

# Post-Quantum Provably-Secure Authentication and MAC from Mersenne Primes

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## ABSTRACT

This paper presents a novel, yet efficient secret-key authentication and MAC, which provide post-quantum security promise, whose security is reduced to the quantum-safe conjectured hardness of Mersenne Low Hamming Combination (MERS) assumption recently introduced by Aggarwal, Joux, Prakash & Santha (CRYPTO 2018). Our protocols are very suitable to weak devices like smart card and RFID tags.

## CCS CONCEPTS

• Security and privacy → Symmetric cryptography and hash functions; Hash functions and message authentication codes; Authentication.

## KEYWORDS

secret-key authentication, MAC, MERS assumption, man-in-the-middle security.

## 1 INTRODUCTION

*Secret key authentication and HB family.* Secret-key authentication protocol is a process by which a *prover* authenticates itself to a *verifier*. In recent years such protocols have become an important mechanism for low-cost device authentication with small computational power such as smart cards or radio-frequency identification (RFID) tags.

For this purpose, Hopper and Blum [HB01] introduced a two-round secret-key authentication protocol, denoted by HB, requiring only bit-wise operations whose security relies on the Learning Parity with Noise (LPN) problem [BFKL94]. Juels and Weis [JW05] pointed out that HB is insecure against active adversary and proposed HB<sup>+</sup> built upon the HB protocol, a three-round secret-key authentication protocol.<sup>1</sup> Soon after, HB<sup>+</sup> was shown vulnerable to a *man-in-the-middle* (MIM) attack proposed by Gilbert, Robshaw, and Silbert [GRS05]. The line of researches [BCD06, DK07, GRS08b, KPV<sup>+</sup>17, HKL<sup>+</sup>12, CKT16] proposed variants of HB/HB<sup>+</sup> and some of them are secure against MIM attacks.

Their underlying problems are the LPN problem and its variants. Several attacks on the LPN problem have been proposed over the last years [LF06, EKM17]. Most of them are variants of the BKW algorithm [BKW03] whose running time is  $2^{O(\frac{k}{\log k})}$ . In addition, [EKM17] introduced an algorithm solving the LPN problem running on quantum setting. They make the HB family very inefficient in

<sup>1</sup>Later, Katz, Shin, and Smith gave simplified security proofs of them [KSS10]

practice either in classical or quantum setting. Moreover, Bernstein and Lange [BL12] discussed the comparison of Lapin [HKL<sup>+</sup>12] and (light-weight) block-ciphers on RFID tags and smart cards. Armknecht, Hamann, and Mikhalev [AHM14] also discussed the hardware limits of low-cost RFID tags in the range of \$0.05–\$0.10. They concluded that all LPN-based authentication protocols are out of the range.

Hence, it is desirable to come up with a new proposal for secret-key authentication and MAC that provides provable security with better efficiency in terms of key-size, communication, and rounds, while providing post-quantum security promise.

*The Mersenne Low-Hamming Combination (MERS) problem and its application.* In 2017, Aggarwal, Joux, Prakash, and Santha proposed the *Mersenne Low Hamming Combination (MERS) problem* [AJPS18, AJPS17]: Given a Mersenne prime in the form  $p = 2^n - 1$  (where  $n$  is prime), samples of the  $MERS_{n,h}$  distribution are constructed as  $(a, b = as + e)$ , where  $a \in \mathbb{Z}_p$  is chosen uniformly at random, the secret  $s$  and the error  $e$  are chosen uniformly at random from the elements in  $\mathbb{Z}_p$  of the Hamming weight  $h$ . The decisional version of the MERS assumption states that any efficient adversary cannot distinguish the  $MERS_{n,h}$  distribution from the uniform distribution over  $\mathbb{Z}_p^2$ . Aggarwal et al. proposed a public-key encryption scheme based on the  $MERS_{n,h}$  problem [AJPS18, AJPS17].

Regarding the practical aspect, MERS assumption provides efficiency due to its reliance on Mersenne primes [BKLM11]. The potential benefit of MERS-based scheme is a subject of several ongoing research [AJPS18, AJPS17, Sze17, FN17]. Unfortunately, because of their constraint that  $n = \Theta(h^2)$  from the correctness of the key-encapsulation mechanisms, the mechanisms in [AJPS18, AJPS17, Sze17, FN17] set  $n = 216091$  or  $756839$ . This impacts the sizes of public key and ciphertext, which are approximately  $n$  bits, 26.41 KiB – 100.39 KiB.

## 1.1 Our contribution

There are three main contributions in this paper:

- **New version of MERS problem** The first contribution of this work is MERS-U, which is the MERS problem assuming that the secret is *uniform*. We formally prove that the MERS-U problem is hard if the MERS problem is hard as in the case of the LWE problem [ACPS09].
- **Two-round authentication with S-MIM security** The second contribution is a two-round authentication protocol secure against sequential man-in-the-middle (S-MIM) attacks with tight reductions to the MERS problem. Our construction need not require  $n = \Theta(h^2)$  as in KEMs/PKEs in [AJPS18, AJPS17, Sze17, FN17] and we can set  $n = \Theta(h)$ , say,  $n = 4h$ .

Thus, we can set  $n = 521$  and  $h = 128$ , and this makes our protocol efficient and compact, say, the communication complexity is at most  $3n = 1563$  bits.

- Message Authentication Code (MAC) The third contribution is to construct a MAC scheme that is existentially unforgeable under chosen message attacks (UF-CMA) assuming that the MERS problem is hard. Our MAC improves upon the key size, communication and computation complexity with respect to prior works [KPV<sup>+</sup>17, DKPW12]. Again, we can set  $n = \Theta(h)$  as in the authentication.

## 1.2 Related Works

*Security notions.* Bellare and Rogaway [BR94] gave the formal security definition of *mutual* authentication schemes. Their security model captures MIM attack and more. Vaudeney [Vau07] gave the formal security and privacy definitions of RFID authentications. In this paper, we only consider *unilateral* authentication scheme and do not consider any corruption. Mol and Tessaro [MT12] gave the security definitions for *unilateral* authentication scheme that captures from passive attacks to MIM attacks. Lyubashevky and Masny [LM13] introduced an interesting notion of security against Man-In-the-Middle (MIM) attacks, which slightly weakens MIM to only allow the attacker to interfere with *non-overlapping sequential sessions*. This seems sufficient for real-world application in which the keys do not allow parallel sessions. Cash, Kiltz, and Tessaro [CKT16] also defined Sequential MIM (S-MIM) security. We adopt the following definition of S-MIM security.

*Authentication from LPN/LWE.* Hopper and Blum [HB01] introduced a secret-key authentication protocol that is proven secure against passive adversaries from the hardness of the LPN problem. Since then, a family of LPN-based authentication protocols has been developed. Juels and Weis [JW05] proposed an efficient three-round variant of HB, called HB<sup>+</sup>, which they proved to be secure against active attacks. Later, Gilbert et al. [GRS05] show that HB<sup>+</sup> is not secure against a MIM attack, resulting in several variants [MP07, DK07]. However, most of these variants lack security proofs [GRS08a]. Recent proposals [GRS08b, KPV<sup>+</sup>17, HKL<sup>+</sup>12, LM13, CKT16] have proofs for active security or variants of MIM security.

LPN-based protocols have gained some popularity since they require only small number of primitive bit-wise operations (e.g. "XOR" and "AND") for their implementation. However, all LPN-based protocols require huge security parameters. [EKM17] estimates the hardness of LPN <sub>$\ell, \tau$</sub> . According to their estimation, for  $\tau = 1/8$ ,  $\ell = 670, 1060, 1410$  corresponds to 128, 192, and 256 bit security assuming that the memory is constrained to  $2^{80}$  bits. If we set  $\tau = 1/20$  as in [KPV<sup>+</sup>17], then  $\ell$  should be larger than 1280 for 128-bit security.

*Authentication from number-theoretic problems.* Concurrently to above, there is another type of protocols based on number-theoretic assumptions, which are DDH-based protocols introduced in [DKPW12, LM13, CKT16]. Unfortunately, like RSA, the DDH implementation is not suitable for low-cost device. Besides that, the DDH assumption is known to be threatened by Shor's algorithm that runs by quantum computer [Sho97].

*Authentication from an "ad-hoc" block-cipher.* The most common and direct approach to construct a secret-key authentication is by using an "ad-hoc" block-cipher such as AES or DES [FDW04]. Their implementations roughly require approximately 4,000 gates. Contrary to PRF from mathematical assumptions, it does not provide provable security proofs. That is, we need to assume that the block cipher is PRF or OW-CCA encryption.

Fortunately, we can weaken the assumption; we just assume that the block cipher is weak PRF. Dodis et al. [DKPW12] show how to construct a three-round authentication from any weak PRFs, which is secure against active attacks. Later, Lyubashevky and Masny [LM13] constructs a three-round authentication from any weak PRFs with MIM security in sequential sessions.

*MAC.* Message Authentication Code (MAC) is one of the most fundamental primitive in cryptography, used to authenticate a message. Similarly to secret-key authentication, most of MAC schemes have been based on PRFs. This is achieved either by using secure block ciphers [Pre97] or number-theoretic constructions as shown in [DKPW12, KPV<sup>+</sup>17]; the latter provides provably (weakly) MIM-secure<sup>2</sup> authentication scheme and MAC based on LPN/LWE and their ring/field variants.

## 1.3 Organization of the Paper

In Section 2, we review the basic notion and notations, secret-key authentication, and MAC. In Section 3, we review the MERS problem and assumption. In Section 4, we construct a two-round secret-key authentication scheme that is secure against passive adversaries. Next, we build an efficient two-round authentication protocol that has special properties (ROR-CMA security) in Section 5. We then build an efficient two-round authentication protocol secure against S-MIM attacks upon it in Section 6, by applying the transformation of [CKT16]. Finally, we obtain a MAC scheme from the MERS problem in Section 7.

## 2 PRELIMINARIES

### 2.1 Notation

We denote by  $\|x\|$  the Hamming weight of an  $n$ -bit string  $x$ , which is the total number of 1's in  $x$ . Let  $\mathfrak{S}_{n,h}$  be the set of all  $n$ -bit strings of Hamming weight  $h$ .

Let  $n$  be a positive integer and let  $p = 2^n - 1$ . We call  $p$  a Mersenne number if  $n$  is prime. If  $p$  is itself a prime number then  $p$  is called a Mersenne prime.<sup>3</sup>

Let  $\mathbb{Z}_p$  be the integer ring modulo  $p$ , where  $p$  is a Mersenne prime. We have the following properties [AJPS18]:

LEMMA 2.1. *For any  $x, y \in \mathbb{Z}_p$ , we have*

- $\|x + y \pmod{p}\| \leq \|x\| + \|y\|$
- $\|x \cdot y \pmod{p}\| \leq \|x\| \cdot \|y\|$
- If  $x \neq 0^n$ ,  $\|-x \pmod{p}\| = n - \|x\|$

The proof of this lemma is in [AJPS18].

