New Semi-Free-Start Collision Attack Framework for Reduced RIPEMD-160

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Abstract. RIPEMD-160 is a hash function published in 1996, which shares similarities with other hash functions designed in this time-period like MD4, MD5 and SHA-1. However, for RIPEMD-160, no (semi-free-start) collision attacks on the full number of steps are known. Hence, it is still used, e.g., to generate Bitcoin addresses together with SHA-256, and is an ISO/IEC standard. Due to its dual-stream structure, even semi-free-start collision attacks starting from the first step only reach 36 steps, which were firstly shown by Mendel et al. at Asiacrypt 2013 and later improved by Liu, Mendel and Wang at Asiacrypt 2017. Both of the attacks are based on a similar freedom degree utilization technique as proposed by Landelle and Peyrin at Eurocrypt 2013. However, the best known semi-free-start collision attack on 36 steps of RIPEMD-160 presented at Asiacrypt 2017 still requires $2^{55.1}$ time and 2^{32} memory. Consequently, a practical semi-free-start collision attack for the first 36 steps of RIPEMD-160 still requires a significant amount of resources. Considering the structure of these previous semi-free-start collision attacks for 36 steps of RIPEMD-160, it seems hard to extend it to more steps. Thus, we develop a different semi-free-start collision attack framework for reduced RIPEMD-160 by carefully investigating the message expansion of RIPEMD-160. Our new framework has several advantages. First of all, it allows to extend the attacks to more steps. Second, the memory complexity of the attacks is negligible. Hence, we were able to give a practical semi-free-start collision attack on 36 steps of RIPEMD-160 with time complexity 2^{41}. Additionally, we describe semi-free-start collision attacks on 37, 38 and 40 (out of 80) steps of RIPEMD-160 with time complexity 2^{49}, 2^{53} and $2^{74.6}$, respectively. To the best of our knowledge, these are the best semi-free-start collision attacks for RIPEMD-160 starting from the first step with respect to the number of steps, including the first practical colliding message pairs for 36 steps of RIPEMD-160.

Keywords: hash function · RIPEMD-160 · freedom degree utilization · semi-free-start collision attack

1 Introduction

In the 1990s, most popular hash functions, like MD4, MD5, RIPEMD-160 [DBP96], and SHA-1 followed a similar design strategy based on round functions involving modular additions, word-wise rotations, and XORs (ARX). For 3 out of the aforementioned hash functions, MD4 [Dob96, WLF+05], MD5 [WY05], and SHA-1 [WYY05, SBK+17] practical collision attacks were shown and thus have been phased out in most applications. However, if we look at RIPEMD-160, no collision attack on the full number of rounds is known. Moreover, RIPEMD-160 is still used in several applications, e.g., to generate Bitcoin addresses together with SHA-256, and is still an ISO/IEC standard. Hence, getting more insight into the security of RIPEMD-160 is of practical interest and importance.

In contrast to MD4, MD5, and SHA-1, the compression function of RIPEMD-160 is of a more complex nature, since the chaining value is duplicated and processed in two branches. Both branches, hereby employ a slightly different round function and also the message expansion follows a different pattern. At the end of the compression function, both branches are merged again to form the 160-bit internal state or final hash value. This increased complexity seems to complicate the analysis, and in contrast to MD4 [Dob96, WLF+05], MD5 [WY05], and SHA-1 [WYY05, SBK+17], collision attacks on RIPEMD-160 do not reach the full number of rounds.

The first results regarding collision attacks on reduced RIPEMD-160 are by Liu, Mendel and Wang at Asiacrypt 2017 [LMW17]. Their attack follows the idea of using a differential characteristic that is sparse on the left branch and dense on the right branch, where message modification is used to fulfill as many conditions as possible on the dense right stream. This strategy allows for collision attacks [LMW17] reaching 30 steps with a time complexity of 2^{70}. Recently, at Crypto 2019 [LDM+18], a different strategy to find collisions was proposed, where the dense part is placed on the left branch the sparse part is placed on the right branch. As a result, they provided the first colliding message pairs for 30 and 31 steps of RIPEMD-160 and a theoretical collision attack up to 34 steps. When considering semi-free-start collisions, attacks on step-reduced versions of RIPEMD-160 reach up to 48 steps [WSL17] if an attacker has the freedom to choose the step where the attack starts and 36 steps [MPS+13, LMW17] if they start from the first step. A summary of collision attacks for RIPEMD-160 is given in Table 1.

