1}$	2797 [4]	2503	invSLMP	10.5
$\mathbb{F}_{2^{221}-3}$	-	8082	invUSLA	-
		9957	invUSL	
$\mathbb{F}_{2^{222}-117}$	-	8385	invUSLA	-
		10278	invUSL	
$\mathbb{F}_{2^{251}-9}$	12202 [16]	11245	invSLa	7.8
		14134	invUSL	
$\mathbb{F}_{2^{255}-19}$	12359 [5, 5-limb]	11854	invUSLA	4.1
	15880 [5, 4-limb]	14228	invUSL	
$\mathbb{F}_{2^{256}-2^{32}-977}$	20209 [19]	12809	invSLa	36.6
		17202	invUSL	
$\mathbb{F}_{2^{266}-3}$	12705 [16]	12413	invUSLA	2.3
		14892	invUSL	
$\mathbb{F}_{2^{382}-105}$	-	33437	invUSLA	-
		39722	invUSL	
$\mathbb{F}_{2^{382}-105}$	-	33699	invUSLA	-
		40922	invUSL	
$\mathbb{F}_{2^{414}-17}$	-	43218	invUSLA	-
		46905	invUSL	
$\mathbb{F}_{2^{511}-187}$	-	72804	invUSL	-
$\mathbb{F}_{2^{512}-569}$	-	73771	invUSL	-
$\mathbb{F}_{2^{521}-1}$	76298 [14]	62244	invUSLA	18.4
		71546	invUSL	
$\mathbb{F}_{2^{607}-1}$		94149	invUSL	-

Table 4: Comparison of timings of various field inversion algorithms on Haswell.

Timings on Haswell, Skylake and Kabylake are shown in Tables 4, 5 and 6 respectively. The timings in the tables are the numbers of cycles. For comparison, we provide the timings of the most efficient (to the best of our knowledge) and publicly available previous implementations. The timings of the previous implementations were obtained by downloading the relevant software and measuring the required cycles on the same platforms where the present implementations have been measured. A '-' in the columns headed 'prevous' indicates that we were unable to find a (reasonably efficient) previous implementation of inversion in the corresponding field. The columns headed 'sup' provide the speed-up percentage. These have been computed using the following formula.

$$\mathsf{sup} = 100 \times \frac{(\text{previous cycle count} - \text{present cycle count})}{\text{previous cycle count}}.$$

For the maa implementations, all applicable algorithms have been implemented and timings recorded. To simplify the presentation, we provide timings for invUSL, and if invUSL is not the fastest, then the timing for the fastest applicable algorithm is provided. The speed-up percentage corresponds to the faster of the two timings.

For the prime $2^{127} - 1$, the maa type implementation is done using invSLMP. This is basically an optimised version of the implementation by Bernstein et al. [4].

Based on Tables 4 to 6, we make the following observations.

1. Among the primes considered in this work, only the prime $2^{255} - 19$ has a previous maax type implementation. For the other primes, we provide the first maax type implementations.

2. For each prime, where a previous maa implementation is available, we report a faster maa implementation. On all three processors, the speed-up percentage varies from 4% to about 30%.

3. Comparing maa and maax type implementations, it may be noted that maax is always faster. The amount of speed-up, however, is not uniform. This is shown in Table 7 which compares the best timing of the maa type implementation with the corresponding maax type implementation for the Skylake and the Kabylake processors.

