Stronger Notions and Constructions for Multi-Designated Verifier Signatures *

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Abstract. Off-the-Record (OTR) messaging is a protocol used to authenticate messages while also giving senders plausible deniability. Multi-Designated Verifier Signatures (MDVS) are a primitive that allows for OTR to be extended to handle group messaging. In group OTR, the sender wants the designated verifiers to be able to authenticate the messages (that is, the signature should be *unforgeable*), but even if some verifiers are corrupt and collude, they should not be able to prove authenticity to any outsiders (that is, the signature should be *source-hiding*). We additionally require *consistency*, meaning that if any one of the designated verifiers can verify an honestly produced signature, then all of them can.

The contributions of this paper are two-fold: stronger definitions, and new constructions meeting those definitions. Existing literature defines and builds limited notions of MDVS, where source-hiding only holds when all verifiers are corrupt. We strengthen source-hiding to support any subset of corrupt verifiers, and give the first formal definition of consistency.

We give two constructions of our stronger notion of MDVS: one from functional encryption, and one from standard primitives such as pseudorandom functions, pseudorandom generators, key agreement and NIZKs. The second construction has somewhat larger signatures, but does not require a trusted setup.

* This research was supported by: the Concordium Blockhain Research Center, Aarhus University, Denmark; the Carlsberg Foundation under the Semper Ardens Research Project CF18-112 (BCM); the European Research Council (ERC) under the European Unions's Horizon 2020 research and innovation programme under grant agreement No 669255 (MPCPRO) and No 803096 (SPEC); the Danish Independent Research Council under Grant-ID DFF-6108-00169 (FoCC);

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1 Introduction

Encrypted and authenticated messaging has experienced widespread adoption in recent years, due to the attractive combination of properties offered by, for example, the Signal protocol [Mar13]. With so many conversations happening over the internet, there is a growing need for protocols offering security to conversation participants. Encryption can be used to guarantee privacy of message

contents, but authenticating messages while maintaining the properties of an in person conversation is more involved. There are two properties of in person conversations related to authenticity that we wish to emulate in the context of digital conversations:

- *Unforgeability*, meaning that the receiver should be convinced that the message actually came from the sender in question, and
- *Source hiding* or *deniability*, meaning that the receiver cannot later prove to a third party that the message came from the sender.

Source hiding is easy to achieve on its own simply by not using any authentication mechanism; however, that makes it impossible to simultaneously achieve unforgeability. Using digital signatures provides unforgeability, but permanently link participants to their statements within the conversation, thus making it impossible to achieve source hiding.

Off-the-Record (OTR) messaging was introduced to offer a solution to this in the two-party case, enabling authentication of messages in such a way that participants can convincingly deny having made certain statements, or even having taken part in the conversation at all [BGB04]. The foundation of the protocol is that the intended sender and receiver derive shared keys and use these to encrypt and authenticate messages that they send to one another. In other words, the protocol deals with encrypted messages accompanied by a message authentication code (MAC) constructed with the shared key. MACs work well in two-party conversations, because for parties S(ender) and R(eciever) with a shared secret key, a MAC attests 'this message comes from S or R'. MACs provide unforgeability, since a party R receiving a message authenticated with such a MAC knows that if this MAC verifies, the message came from S. MACs provide source hiding (deniability) as R cannot convince a third party T that a message and MAC originally came from S (since R could have produced it just as easily). More generally, tools that combine unforgeability and source hiding for two-party communication are known as *Designated Verifier Signatures* (DVSs, proposed by [JSI96] and [Cha96]).

When there are multiple recipients, for example in group messaging, the situation becomes more complicated. The goals are similar: we want any recipient R to be able to verify that a message came from S (*unforgeability*), without being able to prove this to a third party T (*source hiding*). However, with more than one recipient, it is also natural to demand what we call *consistency*: a corrupt S cannot produce an authenticated message that some recipients accept while others reject it. The tool to use here is Multi-Designated Verifier Signatures (MDVSs).

The way source hiding is formalized for multiple recipients in MDVS is as follows: we require that any possible set of corrupt recipients $\mathcal{C}$ can simulate a signature such that, even given their secret keys, a third party cannot distinguish the simulation from a real signature coming from S. Note that a simulated signature should not convince any honest recipient outside $\mathcal{C}$, since otherwise $\mathcal{C}$ could break unforgeability.

It would be unreasonable to make any assumptions in advance about which subset of recipients might be corrupt, so we should clearly ask for source hiding with respect to any subset $\mathcal{C}$ within the set of recipients $\mathcal{D}$. Nevertheless, all previous constructions achieve source hiding only when $\mathcal{C} = \mathcal{D}$[4].

In a nutshell, the contributions of this paper are two-fold. We first provide formal definitions of the strongest flavors of all the relevant properties, and then give several constructions that achieve them all, including source hiding for any corrupt subset of recipients. In the following subsections, we give an overview of previous work and then discuss our results in more detail.

1.1 Flavors of Multi-Designated Verifier Signatures

There are many ways to define MDVS and its properties. Fig 1 summarizes the approaches taken by prior work, compared to our own.

Typically, an MDVS scheme has a public verification algorithm that anyone can use to 'validate' a signature. We use *validation* to mean ascertaining that the signature either comes from the signer, or is a simulation created by the designated verifiers. We reserve the term *verification* for ascertaining that the signature could only have come from the signer. If there is a public validation algorithm, the source hiding property of MDVS schemes demands that validity alone does not link the signature to the signer; only the designated verifiers can know whether it was the signer that created the signature or it was simulated by designated verifiers. There is also is a "strong" notion of MDVS, where a signature reveals nothing about the identity of the signer to anyone other than the designated verifiers; thus, there should be no public validation algorithm. This is formalized by an additional property called privacy of signer's identity (PSI).

Orthogonally to the notions of validation, there are different notions of verification. In some MDVS schemes, even a single designated verifier cannot link a signature to the signer; the designated verifiers need to work together in order to verify a signature. Thus, we have two notions of verification: *local* verification and *cooperative* verification (where *all* designated verifiers need to cooperate in order to verify the signature).

Recall that the source hiding property states that an outsider cannot determine whether a given signature was created by the signer or simulated by the designated verifiers. We have three flavors of such simulatability: *one* designated verifier (out of n) can by himself simulate a signature (as done by [Tia12]), *all* designated verifiers need to collude in order to simulate a signature (all other works on MDVS), or *any subset* of the designated verifiers can simulate a signature (this paper). Of course, the simulated signature should remain indistinguishable from a real one even in the presence of the secrets held by the simulating parties.

[4] One previous work [Tia12] achieves source hiding when $\mathcal{C}$ consists of a single verifier. However, in this construction a simulated signature created by a malicious verifier will look like a real signature for all other designated verifers, hence allowing this designated verifier to forge signatures.

There is also the standard security property of signature schemes, which is *unforgeability*; no one (except the signer) should be able to construct a signature that any verifier will accept as a valid signature from that signer. We have two flavors of unforgeability. The first is *weak* unforgeability, where designated verifiers can forge, but others cannot. The second is *strong* unforgeability, where a designated verifier can distinguish between simulated and real signatures; that is, even other designated verifiers cannot fool a verifier into accepting a simulated signature.[5]

Schemes	PSI	Verification	Simulation	Unforgeability	Signature Size
[JSI96,LV04,LSMP07] [Cho08,Ver08,ZAYS12]	No	Local	All	Weak	$O(1)$
Our work, from standard primitives	No	Local	any subset	Strong	$O(\|\mathcal{D}\|)$
[NSM05,Cho06]	Yes	All	All	Weak	$O(\|\mathcal{D}\|)$
[MW08,SHCL08,Cha11]	Yes	All	All	Weak	$O(1)$
[SKS06,LV07,Ver08,ZAYS12]	Yes	Local	All	Weak	$O(\|\mathcal{D}\|)$
[Tia12]	Yes	Local	One	Weak	$O(1)$
Our work, from FE	Yes	Local	any subset	Strong	$O(\|\mathcal{C}\|)$

Fig. 1. Existing MDVS constructions and their properties. Let $\mathcal{D}$ be the set of designated verifiers, and let $\mathcal{C} \subseteq \mathcal{D}$ be the set of colluding designated verifiers.

1.2 Our Contributions

We propose formal definitions of all the relevant security properties of MDVS in the strongest flavor, including the first formal (game based) definition of consistency, where a corrupt signer can collude with some of the designated verifiers to create an inconsistent signature.

We then give several different constructions of MDVS that achieve these properties, including local verification, any-subset simulation and strong unforgeability. Our constructions, and the tools they require, are mapped out in Fig 2. In particular, these are the first constructions with any-subset simulation and with strong unforgeability, as described in Fig 1. We get these results at the expense of signature sizes that are larger than in some of the earlier constructions. However, this is unavoidable, as shown in Theorem 1 below. We emphasize again that we should care about these stronger properties. Namely, providing source hiding only when all verifiers are corrupt is unreasonable: In reality, for instance, an adversary could coerce a single verifier R into revealing the authenticated message and her secret key. If the adversary can convince himself that R did not

[5] Note that when all designated verifiers are needed for the simulation, then a designated verifier will be able to distinguish a simulation from a real signature based on whether he participated in the simulation of the signature. However, if this is the only way he can distinguish, then the signature scheme has *weak* unforgeability, since the simulated signature is still a valid forgery.

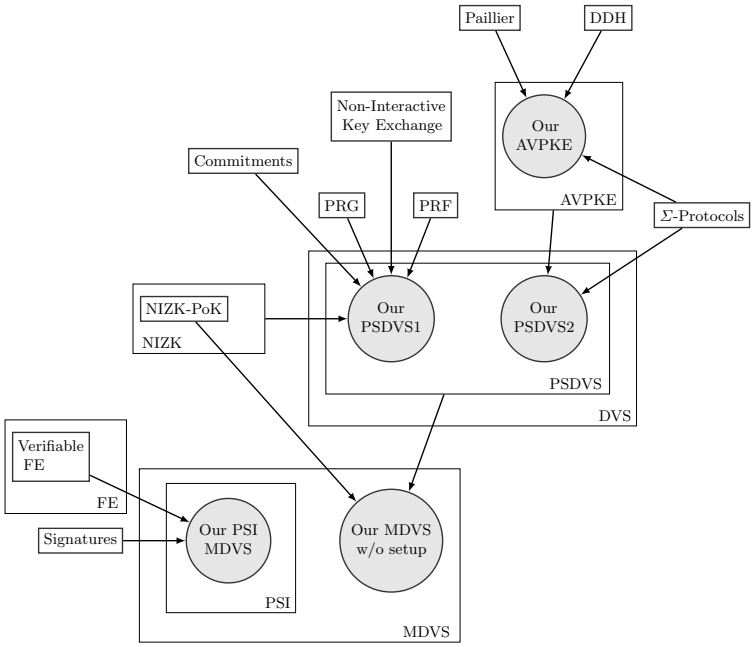

Fig. 2. Our MDVS constructions and building blocks.

get help from *all* other verifiers to create a simulation, he is also convinced that the revealed messages are genuine. On the other hand, when not all designated verifiers take part in the simulation, there is a remaining set of honest verifiers, and we do not want them to be fooled by the fake signature created by $\mathcal{C}$. This is exactly what strong unforgeability guarantees.

Theorem 1. *Any MDVS with any-subset simulation and strong unforgeability must have signature size* $\Omega(|\mathcal{D}|)$.

Remark 1. It may seem from the table that our functional encryption based scheme contradicts the theorem, but this is not the case. It can be instantiated such that signatures can be simulated by collusions up to a certain maximal size $|\mathcal{C}|$, and then signatures will be of size $\Omega(|\mathcal{C}|)$. So, if we want *any* subset to be able to simulate, the signature size is $\Omega(|\mathcal{D}|)$, in accordance with the theorem.

Proof. Imagine that we give all the verifiers' keys to a sender and a receiver; the sender can now encode an arbitrary subset $\mathcal{C} \subseteq \mathcal{D}$ by letting $\mathcal{C}$ construct a simulated signature σ on some default message, and sending it to the receiver. The receiver can infer $\mathcal{C}$ from σ: by strong unforgability, all verifiers' keys outside $\mathcal{C}$ will reject σ, whereas keys in $\mathcal{C}$ will accept, since we require the simulation to look convincing even given the secret keys in $\mathcal{C}$. It follows that σ must consist of enough bits to determine $\mathcal{C}$, which is $\log_2(2^{|\mathcal{D}|}) = |\mathcal{D}|$. □

Why First Ideas Fail.

Using MACs Black-box usage of a standard MAC scheme cannot help us combine unforgeability with consistency. There are two straightforward ways to use a standard MAC scheme in this context: sharing a MAC key among the entire group, and sharing MAC keys pairwise. Sharing a single key does not provide the desired notion of unforgeability, since any member of the group can forge messages from any other member. Sharing keys pairwise does not provide the desired notion of consistency. If recipients R_1 and R_2 are the chosen recipients of a message, and R_1 receives a message he accepts as coming from S, he cannot be sure that R_2 would also accept that message: If S is corrupt, he could include a valid MAC for R_1 and an invalid MAC for R_2.

Using Proofs of Knowledge A standard technique for making designated verifier signatures for a single verifier is to start from an interactive protocol that proves knowledge of either the signer's or the verifier's secret key, and turn this into a signature scheme using the Fiat-Shamir paradigm. It may seem natural to try to build an MDVS from this. However, it turns out to be challenging to achieve strong unforgeability using this technique; a signature cannot consist of a proof of knowledge of the signer's or one of the verifiers' secret keys, since any verifier will be able to convince other verifiers to accept a signature that did not come from the signer. For the same reason, a signature cannot consist of a proof of knowledge of the signer's secret key or some subset of the verifiers' secret keys.

MDVS from Standard Primitives. Our first class of MDVS constructions is based only on standard primitives. With one exception specified below, all of these constructions can be instantiated in the random oracle model and will then require no trusted setup. Without random oracles, we would need to set up a common reference string.

The idea is that the signer creates a DVS signature for each verifier individually, and then proves the consistency of those signatures.[6]. To support such proofs, we define a new primitive called Publicly Simulatable Designated Verifier Signatures (PSDVS) in Section 3.1, which is a single-verifier DVS equipped with extra properties. We then show, in Section 3.2, that a PSDVS together with a non-interactive zero knowledge proof of knowledge (NIZK-PoK) imply an MDVS for any number of signers and verifiers. Finally, we give some constructions of PSDVS. Our first PSDVS construction (in Section 3.3) uses only generic tools, namely psedudorandom functions, non-interactive key exchange (such as Diffie-Hellman), and non-interactive zero-knowledge proofs of knowledge. Our second PSDVS construction (in Section 3.4) aims at better concrete efficiency. It is based on DDH, strong RSA and Paillier encryption, is secure in the random oracle model, and requires a constant number of exponentiations for all

[6] Simply proving that all of the signatures verify would violate the source hiding property; instead, the signer proves that either all of the signatures are real, or they are all simulated, as described in Section 3

operations. This scheme requires the trusted generation of an RSA modulus so that the factorization remains unknown. We also sketch a variant that requires no trusted setup, is secure in the random oracle model, and only requires (a variant of) the DDH assumption. However, this version requires double discrete log proofs, and therefore requires a non-constant number of exponentiations.

In order to support one of our constructions in which the signer sends an encrypted MAC key, we introduce a new building block we call Authenticated and Verifiable Encryption (AVPKE), which may be of independent interest. This is a variant of Paillier encryption with built-in authentication, and as such it is related to the known primitive "signcryption" [Zhe97]. However, our AVPKE scheme has the additional property that we can give efficient zero-knowledge proofs involving the encrypted message, using the algebraic properties of Paillier encryption.

To sign in our PSDVS schemes, the signer and verifiers first must establish a shared symmetric key k. In some cases they can do this non-interactively, using their secret and public keys, while in other cases the signer must send an encrypted key alongside the signature. After this, the signer sends a MAC on the message under key k; this MAC is based on a pseudorandom function.

MDVS from Functional Encryption. Our second construction is based on Verifiable Functional Encryption (VFE). With current state of the art, this requires non-standard computational assumptions and a trusted setup for generating keys. On the other hand, it can be set up to have smaller signatures if we are willing to make a stronger assumption on the number of colluding verifiers. Namely, the signature size is $O(|\mathcal{C}|)$, where $|\mathcal{C}|$ is the size of the largest number of corrupt verifiers we want to tolerate.

If we are going to put a bound on the size of a collusion, it may seem we can use bounded collusion FE, which can be realized from standard assumptions [GVW12,AV19], and then there is no need for our other constructions from standard primitives. However, this is not true. First, use of any kind of FE means we inherently need a trusted setup. Second, bounded collusion FE requires us to fix the bound on collusion size at key generation time; a bound that may later turn out to be too small. Third, ciphertext sizes in bounded collusion FE depend on the bound; thus, choosing a large bound to make sure we can handle the application implies a cost in efficiency. The MDVS signature sizes would depend on some upper bound on number of corrupt parties in the system, as opposed to on the number of recipients for the signature in question.

In a nutshell, the idea behind the functional encryption based construction is to do the proof of knowledge of one of the relevant secret keys "inside the ciphertext". In a little more detail, the idea is to encrypt a list of t *standard* signatures, where t is the maximal size of collusion we want to protect against (that is, $t \geq |\mathcal{C}|$), and the MDVS signature will simply be this ciphertext. To sign, the signer will generate their own standard signature σ_{S} on the message, and then encrypt a list a signatures consisting of σ_{S} followed by $t-1$ dummy values. To verify a signature, a verifier R gets a functional decryption key that will look

at the list of signatures inside the ciphertext and output accept or reject. It will accept if the list contains a valid signature from S or a valid signature from R. Now, if a corrupt set of verifiers $\mathcal{C}$ wants to simulate a signature, they will all sign the message and encrypt the list of these signatures. By security of the encryption scheme, this looks like a real signature, and will indeed verify under all verification keys belonging to verifiers in $\mathcal{C}$. However, no honest verifier will accept it as a signature from S, so we have strong unforgeability.

