Anonyma: Anonymous Invitation-Only Registration in Malicious Adversarial Model*

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In invitation-based systems, a new user can register upon having a threshold number of invitations issued by the existing members. The newcomer hands his invitations to the system administrator who verifies whether the invitations are issued by legitimate members. This causes the administrator to be aware of who is invited by whom. However, the inviter-invitee relationship is privacy-sensitive information and can lead to inference attacks where the invitee's profile (e.g., political view or location) can be extracted through the profiles of his inviters. Addressing this problem, we propose *Anonyma*, an anonymous invitation-based system, where a corrupted administrator, who may even collude with a subset of existing members, is not able to figure out who is invited by whom. We formally define and prove the inviter anonymity as well as unforgeability of invitations against a *malicious and adaptive adversary*. Our design only incurs a constant cost to authenticate a new registration. This is significantly better than similar works where the generation of invitations and verification of new registration cause an overhead linear in the total number of existing members. Besides, *Anonyma* is efficiently scalable in the sense that once a user joins the system, the administrator can instantly, and without re-keying the existing members, issue credentials for the newcomer to be able to act as an inviter. We additionally design *AnonymaX*, an anonymous cross-network invitation-based system empowering third-party authentication where the invitations issued by the members of one system can be used for registering to another system.

Additional Key Words and Phrases: Invitation-Based System, Anonymity, Unforgeability, Integrity, Cross-Network Invitation, Third-party Authentication, Malicious Adversary.

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1 INTRODUCTION

Invitation-based systems [9], where registration is only possible through invitations from current members, are used to limit the number of users, improve service quality, and protect against spammers. In these systems, existing members, known as inviters, play a role in inviting new users, referred to as invitees. The invitee receives invitations from a subset of inviters, submits them to the server/system administrator for verification, and the server accepts or rejects the registration request based on the legitimacy of the invitations. Various services and platforms have implemented

*This is an extension to our prior publication Inonymous [9]

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invitation-only registration, such as Google Inbox, Orkut, Google Wave[1], Spotify[2], Facebook secret groups[3], WhatsApp[4], Telegram[5], and Facebook's trustee-based social authentication [11, 37, 48]. While invitation-based systems offer benefits, they also have security concerns. The administrator's knowledge of who invited whom can lead to privacy breaches, as the relationship between inviters and invitees may reveal personal attributes and characteristics such as location, religious beliefs, sexual orientation, and political views about both the inviter and the invitee [18, 36, 47]. This leakage of information, known as an inference attack, highlights the need to protect the inviter-invitee relation as privacy-sensitive information [9].

Inonymous [9] introduces a solution to protect inviter anonymity and prevent inference attacks in invitation-based systems. It allows invitees to prove their possession of valid invitations without revealing their inviters' identities. Inonymous ensures that only invitees with legitimate inviters can join the system. However, in the presence of an active adversarial model, where parties deviate from the protocol, the privacy guarantees of the design are compromised, and the relationship between inviters and invitees becomes exposed.

Other existing studies, as discussed in detail in Section 7, do not meet the specific requirements of anonymous invitation-only registration simultaneously: inviter anonymity, invitation unforgeability, and scalability. Group signature schemes lack invitation unforgeability, allowing the inviter to generate unlimited signatures (invitations) without detection by the group administrator [8, 16, 33, 64]. Selective-disclosure credential schemes focus on attribute anonymity rather than protecting the identification of honest inviters under collusion between inviters and the group administrator in invitation-based systems [15, 22, 62]. (t, N) threshold ring signature schemes suffer from bandwidth consumption, computational overhead, and performance degradation due to large invitation sizes [10, 12, 41, 46, 49, 52]. E-voting systems burden non-inviters by requiring the participation of all members in each voting round for anonymity, which is unnecessary [26, 43, 51, 58]. Other e-voting approaches face scalability issues as they require re-keying the entire system for each round of voting [5, 54]. Direct Anonymous Attestation (DAA) schemes have linearly increasing invitation letter sizes and do not protect inviter/attester anonymity in cooperative scenarios, making them unsuitable for anonymous invitation-based scenarios [13]. Delegatable Anonymous Credentials (DAC) lack sufficient protection for invitation unforgeability, compromising the integrity of the invitation-based system [2, 19–21, 29, 34, 50].

These limitations emphasize the need for our proposal, *Anonyma*, which overcomes these shortcomings and provides an efficient and provably secure solution for anonymous invitation-based systems under an active and adaptive adversarial model. *Anonyma* builds upon the foundation laid by Inonymous and addresses Inonymous shortcomings in withstanding the active adversarial model. We conduct a comprehensive study to identify Inonymous vulnerabilities in the presence of active adversaries and uncover various non-trivial attack scenarios that can compromise the system's anonymity and unforgeability. We detail these attack scenarios while revealing different design choices of Anonyma. To enhance security, Anonyma intentionally integrates all identified attack vectors into a robust game-based security definition, representing a significant advancement over Inonymous. Additionally, a new set of security proofs is provided to support the effectiveness of the malicious-resistant design. To counteract active attacks, Anonyma introduces several updates to the design, namely, verifiable secret sharing schemes and zero-knowledge proof techniques are leveraged to strengthen the system's defenses while minimizing computational overhead.

[1]http://www.macworld.com/article/1055383/gmail.html
[2]https://community.spotify.com/t5/Accounts/Spotify-Family-Q-amp-A/td-p/988520
[3]https://blog.hootsuite.com/facebook-secret-groups/
[4]ttps://faq.whatsapp.com/en/android/26000123/?category=5245251
[5]https://telegram.org/tour/groups

Furthermore, the design of Anonyma is updated to resist adaptive adversarial corruption, making it a more realistic solution compared to Inonymous which could only protect invitation-unforgeability against static adversaries.
***Anonyma*:** An overview of *Anonyma* is depicted in Figure 1. In Anonyma, the group administrator designates initial members who can generate invitations for their intended invitees. Members can generate multiple invitations, but it is guaranteed that no inviter can generate more than one valid invitation for a particular invitee. Invitees must collect a specific number of invitations (denoted as t) from distinct members. These collected invitations are then aggregated to remove identifiable information about individual inviters, preserving inviter anonymity. The administrator can verify the integrity of the invitations without knowing the identities of the inviters.

Additionally, we propose *AnonymaX*, an anonymous cross-network invitation-based system that empowers the third-party authentication paradigm[6]. It enables mutually trusted domains to rely on each other's authentication mechanisms, similar to Microsoft Active Directory[7]. In *AnonymaX*, users from different administrative domains can invite each other, allowing them to join new domains like Instagram through invitations from members of other networks such as Twitter [9].

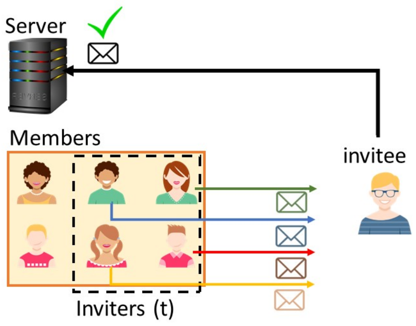


Fig. 1. *Anonyma* overview. The invitee receives the individual invitations, aggregates them into a unified letter, and hands it over to the group administrator. The group administrator can authenticate the letter without knowing the identity of the inviters.

In summary, the contributions of *Anonyma* span both security & privacy aspects, and design efficiency as outlined below.

- **Inviter Anonymity:** No adversarial entity would be able to identify who is invited by whom. This property holds even if the group administrator colludes with a subset of inviters of an invitee and attempts to discover the identity of the other inviters of that invitee. Both group administrators and the inviters are active adversaries who may not follow the protocol specifications. The formal security definition and proof are supplied in Section 6.
- **Invitation Unforgeability:** An adversarial invitee cannot register to the system unless he has t many legitimate invitations. This property holds even if the invitee colludes with $t-1$ inviters (clearly the registration of an invitee with t invitations is legitimate). The formal security definition and proof are supplied in Section 6. The provided definition also implies the following properties.

(1) **Invitation non-exchangability**: This property indicates that invitations issued for a particular invitee are not reusable for another user i.e., each invitation is tied to its intended invitee.

[6]https://www.synopsys.com/blogs/software-security/5-reasons-third-party-authentication/
[7]https://docs.microsoft.com/en-us/previous-versions/windows/it-pro/windows-2000-server/cc977993(v=technet.10)?redirectedfrom=MSDN.

(2) **Preventing double invitation:** This feature indicates that an inviter cannot issue more than one valid invitation for a single invitee. This is essential since otherwise an invitee with insufficient inviters (i.e., $t^{'} < t$) can obtain multiple invitations (e.g., $t - t^{'}$) from one of her inviters and successfully register.

Note that in an invitation-based system, any collection of t members can decide to invite arbitrary many invitees. This is a legitimate property that complies with the invitation-only policy and should not count as a violation of invitation-unforgeability.

- **Short invitations:** Each invitation, as well as the aggregated version, embodies only two group elements regardless of the threshold t and the current number of system members. Additionally, the complexity of communication among all the parties is of constant overhead.
- **Non-Interactive:** The invitation generation is a stand-alone non-interactive procedure that each inviter can run needless to the presence of the invitee, other invites, and the group administrator. The same holds for the verification of the final aggregated invitation that is executed by the group administrator.
- **Light computations:** All the computations in *Anonyma* are light and efficient. An inviter has to carry out a constant number of group exponentiation for invitation generation ($O(1)$). The running time complexity of the invitee for aggregation of the t invitations is $O(t)$ whereas the group administrator verifies the final consolidated invitation in $O(1)$. This is in contrast to the related work [10, 12, 46, 49] where the inviter's computational overhead is $O(N)$ where N is the total number of group members.
- **Scalability:** In *Anonyma*, the set of group members who are eligible to issue invitations (being inviters) is dynamic and the server can, efficiently and without re-keying the existing members, generate credentials for a new user to empower her to invite others. This is significantly better than the related studies [5, 54] where the same action imposes $O(N)$ communication overhead where N is the total number of group members.

2 NOTATIONS AND PRELIMINARIES

Notation: We refer to a Probabilistic Polynomial-Time entity as PPT. TTP stands for Trusted Third Party. $x \in_R X$ and $x \leftarrow X$ both mean x is randomly selected from set X. $\perp$ indicates an empty string. $\equiv_c$ stands for computational indistinguishability. We use $DL_g(y)$ to indicate the discrete logarithm of y in base g.

Negligible Function: Function f is negligible if for $\forall p(.)$ where $p(.)$ is polynomial, there exists integer N s.t. for every $n > N$, $f(n) < \frac{1}{p(n)}$.

Pseudo-Random Generator [6]: A deterministic polynomial time function $P : \{0,1\}^m \rightarrow \{0,1\}^{l(m)}$ (where $l(.)$ is a polynomial) is called Pseudo Random Generator (PRG) if $m < l(m)$ and for any probabilistic polynomial-time distinguisher D there exists a negligible function $negl(.)$ such that:

$$|Pr[x \leftarrow \{0,1\}^m : D(P(x)) = 1] - Pr[y \leftarrow \{0,1\}^{l(m)} : D(y) = 1]| = negl(m) \tag{1}$$

(t,n)-Shamir Secret Sharing Scheme (SSS): The (t,n)-Shamir secret sharing scheme [61] is a tool by which one can split a secret value into n pieces such that any subset of t shares can reconstruct the secret. The scheme works based on polynomial evaluations. Let F_q be a finite field of order q. The secret holder/dealer picks a random polynomial f of degree $t-1$ with coefficients from Z_q:

$$f(x) = \sum_{i=0}^{t-1} a_i \cdot x^i \tag{2}$$

The dealer sets the secret data S as the evaluation of that function at point 0 i.e., $f(0) = a_0 = S$. The share of each participant shall be one point on f e.g., $f(j)$ is the share of j^{th} shareholder. As such, a dealer can generate arbitrary many shares from its secret (i.e., by evaluating function f on a new point). Since each polynomial of degree $t-1$ can be uniquely reconstructed by having t distinct points of that function, t Shamir shareholders are able to reconstruct the secret. Given any t shares $\{(i, s_i)\}_{i=1}^{t}$, the secret reconstruction algorithm works as below.

$$S = f(0) = \sum_{i=1}^{t} s_i \cdot B_i \tag{3}$$

where B_is are Lagrange coefficients defined as

$$B_i = \prod_{j \neq i}^{j=1:t} \frac{j}{j-i} (\text{mod q}) \tag{4}$$

Shamir secret sharing scheme satisfies the following properties: 1) Given t or more than t shares, it can reconstruct the secret S easily; and 2) with knowledge of fewer than t shares, it cannot reconstruct the secret S. Shamir's scheme is *information theoretically secure* relying on no computational assumption.

Shamir shares are homomorphic under addition operation i.e., let $[s_1]$ and $[s_2]$ be shares of S_1 and S_2 (using (t, n)-Shamir secret sharing scheme), then $[s_1] + [s_2]$ constitutes a share of $S_1 + S_2$.

Verifiable Shamir Secret Sharing Scheme Feldman [32] proposes a verifiable secret-sharing scheme that is an extension of Shamir secret sharing. In this version, shareholders can verify the validity of their shares and ensure that the same polynomial is being evaluated. The secret dealer generates a random polynomial f using the standard Shamir secret sharing scheme, as shown in Equation 3, and performs secret share calculation. However, to guarantee the authenticity of these shares, the secret owner distributes commitments to the coefficients of $f(.)$ modulo q, by publishing $F_0 = g^{a_0}, F_1 = g^{a_1}, \cdots, F_{t-1} = g^{a_{t-1}}$. Each shareholder can verify the authenticity of their share s_i using the polynomial commitments provided in Equation 5.

$$\prod_{j=0}^{t-1} F_j^{i^j} = g^{a_0} \cdot g^{a_1 \cdot i} \cdots g^{a_{t-1} \cdot i^{t-1}} = g^{a_0 + a_1 \cdot i + \cdots + a_{t-1} \cdot i^{t-1}} = g^{f(i)} \overset{?}{==} g^{s_i} \tag{5}$$

A share s_i is considered valid only if the equality check in Equation 5 holds.
Multiplicative Homomorphic Encryption: A public key encryption scheme π consists of three algorithms key generation, encryption, and decryption, denoted by $\pi = (\mathsf{EGen}, \mathsf{Enc}, \mathsf{Dec})$. Using EGen, a pair of keys is generated called encryption key ek and decryption key dk. π is called multiplicatively homomorphic if for every a and b, $\mathsf{Enc}_{ek}(a) \otimes \mathsf{Enc}_{ek}(b) = \mathsf{Enc}_{ek}(a \cdot b)$ where a and b belong to the encryption message space and $\otimes$ is an operation over ciphertexts. As an example, in ElGamal encryption [31], $\otimes$ corresponds to group multiplication. Additionally, we have $\mathsf{Enc}_{ek}(a)^c = \mathsf{Enc}_{ek}(a^c)$ where a is a plain message and c is any integer. Throughout the paper, we consider the ElGamal scheme as our underlying encryption scheme.
Signature Scheme: A signature scheme [57] Sig consists of three algorithms key generation, sign and verify denoted by $\mathsf{Sig} = (\mathsf{SGen}, \mathsf{Sign}, \mathsf{SVrfy})$. A pair of keys (sk, vk) is generated via SGen where sk is the signature key and vk is the verification key. The signer signs a message m using sk by computing $\eta = \mathsf{Sign}_{sk}(m)$. Given the verification key vk, a receiver of signature runs $\mathsf{SVrfy}_{vk}(\eta, m)$ to verify.

A signature scheme $\mathsf{Sig} = (\mathsf{SGen}, \mathsf{Sign}, \mathsf{SVrfy})$ is said to be existentially unforgeable under adaptive chosen message attack if $\forall$ probabilistic polynomial time adversary A, there exists a negligible function $negl(.)$ s.t. the following holds [42]:

$$Pr[(sk, vk) \leftarrow \mathsf{SGen}(1^{\lambda}); (m, \sigma) \leftarrow A^{\mathsf{Sign}_{sk}(.)}(vk) \text{ s.t. } m \notin Q \textit{ and } \mathsf{SVrfy}_{vk}(m, \sigma) = accept] = negl(\lambda) \tag{6}$$

$A^{\mathsf{Sign}_{sk}(.)}$ indicates that the adversary has oracle access to the signature algorithm. Q indicates the set of adversary's queries to the signature oracle.