<sup>2</sup>"MIM security" in [DKPW12] is defined by two-phase games. This is  $(\{P, V\}, \{V\})$ -auth security, while the MIM security is  $(\{\}, \{P, V\})$ -auth security using [MT12]'s terminology.

<sup>3</sup>For example,  $n$  can be 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, and so on. Mersenne-756839 employed  $n = 756839$  and Ramstake employed  $n = 216091$  and 756839

Protocol	# $r$	Assumption	Security	Key Size	Comm.
Auth <sub>wprf</sub> [DKPW12]	3	weak PRF	active	$ \mathbb{K}  +  \mathbb{H} $	$2 \mathbb{D}  +  \mathbb{F} $
Auth <sub>Fig2</sub> [LM13]	3	weak PRF	S-MIM	$ \mathbb{K}  +  \mathbb{H} $	$ \mathbb{D}  + 2 \mathbb{F} $
Auth <sub>wprf</sub> [CKT16]	2	weak PRF	S-MIM	$2\ell \mathbb{K}  +  \mathbb{H} $	$ \mathbb{D}  +  \mathbb{F} $
Auth [KPV <sup>+</sup> 17]	2	LPN $_{\ell,\gamma}$	active	$2\ell$	$2\ell + (\ell + 1)\eta$
Lapin [HKL <sup>+</sup> 12]	2	Ring-LPN $_{\ell,\gamma}$	active	$2\ell$	$3\ell$
Auth <sub>LPN</sub> [CKT16]	2	LPN $_{\ell,\gamma}$	S-MIM	$5\ell$	$(\eta + 2)\ell$
Auth <sub>TLPN</sub> [CKT16]	2	LPN $_{\ell,\gamma}$	S-MIM	$(2\eta + 2)\ell$	$2\ell + \eta$
Auth <sub>Field-LPN</sub> [CKT16]	2	Field-LPN $_{\ell,\gamma}$	S-MIM	$4\ell$	$3\ell$
Auth <sub>s-mim</sub> [Sect. 6]	2	MERS $_{n,h}$	S-MIM	$4n$	$3n$

**Table 1: Authentication Protocols based on Weak-PRFs, the LPN-related assumptions, and the MERS assumption. A family of weak PRFs is denoted by  $\mathcal{F} := \{F: \mathbb{K} \times \mathbb{D} \rightarrow \mathbb{F}\}$ . A family of pairwise independent hash functions is denoted by  $\mathcal{H} := \{H: \mathbb{H} \times \mathbb{D} \rightarrow \mathbb{F}\}$ .  $\ell$  and  $\gamma$  defines the dimension and the error rate of the LPN problem.  $\eta = O(\ell)$  defines the number of parallel repetitions.  $n$  and  $h$  are parameters for MERS $_{n,h}$ .**

Protocol	Assumption	Security	Key Size	Comm.
MAC <sub>1</sub> [KPV <sup>+</sup> 17]	LPN $_{\ell,\gamma}$	UF-CMA	$2\ell +  H  +  \pi $	$\ell\eta + \eta + \nu$
MAC <sub>2</sub> [KPV <sup>+</sup> 17]	LPN $_{\ell,\gamma}$	UF-CMA	$(\mu + 1)\ell + \eta +  H  +  \pi $	$\ell\eta + \eta + \nu$
MAC <sub>MERS</sub> [Sect. 7]	MERS $_{n,h}$	UF-CMA	$(\mu + 2)n +  H  +  \pi $	$2n + \nu$

**Table 2: MACs based on the LPN-related assumptions and the MERS assumption.  $\ell$  and  $\gamma$  defines the dimension and the error rate of the LPN problem.  $\eta = O(\ell)$  defines the number of parallel repetitions.  $n$  and  $h$  are parameters for MERS $_{n,h}$ . A family of pairwise independent hash functions is denoted by  $\mathcal{H} := \{H: \mathbb{M} \times \{0, 1\}^\nu \rightarrow \{0, 1\}^\mu\}$ . A family of pairwise independent permutations is denoted by  $\mathcal{P} := \{\pi: \{0, 1\}^z \rightarrow \{0, 1\}^z\}$ , where  $z = \ell\eta + \eta + \nu$  for LPN case and  $z = 2n + \nu$  for MERS case.**

## 2.2 Secret-Key Authentication Syntax

Secret-key authentication protocol  $\text{Auth} = (\text{KeyGen}, P, V)$  is an interactive protocol in which  $P$  and  $V$  share the same secret key  $\text{SK}$  (in the context of RFID, we consider  $P$  as a *tag* and  $V$  as a *reader*). More formally, a secret-key authentication protocol proceeds in two phases:

- **Key-generation algorithm:** The key-generation algorithm  $\text{KeyGen}(1^\kappa)$  is executed on the security parameter  $\kappa$  and outputs a secret key  $\text{SK}$ .
- **Authentication Protocol:** The interactive algorithm between  $P$  and  $V$  takes as input the shared secret key  $\text{SK}$  and is executed  $r$  rounds. And finally,  $V$  outputs either *Accept* or *Reject*.

In this paper, we only consider *two-round random-challenge* secret-key authentication protocols, in which the protocol is run as follows; the verifier chooses a challenge  $c$  from the challenge space  $\mathcal{C}$  uniformly at random and sends it as the first message; the prover receives  $c$ , computes a response  $\tau \leftarrow P_{\text{SK}}(c)$ , and sends it as the second message; the verifier receives  $\tau$  and outputs its decision  $d \leftarrow V_{\text{SK}}(c, \tau)$ .

We say that the authentication protocol has *completeness error*  $\alpha$  if for all secret keys  $\text{SK}$  generated by  $\text{KeyGen}(1^\kappa)$  the honestly executed protocol returns *reject* with probability at most  $\alpha$ . More formally, for all  $1^\kappa \in \mathbb{N}$ ,  $\text{SK} \leftarrow \text{KeyGen}(1^\kappa)$ :

$$\Pr[c \leftarrow_{\mathcal{S}} \mathcal{C}; \tau \leftarrow P_{\text{SK}}(c); d \leftarrow V_{\text{SK}}(c, \tau) : d = \text{Reject}] \leq \alpha.$$

## 2.3 Security Models

As for public-key authentication [FS87], several security notions have been introduced for secret-key authentication. There are three main security models against impersonation attacks that are: *passive*, *active*, and *man-in-the-middle*. All three models proceed in two steps: In the first step, the adversary interacts with  $P$  and  $V$  and then in the second step, it starts interacting only with  $V$  in order to get accepted. The weakest notion, which is the passive security, is when the adversary should not be able to interact with  $V$  after eavesdropping several sessions in the authentication protocol between  $P$  and  $V$ . A stronger notion, which is the active security, is when the adversary should not be able to interact with  $V$  after interacting *arbitrarily* with  $P$  and eavesdropping passively several sessions in the authentication protocol between  $P$  and  $V$ .

Finally, the strongest and most realistic security model of adversary is a *man-in-the-middle attack* (MIM), where the adversary, in the first phase, can *arbitrarily* interact with  $P$  and  $V$  before making verification queries to the reader.

**2.3.1 Passive Security.** As the basic security notion, we review the definition of passive security for *two-round random-challenge* secret-key authentication protocols.

*Definition 2.2 (Passive security).* Let  $\text{Auth} = (\text{KeyGen}, P, V)$  be a *two-round random-challenge* secret-key authentication protocol. Define the security game  $\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa)$  between a challenger and an adversary  $\mathcal{A}$  as in Figure 1. For any adversary  $\mathcal{A}$ , we define its

advantage against Auth as the quantity

$$\text{Adv}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa) := \Pr[\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa) \Rightarrow \text{True}].$$

We say Auth is  $(t, q, \epsilon)$ -passively-secure if for all  $t$ -time adversary  $\mathcal{A}$  querying to  $T$  at most  $q$  times, we have  $\text{Adv}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa) \leq \epsilon$ .

$\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa)$	Oracle $T()$
$\text{SK} \leftarrow_{\S} \text{KeyGen}(1^\kappa)$	$c \leftarrow_{\S} C$
$st \leftarrow \mathcal{A}^{T(\cdot)}(1^\kappa)$	$\tau \leftarrow \text{P}_{\text{SK}}(c)$
$c^* \leftarrow_{\S} C$	<b>return</b> $(c, \tau)$
$\tau^* \leftarrow \mathcal{A}(st, c^*)$	
<b>return</b> $(\text{V}_{\text{SK}}(c^*, \tau^*) = \text{Accept})$	

**Figure 1: Definition of  $\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa)$**

## 2.4 Tag Sparsity Definition and Security

In this section we define an important tool that our construction relies on, which is *tag sparsity* [CKT16].

This is the property of an authentication protocol  $\text{Auth} = (\text{KeyGen}, P, V)$  for which the tag  $\tau$  is composed into two distinct components, which are  $\tau_1 \in \mathcal{T}_1$  and  $\tau_2 \in \mathcal{T}_2$ .

Informally, this notion is based on the fact that for any challenge  $c$ , a secret SK, and a left tag  $\tau_1$ , it is hard for an adversary to find a right tag  $\tau_2$  which makes  $\tau$  accepted.

*Definition 2.3 (Right Tag-Sparsity [CKT16, Definition 4]).* Let  $\text{Auth} = (\text{KeyGen}, P, V)$  be a two-round random-challenge secret-key authentication protocol with tags in  $\mathcal{T}_1 \times \mathcal{T}_2$  and challenge space  $C$ . For  $\epsilon = \epsilon(1^\kappa)$ , we say that Auth has  $\epsilon$ -sparse right tags (or Auth has  $\epsilon$ -right tag sparsity) if

$$\Pr[\tau_2 \leftarrow_{\S} \mathcal{T}_2; d \leftarrow \text{V}_{\text{SK}}(c, (\tau_1, \tau_2)) : d = \text{Accept}] \leq \epsilon$$

for all  $c \in C$ , SK, and  $\tau_1 \in \mathcal{T}_1$ .

**2.4.1 ROR-CMA security.** In our construction we are also considering a new property introduced in [CKT16], called *real-or-random right-tag chosen-message security* (ROR-CMA) suitable to tag-sparsity notion. Roughly speaking, the scheme is ROR-CMA-secure if, given a random challenge  $c^*$ , any efficient adversary cannot distinguish a real prover from the fake prover that returns the random right tag  $\tau_2$  on all challenge except  $c^*$  even if it can finally access to the verification oracle on the challenge  $c^*$  and  $\tau^*$  of its choice. The formal statement follows:

*Definition 2.4 (ROR-CMA security).* Let  $\text{Auth} = (\text{KeyGen}, P, V)$  be a two-round random-challenge secret-key authentication protocol. For  $b \in \{0, 1\}$ , we define the security game  $\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{ror-cma}, b}(\kappa)$  between a challenger and an adversary  $\mathcal{A}$  as in Figure 2. For any adversary  $\mathcal{A}$ , we define its ROR-CMA advantage against Auth as the quantity

$$\text{Adv}_{\text{Auth}, \mathcal{A}}^{\text{ror-cma}}(\kappa) := \left| \Pr[\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{ror-cma}, 0}(\kappa) \Rightarrow 1] - \Pr[\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{ror-cma}, 1}(\kappa) \Rightarrow 1] \right|.$$

We say Auth is  $(t, q, \epsilon)$ -ROR-CMA-secure if for all  $t$ -time adversary  $\mathcal{A}$  issuing at most  $q$  queries to the oracle  $T_b(\cdot)$ , we have  $\text{Adv}_{\text{Auth}, \mathcal{A}}^{\text{ror-cma}}(\kappa) \leq \epsilon$ .