2 Multi-Designated Verifier Signatures

MDVS Algorithms. Let $\mathcal{S}$ be a set of signer identities and $\mathcal{R}$ be a set of verifier identities. A *multi-designated verifier signature* (MDVS) scheme is defined by the following probabilistic polynomial-time algorithms:

$\mathsf{Setup}(1^\kappa) \to (pp, msk)$**:** The setup algorithm, on input the security parameter $\kappa \in \mathbb{N}$, outputs public parameters pp and the master secret key msk.

$\mathsf{SignKeyGen}(pp, msk) \to (spk, ssk)$**:** The signer key generation algorithm, on input the public parameter pp and the master secret key msk, outputs the public key spk and secret key ssk for a signer.

$\mathsf{VerKeyGen}(pp, msk) \to (vpk, vsk)$**:** The verifier key generation algorithm, on input the public parameter pp and the master secret key msk, outputs the public key vpk and secret key vsk for a verifier.

$\mathsf{Sign}(pp, ssk_i, \{vpk_j\}_{j\in\mathcal{D}}, m) \to \sigma$**:** The signing algorithm, on input the public parameters pp, a secret signing key ssk_i, the public keys of the designated verifiers $\{vpk_j\}_{j\in\mathcal{D}}$, and a message m, outputs a signature σ.

$\mathsf{Verify}(pp, spk_i, vsk_j, \{vpk_{j'}\}_{j'\in\mathcal{D}}, m, \sigma) \to d$**:** The verification algorithm, on input the public parameters pp, a public verification key spk_i, a secret key vsk_j of a verifier such that $j \in \mathcal{D}$, the public keys of the designated verifiers $\{vpk_j\}_{j\in\mathcal{D}}$, a message m, and a signature σ, outputs a boolean decision d: $d = 1$ (accept) or $d = 0$ (reject).

$\mathsf{Sim}(pp, spk_i, \{vpk_j\}_{j\in\mathcal{D}}, \{vsk_j\}_{j\in\mathcal{C}}, m) \to \sigma'$**:** The simulation algorithm, on input public parameters pp, a public verification key spk_i, the public keys of the designated verifiers $\{vpk_j\}_{j\in\mathcal{D}}$, the secret keys of the corrupt designated verifiers $\{vsk_j\}_{j\in\mathcal{C}}$, and a message m, outputs a simulated signature σ'.

The different algorithms take many different inputs, which are not all needed for all of our constructions. For instance, the constructions based on standard primitives (Section 3) do not need a master secret key; they allow key pairs for signers and verifiers to be generated locally. Additionally, some of our constructions do not use the signers' and verifiers' public keys in all of the algorithms in which they appear as inputs above. Thus, to simplify the notation we exclude these inputs in later sections whenever they are not needed.

MDVS Properties. Let σ be a signature from signer i on message m and designated for verifiers $\mathcal{D}$. We ask for the following (informal) properties:

Correctness: All verifiers $j \in \mathcal{D}$ are able to verify an honestly generated signature σ.

Consistency: If there exists one verifier $j \in \mathcal{D}$ that accepts the signature σ, then all other designated verifiers (i.e. all $j' \in \mathcal{D} \setminus \{j\}$) also accept the signature.

Unforgeability: An adversary without knowledge of the secret key ssk_i for signer i cannot create a signature σ' that is accepted by any designated verifier as a signature from signer i.

Source hiding: Given a signature σ, any malicious subset of the designated verifiers $\mathcal{C} \subseteq \mathcal{D}$ cannot convince any outsider that σ is a signature from signer i (i.e. the malicious set could have simulated the signature themselves).

(Optionally) Privacy of Identities: Any outsider (without colluding with any designated verifiers) cannot determine the identity of the signer and/or the identities of the designated verifiers.

Throughout our formal definitions we use the following six oracles:

Signer Key Generation Oracle: $\mathcal{O}_{SKG}(i)$
1. If a signer key generation query has previously been performed for i, look up and return the previously generated key.
2. Otherwise, output and store $(spk_i, ssk_i) \leftarrow \mathsf{SignKeyGen}(pp, msk)$.

Verifier Key Generation Oracle: $\mathcal{O}_{VKG}(j)$
1. If a verifier key generation query has previously been performed for j, look up and return the previously generated key.
2. Otherwise, output and store $(vpk_j, vsk_j) \leftarrow \mathsf{VerKeyGen}(pp, msk)$.

Public Signer Key Generation Oracle: $\mathcal{O}_{SPK}(i)$
1. $(spk_i, ssk_i) \leftarrow \mathcal{O}_{SKG}(i)$.
2. Output spk_i.

Public Verifier Key Generation Oracle: $\mathcal{O}_{VPK}(j)$
1. $(vpk_j, vsk_j) \leftarrow \mathcal{O}_{VKG}(j)$.
2. Output vpk_j.

Signing Oracle: $\mathcal{O}_S(i, \mathcal{D}, m)$
1. $(spk_i, ssk_i) \leftarrow \mathcal{O}_{SKG}(i)$.
2. For all $j \in \mathcal{D}$: $vpk_j \leftarrow \mathcal{O}_{VPK}(j)$.
3. Output $\sigma \leftarrow \mathsf{Sign}(pp, ssk_i, \{vpk_j\}_{j \in \mathcal{D}}, m)$.

Verification Oracle: $\mathcal{O}_V(i, j, \mathcal{D}, m, \sigma)$
1. $spk_i \leftarrow \mathcal{O}_{SPK}(i)$.
2. $(vpk_j, vsk_j) \leftarrow \mathcal{O}_{VKG}(j)$.
3. Output $d \leftarrow \mathsf{Verify}(pp, spk_i, vsk_j, \{vpk_{j'}\}_{j' \in \mathcal{D}}, m, \sigma)$.

Definition 1 (Correctness). *Let $\kappa \in \mathbb{N}$ be the security parameter, and let $\mathsf{MDVS} = (\mathsf{Setup}, \mathsf{SignKeyGen}, \mathsf{VerKeyGen}, \mathsf{Sign}, \mathsf{Verify}, \mathsf{Sim})$ be an MDVS scheme.* MDVS *is* correct *if for all $i \in \mathcal{S}$, $m \in \mathcal{M}$, $\mathcal{D} \subseteq \mathcal{R}$ and $j \in \mathcal{D}$, it holds that*

$$\Pr\left[\mathsf{Verify}(pp, spk_i, vsk_j, \{vpk_{j'}\}_{j' \in \mathcal{D}}, m, \sigma) \neq 1\right] = 0,$$

where the inputs to Verify *are generated as follows:*

– $(pp, msk) \leftarrow \mathsf{Setup}(1^\kappa)$;
– $(spk_i, ssk_i) \leftarrow \mathsf{SignKeyGen}(pp, msk, i)$;
– $(vpk_j, vsk_j) \leftarrow \mathsf{VerKeyGen}(pp, msk, j)$ *for* $j \in \mathcal{D}$;
– $\sigma \leftarrow \mathsf{Sign}(pp, ssk_i, \{vpk_j\}_{j\in\mathcal{D}}, m)$.

In Def 1, we require that all the designated verifiers can verify the signature, without considering what happens for parties that are not designated verifiers (i.e. parties who should not be able to verify the signature). Parties that are not designated verifiers are accounted for by the source hiding property.

Definition 2 (Consistency). *Let* $\kappa \in \mathbb{N}$ *be the security parameter, and let* $\mathsf{MDVS} = (\mathsf{Setup}, \mathsf{SignKeyGen}, \mathsf{VerKeyGen}, \mathsf{Sign}, \mathsf{Verify}, \mathsf{Sim})$ *be an MDVS scheme. Consider the following game between a challenger and an adversary* $\mathcal{A}$*:*

$\mathsf{Game}^{con}_{\mathsf{MDVS},\mathcal{A}}(\kappa)$

1. $(pp, msk) \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(m^*, i^*, \mathcal{D}^*, \sigma^*) \leftarrow \mathcal{A}^{\mathcal{O}_{SKG}, \mathcal{O}_{VKG}, \mathcal{O}_{SPK}, \mathcal{O}_{VPK}, \mathcal{O}_V}(pp)$

We say that $\mathcal{A}$ *wins the game if there exist verifiers* $j_0, j_1 \in \mathcal{D}^*$ *such that:*

$$\mathsf{Verify}(pp, spk_{i^*}, vsk_{j_0}, \{vpk_{j'}\}_{j'\in\mathcal{D}^*}, m^*, \sigma^*) = 0,$$
$$\mathsf{Verify}(pp, spk_{i^*}, vsk_{j_1}, \{vpk_{j'}\}_{j'\in\mathcal{D}^*}, m^*, \sigma^*) = 1,$$

where all keys are the honestly generated outputs of the key generation oracles, and $\mathcal{O}_{VKG}$ *is never queried on* j_0 *or* j_1*.*

We say that an MDVS scheme is consistent *if, for all PPT adversaries* $\mathcal{A}$*,*

$$\mathsf{adv}^{con}_{\mathsf{MDVS},\mathcal{A}}(\kappa) = \Pr\left[\mathcal{A} \text{ wins } \mathsf{Game}^{con}_{\mathsf{MDVS},\mathcal{A}}(\kappa)\right] \leq \mathsf{negl}(\kappa).$$

Def 2 states that even a valid signer (i.e. someone who knows a secret signing key) cannot create an inconsistent signature that will be accepted by some designated verifiers and rejected by others. By the correctness property, an honestly generated signature is accepted by all designated verifiers. By design, corrupt designated verifiers can construct an inconsistent signature, since some verifiers will accept it (i.e. those verifiers that created it), while the remaining honest designated verifiers will reject the simulated signature. Thus, we need to ask for $j \neq j_0, j_1$ for all queries j to the oracle $\mathcal{O}_{VKG}$.

Definition 3 (Existential Unforgeability). *Let* $\kappa \in \mathbb{N}$ *be the security parameter, and let* $\mathsf{MDVS} = (\mathsf{Setup}, \mathsf{SignKeyGen}, \mathsf{VerKeyGen}, \mathsf{Sign}, \mathsf{Verify}, \mathsf{Sim})$ *be an MDVS scheme. Consider the following game between a challenger and an adversary* $\mathcal{A}$*:*

$\mathsf{Game}^{euf}_{\mathsf{MDVS},\mathcal{A}}(\kappa)$

1. $(pp, msk) \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(m^*, i^*, \mathcal{D}^*, \sigma^*) \leftarrow \mathcal{A}^{\mathcal{O}_{SKG}, \mathcal{O}_{VKG}, \mathcal{O}_{SPK}, \mathcal{O}_{VPK}, \mathcal{O}_S}(pp)$

We say that $\mathcal{A}$ wins the game if we have all of the following:

- *for all queries i to oracle $\mathcal{O}_{SKG}$, it holds that $i^* \neq i$;*
- *for all queries $(i, \mathcal{D}, m)$ to oracle $\mathcal{O}_S$ that result in signature σ, it holds that $(i^*, \mathcal{D}^*, m^*) \neq (i, \mathcal{D}, m)$;*
- *there exists a verifier $j' \in \mathcal{D}^*$ such that for all queries j to oracle $\mathcal{O}_{VKG}$, it holds that $j' \neq j$ and*

$$\mathsf{Verify}(pp, spk_{i^*}, vsk_{j'}, \{vpk_{j''}\}_{j'' \in \mathcal{D}^*}, m^*, \sigma^*) = 1,$$

where all keys are honestly generated outputs of the key generation oracles.

We say that an MDVS scheme is existentially unforgable *if, for all PPT adversaries $\mathcal{A}$,*

$$\mathsf{adv}^{euf}_{\mathsf{MDVS},\mathcal{A}}(\kappa) = \Pr\left[\mathcal{A} \text{ wins } \mathsf{Game}^{euf}_{\mathsf{MDVS},\mathcal{A}}(\kappa)\right] \leq \mathsf{negl}(\kappa).$$

Def 3 states that an adversary cannot create a signature that any honest verifier will accept as a signature from a signer whose secret signing key the adversary does not know. The adversary will always get the public keys of the involved parties, i.e. signer with identity i^* and the designated verifiers $\mathcal{D}$, through the key generation oracles. He is also allowed to obtain the secret keys of every party except the signer i^* and at least one designated verifier. The reason why we need at least one honest verifier, is that corrupt verifiers can create a simulated signature that will look like a real signature with respect to their own verifier secret keys. However this simulation will be rejected by any honest designated verifier, i.e. the simulation will be a valid forgery for the corrupt verifiers, but not for the honest verifiers.

Definition 4 (Privacy of Identities). *Let $\kappa \in \mathbb{N}$ be the security parameter, and let $\mathsf{MDVS} = (\mathsf{Setup}, \mathsf{SignKeyGen}, \mathsf{VerKeyGen}, \mathsf{Sign}, \mathsf{Verify}, \mathsf{Sim})$ be an MDVS scheme. Consider the following game between a challenger and an adversary $\mathcal{A}$, where all keys are the honestly generated outputs of the key generation oracles:*

$\mathsf{Game}^{pri}_{\mathsf{MDVS},\mathcal{A}}(\kappa)$

1. $(pp, msk) \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(m^*, i_0, i_1, \mathcal{D}_0, \mathcal{D}_1) \leftarrow \mathcal{A}^{\mathcal{O}_{SKG}, \mathcal{O}_{VKG}, \mathcal{O}_{SPK}, \mathcal{O}_{VPK}, \mathcal{O}_S, \mathcal{O}_V}(pp)$
3. $b \leftarrow \{0, 1\}$
4. $\sigma^* \leftarrow \mathsf{Sign}(pp, ssk_{i_b}, \{vpk_j\}_{j \in \mathcal{D}_b}, m^*)$
5. $b' \leftarrow \mathcal{A}^{\mathcal{O}_{SKG}, \mathcal{O}_{VKG}, \mathcal{O}_{SPK}, \mathcal{O}_{VPK}, \mathcal{O}_S, \mathcal{O}_V}(\sigma^*)$

We say that $\mathcal{A}$ wins the game if $b = b'$, and all of the following hold:

- *for all queries i to $\mathcal{O}_{SKG}$, it holds that $i \notin \{i_0, i_1\}$;*
- *for all gueries j to $\mathcal{O}_{VKG}$, it holds that $j \notin \mathcal{D}_0 \cup \mathcal{D}_1$;*
- *for all queries $(i, j, \mathcal{D}, m, \sigma)$ to $\mathcal{O}_V$, it holds that $\sigma^* \neq \sigma$.*

We say that an MDVS scheme has privacy of identities *if, for all PPT adversaries* $\mathcal{A}$*,*

$$\mathsf{adv}^{pri}_{\mathsf{MDVS},\mathcal{A}}(\kappa) = \Pr\left[\mathcal{A} \text{ wins } \mathsf{Game}^{pri}_{\mathsf{MDVS},\mathcal{A}}(\kappa)\right] - \frac{1}{2} \leq \mathsf{negl}(\kappa).$$

We say that an MDVS scheme has additional properties as follows:

- *privacy of the signer's identity (PSI) if* $i_0 \neq i_1$*;*
- *privacy of the designated verifiers' identities (PVI) if* $\mathcal{D}_0 \neq \mathcal{D}_1$ *and* $|\mathcal{D}_0| = |\mathcal{D}_1|$*;*
- *privacy of identities if* $i_0 \neq i_1$*,* $\mathcal{D}_0 \neq \mathcal{D}_1$ *and* $|\mathcal{D}_0| = |\mathcal{D}_1|$*.*

Def 4 states that an adversary cannot distinguish between two different signatures from two different signers (PSI), if he does not know the secret key of any of the signers or designated verifers)as designated verifiers *are* allowed to identify the signer). Furhtermore, it does not help him to see other signatures that he knows are from the signers in question.

In addition, if we have that the set of designated verifiers $\mathcal{D}_0 \neq \mathcal{D}_1$ and $|\mathcal{D}_0| = |\mathcal{D}_1|$, then the MDVS has privacy of designated verifier's identities (PVI), which means that any outsider without knowledge of any secret keys cannot determine the set of designated verifiers.