Zero-knowledge Proof of Knowledge of Discrete Logarithm (ZKPODL): This proof system was initially introduced by Schnorr [59] for proving the knowledge of a discrete logarithm in the group G of prime order q with generator g. That is, for a given $\omega, g \in G$, one can prove the knowledge of $x \in Z_q$ s.t. $x = DL_g(\omega)$ (DL stands for discrete logarithm). Using the method featured by [40], we convert the Σ protocol of Schnorr to a zero-knowledge proof system. More information about Σ protocols and their conversion to a zero-knowledge proof system is provided in Section A.

Zero-Knowledge Proof of Plaintext Knowledge: This proof system is used to prove the plaintext knowledge of a given ciphertext. That is, given ciphertext C that is encrypted under public key pk, a prover proves the knowledge of x and r s.t. $C = \mathsf{Enc}_{pk}(x, r)$. r is the randomness used while encryption. We instantiate such a proof system using the proposal of [38] for the ElGamal encryption scheme.

Zero-Knowledge Proof of Discrete Logarithm Equality: For a group G of prime order q and generators $g_1, g_2, h_1, h_2 \in G$, the ZKP of discrete logarithm equality is a protocol to prove that $h_1 = g_1^{\alpha}$ and $h_2 = g_2^{\alpha}$ where $\alpha \in Z_q$ [24].

Bilinear Map: Consider G_1 and G_2 as multiplicative groups of prime order q. Let g_1 be the generator of G_1. We employ an efficiently computable bilinear map $e : G_1 \times G_1 \longrightarrow G_2$ with the following properties [63]

- Bilinearity: $\forall u, v \in G_1$ and $\forall a, b \in \mathbb{Z}_q : e(u^a, v^b) = e(u, v)^{a \cdot b}$.
- Non-degeneracy: $e(g_1, g_1) \neq 1$.

We adopted Type 1 pairing (symmetric) in favor of simple protocol description and security analysis, nevertheless, this can be translated to an asymmetric pairing type to empower more efficient implementation [23].

Computational Diffie-Hellman Assumption [30]: Given a cyclic group G of prime order q with a generator g, and two randomly selected group elements $h_1 = g^{r_1}, h_2 = g^{r_2}$, the Computational Diffie-Hellman (CDH) assumption is hard relative to G if for every PPT adversary A there exists a negligible function $negl(\lambda)$ where λ is the security parameter, such that:

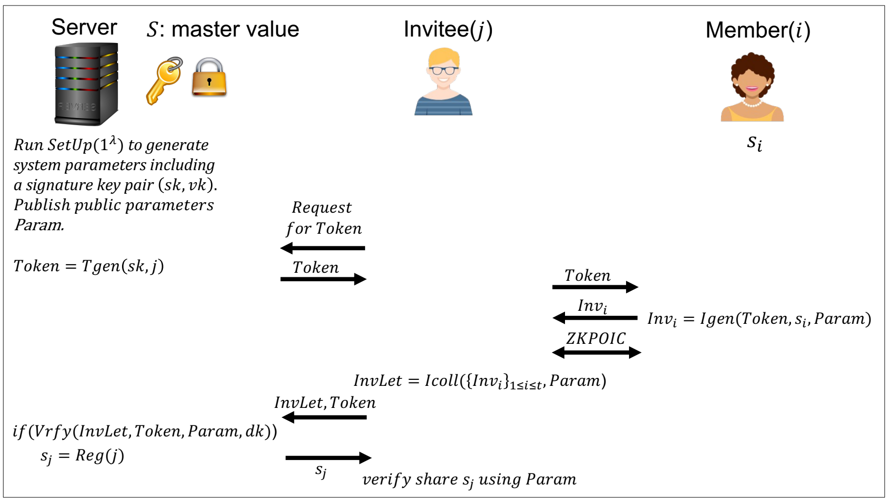
$$\Pr[\ A(G, q, g, h_1, h_2) = g^{r_1 \cdot r_2}] = negl(\lambda)$$

3 CONSTRUCTION

Anonyma consists of the following algorithms: *SetUp, Token generation (Tgen), Invitation generation (Igen), Invitation collection (Icoll), Invitation Verification (Ivrfy)* and *Registration (Reg)*. The summary of each algorithm is explained in Section 3.1 followed by the full construction in Section 3.2. Throughout the paper, we assume that all the parties communicate via secure and authenticated channels. The general interaction between the parties is illustrated in Figure 2.

3.1 Construction Overview

- *SetUp*: The server invokes the *SetUp* algorithm with the input of the security parameter 1^{λ} to initialize the system parameters: a cyclic group G, a master value $S \in_R G$, as well as key pairs for a signature scheme (denoted by sk, vk) and ElGamal key pair (denoted by ek, dk). At the beginning of the system lifetime, the server needs to register at least t initial users so that they can start inviting others. These initial members are given credentials by the server



Server
S: master value
Invitee(j)
Member(i)
s_i
Run $SetUp(1^\lambda)$ to generate system parameters including a signature key pair (sk, vk). Publish public parameters Param.
Request for Token
$Token = Tgen(sk, j)$
Token
Token
Inv_i
$Inv_i = Igen(Token, s_i, Param)$
ZKPOIC
$InvLet = Icoll(\{Inv_i\}_{1 \le i \le t}, Param)$
InvLet, Token
$if(Vrfy(InvLet, Token, Param, dk))$
$s_j = Reg(j)$
s_j
verify share s_j using Param

Fig. 2. *Anonyma* workflow.

to be able to make invitations. Each credential is indeed a share s_i of the server's master value S that is generated using (t, n)-SSS scheme. For the shares to be verifiable (the member can verify whether or not his piece is valid), the server publishes the commitment to the polynomial function generated as part of the SSS scheme. A newcomer can register to the system if she manages to compute a designated function of the master value S. Given that S can be reconstructed only with the combination of t valid shares i.e., the presence of t shareholders, the newcomer cannot register unless with having t distinct inviters. As such, invitation unforgeability is guaranteed. Note that any group of t members has the ability to invite an unlimited number of invitees to the system. However, it is important to clarify that this ability should not be misconstrued as granting complete control of the system to its members. The invitation-only policy serves as a means to foster the system's growth while maintaining trust among its members, rather than imposing restrictions on its expansion.

- $Tgen$: Each newcomer (i.e., invitee) contacts the server to get a token. The server runs the $Tgen$ algorithm to generate a $Token$ and hands it to the invitee. The $Token$ is a server signed certificate that embodies the index of the newcomer (each user is associated with a unique index) as well as a random element from the group G. Tokens shall be used by the inviters to invite their intended invitee. Invitations issued for a particular token cannot be used for another token. This way, the non-exchangeability of the invitations is guaranteed.
- *Igen*: The Invitee contacts each of his t inviters (this communication cannot be observed by the server/administrator) and communicates his $Token$ with them. Provided a valid $Token$, each inviter generates an invitation by executing *Igen*. The invitation consists of two parts: 1) a masked version of the inviter's share s_i, 2) and the masking value encrypted using the server's ek. The $Token$ is integrated into both parts of the invitation. As a part of *Igen*, the inviter has to prove in zero-knowledge that his invitation is well structured. For this sake, we devise a zero-knowledge proof protocol (i.e., Zero-Knowledge Proof of Invitation Correctness (ZKPOIC)). This proof helps in protecting between-inviter anonymity, i.e., inviters who collude with the server do not learn the identity of other inviters. Next, the inviter hands his invitation Inv_i to the invitee.

- *Icoll*: Upon the receipt of t invitations $\{\mathsf{Inv}_i\}_{i=1}^{t}$, the invitee invokes the *Icoll* algorithm through which he aggregates and blinds the invitations into a unified invitation letter $InvLet$. Aggregation and blinding remove any identifiable information about the identity of the inviters and helps in providing inviter anonymity (especially against a corrupted server). Additionally, through aggregation, the masking version of the master value S gets homomorphically reconstructed. We utilize the homomorphic property of both Shamir shares and the ElGamal encryption scheme to enable aggregation. At last, the invitee submits the final invitation letter $InvLet$ together with his $Token$ to the server.
- *Ivrfy*: The server authenticates the invitation letter by running *Ivrfy* and accepts or rejects accordingly. In a nutshell, the $InvLet$ is valid if and only if it contains the master value S.
- *Reg*: If the verification passes successfully (i.e., *Ivrfy* outputs accept), the server runs the *Reg* algorithm to issue credentials for the newcomer to enable him to act as an inviter. This credential is a Shamir share of the server's master value S. The newcomer verifies the validity of his share using the parameters output by the server in the *SetUp* phase and then stores his share for inviting others.

3.2 Full Construction

3.2.1 SetUp: This algorithm is run by the server who inputs the security parameter 1^λ and generates system parameters $Param$ as follows.

- Two primes p and q of length λ such that $q|p-1$.
- g is a generator of a cyclic subgroup G of order q in Z_p^*.
- ElGamal encryption scheme $\pi = (\mathsf{EGen}, \mathsf{Enc}, \mathsf{Dec})$ with the key pair $(ek = h = g^a, dk = a)$ denoting encryption key and decryption key, respectively. dk remains at the server while ek is published.
- A signature scheme $\mathsf{Sig} = (\mathsf{SGen}, \mathsf{Sign}, \mathsf{SVrfy})$. The signature and verification keys (sk, vk) are generated according to SGen. vk is published.
- A pseudo random generator PRG:$\{0,1\}^\lambda \longrightarrow Z_q$.
- A master value $S \longleftarrow Z_q$.
- A collision resistant, fixed length, hash function $(HGen, H)$, with the key $hk \longleftarrow HGen(1^\lambda)$. hk is an implicit input to the hash function $H(.)$.
- A randomly chosen polynomial function $f(y) = a_{t-1}y^{t-1} + ... + a_1 y + a_0$ of degree $t-1$ whose coefficients $a_1, ..., a_{t-1}$ belong to Z_q and $a_0 = S$.
- The server initially registers t users into the system so that they can start inviting outsiders. Each user is associated with a unique index i and shall receive the evaluation of function f on that index, i.e. $s_i = f(i)$. We refer to s_i as the master share of the user with index i.
- The server publishes $F_0 = g^{a_0}, F_1 = g^{a_1}, \cdots, F_{t-1} = g^{a_{t-1}}$ as the commitment to the selected function f. Given $F_0, \cdots, F_{t-1}$, the computation of commitment on $f(i)$ for any i is immediate as given in Equation 20. We will use γ_i to indicate g^{s_i}.

$$\gamma_i = \prod_{j=0}^{t-1} F_j^{i^j} = g^{a_0} \cdot g^{a_1 \cdot i} \cdots g^{a_{t-1} \cdot i^{t-1}} = g^{a_0 + a_1 \cdot i + \cdots + a_{t-1} \cdot i^{t-1}} = g^{f(i)} = g^{s_i} \tag{7}$$

- The server publishes public parameters $Param = (G, p, q, g, ek, vk, (F_0, \cdots, F_{t-1}))$.

3.2.2 Token Generation: Users wishing to register to the system first need to contact the server and obtain a token. The server generates a token through the token generation algorithm shown in Algorithm 3.1. In this procedure, the server initially assigns the user a unique index $j = H(n+1)$ where n is the total number of requested tokens so far. n is initially set to t to account for the initial registered t members at the SetUp phase. Next, the server generates a random group element ω (line 2) and certifies $j||\omega$ using his signing key sk (line 3). Let η be the signature outcome. The tuple (η, j, ω) constitutes the Token (line 4). We remark that the server is not required to record any information regarding the issued tokens. Thus, the generated tokens can simply be discarded and only the total number of generated tokens n needs to be retained. Therefore, we do *not* incur any storage load on the server per token.

Algorithm 3.1: Tgen [Server]

```
Input: $sk, n$
Output: $Token$
1 $j = H(n + 1)$
2 $r \leftarrow Z_q; \omega = g^r$
3 $\eta = \mathsf{Sign}_{sk}(j||\omega)$
4 $Token = (\eta, j, \omega)$
```

3.2.3 Invitation Generation: Invitation generation is run by the inviter to generate an invitation for a token given by the invitee. The procedure is shown in Algorithm 4.2. We assume that invitee and inviter communicate out of band (cannot be observed by the server/administrator), e.g., using a messaging application. Firstly, the inviter checks the authenticity of the token against the server verification key vk (line 1). Then, he samples a random value δ_i from Z_q by applying PRG on the random seed v (lines 2-3). Then, he blinds his master share using δ_i, i.e., $s_i + \delta_i$, and then ties this value to the provided token as $\tau_i = \omega^{s_i+\delta_i}$ (line 4). He also encrypts the masking value ω^{δ_i} as $e\delta_i$ using the server's encryption ek (line 5). To ensure that the inviter is acting honestly (i.e., generating the invitation as instructed in the algorithm), the inviter must prove the correctness of the invitation in zero-knowledge. To enable this, we propose a zero-knowledge proof system for the Proof Of Invitation Correctness *ZKPOIC*. The inviter and invitee engage in *ZKPOIC* (line 7) through which the inviter proves the correctness of his invitation $Inv_i = (\tau_i, e\delta_i)$ to the invitee in zero-knowledge. In the following, we explain our proposed proof system. We first draw a Σ protocol for *POIC* and prove its security. Then, the zero-knowledge variant is immediate using the method proposed in [40, 56].

Algorithm 3.2: Igen [Inviter]

```
Input: $Token = (\eta, j, \omega), s_i, Param$
Output: $Inv_i/\perp$
1 if $Svrfy_{vk}(\eta, j||\omega)$=accept then
2     $v \leftarrow \{0,1\}^\lambda$
3     $\delta_i = \mathsf{PRG}(v)$
4     $\tau_i = \omega^{s_i+\delta_i}$
5     $e\delta_i = \mathsf{Enc}_{ek}(\omega^{\delta_i})$
6     $Inv_i = (\tau_i, e\delta_i)$
7     return $inv_i$ //Inviter authenticates $Inv_i$ through ZKPOIC
8 return $\perp$
```

Σ **Protocol for Proof Of Invitation Correctness (POIC):** The invitation is constructed correctly if the inviter proves the following statements:

(1) The inviter possesses a valid share of the master value S. That is, the inviter holding index i must prove the knowledge of the discrete log of γ_i, i.e., s_i. Note that $\gamma_i = g^{s_i}$ can be computed from $F_0, ..., F_{t-1}$ as explained in Equation 20.
(2) The inviter knows the plaintext of $e\delta_i$, i.e., the knowledge of ω^{δ_i} and r such that $e\delta_i = (e\delta_{i,1} = \omega^{\delta_i} \cdot h^r, e\delta_{i,2} = g^r)$.
(3) The randomness δ_i used in the creation of τ_i is correctly encrypted in $e\delta_i$. This can be captured by proving that $\tau_i \cdot e\delta_{i,1}^{-1} \cdot h^r$ $(= \omega^{s_i})$ and $\gamma_i = g^{s_i}$ have the same discrete logarithm s_i. The former is true due to Equation 8.

$$\tau_i \cdot e\delta_{i,1}^{-1} \cdot h^r = \omega^{\delta_i + s_i} \cdot \omega^{-\delta_i} \cdot h^{-r} \cdot h^r = \omega^{s_i} \tag{8}$$

To enable zero-knowledge proof of the aforementioned statements, we devise a Σ-protocol (P, V) as depicted in Figure 3. We refer to this proof system by **Proof of Invitation Correctness**, or *POIC*. POIC captures the relation R indicated in Equation 9 which embodies four different predicates.

$$R = \{(\overbrace{(\tau_i, e\delta_i = (e\delta_{i,1}, e\delta_{i,2}), \gamma_i, \omega),}^{\text{Public inputs}} \overbrace{(s_i, r, \delta_i)}^{\text{Private inputs}})| \tag{9}$$
$$DL_g(\gamma_i) = s_i \ \wedge \tag{10}$$
$$e\delta_{i,1} = \omega^{\delta_i} \cdot h^r \ \wedge \tag{11}$$
$$DL_g(e\delta_{i,2}) = r \ \wedge \tag{12}$$
$$DL_\omega(\tau_i \cdot e\delta_{i,1}^{-1} \cdot h^r) = DL_g(\gamma_i) = s_i\} \tag{13}$$

For the proof of relation R in Equation 9, we incorporate the Schnorr protocol [42] for the proof of discrete logarithm knowledge (Equations 10 and 12), proof of plaintext knowledge as proposed in [38] (Equation 11), and the proof of discrete logarithm equality [24] (Equation 13).