It should be noted that for the two semi-free-start collision attacks for reduced RIPEMD-160 starting from the first step, both share the same differential characteristic [MPS+13, LMW17]. Moreover, the underlying idea of the two semi-free-start collision attacks is almost the same. Specifically, the dense parts with many differential conditions on both branches are firstly fixed. Then, the remaining free message words are utilized to achieve efficient merging at the initial value, which is following the idea from [LP13].

Up until now, no practical semi-free-start collision attack on the first 36 steps of RIPEMD-160 is achieved. Moreover, a semi-free-start collision attack on more steps of RIPEMD-160 starting from the first step was out of reach as well. Thus, we are motivated to further investigate the semi-free-start collision resistance of reduced RIPEMD-160. To do so, we place the dense differential characteristic on the left branch and the sparse differential characteristic on the right branch in order to make the new semi-free-start collision attack framework work efficiently, which follows a similar spirit as in [LDM+18]. The contribution of this paper is summarized below.

1.1 Our Contributions

With a new freedom degree utilization strategy, we develop a semi-free-start collision attack framework for reduced RIPEMD-160. Different from previous semi-free-start collision attack frameworks [MPS+13, LMW17] for RIPEMD-160 which require a costly freedom degrees consumption to achieve efficient merging at the initial value, no merging phase is needed under the new attack framework. With such a new framework, we were able to extend the semi-free-start collision attacks on reduced RIEPMD-160 to more steps. In addition, there are negligible memory requirements. Most importantly, combined with the use of automated techniques [MNS11, MNS13, EMS14] to solve the nonlinear differential characteristic for RIPEMD-160, improved semi-free-start collision attacks for reduced RIPEMD-160 are obtained, as specified below.

- The first practical semi-free-start collision attack on 36 steps of RIPEMD-160 is achieved with time complexity 2^{41}.
- Semi-free-start collision attack on 37 steps is achived with time complexity 2^{49}.

- Semi-free-start collision attack on 38 steps is achieved with time complexity 2^{53}.
- Semi-free-start collision attack on 40 steps is achieved with time complexity $2^{74.6}$.

1.2 Organization

This paper is organized as follows. The notation, and description of RIPEMD-160 is given in Section 2. Then, we describe our semi-free-start collision attack framework for reduced RIPEMD-160 in Section 3. Next, we discuss how to get a desirable differential characteristic in Section 4. Section 5 presents the application of our semi-free-start collision attack to the discovered differential characteristics. Finally, the paper is concluded in Section 6.

Table 1: Summary of preimage and collision attack on RIPEMD-160

Target	Attack Type	Steps	Time	Memory	Ref.
comp. function	preimage	31	2^{148}	2^{17}	[OSS12]
hash function	preimage	31	2^{155}	2^{17}	[OSS12]
comp. function	semi-free-start collision	36[a]	low	negligible	[MNSS12]
		42[a]	$2^{75.5}$	2^{64}	[MPS+13]
		48[a]	$2^{76.4}$	2^{64}	[WSL17]
		36	$2^{70.4}$	2^{64}	[MPS+13]
		36	$2^{55.1}$	2^{32}	[LMW17]
		36	2^{41}	negligible	Section 5.1
		37	2^{49}	negligible	Section 5.2
		38	2^{53}	negligible	Section 5.2
		40	$2^{74.6}$	negligible	Section 5.2
hash function	collision	30	$2^{35.9}$	2^{32}	[LDM+18]
		31	$2^{41.5}$	2^{32}	[LDM+18]
		33	$2^{67.1}$	2^{32}	[LDM+18]
		34	$2^{74.3}$	2^{32}	[LDM+18]

[a] An attack starting at an intermediate step.