Definition 5 (Source hiding). *Let* $\kappa \in \mathbb{N}$ *be the security parameter, and let* $\mathsf{MDVS} = (\mathsf{Setup}, \mathsf{SignKeyGen}, \mathsf{VerKeyGen}, \mathsf{Sign}, \mathsf{Verify}, \mathsf{Sim})$ *be an MDVS scheme. Let* Sim *be a simulator. Consider the following game between a challenger and an adversary* $\mathcal{A}$*, where all keys are honestly generated outputs of the key generation oracles:*

$\mathsf{Game}^{sh}_{\mathsf{MDVS},\mathsf{Sim},\mathcal{A}}(\kappa)$

1. $(pp, msk) \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(i^*, \mathcal{D}^*, m^*, \mathcal{C}^*) \leftarrow \mathcal{A}^{\mathcal{O}_{SKG},\mathcal{O}_{VKG},\mathcal{O}_{SPK},\mathcal{O}_{VPK},\mathcal{O}_S,\mathcal{O}_V}(pp)$
3. $b \leftarrow_\$ \{0,1\}$
4. $\sigma_0 \leftarrow \mathsf{Sign}(pp, ssk_{i^*}, \{vpk_j\}_{j\in\mathcal{D}^*}, m^*)$
5. $\sigma_1 \leftarrow \mathsf{Sim}(pp, spk_{i^*}, \{vpk_j\}_{j\in\mathcal{D}^*}, \{vsk_j\}_{j\in\mathcal{C}^*}, m^*)$
6. $b' \leftarrow \mathcal{A}^{\mathcal{O}_{SKG},\mathcal{O}_{VKG},\mathcal{O}_{SPK},\mathcal{O}_{VPK},\mathcal{O}_S,\mathcal{O}_V}(\sigma_b)$

We say that $\mathcal{A}$ *wins the game if* $b' = b$*, and all of the following hold:*

- $|\mathcal{C}^*| \leq t$ *and* $\mathcal{C}^* \subseteq \mathcal{D}^*$*;*
- *for all queries* i *to oracle* $\mathcal{O}_{SKG}$ *it holds that* $i^* \neq i$*;*
- *for all queries* j *to oracle* $\mathcal{O}_{VKG}$ *it holds that* $j \notin \mathcal{D}^* \backslash \mathcal{C}^*$*;*
- *for all queries* $(i, j, \mathcal{D}, m, \sigma)$ *to* $\mathcal{O}_V$ *it holds that* $\sigma_b \neq \sigma$*.*

We say that an MDVS scheme is t-source hiding *if, for all PPT adversaries* $\mathcal{A}$*,*

$$\mathsf{adv}^{sh}_{\mathsf{MDVS},\mathsf{Sim},\mathcal{A}}(\kappa) = \mathsf{Pr}\left[\mathcal{A} \text{ wins } \mathsf{Game}^{sh}_{\mathsf{MDVS},\mathsf{Sim},\mathcal{A}}(\kappa)\right] - \frac{1}{2} \leq \mathsf{negl}(\kappa).$$

If a scheme allows t *to take arbitrary values, we say that it is* source hiding*.*

Def 5 states that any adversary that corrupts a subset (of size t) of the designated verifiers $\mathcal{C}^*$, cannot determine whether the received signature was created by real signer i^* or simulated by the corrupt verifiers $\mathcal{C}^*$. The adversary is not allowed to see the secret keys for the designated verifiers that are in $\mathcal{D}^* \backslash \mathcal{C}^*$. This is due to the fact that the adversary decides to corrupt the designated verifiers in $\mathcal{C}^*$ (getting their secret information), and then sees a signature, which is either created by this set of corrupt verifiers or by the real signer. If the adversary was allowed to get secret keys of additional parties in $\mathcal{D}^*$ (which are not in $\mathcal{C}^*$), then he would be able to distinguish trivially, since any honest designated verifiers (i.e. any $j \in \mathcal{D}^* \backslash \mathcal{C}^*$) can distinguish simulated signatures from real signatures (from the unforgability property).

Relation to previous definitions. Our definition of MDVS is consistent with previous work in this area, but with some differences: our syntax of MDVS follows closely the one introduced by [LV04] but we allow for a master secret key in the case where the keys are genereated by a trusted party (like in one of our constructions); our definitions of security are adapted from those in [LV04,ZAYS12] to capture the flexibility introduced by allowing any subset of designated verifiers to simulate a signature, thus providing better deniability properties; we formalize consistency as an additional and desirable requirement.

3 Standard Primitive-Based MDVS Constructions

In this section we show how to create an MDVS scheme that uses only standard primitives, such as key exchange, commitments, pseudorandom functions and generators, and non-interactive zero knowledge proofs.

On a high level, one way to build an MDVS is for the signer to use a separate DVS with each verifier; the MDVS signature would then consist of a vector of individual DVS signatures. This gives us almost everything we need — the remaining issue is consistency. Each verifier can verify one of the DVS signatures, but is not convinced that all of the other verifiers will come to the same conclusion.

A solution to this consistency issue is to include as part of the MDVS signature a zero knowledge proof that all of the DVS signatures verify. However, this introduces a new issue with source-hiding. Now, unless *all* of the verifiers are corrupt, a corrupt set of verifiers will not be able to simulate a signature. In order to produce such a convincing zero knowledge proof as part of the signature, they would need to forge signatures for the other verifiers in the underlying DVS scheme, which they should not be able to do.

So, instead of using a zero knowledge proof of knowledge that all of the DVS signatures verify, we use a proof that *either* all of the DVS signatures verify, *or they are all simulated*. Then, a corrupt set of verifiers can simulate all of the underlying DVS signatures — with the caveat that the signatures they simulate for themselves should be convincing simulations even in the presence of their secret keys — and, instead of proving that all of the signatures verify, they prove that all of the signatures are simulations.

In order to support such proofs, in Section 3.1 we introduce a new primitive called a Provably Simulatable DVS (PSDVS). Then, in Section 3.2 we show how to compose PSDVS instances into an MDVS. In Section 3.3 we build a PSDVS out of generic standard primitives. In Section 3.4 we build a more efficient PSDVS out of concrete instantiations of those primitives.

3.1 New Primitive: Provably Simulatable Designated-Verifier Signatures (PSDVS)

Designated Verifier Signatures (DVS) have a simulation algorithm Sim which is used to satisfy the source-hiding property (the single-verifier equivalent of Def 5). Given the signer's public key, the verifier's secret key and a message m, Sim should return a signature which is indistinguishable from a real signature. A *Provably Simulatable DVS (PSDVS)*, in addition to correctness and existential unforgeability, must have two notions of transcript simulation: *public* simulation and *verifier* simulation. Furthermore, for each of these, it should be possible to produce a zero knowledge proof that the signature produced is a simulation. Additionally, it should be possible to similarly prove that a real signature is, in fact, real. This makes a PSDVS well suited for use in an MDVS which uses a zero knowledge proof of knowledge to enforce consistency.

More formally, a PSDVS consists of the standard DVS algorithms Setup, SignKeyGen, VerKeyGen, Sign, Verify, as well as five additional algorithms: RealSigVal to validate real signatures, and PubSigSim, PubSigVal, VerSigSim and VerSigVal to simulate signatures and to validate such simulations.

Definition 6. *A PSDVS must satisfy the standard notions of correctness and existential unforgeability. Additionally, it should satisfy* PubSigSim indstinguishability *(Def 7),* PubSigSim correctness *(Def 8),* PubSigSim soundness *(Def 9),* VerSigSim indstinguishability *(Def 10),* VerSigSim correctness *(Def 11),* VerSigSim soundness *(Def 12),* provable signing correctness *(Def 13), and* provable signing soundness *(Def 14).*

Provable Public Simulation As in *PSI* (Def 4), anyone should be able to produce a signature that is indistinguishable from a real signature. Additionally, the party simulating the signature should be able to produce a proof that this is *not* a real signature. This proof will be incorporated into the MDVS proof of consistency; the underlying PSDVS signatures should be proven to be real, or all be proven to be fake.

In other words, we require two additional algorithms, as follows:

1. $\mathsf{PubSigSim}(spk, vpk, m) \rightarrow (\sigma, \pi)$
2. $\mathsf{PubSigVal}(spk, vpk, m, \sigma, \pi) \rightarrow d \in \{0, 1\}$

Definition 7 (PubSigSim Indistinguishability). *We say that the PSDVS has* PubSigSim Indistinguishability *if* PubSigSim *produces a signature σ that is indistinguishable from real. More formally, an adversary should not be able to win the following game with probability non-negligibly more than half:*

$\mathsf{Game}^{PubSigSim\text{-}Ind}_{\mathsf{PVDVS},\mathcal{A}}(\kappa)$

1. $pp \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(spk, ssk) \leftarrow \mathsf{SignKeyGen}(1^\kappa, pp)$
3. $(vpk, vsk) \leftarrow \mathsf{VerKeyGen}(1^\kappa, pp)$
4. $m^* \leftarrow \mathcal{A}^{\mathcal{O}_S, \mathcal{O}_V}(spk, vpk)$
5. $b \leftarrow_\$ \{0, 1\}$
6. $\sigma_0 \leftarrow \mathsf{Sign}(ssk, vpk, m^*)$
7. $(\sigma_1, \pi) \leftarrow \mathsf{PubSigSim}(spk, vpk, m^*)$
8. $b' \leftarrow \mathcal{A}^{\mathcal{O}_S, \mathcal{O}_V}(spk, vpk, m^*, \sigma_b)$

We say that $\mathcal{A}$ *wins the* PubSigSim-Ind *game if* $b = b'$ *and for all queries* (m, σ) *to* $\mathcal{O}_V$*, it holds that* $(m, \sigma) \neq (m^*, \sigma_b)$*.*

Definition 8 (PubSigSim Correctness). *We say that the PSDVS has* PubSigSim Correctness *if for all* $pp \leftarrow \mathsf{Setup}(1^\kappa)$*;* $(spk, ssk) \leftarrow \mathsf{SignKeyGen}(1^\kappa, pp)$*;* $(vpk, vsk) \leftarrow \mathsf{VerKeyGen}(1^\kappa, pp)$*;* $m \in \{0, 1\}^*$*;* $(\sigma, \pi) \leftarrow \mathsf{PubSigSim}(spk, vpk, m)$*;*

$$\Pr[\mathsf{PubSigVal}(spk, vpk, m, \sigma, \pi) = 1] = 1.$$

Definition 9 (PubSigSim Soundness). *We say that the PSDVS has* PubSigSim Soundness *if it is hard to construct a signature* σ *which is accepted by the verifier algorithm and at the same time can be proven to be a simulated signature. More formally, an adversary should not be able to win the following game with non-negligible probability:*

$\mathsf{Game}^{PubSigSim\text{-}Sound}_{\mathsf{PVDVS},\mathcal{A}}(\kappa)$

1. $pp \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(spk, ssk) \leftarrow \mathsf{SignKeyGen}(1^\kappa, pp)$
3. $(vpk, vsk) \leftarrow \mathsf{VerKeyGen}(1^\kappa, pp)$
4. $(m^*, \sigma^*, \pi^*) \leftarrow \mathcal{A}(ssk, spk, vpk)$

We say that $\mathcal{A}$ *wins the* PubSigSim-Sound *game if* $\mathsf{Verify}(vsk, m^*, \sigma^*) = 1$ *and* $\mathsf{PubSigVal}(spk, vpk, m^*, \sigma^*, \pi^*) = 1$*.*

Provable Verifier Simulation As in *source hiding* (Def 5), a verifier should be able to produce a signature that is indistinguishable from a real signature, *even given its secret key.* Additionally, the verifier should be able to produce a proof that the signature is *not* a real signature (that is, that the verifier, and not the signer, produced it). This proof will be incorporated into the MDVS proof of consistency.

In other words, we require two additional algorithms, as follows:

1. $\mathsf{VerSigSim}(spk, vpk, vsk, m) \rightarrow (\sigma, \pi)$
2. $\mathsf{VerSigVal}(spk, vpk, m, \sigma, \pi) \rightarrow d \in \{0, 1\}$

Definition 10 (VerSigSim Indistinguishability). *We say that the PSDVS has* VerSigSim Indistinguishability *if* VerSigSim *produces a signature* σ *that is indistinguishable from real. More formally, an adversary should not be able to win the following game with probability non-negligibly more than half:*

$\mathsf{Game}^{VerSigSim\text{-}Ind}_{\mathsf{PVDVS},\mathcal{A}}(\kappa)$

1. $pp \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(spk, ssk) \leftarrow \mathsf{SignKeyGen}(1^\kappa, pp)$
3. $(vpk, vsk) \leftarrow \mathsf{VerKeyGen}(1^\kappa, pp)$
4. $m^* \leftarrow \mathcal{A}^{\mathcal{O}_S}(spk, vpk, vsk)$
5. $b \leftarrow_\$ \{0, 1\}$
6. $\sigma_0 \leftarrow \mathsf{Sign}(ssk, vpk, m^*)$
7. $(\sigma_1, \pi) \leftarrow \mathsf{VerSigSim}(spk, vsk, m^*)$
8. $b' \leftarrow \mathcal{A}^{\mathcal{O}_S}(spk, vpk, vsk, m^*, \sigma_b)$

We say that $\mathcal{A}$ *wins the* VerSigSim-Ind *game if* $b = b'$.

Definition 11 (VerSigSim Correctness). *We say that the PSDVS has* VerSigSim Correctness *if for all* $pp \leftarrow \mathsf{Setup}(1^\kappa)$, $(spk, ssk) \leftarrow \mathsf{SignKeyGen}(1^\kappa, pp)$, $(vpk, vsk) \leftarrow \mathsf{VerKeyGen}(1^\kappa, pp)$, $m \in \{0,1\}^*$, $(\sigma, \pi) \leftarrow \mathsf{VerSigSim}(spk, vpk, vsk, m)$,

$$\Pr[\mathsf{VerSigVal}(spk, vpk, m, \sigma, \pi) = 1] = 1.$$

Definition 12 (VerSigSim Soundness). *We say that the PSDVS has* VerSigSim Soundness *if the signer is not able to produce* σ *and* π *that pass the validation check* VerSigVal*, i.e.* π *is a proof that* σ *was not produced by the signer. More formally, an adversary should not be able to win the following game with non-negligible probability:*

$\mathsf{Game}^{VerSigSim\text{-}Sound}_{\mathsf{PVDVS},\mathcal{A}}(\kappa)$

1. $pp \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(spk, ssk) \leftarrow \mathsf{SignKeyGen}(1^\kappa, pp)$
3. $(vpk, vsk) \leftarrow \mathsf{VerKeyGen}(1^\kappa, pp)$
4. $(m^*, \sigma^*, \pi^*) \leftarrow \mathcal{A}(ssk, spk, vpk)$

$\mathcal{A}$ *wins the* VerSigSim-Sound *game if* $\mathsf{VerSigVal}(spk, vpk, m^*, \sigma^*, \pi^*) = 1$.

Provable Signing Lastly, we require a provable variant of signing, so that the signer is able to produce a proof that a signature is real. In other words, we require the signing algorithm $\mathsf{Sign}(spk, ssk, vpk, m) \to (\sigma, \pi)$ to output π as well. We also require one additional validation algorithm, as follows:

$$\mathsf{RealSigVal}(spk, vpk, m, \sigma, \pi) \to d \in \{0, 1\}$$

Definition 13 (Provable Signing Correctness). *We say that the PSDVS has* Provable Signing Correctness *if* $\forall pp \leftarrow \mathsf{Setup}(1^\kappa)$, $(spk, ssk) \leftarrow \mathsf{SignKeyGen}(1^\kappa, pp)$, $(vpk, vsk) \leftarrow \mathsf{VerKeyGen}(1^\kappa, pp)$, $m \in \{0,1\}^*$, $(\sigma, \pi) \leftarrow \mathsf{Sign}(spk, ssk, vpk, m)$,

$$\Pr[\mathsf{RealSigVal}(spk, vpk, m, \sigma, \pi) = 1] = 1.$$

Definition 14 (Provable Signing Soundness). *We say that the PSDVS has* Provable Signing Soundness *if the proof of correctness* π *produced by* Sign *does not verify unless* σ *verifies. More formally, an adversary should not be able to win the following game with non-negligible probability:*

$\mathsf{Game}^{Sign\text{-}Sound}_{\mathsf{PVDVS},\mathcal{A}}(\kappa)$

1. $pp \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(spk, ssk) \leftarrow \mathsf{SignKeyGen}(1^\kappa, pp)$
3. $(vpk, vsk) \leftarrow \mathsf{VerKeyGen}(1^\kappa, pp)$
4. $(m^*, \sigma^*, \pi^*) \leftarrow \mathcal{A}(ssk, spk, vpk)$

We say that $\mathcal{A}$ wins the Sign-Sound *game if* $\mathsf{RealSigVal}(spk, vpk, m^*, \sigma^*, \pi^*) = 1$ *and* $\mathsf{Verify}(spk, vsk, m^*, \sigma^*) = 0$.

Note that none of these proofs are parts of the signature. If included in the signature, such proofs would allow an adversary to distinguish a simulation from a real signature.

3.2 Standard Primitive-Based MDVS Construction

Given a PSDVS, as defined in Section 3.1, we can build an MDVS. The transformation is straightforward: the signer uses the PSDVS to sign a message for each verifier, and proves consistency using a non-interactive zero knowledge proof of knowledge. The proof of consistency will claim that either all of the PSDVS signatures verify, or all of them are simulated. Const 1 describes this transformation.

Construction 1 *Let* $\mathsf{PSDVS} = (\mathsf{Setup}, \mathsf{SignKeyGen}, \mathsf{VerKeyGen}, \mathsf{Sign}, \mathsf{Verify}, \mathsf{RealSigVal}, \mathsf{PubSigSim}, \mathsf{PubSigVal}, \mathsf{VerSigSim}, \mathsf{VerSigVal})$ *be a provably simulatable designated verifier signature scheme, and* $\mathsf{NIZK\text{-}PoK} = (\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ *be a non-interactive zero knowledge proof of knowledge system and $\mathcal{R}_{cons}$ a relation that we will define later in the protocol.*

$\mathsf{Setup}(1^\kappa)$**:**

1. $crs \leftarrow \mathsf{NIZK\text{-}PoK.Setup}(1^\kappa, \mathcal{R}_{cons})$.
2. $\mathsf{PSDVS}.pp \leftarrow \mathsf{PSDVS.Setup}(1^\kappa)$.

Output $(crs, \mathsf{PSDVS}.pp)$ as the public parameters pp.

$\mathsf{SignKeyGen}(1^\kappa, pp)$**:** $(spk_i, ssk_i) \leftarrow \mathsf{PSDVS.SignKeyGen}(1^\kappa, \mathsf{PSDVS}.pp)$.

Output (spk_i, ssk_i) as signer S_i's public/secret key pair.

$\mathsf{VerKeyGen}(1^\kappa, pp)$**:** $(vpk_j, vsk_j) \leftarrow \mathsf{PSDVS.VerKeyGen}(1^\kappa, \mathsf{PSDVS}.pp)$.

Output (vpk_j, vsk_j) as verifier R_j's public/secret key pair.

$\mathsf{Sign}(pp, ssk_i, \{vpk_j\}_{j\in\mathcal{D}}, m)$**:**

1. For every verifier $j \in \mathcal{D}$, compute a signature and proof of signature validity as $(\sigma_j, \pi_j) \leftarrow \mathsf{PSDVS.Sign}(ssk_i, vpk_j, m)$.
2. Create a proof π of consistency, i.e a proof of knowledge of $\{\pi_j\}_{j\in\mathcal{D}}$ such that either all signatures are real (as demostrated by $\{\pi_j\}_{j\in\mathcal{D}}$), or all signatures are fake (as could be demonstrated by the proofs produced by $\mathsf{PSDVS.PubSigSim}$ or $\mathsf{PSDVS.VerSigSim}$).