Completeness: To prove that completeness holds, observe that if the prover P follows the protocol honestly, then due to the Equations 14, 15, 16, and 17, the verifier V accepts.

$$A \cdot \gamma_i^e = g^{s'} \cdot (g^{s_i})^e = g^{s' + e \cdot s_i} = g^{Z_1} \tag{14}$$

$$B_1 \cdot e\delta_{i,1}^e = (\omega^{\delta'} \cdot h^{r'}) \cdot (\omega^{\delta} \cdot h^r)^e = \omega^{\delta' + e \cdot \delta_i} \cdot h^{r' + e \cdot r} = \omega^{Z_2} \cdot h^{Z_3} \tag{15}$$

$$B_2 \cdot e\delta_{i,2}^e = (g^{r'}) \cdot (g^r)^e = g^{r' + e \cdot r} = g^{Z_3} \tag{16}$$

$$\begin{aligned} &C \cdot \tau_i^e \cdot B^{-1} \cdot e\delta_{i,1}^{-e} \cdot h^{Z_3} = \\ &(\omega^{s' + \delta'}) \cdot (\omega^{e \cdot s_i + e \cdot \delta_i}) \cdot (\omega^{-\delta'} \cdot h^{-r'}) \cdot (\omega^{-e \cdot \delta_i} \cdot h^{-e \cdot r}) \cdot h^{r' + e \cdot r} = \\ &\omega^{(s' + e \cdot s_i)} = \omega^{Z_1} \end{aligned} \tag{17}$$

We prove the special soundness and special honest verifier zero-knowledge properties in Section B. We additionally present the security properties of a zero-knowledge proof system for POIC that is achieved using the method given in [40, 56].

3.2.4 Invitation Collection: Upon receipt of t invitations, the invitee runs the Invitation Collection ($Icoll$) procedure as indicated in Algorithm 3.3. The invitee aggregates τ_i values as $\prod_{i=1}^{t} \tau_i^{B_i}$ (line 3). He operates similarly for $e\delta_i$ values as $\prod_{i=1}^{t} e\delta_i^{B_i}$ (line 4). B_is are the Lagrange coefficients (as defined in Equation 4) used for the reconstruction of the master value S from the Shamir shares. Next, the invitee randomizes both aggregates T and $e\Delta$ by adding a random value of his own choice, i.e., δ^*. The randomization cancels out the effect of the Lagrange coefficients and makes the final aggregates, i.e., T and $e\Delta$, independent of the B_i values. Recall that the Lagrange coefficients are

$$
\begin{array}{lcl}
\underline{\text{Prover } (s_i, r, \delta_i)} & & \underline{\text{Verifier}} \\
s', r', \delta' \leftarrow Z_q & & \\
A = g^{s'} & & \\
B_1 = \omega^{\delta'} \cdot h^{r'} & & \\
B_2 = g^{r'} & & \\
C = \omega^{s'+\delta'} & \xrightarrow{A, B=(B_1, B_2), C} & \\
 & \xleftarrow{e} & e \leftarrow Z_q \\
Z_1 = s' + e \cdot s_i & & \\
Z_2 = \delta' + e \cdot \delta_i & & \\
Z_3 = r' + e \cdot r & & \\
 & \xrightarrow{Z_1, Z_2, Z_3} & \\
 & & \text{if} \, (A \cdot \gamma_i^e == g^{Z_1} \\
 & & \wedge B_1 \cdot e\delta_{i,1}^e == \omega^{Z_2} \cdot h^{Z_3} \\
 & & \wedge B_2 \cdot e\delta_{i,2}^e == g^{Z_3} \\
 & & \wedge C \cdot \tau_i^e \cdot B^{-1} \cdot e\delta_{i,1}^{-e} \cdot h^{Z_3} == \omega^{Z_1}) \\
 & & \text{Accept}
\end{array}
$$

Fig. 3. Σ protocol of Proof of Invitation Correctness for the common input $\tau_i, e\delta_i = (e\delta_{i,1}, e\delta_{i,2}), \gamma_i, \omega$. The prover has the private input (s_i, r, δ_i).

dependent on the inviters' indices and by hiding them we aim at protecting inviter anonymity. The final invitation letter *InvLet* shall be the pair $(T, e\Delta)$. The invitee submits the invitation letter and the token to the server.

In Equations 18 and 19, we expand the result of T and $e\Delta$, which leads to the following observations.

$$
\begin{aligned}
T = \omega^{\delta^*} \cdot \prod_{i=1}^{t} \tau_i^{B_i} = \omega^{\delta^*} \cdot \prod_{i=1}^{t} \omega^{B_i \cdot s_i + B_i \cdot \delta_i} &= \omega^{\delta^* + \sum_{i=1}^{t} B_i \cdot s_i + \sum_{i=1}^{t} B_i \cdot \delta_i} \\
&= \omega^{S + \delta^* + \sum_{i=1}^{t} B_i \cdot \delta_i} = \omega^{S+\Delta}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
e\Delta = \mathsf{Enc}_{ek}(\omega^{\delta^*}). \prod_{i=1}^{t} e\delta_i^{B_i} &= \mathsf{Enc}_{ek}(\omega^{\delta^*}). \prod_{i=1}^{t} \mathsf{Enc}_{ek}(\omega^{B_i \cdot \delta_i}) \\
&= \mathsf{Enc}_{ek}(\omega^{\delta^* + \sum_{i=1}^{t} B_i \cdot \delta_i}) = \mathsf{Enc}_{ek}(\omega^{\Delta})
\end{aligned}
\tag{19}
$$

The first observation is that T has the master value S embedded in its exponent. Intuitively, the presence of S in the exponent proves that the invitee has t distinct invitations. Otherwise, the reconstruction of S would be impossible (we elaborate on this in Section 6 and formally prove the unforgeability of invitations). Another observation is that the computation of both T and $e\Delta$ depends on the token ω. Hence, as desired, the resultant *InvLet* is now bound to the given token. This would help with the non-exchangeability of the invitations. At last, T contains a masked version of master value, i.e., $S + \Delta$, in the exponent whereas $e\Delta$ embodies the corresponding masking value Δ. The encryption $e\Delta$ of the masking value shall be used at the server for verification purposes (see invitation verification below).

Algorithm 3.3: Icoll [Invitee]

```
Input: ${Inv_i = (\tau_i, e\delta_i) | 1 \leq i \leq t}, Param$
Output: $InvLet$
1 $r \leftarrow \{0,1\}^\lambda$
2 $\delta^* = \mathsf{PRG}(r)$
3 $T = \omega^{\delta^*} \cdot \prod_{i=1}^{t} \tau_i^{B_i}$
4 $e\Delta = \mathsf{Enc}_{ek}(\omega^{\delta^*}) \cdot \prod_{i=1}^{t} e\delta_i^{B_i}$
5 $InvLet = (T, e\Delta)$
```

3.2.5 Invitation Verification: Once the invitee hands his invitation letter $InvLet$ together with the corresponding token $Token$ to the server, the server executes the invitation verification procedure shown in Algorithm 3.4. As the first step, the server authenticates the $Token$, i.e., whether it is signed under the server's signature key sk (line 1). Next, the validity of the invitation letter $InvLet$ must be checked. For that, the server decrypts $e\Delta$ using its decryption key dk and obtains ω^Δ (line 2). Recall that Δ was used to mask the master value S in $T = \omega^{S+\Delta}$. Thus, if T and $e\Delta$ are constructed correctly, we expect that $\omega^S \cdot \omega^\Delta = T$ (line 3). If all the verification steps are passed successfully, then the server accepts the user's membership request.

Algorithm 3.4: *Ivrfy* [Server]

```
Input: $InvLet = (T, e\Delta), Token = (\eta, j, \omega), Param, dk$
Output: $reject/accept$
1 if $_{vk}(\eta, j||\omega)$=accept then
2     $\omega^\Delta = Dec_{dk}(e\Delta)$
3     if $\omega^S \cdot \omega^\Delta = T$ then
4         return accept
```

3.2.6 Registration: The server invokes the registration procedure (Algorithm 3.5) for users who pass the verification phase (Algorithm 3.4). The input to Algorithm 3.5 is the index j of the newcomer, and the output is a Shamir share Φ_j of the master value S, where Φ_j is the evaluation of polynomial f at point j (line 1). Note that the index j is the index included in the user's $Token = (\eta, j, \omega)$. The server delivers Φ_j to the user who can then start making invitations as an inviter. The user authenticates his share by comparing the commitment γ_j (as given in Equation 20) against its own share, i.e., g^{Φ_j}. If they are equal, the user accepts and stores the share.

Algorithm 3.5: Reg [Server]

```
Input: $j$
Output: $\Phi_j$
1 $\Phi_j = f(j)$
```

4 *ANONYMAX*: ANONYMOUS CROSS-NETWORK INVITATION-BASED SYSTEM

AnonymaX is an anonymous cross-network invitation-based system, which enables third-party authentication[8]. That is, mutually trusted domains can rely on each other's authentication mechanism. Essentially, if a user is authenticated by one domain, its authentication is accepted by all other domains that trust the authenticating domain. Users of different trusting administrative domains can be inviters of each other. In specific, a user can join one domain, e.g., Twitter, by obtaining invitations from members of another network, e.g., Facebook. Facebook and Twitter in the preceding example are called *authenticating* domain/network (AN) and *registration* domain/network (RN), respectively.

Failed Approaches: One simple but cumbersome solution to empower a cross-network invitation-based system is to follow the regular invitation-based system, i.e. each time a user wants to join the *registration* domain, the *authenticating* server authenticates that particular user and communicates the authentication result to the *registration* server. However, this solution requires the two servers to keep in contact with each other and imposes unnecessary overhead on the *authenticating* server.

An alternative approach proposed by Inonymous [9] (our prior work), is that the *authenticating* server would publish the commitment over the master value S as g^S to the *registration* servers. Subsequently, *registration* servers would follow a different verification method (relying on bilinear maps) to authenticate invitations on their own. While this solution works for the honest but curious adversarial model, it fails in providing invitation unforgeability against a malicious adversary, which is explained next. Consider $registration_1$ and $registration_2$ as two *registration* servers. The corrupted $registration_1$ wants to join $registration_2$ as an invitee without enough inviters. $registration_1$ receives a token with the random value ω from $registration_2$ and then issues the same token to the users who want to join its service. As such, $registration_1$ can reuse the invitation letters of his users to craft a valid invitation letter to join $registration_2$. We address this issue in *AnonymaX* by making the *registration* servers prove in zero-knowledge (using an interactive proof) that they know the discrete logarithm $DL(w)$ of their issued tokens during the $Tgen$ protocol. As such, no *registration* server can issue tokens that are not generated by itself.

4.1 *AnonymaX* Construction

In the description below, we assume there is one *authenticating* network provider AN and multiple *registration* servers denoted by RN_j.

4.1.1 *SetUp.*

The parameters published by the *authenticating* network provider is denoted by $Param_{\text{AN}}$ and comprises the following items:

- Two primes p and q of length λ such that $q|p-1$.
- g is a generator of a cyclic subgroup G of order q in Z_p^*.
- A master value $S \leftarrow Z_q$.
- A randomly chosen polynomial function $f(y) = a_{t-1}y^{t-1} + \ldots + a_1 y + a_0$ of degree $t-1$ whose coefficients $a_1, \ldots, a_{t-1}$ belong to Z_q and $a_0 = S$.
- The server initially registers t users into the system so that they can start inviting outsiders. Each user is associated with a unique index i and shall receive the evaluation of function f on that index, i.e. $s_i = f(i)$. We refer to s_i as the master share of the user with index i.
- The server publishes $F_0 = g^{a_0}, F_1 = g^{a_1}, \cdots, F_{t-1} = g^{a_{t-1}}$ as the commitment to the selected function f. Given $F_0, \cdots, F_{t-1}$, the computation of commitment on $f(i)$ for any i is immediate

[8]https://www.synopsys.com/blogs/software-security/5-reasons-third-party-authentication/

as given in Equation 20. We will use γ_i to indicate g^{s_i}.

$$\gamma_i = \prod_{j=0}^{t-1} F_j^{i^j} = g^{a_0} \cdot g^{a_1 \cdot i} \cdots g^{a_{t-1} \cdot i^{t-1}} = g^{a_0 + a_1 \cdot i + \cdots + a_{t-1} \cdot i^{t-1}} = g^{f(i)} = g^{s_i} \tag{20}$$

- A bilinear map $e\colon G \times G \longrightarrow G_2$ where G and G_2 are multiplicative groups of prime order q.
- The server publishes public parameters $Param_{\mathsf{AN}} = (G, p, q, g, e, (F_0, \cdots, F_{t-1}))$.

Upon the receipt of the $Param_{\mathsf{AN}}$ from the *authenticating* server, a *registration* server RN_j generates the following parameters denoted by $Param_{\mathsf{RN}_j}$.

- ElGamal encryption scheme $\pi = (\mathsf{EGen}, \mathsf{Enc}, \mathsf{Dec})$ with the key pair $(ek_{\mathsf{RN}_j} = h = g^a, dk_{\mathsf{RN}_j} = a)$ denoting encryption key and decryption key, respectively. dk_{RN_j} remains at the server while ek_{RN_j} is published.
- A signature scheme $\mathsf{Sig} = (\mathsf{SGen}, \mathsf{Sign}, \mathsf{SVrfy})$. The signature and verification keys $(sk_{\mathsf{RN}_j}, vk_{\mathsf{RN}_j})$ are generated according to SGen. vk_{RN_j} is published.
- A pseudo random generator PRG:$\{0,1\}^\lambda \longrightarrow Z_q$.
- A collision resistant, fixed length, hash function $(HGen, H)$, with the key $hk \longleftarrow HGen(1^\lambda)$. hk is an implicit input to the hash function $H(.)$.
- A counter n_{RN_j} which is initially zero and gets incremented per token generation.
- The server publishes public parameters $Param_{\mathsf{RN}_j} = (ek_{\mathsf{RN}_j}, vk_{\mathsf{RN}_j})$.

4.1.2 Token Generation. Each invitee willing to join RN_j shall obtain a token from RN_j. During the token generation, the *registration* server follows Algorithm 3.1 and additionally must prove in zero-knowledge that it knows the discrete logarithm of the ω embodied in the token. This is to protect inviter anonymity against the corrupted *registration* server. As such, after the issuance of a token, the *registration* server runs an instance of *ZKPODL* protocol (given in section 2) with the invitee. The modified procedure is provided in Algorithm 4.1.

Algorithm 4.1: XTgen [*registration* Server RN_j]

```
Input: $Param_{\mathsf{AN}}, Param_{\mathsf{RN}_j}, sk_{\mathsf{RN}_j}$
Output: $Token$
1 $i = H(n_{\mathsf{RN}_j} + 1)$
2 $r \longleftarrow Z_q$; $\omega = g^r$
3 $\eta = \mathsf{Sign}_{sk_{\mathsf{RN}_j}}(i||\omega)$
4 $Token = (\eta, i, \omega)$
5 $Run\ \ ZKPODL((G, q, g, \omega), r)$
```

Upon successful proof, the invitee accepts the token.