2 Preliminaries

In this section, we will introduce the notations used in this paper and the specification of RIPEMD-160.

2.1 Notation

1. $\gg, \lll, \ggg, \oplus, \vee, \wedge$ and $\neg$ represent the logic operations *shift right, rotate left, rotate right, exclusive or, or, and, negate*, respectively.
2. $\boxplus$ and $\boxminus$ represent addition and subtraction modulo 2^{32}.
3. $M = (m_0, m_1, ..., m_{15})$ and $M' = (m'_0, m'_1, ..., m'_{15})$ represent two 512-bit message blocks split into 32-bit words m_i and m'_i.
4. K^l_j and K^r_j represent the constant used for left (l) and right (r) branch at round j.
5. Φ^l_j and Φ^r_j represent the 32-bit Boolean function for the left (l) and right (r) branch at round j.

6. s_i^l and s_i^r represent the rotation constant used at the left (l) and right (r) branch during step i.
7. $\pi_1(i)$ and $\pi_2(i)$ represent the index of the message word used at the left (l) and right (r) branch during step i.
8. X_i, Y_i represent the 32-bit internal state of the left (l) and right (r) branch updated during step i.
9. $X_{i,k}$ and $Y_{i,k}$ represent the $(k+1)$-th bit of X_i and Y_i, where the least significant bit is the 1st bit and the most significant bit is the 32nd bit. For example, $X_{i,0}$ represents the least significant bit of X_i.
10. MIN(a,b) represents the minimal value of a and b. MIN$(a,b) = a$ if $a \leq b$ and MIN$(a,b) = b$ if $a > b$.

We also adopt the concept of generalized conditions of De Cannière and Rechberger [DR06] presented in Table 2.

Table 2: Generalized conditions [DR06]

(x, x^*)	(0,0)	(1,0)	(0,1)	(1,1)	(x, x^*)	(0,0)	(1,0)	(0,1)	(1,1)
?	✓	✓	✓	✓	3	✓	✓	–	–
–	✓	–	–	✓	5	✓	–	✓	–
x	–	✓	✓	–	7	✓	✓	✓	–
0	✓	–	–	–	A	–	✓	–	✓
u	–	✓	–	–	B	✓	✓	–	✓
n	–	–	✓	–	C	–	–	✓	✓
1	–	–	–	✓	D	✓	–	✓	✓
♯	–	–	–	–	E	–	✓	✓	✓

- x represents one bit of the first message and x^* represents the same bit of the second message.

2.2 Description of RIPEMD-160

RIPEMD-160 is a 160-bit hash function based on the Merkle-Damgård construction. So it is iterating a compression function H that takes as input a 512-bit message block M_i and a 160-bit chaining variable CV_i. We refer to [DBP96] for a detailed description of the RIPEMD-160 hash function and focus on the compression function next. The RIPEMD-160 compression function consists of two different parallel branches, which we call left and right branch, indicated by the use of X_i and Y_i, respectively. The compression function is segregated into 5 rounds of 16 steps each in both branches, leading to a total of 80 steps per branch.

2.2.1 Initialization

The compression function starts with an initialization, where the 160-bit chaining variable CV_i at the input is divided into five 32-bit words h_j $(j = 0, 1, 2, 3, 4)$. Those five words h_j are used to initialize the state of the two branches:

$$X_{-4} = h_0^{\ggg 10}, \quad X_{-3} = h_4^{\ggg 10}, \quad X_{-2} = h_3^{\ggg 10}, \quad X_{-1} = h_2, \quad X_0 = h_1.$$
$$Y_{-4} = h_0^{\ggg 10}, \quad Y_{-3} = h_4^{\ggg 10}, \quad Y_{-2} = h_3^{\ggg 10}, \quad Y_{-1} = h_2, \quad Y_0 = h_1.$$

The initial value (CV_0) corresponds to:

$X_{-4} = Y_{-4} =$ `0xc059d148`, $X_{-3} = Y_{-3} =$ `0x7c30f4b8`, $X_{-2} = Y_{-2} =$ `0x1d840c95`,

$X_{-1} = Y_{-1} =$ `0x98badcfe`, $X_0 = Y_0 =$ `0xefcdab89`.