More formally, for this NIZK-PoK we define a relation for a statement $u = (spk_i, \{vpk_j\}_{j\in\mathcal{D}}, \{\sigma_j\}_{j\in\mathcal{D}})$ and the witness $w = \{\pi_j\}_{j\in\mathcal{D}}$:

$$\mathcal{R}_{cons} = \Big\{ u = (spk_i, \{vpk_j\}_{j\in\mathcal{D}}, \{\sigma_j\}_{j\in\mathcal{D}}), w = \{\pi_j\}_{j\in\mathcal{D}} : \Big(\bigwedge_{j\in\mathcal{D}} \mathsf{PSDVS.RealSigVal}(spk_i, vpk_j, m, \sigma_j, \pi_j) = 1 \Big) \bigvee \Big(\bigwedge_{j\in\mathcal{D}} \big(\mathsf{PSDVS.VerSigVal}(spk_i, vpk_j, m, \sigma_j, \pi_j) = 1 \vee \mathsf{PSDVS.PubSigVal}(spk_i, vpk_j, m, \sigma_j, \pi_j) = 1 \big) \Big) \Big\} \quad (1)$$

Let $\pi \leftarrow \mathsf{NIZK\text{-}PoK.Prove}(crs, u = (spk_i, \{vpk_j\}_{j\in\mathcal{D}}, \{\sigma_j\}_{j\in\mathcal{D}}), w = \{\pi_j\}_{j\in\mathcal{D}})$.

3. $\sigma = (\{\sigma_j\}_{j\in\mathcal{D}}, \pi)$

Output σ as the signature.

$\mathsf{Verify}(pp, vsk_j, m, \sigma = (\{\sigma_j\}_{j\in\mathcal{D}}, \pi))$:

1. Let $d_\pi \leftarrow \mathsf{NIZK\text{-}PoK.Verify}(crs, u = (spk_i, \mathcal{D}, \{vpk_j\}_{j\in\mathcal{D}}, \{\sigma_j\}_{j\in\mathcal{D}}), \pi)$,
2. $d \leftarrow \mathsf{PSDVS.Verify}(vsk_j, m, \sigma_j) \wedge d_\pi$

Output d as the verification decision.

$\mathsf{Sim}(pp, spk_i, \{vpk_j\}_{j\in\mathcal{D}}, \{vsk_j\}_{j\in\mathcal{C}}, m)$:

1. For $j \in \mathcal{D} \cap \mathcal{C}$: $(\sigma_j, \pi_j) \leftarrow \mathsf{VerSigSim}(spk_i, vpk_j, vsk_j, m)$
2. For $j \in \mathcal{D} \backslash \mathcal{C}$: $(\sigma_j, \pi_j) \leftarrow \mathsf{PubSigSim}(spk_i, vpk_j, m)$
3. Use these signatures and proofs to produce the NIZK π for relation $\mathcal{R}_{cons}$.
4. $\sigma = (\{\sigma_j\}_{j\in\mathcal{D}}, \pi)$.

Output σ as the signature.

Theorem 2. *Assume* PSDVS *is a secure provably simulatable designated verifier signature scheme and* NIZK-PoK *is a secure non-interactive zero knowledge proof of knowledge system. Then Const 1 is a correct and secure MDVS scheme (without privacy of identities (Def 4)).*

Proof. Correctness is apparent by inspection. We show consistency, unforgeability and source hiding separately.

Claim. Const 1 is consistent, as per Def 2.

Assume that Const 1 is inconsistent; then there exists an adversary $\mathcal{A}$ that can produce a message m^* and signature $\sigma^* = (\{\sigma_j^*\}_{j\in\mathcal{D}^*}, \pi^*)$ such that σ^* verifies for some, but not all, of the intended recipients $\mathcal{D}^*$. We can then use $\mathcal{A}$ to create another adversary $\mathcal{B}$ that can break either the security of the underlying PSDVS, or the security of the NIZK-PoK.

$\mathcal{B}$ receives ssk, spk, vpk regardless of whether it's playing the *PubSigSim-Sound, VerSigSim-Sound* or *Sign-Sound* games.

It randomly chooses identities to assign the given signer and verifier keys to; it generates the other signer and verifier keys honestly. It answers signing oracle queries honestly, since in all these cases it has the signer secret key. It

answers key generation keys honestly as well, unless asked for ssk or the secret key corresponding to vpk; then, it aborts. However, since at least one signer and one verifier secret key must remain unqueried by $\mathcal{A}$, the probability of an abort is not overwhelming. Eventually, it gets $\mathcal{D}^*$ and $(m^*, \sigma^* = (\{\sigma_j^*\}_{j \in \mathcal{D}}, \pi^*))$ from $\mathcal{A}$; by assumption, with non-negligible probability, σ^* is inconsistent.

Assume without loss of generality that $j_0, j_1 \in \mathcal{D}^*$, $\mathsf{Verify}(vsk_{j_0}, m^*, \sigma^*) = 0$, and $\mathsf{Verify}(vsk_{j_1}, m^*, \sigma^*) = 1$. Assume, also without loss of generality, that $\mathsf{NIZK\text{-}PoK.Verify}$ is deterministic; then, the decision d_π regarding the validity of the zero knowledge proof of knowledge π^* must be the same for verifiers R_{j_0} and R_{j_1}, and so it must be that $d_\pi = 1$. It follows that in order for $\mathsf{Verify}(vsk_{j_0}, m^*, \sigma^*) = 0$ we need $\mathsf{PSDVS.Verify}(vsk_{j_0}, m^*, \sigma_{j_0}^*) = 0$, and in order for $\mathsf{Verify}(vsk_{j_1}, m^*, \sigma^*) = 1$ we need $\mathsf{PSDVS.Verify}(vsk_{j_1}, m^*, \sigma_{j_1}^*) = 1$. Then, if $d_\pi = 1$, either π^* violates the soundness of $\mathsf{NIZK\text{-}PoK}$, or one of the following must be true:

1. $\mathsf{PSDVS.RealSigVal}(spk, vpk_{j_0}, m^*, \sigma_{j_0}^*, \pi_{j_0}) = 1$. If this is the case, then $\mathcal{B}$ returns $(m^*, \sigma_{j_0}^*, \pi_{j_0})$ (the last of which is extractable from the knowledge soundness property of π^*) as a break of Provable Signing Soundness of the PSDVS, since we know that $\mathsf{PSDVS.Verify}(vsk_{j_0}, m^*, \sigma_{j_0}^*) = 0$.
2. $\mathsf{PSDVS.VerSigVal}(spk, vpk_{j_1}, m^*, \sigma_{j_1}^*, \pi_{j_1}) = 1$. If this is the case, then $\mathcal{B}$ returns $(m^*, \sigma_{j_1}^*, \pi_{j_1})$ (the last of which is extractable from the knowledge soundness property of π^*) as a break of VerSigSim Soundness, since the adversary was not given the secret key corresponding to vpk.
3. $\mathsf{PSDVS.PubSigVal}(spk, vpk_{j_1}, m^*, \sigma_{j_1}^*, \pi_{j_1}) = 1$. If this is the case, then $\mathcal{B}$ returns $(m^*, \sigma_{j_1}^*, \pi_{j_1})$ (the last of which is extractable from the knowledge soundness property of π^*) as a break of PubSigSim Soundness, since we know that $\mathsf{PSDVS.Verify}(vsk_{j_1}, m^*, \sigma_{j_1}^*) = 1$.

Claim. Const 1 is existentially unforgeable, as per Def 3.

This holds since any forgery of Const 1 either includes a forgery of the underlying PSDVS, or includes a fresh proof of knowledge on a statement for which the adversary does not have a witness, which is impossible by the knowledge soundness property of our NIZK proof of knowledge.

Claim. Const 1 is source-hiding, as per Def 5.

The simulation algorithm Sim, described above, is defined to use public and verifier simulation to produce the individual PSDVS signatures, and to prove that all PSDVS signatures are simulations instead of proving that they all verify.

Below we describe a sequence of games; in Game 0, it is impossible for the adversary to distinguish $b = 0$ from $b = 1$, since its view in the two cases are identically distributed. In each subsequent game, the advantage of the adversary is at most negligibly greater than in the previous one; in the final game, the adversary will find itself playing exactly the game described in Def 5.

Game 0**:** This gives $\mathcal{A}$ a real signature no matter what the value of b is.

$\mathcal{A}$ can have no advantage in this game.

Game 1**:** This game is the same as the previous game, except that if $b = 1$, the NIZK-PoK is simulated (and thus does not require the witnesses $\{\pi_j\}_{j \in \mathcal{D}^*}$). If $\mathcal{A}$ distinguishes $b = 0$ from $b = 1$ with non-negligibly greater advantage than in the previous game, $\mathcal{B}$ can use $\mathcal{A}$ to break the zero-knowledge property of the NIZK-PoK.

Game 2.j **for** $j \in \mathcal{D}^* \cap \mathcal{C}^*$**:** This game is the same as the previous game, except that if $b = 1$, R_j's portion of the signature is replaced with a verifier simulation; that is, $(\sigma_j, \pi_j) \leftarrow \mathsf{VerSigSim}(spk_{i^*}, vpk_j, vsk_j, m^*)$.
If $\mathcal{A}$ distinguishes $b = 0$ from $b = 1$ with non-negligibly greater advantage than in the previous game, $\mathcal{B}$ can use $\mathcal{A}$ to break the VerSigSim indistinguishability property.
Note that the statement the NIZK-PoK is proving no longer holds; however, the NIZK-PoK simulator must still produce a simulated NIZK-PoK that is indistinguishable from a real one, since otherwise the NIZK-PoK simulator would be useable to distinguish a signature produced by VerSigSim from a signature produced by Sign.

Game 3.j **for** $j \in \mathcal{D}^* \backslash \mathcal{C}^*$**:** This game is the same as the previous game, except that if $b = 1$, R_j's portion of the signature is replaced with a public simulation; that is, $(\sigma_j, \pi_j) \leftarrow \mathsf{PubSigSim}(spk_{i^*}, vpk_j, m^*)$.
If $\mathcal{A}$ distinguishes $b = 0$ from $b = 1$ with non-negligibly greater advantage than in the previous game, $\mathcal{B}$ can use $\mathcal{A}$ to break the PubSigSim indistinguishability property.

Game 4**:** This game is the same as the previous game, except that if $b = 1$, we replace the simulated π with a real proof; since all the individual signatures are now simulated, such a valid proof can be computed again. Note that now, if $b = 1$, we are executing exactly the simulation procedure Sim described above, and thus this is exactly the game described in Def 5.
If $\mathcal{A}$ distinguishes $b = 0$ from $b = 1$ with non-negligibly greater advantage than in the previous game, $\mathcal{B}$ can use $\mathcal{A}$ to break the zero-knowledge property of the NIZK-PoK.

3.3 Standard Primitive-Based PSDVS Construction

We can build a PSDVS from a special message authentication code (MAC) which looks uniformly random without knowledge of the secret MAC key — such a MAC can be built from any pseudorandom function. A signature on a message m will be a MAC on (m, t), where t is some random tag. Proving that the signature verifies simply involves proving knowledge of a MAC key that is consistent with the MAC and some global public commitment to the MAC key. A public proof that the signature is simulated and does not verify would involve proving that the MAC was pseudorandomly generated. A verifier's proof that the signature is simulated would involve proving that the tag was generated in a way that only the verifier could use (e.g. from a PRF to which only the verifier knows the key).

Of course, this is not ideal, since MACs require knowledge of a shared key; in order to use MACs, we would need to set up shared keys between every possible

pair of signer and verifier. However, we can get around this using non-interactive key exchange (NIKE). Each signer and verifier publishes a public key, and any pair of them can agree on a shared secret key by simply using their own secret key and the other's public key.

Const 2 describes this construction in more detail.

Construction 2 *Let:*

- $\mathsf{COMM} = (\mathsf{Setup}, \mathsf{Commit}, \mathsf{Open})$ *be a commitment scheme,*
- $\mathsf{PRF} = (\mathsf{KeyGen}, \mathsf{Compute})$ *be a length-preserving pseudorandom function,*
- PRG *be a length-doubling pseudorandom generator,*
- $\mathsf{NIZK} = (\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ *be a non-interactive zero knowledge proof system, and*
- $\mathsf{NIKE} = (\mathsf{KeyGen}, \mathsf{KeyExtract}, \mathsf{KeyMatch})$ *be a non-interactive key exchange protocol.* $\mathsf{KeyMatch}$ *is an additional algorithm that checks if a public key and a secret key match.* $\mathsf{KeyMatch}$ *is not typically defined as a part of a* NIKE *scheme; however, such an algorithm always exists.*

We consider the public parameters for the uderlying primitives $\mathsf{COMM}, \mathsf{PRF}, \mathsf{PRG}, \mathsf{NIKE}$ *together with the three common reference strings* (crs_1, crs_2, crs_3) *corresponding to the relations* $\mathcal{R}_1, \tilde{\mathcal{R}}_2, \tilde{\mathcal{R}}_3$ [7] *necessary to compute* NIZK *proofs as part of the public parameters of the PSDVS. Note that relations denoted* $\tilde{\mathcal{R}}$ *refer to statements of fake-ness, whereas relations denoted* $\mathcal{R}$ *refer to statements of real-ness.*

$\mathsf{Setup}(1^\kappa)$**:** $crs_i \leftarrow \mathsf{NIZK.Setup}(1^\kappa, \mathcal{R}_i), i = 1, 2, 3.$

Output $\{crs_i\}_i$ and the other public values as the public parameters pp.

$\mathsf{SignKeyGen}(1^\kappa, pp)$**:**

1. $(\mathsf{NIKE}.pk_\mathsf{S}, \mathsf{NIKE}.sk_\mathsf{S}) \leftarrow \mathsf{NIKE.KeyGen}(1^\kappa)$.
2. $ssk = \mathsf{NIKE}.sk_\mathsf{S}$.
3. $spk = \mathsf{NIKE}.pk_\mathsf{S}$.

Output ssk as the signer's secret key and spk as the signer's public key.

$\mathsf{VerKeyGen}(1^\kappa, pp)$**:**

1. $(\mathsf{NIKE}.pk_\mathsf{R}, \mathsf{NIKE}.sk_\mathsf{R}) \leftarrow \mathsf{NIKE.KeyGen}(1^\kappa)$.
2. $k_\mathsf{R} \leftarrow \mathsf{PRF.KeyGen}(1^\kappa)$. (Informally, this key will be used by the verifier to simulate signatures using $\mathsf{VerSigSim}$.)
3. Choose randomness (i.e. decommitment value) r_R at random.
4. $c_\mathsf{R} = \mathsf{COMM.Commit}(ck, k_\mathsf{R}; r_\mathsf{R})$. (Informally, this commitment will be used by the verifier to support its proofs of fake-ness.)
5. $vsk = (\mathsf{NIKE}.sk_\mathsf{R}, k_\mathsf{R}, r_\mathsf{R})$.
6. $vpk = (\mathsf{NIKE}.pk_\mathsf{R}, c_\mathsf{R})$.

Output vsk as the verifier's secret key and vpk as the verifier's public key.

$\mathsf{Sign}(ssk = \mathsf{NIKE}.sk_\mathsf{S}, vpk = (\mathsf{NIKE}.pk_\mathsf{R}, c_\mathsf{R}), m)$**:**

The signer computes a shared key with the designated verifier and proceeds to sign the message m:

[7] The three relations will be defined later in the protocol description.

1. $k_{shared} = \mathsf{NIKE.KeyExtract}(\mathsf{NIKE}.sk_{\mathsf{S}}, \mathsf{NIKE}.pk_{\mathsf{R}})$. (Informally, this key will be used as a MAC key.)
2. Choose t at random.
3. $\sigma \leftarrow (t, \mathsf{PRF}_{k_{shared}}((m,t)))$.
4. $\pi \leftarrow \mathsf{NIZK.Prove}(crs_1, u, w)$ where $u = ((\sigma_1,\sigma_2), \mathsf{NIKE}.pk_{\mathsf{S}}, \mathsf{NIKE}.pk_{\mathsf{R}}, m)$ and $w = (\mathsf{NIKE}.sk_{\mathsf{S}}, k_{shared}))$
   We define the relation $\mathcal{R}_1$ indexed by NIKE public parameters and PRF for a statement u and witness w:

$$\begin{aligned}\mathcal{R}_1 = \{(u = (\sigma_1,\sigma_2, \mathsf{NIKE}.pk_{\mathsf{S}}, \mathsf{NIKE}.pk_{\mathsf{R}}, m), w = (\mathsf{NIKE}.sk_{\mathsf{S}}, k_{shared})) :\\ \mathsf{KeyMatch}(\mathsf{NIKE}.pk_{\mathsf{S}}, \mathsf{NIKE}.sk_{\mathsf{S}}) = 1\\ \wedge\ k_{shared} = \mathsf{NIKE.KeyExtract}(\mathsf{NIKE}.sk_{\mathsf{S}}, \mathsf{NIKE}.pk_{\mathsf{R}})\\ \wedge\ \sigma_2 = \mathsf{PRF}_{k_{shared}}((m,\sigma_1))\}\end{aligned}$$

Output σ as the signature, and π as the proof of real-ness.

$\mathsf{Verify}(spk = \mathsf{NIKE}.pk_{\mathsf{S}}, vsk = (\mathsf{NIKE}.sk_{\mathsf{R}}, k_{\mathsf{R}}, r_{\mathsf{R}}), m, \sigma = (\sigma_1,\sigma_2))$**:**

1. $k_{shared} = \mathsf{NIKE.KeyExtract}(\mathsf{NIKE}.sk_{\mathsf{R}}, \mathsf{NIKE}.pk_{\mathsf{S}})$. (Informally, this key will be used as a MAC key.)
2. If $\mathsf{PRF}_{k_{shared}}((m,\sigma_1)) = \sigma_2$, set $d = 1$. Otherwise, set $d = 0$.

Output d as the verification decision.