$\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{ror-cma}, b}(\kappa)$	Oracle $T_b(c)$
$\text{SK} \leftarrow_{\S} \text{KeyGen}(1^\kappa)$	$(\tau_1, \tau_2^1) \leftarrow_{\S} \text{P}_{\text{SK}}(c); \tau_2^0 \leftarrow_{\S} \mathcal{T}_2$
$c^* \leftarrow_{\S} C$	<b>if</b> $c = c^*$ <b>then</b>
$(\tau^*, \text{state}) \leftarrow_{\S} \mathcal{A}^{T_b(\cdot)}(1^\kappa, c^*)$	<b>return</b> $\tau := (\tau_1, \tau_2^1)$
$d \leftarrow_{\S} \text{V}_{\text{SK}}(c^*, \tau^*)$	<b>else</b>
<b>return</b> $\mathcal{A}(\text{state}, d)$	<b>return</b> $\tau := (\tau_1, \tau_2^b)$

**Figure 2: Definition of  $\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{ror-cma}, b}(\kappa)$**

## 2.5 Security against Sequential Man-in-the-Middle Adversary

In this paper, we target a weaker notion of the man-in-the-middle security, which is Sequential MIM (S-MIM) security, of [LM13, CKT16]; in which the adversary can first interact *sequentially* with P and V in independent sessions and then makes verification queries to V in order to make the latter accept.

Cash, Kiltz, and Tessaro [CKT16] defined S-MIM security notion for two-round random-challenge secret-key authentication protocols. We invoke the adversary  $\mathcal{A}$  who access to three oracles:  $C$ ,  $P$ , and  $V$ . To synchronize the sessions, each of these oracles use a variable sid associated to a given session. For every session,  $\mathcal{A}$  invokes  $C()$  to get a new random challenge  $c$ , and then invokes the oracle  $P()$  on input  $c'$  that runs  $\text{P}_{\text{SK}}(c')$  and returns a response  $\tau$ . Finally, given  $\tau'$  from  $\mathcal{A}$ ,  $V()$  checks whether  $\tau'$  is a valid response on a session challenge  $c[\text{sid}]$  or not, and then increases the session number sid.  $\mathcal{A}$  wins if it makes  $V$  accepts in some session and has changed at least one of messages in the session sent by P and V.

*Definition 2.5 (S-MIM security [CKT16, Section 2]).* Let  $\text{Auth} = (\text{KeyGen}, P, V)$  be a two-round random-challenge secret-key authentication protocol. Define the security game  $\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{s-mim}}(\kappa)$  between a challenger and an adversary  $\mathcal{A}$  as in Figure 3. For any adversary  $\mathcal{A}$ , we define its S-MIM advantage against Auth as the quantity

$$\text{Adv}_{\text{Auth}, \mathcal{A}}^{\text{s-mim}}(\kappa) := \Pr[\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{s-mim}}(\kappa) \Rightarrow \text{True}].$$

We say Auth is  $(t, q, \epsilon)$ -S-MIM-secure if for all  $t$ -time adversary  $\mathcal{A}$  invoking at most  $q$  sessions, we have  $\text{Adv}_{\text{Auth}, \mathcal{A}}^{\text{s-mim}}(\kappa) \leq \epsilon$ .

Let  $\text{Auth}' = (\text{KeyGen}', P', V')$  be two-round random-challenge authentication protocol with challenge space  $C$  and split tag space  $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$ . We assume that  $\mathcal{T}_2 = \mathbb{F}$  is a finite field with addition  $+$  and multiplication  $\circ$ . Let  $H := \{H_{KH} : \mathcal{T}_1 \rightarrow \mathbb{F}\}$  be a family of pairwise independent hash functions. Cash et al. [CKT16] turn  $\text{Auth}'$  satisfying ROR-CMA security into  $\text{Auth} = (\text{KeyGen}, P, V)$  as follows:

- **Public parameters:** The same as  $\text{Auth}'$ .

$\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{s-mim}}(\kappa)$ $\text{sid} \leftarrow 0$ $\text{SK} \leftarrow_{\S} \text{KeyGen}(1^\kappa)$ $\text{run } \mathcal{A}^{C(\cdot), P(\cdot), V(\cdot)}(1^\kappa)$ $\text{return } \left( \begin{array}{l} \exists i: (c[i], \tau[i]) \neq (c'[i], \tau'[i]) \\ \wedge d[i] = \text{Accept} \end{array} \right)$	$\text{Oracle } C()$ $\text{if } c[\text{sid}] = \perp \text{ then}$ $c[\text{sid}] \leftarrow_{\S} C$ $\text{return } c[\text{sid}]$
$\text{Oracle } P(c')$ $\text{if } c'[\text{sid}] = \perp \text{ then}$ $c'[\text{sid}] \leftarrow c'$ $\tau[\text{sid}] \leftarrow_{\S} P_{\text{SK}}(c')$ $\text{return } \tau[\text{sid}]$	$\text{Oracle } V(\tau')$ $\tau'[\text{sid}] \leftarrow \tau', c \leftarrow_{\S} C()$ $d[\text{sid}] \leftarrow_{\S} V_{\text{SK}}(c, \tau'[\text{sid}])$ $\text{sid} \leftarrow \text{sid} + 1$ $\text{return } d[\text{sid}]$

Figure 3: Definition of  $\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{s-mim}}(\kappa)$

- **Key generation:** The key-generation algorithm  $\text{KeyGen}$  picks  $K_H \leftarrow_{\S} \mathcal{K}_H, K_F \leftarrow_{\S} \mathbb{F} \setminus \{0\}$ , and  $K' \leftarrow_{\S} \text{KeyGen}'(1^\kappa)$ . The key is  $K := (K_H, K_F, K')$ .
- **Challenge:** The challenge is  $c \leftarrow_{\S} C$ .
- **Response:** The response is  $\sigma = (\sigma_1, \sigma_2)$ ; the prover first computes  $\tau = (\tau_1, \tau_2) \leftarrow_{\S} P'_{K'}(c)$  and

$$\sigma = (\sigma_1, \sigma_2) := \left( \tau_1, \tau_2 \circ K_F + H_{K_H}(\tau_1) \right) \in \mathcal{T}_1 \times \mathbb{F}.$$

- **Verification:** Given a challenge  $c$  and response  $\sigma = (\sigma_1, \sigma_2)$ , the verifier first computes

$$\tau = (\tau_1, \tau_2) := \left( \sigma_1, (\sigma_2 - H_{K_H}(\sigma_1)) \circ K_F^{-1} \right)$$

and returns the decision  $d \leftarrow_{\S} V'_{K'}(c, \tau)$ .

**THEOREM 2.6** ([CKT16, THEOREM 5]). *Suppose that  $H$  is  $\delta$ -almost universal and that  $\text{Auth}'$  is  $(t, r, \epsilon)$ -ROR-CMA-secure, satisfies  $\beta$ -right tag sparsity, and has completeness error  $\alpha$ . Then  $\text{Auth}$  is  $(t', r, r \cdot (\epsilon + r/|C| + \beta\delta|\mathbb{F}|/(|\mathbb{F}| - 1) + r\alpha)$ -S-MIM-secure, where  $t' \approx t$ .*

## 2.6 Message Authentication Codes

A MAC scheme is a tuple of three probabilistic polynomial-time algorithms  $\text{MAC} = (\text{KeyGen}, \text{Tag}, \text{Verify})$  over  $(\mathcal{K}, \mathcal{M}, \mathcal{T})$  where  $\mathcal{K}$ ,  $\mathcal{M}$ , and  $\mathcal{T}$  are key space, message space, and tag space, respectively:

- **Key-generation algorithm:** The probabilistic key-generation algorithm  $\text{KeyGen}$  gives secret key  $\text{SK}$  on input a security parameter  $\kappa$ .
- **Tag-generation algorithm:** The probabilistic authentication algorithm  $\text{Tag}$  takes as inputs the secret key  $\text{SK}$ , the message  $m$  and then outputs a tag  $\sigma$ .
- **Verification algorithm:** The deterministic verification algorithm  $\text{Verify}$  takes as inputs a secret key  $\text{SK}$ , a message  $m$  and a tag  $\sigma$  and outputs either  $\text{Accept}$  or  $\text{Reject}$ .

*Completeness.* We say that  $\text{MAC}$  has a completeness error  $\alpha$ , if for all  $m \in \mathcal{M}$  and  $1^\kappa \in \mathbb{N}$ :

$$\Pr[\text{SK} \leftarrow_{\S} \text{KeyGen}(1^\kappa); \sigma \leftarrow_{\S} \text{Tag}(\text{SK}, m); d \leftarrow \text{Verify}(\text{SK}, m, \sigma) : d = \text{Reject}] \leq \alpha.$$

We often say that  $\text{MAC}$  is *perfectly correct* if  $\alpha = 0$ .

*UF-CMA security.* The standard security notion for MAC scheme is *unforgeability under chosen-message attacks* (UF-CMA), captured by the experiment described in Figure 4.

**Definition 2.7.** Let  $\text{MAC} = (\text{KeyGen}, \text{Tag}, \text{Verify})$  be a MAC scheme. We define the security game  $\text{Exp}_{\text{MAC}, \mathcal{A}}^{\text{uf-cma}}(\kappa)$  between a challenger and an adversary  $\mathcal{A}$  as in Figure 4. For any adversary  $\mathcal{A}$ , we define UF-CMA advantage against  $\text{MAC}$  as the quantity

$$\text{Adv}_{\text{MAC}, \mathcal{A}}^{\text{uf-cma}}(\kappa) := \Pr[\text{Exp}_{\text{MAC}, \mathcal{A}}^{\text{uf-cma}}(\kappa) \Rightarrow \text{True}].$$

We say that a MAC is  $(t, q, \epsilon)$ -UF-CMA-secure if for all  $t$ -time adversary  $\text{Adv}_{\text{MAC}, \mathcal{A}}^{\text{uf-cma}}(\kappa)$  issuing at most  $q$  queries to the oracles  $T(\cdot)$  and  $V(\cdot, \cdot)$ , we have  $\text{Adv}_{\text{MAC}, \mathcal{A}}^{\text{uf-cma}}(\kappa) \leq \epsilon$ .