field	maa				maax			
	previous	this work	algorithm	sup	previous	this work	algorithm	sup
$\mathbb{F}_{2^{127}-1}$	2505 [4]	2263	invSLMP	9.7	-	2154	invxSLMP	-
$\mathbb{F}_{2^{221}-3}$	-	7949	invUSLA	-	-	7728	invxSLPMP	-
		8936	invUSL					
$\mathbb{F}_{2^{222}-117}$	-	8033	invUSLA	-	-	7967	invxSLPMP	-
		9682	invUSL					
$\mathbb{F}_{2^{251}-9}$	13632 [16]	11783	invSLa	13.6	-	8784	invxSLPMP	-
		13731	invUSL					
$\mathbb{F}_{2^{255}-19}$	13223 [5, 5-limb]	12671	invUSLA	4.0	12170 [22]	9301	invxSLPMP	23.6
	13901 [5, 4-limb]	13784	invUSL					
$\mathbb{F}_{2^{256}-2^{32}-977}$	18391 [19]	13242	invSLa	28.0	-	11501	invxSLPMP	-
		17212	invUSL					
$\mathbb{F}_{2^{266}-3}$	14472 [16]	13350	invUSLA	7.8	-	12938	invxSLPMP	-
		17964	invSLa					
$\mathbb{F}_{2^{382}-105}$	-	30419	invUSLA	-	-	24549	invxSLPMP	-
		37397	invSLa					
$\mathbb{F}_{2^{383}-187}$	-	30680	invUSLA	-	-	24628	invxSLPMP	-
		37731	invUSL					
$\mathbb{F}_{2^{414}-17}$	-	38096	invUSLA	-	-	30972	invxSLPMP	-
		42105	invUSL					
$\mathbb{F}_{2^{511}-187}$	-	66039	invUSL	-	-	47062	invxSLPMP	-
$\mathbb{F}_{2^{512}-569}$	-	66808	invUSL	-	-	55409	invxSLPMP	-
$\mathbb{F}_{2^{521}-1}$	64924 [14]	54790	invUSLA	15.6	-	53828	invxSLMP	-
		63938	invUSL					
$\mathbb{F}_{2^{607}-1}$	-	83587	invUSL	-	-	74442	invxSLMP	-
$\mathbb{F}_{2^{751}-165}$	-	-	-	-	-	126982	invxSLPMP	-
$\mathbb{F}_{2^{832}-143}$	-	-	-	-	-	166969	invxSLPMP	-
$\mathbb{F}_{2^{896}-213}$	-	-	-	-	-	206759	invxSLPMP	-
$\mathbb{F}_{2^{960}-167}$	-	-	-	-	-	249971	invxSLPMP	-
$\mathbb{F}_{2^{1024}-105}$	-	-	-	-	-	303516	invxSLPMP	-
$\mathbb{F}_{2^{1088}-89}$	-	-	-	-	-	361644	invxSLPMP	-

Table 5: Comparison of timings of various field inversion algorithms on Skylake.

Next we focus on three important primes. These primes are considered important, since some well known curve based cryptosystems are based on these primes.

Case $2^{127}-1$: For maa type implementations, the speed-up percentage obtained is about 10% on all three processors. Further, we provide the first maax type implementation.

Case $2^{255}-19$: For maa type implementations, the speed-up percentage obtained is about 4% on all three processors. For maax type implementation, the speed-up percentage obtained is about 23% on the Skylake and the Kabylake processors.

Case $2^{256}-2^{32}-977$: For maa type implementations, the speed-up percentage obtained is 36% on the Haswell processor, and about 28% on the Skylake and the Kabylake processors. Further, we provide the first maax type implementation.

Based on the above, we conclude that in comparison to previous work, we provide the fastest implementations for all the primes that are considered in this work. The speed-up is significant for some