$\mathsf{RealSigVal}(spk, vpk, m, \sigma, \pi)$**:**

Output $d \leftarrow \mathsf{NIZK.Verify}(crs_1, \sigma, \pi)$ as the validation decision.

$\mathsf{PubSigSim}(m)$**:**

1. Choose a PRG seed seed.
2. Choose σ_1 and σ_2 pseudorandomly by running PRG on seed.
3. $\sigma \leftarrow (\sigma_1,\sigma_2)$.
4. Let $\pi \leftarrow \mathsf{NIZK.Prove}(crs_2, u = \sigma, w = \mathsf{seed})$.
   We define the relation $\tilde{\mathcal{R}}_2$ indexed by the PRG for a statement $u = (\sigma = (\sigma_1,\sigma_2))$ and the witnesses $w = \mathsf{seed}$:

$$\tilde{\mathcal{R}}_2 = \{(u = \sigma; w = \mathsf{seed}) : u = \mathsf{PRG}(w)\} \tag{2}$$

Output σ as the simulated signature, and π as the proof of fake-ness.

$\mathsf{PubSigVal}(spk, vpk, m, \sigma = (\sigma_1,\sigma_2), \pi)$**:**

Output $d \leftarrow \mathsf{NIZK.Verify}(crs_2, \sigma, \pi)$ as the validation decision.

$\mathsf{VerSigSim}(spk = \mathsf{NIKE}.pk_{\mathsf{S}}, vpk = (\mathsf{NIKE}.pk_{\mathsf{R}}, c_{\mathsf{R}}), vsk = (\mathsf{NIKE}.sk_{\mathsf{R}}, k_{\mathsf{R}}, r_{\mathsf{R}}), m)$**:**

The verifier can fake a signature using its PRF key k_{R}.

1. $k_{shared} = \mathsf{NIKE.KeyExtract}(\mathsf{NIKE}.sk_{\mathsf{R}}, \mathsf{NIKE}.pk_{\mathsf{S}})$.
2. Choose r at random.
3. $t \leftarrow \mathsf{PRF}_{k_{\mathsf{R}}}(r)$.
4. $\sigma \leftarrow (t, \mathsf{PRF}_{k_{shared}}((m,t)))$.
5. Let $\pi \leftarrow \mathsf{NIZK.Prove}(crs_3, u = (c_{\mathsf{R}}, \sigma_1), w = (k_{\mathsf{R}}, r_{\mathsf{R}}, r))$.
   We define the relation $\tilde{\mathcal{R}}_3$ indexed by the NIKE public parameters and PRF for statements u and witnesses w:

$$\begin{aligned}\tilde{\mathcal{R}}_3 = \{(u = (c_{\mathsf{R}}, \sigma_1), w = (k_{\mathsf{R}}, r_{\mathsf{R}}, r)) :\\ k_{\mathsf{R}} = \mathsf{COMM.Open}(c_{\mathsf{R}}, r_{\mathsf{R}}) \wedge\ \sigma_1 = \mathsf{PRF}_{k_{\mathsf{R}}}(r)\}\end{aligned}$$

Output σ as the simulated signature and π as the proof of fake-ness.

$\mathsf{VerSigVal}(spk, vpk, m, \sigma, \pi)$**:**

Output $d \leftarrow \mathsf{NIZK.Verify}(crs_3, (\sigma, c_{\mathsf{R}}, m), \pi)$ as the validation decision.

Theorem 3. *If the schemes* PRG, PRF, NIKE, NIZK *are secure, then Const 2 is a correct and secure PSDVS scheme as per Def 6.*

Proof. Correctness. It is straightforward to verify that any honestly generated signature will pass the verification test.

Existential Unforgeability. We can reduce the unforgeability of the PVDVS to the pseudorandomness of the underlying PRF. Suppose there exists a forger $\mathcal{A}$ having non-negligible advantage in winning the existential unforgeability game 3 for a single signer and a single verifier (see Def 3). We can use this forger to build a distinguisher $\mathcal{B}$ that is able to break the pseudo-randomness property of the PRF. $\mathcal{B}$ runs $\mathcal{A}$ and simulate the signing queries m of $\mathcal{A}$ by picking a random tag t and forwarding the query (m, t) to its evaluation oracle that outputs either the evaluation $\mathsf{PRF}(m, t)$, either a truly random value. In the first case, the forger $\mathcal{A}$ has the same view as in the game 3. Given that $\mathcal{A}$ is able to forge a signature on a fresh message m^* in the first case, but not in the second, then this implies that the adversary $\mathcal{B}$ can distinguish between the two, breaking the pseudo-randomness of the PRF.

PubSigSim Indistinguishability (Def 7). By the security property of the PRG (indistinguishability from real randomness), the advantage of an adversary $\mathcal{A}$ in the game $\mathsf{Game}^{PubSigSim\text{-}Sound}_{\mathsf{PVDVS},\mathcal{A}}(\kappa)$ is bounded by the advantage of an adversary $\mathcal{B}$ in distinguishing between PRG and a truly random generator. Note that we can apply the property only for the first half of the signature, adversary $\mathcal{A}$ should not be able to distinguish between a random t and σ_1 generated as $(\sigma_1, \sigma_2) \leftarrow \mathsf{PRG}(\mathsf{seed})$.

PubSigSim Correctness (Def 8). By the completeness of the NIZK scheme, any proof π generated honestly by running the PubSigSim, i.e $\pi \leftarrow \mathsf{NIZK.Prove}(crs_2, u = \sigma, w)$ will be validated by $\mathsf{PubSigVal}(spk, vpk, m, \sigma, \pi)$ that runs NIZK.Verify $(crs_2, u = \sigma, \pi)$ algorithm.

PubSigSim Soundness (Def 9). Suppose there is an adversary $\mathcal{A}$ that wins the game of *PubSigSim-Sound* game with non negligible probability. Then, from an output (m^*, σ^*, π^*) of $\mathcal{A}(ssk, spk, vpk)$ we have from the soundness of the NIZK, since PubSigVal outputs 1, that there is a value seed such that $\sigma^* = (\sigma_1^*, \sigma_2^*) \leftarrow \mathsf{PRG}(\mathsf{seed})$ and also $\sigma^* = (\sigma_1^*, \mathsf{PRF}(m^*, \sigma_1^*))$ from the verification check of the signature $\mathsf{Verify}(vsk, m^*, \sigma^*) = 1$. This implies there exists a collision $\mathsf{PRG}(\mathsf{seed}) = (\sigma_1^*, \mathsf{PRF}(m^*, \sigma_1^*))$ breaking the pseudorandomness of the underlying primitives PRG and PRF. In the first case, given a signature $\sigma^* = (\sigma_1^*, \mathsf{PRF}(m^*, \sigma_1^*))$ that verifies, $\mathcal{A}$ should not be able to find a preimage seed^* for σ^* with respect to the PRG with advantage significantly better than for a truly random function, without breaking the pseudorandomness of the PRG. Otherwise, from computing an output of the pseudorandom generator $\mathsf{PRG}(\mathsf{seed}) = (\sigma_1^*, \sigma_2^*)$, $\mathcal{A}$ should not be able to find a (fixed prefix) preimage (t, m^*) of PRF such that $t = \sigma_1^*$. This is indeed infeasable without breaking the pseudorandom property of the PRF.

VerSigSim Indistinguishability (Def 10). This follows from the pseudorandomness of our PRF. Remark that both a real signature and a verifier-simulated signature pass the verification test, the only difference is in how the tag t is generated, truly random, or as $t \leftarrow \mathsf{PRF}_{k_\mathsf{R}}(r)$.

VerSigSim Correctness (Def 11). As in the case of PubSigSim correctness, this holds by considering the completeness of the NIZK scheme, since an honnest proof generated by VerSigSim, will be validated by VerSigVal that simply runs NIZK.Verify algorithm.

VerSigSim Soundness (Def 12.) This follows from the properties of the underlying COMM and NIZK schemes. Consider an adversary $\mathcal{A}$ that is able to win the game of VerSigSim-Sound, meaning that it produces a couple (σ, π) validated by VerSigVal. Then, if the NIZK scheme is assumed to be sound, the following should hold: $\mathcal{A}$ is able to compute an opening k_R of c_R, breaking the hidding of the commitment scheme COMM or $\mathcal{A}$ is able to find a preimage r for $\mathsf{PRF}_{k_\mathsf{R}}$, i.e $\sigma_1 = \mathsf{PRF}_{k_\mathsf{R}}(r)$ which breaks the pseudorandomness of the PRF.

Provable Signing Correctness (Def 13). As in the case of PubSigSim and VerSigSim, this follows by definition of RealSigVal and the completeness of the NIZK.

Provable Signing Soundness (Def 14). This holds if we assume soundness of the NIZK proof togheter with NIKE security properties and pseudorandomness of the PRF. Assuming that the NIZK is sound, then an adversary winning the game RealSigVal, is able either to break the soundness of the NIKE scheme or to find a preimage of the PRF. Finding a preimage is infeasible, given the pseudorandomness property of the PRF.

3.4 DDH and Paillier-Based PSDVS Construction

The goal of this section is to construct a PSDVS scheme based on DDH and the security of Paillier encryption. The idea in the PSDVS construction is that the authenticator for a message m will be $\mathsf{H}(m,t)^k$ in a group G where t is a nonce, k is a key known to both parties and H is a hash function modeled as a random oracle. The construction requires that certain properties of the key can be proved in zero-knowledge, and we can do this efficiently using standard Σ-protocols because the key is in the exponent. However, naive use of this idea would mean that a sender needs to store a key for every verifier he talks to, and the set-up must generate correlated secret keys for the parties. To get around this, we will instead let the sender choose k on the fly and send it to the verifier, encrypted using a new variant of Paillier encryption. In the following subsection we describe and prove this new encryption scheme, and then we specify the actual PSDVS construction. Paillier-style encryption comes in handy since its algebraic properties are useful in making our ZK proofs efficient.

Paillier-based Authenticated and Verifiable Encryption An authenticated and verifiable encryption scheme (AVPKE) involves a sender S and a verifier R. Such a scheme comes with probabilistic polynomial time algorithms

for setup which outputs public parameters: $\mathsf{Setup}(1^\kappa) = pp$; Key Generation which outputs public and secret keys for S and R: $\mathsf{KeyGen}_\mathsf{S}(pp) = (sk_\mathsf{S}, pk_\mathsf{S})$, $\mathsf{KeyGen}_\mathsf{R}(pp) = (sk_\mathsf{R}, pk_\mathsf{R})$; and finally for encryption and decryption of message k as follows: $E_{sk_\mathsf{S}, pk_\mathsf{R}}(k) = c$ while $D_{sk_\mathsf{R}, pk_\mathsf{S}}(c)$ outputs either reject or a message. We require, of course, that $D_{sk_\mathsf{R}, pk_\mathsf{S}}(E_{sk_\mathsf{S}, pk_\mathsf{R}}(k)) = k$ for all messages k.

Intuitively, the idea is that given only the public key of R and his own secret key, S can encrypt a message k in such a way that on reception, R can check that k comes from S, no third party knows k and finally, the encryption is verifiable in that it allows S to efficiently prove in zero-knowledge that k satisfies certain properties.

To help understand our concrete construction of an AVPKE scheme, we recall that standard Paillier encryption of a message k under the public key n is defined as $(n+1)^k v^n \bmod n^2$ where $v \in_\mathsf{R} \mathbb{Z}_n^*$. In [DJ03], it was suggested that first, v can be chosen as $\pm\hat{g}^s$ for a random s and a $\hat{g}$ of large order modulo n – or equivalently, a random number of Jacobi symbol 1 mod n. This is no security problem, as the Jacobi symbol of v can anyway be efficiently computed from the ciphertext. Further, they suggested that v can be chosen similarly as in ElGamal encryption, if the sender sends along a random power of $\hat{g}$. They also showed that the resulting encryption scheme is still CPA secure under the same assumption. In this way all users can share the same modulus, which comes in very handy in our setting.

We add an authentication mechanism to this encryption scheme and get the following AVPKE scheme.

Construction 3 *Let:*

- Ggen *be a* Group Generator*, a probabilistic polynomial time algorithm which on input* 1^κ *outputs the description of a cyclic group* G *and a generator* g*, such that the order of* G *is a random* κ*-bit RSA modulus* n*, which is the product of so-called safe primes, that is* $n = pq$ *where* $p = 2p' + 1, q = 2q' + 1$ *and* p', q' *are also primes. Finally, we need the algorithm to output an element* $\hat{g} \in \mathbb{Z}_n^*$ *of order* $p'q'$*.*
- $\mathsf{NIZK} = (\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ *be a simulation-sound non-interactive zero knowledge proof system. In this section, we will use* Σ*-protocols made non-interactive using the Fiat-Shamir heuristic, so in this case* Setup *is empty and there is no common reference string.*

Ggen can be constructed using standard techniques. For instance, first generate n using standard techniques, then repeatedly choose a small random number r until $P = 2rn + 1$ is a prime. Let g' be a generator of $\mathbb{Z}_P^*$. Then let G be the subgroup of $\mathbb{Z}_P^*$ generated by $g = g'^{2r} \bmod P$.[8] Finally, to construct the element $\hat{g}$, let $u \in_\mathsf{R} \mathbb{Z}_n$ and set $\hat{g} = u^2 \bmod n$. Indeed, this is a random square, and since the subgroup of squares modulo n has only large prime factors in its order (p' and q'), a random element is a generator with overwhelming probability.

[8] The group G will be more prominently used in the construction of the PSDVS scheme.

Trusted Set-up. Run Ggen to generate a modulus n and $\hat{g} \in \mathbb{Z}_n^*$ as explained above. Set $pp = (n, \hat{g})$.

Key Generation. Set the secret key of S to be $sk_\mathsf{S} \in_\mathsf{R} \mathbb{Z}_n$, and $pk_\mathsf{S} = \hat{g}^{sk_\mathsf{S}}$. For R, set $\alpha_1, \alpha_2 \in_\mathsf{R} \mathbb{Z}_n$ and set $sk_\mathsf{R} = (\alpha_1, \alpha_2)$ and $pk_\mathsf{R} = (\beta_1, \beta_2) = (\hat{g}^{\alpha_1}, \hat{g}^{\alpha_2})$. Public key values are statistically indistinguishable from random elements in the group generated by $\hat{g}$ since n is a sufficiently good "approximation" to the order $p'q'$ of $\hat{g}$.

Encryption. The encryption of message $k \in \mathbb{Z}_n$ under keys $sk_\mathsf{S}, pk_\mathsf{R} = (\beta_1, \beta_2)$ is defined as follows:

$$E_{sk_\mathsf{S}, pk_\mathsf{R}}(k, r, b_1, b_2) =$$
$$((-1)^{b_1}\hat{g}^r \bmod n,\ (n+1)^k((-1)^{b_2}\beta_1^{sk_\mathsf{S}}\beta_2^r \bmod n)^n \bmod n^2),\ \pi_{valid}$$

where $r \in_\mathsf{R} \mathbb{Z}_n$, $b_1, b_2 \in_\mathsf{R} \{0,1\}$ and π_{valid} is a non-interactive zero-knowledge proof of knowledge: given public data $n, \hat{g}, (c_1, c_2)$ the prover shows knowledge of a witness (k, r, b_1, v) such that $c_1 = (-1)^{b_1}\hat{g}^r$ and $c_2 = (n+1)^k v^n \bmod n^2$. An honest prover can use $v = (-1)^{b_2}\beta_1^{sk_\mathsf{S}}\beta_2^r \bmod n$.
The factor $(-1)^{b_1}$ is only in the ciphertext for technical reasons: it allows π_{valid} to be efficient.

Decryption. We get a ciphertext (c_1, c_2, π_{valid}) and $pk_\mathsf{S}, sk_\mathsf{R} = (\alpha_1, \alpha_2)$ as input. Check that c_1, c_2 have Jacobi symbol 1 modulo n, and check π_{valid}. Output reject if the check fails. Otherwise, compute $u = pk_\mathsf{S}^{\alpha_1} c_1^{\alpha_2} \bmod n$ and check that $(c_2 u^{-n})^n \bmod n^2 = \pm 1$. $\mathbb{Z}_{n^2}^*$ contains a unique subgroup of order n, generated by $n+1$. So here we are verifying that – up to a sign difference – $c_2 u^{-n} \bmod n^2$ is in the subgroup generated by $n+1$. If the check fails, output reject. Otherwise, compute k such that $(n+1)^k = \pm c_2 u^{-n} \bmod n^2$.[9]

An AVPKE scheme should allow anyone to make "fake" ciphertexts that look indistinguishable from real encryptions, given only the system parameters. Further, R should be able to use his own secret key and the public key of S to make ciphertexts with exactly the same distribution as real ones. This is indeed true for our scheme:

Fake encryption. Let $r \in_\mathsf{R} \mathbb{Z}_n, b, b' \in_\mathsf{R} \{0,1\}$ and $v \in \mathbb{Z}_n^*$ be a random square, and let

$$E_{fake}(k, r, b, b', v) = ((-1)^b \hat{g}^r \bmod n, (n+1)^k((-1)^{b'}v)^n \bmod n^2), \pi_{valid}$$

where π_{valid} is constructed following the NIZK prover algorithm.

R's equivalent encryption.

$$E_{sk_\mathsf{R}, pk_\mathsf{S}}(k, r, b_1, b_2) =$$
$$((-1)^{b_1}\hat{g}^r \bmod n,\ (n+1)^k((-1)^{b_2}(pk_\mathsf{S}^{\alpha_1}\hat{g}^{r\alpha_2} \bmod n)^n \bmod n^2),\ \pi_{valid}$$

where $r \in_\mathsf{R} \mathbb{Z}_n$, $b_1, b_2 \in_\mathsf{R} \{0,1\}$ and π_{valid} is constructed following the NIZK prover algorithm.

[9] k can be computed using the standard "discrete log" algorithm from Paillier decryption.

In the following, we will sometimes suppress the randomness from the notation and just write, e.g., $E_{sk_{\mathsf{S}},pk_{\mathsf{R}}}(k)$.