2.2.2 Message Expansion

Each 512-bit input message block is divided into 16 32-bit message words m_i. The words m_i will be used for a single step in a permuted order π_1 and π_2 for left branch and right branch, respectively.

2.2.3 Step Function

At step i of round j, the internal state is updated in the following way.

$$\begin{aligned} LQ_i &= X_{i-5}^{\lll 10} \boxplus \Phi_j^l(X_{i-1}, X_{i-2}, X_{i-3}^{\lll 10}) \boxplus m_{\pi_1(i)} \boxplus K_j^l, \\ X_i &= X_{i-4}^{\lll 10} \boxplus (LQ_i)^{\lll s_i^l}, \\ RQ_i &= Y_{i-5}^{\lll 10} \boxplus \Phi_j^r(Y_{i-1}, Y_{i-2}, Y_{i-3}^{\lll 10}) \boxplus m_{\pi_2(i)} \boxplus K_j^r, \\ Y_i &= Y_{i-4}^{\lll 10} \boxplus (RQ_i)^{\lll s_i^r}, \end{aligned}$$

where $i = (1, 2, 3, ..., 80)$ and $j = (0, 1, 2, 3, 4)$. The details of the Boolean functions and round constants for RIPEMD-160 are given in Table 3.

Table 3: Boolean Functions and Round Constants in RIPEMD-160

Round j	ϕ_j^l	ϕ_j^r	K_j^l	K_j^r	Function	Expression
0	XOR	ONX	0x00000000	0x50a28be6	$XOR(x, y, z)$	$x \oplus y \oplus z$
1	IFX	IFZ	0x5a827999	0x5c4dd124	$IFX(x, y, z)$	$(x \wedge y) \oplus (\neg x \wedge z)$
2	ONZ	ONZ	0x6ed9eba1	0x6d703ef3	$IFZ(x, y, z)$	$(x \wedge z) \oplus (y \wedge \neg z)$
3	IFZ	IFX	0x8f1bbcdc	0x7a6d76e9	$ONX(x, y, z)$	$x \oplus (y \vee \neg z)$
4	ONX	XOR	0xa953fd4e	0x00000000	$ONZ(x, y, z)$	$(x \vee \neg y) \oplus z$

2.2.4 Finalization

The finalization is performed after all 80 steps have been executed in both branches. The five 32-bit words h'_j $(j = 0, 1, 2, 3, 4)$ composing the output chaining variable are computed in the following way involving also the chaining value at the input of the compression function h_j $(j = 0, 1, 2, 3, 4)$:

$$\begin{aligned} h_0^{'} &= h_1 \boxplus X_{79} \boxplus Y_{78}^{\lll 10}, \\ h_1^{'} &= h_2 \boxplus X_{78}^{\lll 10} \boxplus Y_{77}^{\lll 10}, \\ h_2^{'} &= h_3 \boxplus X_{77}^{\lll 10} \boxplus Y_{76}^{\lll 10}, \\ h_3^{'} &= h_4 \boxplus X_{76}^{\lll 10} \boxplus Y_{80}, \\ h_4^{'} &= h_0 \boxplus X_{80} \boxplus Y_{79}. \end{aligned}$$

3 Semi-Free-Start Collision Attack Framework

In this section, we will present the details of the new semi-free-start collision attack framework. For this framework, the message difference is inserted only at m_{12}, which is used to update X_{13} and Y_{16}. Then, we carefully build the differential characteristic on the right branch to make it hold with a probability as high as possible. The dense part of the differential characteristic on the left branch will be solved by using automated techniques detailed in [MNS11, MNS13, EMS14]. The pattern of the constructed t-step differential characteristic can be depicted in Figure 1. Following such a strategy to construct the differential characteristic, we can obtain Observation 1.