$\text{Exp}_{\text{MAC}, \mathcal{A}}^{\text{uf-cma}, b}(\kappa)$ $Q_T, Q_V \leftarrow \emptyset$ $\text{SK} \leftarrow_{\S} \text{KeyGen}(1^\kappa)$ $\text{run } \mathcal{A}^{T(\cdot), V(\cdot, \cdot)}(1^\kappa)$ $\text{return } \left( \begin{array}{l} \exists (m, \sigma) \in Q_V \text{ s.t. } m \notin Q_T \\ \wedge \text{Verify}(\text{SK}, m, \sigma) = \text{Accept} \end{array} \right)$	$\text{Oracle } T(m)$ $Q_T \leftarrow Q_T \cup \{m\}$ $\sigma \leftarrow_{\S} \text{Tag}(\text{SK}, m)$ $\text{return } \sigma$
	$\text{Oracle } V(m, \sigma)$ $Q_V \leftarrow Q_V \cup \{(m, \sigma)\}$ $\text{return } \text{Verify}(\text{SK}, m, \sigma)$

Figure 4: Definition of  $\text{Exp}_{\text{MAC}, \mathcal{A}}^{\text{uf-cma}}(\kappa)$

## 2.7 Hash Functions

Our construction relies on pairwise-independent hash functions and is defined as following:

**Definition 2.8** (*Pairwise-independent hash functions*). A function  $h: \mathcal{K} \times \mathcal{N} \rightarrow \mathcal{M}$  is called pairwise-independent hash function if for  $x_1 \neq x_2 \in \mathcal{N}, y_1, y_2 \in \mathcal{M}$ ,

$$\Pr_{\text{SK} \leftarrow \mathcal{K}} [h_{\text{SK}}(x_1) = y_1 \wedge h_{\text{SK}}(x_2) = y_2] \leq \frac{1}{|\mathcal{M}|^2}.$$

*Concrete Construction.* We now consider the following construction of pairwise independent function based on ring of integers modulo prime  $(\mathbb{Z}_p)$ :

**LEMMA 2.9.** *For every  $n \in \mathbb{N}$ , define:  $h: \mathbb{Z}_p^2 \times \mathbb{Z}_p \rightarrow \mathbb{Z}_p$  by  $h_{a,b}(x) = a \cdot x + b$ . Then the function  $h$  is pairwise-independent. That is, for all  $x_1 \neq x_2$  and  $y_1, y_2 \in \mathbb{Z}_p$ ,*

$$\Pr_{(a,b) \leftarrow \mathbb{Z}_p^2} [h_{a,b}(x_1) = y_1 \wedge h_{a,b}(x_2) = y_2] \leq 1/p^2.$$

The proof can be found in [Rub12]

## 3 THE MERS PROBLEM

Aggarwal et al. introduced a new assumption [AJPS18] mimicking NTRU over integers, relying on the properties of Mersenne primes in the ring  $\mathbb{Z}_p$  instead of polynomial rings  $\mathbb{Z}_q[x]/(x^n - 1)$ . The conjecture is based on the observation that given any number  $a \in \mathbb{Z}_p$ , then we obtain this following property: if we multiply

$a$  by any number  $b = 2^x$  and  $x \in [0, n-1]$ , then the result  $c = a \cdot b$  is just a cyclic shift. Since the reduction modulo Mersenne prime does not increase Hamming weights of  $a$ , then the result  $c$  looks pseudorandom (i.e., it is hard to distinguish  $c$  from any random integer modulo Mersenne prime).

For two integers  $n > h$  and for  $n$ -bit Mersenne prime  $p = 2^n - 1$ , and for integer  $s \in \mathbb{Z}_p$ , we define a distribution  $O_{s,n,h}$  as follows: choose  $a \leftarrow \mathbb{Z}_p$  and  $e \leftarrow \mathfrak{S}_{n,h}$ , return  $(a, a \cdot s + e \bmod p)$ . We also define a uniform distribution  $\mathcal{U}$  as follows: choose  $(a, b) \leftarrow \mathbb{Z}_p^2$ .<sup>4</sup>

Let us define the *Mersenne Low-Hamming Combination Assumption* (the MERS assumption).

**Definition 3.1 (MERS problem).** For two positive integers  $n > h$  and for an adversary  $\mathcal{A}$ , we introduce the  $\text{MERS}_{n,h}$  advantage as the quantity:

$$\text{Adv}_{\mathcal{A}}^{\text{MERS}_{n,h}}(\kappa) := \left| \Pr[\mathcal{A}^{O_{s,n,h}} \Rightarrow \text{True}] - \Pr[\mathcal{A}^{\mathcal{U}} \Rightarrow \text{True}] \right|,$$

where  $s \leftarrow_{\mathfrak{S}} \mathfrak{S}_{n,h}$ . We say that the  $\text{MERS}_{n,h}$  problem is  $(t, q, \epsilon)$ -hard if all  $t$ -time attacker  $\mathcal{A}$  with time complexity  $t$ , making at most  $q$  queries, we have  $\text{Adv}_{\mathcal{A}}^{\text{MERS}_{n,h}}(\kappa) \leq \epsilon$ .

The original definition [AJPS18, Definition 5] allows an adversary to query at most twice. We generalize the assumption by allowing polynomially-many queries.

### 3.1 MERS Problem with Uniform Secret

We next define the  $\text{MERS-U}_{n,h}$  problem with an  $n$ -bit Mersenne prime  $p = 2^n - 1$  and integer  $h \in \{0, \dots, n\}$ .

**Definition 3.2 (MERS problem with uniform secret).** For two positive integers  $n > h$  and for an adversary  $\mathcal{A}$ , we define the  $\text{MERS-U}_{n,h}$  advantage as the quantity:

$$\text{Adv}_{\mathcal{A}}^{\text{MERS-U}_{n,h}}(\kappa) := \left| \Pr[\mathcal{A}^{O_{s,n,h}} \Rightarrow \text{True}] - \Pr[\mathcal{A}^{\mathcal{U}} \Rightarrow \text{True}] \right|, \quad (1)$$

where  $s \leftarrow_{\mathfrak{S}} \mathbb{Z}_p$ . We say that the  $\text{MERS-U}_{n,h}$  problem is  $(t, q, \epsilon)$ -hard if all attacker  $\mathcal{A}$  with time complexity  $t$ , making at most  $q$  queries, we have  $\text{Adv}_{\mathcal{A}}^{\text{MERS-U}_{n,h}}(\kappa) \leq \epsilon$ .

It is easy to show that if  $\text{MERS}_{n,h}$  is hard, then  $\text{MERS-U}_{n,h}$  is also hard (by a simple randomization of the secret  $s$ ). We note that the converse is also true.

**PROPOSITION 3.3.** *If the  $\text{MERS-U}_{n,h}$  problem is  $(t', q+1, \epsilon')$ -hard, then the  $\text{MERS}_{n,h}$  problem is  $(t, q, \epsilon')$ -hard, where  $t' \approx t$  and  $\epsilon' \approx \epsilon$ .*

**PROOF.** We show a reduction algorithm by following the reduction in [ACPS09, Lemma 2]. Consider the following conversion, which will map  $O_{s,n,h}$  (and  $\mathcal{U}$ ) into  $O_{\bar{e},n,h}$  where  $\bar{e} \leftarrow \mathfrak{S}_{n,h}$  (and  $\mathcal{U}$ ), respectively: It takes a sample  $(\bar{a}, \bar{b})$  with  $\bar{a} \neq 0$  from the oracle of  $\text{MERS-U}_{n,h}$ . It then converts a sample  $(a, b)$  into  $(a', b')$ , where  $a' := -\bar{a}^{-1} \cdot a$  and  $b' := b + a' \cdot \bar{b}$ .

- Suppose that  $\bar{b} = \bar{a} \cdot s + \bar{e}$ , where  $s \leftarrow \mathbb{Z}_p$  and  $\bar{e} \leftarrow \mathfrak{S}_{n,h}$ . In this case,  $a'$  is uniformly distributed since  $a$  is uniformly distributed and the map  $a \mapsto -\bar{a}^{-1}a$  is one-to-one. Moreover, if  $b = as + e$  with  $e \in \mathfrak{S}_{n,h}$ , then  $b' = b + a' \cdot \bar{b} = as + e + a'(\bar{a}s + \bar{e}) = as + e + a'\bar{a}s + a'\bar{e} = a'\bar{e} + e$  since  $a'\bar{a} \equiv -a$

<sup>4</sup>In the original definition,  $a$  is chosen from  $\{0, 1\}^n$ . This change introduces only negligible distance

(mod  $p$ ). Thus, the converted samples are identified with the samples from  $O_{\bar{e},n,h}$ .

- On the other hand, if the oracle is  $\mathcal{U}$ , then the converted samples are also distributed according to the uniform distribution.

Therefore, the conversion algorithm converts the oracle  $O_{s,n,h}$  (and  $\mathcal{U}$ ) into  $O_{\bar{e},n,h}$  where  $\bar{e} \leftarrow \mathfrak{S}_{n,h}$  (and  $\mathcal{U}$ ), respectively. This completes the proof.  $\square$

### 3.2 Hardness and Concrete Parameters

**Meet-in-the-Middle attack.** de Boer et al. [dBdW18] presented a meet-in-the-middle attack for solving the MERS problem. Their classical attack runs in the time  $\tilde{O}\left(\binom{n-1}{h-1}^{1/2}\right)$ . The quantum version runs in the time  $\tilde{O}\left(\binom{n-1}{h-1}^{1/3}\right)$ . They corresponds to roughly  $\frac{1}{4}h \lg n$  and  $\frac{1}{6}h \lg n$  bits security, respectively.

**LLL-attack.** The authors of [BCGN17, dBdW18] presented an LLL-based algorithm for solving the ratio version of MERS assumption<sup>5</sup> and the MERS problem used in the present paper. For small  $h = O(\sqrt{n})$ , the running time of the LLL attack is  $O(2^{2h})$  on Turing machine and  $O(2^h)$  on quantum machine. As claimed in [AJPS18], attacks against MERS cannot exceed the complexity of the order  $2^h$  where  $h$  is the hamming weight parameter.

Assuming that, when considering the security and implementation of our protocols, one should choose the parameter  $h$  at least half of the desired security level  $\kappa$ .

**Primality of  $n$  in Mersenne primes.** Agrawal discussed that  $p = 2^n - 1$  and  $n$  should be primes to avoid an attack on composite  $n$ . For the details, see Agrawal et al. [AJPS18].

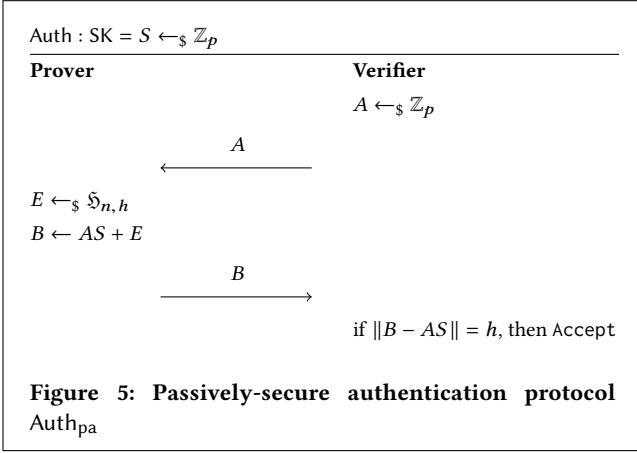
**Parameters.** Assuming the attacks and constraints above, we choose parameter values as  $(\kappa, h, n) = (256, 128, 521)$ . It will serve classical 256-bit sec. and quantum 192-bit sec.