By simple inspection of the scheme it can be seen that:

Lemma 1. *For all k, $D_{sk_{\mathsf{R}},pk_{\mathsf{S}}}(E_{sk_{\mathsf{S}},pk_{\mathsf{R}}}(k,r,b_1,b_2)) = k$. Furthermore, encryption by S and by R returns the same: for all r, b_1, b_2 we have $E_{sk_{\mathsf{S}},pk_{\mathsf{R}}}(k,r,b_1,b_2) = E_{sk_{\mathsf{R}},pk_{\mathsf{S}}}(k,r,b_1,b_2)$*

Lemma 2. *If DDH in $\langle\hat{g}\rangle$ is hard, then $(k, E_{sk_{\mathsf{S}},pk_{\mathsf{R}}}(k,r,b_1,b_2))$ is computationally indistinguishable from $(k, E_{fake}(k,r',b,b',v))$, for any fixed k and random $r, b_1, b_2, r', b, b', v$, as long as the discrete log of β_2 to the base $\hat{g}$ is unknown.*

Proof. This follows immediately from that fact that $(\hat{g}^r, \beta_2^r)$ is indistinguishable from $(\hat{g}^{r'}, v)$ by assumption.

Definition 15. *Consider the following experiment for an AVPKE scheme and a probabilistic polynomial time adversary A: Run the set-up and key generation and run $A^{\mathcal{O}_E,\mathcal{O}_D}(pp, pk_{\mathsf{R}}, pk_{\mathsf{S}})$. Here, $\mathcal{O}_E$ takes a message k as input and returns $E_{sk_{\mathsf{S}},pk_{\mathsf{R}}}(k)$, while $\mathcal{O}_D$ takes a ciphertext and returns the result of decrypting it under $pk_{\mathsf{S}}, sk_{\mathsf{R}}$ (reject or a message). A wins if it makes $\mathcal{O}_D$ accept a ciphertext that was not obtained from $\mathcal{O}_E$. The scheme is* authentic *if any A wins with negligible probability.*

Lemma 3. *If the DDH problem in $\langle\hat{g}\rangle$ is hard, the AVPKE scheme defined above is authentic.*

Proof. Assume for contradiction that we have adversary A who wins the authenticity game with non-negligible probability. We now stepwise transform the game into an algorithm that will solve the computational Diffie-Hellman problem in $\langle\hat{g}\rangle$, which will certainly contradict the assumption. Let Game 0 be the original authenticity game. In Game 1, we replace $\mathcal{O}_E$ by $\mathcal{O}_E^{fake}$ which on input k from A returns $E_{fake}(k)$. At the same time, we replace $\mathcal{O}_D$ by an alternative version $\mathcal{O}_D^{alt}$ that does not use the α_2 part of sk_{R}, namely instead of computing $u = pk_{\mathsf{S}}^{\alpha_1} c_1^{\alpha_2}$ it extracts r from the proof in the ciphertext and computes instead $u = pk_{\mathsf{S}}^{\alpha_1} \beta_2^r$ which is the same value (up to a sign) and hence leads to an equivalent decryption. In addition, if it gets an encryption $E_{fake}(k)$ produced by the fake encryption oracle, it always returns k. Since now the discrete log of β_2 base $\hat{g}$ is not used, we can use our assumption and Lemma 2 to conclude that A's winning probability in the new game is essentially the same. In Game 2, we guess the index j of the first call to $\mathcal{O}_D$ where A gets an accept for a forged ciphertext, which we can do with an inverse polynomial probability, since A is poly-time. If A does not win the game at the j'th call, we declare that it loses. Clearly A still wins Game 2 with non-negligible probability. In Game 3, we replace $\mathcal{O}_D^{alt}$ by $\mathcal{O}_D^{sim}$ which, up to (but not including) call j, responds to a ciphertext that was output by $\mathcal{O}_E^{fake}$ with the message that was encrypted and rejects anything else. Observe that in the event where A wins Game 2, $\mathcal{O}_D^{sim}$ simulates $\mathcal{O}_D^{alt}$ perfectly, so A wins Game 3 with the same probability as Game 2.

Now, observe that we can execute Game 3 up to step j and get A's ciphertext for the j'th call without access to the secret keys of S and R. We can therefore take group elements $\hat{g}, pk_{\mathsf{S}} = \hat{g}^{sk_{\mathsf{S}}}, \beta_1 = \hat{g}^{\alpha_1}$ as input to the CDH problem (where the goal is to find $\hat{g}^{sk_{\mathsf{S}}\alpha_1}$), and execute Game 3 with A on this input. Assume A wins, and let c^* be the ciphertext submitted in the j'th call to $\mathcal{O}_D^{sim}$. Recall that c^* is of form (c_1, c_2, π_{valid}). By simulation soundness of the NIZK used, we can extract the witness claimed in the proof, and so we get r and k, v such that $c_1 = \pm\hat{g}^r$ and $c_2 = (n+1)^k v^n \bmod n^2$.

We can assume that R's decryption algorithm would accept c^*, this implies that v must have Jacobi symbol 1. Also, since the algorithm computes $u = pk_{\mathsf{S}}^{\alpha_1} c_1^{\alpha_2} \bmod n$ and checks that $(c_2 u^{-n})^n \bmod n^2 = \pm 1$, we can assume this check is satisfied. Inserting the expression we have for c_2 we get

$$\pm 1 = (c_2 u^{-n})^n \bmod n^2 = ((n+1)^k v^n u^{-n})^n \bmod n^2 = ((vu^{-1} \bmod n)^n)^n \bmod n^2$$

Here, we have used the fact that $\mathbb{Z}_{n^2}^*$ is the direct product of a group G of order n and a group H of order $\phi(n)$ isomorphic to $\mathbb{Z}_n^*$ under the isomorphism $x \mapsto x^n \bmod n^2$. Since $(vu^{-1} \bmod n)^n \bmod n^2 \in H$ and raising to power n is a 1-1 mapping on H, it follows that $vu^{-1} \bmod n = \pm 1$. Inserting the expression for u, we get

$$\pm v = u = pk_{\mathsf{S}}^{\alpha_1} c_1^{\alpha_2} = pk_{\mathsf{S}}^{\alpha_1} \hat{g}^{r\alpha_2} = \hat{g}^{sk_{\mathsf{S}}\alpha_1} \beta_2^r$$

In conclusion, we can flip a coin and submit plus or minus $v\beta_2^{-r} \bmod n$ as a solution to the CDH problem and this will be correct with half the probability with which A wins Game 3.

We proceed to show that the AVPKE scheme hides the message encrypted even if adversary knows the secret key of the sender, and even if a decryption oracle is given. This is essentially standard CCA security.

Definition 16. *Consider the following experiment for an AVPKE scheme and a probabilistic polynomial time adversary A: Run the set-up and key generation and run $A^{\mathcal{O}_E}(pp, pk_{\mathsf{R}}, sk_{\mathsf{S}})$. Here, $\mathcal{O}_E$ takes two messages k_0, k_1 as input, selects a bit η at random and returns $c^* = E_{sk_{\mathsf{S}}, pk_{\mathsf{R}}}(k_\eta)$. $\mathcal{O}_D$ takes a ciphertext and returns the result of decrypting it under $pk_{\mathsf{S}}, sk_{\mathsf{R}}$ (reject or a message). A may submit anything different from c^* to $\mathcal{O}_D$ and outputs a bit η' prime at the end. It wins if $\eta' = \eta$. The scheme is* private *if any A wins with negligible advantage over 1/2.*

In the following we will use the assumption underlying the Paillier encryption scheme, sometimes known as the *Composite degree residuosity assumption* (CDRA): a random element x in $\mathbb{Z}_{n^2}^*$ where $x \bmod n$ has Jacobi symbol 1 is computationally indistinguishable from $y^n \bmod n^2$ where $y \in \mathbb{Z}_n^*$ is random of Jacobi symbol 1[10].

[10] The original CDRA assumption does not have the restriction to Jacobi symbol 1, but since the Jacobi symbol is easy to compute without the factors of n, the two versions are equivalent.

Lemma 4. *Assume that DDH in $\langle \hat{g} \rangle$ is hard and that CDRA holds. Then the AVPKE scheme satisfies Definition 16.*

Proof. Assume for contradiction that adversary A wins the game with probability non-negligibly larger than 1/2. Let Game 0 be the original game. We change the game stepwise to get new games that can be used to either solve DDH or break CDRA. In Game 1, we replace $\mathcal{O}_D$ by an alternative version $\mathcal{O}_D^{alt}$ that does not use the α_2 part of sk_{R}, namely instead of computing $u = pk_{\mathsf{S}}^{\alpha_1} c_1^{\alpha_2}$ it extracts r from the proof in the ciphertext and computes instead $u = pk_{\mathsf{S}}^{\alpha_1} \beta_2^r$ which is the same value (up to a sign) and hence leads to an equivalent decryption. A wins Game 1 with exactly the same probability. In Game 2, we replace $\mathcal{O}_E$ by $\mathcal{O}_E^{sim}$, defined as follows: take $a, b \in \langle \hat{g} \rangle$ as input, where $a = \hat{g}^r, b = \beta_2^r$. let $\eta \in_{\mathsf{R}} \{0,1\}, r \in_{\mathsf{R}} \mathbb{Z}_n$ and return $(\pm a, (n+1)^{k_\eta} (\pm \beta_1^{sk_{\mathsf{S}}} b)^n \bmod n^2, \pi_{valid})$, where π_{valid} is a simulated proof. Except for the simulation, this is exactly Game 1, so A wins with essentially the same probability – note that by simulation soundness, the witness extraction used by $\mathcal{O}_D^{alt}$ still works. In Game 3, we make a, b be random group elements in $\langle \hat{g} \rangle$. Now, A's winning probability remains essentially the same since otherwise, we could use the difference between Game 2 and 3 to solve DDH. Now note that in Game 3 $(\pm \beta_1^{sk_{\mathsf{S}}} b)^n \bmod n^2$ is in fact a uniformly random element of form $y^n \bmod n^2$, where the Jacobi symbol of $y \bmod n$ is 1. In Game 4, we replace this by x chosen uniformly in $\mathbb{Z}_{n^2}^*$ subject to $x \bmod n$ having Jacobi symbol 1. In Game 4, c^* has no information on η, so here A wins with probability 1/2. This means that we can use the difference between Game 3 and 4 to break CDRA, and we have a contradiction.

We will say that an AVPKE scheme is *secure* if it is authentic, private and supports equivalent encryption by R and indistinguishable fake encryptions.

The PSDVS Scheme We now return to the promised PSDVS scheme.

Construction 4 *Let:*

- Ggen *be a* Group Generator*, a probabilistic polynomial time algorithm which on input 1^κ outputs $G, g, n, \hat{g}$ exactly as in the previous AVPKE construction.*
- H *be a hash function which we model as a random oracle. We assume it maps onto the group G.*
- $\mathsf{NIZK} = (\mathsf{Setup}, \mathsf{Prove}, \mathsf{Verify})$ *be a simulation-sound non-interactive zero knowledge proof system. In this section, we will use Σ-protocols made non-interactive using the Fiat-Shamir heuristic, so in this case* Setup *is empty and there is no common reference string.*

$\mathsf{Setup}(1^\kappa)$**:** Let $(G, g, \hat{g}, n) \leftarrow \mathsf{Ggen}(1^\kappa)$ and let $h \in_{\mathsf{R}} G$. Set $pp = (G, g, \hat{g}, n, h)$. Return pp as the public parameters.

$\mathsf{SignKeyGen}(1^\kappa, pp)$**:** Run key generation for the AVPKE scheme as defined above, to get keys $ssk = sk_{\mathsf{S}}, spk = pk_{\mathsf{S}}$ for the signer S.

Output ssk as the signer's secret key and spk as the signer's public key.

$\mathsf{VerKeyGen}(1^\kappa, pp)$**:**

1. Run key generation for the AVPKE scheme as defined above, to get keys $sk_{\mathsf{R}}, pk_{\mathsf{R}}$ for the verifier R. (These keys will be used to sign messages and verify signatures.)
2. $k_{\mathsf{R}} \in_{\mathsf{R}} \mathbb{Z}_n$. (This key will be used by the verifier to simulate signatures using $\mathsf{VerSigSim}$.)
3. Let $r_{\mathsf{R}} \in_{\mathsf{R}} \mathbb{Z}_n$ and let $c_{\mathsf{R}} = g^{k_{\mathsf{R}}} h^{r_{\mathsf{R}}}$. (This commitment will be used by the verifier to support its proofs of fake-ness.)
4. $vsk = (sk_{\mathsf{R}}, k_{\mathsf{R}}, r_{\mathsf{R}})$, $vpk = (pk_{\mathsf{R}}, c_{\mathsf{R}})$.

Output vsk as the verifier's secret key and vpk as the verifier's public key.

$\mathsf{Sign}(ssk = sk_{\mathsf{S}}, pk_{\mathsf{R}}, m)$**:**

1. Let $t \in_{\mathsf{R}} G, r \in_{\mathsf{R}} \mathbb{Z}_n, b_1, b_2 \in_{\mathsf{R}} \{0,1\}, s \in_{\mathsf{R}} \mathbb{Z}_n^*, k_s \in_{\mathsf{R}} \mathbb{Z}_n$.
2. $\sigma \leftarrow (t, \mathsf{H}(m,t)^{k_s}, E_{sk_{\mathsf{S}}, pk_{\mathsf{R}}}(k_s, r, b_1, b_2))$.
3. $\pi \leftarrow \mathsf{NIZK.Prove}(u = (\sigma = (\sigma_1, \sigma_2, \sigma_3), pk_V, pk_S, m), w = (sk_S, k_s, r, b_1, b_2))$ be a zero knowledge proof of knowledge of witness w such that:

$$\sigma_2 = \mathsf{H}(m, \sigma_1)^{k_s} \wedge \sigma_3 = E_{sk_{\mathsf{S}}, pk_{\mathsf{R}}}(k_s, r, b_1, b_2)$$

Output σ as the signature, and π as the proof of real-ness. Recall that the ciphertext by construction already contains a proof implying that the ciphertext contains a well defined plaintext. The role of the tag t in the signature is to let the verifier give a proof for his way to simulate a signature, this will become clear below.

$\mathsf{Verify}(vsk = (k_s, k_{\mathsf{R}}, r_{\mathsf{R}}), m, \sigma = (\sigma_1, \sigma_2, \sigma_3))$**:**

1. Decrypt σ_3 under $sk_{\mathsf{R}}, pk_{\mathsf{S}}$. If this fails, set $d = 0$ and abort. Otherwise, let k_s be the decryption result.
2. If $\sigma_2 = \mathsf{H}(m, \sigma_1)^{k_s}$, set $d = 1$. Otherwise, set $d = 0$.

Output d as the verification decision.

Informally, forging a signature is hard since the verifier rejects forged ciphertexts and valid ones hide the k_s value inside, by properties of the AVPKE scheme. The only other option is to reuse an existing k_s in a new signature, which is hard if CDH is hard in G. Namely, you have to raise a random group element (output by the random oracle) to the secret exponent k_s.

$\mathsf{PubSigSim}(m)$**:**

1. Let $k, k' \in_{\mathsf{R}} \mathbb{Z}_n$, such that $k \neq k'$.
2. Let $t \in_{\mathsf{R}} G$, $r \in_{\mathsf{R}} \mathbb{Z}_n, b, b' \in_{\mathsf{R}} \{0,1\}$, $v \in_{\mathsf{R}} \mathbb{Z}_n$, such that v has Jacobi symbol 1.
3. $\sigma \leftarrow (t, \mathsf{H}(m,t)^k, E_{fake}(k', r, b, b', v))$.
4. Let $\pi \leftarrow \mathsf{NIZK.Prove}(u = \sigma = (\sigma_1, \sigma_2, \sigma_3), w = (k, k'))$ be a zero-knowledge proof of knowledge such that:

$$\sigma_2 = \mathsf{H}(m, \sigma_1)^k \wedge \sigma_3 = E_{fake}(k', \cdot, \cdot, \cdot, \cdot) \wedge k \neq k'.$$

Output σ as the simulated signature, and π as the proof of fake-ness. The notation $E_{fake}(k', \cdot, \cdot, \cdot, \cdot)$ means that the proof only has to establish that the plaintext inside the encryption is some value k' different from k.

Informally, a simulated signature looks like a real one since fake encryptions are indistinguishable from real ones and by privacy of the encryption scheme, one cannot decide efficiently if $k = k'$ or not. Clearly, a fake signature as defined is always rejected by the verifier.

$\mathsf{VerSigSim}(spk = pk_{\mathsf{S}}, vpk = (c_{\mathsf{R}}, pk_{\mathsf{R}}), vsk = (k_s, k_{\mathsf{R}}, r_{\mathsf{R}}), m)$**:**

1. Let $r_t \in_{\mathsf{R}} \mathbb{Z}_n$, $t = g^{k_{\mathsf{R}}} h^{r_t}$, $k_s \in_{\mathsf{R}} \mathbb{Z}_n$, $b_1, b_2 \in_{\mathsf{R}} \{0,1\}$ and $v \in_{\mathsf{R}} \mathbb{Z}_n^*$ of Jacobi symbol 1.
2. $\sigma \leftarrow (t, \mathsf{H}(m,t)^{k_s}, E_{sk_{\mathsf{R}}, pk_{\mathsf{S}}}(k_s, r, b_1, b_2))$.
3. Let $\pi \leftarrow \mathsf{NIZK.Prove}(u = ((\sigma_1, \sigma_2, \sigma_3), c_{\mathsf{R}}, m), w = (k_{\mathsf{R}}, r_{\mathsf{R}}, r_t))$ be a zero-knowledge proof of knowledge of witness $w = (k_{\mathsf{R}}, r_{\mathsf{R}}, r_t)$ such that:

$$\sigma_1 = g^{k_{\mathsf{R}}} h^{r_t} \wedge c_{\mathsf{R}} = g^{k_{\mathsf{R}}} h^{r_{\mathsf{R}}}.$$

Output σ as the simulated signature and π as the proof of fake-ness.
Informally, the simulated signature has exactly the same distribution as a real signature, and cannot be distinguished even given the verifier's key: for every t here exists a r_t that the verifier could have used to generate it. Only the verifier can prove fake-ness since anyone else has no information on k_{R}, and so giving a proof would, with overwhelming probability, require opening c_{R} to a value different from k_{R}, which is infeasible if discrete log is hard in G.