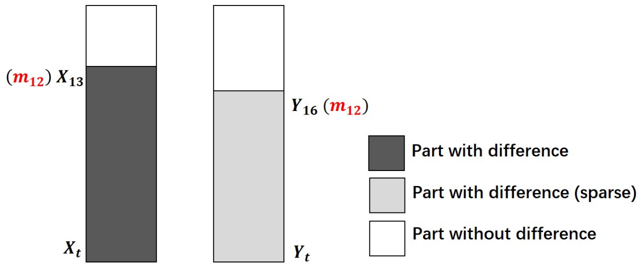

Figure 1: Attack on t steps of RIPEMD-160 by inserting difference at m_{12}

X_{13}	X_{14}	X_{15}	X_{16}	X_{17}
m_{12}	m_{13}	m_{14}	m_{15}	m_7

X_{18}	X_{19}	X_{20}	X_{21}	X_{22}	X_{23}	X_{24}	X_{25}	X_{26}	X_{27}	X_{28}	X_{29}	X_{30}	X_{31}	X_{32}
m_4	m_{13}	m_1	m_{10}	m_6	m_{15}	m_3	m_{12}	m_0	m_9	m_5	m_2	m_{14}	m_{11}	m_8

X_{33}	X_{34}	X_{35}	X_{36}	X_{37}	X_{38}	X_{39}	X_{40}
m_3	m_{10}	m_{14}	m_4	m_9	m_{15}	m_8	m_1

Figure 2: Partial information of the message expansion of RIPEMD-160

Observation 1. Since X_{13} is the first internal state with difference on the left branch, there will be bit conditions on $X_{12} \oplus X_{11}^{\lll 10}$, $X_{13} \oplus X_{12}^{\lll 10}$, $X_{14} \oplus X_{12}^{\lll 10}$. To make the total number of these bit conditions small, the total number of active bits in X_{13} and X_{14} should be as small as possible.

Moreover, considering the specifics of the message expansion of RIPEMD-160, one more important observation can be obtained, which will play an important role in our semi-free-start attack framework. The Observation 2 is specified below.

Observation 2. For the left branch, X_{17} is updated with m_7 in the second round. Besides, m_7 is used to update X_{42} in the third round.

For a better understanding of this paper, we also present partial information of the message expansion, as illustrated in Figure 2.

3.1 Specification of the semi-free-start collision attack framework

Based on the above strategy to construct the differential characteristic as well as the the observation of the message expansion of RIPEMD-160, an efficient semi-free-start collision attack framework can be discovered, as illustrated in Figure 3. Suppose our aim is to mount semi-free-start collision attack on t steps of RIPEMD-160. On the whole, the attack procedure can be divided into 3 steps as follows.

Step 1: **Finding a starting point.** Find a solution (starting point) for X_i ($13 \leq i \leq t$). With single-step message modification, randomly choose values for X_i ($13 \leq i \leq 32$) while keeping the conditions on them hold. Based on Observation 2, all message words except m_7 will be fixed. The remaining work is to ensure the conditions on X_i ($33 \leq i \leq t$) hold. Generally, the conditions on this part can be partially satisfied with dedicated multi-step message modification. However, it will require some manual work. As will be shown, finding a starting point is not the bottleneck of our attack framework. Therefore, we remove the dedicated hand-tuned multi-step message modification and use a non-optimized method to satisfy the conditions on X_i ($33 \leq i \leq t$) for simplicity.