## 4 PASSIVELY-SECURE AUTHENTICATION BASED ON MERS

In this section we introduce our new two-round authentication protocol based on  $\text{MERS}_{n,h}$  problem with passive security. Our  $\text{Auth}_{\text{pa}}$  is defined as follows:

- **Public parameters:** The authentication protocol has the following public parameters that depend on the security parameter  $\kappa$ .
  - $n \in \mathbb{N}$ : the length of  $A$ ,  $S$ , and  $E$
  - $h \in \mathbb{N}$ : the Hamming weight of  $E$
- **Key generation:** The key-generation algorithm  $\text{KeyGen}(1^\kappa)$  outputs  $\text{SK} = S \leftarrow_{\mathfrak{S}} \mathbb{Z}_p$ .
- **Authentication protocol:** To be authenticated by verifier, a prover follows the two-round authentication protocol shown on Figure 5.

<sup>5</sup>The Mersenne Low Hamming Ratio Assumption states that, given an  $n$ -bit Mersenne prime  $p = 2^n - 1$  and an integer  $h$ , any PPT adversary cannot distinguish between  $F/G \bmod p$  with  $F, G \leftarrow_{\mathfrak{S}} \mathfrak{S}_{n,h}$ , and  $R \leftarrow \mathbb{Z}_p$  with non-negligible advantage.



**THEOREM 4.1.** *If the MERS- $U_{n,h}$  problem is  $(t, q, \epsilon)$ -hard and  $\frac{1}{p} \sum_{i=0}^{2h} \binom{n}{i}$  is negligible in  $\kappa$ , then Auth<sub>pa</sub> is passively-secure authentication.*

The proof results straightforwardly from the MERS- $U_{n,h}$  assumption:  
We have

$$\Pr[\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa) \Rightarrow \text{True}] \leq \epsilon + \frac{1}{p} \sum_{i=0}^{2h} \binom{n}{i}.$$

The security proof is obtained by following the proof of [KSS10, Theorem 2].

**PROOF.** Let  $\mathcal{A}$  be an adversary against passive security of Auth<sub>pa</sub>. Let us consider the following reduction algorithm  $\mathcal{B}$  solving MERS- $U_{n,h}$  by using  $\mathcal{A}$ : In the learning phase,  $\mathcal{B}$  sends a sample  $(a, b)$  from its oracle as a transcript  $(A, B)$ . In the impersonating phase,  $\mathcal{B}$  gets a sample  $(\bar{a}, \bar{b})$  from its oracle, sends  $A := -\bar{a}$  to  $\mathcal{A}$ , and receives  $B$  from  $\mathcal{A}$ . It outputs 1 if  $\|\bar{b} + B\| \leq 2h$  and 0 otherwise.

If  $\mathcal{B}$ 's oracle is  $\mathcal{U}$ , then  $\mathcal{B}$  outputs 1 with probability exactly  $\frac{1}{p} \cdot \sum_{i=0}^{2h} \binom{n}{i}$ , since  $\bar{b}$  is uniformly distributed and independent of everything else.

Next, suppose that  $\mathcal{B}$ 's oracle is  $\mathcal{O}_{s,n,h}$ . In this case, the simulation of the learning phase is perfect, where the secret key is  $S = s$ . Therefore, the event that  $\|B - A \cdot S\| = h$  holds with probability is exactly  $\Pr[\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa) \Rightarrow \text{True}]$ . We note that if  $\|B - A \cdot S\| = h$  holds, then  $\|B + \bar{a} \cdot s\| = h$  also holds. Meanwhile,  $\|\bar{b} - \bar{a}s\| = h$  since the oracle is  $\mathcal{O}_{s,n,h}$ . Thus, with probability at least  $\Pr[\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa) \Rightarrow \text{True}]$ ,  $\|\bar{b} + B\| = \|\bar{b} - \bar{a}s + \bar{a}s + B\| \leq \|\bar{b} - \bar{a}s\| + \|\bar{a}s + B\| = 2h$  holds.

Therefore, we have

$$\begin{aligned} \Pr[s \leftarrow \mathbb{Z}_p : \mathcal{D}^{\mathcal{O}_{s,n,h}}() = 1] - \Pr[\mathcal{D}^{\mathcal{U}}() = 1] \\ \geq \Pr[\text{Exp}_{\text{Auth}, \mathcal{A}}^{\text{pa}}(\kappa) \Rightarrow \text{True}] - \frac{1}{p} \sum_{i=0}^{2h} \binom{n}{i} \end{aligned}$$

and this yields the theorem as we wanted.  $\square$

*Active attack against Auth<sub>pa</sub>.* The active attack against Auth<sub>pa</sub> based on MERS<sub>n,h</sub> is quite similar to the active attack against

HB<sup>+</sup> [GRS05]. It consists for an arbitrary fixed  $A$ , the adversarial verifier can send fixed  $A$  repeatedly and obtain

$$B_1 \equiv AS + E_1 \pmod{p}, \dots, B_k \equiv AS + E_k \pmod{p},$$

where  $E_i$ 's Hamming weight is at most  $h$ . The adversary takes majority of  $B_i$  values to get noise-free equation; it can determine  $AS$ 's bits from LSB to MSB by taking the majority. It then learns  $AS \pmod{p}$  and obtains  $S$  by computing  $A^{-1}$ .

## 5 ROR-CMA-SECURE AUTHENTICATION BASED ON MERS

Our Auth<sub>rOR</sub> is defined as follows:

- **Public parameters:**  $n$  and  $h$  as in section 4.
- **Key generation:** The key-generation algorithm  $\text{KeyGen}_{\text{rOR}}(1^\kappa)$  outputs  $\text{SK} = (S_1, S_2) \leftarrow_{\mathcal{S}} \mathbb{Z}_p^2$ .
- **Authentication protocol:** To be authenticated by V, P follows the 2-round authentication protocol shown on Figure 6.

**THEOREM 5.1.** *Auth<sub>rOR</sub> has  $\binom{n}{h}/p$ -sparse right tags.*

**PROOF.** For any secret  $(S_1, S_2)$ , challenge  $A$ , and left tag  $R \neq 0$ , we have  $\Pr[\text{V}_{(S_1, S_2)}(A, (R, B)) \Rightarrow \text{Accept} : B \leftarrow_{\mathcal{S}} \mathbb{Z}_p] = |\mathfrak{S}_{n,h}|/p = \binom{n}{h}/p$ .  $\square$

**THEOREM 5.2.** *If the MERS- $U_{n,h}$  problem is  $(t, q, \epsilon)$ -hard, then Auth<sub>rOR</sub> is  $(t', q, \epsilon)$ -ROR-CMA-secure, where  $t' \approx t$ .*

**PROOF OF THEOREM 5.2.** We follow the proof of the ROR-CMA security of the LPN-based authentication scheme in Cash, Kiltz, and Tessaro [CKT16, Theorem 7].

The security of the MERS-based Auth<sub>rOR</sub> essentially builds on the ROR-CMA notion. Let us consider an adversary  $\mathcal{A}$  who plays the security game  $\text{Exp}_{\text{Auth}_{\text{rOR}}, \mathcal{A}}^{\text{ror-cma}, b}(\kappa)$ . We build an adversary  $\mathcal{B}$  who solves the MERS<sub>n,h</sub> problem, where  $n$  and  $h$  are known, by using  $\mathcal{A}$  as in Figure 7.

Assume that  $S_1$  is the secret of the MERS- $U_{n,h}$  problem.  $\mathcal{B}$  chooses  $S'_2 \leftarrow_{\mathcal{S}} \mathbb{Z}_p$  and  $A^* \leftarrow_{\mathcal{S}} \mathbb{Z}_p$ . It implicitly defines  $S_2 := -A^* \cdot S_1 + S'_2 \pmod{p}$ . Since  $S'_2$  is uniform over  $\mathbb{Z}_p$ ,  $S_2$  is also. In addition, we have

$$B^* - R^* \cdot (S_1 \cdot A^* + S_2) \equiv B^* - R^* \cdot S'_2 \pmod{p}.$$

Thus, the decision by  $\mathcal{B}$  is always correct.

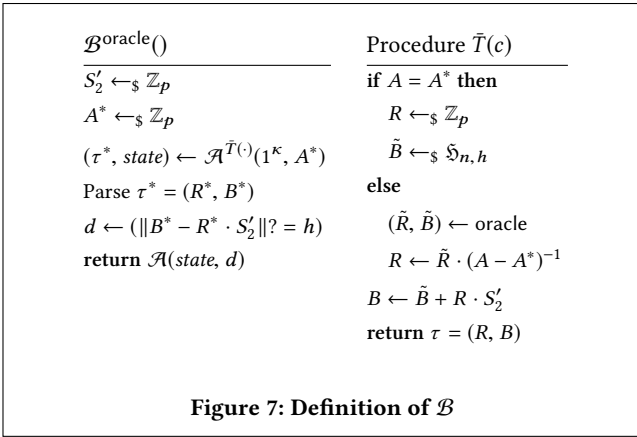
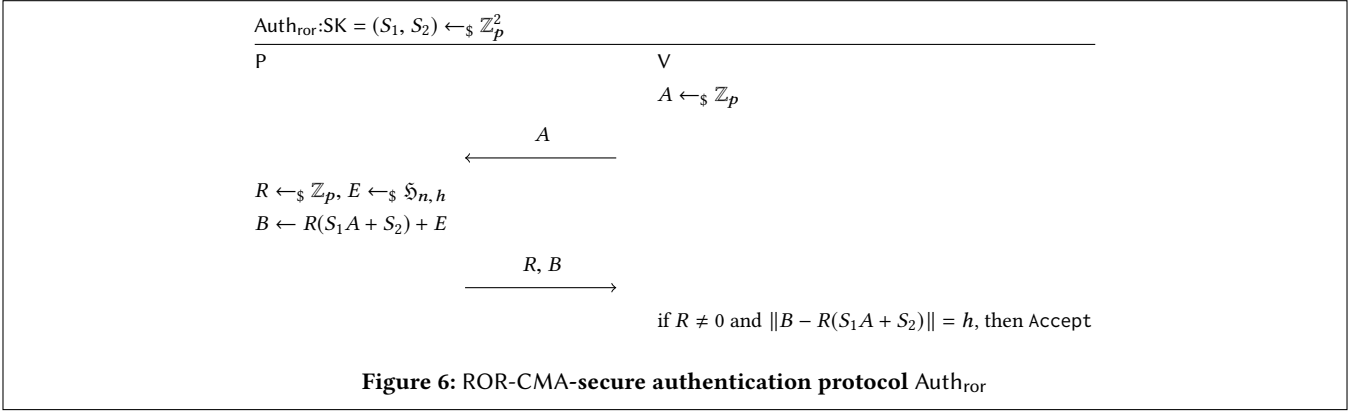
We assume that oracle returns  $(\tilde{R}, \tilde{B}) = \tilde{R}S_1 + E$ , where  $E \leftarrow_{\mathcal{S}} \mathfrak{S}_{n,h}$  or  $\mathbb{Z}_p$ .