4.1.3 Invitation Generation. The invitee needs to collect invitations from the members of the *authenticating* network AN to be used in the registration of a particular *registration* network RN_j. Inviters issue invitations as in the regular invitation procedure given in Algorithm 4.2. However, the inviters should verify the tokens against the verification key of the issuing server i.e., RN_j. Also, the inviters shall use the encryption key of the *registration* network ek_{RN_j} to encrypt their masking values. As such, in Algorithms 4.2 and 3.3, the inviter uses ek_{RN_j} and vk_{RN_j}, i.e. $Param_{\mathsf{RN}_j}$ as input. Therefore, the invitation letters received by the RN_j are of the form $InvLet = (T, e\Delta)$ where $e\Delta$ is an encrypted masking value under ek_{RN_j}.

Algorithm 4.2: XIgen [Inviter]

```
Input: $Token = (\eta, j, \omega)$, $s_i$, $Param_{\mathsf{RN}_j}$
Output: $Inv_i / \bot$
1 if $Svrfy_{vk_{\mathsf{RN}_j}}(\eta, j||\omega)$=$accept$ then
2     $v \leftarrow \{0,1\}^{\lambda}$
3     $\delta_i = \mathsf{PRG}(v)$
4     $\tau_i = \omega^{s_i + \delta_i}$
5     $e\delta_i = \mathsf{Enc}_{ek_{\mathsf{RN}_j}}(\omega^{\delta_i})$
6     $Inv_i = (\tau_i, e\delta_i)$
7     return $inv_i$ //Inviter authenticates $Inv_i$ through $ZKPOIC$
8 return $\bot$
```

4.1.4 Invitation Collection. Each invitation is encrypted using an identical encryption key ek_{RN_j}, and they all possess a portion of the same master value S. As a result, the set of homomorphic operations outlined in *Icoll* i.e., Algorithm 3.3 can be followed by the invitee without any alterations.

4.1.5 Invitation Verification. The verification routine run by the *registration* server RN_j, which is also shown in Algorithm 4.3, relies on the existence of a bilinear map $e\colon G \times G \longrightarrow G_2$ where G and G_2 are multiplicative groups of prime order q. The bilinear map description is available in the public parameters of *authenticating* server i.e., $Param_{\mathsf{AN}}$. The only difference between Algorithm 4.3 and Algorithm 3.4 is at the second verification step i.e., line 3. The correctness holds by the bilinearity of the bilinear map e, as in Equation 21.

$$\begin{aligned} e(\omega, g^S) \cdot e(\omega^{\Delta}, g) &= e(\omega, g)^S \cdot e(\omega, g)^{\Delta} = e(w, g)^{S+\Delta} = e(w^{S+\Delta}, g) \\ &= e(T, g) \end{aligned} \tag{21}$$

Algorithm 4.3: XIVerify [*registration* Server RN_j]

```
Input: $InvLet = (T, e\Delta), Token = (\eta, i, \omega), Param_{\mathsf{AN}}, Param_{\mathsf{RN}_j}, dk_{\mathsf{RN}_j}$
Output: $reject/accept$
1 if $\mathsf{SVrfy}_{vk_{\mathsf{RN}_j}}(\eta, i||\omega)$=$accept$ then
2     $\omega^{\Delta} = Dec_{dk_{\mathsf{RN}_j}}(e\Delta)$
3     if $e(\omega, g^S) \cdot e(\omega^{\Delta}, g) = e(T, g)$ then
4         return accept
```

5 PERFORMANCE

5.1 Running Time

In this section, we analyze the running time of each algorithm of *Anonyma*.

Simulation setting: The execution time was measured using a standard laptop equipped with an Intel Core i5 CPU running at 1.6 GHz and 8 GB of 1600 MHz DDR3 memory. The simulation setup consists of 100 registered members and 100 invitees. Each invitee receives a specific number of invitations, determined by a threshold, from randomly selected inviters among the 100 initially registered members. The DSA signature scheme [44] is utilized with a key length of 2048 bits. The

	SetUp	Tgen	Igen	Ivrfy	Reg
Server	2265 ms	2.28 ms	-	1.71 ms	0.02 ms
Inviter	-	-	6.26 ms	-	-

Table 1. Running time of the server and the inviter. (ms: milliseconds)

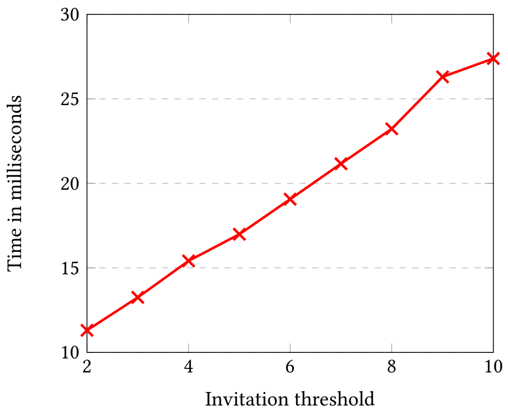


Fig. 4. Invitee running time for *Icoll*.

experimental results obtained under these conditions are collected and summarized in Table 1 and Figure 4. More detailed explanations are provided subsequently.

Server: The server takes 2265 milliseconds to complete the *SetUp* phase, which needs to be executed only once throughout the lifetime of the system. The *Token Generation* algorithm requires 2.28 milliseconds to run. The *Invitation Verification* procedure takes 1.71 milliseconds. The *Registration* process for each new member requires 0.02 milliseconds.

Inviter: The inviter's involvement is limited to executing the *Invitation Generation* algorithm, which takes approximately 6.26 milliseconds to complete.

Invitee: The invitee carries out the invitation collection ($Icoll$) procedure, which has a running time that is directly proportional to the number of required invitations i.e., t. Specifically, the running time of $Icoll$ can be expressed as $t \cdot time_{Auth} + (t-1) \cdot time_{Agg}$, where t represents the threshold, $time_{Auth}$ is the time taken to authenticate each invitation, and $time_{Agg}$ is the time required for the homomorphic aggregation of two individual invitations. Experimental data reveals that $time_{Auth} \approx 5ms$ and $time_{Agg} \approx 6ms$. The observed results, depicted in Figure 4, support this relationship.

5.2 Communication Complexity

The communication complexity signifies the amount of data exchanged among the parties. *Anonyma* features an asymptotic communication complexity that is constant in the security parameter. The concrete values are also measured as the number of exchanged bits and are summarized in Table 3 and explained next. $Icoll$ and $Ivrfy$ are both non-interactive and hence incur no communication complexity. In the $Tgen$ protocol, the transmission of $Token$ (one group element of size 1024 bits and a signature of size 2048 bits) incurs 3072 bits$\approx$ 0.38 KB. For the invitation generation protocol $Igen$, the inviter exchanges the invitation (consisting of 3 group elements) together with the ZKPOIC (with 8 group elements) which on aggregate yields 11264 bits ($\approx$ 1.4 KB) data transfer. The registration

	Tgen	Igen	Icoll	Reg
Server	0.38 KB	-	-	0.9 KB
Invitee	0.38 KB	1.4 KB	-	0.9 KB
Inviter/Group member	-	1.4 KB	-	-

Table 2. The communication complexity of protocols in *Anonyma* for each involved party. Values are also measured as the number of exchanged bits. KB stands for KilloBytes.

	Server	Inviter	Invitee
Storage	$2.61 + 0.26 \cdot t$ KB	0.13 KB	0 KB

Table 3. The storage overhead of *Anonyma*'s entities. KB stands for KilloBytes. t is the invitation threshold.

protocol involves the invitee submitting the invitation letter (with 3 group elements) together with the token (of size 3072 bits) to the server which results in 6144 bits ($\approx$ 0.77 KB) data exchange. The server then responds by a share of the master secret, which is of size 1024 bits ($\approx$ 0.13 KB). This results in an aggregate communication overhead of approximately 0.9KB for both the invitee and the server.

5.3 Storage

The storage requirement of each entity is measured based on the number of bits that the party needs to persistently retain locally. The server holds two pairs of keys for signature and encryption, hence requiring $\approx$ 20896 bits ($\approx$ 2.61 KB) of storage. Moreover, the server saves the description of the polynomial of degree $t-1$ with t coefficients and their corresponding commitments which results in $\approx 2t \cdot 1024$ bits ($\approx 0.26 \cdot t$ KB) of storage requirement at the server. The inviter only needs to keep its share of the master value which is of size 1024 bits ($\approx$ 0.13 KB). For the invitee, no local storage is required.

6 SECURITY

In this section, we provide security definitions for inviter anonymity and invitation unforgeability, and then prove the security of *Anonyma* (Sections 6.1 and 6.2). In Section 6.3, we prove the security of *AnonymaX* for which we supply a new security definition capturing invitation unforgeability in the cross-network invitation based systems. The formal proof of special soundness and honest verifier zero-knowledge property of our proposed Σ protocol for proof of invitation correctness (POIC), as well as the ideal functionality F^{R}_{POIC} corresponding to the zero-knowledge version of POIC are provided in the Appendix, section B.

6.1 Inviter Anonymity

An invitation-based system protects inviter anonymity if an invitee with t inviters can authenticate himself to the server without disclosing the identity of his inviters to the server. In the extreme situation where a corrupted server also manages to control $t-1$ inviters of an invitee, the inviter anonymity should guarantee that the identity of the remaining non-colluding inviter remains protected against the server and other inviters. The coalition of the server and $t-1$ inviters is the most powerful adversary against inviter anonymity. In the following, we present the formal definition of inviter anonymity as well as a formal security proof of inviter anonymity of *Anonyma*.

6.1.1 Security Definition: We model inviter anonymity as a game denoted by $\mathsf{InvAnonym}_{\mathsf{A}}(\lambda)$ played between a challenger and an adversary. The challenger acts as the invitee as well as the uncorrupted members of the system. On the other hand, the adversary plays as the server as well as arbitrary many corrupted members. The challenger is to register the invitee into the system while $t-1$ inviters of the invitee are controlled by the adversary and the remaining inviter is under the control of the challenger. As such, the adversary issues $t-1$ invitations on behalf of the corrupted inviters. Then, the adversary selects two indexes u_0, u_1 corresponding to two uncorrupted members. The challenger selects one of them randomly as u_b, where $b \in \{0,1\}$, to be the other inviter. The challenger issues an invitation from u_b for the invitee and combines it with the $t-1$ invitations issued by the adversary. The final invitation letter is submitted to the adversary (who also plays the role of the server). The challenge of the adversary is to guess a bit b indicating the index of the uncorrupted inviter. If the adversary cannot guess that index with more than a negligible advantage, then the system provides inviter anonymity. The formal definition follows.

Inviter Anonymity Experiment $\mathsf{InvAnonym}_{\mathsf{A}}(\lambda)$

(1) The adversary is given the security parameter 1^{λ}. It acts as the server and hands over $Param$ to the challenger.
(2) The adversary registers arbitrary many users to the system. The adversary instructs the challenger to register honest users through the Reg protocol. U_h and U_c contain the indices of the honest and corrupted members, respectively.
(3)(a) The adversary outputs the index of two honest inviters $u_0, u_1 \in U_h$.
  (b) The adversary, acting as the server, generates a token $Token$ for the invitee with index $j^* \in U_h$.
  (c) The adversary specifies a set of $t-1$ indices $I_c \subset U_c$ to be the corrupted inviters. For every $i \in I_c$, the adversary engages with the challenger in the execution of the $Igen$ protocol using $Token$ as the input. As the result, the invitee (i.e., the challenger) obtains a set of $t-1$ invitations denoted by $\{Inv_i\}_{i \in I_c}$.
(4)(a) The challenger selects a bit value $b \leftarrow \{0,1\}$. The challenger runs the $Igen$ protocol over $Token$ to issue an invitation from u_b for the invitee. Let Inv_b be the result.
  (b) The challenger runs $Icoll$ using $\{Inv_i\}_{i \in I_c} \cup Inv_b$ and $Param$ and generates an invitation letter $InvLet$. The challenger attempts to register to the system by sending $InvLet$ to the adversary.
(5) The adversary guesses a bit b'.
(6) The output of the game is 1 if $b == b'$, 0 otherwise.

Definition 6.1. An invitation-based system has inviter anonymity if for every probabilistic polynomial time adversary A there exists a negligible function $negl(.)$ such that:

$$Pr[\mathsf{InvAnonym}_{\mathsf{A}}(\lambda) = 1] = \frac{1}{2} + negl(\lambda)$$

At a high level, in *Anonyma*, the anonymity of the inviter holds due to the soundness of the proposed ZKPOIC (zero-knowledge proof of invitation correctness) and the security of the pseudo-random number generator (i.e., PRG). Below, to give an insight into how ZKPOIC can protect inviter anonymity, we draw a situation where the lack of ZKPOIC would immediately break inviter anonymity. Then, by relying on the F^{R}_{POIC} hybrid model for our proof, we relate the inviter anonymity of *Anonyma* to the security of the deployed PRG.

Recall that, as defined in the game, the adversary controls the server and $t-1$ inviters of the honest invitee. Due to the employed ZKPOIC, the invitee is assured that the inviters are not able to deviate from the protocol descriptions and hence would have to use their real master shares for the invitation generation. This implies that the master value S shall be reconstructed correctly as the output of $Icoll$. Therefore, as the result of the registration of the invitee (step 4.b from $\mathsf{InvAnonym}_{\mathsf{A}}(\lambda)$ experiment), the server obtains $InvLet = (T = \omega^{S+\Delta}, e\Delta)$ out of which the adversary can learn S and Δ. According to the Shamir secret sharing scheme, although the adversary knows $t-1$ shares that are used for the reconstruction of S, the remaining contributing shareholder can be any of the existing members, and hence the inviter anonymity is guaranteed. Now, consider that the inviters are not required to prove the correctness of their invitations. The $t-1$ corrupted inviters use zeros instead of their real master shares for invitation generation, i.e., $s_i = 0$ for $i \in I_c$. Then, the server obtains $w^{S'}$ with the following value: $S' = s_{u_b}.B_{u_b} + \sum_{i \in I_c} s_i.B_i = s_{u_b}.B_{u_b}$. The adversary can simply try the combinations of master shares s_{u_0} and s_{u_1} with B_0 and B_1, respectively, and figure out the remaining inviter's index (in practice, the possible number of values is linear in the number of non-colluding inviters, which is the number of registered users). This trivially breaks inviter anonymity, which follows from the lack of ZKPOIC.

As we discussed before, due to ZKPOIC all the invitations issued for the invitee are guaranteed to be well-structured (and their correctness are proven during $Igen$). Thus, the execution of $Icoll$ by the invitee would lead to a valid invitation letter of the form $InvLet = (\omega^{S+\Delta}, e\Delta)$ where S is the server's master value, $e\Delta$ is the encryption of ω^{Δ} and $\Delta = \delta^* + \sum_{i \in I_c} B_i \cdot \delta_i + B_{u_b} \cdot \delta'$ (δ^* is the masking value added by the invitee, δ' is the non-colluding inviter's masking value resulted from a *PRG* and B_{u_b} is the Lagrange coefficient computed based on the index of the non-colluding inviter). The adversary may get some idea about the identity of the non-colluding inviter by extracting the Lagrange coefficients from the Δ value (Lagrange coefficients are the function of inviters' indices). Two cases may occur. If the random values δ' and δ^* are selected truly at random, then we know that Δ is also a random value and conveys nothing about the Lagrange coefficient B_{u_b}. Though, if δ' and δ^* are the output of a *PRG* then the adversary may have advantages to extract the Lagrange coefficients. We denote the adversary's advantage by ϵ. If ϵ is non-negligible, it implies that we can distinguish between a *PRG* and a random number generator hence we break the security of the *PRG*. In the following, we provide formal proof.