field	maa				maax			
	previous	this work	algorithm	sup	previous	this work	algorithm	sup
$\mathbb{F}_{2^{127}-1}$	2418 [4]	2185	invSLMP	9.6	-	2078	invxSLMP	-
$\mathbb{F}_{2^{221}-3}$	-	7668	invUSLA	-	-	7457	invxSLPMP	-
		8616	invUSL					
$\mathbb{F}_{2^{222}-117}$	-	7745	invUSLA	-	-	7685	invxSLPMP	-
		9338	invUSL					
$\mathbb{F}_{2^{251}-9}$	13148 [16]	11395	invSLa	13.3	-	8465	invxSLPMP	-
		13238	invUSL					
$\mathbb{F}_{2^{255}-19}$	12753 [5, 5-limb]	12217	invUSLA	4.2	11613 [22]	8971	invxSLPMP	22.8
	13318 [5, 4-limb]	13290	invUSL					
$\mathbb{F}_{2^{256}-2^{32}-977}$	17682 [19]	12772	invSLa	27.8	-	11081	invxSLPMP	-
		16594	invUSL					
$\mathbb{F}_{2^{266}-3}$	13807 [16]	12704	invUSLA	8.0	-	12328	invxSLPMP	-
		13963	invUSL					
$\mathbb{F}_{2^{382}-105}$	-	29167	invUSLA	-	-	23522	invxSLPMP	-
		34010	invUSL					
$\mathbb{F}_{2^{383}-187}$	-	29205	invUSLA	-	-	23591	invxSLPMP	-
		34154	invUSL					
$\mathbb{F}_{2^{414}-17}$	-	36553	invUSLA	-	-	29766	invxSLPMP	-
		40442	invUSL					
$\mathbb{F}_{2^{511}-187}$	-	63515	invUSL	-	-	45014	invxSLPMP	-
$\mathbb{F}_{2^{512}-569}$	-	64254	invUSL	-	-	53450	invxSLPMP	-
$\mathbb{F}_{2^{521}-1}$	59641 [14]	52855	invUSLA	11.4	-	51909	invxSLMP	-
		61665	invUSL					
$\mathbb{F}_{2^{607}-1}$	-	80652	invUSL	-	-	71821	invxSLMP	-
$\mathbb{F}_{2^{751}-165}$	-	-	-	-	-	122388	invxSLPMP	-
$\mathbb{F}_{2^{832}-143}$	-	-	-	-	-	160966	invxSLPMP	-
$\mathbb{F}_{2^{896}-213}$	-	-	-	-	-	199332	invxSLPMP	-
$\mathbb{F}_{2^{960}-167}$	-	-	-	-	-	241601	invxSLPMP	-
$\mathbb{F}_{2^{1024}-105}$	-	-	-	-	-	292791	invxSLPMP	-
$\mathbb{F}_{2^{1088}-89}$	-	-	-	-	-	349019	invxSLPMP	-

Table 6: Comparison of timings of various field inversion algorithms on Kabylake.

field	Skylake			Kabylake		
	maa	maax	speed-up %	maa	maax	speed-up %
$\mathbb{F}_{2^{127}-1}$	2263	2154	4.8	2185	2078	4.9
$\mathbb{F}_{2^{221}-3}$	7949	7728	2.5	7668	7457	2.8
$\mathbb{F}_{2^{222}-117}$	8033	7967	0.8	7745	7685	0.8
$\mathbb{F}_{2^{251}-9}$	11783	8784	25.5	11395	8465	25.7
$\mathbb{F}_{2^{255}-19}$	12671	9301	26.6	12217	8971	26.6
$\mathbb{F}_{2^{256}-2^{32}-977}$	13242	11501	13.1	12772	11081	12.2
$\mathbb{F}_{2^{266}-3}$	13350	12938	3.1	12704	12328	3.0
$\mathbb{F}_{2^{382}-105}$	30419	24549	19.3	29167	23522	19.4
$\mathbb{F}_{2^{383}-187}$	30680	24628	19.7	29205	23591	19.2
$\mathbb{F}_{2^{414}-17}$	38096	30972	18.7	36553	29766	18.6
$\mathbb{F}_{2^{511}-187}$	66039	47062	28.7	63515	45014	29.1
$\mathbb{F}_{2^{512}-569}$	66808	58911	11.8	64254	56827	11.6
$\mathbb{F}_{2^{521}-1}$	54790	53828	1.8	52855	51909	1.8
$\mathbb{F}_{2^{607}-1}$	83587	74442	10.9	80652	71821	10.9

Table 7: Comparison between maa and maax type implementations on the Skylake and the Kabylake processors.

important primes. For the latest processors, the maax type implementations are faster than maa type implementations. Apart from the prime $2^{255}-19$, for all the other primes we provide the first maax type implementations.

11 Conclusion

In this paper, we have considered efficient inversion over (pseudo-)Mersenne prime order fields. Our contributions have been two fold. On the theoretical side, we provide general forms of various reduction algorithms along with rigorous proofs of correctness. On the practical side, we provide efficient assembly implementation for a wide range of Intel processors. Our implementations are faster than previous work. We have made all our source codes publicly available so that they can be used to replace inversion routines in existing softwares.

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