Let us consider  $\tilde{T}(\cdot)$ , the simulation of  $T(\cdot)$ . If  $A = A^*$ , then the simulation is perfect, since  $S'_2 = S_1 A^* + S_2 \pmod{p}$  and  $B = R \cdot S'_2 + \tilde{B}$  where  $\tilde{B} \leftarrow_{\mathcal{S}} \mathfrak{S}_{n,h}$ . Otherwise, that is, if  $A \neq A^*$ , we have

$$\begin{aligned} B &= \tilde{B} + R \cdot S'_2 \\ &= \tilde{R}S_1 + E + R \cdot S'_2 \\ &= R \cdot (A - A^*)S_1 + E + R \cdot S'_2 \\ &= R \cdot (AS_1 - A^*S_1 + S'_2) + E \\ &= R \cdot (AS_1 + S_2) + E, \end{aligned}$$

where  $E$  is chosen from  $\mathfrak{S}_{n,h}$  or  $\mathbb{Z}_p$  uniformly at random.

If  $E$  is chosen from  $\mathfrak{S}_{n,h}$ , then  $(R, B)$  is distributed as a response computed by the honest prover with secret key  $(S_1, S_2)$ . On the other hand, if  $E$  is chosen from  $\mathbb{Z}_p$ , then  $(R, B)$  is uniformly distributed



over  $\mathbb{Z}_p^2$ . Therefore,  $\mathcal{B}$ 's simulations are perfect in both cases. This completes the proof.  $\square$

S-MIM attack against  $\text{Auth}_{\text{ror}}$ . Flip  $B$ 's two bits. With probability  $\approx 1/h(n-h)$ , it will modify  $E$  while keeping its Hamming weight.

## 6 S-MIM-SECURE AUTHENTICATION BASED ON MERS

Now we turn our ROR-CMA-secure protocol into a S-MIM-secure protocol by using the transformation described in Section 2.5 by using the pairwise independent hash function in Section 2.7.

We set  $\mathbb{F} := \mathbb{Z}_p$  and employ the family of pairwise independent hash functions  $\{H_{K_1, K_2}: \mathbb{Z}_p \rightarrow \mathbb{Z}_p \mid K_1, K_2 \in \mathbb{Z}_p\}$ , where  $H_{K_1, K_2}(R) = K_1 \cdot R + K_2$ . Applying the transformation, the key consists of  $K = (S_1, S_2, K_F, K_1, K_2)$ . The response to a challenge  $c$  is computed as  $\sigma = (\sigma_1, \sigma_2)$ , where

$$\sigma_1 = R \text{ and } \sigma_2 = \underbrace{(R \cdot (S_1 \cdot A + S_2) + E) \cdot K_F + K_1 \cdot R + K_2}_{=\tau_2}.$$

We can apply the compression technique in [CKT16]. Prover sends  $\sigma = (R, Z)$ , where

$$\begin{aligned} Z &= (R \cdot (S_1 \cdot A + S_2) + E) \cdot K_F + (K_1 \cdot R + K_2) \\ &= R(S_1 K_F \cdot A + S_2 K_F + K_1) + K_F \cdot E + K_2 \\ &= R(X_1 \cdot A + X_2) + X_3 \cdot E + X_4, \end{aligned}$$

by substituting  $X_1 = S_1 K_F$ ,  $X_2 = S_2 K_F + K_1$ ,  $X_3 = K_F$ , and  $X_4 = K_2$ . The verifier also checks if

$$R \neq 0 \wedge \|(Z - R(X_1 A + X_2) - X_4) \cdot X_3^{-1}\| = h$$

or not. (We can choose them as  $X_1 \leftarrow_{\S} \mathbb{Z}_p^*$ ,  $X_2 \leftarrow \mathbb{Z}_p$ ,  $X_3 \leftarrow \mathbb{Z}_p^*$ , and  $X_4 \leftarrow \mathbb{Z}_p$ .)

The compressed authentication systems, denoted by  $\text{Auth}_{\text{s-mim}}$ , is summarized as follows:

- **Public parameters:**  $n$  and  $h$  as in section 4.
- **Key generation:** The key-generation algorithm  $\text{KeyGen}$  outputs  $\text{SK} = (X_1, X_2, X_3, X_4) \leftarrow_{\S} \mathbb{Z}_p^* \times \mathbb{Z}_p \times \mathbb{Z}_p^* \times \mathbb{Z}_p$ .
- **Challenge:** The challenge is  $A \leftarrow_{\S} \mathbb{Z}_p$ .
- **Response:** The response is  $\sigma = (R, Z)$  with  $Z := R \cdot (X_1 A + X_2) + X_3 E + X_4$ , where  $R \leftarrow_{\S} \mathbb{Z}_p$  and  $E \leftarrow_{\S} \mathfrak{S}_{n,h}$ .
- **Verification:** Given a challenge  $A$  and response  $\sigma = (R, Z)$ , the verifier accepts if and only if  $R \neq 0$  and  $\|(Z - R(X_1 A + X_2) - X_4) \cdot X_3^{-1}\| = h$ .

Combining Theorem 5.1, Theorem 5.2, and Theorem 2.6, we get the following corollary.

**COROLLARY 6.1.** *If  $\text{MERS-U}_{n,h}$  is  $(t, q, \epsilon)$ -hard, then  $\text{Auth}_{\text{s-mim}}$  is  $(t', q, \epsilon')$ -S-MIM-secure, where  $t' \approx t$  and  $\epsilon' = q \cdot (\epsilon + q/p + \binom{n}{h}) / (p-1)$ .*

## 7 MAC FROM MERS

In this section, we introduce MAC based on MERS-U. Our construction is an analogue to that in [KPV<sup>+</sup>17]. The scheme  $\text{MAC} = (\text{KeyGen}, \text{Tag}, \text{Verify})$  is summarized as follows:

- **Public parameters:** The public parameters  $p(1^\kappa)$  on the security parameter  $\kappa$ , outputs the public parameters  $n$  and  $h$  as in section 4.
- **Key generation:** The algorithm  $\text{KeyGen}$ , given public parameters  $p$ , samples  $s'_0, s_0, s_1, \dots, s_\mu \leftarrow_{\S} \mathbb{Z}_p$ ,  $h: \{0, 1\}^* \times \{0, 1\}^\mu \rightarrow \{0, 1\}^\mu$ , and pairwise-independent permutation  $\pi$  over  $\mathbb{Z}_p \times \mathbb{Z}_p \times \{0, 1\}^\mu$ , and outputs  $\text{SK} := (s'_0, s_0, s_1, \dots, s_\mu, h, \pi)$ .



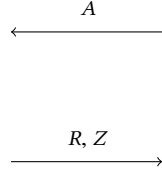
$\text{Auth}_{\text{s-mim}}:\text{SK} = (X_1, X_2, X_3, X_4) \leftarrow_{\S} \mathbb{Z}_p^* \times \mathbb{Z}_p \times \mathbb{Z}_p^* \times \mathbb{Z}_p$

**Prover**

**Verifier**

$A \leftarrow_{\S} \mathbb{Z}_p$

$R \leftarrow_{\S} \mathbb{Z}_p, E \leftarrow_{\S} \mathfrak{S}_{n,h}$   
 $Z \leftarrow R(X_1A + X_2) + X_3E + X_4$



if  $R \neq 0$  and  $\|(Z - R(X_1A + X_2) - X_4) \cdot X_3^{-1}\| = h$ , then Accept

**Figure 8: S-MIM-secure authentication protocol  $\text{Auth}_{\text{s-mim}}$**

- **Tagging:** The algorithm Tag, given a secret key SK and a message  $m \in \mathcal{M}$ . This probabilistic authentication algorithm proceeds as follows:
  - Samples  $R \leftarrow_{\S} \mathbb{Z}_p, E \leftarrow_{\S} \mathfrak{S}_{n,h}$  and  $\beta \leftarrow_{\S} \{0, 1\}^v$ .
  - Compute  $A := h(m, \beta)$ .
  - Compute  $S_A = s_0 + \sum_{i=1}^{\mu} A[i] \cdot s_i$ .
  - Compute  $B := R \cdot S_A + E + s'_0$ .
  - Output  $\sigma = \pi(R, B, \beta)$ .
- **Verification:** The algorithm Verify proceeds as follows:
  - Parse  $\pi^{-1}(\sigma)$  as  $(R, B, \beta)$ . If  $R = 0$ , then Reject.
  - Compute  $A := h(m, \beta)$  and  $S_A := s_0 + \sum_{i=1}^{\mu} A[i] \cdot s_i$ .
  - If  $\|B - (R \cdot S_A + s'_0)\| = h$  then return Accept, otherwise Reject.

Our scheme is perfectly correct.

In what follows, we let  $\alpha_{n,h} := \binom{n}{h}/p$ .

**THEOREM 7.1.** *If the MERS- $U_{n,h}$  problem is  $(t, Q, \epsilon)$ -hard, then MAC is  $(t', Q, \epsilon')$ -UF-CMA-secure, where  $t \approx t'$  and*

$$\epsilon = \min \{ \epsilon'/2 - Q^2/2^{\mu}, \epsilon'/(8\mu Q_{\text{Verify}}) - Q_{\text{Verify}}\alpha_{n,h} \},$$

where  $Q_{\text{Verify}} \leq Q$  is the number of verification queries.

We obtain our main theorem by combining two lemmas Theorem 7.6 and Theorem 7.3. In what follows, we say a forgery  $(m, \sigma)$  is *fresh* if the  $A$  contained in  $(m, \sigma)$  is different from all  $A$ 's contained in all the previous queries to  $V$  and  $T$ . For our proof, we are distinguishing two cases: the case where the probability that  $A$  is fresh is sufficiently low as  $\Pr[\text{Fresh}] \leq \epsilon'/2$ , or the complement case where  $\Pr[\text{Fresh}] > \epsilon'/2$ .

## 7.1 Proof of Lemmas

Before proving our main theorem, we review a useful lemma for fresh case.

**LEMMA 7.2.** *Consider the two games Real and Rand between a challenger and an adversary  $\mathcal{B}$  defined in Figure 9. Assume that the MERS- $U_{n,h}$  problem is  $(t, Q, \epsilon)$ -hard. Then, for all  $(t', Q)$ -adversary  $\mathcal{B}$  with  $t' \approx t$ , we have*

$$|\Pr[\text{Real}_{\mathcal{B}}(\kappa) \Rightarrow 1] - \Pr[\text{Rand}_{\mathcal{B}}(\kappa) \Rightarrow 1]| \leq 2\mu\epsilon.$$

The proof of Theorem 7.2 is in Appendix A.