THEOREM 6.2. *Anonyma provides inviter anonymity in F^R_{POIC} hybrid model (as defined in Equation 40), assuming that PRG is a secure pseudo-random number generator.*

Proof: We reduce the inviter anonymity of *Anonyma* to the security of the employed *PRG*. If there exists a PPT adversary A who breaks the inviter anonymity of *Anonyma* with non-negligible advantage, then we can construct a PPT adversary B who distinguishes between a random number generator and a pseudo-random number generator with the same advantage as A. Assume A's success probability is

$$Pr[\mathsf{InvAnonym}_{\mathsf{A}}(\lambda) = 1] = \frac{1}{2} + \epsilon(\lambda) \tag{22}$$

B runs A as its subroutine to distinguish the pseudo-random number generator from the truly random number generator. B is given a vector of values in Z_q denoted by $\overrightarrow{\delta} = (\delta', \delta'')$ and aims at specifying whether $\overrightarrow{\delta}$ is selected truly at random or is the output of a *PRG*. B invokes A as its subroutine and emulates the game of inviter anonymity for A as follows. If A succeeds then B realizes that $\overrightarrow{\delta}$ is pseudo-random, otherwise random.

(1) B is given the security parameter 1^{λ} and a vector of two values denoted by $\overrightarrow{\delta} = \{\delta', \delta''\}$ s.t. $\delta', \delta'' \in Z_q$. Adversary A outputs $Param$ including ek, vk, and $(F_0, \cdots, F_{t-1})$.
(2) A registers its own users. U_c contains the indices of corrupted members. Also, A instructs B to register users into the system. Let U_h indicate the set of indices registered by B. For each

user $i \in U_h$, B obtains a share s_i and verifies its correctness by checking whether g^{s_i} is equal to $\prod_{j=0}^{t-1} F_j^{(i^j)}$.

(3)(a) A outputs two indices $u_0, u_1 \in U_h$.

(b) A outputs a token $Token = (\eta, j^*, \omega)$. j^* is the index of the invitee in U_h.

(c) A specifies a set of $t-1$ incides $I_c \subset U_c$ to be the corrupted inviters. For every $i \in I_c$, and $Token$, A engages in $Igen$ with the callenger. A outputs $Inv_i = (\tau_i, e\delta_i)$ and contacts F^R_{POIC} with the input of $(\tau_i, e\delta_i, \gamma_i, \omega)(s_i, r_i, \delta_i)$. B acting as F^R_{POIC}, accepts or rejects A's proof by verifying whether $(\tau_i, e\delta_i, \gamma_i, \omega)$ and (s_i, r_i, δ_i) fit into the relation R as defined in Equation 9. Note that at this step B can learn all the $t-1$ master shares of corrupted inviters, i.e., $\{s_i\}$ for $i \in I_c$.

(4)(a) B selects a random bit b. B uses the $Token$ and runs $Igen$ to create an invitation letter from u_b as $Inv_{u_b} = (\tau_{u_b}, e\delta_{u_b}) = (\omega^{s_{u_b}+\delta'}, Enc_{ek}(\omega^{\delta'}))$ (δ' is given from the distinguish-ability game of PRG)).

(b) B runs $Icoll$ over $\{Inv_i\}_{i\in I_c} \cup Inv_b$, sets $\delta^* = \delta''$ and computes
$T = \omega^{\delta^*} \cdot \tau_{u_b}^{B_{u_b}} \cdot \prod_{i \in I_c} \tau_i^{B_i}$
and
$e\Delta = Enc_{ek}(\omega^{\delta^*}) \cdot Enc_{ek}(\omega^{\delta'})^{B_{u_b}} \cdot \prod_{i\in I_c} e\delta_i^{B_i}$.
The value of $e\Delta$ will be equal to $Enc_{ek}(\omega^{\delta^*+\delta'\cdot B_{u_b}+\sum_{i\in I_c}\delta_i \cdot B_i})$. B_i and B_{u_b} denote the Lagrange coefficients as defined in Equation 4. B submits $InvLet = (T, e\Delta)$ to the adversary A.

(5) A outputs a bit b'.

(6) If $b = b'$ then B outputs 0, otherwise 1.

Note that B follows all the steps as indicated in the $\mathsf{InvAnonym}_\mathsf{A}(\lambda)$ game and hence is indistinguishable from a real challenger. This means that B also runs in polynomial time (as there is no rewind). B ties the $\mathsf{InvAnonym}_\mathsf{A}(\lambda)$ game to the security of PRG by embedding δ' and δ'' (the challenge of PRG game) as the randomness δ_{u_b} (used by the non-colluding inviter for the invitation generation), and the value of δ^* (used by the invitee in $Icoll$ execution), respectively. Below is the success probability analysis of the reduction.

Let $\overrightarrow{\delta}$ be a truly random vector. Once the adversary decrypts $e\Delta$ he obtains

$$\omega^{\delta''+\Gamma}$$

where

$$\Gamma = \delta' \cdot B_{u_b} + \sum_{i=1}^{t-1} \delta_i \cdot B_i$$

Γ is a function of inviters indices due to the presence of Lagrange coefficients whereas δ'' is a random value completely independent of inviters' indices. If $\overrightarrow{\delta}$ is a random vector then δ'' is also a random value from $\mathbb{Z}_q$. Therefore, in $\omega^{\delta''+\Gamma}$, Γ is indeed masked with δ'' ($\delta'' + \Gamma \mod q$ is a completely random element of $\mathbb{Z}_q$). By this masking, Δ (i.e., $\delta''+\Gamma$) becomes completely independent of the Lagrange coefficients and A has no advantage to infer the identity of the uncorrupted inviter. Thus, A's advantage is exactly $\frac{1}{2}$ i.e.,

$$Pr[B(\overrightarrow{\delta} \leftarrow Z_q) = 1] = Pr[b = b'] = \frac{1}{2} \tag{23}$$

but if $\overrightarrow{\delta}$ is the output of a *PRG* then

$$Pr[r \leftarrow \{0,1\}^\lambda : B(\overrightarrow{\delta} = PRG(r)) = 1] = Pr[b = b'] = \frac{1}{2} + \epsilon(\lambda) \tag{24}$$

where $\frac{1}{2}+\epsilon(\lambda)$ is the success probability of A (as assumed in our proof in Equation 22). By combining Equations 23 and 24 we have

$$|Pr[r \leftarrow \{0,1\}^{\lambda} : B(\overrightarrow{\delta} = PRG(r)) = 1] - Pr[B(\overrightarrow{\delta} \leftarrow Z_q) = 1]| = \epsilon(\lambda) \tag{25}$$

Equation 25 corresponds to the security definition of *PRG* (see Equation 1). Thus, if $\epsilon(\lambda)$ is non-negligible, then the distinguisher B can distinguish a *PRG* from a random generator. This contradicts with the security definition of *PRG*. Therefore, $\epsilon(\lambda)$ must be negligible according to the *PRG* definition. This concludes the security proof of inviter anonymity of *Anonyma*. ■

6.2 Invitation Unforgeability

In an invitation-based system, the invitation unforgeability indicates that people who do not have enough inviters (less than t) should not be able to join the system. Hence, no adversary can forge invitations on his own. Unforgeability results from the fulfillment of the following properties, namely, the *invitation non-exchangability* and *prevention of double-invitations*. The former means that invitations are bound to their invitee and are not exchangable from one to another while the latter indicates that an inviter is only able to make one single (and not more) valid invitation for each invitee. Next, we present a formal definition for invitation unforgeability, as well as the said constituent properties, together with a formal security proof of *Anonyma*.

6.2.1 Security Definition: We define the following game denoted by $InvUnforge_A(\lambda)$ running between a challenger and an adversary. The adversary controls a set of $t-1$ members who can adaptively register with the system. The rest of the users denoted by I_h are controlled by the challenger. Also, the adversary has oracle access to the token generation $Tgen(sk, n)$, invitation generation $Igen(., s_i, Param)$ for $i \in I_h$, and invitation verification $Ivrfy(., ., Param, dk)$ algorithms. Finally, the adversary wins the game if it manages to register to the system successfully, using a token that was not queried from the invitation generation oracle. The success of the adversary asserts that the invitations are forgeable. Otherwise, the system provides invitation unforgeability. We remark that by giving the adversary oracle access to the invitation generation algorithm we aim to capture the *non-exchangability* of invitations. This oracle access is equivalent to having an adversary who eavesdrops on the communication of other invitees and inviters and wishes to forge an invitation over its token. The *prevention of double-invitations* is modeled in the fact that the adversary has the control of $t-1$ inviters and can internally execute their invitation generation function any number of times to possibly make a double-invitation.

Invitation Unforgeability experiment $InvUnforge_A(\lambda)$:

(1) The challenger runs the setup algorithm and outputs $Param$ to the adversary. **Learning phase:** The next steps (2-5) are the learning phase of the adversary and can be run in an arbitrary order.

(2)(a) The adversary registers $t-1$ users to the system adaptively, at any phase of the game, via Reg protocol. I_c denotes the index set of registered corrupted members.

(b) The adversary instructs the challenger to register honest users to the system. I_h contains the index of registered honest members.

(3) The adversary asks the challenger to issue a token, polynomially many times. Q^{Token} holds the set of tokens queried by the adversary.

(4) The adversary queries invitation verification function on the invitations of his own choice. The challenger runs the $Ivrfy$ algorithm and responds accordingly. This step can be repeated polynomially many times.

(5) The adversary has oracle access to the $Igen$ algorithm. That is, the adversary asks the challenger to use a particular token and generate an invitation from an honest member. As such, the adversary specifies the index $i \in I_h$ of an honest member together with a valid $Token_j \in Q^{Token}$. Then, the challenger issues an individual invitation by running $Inv_{i,j} = Igen(Token_j, s_i, Param)$ and gives the output to the adversary. Let $Q^{Inv} = \{(Token_j, Inv_{i,j})\}$ be the set of tokens together with the individual invitations queried by the adversary. This step can be repeated polynomially many times.

Challenge phase:

(6) The adversary outputs an invitation letter $InvLet$ for a valid token $Token' \in Q^{Token}$ for which no query exists in Q^{Inv}.

(7) If the output of $Ivrfy(InvLet, Token', Param, dk)$ is accepted then the game's output is 1 indicating the adversary's success, 0 otherwise.

Definition 6.3. An invitation-based system has invitation unforgeability if for every probabilistic polynomial time adversary A there exists a negligible function $negl(.)$ such that:

$$Pr[InvUnforge_A(\lambda) = 1] = negl(\lambda)$$

At a high level, in order for the adversary to be able to win the game, it has to compute ω^{*S} for some token $Token' = (\tau, j, \omega^*)$ where $Token'$ does not belong to Q^{Inv}. Through the oracle accesses, the adversary learns a set of individual invitations $Q^{Inv} = \{(Token_j, Inv_{i,j})\}$ where $Inv_{i,j} = (\tau_{i,j} = \omega_j^{s_i+\delta_{i,j}}, e\delta_{i,j} = Enc_{ek}(\omega_j^{\delta_{i,j}}))$. The $Inv_{i,j}$ carries no useful information regarding the master value S to the adversary as the master share s_i is masked through a random value $\delta_{i,j}$. There is no way for the adversary to get to learn $\delta_{i,j}$ unless with decryption of $e\delta_{i,j}$ which is not possible as the adversary lacks the decryption key dk. Alternatively, the adversary may attempt to combine invitations issued under different tokens to obtain a valid invitation under a new token $Token'$. This is impossible due to the CDH problem. That is, given $\tau_{i,j}$ $(= \omega_j^{s_i+\delta_{i,j}})$ and $\omega^*(= g^x)$, the adversary must compute $\tau_{i,j}^x$ which corresponds to a solution to the CDH problem. Similarly, the knowledge of ω^* and $F_0 = g^S$ (from $Param$) does not help in making a valid invitation letter since computing ω^{*S} is equivalent to solving the CDH problem. That is, given $\omega^* = g^x$ and $F_0 = g^S$, the adversary shall compute $g^{x \cdot S} = \omega^S$. In the theorem and proof given below, we reduce $InvUnforge_A(\lambda)$ game to the CDH problem.

Theorem 6.4. *Anonyma satisfies invitation unforgeability as defined in Definition 6.3, in F^R_{POIC} hybrid model, given that the signature scheme Sig is existentially unforgeable under chosen message* *attack, the encryption scheme π is CPA secure, and Computational Diffie-Hellman problem is hard relative to group G.*

Proof: If there exists a PPT adversary A who breaks the invitation unforgeability with the non-negligible advantage then we can construct a PPT adversary B who solves the CDH problem with non-negligible advantage.

Let ϵ denote the probability of success of A. B interacts with the CDH challenger and also runs A as its subroutine. B is given $G, q, g, X = g^x, Y = g^y \in G$ for which B is supposed to compute $Z = g^{x \cdot y}$.

(1) B runs the setup algorithm and generates the encryption and signature key pairs (ek, dk) (sk, vk) as normal. To set up Shamir secret sharing scheme, B performs as follows. B sets $F_0 = Y$ (recall that F_0 is the commitment to master value S thus $F_0 = g^{f(0)} = g^S$; this implies that B does not know the master value S since it is the discrete logarithm of Y (i.e., y), which is selected by the CDH challenger). B selects a set denoted by I_c consisting of $t-1$ random indices to be the indices of the members that are to be registered by the adversary. That is, whenever the adversary attempts to register a corrupted member via *Reg* protocol, B uses one of the indices in the I_c for the registration. This is only needed for B to prepare the system parameters i.e., commitments to the Shamir coefficients, however, this act is not visible to the adversary, hence adversary's view would remain indistinguishable from a real game, as the index of the members it registers to the system are still random (this aligns with the original protocol in which indices are random values calculated as the hash of the total number of issued tokens). B selects $t-1$ random values $s_{i \in I_c} \leftarrow Z_q$ to be the master shares of the corrupted members. Also, B computes $\gamma_i = g^{s_i}$ for $i \in I_c$. Recall that the share of the master value for the i^{th} user is $f(i)$, thus by setting the master shares of corrupted parties, B fixes $t-1$ points of polynomial f as $f(i) = s_i$ for $i \in I_c$. These $t-1$ points together with F_0, which is indeed $g^{f(0)}$, will fix polynomial f since the degree of f is $t-1$. Next, B interpolates Y i.e., $(g^{f(0)})$ and $\{(i, \gamma_i)\}_{i \in I_c}$, and computes the commitments $F_1, \cdots, F_{t-1}$ (where $F_1 = g^{a_1}, \cdots, F_{t-1} = g^{a_{t-1}}$) over the coefficients of polynomial f [60] (where $f = S + a_1 \cdot x + ... + a_{t-1} \cdot x^{t-1}$).
Note that B does not obtain the exact coefficients of the polynomial f (i.e., a_i values) but only computes the commitments $F_i = g^{a_i}$. This is sufficient for B to simulate the role of the server since it only needs to publish the commitments of the polynomial and not the exact coefficients.
B outputs $param = (G, q, g, ek, vk, (F_0, ..., F_{t-1}))$, as well as the security parameter 1^λ. Note that B also records the master shares of corrupted members i.e., $\{(i, f(i))\}_{i \in I_c}$ to use in the registration phase.

(2)(a) A registers a corrupted user to the system. B picks one of the elements in I_c and its corresponding share which were both computed during the setup protocol to A.

(b) A instructs B to register an honest user to the system. Note that B cannot generate the master shares of honest users since it does not know the coefficients of the function f. However, since it is a local calculation for B, this shortage remains unnoticed to A. B assigns a random index to the honest user and adds it to the I_h.

(3) A has oracle access to the token generation $Tgen$. Initially, B draws a random value $q^* \in [1, P(\lambda)]$ where $P(\lambda)$ is the upper bound on the number of adversary's queries to $Tgen$. B answers the queries of A for $Tgen$ as follows. For the q^{*th} query, picks a random index j^* that does not belong to I_c and I_h, B sets $Token^* = (Sign_{sk}(j^* || X), j^*, X)$ (X was given to B from the CDH game) and inserts $(j^*, X, \perp)$ into Q^{Token}. Otherwise, B selects a random index

$j \notin I_c \cup I_h$, and a random $r_j \in_R Z_q$, sets $\omega_j = g^{r_j}$ and outputs $Token_j = (Sign_{sk}(j||\omega_j), j, \omega_j)$. B records (j, ω_j, r_j) inside Q^{Token}.