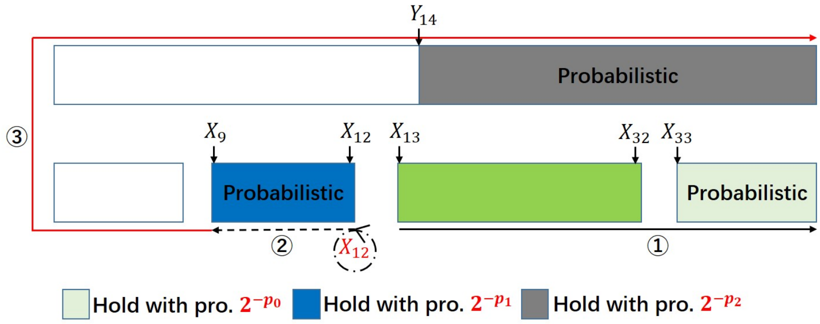

Figure 3: Semi-free-start collision attack framework for RIPEMD-160

Step 2: **Filtering invalid X_{12}.** Suppose there are n bit conditions on X_{12}. Then, for a fixed starting point, n bits of X_{12} will be fixed, thus leaving 2^{32-n} possible values for X_{12} in total. For each possible value of X_{12}, we can compute m_7 as follows:

$$m_7 = (X_{17} \boxminus X_{13}^{\lll 10})^{\ggg 7} \boxminus XOR(X_{16}, X_{15}, X_{14}^{\lll 10}) \boxminus X_{12}^{\lll 10} \boxminus K_0^l.$$

Consequently, for each possible value of X_{12}, all message words will become fixed. Then, we compute backward until X_9 and check the bit conditions on $X_{12} \oplus X_{11}^{\lll 10}$ as well as the conditions on

$$LQ_i = (X_i \boxminus X_{i-4}^{\lll 10})^{\ggg s_i^l} \ (13 \le i \le 16),$$

which are used to ensure the correct propagation of the modular difference of X_i $(13 \le i \le 16)$. If these conditions hold, move to Step 3. Otherwise, choose another possible value for X_{12} and repeat. If all 2^{32-n} possible values are used up, start generating a new starting point and repeat Step 2.

Step 3: **Verifying the right branch.** Until this phase, all message words are fixed. Then, for the left branch, we can compute backward to obtain the initial value. At last, we compute forward to compute the internal states on the right branch. If the conditions on the right branch do not hold, return Step 2. Otherwise, a semi-free-start collision is found.

3.2 Generating a starting point

Note that when all possible values for X_{12} are used up, we have to generate another starting point, i.e. another solution for X_i $(13 \le i \le t)$. Actually, after one starting point is obtained, a new starting point can be derived from it in negligible time, thus explaining why Step 1 is not the bottleneck of our attack framework.

In the following, we will expand on how to derive a new starting point from an existing one. For a better understanding of the next parts, we strongly suggest the readers can refer to the message expansion of RIPEMD-160 in Figure 2 since the next parts are highly related with it.

There are two strategies to derive a new starting point for two different cases. Suppose the conditions on X_{13} hold with probability 2^{-p_3} and the conditions on X_i $(36 \le i \le t)$ hold with probability 2^{-p_4}. The two strategies are as follows.

3.2.1 Strategy 1

This strategy is suitable for the case when $p_4 \le p_3$. The procedure to generate a new starting point can be described below.

Strategy 1. Randomly choose a value for X_i ($13 \leq i \leq 15$) while keeping the conditions on them hold. Then, modify m_4, m_{13} and m_1 as follows to keep X_{18}, X_{19} and X_{20} stay the same.

$$\begin{aligned} m_4 &= (X_{18} \boxminus X_{14}^{\lll 10})^{\ggg s^l_{18}} \boxminus IFX(X_{17}, X_{16}, X_{15}^{\lll 10}) \boxminus X_{13}^{\lll 10} \boxminus K^l_1, \\ m_{13} &= (X_{19} \boxminus X_{15}^{\lll 10})^{\ggg s^l_{19}} \boxminus IFX(X_{18}, X_{17}, X_{16}^{\lll 10}) \boxminus X_{14}^{\lll 10} \boxminus K^l_1, \\ m_1 &= (X_{20} \boxminus X_{16}^{\lll 10})^{\ggg s^l_{20}} \boxminus IFX(X_{19}, X_{18}, X_{17}^{\lll 10}) \boxminus X_{15}^{\lll 10} \boxminus K^l_1. \end{aligned}$$

In this way, X_i ($16 \leq i \leq 35$) will stay the same. However, since X_{36} is updated with m_4, we have to recompute new values for X_i ($36 \leq i \leq t$) and verify whether the conditions on them can still hold. If they do not hold, start choosing another value for X_i ($13 \leq i \leq 15$) while keeping the conditions on them hold and repeat the above precedure until the conditions on X_i ($36 \leq i \leq t$) hold. Consequently, the time to generate a new starting point is about 2^{p_4} computations.