**Real $_{\mathcal{B}}(\kappa)$ , Rand $_{\mathcal{B}}(\kappa)$**

$L := \emptyset$   
 $s'_0, s_0, s_1, \dots, s_{\mu} \leftarrow_{\S} \mathbb{Z}_p$   
 $d \leftarrow \mathcal{B}^{\text{Eval}(\cdot), \text{Chal}(\cdot, \cdot)}(1^{\kappa})$   
**return**  $d \wedge (A^* \notin L)$

**Oracle Chal $(R^*, A^*)$  // one query**

$S_{A^*} := s_0 + \sum_{j=1}^{\mu} A^*[j] \cdot s_j$   
 $B^* := s'_0 + R^* \cdot S_{A^*}$   
**return**  $B^*$

**Oracle Eval $(A)$**

**if**  $A \in L$  **then**  
**return**  $\perp$   
 $L \leftarrow L \cup \{A\}$   
 $S_A := s_0 + \sum_{j=1}^{\mu} A[j] \cdot s_j$   
 $R \leftarrow_{\S} \mathbb{Z}_p; E \leftarrow_{\S} \mathfrak{S}_{n,h}$   
**if** Real **then**  
 $B := s'_0 + R \cdot S_A + E$   
**if** Rand **then**  
 $B \leftarrow_{\S} \mathbb{Z}_p$   
**return**  $\tau = (R, B)$

**Figure 9: Definition of Real and Rand**

### 7.1.1 Fresh Case.

**LEMMA 7.3.** *Suppose that there exists an adversary  $\mathcal{A}$  that breaks  $(t', Q, \epsilon')$ -UF-CMA-security of MAC. If the probability that the first forgery found by the adversary is more likely to be fresh:  $\Pr[\text{Fresh}] > \epsilon'/2$ , then we have another  $(t, Q, \epsilon)$ -adversary  $\mathcal{B}$  that breaks MERS- $U_{n,h}$  with*

$$t \approx t' \text{ and } \epsilon \geq \epsilon'/(4\mu Q_{\text{Verify}}) - Q_{\text{Verify}}\alpha_{n,h}.$$

where  $Q_{\text{Verify}} \leq Q$  is the number of verification queries.

**PROOF OF THEOREM 7.3.** We define the following games:

- Let  $G_0$  be the original security game  $\text{Exp}^{\text{uf-cma}}$ .
- Let  $G_j$  for  $j = 1, \dots, Q_{\text{Verify}}$  denote the games where the adversary is allowed to ask only  $j$  verification queries.
- We also define  $G'_j$  as same as the game  $G_j$  except that the tag oracle will use random  $R, B, \beta$  to compute  $\sigma$  instead of the real computation.

As [KPV<sup>+</sup>17], we have

$$\epsilon'/2 < \Pr[\text{Fresh}] = \Pr[G_0 = 1] \leq \sum_j^{Q_{\text{Verify}}} \Pr[G_j = 1].$$

Thus, what we should do is bounding  $\Pr[G_j = 1]$ .

CLAIM 7.4. Assume that  $\mathcal{A}$  is a  $(t, Q)$ -adversary. for all  $j$ , there exists a  $(t', Q)$ -adversary  $\mathcal{B}$  such that  $t' \approx t$  and

$$|\Pr[G_j = 1] - \Pr[G'_j = 1]| \leq |\Pr[\text{Real}_{\mathcal{G}}(\kappa) \Rightarrow 1] - \Pr[\text{Rand}_{\mathcal{G}}(\kappa) \Rightarrow 1]|.$$

PROOF. We construct  $\mathcal{B}$  as follows:

- (1)  $\mathcal{B}$  samples  $h$  and  $\pi$ .
- (2)  $\mathcal{B}$  runs  $\mathcal{A}$  on input  $1^\kappa$  and simulates the oracles as follows:
  - $T(m)$ :
    - (a) sample a random  $\beta \leftarrow_{\mathcal{S}} \{0, 1\}^v$  and compute  $A = h(m, \beta)$ .
    - (b) query  $A$  to oracle Eval and obtain a pair  $(R, B)$ .
    - (c) return  $\sigma := \pi(R, B, \beta)$ .
  - $V(m, \sigma)$ :
    - (a) if  $(m, \sigma)$  is previously returned to  $\mathcal{A}$ , then  $\mathcal{B}$  returns Accept.
    - (b) if  $(m, \sigma)$  is not  $j$ -th verification query, then  $\mathcal{B}$  returns Reject.
    - (c) if  $(m, \sigma)$  is the  $j$ -th verification query; we call it  $(m^*, \sigma^*)$ . let  $(R^*, B^*, \beta^*) := \pi^{-1}(\sigma^*)$ ; compute  $A^* := h(m^*, \beta^*)$ ; send  $(R^*, A^*)$  to oracle Chal and obtain  $B'$ . If  $\|B^* - B'\| = h$ , then return Accept. otherwise, return Reject.

The  $j$ -th verification query is fresh by the definition. In addition, since the oracle Chal returns  $B' := s'_0 + R^* \cdot S_{A^*}$ , this simulated verification procedure correctly checks the Hamming weight of  $\|B^* - (s'_0 + R^* \cdot S_{A^*})\|$  as the correct verification. Therefore, the simulation is perfect if  $A^*$  is fresh as we wanted.  $\square$

CLAIM 7.5. for all  $j$ ,

$$\Pr[G'_j = 1] \leq \alpha_{n,h}$$

PROOF. Fix a value  $j \in \{1, \dots, Q_{\text{Verify}}\}$ . In game  $G'_j$ , the adversary obtains no information on  $(s'_0, s_0, s_1, \dots, s_\mu)$  from the tagging oracle  $T(\cdot)$  because the oracle returns random values  $(R, B)$ . Therefore, the value  $X := B^* - B' = B^* - (R^* \cdot S_{A^*} + s'_0)$  should be uniformly at random over  $\mathbb{Z}_p$ , since  $s'_0$  is kept secret. Thus, the probability that the verification  $\|B^* - B'\| = h$  passes is at most

$$\Pr[X \leftarrow \mathbb{Z}_p : \|X\| = h] = \binom{n}{h}/p = \alpha_{n,h}.$$

$\square$

Combining those two claims, we obtain the following result: If  $\mathcal{A}$  is  $(t, Q)$ -adversary, then there is a  $(t', Q)$ -adversary  $\mathcal{B}$  such that  $t' \approx t$  and

$$\begin{aligned} \Pr[G_j = 1] &\leq \Pr[G'_j = 1] + |\Pr[G_j = 1] - \Pr[G'_j = 1]| \\ &\leq \alpha_{n,h} + |\Pr[\text{Real}_{\mathcal{G}}(\kappa) \Rightarrow 1] - \Pr[\text{Rand}_{\mathcal{G}}(\kappa) \Rightarrow 1]| \end{aligned}$$

as we wanted. Applying Theorem 7.2, we have

$$\Pr[G_j = 1] \leq \alpha_{n,h} + 2\mu\epsilon$$

under the assumption that the MERS- $U_{n,h}$  problem is  $(t, Q, \epsilon)$ -hard. Therefore, we have

$$\epsilon'/2 \leq \sum_j^{Q_{\text{Verify}}} \Pr[G_j = 1] \leq Q_{\text{Verify}}\alpha_{n,h} + 2Q_{\text{Verify}}\mu\epsilon.$$

This yields

$$\epsilon \geq \epsilon'/(4Q_{\text{Verify}}\mu) - Q_{\text{Verify}}\alpha_{n,h}$$

as we wanted.  $\square$

### 7.1.2 Non-Fresh Case.

LEMMA 7.6. Let  $\mu = v$ . Suppose that there exists an adversary  $\mathcal{A}$  that breaks  $(t', Q, \epsilon')$ -UF-CMA-security of MAC. If the probability that the first forgery found by the adversary is more likely to be non-fresh, that is,  $\Pr[\text{Fresh}] \leq \epsilon'/2$ , then we have  $\mathcal{B}$  that breaks the  $(t, Q, \epsilon)$ -hardness of the MERS- $U_{n,h}$  problem, where

$$t \approx t' \text{ and } \epsilon \geq \epsilon'/2 - Q^2/2^\mu.$$

PROOF. This proof is similar to the proof of the ROR-CMA security in section 5.

Let us construct an adversary  $\mathcal{B}^{\text{oracle}}$  who will distinguish between two oracles  $\mathcal{O}$  and  $\mathcal{U}$ .

$\mathcal{B}$  samples  $\pi, h, s'_0, s_1, \dots, s_\mu$  except  $s_0$  as defined in KeyGen. It then runs  $\mathcal{A}$  and simulates the oracles as follows:

- $T(m)$ : On a query  $m$ ,
  - (1) Sample  $\beta$  and compute  $A := h(m, \beta)$
  - (2) Call the oracle and obtain  $(\tilde{R}, \tilde{B})$
  - (3) Compute  $B := \tilde{B} + \tilde{R} \cdot (\sum_{i=1}^\mu A[i] \cdot s_i) + s'_0$
  - (4) Return  $\sigma := \pi(\tilde{R}, B, \beta)$
- $V(m, \sigma)$ : On a query  $(m, \sigma)$ ,  $\mathcal{B}$  always answers Reject.

Finally,  $\mathcal{B}^{\text{oracle}}$  outputs 1 if any query to  $T$  or  $V$  contains  $\beta$  that has appeared in a previous query to  $T$  or  $V$ . It outputs 0 otherwise.

We note that if oracle =  $\mathcal{O}_{s,n,h}$ , then  $\tilde{B} = \tilde{R} \cdot s + e$ , where  $e \leftarrow_{\mathcal{S}} \mathfrak{S}_{n,h}$  and the simulation of  $T$  is perfect by letting  $s_0 := s$ .

CLAIM 7.7. If oracle =  $\mathcal{O}_{s,n,h}$ , then the probability that  $\mathcal{B}^{\text{oracle}}$  outputs 1 is  $\geq \epsilon'/2$

PROOF. The proof is the same as that in [KPV<sup>+</sup>17, Proof of Claim 4.5]. The simulation of  $T$  is perfect. In addition, until  $\mathcal{A}$  makes a valid forgery, the simulation of  $V$  is also perfect. The probability that  $\mathcal{A}$  output his first forgery which is *not* fresh is simply lower bounded by  $\epsilon' - \epsilon'/2 = \epsilon'/2$ . Thus, we obtain the lower bound in the claim.  $\square$

CLAIM 7.8. If oracle =  $\mathcal{U}$ , then the probability that  $\mathcal{B}^{\text{oracle}}$  outputs 1 is at most  $\leq Q^2/2^\mu$ .

PROOF. The proof is the same as that in [KPV<sup>+</sup>17, Proof of Claim 4.6].

We have  $A_i = A_j$  if and only if  $h(m_i, \beta_i) = h(m_j, \beta_j)$ . Now we will upper bound the probability that an adversary find such collision which imply the same probability that  $\mathcal{B}^{\text{oracle}}$  outputs 1, assuming that an adversary makes at most  $Q$  queries and fixing that up to the  $(i-1)$ -th query by which we assume that all the  $\mathcal{A}'$ 's were distinct. Then we obtain two cases of collision:

- The probability of collision that the  $i$ -th query in which  $\beta_i$  will collide with a previous  $\beta_j$  is at most  $(i-1)/2^v$ .
- If the first collision does not happen then the probability of collision in  $h(m_i, \beta_i) = h(m_j, \beta_j)$  will be  $(i-1)/2^\mu$ .