(4) The adversary queries the invitation verification function on the invitation letters and tokens of his own choice i.e., $InvLet = (T, e\Delta)$ and $Token = (\eta, j, \omega_j)$. B first authenticates the token against the signature verification key. If not verified, B outputs $reject$ to A, otherwise,
- If $Token == Token^*$, then B aborts the experiment.
- If $Token \neq Token^*$, B proceeds as follows. Due to the lack of master value S, B has to run different than the normal $Ivrfy$ algorithm. B decrypts $e\Delta$ as $\omega_j^{\Delta} = Dec_{dk}(e\Delta)$. Next, B retrieves the record of (j, ω_j, r_j) corresponding to ω_j from Q^{Token} and checks whether $F_0^{r_j} \cdot \omega_j^{\Delta} \overset{?}{=} T$ and responds to A accordingly. The right side of this equality check is the same as line 3 of $Ivrfy$ Algorithm (Algorithm 3.4) since

$$F_0^{r_j} \cdot \omega_j^{\Delta} = g^{S \cdot r_j} \cdot \omega_j^{\Delta} = g^{r_j \cdot S} \cdot \omega_j^{\Delta} = \omega_j^{S} \cdot \omega_j^{\Delta} \tag{26}$$

(5) A has oracle access to $Igen$ algorithm. A outputs an index i of an honest member together with a $Token_j = (\eta, j, \omega_j)$. B first authenticates the token against the signature verification key. If successful, then, it attempts to issue an invitation. Notice that B cannot generate the invitation by following $Igen$ since it does not have the master share of honest users i.e., s_i for $i \in I_h$. B performs differently to compute a valid invitation as explained next. B retrieves the record (j, ω_j, r_j) from Q^{Token} (if the token is valid and has a correct signature from the server then it must be already queried by the adversary and hence should exist in Q^{Token}, otherwise, the signature forgery happens which is not possible due to the security of the underlying signature scheme). B computes $\gamma_i = \prod_{j=0}^{t} F_j^{i^j}$ as well as selects a random value $\delta_i \in_R Z_q$. Then, B constructs $\tau_{i,j} = \gamma_i^{r_j} \cdot g^{\delta_i \cdot r_j}$ where r_j is the discrete logarithm of ω_j in base g. It is immediate that $\tau_{i,j}$ is well-structured since

$$\tau_{i,j} = \gamma_i^{r_j} \cdot g^{\delta_i \cdot r_j} = (g^{s_i})^{r_j} \cdot (g^{\delta_i})^{\cdot r_j} = (g^{r_j})^{s_i} \cdot (g^{r_j})^{\delta_i} = \omega_j^{s_i} \cdot \omega_j^{\delta_i} = \omega_j^{s_i+\delta_i} \tag{27}$$

B constructs $e\delta_{i,j}$ as $Enc_{ek}(\omega_j^{\delta_i})$ and outputs $Inv_{i,j} = (\tau_{i,j}, e\delta_{i,j})$ to A.
Finally, B acts as F_{POIC}^{R} and waits for A's message asking verification of $(\tau_{i,j}, e\delta_{i,j}, \gamma_i, \omega_j)$ for which B responds $accept$ to A. B keeps the set of individual invitations and their tokens queried by A in $Q^{Inv} = \{(Token_j, Inv_{i,j})\}$.

(6) The adversary outputs an invitation letter $InvLet = (T, e\Delta)$ for a token $Token'$ for which no query has been made from $Igen$ i.e., $Token' \notin Q^{Inv}$.

(7) B verifies whether the $Token'$ is correctly signed under sk. If not, B outputs $\perp$ to CDH challenger. Otherwise:
- If $Token' \neq Token^*$, B outputs $\perp$ to the CDH game.
- If $Token' == Token^*$, B outputs $T \cdot Dec_{dk}(e\Delta)^{-1}$ to the CDH challenger. In fact, if A constructs $InvLet$ correctly, we expect that $T = {\omega^*}^{S+\Delta}$ and $e\Delta = Enc({\omega^*}^{\Delta})$. Given that $X = g^x = \omega^*$ and $Y = g^y = g^S$, we have

$$T \cdot Dec_{dk}(e\Delta)^{-1} = (\omega^*)^{S+\Delta} \cdot (\omega^*)^{\Delta^{-1}} = (\omega^*)^S = (g^x)^y = g^{xy} \tag{28}$$

$g^{x \cdot y}$ is the solution to the given CDH problem.

This is immediate that B runs in polynomial time. The index q^* chosen by B at step 3 represents a guess as to which $Tgen$ oracle query of A will correspond to the token of eventual invitation letter forgery output by A. If this guess is correct, then A's view while running with B is identical to $InvUnforge_A(\lambda)$ game.

When B guesses correctly and A outputs a forgery, then B can solve the given instance of CDH. Assume that A's advantage in $InvUnforge_A(\lambda)$ game is ϵ. The probability that B wins is

$$\begin{aligned} Pr[\text{B wins}] &= Pr[B(G, q, g, X = g^x, Y = g^y) = g^{x \cdot y}] \\ &= Pr[\text{A wins} \wedge (Token' = Token^*)] \\ &= Pr[\text{A wins } | Token' = Token^*] \cdot Pr[Token' = Token^*] \\ &\geq \epsilon \cdot \frac{1}{Poly(\lambda)} \end{aligned} \tag{29}$$

The last equality holds since the number of queries made by A is at most $P(\lambda)$ (P is polynomial in 1^λ), hence, the probability $Token^* = Token'$ is $\frac{1}{Poly(\lambda)}$. Note that due to the signature unforgeability, A cannot create a valid token outside of the set of queried tokens i.e., $\notin Q^{Token}$.

Assuming that ϵ is non-negligible, B also wins with non-negligible probability. This contradicts the hardness of the CDH problem. Hence A's success probability in $InvUnforge_A(\lambda)$ must be negligible. This concludes the proof. ■

6.3 Security of *AnonymaX*

6.3.1 Inviter Anonymity. The inviter anonymity of *AnonymaX* can be defined identically to the experiment of $\mathsf{InvAnonym}_A(\lambda)$. The challenger shall control the honest members, i.e. U_h and invitee whereas the adversary will have the control of the *authenticating* server AN and all the *registration* servers RN_j together with the corrupted members I_c which constitute $t-1$ inviters of the honest invitee. *AnonymaX* meets inviter anonymity due to the similar proof supplied for *Anonyma.* Without loss of generality and for the sake of simplicity, we consider only one *registration* server to exist, though the extension of proof for multiple *registration* servers is straightforward. In particular, the following theorem holds for *AnonymaX* with one *authenticating* server and one *registration* server.

THEOREM 6.5. *AnonymaX provides inviter anonymity in F^R_{POIC} hybrid model (as defined in Equation 40), assuming that the encryption scheme π is CPA-secure and the utilized PRG is a secure pseudo-random number generator.*

Proof Sketch: Given that a PPT adversary A' can break the inviter anonymity game for *AnonymaX* with non-negligible advantage, we can construct an adversary B' to distinguish between a PRG and a truly random number generator. The internal code of adversary B' shall be identical to the simulator B in proof of Theorem 6.2. The sole distinction lies in the *SetUp* phase, where the challenger generates two encryption keys, ek_{RN} and ek_{AN} for the *registration* and *authenticating* servers respectively. However, in the experiment, only ek_{RN} will be utilized.

6.3.2 Invitation Unforgeability. Invitation unforgeability guarantees that a corrupted invitee with an insufficient number of inviters would not be able to join the system. In a cross-network invitation-based system, the invitation unforgeability should additionally hold for the *registration* service. That is, if Alice does not have enough inviters from the *authenticating* system, she should not be able to successfully register to the *registration* service.

It is important to acknowledge that invitation unforgeability cannot be defined in a cross-network invitation-based system when AN acts against RN. In this scenario, AN can generate valid invitations for the invitee of its choice to join the *registration* service. This becomes trivial since AN can register an arbitrary number of users into its own system (*authenticating* system). Consequently, any subset of t registered users within the *authenticating* service will have the ability to issue invitations and register an arbitrary number of users into the *registration* service. It is essential to recognize that this limitation is not specific to our design but is inherent in any cross-network invitation-based system.

In the invitation unforgeability game as defined in $XInvUnforge_A(\lambda)$, we consider N *registration* servers which all accept invitations from the members of one *authenticating* server. The adversary plays on behalf of $t-1$ corrupted users of the *authenticating* service and a subset of *registration* servers. The challenger controls the honest users, i.e. I_h of the *authenticating* service together with AN and some of the uncorrupted *registration* servers. At the end of the game, the mission of the adversary as a corrupted invitee with an insufficient number of inviters is to successfully register to one of the honest *registration* servers RN_{j^*} controlled by the challenger.

Invitation Unforgeability experiment for cross-network invitation based system: In $XInvUnforge_A(\lambda)$, we index *registration* servers as RN_j, where $1 \leq j \leq N$, and the *authenticating* server as AN. The set of servers controlled by the adversary is denoted as set C. We assume H indicates the set of un-corrupted *registration* servers. The set of members of *authenticating* service is denoted by U_{AN}. I_c represents the set of $t-1$ corrupted members in the *authenticating service* whereas I_h contains the indices of the honest members. We have $U_{\mathsf{AN}} = I_h \cup I_c$. We prefix the algorithms with its executing entity, e.g. we write $\mathsf{RN}_j.Tgen$ to show the invocation of the token generation algorithm at the registration server j.

Invitation Unforgeability experiment $XInvUnforge_A(\lambda)$ for cross-network invitation based system:

(1) The challenger runs $Setup$ for all $\mathsf{RN}_j \in H$ and outputs $Param_{\mathsf{RN}_j}$ to the adversary. The adversary outputs $Param_{\mathsf{RN}_j}$ for $\mathsf{RN}_j \in C$.
Learning Phase: The next steps (2-5) are the learning phase of the adversary and can be run in an arbitrary order.

(2)(a) The adversary registers a corrupted user i to the *authenticating* system. The adversary repeats this part $t-1$ times. Let I_c denote the index of corrupted registered members.
(b) The adversary instructs the challenger to register an honest user i to the *authenticating* system where $i \in I_h$.

(3) The adversary has Oracle access to the $\mathsf{RN}_j.Tgen$ for $\mathsf{RN}_j \in H$. Also, the adversary generates a $Token$ for a user $i \in I_h$ from RN_j where $\mathsf{RN}_j \in C$ and hands it over to the challenger.

(4) The adversary has oracle access to $\mathsf{RN}_j.XIvrfy$ for $\mathsf{RN}_j \in H$.

(5) The adversary has oracle access to the $Igen$ algorithm. That is, the adversary specifies the index l of an honest member i.e., $l \in I_h$ and a server $\mathsf{RN}_j \in H \cup C$ together with a $Token$ issued by RN_j.
The challenger generates an individual invitation by running $Inv_l = Igen(Token, s_l, Param_j)$ and gives the output Inv_l to the adversary. Let $Q^{Inv}_{\mathsf{RN}_j} = \{(Token, Inv_l)\}$ be the set of tokens together with the individual invitations queried by the adversary to be generated by the l^{th} user for the j^{th} registration service.

(6) **Challenge Phase:** The adversary outputs an invitation letter $InvLet$ together with token $Token^{'}$ for the registration in RN_{j^*} server where $\mathsf{RN}_{j^*} \in H$. There should not be any issued invitation using $Token^{'}$ in $Q^{Inv}_{\mathsf{RN}_{j^*}}$.

(7) If the output of $XIvrfy(InvLet, Token^{'}, Param_{\mathsf{AN}}, Param_{\mathsf{RN}_{j^*}}, dk_{\mathsf{RN}_{j^*}})$ is accepted, then the game's output is 1 indicating the adversary's success, 0 otherwise.

Definition 6.6. A cross-network invitation-based system has invitation unforgeability if for every probabilistic polynomial time adversary A there exists a negligible function $negl(.)$ such that:

$$Pr[XInvUnforge_A(\lambda) = 1] = negl(\lambda)$$

THEOREM 6.7. *AnonymaX satisfies invitation unforgeability as defined in Definition 6.6, in F^R_{POIC} and F^R_{PODL} hybrid model, given that the signature scheme Sig is existentially unforgeable under chosen message attack, and Computational Diffie-Hellman problem is hard relative to group G.*

Proof Sketch, The reduction idea between CDH problem and $XInvUnforge_A(\lambda)$ of *AnonymaX* is similar to $InvUnforge_A(\lambda)$. However, in *AnonymaX*, the simulator B additionally can extract the CDH solution during the $Tgen$ and $XIvrfy$ which we explain below. B is given $X = g^x$ and $Y = g^y$ from the CDH challenger and sets Y as the commitment to the master value S. B also guesses at which query of $Tgen$ A will succeed to forge a valid $InvLet$. B sets the value ω of that token to X. B can solve the CDH challenge if

- A creates a token with the value of X for which A must prove the knowledge of the discrete logarithm x. Then B outputs Y^x as the CDH solution.
- The adversary A queries $XIvrfy$ with a valid invitation letter $InvLet$ over the token with $\omega = X$, then B extracts the CDH solution. The $InvLet$ is of the form $\omega^S = g^{xy}$ which is the solution to the CDH problem.
- A submits a valid invitation letter using the token X. That is of the form $\omega^S = g^{xy}$ which is the solution to the CDH problem.

A may also win by forging a token (i.e., a signature) on behalf of an honest *registration* server. However, since the signature scheme is secure, the probability of signature forgery is negligible.

Formal Proof: If there exists a PPT adversary A who breaks the invitation unforgeability of *AnonymaX* with non-negligible advantage, then we can construct a PPT adversary B who solves the CDH problem with non-negligible advantage.

Let ϵ denote the probability of success of A. B interacts with the CDH challenger and also runs A as its subroutine. B is given the security parameter 1^λ, $G, q, g, X = g^x, Y = g^y \in G$ for which B is supposed to compute Z s.t. $Z = g^{x \cdot y}$.

(1) For every $\mathsf{RN}_j \in H$, B runs the setup algorithm and generates the encryption and signature key pairs $(ek_{\mathsf{RN}_j}, dk_{\mathsf{RN}_j})$ $(sk_{\mathsf{RN}_j}, vk_{\mathsf{RN}_j})$ as normal. $Param_{\mathsf{RN}_j}$ will be $(ek_{\mathsf{RN}_j}, vk_{\mathsf{RN}_j})$.
Similarly, B sets up an encryption and signature key pairs for AN as $(ek_{\mathsf{AN}}, dk_{\mathsf{AN}})$ and $(sk_{\mathsf{AN}}, vk_{\mathsf{AN}})$, respectively. As for the initialization of the Shamir secret sharing scheme, B performs as follows. B sets $F_0 = Y$ (recall that F_0 is the commitment to master value S thus $F_0 = g^{f(0)} = g^S$; this implies that B does not know the master value S since it is the discrete logarithm of Y (i.e., y), which is selected by the CDH challenger). B selects $t-1$ random index i as the index of corrupted members and saves them in I_c. It also draws random values $s_{i \in I_c} \leftarrow Z_q$ to be the master shares of the corrupted members. Also, B computes $\gamma_i = g^{s_i}$ for $i \in I_c$. Recall that the share of the master value for the i^{th} user is $f(i)$, thus by setting the master shares of corrupted parties, B fixes $t-1$ points of polynomial f as $f(i) = s_i$ for $i \in I_c$. These $t-1$ points together with F_0, which is indeed $g^{f(0)}$, will fix polynomial f since the degree of f is $t-1$. Next, B interpolates Y i.e., $(g^{f(0)})$ and $\{(i, \gamma_i)\}_{i \in I_c}$, and computes the commitments $F_1, \cdots, F_{t-1}$ (where $F_1 = g^{a_1}, \cdots, F_{t-1} = g^{a_{t-1}}$) over the coefficients of polynomial f [60] (where $f = S + a_1 \cdot x + ... + a_{t-1} \cdot x^{t-1}$). Note that B does not obtain the exact coefficients of the polynomial f (i.e., a_i values) but only computes the commitments $F_i = g^{a_i}$. This is sufficient for B to simulate the role of the AN since it only needs to publish the commitments of the polynomial and not the exact coefficients.