3.2.2 Strategy 2

This strategy is suitable for the case when $p_3 < p_4$. The procedure to generate a new starting point can be described below.

Strategy 2. Randomly choose a value for X_i ($14 \leq i \leq 15$) while keeping the conditions on them hold. Then, we compute X_{13} by using X_i ($14 \leq i \leq 18$) and m_4 as follows.

$$X_{13} = ((X_{18} \boxminus X_{14}^{\lll 10})^{\ggg s^l_{18}} \boxminus IFX(X_{17}, X_{16}, X_{15}^{\lll 10}) \boxminus m_4 \boxminus K^l_1)^{\ggg 10}.$$

Next, we verify the conditions on X_{13} and $LQ_{17} = (X_{17} \boxminus X_{13}^{\lll 10})^{\ggg s^l_{17}}$. If they do not hold, start randomly choosing another valid value for X_i ($14 \leq i \leq 15$) and repeat until the conditions on X_{13} and $LQ_{17} = (X_{17} \boxminus X_{13}^{\lll 10})^{\ggg s^l_{17}}$ hold. If they hold, modify m_{13} and m_1 as follows to keep X_{19} and X_{20} stay the same.

$$\begin{aligned} m_{13} &= (X_{19} \boxminus X_{15}^{\lll 10})^{\ggg s^l_{19}} \boxminus IFX(X_{18}, X_{17}, X_{16}^{\lll 10}) \boxminus X_{14}^{\lll 10} \boxminus K^l_1, \\ m_1 &= (X_{20} \boxminus X_{16}^{\lll 10})^{\ggg s^l_{20}} \boxminus IFX(X_{19}, X_{18}, X_{17}^{\lll 10}) \boxminus X_{15}^{\lll 10} \boxminus K^l_1. \end{aligned}$$

In this way, X_i ($16 \leq i \leq 39$) will stay the same. Thus, for the attack on fewer than 40 steps, the time to generate a new starting point is about 2^{p_3} computations.

For the attack on 40 steps of RIPEMD-160, since X_{40} is updated with m_1, we have to recompute a new value for X_{40} and check its conditions. If they do not hold, start choosing another new valid value for X_i ($14 \leq i \leq 15$) and repeat until a valid starting point is found. For the attack on 40 steps of RIPEMD-160, we only need to check whether LQ_{40} can satisfy its corresponding equation. As will be shown in the 40-step differential characteristic, such a probability is close to 1 and therefore the time to generate a starting point is also about 2^{p_3} computations.

Indeed, the above two strategies to generate a new starting point imply that only one solution X_i ($16 \leq i \leq 35$) is needed. For such a solution, m_7, m_4, m_{13} and m_1 are not fixed. When the case is $p_4 \leq p_3$, we directly use **Strategy 1** to generate a starting point. When the case is $p_3 < p_4$, we first exhaust all valid values for X_{36} and compute the corresponding m_4 (m_4 is used to update X_{36}) as well as X_i ($37 \leq i \leq t \leq 39$). Record m_4 which can make the conditions on X_i ($37 \leq i \leq t \leq 39$) hold. Then, **Strategy 2** can be applied to find a starting point.

Obviously, finding a solution for X_i ($16 \leq i \leq 35$) cannot be the bottleneck since only three internal states X_{33}, X_{34} and X_{35} cannot hold trivially. In our implementation, when the number of conditions on X_{33}, X_{34} and X_{35} are small, we simply make them hold probabilistically. In other words, we repeat finding a solution for X_i ($16 \leq i \leq 32$) with single-step message modification until the conditions on X_i ($33 \leq i \leq 35$) hold.