Theorem 4. *If the AVPKE scheme is secure, and under the DDH assumption, Const 4 is a secure PSDVS scheme.*

Remark 2. To get a concrete instantiation of the PSDVS scheme, we need to instantiate the AVPKE scheme (as explained above) and also the NIZKs assumed in Const 4 (as explained in the Appendix A). This way, we get an instantiation in the random oracle model, secure under the strong RSA, the DDH and the CDRA assumptions.

Proof. In this proof we refer to the definitions in Section 3.1.
Unforgeability. Suppose adversary A wins the unforgability game (in presence of a signing and verification oracle), and let $\sigma^* = (\sigma_1^*, \sigma_2^*, \sigma_3^*)$ be the forged signature, and m^* the message in question. If σ_3^* is a ciphertext that was not output by the signing oracle, then A can be used to break authenticity of the AVPKE scheme, since R only accepts a signature if decryption of σ_3^* does not reject. So we may assume that σ_3^* is a valid ciphertext $E_{sk_{\mathsf{S}}, pk_{\mathsf{R}}}(k)$ that was used in a genuine signature $\sigma = (t, \mathsf{H}(m,t)^k, E_{sk_{\mathsf{S}}, pk_{\mathsf{R}}}(k))$. Since A wins, $m \neq m^*$, so we can assume that $\mathsf{H}(m,t)$ and $\mathsf{H}(m^*, \sigma_1^*)$ are independent random variables output by the random oracle, and furthermore since R accepts the forged signature, we have $\sigma_2^* = \mathsf{H}(m, t^*)^k$.

This means we can use A to solve CDH in $\langle \hat{g} \rangle$ which contradicts the assumption that DDH (and hence CDH) is hard: Given a random CDH challenge $\hat{h}, \hat{h}^a, \hat{h}^b$, we will guess which genuine signature and which calls to the random oracle will be involved in the forgery. We will program the random oracle so that $\mathsf{H}(t,m) = \hat{h}$ and set $\sigma_2 = \hat{h}^a$. This means that we are implicitly claiming that the exponent used in the signature is a. This does not match the encryption $E_{sk_{\mathsf{S}}, pk_{\mathsf{R}}}(k)$, but we program the verification oracle to accept the signature anyway, and by privacy of the encryption scheme, the inconsistency does not affect

A's behavior significantly. Finally, we set $\mathsf{H}(t^*, m^*) = \hat{h}^b$, and it is now clear that if A wins, we have $\sigma_2^* = (\hat{h}^b)^a = \hat{h}^{ab}$.
Provable Public Simulation. PubSigSim indistinguishability follows from privacy of the AVPKE scheme: if an adversary A wins the PubSigSim-Ind game, we can use A to construct an adversary that wins the privacy game. The adversary can use sk_{S} to emulate the signing oracle and the use the decryption oracle to emulate the (less powerful) verification oracle. PubSigSim correctness and PubSigSim soundness follow immediately from completeness and soundness of the NIZK used.
Provable Verifier Simulation. VerSigSim indistinguishability is clear, since the signature produced by VerSigSim has exactly the same distribution as regular signatures. VerSigSim correctness follows from completeness of the NIZK used. Finally, for VerSigSim soundness, assume that a corrupt S could produce a proof that VerSigVal would accept. By soundness of the NIZK used, this would mean that we could extract from the proof data such that both the tag t in the signature and c_{R} could be opened as commitments to the same value, say x. However, R already knows from the key generation how to open c_{R} to the value k_{R}. The adversary gets no information on k_{R} during the game because commitments are perfectly hiding, and so $x \neq k_{\mathsf{R}}$ with overwhelming probability. This means we can use the adversary to break the binding property of the commitment scheme. Sign. It is easy to see that Provable Signing Correctness and Soundness follow immediately from completeness and soundness of the NZIK used in Sign.

In Appendix A we list, for completeness, the standard Σ-protocols we need to instantiate our construction.

3.5 Sketch of a PSDVS Scheme Based on Prime Order Groups

In this subsection we sketch a variant of the previous scheme where we need only prime order groups and no trusted set-up. So we let G be a group of prime order p with generator g, where $p = 2q + 1$ and q is also prime. We let H denote the subgroup of Z_p^* of order q. H is also the group of squares modulo p. We let h be a generator of H. These parameters can be generated in public and can be verified by anyone, so no trusted set-up with secret trapdoor is required.

We make public keys for S and R as follows: $pk_{\mathsf{S}} = h^{sk_{\mathsf{S}}}$, $pk_{\mathsf{R}} = h^{sk_{\mathsf{R}}}$, $sk_{\mathsf{S}}, sk_{\mathsf{R}} \in_{\mathsf{R}} \in Z_q$.

The parties can compute a shared key $k = h^{sk_{\mathsf{S}} sk_{\mathsf{R}}}$ from only the public keys, which is pseudorandom for anyone else if DDH in H is hard.

We let the signature on m be $\mathsf{H}(m, t)^k$, which can be verified by R in the obvious way.

To prove in ZK that a signature $\sigma = (\sigma_1, \sigma_2)$ is valid, one proves knowledge of sk_{S}, such that $g^{pk_{\mathsf{S}}} = g^{h^{sk_{\mathsf{S}}}}$ and $\sigma_2 = \mathsf{H}(m, \sigma_1)^k = \mathsf{H}(m, \sigma_1)^{pk_{\mathsf{R}}^{sk_{\mathsf{S}}}}$ share the same "level-2 " discrete log, which can be done using a standard protocol which, however, requires a number of exponentiations linear in the security parameter.

For PubSigSim, one generates a fake signature by choosing a random element $e \in \mathbb{Z}_p^*$ and letting the simulated signature be $\mathsf{H}(m, t)^{-e^2 \bmod p}$. $-e^2 \bmod p$ is not

a square modulo p and hence is not in H. It will therefore not be accepted, since the correct k is in H by construction. This is indistinguishable from a genuine signature if we make a nonstandard variant of the DDH assumption saying that g raised to a random square is indistinguishable from g raised to a random non-square. One can show in ZK that $\mathsf{H}(m,t)^{-e^2 \bmod p}$ has the right form by showing that the discrete log of its inverse is a square, which can be done with standard Σ protocols.

The case of $\mathsf{VerSigSim}$ is handled in exactly the same way as the previous construction.

4 FE-based Construction

In this section, we present an MDVS scheme based on functional encryption. At a high level, we are first given a digital signature scheme (DS) and a functional encryption scheme (FE). The keys of the signer with identity i are a secret DS signing key sk_i and corresponding public DS verification key vk_i. An MDVS signature CT is a FE ciphertext obtained by encrypting the plaintext that consists of the message m, the signer's DS verification key vk_i, a set of designated verifier public keys $\{vpk_j\}_{j\in\mathcal{D}}$, and the signer's DS signature σ on the message using the secret DS signing key sk_i, i.e. $\mathsf{CT} = \mathsf{FE.Enc}(pp, (m, vk_i, \{vpk_j\}_{j\in\mathcal{D}}, \sigma))$. The verifier with identity j gets a public key $vpk_j = j$, a secret DS key pair (sk_j, vk_j), and an FE secret key dk_j. dk_j is a secret key for the function that checks whether vpk_j is among the designated verifier public keys, and then checks whether the DS signature σ inside the ciphertext CT is either a valid signature under the signer's verification key vk_i, or under the current verifier's verification key vk_j. However, this basic scheme does not fulfill all the MDVS properties, and we therefore need to tweak it a little.

From One to Many DS Signatures. In order to ensure that any subset of valid verifiers cannot convince an outsider of the origin of the MDVS signature, we need to replace the one DS signature in the ciphertext with a set of DS signatures. The reason is that any valid verifier can prove to an outsider that "it was either me or the signer that constructed the signature". Thus, if more than one verifier proves this about the same MDVS signature, then the signature must have come from the signer.

To resist this kind of "intersection attack", we allow the ciphertext to contain a set Σ of DS signatures, and change the corresponding FE secret keys to check if there exists a DS signature in the set that either verifies under the signer's or the verifier's DS verification key. Now, an outsider will no longer be convinced that it was the signer who constructed the MDVS signature, since each of the colluding verifiers could have constructed a DS signature that verifies under their own verification key, and then encrypted this set together with the public verification key of the signer.

Achieving consistency. In order to achieve consistency, we need security against a malicious encryptor in the FE scheme. We need to ensure that any (possibly maliciously generated) ciphertext is consistent with one specific message across decryption with different functions. Otherwise, a malicious MDVS signer may be able to construct a ciphertext (i.e. a signature) that will be valid for one designated verifier but not valid for another, thereby breaking the consistency property. Security against an malicious encryptor is a property of verifiable functional encryption (VFE), which was introduced by Badrinarayanan et. al [BGJS16]. However, it turns out that we do not need the full power of VFE, which also includes precautions against a malicious setup. Thus, we define a weaker notion of VFE, and substitute the standard FE scheme with this new scheme allowing us to achieve the MDVS consistency property.

In Section 4.1 we introduce ciphertext verifiable functional encryption, followed by our MDVS construction based on functional encryption which is presented in Section 4.2.

4.1 Verifiable Functional Encryption.

A functional encryption scheme (FE) starts with an authority that generates the public parameters pp and a master secret key msk. Then the holder of the master secret key can generate a decryption key dk_f associated with some function f that belongs to some predefined function family. Anyone can generate an encryption CT of some value x, using only the public parameters, and the party that has been delegated the decryption key dk_f can decrypt the ciphertext CT to obtain $f(x)$.

The standard security properties of functional encryption consider only the case where an adversary holds a set of decryption keys $dk_{f_1}, \ldots, dk_{f_q}$, and wants to learn more than it is allowed to about some encrypted message. The security property says that given an encryption of x, the adversary should only learn $f_1(x), \ldots, f_q(x)$.

However, in some settings, we additionally need security against a malicious encryptor, and possibly a malicious authority. To achieve this, Badrinarayanan et. al [BGJS16] introduced verifiable functional encryption (VFE), which is an FE scheme extended with verification algorithms that check the validity of the ciphertexts and decryption keys.

We require security only against a malicious encryptor, allowing us to define a weaker notion of verifiability for functional encryption that handles malicious encryptors and decryptors, while assuming an honest authority.

Verifiable Functional Encryption. Let $\mathcal{F} = \mathcal{F}_\kappa$ be a function family, $\mathcal{M} = \mathcal{M}_\kappa$ the message space, and $\mathcal{Y} = \mathcal{Y}_\kappa$ the output space, such that $\mathcal{F} : \mathcal{M} \to \mathcal{Y}$. Let $\mathcal{C} = \mathcal{C}_\kappa$ be the ciphertext space. Then, we define a *ciphertext verifiable functional encryption* (VFE) scheme for function family $\mathcal{F}$ by the following algorithms:

$\mathsf{Setup}(1^\kappa) \to (pp, msk)$**:** The PPT algorithm Setup, on input the security parameter κ, outputs the public parameters pp and the master secret key msk.

$\mathsf{KeyGen}(msk, f) \to dk_f$**:** The PPT key generation algorithm KeyGen, on input the master secret key msk and a function $f \in \mathcal{F}$, outputs a secret key dk_f.

$\mathsf{Enc}(pp, m) \to \mathsf{CT}$**:** The PPT encryption algorithm Enc, on input the public parameters pp and a message m, outputs a ciphertext CT.

$\mathsf{Dec}(dk_f, \mathsf{CT}) \to y'$**:** The decryption algorithm Dec, on the decryption key dk_f and the ciphertext CT, outputs $y' \in \mathcal{Y} \cup \{\perp\}$.

$\mathsf{Verify}(pp, \mathsf{CT}) \to d$**:** The public verification algorithm Verify, on input the public parameters pp and the ciphertext CT, outputs a boolean decision $d = 0$ (reject) or $d = 1$ (accept).

Properties. The scheme will have the standard correctness and IND-CPA security of a FE scheme, as described in the following definitions.

Definition 17 (Correctness). *Let $\kappa \in \mathbb{N}$ be the security parameter, and let $\mathcal{F} : \mathcal{M} \to \mathcal{Y}$ be a function family. Let $\mathsf{VFE} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ be a FE scheme for function family $\mathcal{F}$. For all messages $m \in \mathcal{M}$, all functions $f \in \mathcal{F}$ we have*

$$Pr\left[\mathsf{Dec}(\mathsf{KeyGen}(msk, f), \mathsf{Enc}(pp, m)) \neq f(m)\right] \leq \mathsf{negl}(\kappa),$$

where $(pp, msk) \leftarrow \mathsf{Setup}(1^\kappa)$, and the probability is taken over the random choices of all algorithms.

Definition 18 (IND-CPA Security). *Let $\kappa \in \mathbb{N}$ be the security parameter, and let $\mathcal{F} : \mathcal{M} \to \mathcal{Y}$ be a function family. Let $\mathsf{VFE} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ be a FE scheme for function family $\mathcal{F}$. Consider the following game between a challenger and an adversary $\mathcal{A}$:*

$\mathsf{Game}^{\mathsf{IND-CPA}}_{\mathsf{VFE},\mathcal{F},\mathcal{A}}(\kappa)$

1. $(pp, msk) \leftarrow \mathsf{Setup}(1^\kappa)$
2. $(m_0, m_1) \leftarrow \mathcal{A}^{\mathcal{O}_G}(pp)$
3. $b \leftarrow_\$ \{0, 1\}$
4. $c \leftarrow \mathsf{Enc}(pp, m_b)$
5. $b' \leftarrow \mathcal{A}^{\mathcal{O}_G}(c)$

The key generation oracle is defined $\mathcal{O}_G(f_i) := \mathsf{KeyGen}(msk, f_i)$.

We say that $\mathcal{A}$ wins the IND-CPA game *if $b = b'$, $|m_0| = |m_1|$, and $f_i(m_0) = f_i(m_1)$ for all queries $f_i \in \mathcal{F}$ to oracle $\mathcal{O}_G$. We say a FE scheme satisfies the IND-CPA security property if, for all PPT $\mathcal{A}$,*

$$\mathsf{adv}^{\mathsf{IND-CPA}}_{\mathsf{VFE},\mathcal{F},\mathcal{A}}(\kappa) = Pr[\mathcal{A} \text{ wins the game}] - \frac{1}{2} \leq \mathsf{negl}(\kappa).$$

We also need security against a malicious encryptor. Here we simplify the verifiability of [BGJS16] to verifiability of ciphertexts:

Definition 19 (Ciphertext Verifiability). *A scheme VFE for function family $\mathcal{F}$ is ciphertext verifiable, if, for all $\mathsf{CT} \in \{0,1\}^*$, there exists $x \in \mathcal{M}$ such that for all $f \in \mathcal{F}$ and $dk_f \leftarrow \mathsf{KeyGen}(msk, f)$, if $\mathsf{Verify}(pp, \mathsf{CT}) = 1$, then*

$$Pr\left[\mathsf{Dec}(dk_f, \mathsf{CT}) = f(x)\right] = 1,$$

where $(pp, msk) \leftarrow \mathsf{Setup}(1^\kappa)$.

Def 19 states that for all ciphertexts constructed by a malicious encryptor, it must hold that if the ciphertext CT passes the verification algorithm, then there exists a unique input x that can be associated with ciphertext CT, meaning that for all functions $f \in \mathcal{F}$ the decryption of CT will yield $f(x)$.

4.2 The MDVS Construction.

Construction 5 *Let* $\mathsf{SIGN} = (\mathsf{KeyGen}, \mathsf{Sign}, \mathsf{Verify})$ *be a standard digital signature scheme and let* $\mathsf{VFE} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Verify})$ *be a functional encryption scheme secure with ciphertext verifiability. Then we define a MDVS scheme* $\mathsf{FEMDVS} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Sign}, \mathsf{Verify}, \mathsf{Sim})$ *as follows:*

$\mathsf{Setup}(1^\kappa)$**:** $(pp^{\mathsf{FE}}, msk^{\mathsf{FE}}) \leftarrow \mathsf{VFE.Setup}(1^\kappa)$.

Output public parameter $pp = pp^{\mathsf{FE}}$ and master secret key $msk = msk^{\mathsf{FE}}$.

$\mathsf{SignKeyGen}(i)$**:** $(sk_i, vk_i) \leftarrow \mathsf{SIGN.KeyGen}(1^\kappa)$.

Output the signer's secret key $ssk_i = sk_i$ and public key $spk_i = vk_i$.[11]

$\mathsf{VerKeyGen}(msk, j)$**:**

1. $vpk_j = j$,
2. $(sk_j, vk_j) \leftarrow \mathsf{SIGN.KeyGen}(1^\kappa)$,
3. $dk_j \leftarrow \mathsf{VFE.KeyGen}(msk^{\mathsf{FE}}, f_j)$, where f_j is defined as follows.

Function f_j

Input: $m, vk_i, \{vpk_j\}_{j\in\mathcal{D}}, \Sigma$;
Const: vpk_j, vk_j;
1. If $vpk_j \notin \{vpk_j\}_{j\in\mathcal{D}}$: output $\perp$;
2. If $\exists\sigma \in \Sigma : \mathsf{SIGN.Verify}(vk_i, m, \sigma) = 1$ OR $\mathsf{SIGN.Verify}(vk_j, m, \sigma) = 1$:
   output $(m, vk_i, \{vpk_j\}_{j\in\mathcal{D}})$;
3. Else output $\perp$;

Output the verifiers secret key $vsk_j = (sk_j, dk_j)$ and public key $vpk_j = j$.[12]

$\mathsf{Sign}(pp, ssk_i, \{vpk_j\}_{j\in\mathcal{D}}, m)$**:**

1. $\sigma \leftarrow \mathsf{SIGN.Sign}(sk_i, m)$
2. $\mathsf{CT} = \mathsf{VFE.Enc}(pp^{\mathsf{FE}}, (m, vk_i, \{vpk_j\}_{j\in\mathcal{D}}, \{\sigma, \perp, \cdots, \perp\}))$

Output the ciphertext CT as the signature.