B outputs $param = (G, q, g, ek_{\mathsf{AN}}, vk_{\mathsf{AN}}, (F_0, ..., F_{t-1}))$, as well as the security parameter 1^λ to the adversary. Note that B also records the master shares of corrupted members i.e., $\{(i, f(i))\}_{i \in I_c}$ to use in the registration phase.

(2)(a) A registers a corrupted user to the *authenticating* system. B selects an index i from I_c together with $f(i)$ (which were both computed during the *SetUp* protocol) and sends them to A. As the selection of the index i is random, it mimics the computation of a hash function and, hence is distinguishable from the real protocol execution in the adversary's view.

(b) A instructs B to register an honest user to the *authenticating* system. Note that B cannot generate the master shares of honest users since it does not know the coefficients of the function f. However, since it is a local calculation for B, this shortage remains unnoticed to A. B records the index of the honest user inside I_h.

(3) A has oracle access to token generation i.e., $\mathsf{AN}.Tgen$ and $\mathsf{RN}_j.Tgen \in H$. B keeps the set of tokens queried by A for each server inside Q^{Token}_{AN} or $Q^{Token}_{\mathsf{RN}_j}$. Initially, B draws random values $j^* \in [1, N]$ (to be the guess over the index of the honest *registration* server for which the adversary comes up with the invitation letter forgery) and $q^* \in [1, P(\lambda)]$ where $P(\lambda)$ is the upper-bound on the number of adversary's queries to $Tgen$ for each of the servers. B also picks a random index l^* where l^* is not in any prior queries to any of the *registration* servers $\mathsf{RN}_{j \in [1,N]}.Tgen$.

- If j is equal to j^*, and if this is the q^* query to $\mathsf{RN}_{j^*}.Tgen$ then B returns
$$Token^* = (Sign_{sk_{\mathsf{RN}_{j^*}}}(l^* || X), l^*, X)$$
X was given to B from the CDH game. B plays the role of F^R_{PODL}, receives the verification request of (G, q, g, X) from the adversary and outputs *accept* to the adversary. B inserts $(X, \perp)$ into $Q^{Token}_{j^*}$.
- If $j \neq j^*$ or this is not the q^* query to $\mathsf{RN}_{j^*}.Tgen$, and assuming this is the q^{th} query of adversary to $\mathsf{RN}_j.Tgen$, B selects a random index l and a random $r \in_R Z_q$, sets $\omega = g^r$ and outputs $Token = (Sign_{sk_{S_j}}(l || \omega), l, \omega)$. B plays the role of F^R_{PODL}, receives the verification request of (G, q, g, ω) from the adversary and outputs *accept* to the adversary. B inserts (ω, r) to $Q^{Token}_{\mathsf{RN}_j}$.

The adversary may generate a token $Token = (\eta, l, \omega)$ for a user $l \in I_h$ from $\mathsf{RN}_j \in C$ and hand it over to the challenger. The adversary contacts F^R_{PODL} i.e., the challenger B and hands over $((G, q, g, \omega), r)$. B checks whether $g^r = \omega$ and accepts or rejects the token accordingly. Also, B verifies the signature η against the verification key of RN_j and accepts or rejects the token accordingly. If the verification passed successfully, B stores (ω, r) in $Q^{Token}_{\mathsf{RN}_j}$. If $\omega == X$ (the CDH challenge), and the token is accepted, then B outputs Y^r to the CDH challenger.

(4) The adversary queries $\mathsf{RN}_j.XIvrfy(InvLet, Token, Param_{\mathsf{AN}}, dk_{\mathsf{RN}_j})$ for $\mathsf{RN}_j \in H$ on the invitation letters and tokens of his own choice i.e., $InvLet = (T, e\Delta)$ and $Token = (\eta, l, \omega)$. B runs the $XIvrfy$ algorithm and responds accordingly. If the output of $XIvrfy$ is not reject and if $j = j^*$ and $Token = Token^*$ (i.e., $\omega = X$), then B outputs $T \cdot Dec_{dk_{S_{j^*}}}(e\Delta)^{-1}$ to the CDH game.

$$T \cdot Dec_{dk_{S_{j^*}}}(e\Delta)^{-1} = X^{S+\Delta} \cdot X^{-\Delta} = X^S = g^{xS} = g^{xy} \tag{30}$$

(5) A has oracle access to $Igen$ algorithm. A asks the challenger to generate an invitation from the honest member i for the *registration* server $\mathsf{RN}_j \in H \cup C$ using a $Token = (\eta, l, \omega)$. B first authenticates the token against the signature verification key of RN_j. If not verified, B outputs $reject$ to A. Also, if $\omega = X$, then B aborts. Otherwise, B attempts to issue an invitation. Notice that B cannot generate the invitation by following $Igen$ since it does not have the master

share of honest users i.e., s_i for $i \in I_h$. B performs differently to compute a valid invitation as explained next. B computes $\gamma_i = \prod_{v=0}^{t} F_v^{i^v} = g^{s_i}$ (the second equality holds due to Equation 20) as well as selects a random value $\delta_i \in_R Z_q$. Then, B constructs $\tau_i = \gamma_i^r \cdot g^{\delta_i \cdot r}$ where r is the discrete logarithm of ω in base g. It is immediate that τ_i (to be the first component of the invitation letter) is well-structured since

$$\tau_i = \gamma_i^r \cdot g^{\delta_i \cdot r} = (g^{s_i})^r \cdot (g^{\delta_i})^{\cdot r} = (g^r)^{s_i} \cdot (g^r)^{\delta_i} = \omega^{s_i} \cdot \omega^{\delta_i} = \omega^{s_i + \delta_i} \tag{31}$$

B constructs $e\delta_i$ as $Enc_{ek_{\mathsf{RN}_j}}(\omega^{\delta_i})$ and outputs $Inv_i = (\tau_i, e\delta_i)$ to A.

Finally, B acts as F_{POIC}^R and waits for A's message asking verification of $(\tau_i, e\delta_i, \gamma_i, \omega)$ for which B responds *accept* to A. B keeps the set of individual invitations and their tokens queried by A for each server RN_j in $Q_{\mathsf{RN}_j}^{Inv} = \{(Inv_i, Token)\}$.

(6) The adversary outputs an invitation letter $InvLet = (T, e\Delta)$ for a valid token $Token'$ issued by $\mathsf{RN}_{j'} \in H$ i.e., $Token' \in Q_{\mathsf{RN}_j'}^{Token}$ for which no query has been made from $\mathsf{RN}_{j'}.Igen$ i.e., $Token' \notin Q_{\mathsf{RN}_{j'}}^{Inv}$.

(7) B verifies whether the $Token'$ is correctly signed under $sk_{\mathsf{RN}_j'}$. If not, B outputs $\perp$ to the CDH challenger. Otherwise:
- If $j' \neq j^*$ or $Token' \neq Token^*$ B outputs $\perp$ to the CDH game.
- If $j' = j^*$ and $Token' = Token^*$, B outputs $T \cdot Dec_{dk_{\mathsf{RN}_{j^*}}}(e\Delta)^{-1}$ to the CDH challenger. In fact, if A constructs $InvLet$ correctly, we expect that $T = X^{S+\Delta}$ and $e\Delta = Enc_{ek_{\mathsf{RN}_{j^*}}}(X^{\Delta})$. Given that $X = g^x$ and $Y = g^y = g^S$, we have

$$T \cdot Dec_{dk_{\mathsf{RN}_{j^*}}}(e\Delta)^{-1} = (X)^{S+\Delta} \cdot (X)^{\Delta^{-1}} = (X)^S = (g^x)^y = g^{xy} \tag{32}$$

$g^{x \cdot y}$ is the solution to the given CDH problem.

This is immediate that B runs in polynomial time. In step 3, B predicts the token for which the adversary, A, will create a forged invitation letter to win the game. This prediction consists of a server index j^* and a query index q^*, representing the server RN_{j^*} and the corresponding query index for $\mathsf{RN}_{j^*}.Tgen$. If B's guess is accurate, then the view of A during their interaction with B will be indistinguishable from the $XInvUnforge_A(\lambda)$ game.

When B guesses correctly and A outputs a forgery, then B can solve the given instance of CDH. Assume that A's advantage in $XInvUnforge_A(\lambda)$ game is ϵ. The probability that B wins is

$$\begin{aligned} Pr[\text{B wins}] &= Pr[B(G, q, g, X = g^x, Y = g^y) = g^{x \cdot y}] \\ &= Pr[\text{A wins} \wedge (Token' = Token^* \text{ AND } j' = j^*)] \\ &= Pr[\text{A wins } | Token' = Token^* \text{ AND } j' = j^*] \cdot Pr[Token' = Token^* \text{ AND } j' = j^*] \\ &\geq \epsilon \cdot \frac{1}{Poly(\lambda)} \cdot \frac{1}{N} \end{aligned} \tag{33}$$

The last equality holds since the number of queries made by A is at most $Poly(\lambda)$ (i.e., polynomial in λ), and there are N *registration* servers (honest and corrupted), hence, the probability $Token^* = Token'$ and $j' = j^*$ is at least $\frac{1}{Poly(\lambda)} \cdot \frac{1}{N}$.

A may attempt to forge a token on behalf of $\mathsf{RN}_{j'}$ for which it has obtained an individual invitation from an honest user for the registration in one of the corrupted *registration* servers. However, due to the signature unforgeability, A cannot create a valid token outside of the set of queried tokens i.e., $\notin Q_{\mathsf{RN}_j}^{Token}$ for all $\mathsf{RN}_j \in H$. Also, all the queries to $Igen(Token = (\eta, i, \omega), s_l, Param_{\mathsf{RN}_j})$ where $l \in I_h$ and $\mathsf{RN}_j \in C$ are answered if the given $Token$ is generated by the corrupted server RN_j correctly i.e., $Token \in Q^{Tgen}_{\mathsf{RN}_j}$ which means that the adversary has passed ZKPODL successfully (knows the DL of the ω). The presence of zero-knowledge proof will prevent the adversary from using a token of an honest server since the adversary does not know the DL of ω due to the hardness of the discrete logarithm assumption. Without ZKPODL, the adversary can win the $XInvUnforge_A(\lambda)$ [9].

Assuming that ϵ is non-negligible, B also wins with non-negligible probability. This contradicts the hardness of the CDH problem. Hence A's success probability in $InvUnforge_A(\lambda)$ must be negligible. This concludes the proof. ■

7 RELATED WORK

In this section, we explore relevant studies that closely resemble invitation-only registration systems and specifically address the confidentiality of the relationship between the inviter and the invitee, as well as invitation unforgeability. We also highlight their limitations when applied to anonymous invitation-based systems.

7.1 Group Signatures and Linkable Group Signatures

In group signature schemes [3, 7, 8, 15–17, 25, 27, 33, 45, 64], a group of users is granted signature power by a group authority in such a way that any member of the group can anonymously sign on behalf of the group. The group authority also holds enough authority to revoke user anonymity [3, 53]. One can consider the adoption of group signatures for the invitation-based system in which the signers are the inviters and the signature corresponds to an invitation. However, such adoption does not preserve invitation unforgeability [25]. In specific, there is no limit on the number of signatures (invitations) that one (inviter) can generate for an invitee hence no invitation unforgeability is guaranteed. Furthermore, the traceability of signatures by the group authority [8, 16, 33, 64] is against the anonymity requirement of invitation-based systems. In the dynamic variant of group signatures, the authority's power in anonymity revocation is delegated to a separate entity called tracer [4]. However, this setting still demands trust assumption on the tracer which is not desirable for the signer/inviter anonymity.

7.2 Selective-Disclosure Credentials with Threshold Issuance

Selective-disclosure credentials with threshold issuance [62] consist of a set of distributed and mutually un-trusted authorities who can jointly issue credentials on users' private/public attributes. Users later can use their credentials to anonymously reveal the possession of certain attributes to verifiers. Credentials enjoy unforgeability in the face of a small subset of corrupted issuing authorities. Moreover, users can randomize their credentials and protect their attributes against the conspiracy of the verifier and all the authorities. One can adopt such a scheme to enable an invitation-based system where the issuing authorities correspond to the inviters, the user is the invitee, the issued credentials constitute invitations, and the verifier has the role of the group administrator. While such integration achieves invitation-unforgeability, it falls short in preserving the anonymity of the inviter-invitee relationship. Notice that the notion of anonymity in selective-disclosure credential [62] concerns the attribute anonymity (i.e., being able to anonymously prove the possession of an attribute without explicitly revealing the credentials) whereas the concept of

[9]A takes ω from one of the tokens $Token = (\eta, l, \omega)$ in $Q^{Tgen}_{\mathsf{RN}_{j^*}}$ and then generates a valid token $Token'' = (Sign_{sk_{\mathsf{RN}_j}}(\omega), i, \omega)$ at step 4 from a corrupted server $\mathsf{RN}_j \in C$. Next, A queries $Igen(Token'', s_l, Param_{\mathsf{RN}_j})$ for $l \in I_h$ and obtains a valid invitation $Inv_l = (\tau_l, e\delta_l)$. Given Inv_l and $dk_{sk_{\mathsf{RN}_j}}$, the adversary would be able to open $e\delta_l$ to ω^{δ_l} hence can construct its t^{th} valid individual invitation as $Inv_t = (\tau_l, Enc_{ek_{\mathsf{RN}_{j^*}}}(\omega^{\delta_l}))$ for a $Token = (\eta, l, \omega) \in Q^{Tgen}_{\mathsf{RN}_{j^*}}$. The adversary combines Inv_t with $t-1$ invitations issued by the $t-1$ corrupted inviters under its control and hands over an intact $InvLet$ to B.

anonymity in an invitation-based system regards inviter-invitee relationship, i.e., to provably ensure that no verifier (i.e., the group administrator in the invitation-based scenario) can collude with a subset of issuing authorities (i.e., colluding inviters) and identify other non-colluding authorities (i.e., verifiers) involved in the credential issuance. The latter definition has no clear and provable measurement in the selective-disclosure credential schemes as it is an orthogonal and irrelevant concern in this context. This concern is relevant to all selective disclosure credential schemes like [15, 22] and [62] regardless of their issuance type, whether they are centralized or decentralized.

7.3 Electronic Voting (e-voting)

Electronic voting systems consist of a set of voters, some candidates to be voted, and one/multiple authorities that handle tallying. An e-voting system must ensure that only authorized users participate in the voting and that each voter casts only one vote. More importantly, the content of the individual votes must be kept private, i.e., no vote can be traced back to its voter. In the literature, this property is known as vote privacy, anonymity, and untraceability. E-voting techniques are similar to anonymous invitation-only systems in many aspects. The role of voters is analogous to the inviters. Each round of the election with the Yes/No votes for a candidate can be treated as inviters casting their invitations for the registration of a newcomer. A Yes vote indicates inviting the candidate/newcomer and a *No* vote implies not inviting. Preserving the privacy of the vote is equivalent to inviter anonymity. Likewise, the prevention of double voting resembles invitation unforgeability.

Despite the aforementioned similarities, e-voting proposals fall short in satisfying inviter anonymity, invitation unforgeability, and scalability simultaneously. To illustrate this incompatibility, we first classify the e-voting techniques into two main categories: 1- explicit vote casting, 2-anonymous vote casting. Under each category, we identify the subtleties of transplanting the e-voting solution into invitation-only systems.