Then similarly to the proof in [KPC<sup>+</sup>11] we obtain  $\sum_{n=1}^Q ((i-1)/2^v + (i-1)/2^\mu) \leq Q^2/2^\mu$  where  $\mu = v$ .  $\square$

Combining two claims, we have

$$\epsilon \geq \epsilon'/2 - Q^2/2^\mu$$

as we wanted. □

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## A PROOF OF LEMMA 7.2

The proof is almost same as that in [KPV<sup>+</sup>17].

For  $i = 0, \dots, \mu$  and  $A \in \{0, 1\}^\mu$ , we define  $A[1..i]$  as the  $i$ -bit string  $A_1 \dots A_i \in \{0, 1\}^i$ . (We let  $A[1..0] = \perp$ .) For  $i = 0, \dots, \mu$ ,  $\text{RF}_i, \text{RF}'_i: \{0, 1\}^i \rightarrow \mathbb{Z}_p$  be two random functions. (If  $i = 0$ , then  $\text{RF}_0(\perp) = b'$  for some random  $b' \leftarrow_{\S} \mathbb{Z}_p$ .)

We define the line of games as follows:

- $G_0$ : this game is the same as Real except that
  - in the beginning, we sample  $2\mu$  elements  $s_{1,0}, \dots, s_{\mu,0}, s_{1,1}, \dots, s_{\mu,1}$  from  $\mathbb{Z}_p$  instead of  $\mu+1$  elements  $s_0, s_1, \dots, s_\mu$  from  $\mathbb{Z}_p$ .
  - in the computation of  $S_A$ , we compute  $S_A := \sum_{j=1}^\mu s_{j,A[j]}$  instead of  $S_A := s_0 + \sum_{j=1}^\mu A[j] \cdot s_j$ . (We also replace the computation of  $S_{A^*}$ .)
- $G_{1,i}$  for  $i = 0, \dots, \mu$ : this game is the same as  $G_0$  except that
  - in the oracle Chal, we let  $s'_0 := \text{RF}_i(A^*[1..i])$
  - in the oracle Eval, we compute  $B := \text{RF}_i(A[1..i]) + RS_A + E$  instead of  $B := s'_0 + RS_A + E$ .
- $G_2$ : this game is the same as  $G_{1,\mu}$  except that
  - in the oracle Chal, we sample  $B^* \leftarrow_{\S} \mathbb{Z}_p$  instead of  $B^* := s'_0 + R^* \cdot S_{A^*}$
  - in the oracle Eval, we compute  $B := \text{RF}_\mu(A)$  instead of  $B := \text{RF}_\mu(A) + RS_A + E$ .

LEMMA A.1.  $\Pr[G_0 = 1] = \Pr[\text{Real} \Rightarrow 1]$

PROOF. In  $G_0$ , we replace the computation of  $S_A$ . We note that if we set  $s_0 := \sum_{j=1}^\mu s_{j,0}$  and  $s_j := s_{j,1} - s_{j,0}$ , we have  $S_A = s_0 + \sum_{j=1}^\mu A[j] \cdot s_j = \sum_{j=1}^\mu s_{j,A[j]}$ . In addition, if we choose  $s_{j,k}$  uniformly at random, then  $s_0, s_1, \dots, s_\mu$  are also distributed according to the uniform distribution over  $\mathbb{Z}_p$ . Hence, the two games are equivalent.  $\square$

LEMMA A.2. We have  $\Pr[G_0 = 1] = \Pr[G_{1,0} = 1]$ .

PROOF.  $G_0$  is the same as  $G_{1,0}$ , since  $s'_0$  can be interpreted as  $\text{RF}_0(\perp)$  [KPV<sup>+</sup>17].  $\square$

LEMMA A.3. Let  $\mathcal{B}$  be a  $(t, Q)$ -adversary. For all  $i \in \{0, \dots, \mu-1\}$ , there exists a  $(t', Q)$ -adversary  $\mathcal{D}$  such that

$$t' \approx t \text{ and } |\Pr[G_{1,i} = 1] - \Pr[G_{1,i+1} = 1]| \leq 2 \cdot \text{Adv}_{\mathcal{D}}^{\text{MERS-U}_{n,h}}(\kappa).$$

PROOF. Notice that for arbitrarily fixed  $b \in \{0, 1\}$  and two random functions  $\text{RF}_i$  and  $\text{RF}'_i$ , we can define a new random function  $\text{RF}_{i+1}$  by

$$\text{RF}_{i+1}(A[1\dots i+1]) := \begin{cases} \text{RF}_i(A[1\dots i]) & \text{if } A[i+1] = b \\ \text{RF}_i(A[1\dots i]) + \text{RF}'_i(A[1\dots i]) & \text{o.w.} \end{cases}$$

Our adversary  $\mathcal{D}$  guesses  $E \leftarrow_{\S} \{0, 1\}$  as the prediction of  $A^*[i+1]$  and simulate the oracles by using the above observation. We construct a distinguisher  $\mathcal{D}$  as follows:

- (1) Given  $1^\kappa$ ,  $\mathcal{D}$  prepares parameter values as follows:
  - Sample  $b \leftarrow \{0, 1\}$  and initialize  $L := \emptyset$  and  $L_i := \emptyset$ .
  - Choose  $s_{j,\beta} \leftarrow \mathbb{Z}_p$  for all  $j \in [1, \mu]$  and  $\beta \in \{0, 1\}$  except for  $s_{i+1,1-b}$ .
  - Query to its oracle for  $Q$  times and obtain the answers  $(R_j, B'_j)$  for  $j \in [Q]$ .
- (2)  $\mathcal{D}$  runs  $\mathcal{B}$  and simulates Eval and Chal as follows:
  - Simulation of Eval on input  $A \in \{0, 1\}^\mu$ :
    - (a) Update  $L := L \cup \{A\}$
    - (b) If  $A[i+1] = b$ , then  $R \leftarrow_{\S} \mathbb{Z}_p$ ,  $E \leftarrow_{\S} \mathfrak{S}_{n,h}$ , compute  $B := \text{RF}_i(A[1\dots i]) + R \cdot (\sum_{j=1}^\mu s_{j,A[j]}) + E$  and return  $(R, B)$ .
    - (c) Else, that is, if  $A[i+1] = 1-b$ , then
      - (i) If  $L_i$  contains  $(A[1\dots i], (R_j, B'_j))$  for some  $j$ , then let  $(R, B') := (R_j, B'_j)$ .
      - (ii) Else, use a next fresh pair, that is,  $(R, B') := (R_j, B'_j)$  for the first  $j$ . Add  $(A[1\dots i], (R_j, B'_j))$  to the list  $L_i$ .
      - (iii) Compute  $B := \text{RF}_i(A[1\dots i]) + R \cdot (\sum_{j=1, j \neq i+1}^\mu s_{j,A[j]}) + B'$  and return  $(R, B)$ .
  - Simulation of Chal on input  $R^*$  and  $A^*$ :
    - (a) If  $A^*[i+1] \neq b$ , abort.
    - (b) Else, define  $S_{A^*} := \sum_{j=1}^\mu s_{j,A^*[j]}$ .
    - (c) Return  $B^* := R^* \cdot S_{A^*} + \text{RF}_i(A^*[1\dots i])$ .
- (3) Finally,  $\mathcal{B}$  will outputs its decision  $d$  and stops.  $\mathcal{D}$  outputs  $d \wedge (A^* \notin L)$ .

Suppose that the guess  $b$  is correct. This happens with probability  $1/2$ . If so,  $\mathcal{D}$  perfectly simulates Chal, since  $\text{RF}_{i+1}(A^*[1\dots i+1]) = \text{RF}_i(A^*[1\dots i])$  if  $A^*[i+1] = b$ . We next analyze the simulation of Eval: If  $A[i+1] = b$ , then we have  $\text{RF}_{i+1}(A[1\dots i+1]) = \text{RF}_i(A[1\dots i])$ . Thus, the distributions of  $Z$  are equal each other. Otherwise, that is, if  $A[i+1] = 1-b$ , then we consider two cases: If the oracle outputs  $B' := Rs + E$  with  $E \leftarrow_{\S} \mathfrak{S}_{n,h}$ , then we have

$$\begin{aligned} B &:= \text{RF}_i(A[1\dots i]) + R \cdot \left( \sum_{j=1, j \neq i+1}^\mu s_{j,A[j]} \right) + R \cdot s + E \\ &= \text{RF}_i(A[1\dots i]) + R \cdot \left( \sum_{j=1}^\mu s_{j,A[j]} \right) + E \end{aligned}$$

by letting  $s_{i+1,1-b} := s$ . Therefore, if the oracle is  $\mathcal{O}_{s,n,h}$ , then  $\mathcal{D}$  perfectly simulates  $G_i$ . On the other hand, if the oracle is  $\mathcal{U}$ , that

is,  $B' = Rs + E + U$  with  $E \leftarrow_{\mathcal{S}} \mathfrak{S}_{n,h}$  and  $U \leftarrow_{\mathcal{S}} \mathbb{Z}_p$ , then we have

$$\begin{aligned} B &:= \text{RF}_i(A[1\dots i]) + R \cdot \left( \sum_{j=1, j \neq i+1}^{\mu} s_{j,A[j]} \right) + R \cdot s + E + U \\ &= \text{RF}_i(A[1\dots i]) + U + R \cdot \left( \sum_{j=1}^{\mu} s_{j,A[j]} \right) + E. \end{aligned}$$

By letting  $U := \text{RF}'_i(A[1\dots i])$ , we observe that  $\mathcal{D}$  perfectly simulates  $G_{i+1}$ .

Therefore, we have

$$t' \approx t \text{ and } |\Pr[G_{1,i} = 1] - \Pr[G_{1,i+1} = 1]| = 2 \cdot \text{Adv}_{\mathcal{D}}^{\text{MERS-U}_{n,h}(\kappa)}$$

as we wanted.  $\square$

LEMMA A.4. *We have  $\Pr[G_{1,\mu} = 1] = \Pr[G_2 = 1]$ .*

PROOF. This is almost obvious. Notice that every query  $A$  to Eval and Chal should be fresh. Thus, in both cases,  $\text{RF}_{\mu}(A)$  makes  $B$  (and  $B^*$ ) random.  $\square$

LEMMA A.5. *We have  $\Pr[G_2 = 1] = \Pr[\text{Rand} \Rightarrow 1]$ .*

PROOF. In  $G_2$ , all returned values  $(R, B)$  from Eval and  $B^*$  from Chal are fresh and random if  $A^* \notin L$ . We also know that in Rand, all values are fresh and random if  $A^* \notin L$ , because  $s'_0$  is random and kept secret. Therefore, there are no difference between  $G_2$  and Rand if  $A^* \notin L$ . This completes the proof.  $\square$