$\mathsf{Verify}(spk_i, vsk_j, \{vpk_j\}_{j\in\mathcal{D}}, m, \mathsf{CT})$**:**

1. Check whether $\mathsf{VFE.Verify}(pp^{\mathsf{FE}}, \mathsf{CT}) = 1$. If not, output 0.
2. Compute $(m', vk_i', \{vpk_j\}_{j\in\mathcal{D}'})\backslash\perp \leftarrow \mathsf{VFE.Dec}(dk_j, \mathsf{CT})$. If the output is $\perp$, output 0.

[11] We assume that the mapping $i \to (ssk_i, spk_i)$ is unique in the system. This can be achieved without loss of generality by pseudorandomly generating the randomness required in the key generation process from the identity i and the master secret key.

[12] We assume that the mapping $j \to (vsk_j, vpk_j)$ is unique in the system. This can be achieved wlog by pseudorandomly generating the randomness required in the key generation process from the identity j and the master secret key.

3. Check $m' = m$, $vk'_i = vk_i$ (with $spk_i = vk_i$), and $\mathcal{D}' = \mathcal{D}$. If all hold, output 1. Otherwise output 0.

$\mathsf{Sim}(pp, spk_i, \{vpk_j\}_{j\in\mathcal{D}}, \{vsk_j\}_{j\in\mathcal{C}}, m)$**:**

1. For each $j \in \mathcal{C}$, $vsk_j = (sk_j, dk_j)$.
2. Compute $\sigma_j \leftarrow \mathsf{SIGN.Sign}(sk_j, m^*)$.
3. Let $\Sigma = \{\sigma_j\}_{j\in\mathcal{C}^*}$, add default values to get the required size.
4. Output $\mathsf{CT} = \mathsf{VFE.Enc}(pp^{\mathsf{FE}}, (m^*, spk_i, \{vpk_j\}_{j\in\mathcal{D}}, \Sigma))$;

Theorem 5. *Assume that* VFE *is a IND-CPA secure functional encryption scheme with ciphertext verifiability, and* SIGN *is an existential unforgeable digital signature scheme. Then Const 5 is a correct and secure MDVS scheme (with privacy of identities).*

Proof. Correctness: Follows directly from an inspection of the algorithms and the correctness of the functional encryption scheme.

Consistency: Assume that there exist an adversary that produces an inconsistent signature: $(i^*, \{vpk_j\}_{j\in\mathcal{D}^*}, m^*, \mathsf{CT}^*)$, such that there exists $j_1, j_2 \in \mathcal{D}^*$ (for which the adversary does not have the corresponding secret keys) such that:

$$\mathsf{Verify}(spk_{i^*}, vsk_{j_1}, \{vpk_j\}_{j\in\mathcal{D}^*}, m^*, \mathsf{CT}^*) = 1,$$
$$\mathsf{Verify}(spk_{i^*}, vsk_{j_2}, \{vpk_j\}_{j\in\mathcal{D}^*}, m^*, \mathsf{CT}^*) = 0,$$

where $spk_{i^*} = vk_{i^*}$, $vsk_{j_1} = (sk_{j_1}, dk_{j_1})$, and $vsk_{j_2} = (sk_{j_2}, dk_{j_2})$.

Since the verification for j_1 yields 1, then both j_1 and j_2 will verify the ciphertext: $\mathsf{VFE.Verify}(pp^{\mathsf{FE}}, \mathsf{CT}) = 1$, meaning that (thanks to ciphertext verifiability) there exists a unique encrypted message $(m, vk_i, \{vpk_j\}_{j\in\mathcal{D}}, \Sigma)$ in CT that is consistent across decryption with different functions.

Then j_1 will decrypt: $(m, vk_i, \{vpk_j\}_{j\in\mathcal{D}}) \leftarrow \mathsf{VFE.Dec}(dk_j, \mathsf{CT})$, which is equal to $(m^*, vk_{i^*}, \{vpk_j\}_{j\in\mathcal{D}^*})$. On the other hand j_2 will decrypt

$$(m', vk'_i, \{vpk_j\}_{j\in\mathcal{D}'}) \text{ or } \perp \leftarrow \mathsf{VFE.Dec}(dk_j, \mathsf{CT})$$

If the output was $\perp$, then either $j_2 \notin \mathcal{D}$ or there does not exists a valid signature in Σ. Otherwise the output was $(m', vk'_i, \{vpk_j\}_{j\in\mathcal{D}'})$ which is different from $(m, vk_i, \{vpk_j\}_{j\in\mathcal{D}})$ in at least one component. Thus, the output of the decrypt of j_1 and j_2 is not consistent with a unique message, which contradicts the fact that $\mathsf{VFE.Verify}(pp^{\mathsf{FE}}, \mathsf{CT}) = 1$ (i.e. there exists a unique encrypted message).

Thus, an adversary that violated the consistency of the MDVS scheme, violates the ciphertext verifiability of the FE scheme.

Existential Unforgeability: Assume that the adversary produces a valid forgery: $(i^*, \{vpk_j\}_{j\in\mathcal{D}^*}, m^*, \mathsf{CT}^*)$, where CT^* is the signature (a FE ciphertext) on message m^*, from signer i^* designated to the verifiers in the set $\mathcal{D}^*$.

Then there exists a designated verifier $j \in \mathcal{D}^*$, who has not been corrupted by the adversary, such that the FE decryption will yield m', vk'_i and $\mathcal{D}'$, where $m' = m^*$, $vk'_i = vk_{i^*}$, and $\mathcal{D}' = \mathcal{D}^*$ (otherwise it would not be a valid forgery).

The only thing left in the FE ciphertext is the set of signatures Σ. In order for CT^* to be a forgery, then there must exist a digital signature $\sigma \in \Sigma$ such that

$$\mathsf{SIGN.Verify}(vk_i', m', \sigma) = 1 \text{ OR } \mathsf{SIGN.Verify}(vk_j, m', \sigma) = 1.$$

This means that the adversary must create a digital signature forgery for either signer i or "signer" j, without knowing the corresponding secret signing keys. This contradicts the assumption that the digital signature satisfies existential unforgeability.

Privacy of Identities: The adversary receives a signature CT^*, which is an FE encryption of one of the two following messages:

1. $(m^*, vk_{i_0}, \{vpk_j\}_{j\in\mathcal{D}_0}, \{\mathsf{SIGN.Sign}(sk_{i_0}, m^*), \bot, \cdots, \bot\})$,
2. $(m^*, vk_{i_1}, \{vpk_j\}_{j\in\mathcal{D}_1}, \{\mathsf{SIGN.Sign}(sk_{i_1}, m^*), \bot, \cdots, \bot\})$.

In the PSI game, the adversary is not allowed to ask for verification keys for any of the designated verifiers in $\mathcal{D}_0$ or $\mathcal{D}_1$. This means that for all verification keys (i.e. FE decryption keys) the adversary can ask for, we have that the underlying function f_j evaluated on the two messages will yield $\bot$, since j is not in any of the two sets of designated verifiers.

Thus, privacy of the identities (PSI and PVI) follows directly from the IND-CPA security of the functional encryption scheme.

Source Hiding: The source hiding property follows directly from the IND-CPA security of the functional encryption scheme. The messages in the real and simulated version have the same length, since we add default values to Σ to ensure we have the same number of elements in both cases.

For all functions f_j that the adversary gets an FE decryption key for, we argue that evaluation on each of the two messages results in the same output. First we look at the decryption keys dk_j for $j \notin \mathcal{D}^*$ (i.e. not a designated verifier). In both the real and the simulated case the decryption will yield $\bot$, since j is not a designated verifier.

Next, we look at the decryption keys dk_j for $j \in \mathcal{D}^*$. In this case j must also be in the corruption set $\mathcal{C}^*$. Thus, in the real case the set Σ contains σ_{i^*} a digital signature of message m^* under the signing key of signer i^*, and the decryption will yield $(m^*, vk_{i^*}, \{vpk_j\}_{j\in\mathcal{D}^*})$. In the simulated case the set Σ contains σ_j a digital signature of message m^* under the signing key of party j. Thus, the decryption will again yield $(m^*, vk_{i^*}, \{vpk_j\}_{j\in\mathcal{D}^*})$, since function f_j does not differentiate whether it was the verification vk_{i^*} or vk_j that was used to verify the digital signature. □

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A Instantiation of Non-Interactive ZK Proofs

We list here, for completeness, the standard Σ-protocols we need. We assume the same set-up as above, that is, a group G of order an RSA modulus n is given (a product of safe primes), as well as a generator g of G, and a generator $\hat{g}$ of the subgroup of squares mod n.

The protocols we list here are well-known or simple variations of know protocols. It is easy to show the standard completeness, soundness and honest verifier ZK properties, so we will not present these proofs, but recall some ideas where these may be less well known. The protocols can be turned into non-interactive ZK proofs of knowledge in the standard way in the random oracle model using the Fiat-Shamir heuristic.

All proofs have negligible soundness error, so we always need only 1 iteration of each protocol.

A general technical remark: some of the protocols are usually designed for use in a group of prime order, while here we use them in a group of order n. The only difficulty this could lead to is that the proofs of soundness requires us to invert various non-zero numbers modulo the group order. This could in principle

fail modulo n, but this would lead to finding a non-trivial factor of n (which is generated in a trusted manner as part of the setup) and so can only happen with negligible probability if factoring is hard, which we have to assume throughout anyway.

Protocols for the AVPKE scheme. Some number theoretic background first: because $n = pq = (2p' + 1)(2q' + 1)$ is a product of safe primes, the subgroup of squares in $\mathbb{Z}_n^*$ has order $p'q'$ and $\hat{g}$ is chosen to be a generator. It lies inside the subgroup of numbers of Jacobi symbol 1 which has order $2p'q'$ and is generated by $\hat{g}$ and -1.

The first protocol has as public input $\hat{g}, \hat{h} \in \mathbb{Z}_n^*$ and we assume $\hat{h}$ Jacobi symbol 1, this can be easily checked by the verifier. Recall that an honest prover knows r such that $\hat{h} = \hat{g}^r$. The protocol goes as follows:

Protocol Composite order discrete log.

1. P chooses $s \in_R \{0,1\}^{3\kappa}$ and sends $a = \hat{g}^s$ to V (interpreting s as a binary number).
2. V sets $e \in_R \mathbb{Z}_n$ and sends it to P.
3. P returns $z = s + er$ to V, and V checks that $\hat{g}^z = a\hat{h}^e \bmod n$.

This protocol is easily seen to be complete and statistical honest verifier zero-knowledge (note that s is chosen to be exponentially larger than er so z is statistically close to a random 3κ bit number. Soundness is more tricky. *Assuming* that $\hat{h}$ is in the group generated by $\hat{g}$, then the protocol is a proof of knowledge of r, under the strong RSA assumption modulo n, as shown by Fujisaki and Okamoto [FO97]. Now, since $\hat{h}$ has Jacobi symbol 1, either $\hat{h}$ or $-\hat{h}$ is in the subgroup, so the protocol proves that P knows the discrete log of $\hat{h}$ or $-\hat{h}$.

For the second protocol we have public input $c = (n+1)^k v^n \bmod n^2$ and $\beta = \gamma^k$ for some $\gamma \in G$. The prover knows k and v. The protocol goes as follows:

Protocol Plaintext and discrete log knowledge.

1. P chooses $s \in_R \mathbb{Z}_n, u \in_R \mathbb{Z}_n^*$ and sends $a = (n+1)^s u^n \bmod n^2$, $\alpha = \gamma^s$ to V.
2. V sets $e \in_R \mathbb{Z}_n$ and sends it to P.
3. P returns $z = s + er \bmod n, w = uv^e \bmod n$, and V checks that $(n+1)^z w^n = ac^e \bmod n^2$ and that $\gamma^z = \alpha\beta^e$.

It is trivial to prove this protocol complete, sound and honest-verifier zero-knowledge.

To to the proof π_{valid} in the AVPKE scheme, we run the Composite order discrete log and the Plaintext and discrete knowledge protocols where in the latter we omit γ, β, α.

Protocols for the PVDVS scheme. The first protocol works with the well-known Pedersen commitments, where a commitment to x with randomness r is of form $g^x h^r$ where $g, h \in G$. These commitments are perfectly hiding and computationally binding if discrete log in G is hard. The protocol shows that two Pedersen

commitments contain the same value. The public input is $c_1 = g_1^x h_1^{r_1}, c_2 = g_2^x h_2^{r_2}$. The protocol works as follows:

Protocol Commitment equality.

1. P sets $y, s_1, s_2 \in_{\mathsf{R}} \mathbb{Z}_n$, and sends $a_1 = g_1^y h_1^{s_1}, a_2 = g_2^y h_2^{s_2}$ to V
2. V sets $e \in_{\mathsf{R}} \mathbb{Z}_n$ and sends to P.
3. P sends $z = y + ex \bmod n$ and $u_1 = s_1 + er_1 \bmod n, u_2 = s_2 + er_2 \bmod n$ to V.
4. V checks that $g_1^z h_1^{u_1} = a_1 c_1^e$ and that $g_2^z h_2^{u_2} = a_2 c_2^e$.

The NIZK in VerSigSim uses this protocol.

For the NIZK in PubSigSim, recall that we have $\mathsf{H}(m,t)^k$, $E_{fake}(k', r, b, b', v)$ as input, and we want to demonstrate that $k \neq k'$. The prover includes $\mathsf{H}(m,t)^{k'}$ in the proof, and runs the Plaintext and discrete log knowledge protocol to demonstrate that the exponent k' is also present in the encryption. For this, we consider only the last part of the ciphertext, which is a Paillier encryption of k', as this part already uniquely determines k'. The prover can use k' and $(-1)^{b'} v$ as witness. Finally, the verifier checks that $\mathsf{H}(m,t)^{k'} \neq \mathsf{H}(m,t)^k$. Soundness and completeness of this should be clear. For zero-knowledge, the simulator would produce $\mathsf{H}(m,t)^{k''}$ for random k'', and simulate the Plaintext and discrete log knowledge protocol. Of course the statement in question is now false, but by privacy of the encryption scheme, the simulated public data is indistinguishable from the case where the statement is true, so the simulator must still produce an indistinguishable transcript.

Finally, for the proof of real signature in Sign, we have to show that the ciphertext encrypting k is completely well formed so that R will accept, and that this exponent is used in the MAC. For this we use the following protocol. The public input is $\delta = \gamma^k$ for a publicly known $\gamma \in G$ (namely a hash-value), $pk_{\mathsf{S}} = \hat{g}^{sk_{\mathsf{S}}} \bmod n$ and

$$C = E_{sk_{\mathsf{S}}, pk_{\mathsf{R}}}(k, r, b_1, b_2) = ((-1)^{b_1} \hat{g}^r \bmod n,\ (n+1)^k ((-1)^{b_2} \beta_1^{sk_{\mathsf{S}}} \beta_2^r \bmod n)^n \bmod n^2),$$

for publicly known $\beta_1, \beta_2 \in Z_{n^2}^*$, which in our context come from R's public key pk_{R}. The prover's secret witness is $sk_{\mathsf{S}}, k, r, b_1, b_2$.

Protocol Valid signature.

1. P chooses $s \in \mathbb{Z}_n$, $x, y \in \{0,1\}^{3\kappa}$, $c_1, c_2 \in \{0,1\}$. He then sends to V: $a = E_{sk_{\mathsf{S}}+x, pk_{\mathsf{R}}}(s, y, c_1, c_2)$, $\xi = \hat{g}^x \bmod n$ and $\omega = \gamma^s$.
2. V sets $e \in_{\mathsf{R}} \mathbb{Z}_n$ and sends it to P.
3. P returns $z_s = s + er \bmod n$, $z_x = x + esk_{\mathsf{S}}$, $z_y = y + er$, $d_1 = (c_1 + eb_1) \bmod 2$, $d_2 = (c_2 + eb_2) \bmod 2$. V checks that $E_{z_x, pk_{\mathsf{R}}}(z_s, z_y, d_1, d_2) = aC^e$, that $\hat{g}^{z_x} = \xi pk_{\mathsf{S}}^e$ and that $\gamma^{z_s} = \omega \delta^e$.

It is tedious but straightforward to check completeness. For (statistical) honest verifier ZK we use that x, y are both exponentially lager than esk_{S} and er,

respectively. For soundness, assume we get acceptable answers $(z_s, z_x, z_y, d_1, d_2)$ and $(z'_s, z'_x, z'_y, d'_1, d'_2)$ to challenges e, e'. This will imply that we get equations:

$$E_{z_x - z'_x, pk_{\mathsf{R}}}(z_s - z'_s, z_y - z'_y, (d_1 - d'_1) \bmod 2, (d_2 - d'_2) \bmod 2) = C^{e-e'}$$

$$\hat{g}^{z_x - z'_x} = pk_{\mathsf{S}}^{e-e'}, \quad \gamma^{z_s - z'_s} = \delta^{e-e'}$$

Now, note that the protocol is in fact (implicitly) using the Composite order discrete log protocol to prove knowledge of the discrete logs of plus or minus the numbers $pk_{\mathsf{S}} = \hat{g}^{sk_{\mathsf{S}}} \bmod n$ and $(-1)^{b_1}\hat{g}^r \bmod n$, where the latter occurs inside the ciphertext.

The soundness proof for that protocol from [FO97] argues if we get acceptable answers to two challenges e, e', then under the strong RSA assumption, it must be that $e - e'$ divides the difference between the answers. So under this assumption we get that $e - e'$ divides $z_x - z'_x$ and $z_y - z'_y$. It is now straightforward to extract a valid witness, essentially by dividing by $e - e'$ in the exponent on both sides.