(1) Explicit vote casting: The voter authenticates himself to the authorities explicitly and immediately casts his private ballot. The ballot is shielded using either a threshold encryption scheme whose decryption key is divided between multiple authorities [43, 58], or secret sharing schemes where multiple authorities obtain one share of the ballot [51]. Before tallying the votes, the identifiable information shall be removed from the individual votes either by shuffling them through mix-net [26] or by homomorphically aggregating them [51]. In the context of the invitation-only system, this type of proposal has performance problems. That is, to preserve the inviter's anonymity (namely, hiding the identity of voters with the Yes vote), all the members should participate in the voting (including those who will cast a No vote). Otherwise, the real inviters will be revealed to the voting authorities. This imposes an unnecessary load on the non-inviters (voters with *No* votes). In contrast, in *Anonyma*, the entire invitation procedure is carried out only by the invitee and his inviters.
(2) Anonymous vote casting: This technique relies on one-time pseudonyms together with an anonymous communication channel. A voter hands over their credential (e.g., social security number – SSN) to the voting authority. Then, through a blind signature scheme, the voting authority issues a signature on the voter's pseudonym (that is also bonded to the voter's SSN). The pseudonym is a one-time value and untraceable to the real identity, i.e., SSN. Later on, a voter casts a vote under his pseudonym and via an anonymous communication channel to the voting authority. Voters attempting to vote twice will have to risk the disclosure of their real identities (i.e., SSN) [5, 54]. When we integrate this solution to the invitation-only system, the main problem is that the pseudonyms are one-time hence a user cannot use the same pseudonym for multiple elections (i.e., to invite different users). Otherwise, his

identity will be disclosed. To cope with this issue, upon the arrival of each newcomer, the authority has to issue new pseudonyms for all the existing members to enable them to act as the inviter for the newcomer (regardless of whether they are the inviters of the newcomer or not). This is certainly not an efficient solution as the load of the authority scales linearly with the number of joining members. Moreover, all the existing members also have to work linearly in the number of joining users. Alternatively, the authority should issue multiple pseudonyms (instead of one) for each member. However, this is not clear how to ensure that an inviter will only be able to use one pseudonym for each newcomer. In other words, the inviter should not be able to use all of his pseudonyms to make t valid invitations for just a single invitee since otherwise it would violate the invitation unforgeability (in which the invitations must be issued by t *distinct* inviters).

7.4 (t,N) Threshold Ring Signature

Ring signature schemes, first introduced by [55], are variants of group signature schemes without any group manager and with unconditional user anonymity. Users are not fixed to a specific group rather a signer can blend with an ad-hoc group of users to hide its identity when signing a message. To do so, the signer utilizes a set of users' public keys, without their knowledge, and creates a ring from those keys during the signing procedure. The user then sends the message along with the ring, to the verifier who can verify that one of the ring members generated it [1].

In a (t, N) threshold ring signature scheme [10, 12, 41, 46, 49, 52], the signature of t members of the group are needed to create a valid signature on a message. This fact can be verified by the signature verifier as well.

An invitation-based system can be instantiated from a threshold ring signature, assuming that the signers are the inviters and that a valid signature constitutes a valid invitation for a newcomer. The important shortcoming of such schemes is that their running time complexity for the generation of an invitation is at least linearly dependent on the size of the system, i.e., the total number of existing members [10, 12, 46, 49]. In some other cases, the dependency is exponential [41]. The same issue applies to the length of the signature (invitation letter) as it is comprised of $O(N)$ group elements where N is the total number of existing members. This considerably degrades the system's performance. In contrast, the performance of *Anonyma* is only influenced by the threshold of t and is independent of the size of the system. That is, the invitation generation complexity is $O(t)$, and the invitation verification is done in $O(1)$. Also, the invitation length is $O(1)$.

7.5 Direct Anonymous Attestation

The Direct Anonymous Attestation (DAA) scheme [13] can be seen as a group signature scheme without the revocable anonymity feature that allows users to decide whether a signature should be linkable to another signature. DAA can detect if a signature was produced using a known key, and it is used as a method for remote authentication of a hardware module called Trusted Platform Module (TPM) to a remote third-party verifier. At a high level, the hardware module obtains the issuer's signature on a secret message using a two-party protocol, and the signature is called attestation. Later, the module utilizes zero-knowledge proof to prove possession of a valid signature from the issuer to a verifier, where the proof may rely on randomness provided by the verifier.

The DAA scheme shares similarities with an invitation-based system, where the issuer acts as the group admin, the TPM constitutes an inviter, and an invitee acts as an intermediary to collect proof of possession of attestation from inviters and hands them to the group admin. However, the DAA scheme [13] does not fully align with *Anonyma*'s system model where a DAA scheme requires that the issuer and verifier be separate entities to maintain TPM/invitee anonymity i.e., to ensure that no single entity has access to both the identifying and verifying information, making it more difficult to de-anonymize the TPM. This makes DAA not applicable to anonymous invitation-based scenarios where the issuer and verifier are controlled by the same entity i.e. the group administrator. Additionally, the size of the invitation letter when using DAA is linear in the invitation threshold, whereas in *Anonyma*, the invitation letter is a constant size.

7.6 Delegatable Anonymous Credentials

Anonymous Credentials (AC) systems [14, 15, 28, 39] enable users to demonstrate ownership of attributes stored in a credential without disclosing additional information. Delegatable Anonymous Credentials (DAS) systems [2, 19–21, 29, 34, 50], a more advanced form of AC, allow credential holders to delegate their credentials to other users, mimicking the hierarchies frequently seen in public-key infrastructures (PKI). DACs offer greater privacy protection than standard AC systems because they conceal the identities of issuers and delegators. Even when all other user information is protected, an issuer's identity can potentially reveal information about the user's identity. In a DAC system, a root credential issuer generates credentials for requesting users based on various attributes. These credentials are linked to the receiver's public key, and the knowledge of the corresponding secret key is crucial for the zero-knowledge proof of ownership. However, the system permits users to substitute their embedded public keys in the credentials with the delegatee's public key, enabling another user to acquire possession of the credential. Credential verifiers need the root issuer's public key and the credential owner's public key (not the delegator's public key) to verify the ownership of the credentials claimed by a user.

When creating an anonymous invitation-based system with a DAC, the group administrator assumes the roles of the root issuer and the credential verifier. Inviters or existing users receive credentials from the group administrator, which they can delegate to invite other users. However, a drawback of this design is that invitees with insufficient invitations may exchange their credentials to meet the required invitation threshold, thereby compromising the invitation-based system's invitation unforgeability (which differs from the unforgeability property of DACs, which ensures users cannot generate credentials not originally issued by the root issuer). This can be achieved using the delegation property of credentials in DACs. For example, a user with only one invitation could collude with another invitee who possesses $t - 1$ credentials, and both parties could delegate their credentials to each other to achieve t valid invitations for each of them individually. It's important to note that delegation doesn't void the possession of the previous owner but instead extends it to the new user.

8 CONCLUSION

Anonyma is a secure and anonymous invitation-based system that provides inviter anonymity and invitation unforgeability. These properties are rigorously defined and proven in the context of a malicious and adaptive adversarial model. Inviter anonymity in *Anonyma* relies on the security of the underlying pseudo-random number generator, while invitation unforgeability relies on the computational Diffie-Hellman assumption.

Anonyma is designed for efficiency, with constant computational overhead for inviters and the system administrator, and an overhead of $O(t)$ for the invitee. Invitations in *Anonyma* are short and of constant size, containing only two group elements. The design also features standalone protocols for invitation generation, aggregation, and verification. These protocols are non-interactive and executed locally by each entity (i.e., inviter, invitee, and group administrator) without the need for interaction with the other party. This asynchronous design is a vital feature for the feasibility of the design in practical deployment.

Anonyma also merits scalability, allowing the dynamic addition of new members without re-keying existing ones. Our asymptotic and numerical analysis proved that authorizing a new user incurs constant computation and communication complexity.

Additionally, a cross-network version of *Anonyma*, called *AnonymaX*, is proposed, enabling third-party authentication. *AnonymaX* mirrors the scalability and efficiency of *Anonyma*. Separate proofs are provided for invitation unforgeability and inviter anonymity in the cross-network version.

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A DEFINITIONS

Computational Indistinguishability: Let $X = \{(in, \lambda)\}_{in \in \{0,1\}^*, \lambda \in \mathbb{N}}$ and $Y = \{(a, \lambda)\}_{in \in \{0,1\}^*, \lambda \in \mathbb{N}}$ be two series of random variables which are indexed with in and λ where in is the input and λ is the security parameter. The two distributions are computationally indistinguishable i.e., $X \equiv_c Y$ if the following holds: $\forall$ D (a non-uniform polynomial-time distinguisher), $\exists$ a negligible function $negl(.)$ s.t. $\forall in \in \{0,1\}^*$ and $\forall \lambda \in \mathbb{N}$ [40]:

$$|Pr[D(X(in, \lambda)) = 1] - Pr[D(Y(in, \lambda)) = 1]| \leq negl(\lambda) \tag{34}$$

Secure Multi-Party Computation [35]: Consider function $F(in_1, ..., in_N)$ $= (f_1(in_1, ..., in_N), \cdots, f_N(in_1, ..., in_N))$ that receives inputs in_i from i^{th} party to whom delivers $f_i(in_1, ..., in_N)$. F shall be run by a trusted third party. We refer to such execution as the *IDEAL* world. Assume γ^F is a multi-party protocol that computes F. The execution of γ^F by the interaction of parties constitutes the *REAL* world. γ^F is said to securely realize F if the following holds. That is, for every PPT adversary A in protocol γ^F with auxiliary input $aux \in \{0,1\}^*$ and controlling parties specified in P_c, there exists a PPT simulator Sim for the ideal functionality F, that $\forall$ security parameter λ:

$$\{IDEAL_{F,Sim(aux),P_c}(in_1, ..., in_N, \lambda)\}\} \equiv_c \{REAL_{\gamma^F,A(aux),P_c}(in_1, ..., in_N, \lambda)\} \tag{35}$$

$IDEAL_{F,Sim(aux),P_c}(in_1, ..., in_N, \lambda)$ represents the output of parties in interaction with ideal functionality F while Sim is controlling parties specified in set P_c. Similarly, $REAL_{\gamma^F,A(aux),P_c}(in_1, ..., in_N, \lambda)$ asserts the output of the parties interacting in protocol γ^F.

Hybrid Model: Assume θ is a multiparty protocol that makes use of a sub-protocol γ^F. γ^F in turn securely realizes the ideal functionality F. The hybrid model allows proving the security of θ by replacing γ^F with F. As such, for any execution of γ^F in the proof, parties contact a trusted third party running the ideal functionality F. This would be called F-hybrid model [40].
Sigma protocol: A Σ protocol is a three rounds proof system (P, V) for a relation R which satisfies the following properties [40]:

- **Completeness:** An honest prover P holding a private input w, where $(x, w) \in R$, can always convince an honest verifier V.
- **Special soundness:** There exists a polynomial time machine A that for every pair of accepting transcripts (a, e, z) and (a, e', z') (where $e \neq e'$) of an statement x, A extracts witness w s.t. $(x, w) \in R$
- **Special honest verifier zero knowledge:** There exists a PPT machine S which given statement x and e can generate an accepting transcript (a, e, z) whose distribution is the same as the transcript of the real interaction of P and V. More formally, $\forall (x, w) \in R$ and $e \in \{0,1\}^t$

$$\{S(x, e)\} \equiv_c \{(P(x, w), V(x, e))\} \tag{36}$$

The output of simulator S is denoted by $\{S(x, e)\}$. $\{(P(x, w), V(x, e))\}$ indicates the output transcript of an execution between P (holding inputs x and w) and V (with inputs x and random tape e).

Zero-knowledge proof of knowledge from Σ protocols: Following the method given in [40, 56], it is proven that one can efficiently construct a zero-knowledge proof of knowledge (ZKPOK) system from any sigma protocol. We refer to [40] for more details of such construction. Applying this method on a Σ protocol Π (defined for the relation R) will result in construction that securely realizes the ideal functionality F_Π^R (defined in Equation 37) in the presence of malicious prover and verifier.

$$F_\Pi^R((x, w), x) = (\perp, R(x, w)) \tag{37}$$

where x refers to the statement whose correctness is to be proven and w indicates the witness. The ideal functionality F_Π^R that is run by a trusted party, receives a common input x from the prover and the verifier. Also, the prover inputs F with the private input w from the prover. F_Π^R outputs to the verifier whether x and w fit into the relation R.

B PROOF OF INVITATION CORRECTNESS

B.1 Soundness:

Consider two valid transcripts $(A, B = (B_1, B_2), C, e, Z_1, Z_2, Z_3)$ and $(A, B = (B_1, B_2), C, e^*, Z_1^*, Z_2^*, Z_3^*)$, where $e \neq e^*$, $Z_1 \neq Z_1^*$, $Z_2 \neq Z_2^*$, and $Z_3 \neq Z_3^*$, then we extract δ_i, r and s_i as explained below. Since both transcripts are accepting we have $A \cdot \gamma_i^e = g^{Z_1}$ and $A \cdot \gamma_i^{e^*} = g^{Z_1^*}$. We divide both sides of equality and obtain

$$g^{Z_1 - Z_1^*} = \gamma_i^{e-e^*} = g^{s_i \cdot (e-e^*)} (\bmod p) \tag{38}$$

Thus, $Z_1 - Z_1^* \equiv s_i \cdot (e - e^*)\ mod\ q$. It follows that $s_i = \frac{Z_1 - Z_1^*}{e-e^*}$. To extract δ_i and r we proceed as follows. We know that $B_1 \cdot e\delta_{i,1}^e = \omega^{Z_2} \cdot h^{Z_3}$ as well as $B_1 \cdot e\delta_{i,1}^{e^*} = \omega^{Z_2^*} \cdot h^{Z_3^*}$. Dividing both sides of equalities results in

$$\begin{aligned} e\delta_{i,1}^{e-e^*} &= \omega^{Z_2 - Z_2^*} \cdot h^{Z_3 - Z_3^*} (\bmod p) \\ \omega^{\delta_i (e-e^*)} \cdot h^{r(e-e^*)} &= \omega^{Z_2 - Z_2^*} \cdot h^{Z_3 - Z_3^*} (\bmod p) \end{aligned} \tag{39}$$

As such, it follows that $\delta_i = \frac{Z_2 - Z_2^*}{e-e^*} \bmod q$ and $r = \frac{Z_3 - Z_3^*}{e-e^*} \bmod q$.

B.2 Special honest verifier zero-knowledge:

We construct a PPT simulator Sim which is given $\tau_i, e\delta_i = (e\delta_{i,1}, e\delta_{i,2}), \gamma_i, \omega$ and e and generates an accepting transcript. It selects Z_1, Z_2, Z_3 at random and constructs $A = \frac{g^{Z_1}}{\gamma_i^e} \bmod p$ and $B_1 = \frac{\omega^{Z_2} \cdot h^{Z_3}}{e\delta_{i,1}^e} \bmod p$ and $B_2 = \frac{g^{Z_3}}{e\delta_{i,2}^e} \bmod p$ and $C = \tau_i^{-e} \cdot B \cdot e\delta_{i,1}^e \cdot h^{-Z_3} \cdot \omega^{Z_1} \bmod p$. Sim outputs $(A, B = (B_1, B_2), C, e, Z_1, Z_2, Z_3)$. It is immediate that the probability distribution of $(A, B, C, e, Z_1, Z_2, Z_3)$ and a real conversation between the honest prover and honest verifier are identical.

B.3 Zero-knowledge POIC (ZKPOIC):

We apply the method given in [40] to our Σ protocol to convert it to a zero-knowledge proof system. Let F_{POIC}^R (given in Equation 40) demonstrate the security guarantees of the resultant ZKPOIC over the relation R that we defined in Equation 9.

$$F_{POIC}^R((X, W), X) = (\perp, R(X, W)) \tag{40}$$

F_{POIC}^R shall be run by a trusted third party. X refers to the statement whose correctness is to be proven, i.e., $X = (\tau_i, e\delta_i, \gamma_i, \omega)$ contains the content of an individual invitation letter $(\tau_i, e\delta_i)$ as well as the commitment to the inviter's master share, i.e., γ_i, and the token ω. The witness W, which is only known to the prover, is (s_i, r, δ_i). The ideal functionality F_{POIC}^R receives a common input X from the prover and the verifier as well as the private input W from the prover. F_{POIC}^R outputs to the verifier whether X and W fit into the relation R.