As shown in **Strategy 2**, we fix the value for m_4 to keep the internal states X_i ($36 \leq i \leq t \leq 39$) stay the same. In this case, the freedom degree to generate a new starting point is provided by the free bits of X_{14} and X_{15}. When the right branch holds with a relatively low probability, i.e. like the 40-step differential characteristic, sufficient number of starting points are needed. Therefore, we can also use the freedom degrees of m_4. Specifically, we can first store all valid values for m_4 which can make the conditions on X_i ($36 \leq i \leq t \leq 39$) hold in an array. This can be achieved by exhausting all valid values for X_{36} and compute X_i ($37 \leq i \leq t \leq 39$) as well as check the conditions on them for a fixed solution for X_i ($16 \leq i \leq 35$). Then, instead of only randomly choosing a valid value for X_{14} and X_{15}, we can also randomly choose a valid value for m_4 from this array. In a word, to generate a new starting point, the freedom degrees can be provided by X_{14}, X_{15} and m_4. Such a slightly modified **Strategy 2** will require some memory to store all valid m_4.

3.3 Complexity Evaluation

Although no differential characteristic is presented now, we can give a rough estimation of the time complexity of the semi-free-start collision attack on t ($36 \leq t \leq 40$) steps of RIPEMD-160 before considering a specific differential characteristic. This is owing to the efficiency of our semi-free-start collision attack framework.

Specifically, when a starting point is found, we can exhaust all valid values for X_{12} and initially filter them by checking the conditions on X_{11} and LQ_i ($13 \leq i \leq 16$). When all possible value for X_{12} are used up for a starting point, we can efficiently generate a new starting point with time $\text{MIN}(2^{p_3}, 2^{p_4})$, where p_3 and p_4 are defined in Section 3.2.

As shown in Figure 3, suppose the conditions on $X_{12} \oplus X_{11}^{\lll 10}$ and LQ_i ($13 \leq i \leq 16$) hold with probability 2^{-p_1}, and the fully probabilistic right branch holds with probability 2^{-p_2}. Moreover, we also suppose there are n bit conditions on X_{12}. Then, for each starting point, we will verify the right branch with different m_7 for about 2^{32-n-p_1} times. The time complexity of this phase can be estimated as:

$$T_0 = \frac{4}{80} \cdot 2^{32-n} + \frac{13+t}{80} \cdot 2^{32-n-p_1}.$$

As will be shown, p_1 will be very small in our discovered differential characteristic, i.e. $p_1 \approx 2$. Therefore, we roughly estimate the time complexity of this phase as

$$T_0 = \frac{4}{80} \cdot 2^{32-n} + \frac{13+t}{80} \cdot 2^{32-n-p_1} \approx 2^{32-n-p_1}.$$

However, the right branch holds with probability 2^{-p_2}. Thus, it is expected to verify the right branch for about 2^{p_2} times under our semi-free-start collision attack framework in order to find a semi-free-start collision. Since each starting point can only provide about 2^{32-n-p_1} times of checking, we need to have about $2^{p_2-(32-n-p_1)} = 2^{p_1+p_2+n-32}$ starting points. Suppose only one solution for X_i ($16 \leq i \leq 35$) is enough, which means m_4, X_{14} and X_{15} can provide sufficient freedom degrees to generate $2^{p_1+p_2+n-32}$ starting points. Then, apart from the initial starting point, each starting point can be generated with time $\text{MIN}(2^{p_3}, 2^{p_4})$. Thus, the total time complexity of our semi-free-start collision attack on t steps of RIPEMD-160 is

$$\begin{aligned} T &= \text{MIN}(2^{p_3}, 2^{p_4}) \times 2^{p_1+p_2+n-32} + 2^{p_1+p_2+n-32} \times 2^{32-n-p_1} \\ &= \text{MIN}(2^{p_3}, 2^{p_4}) \times 2^{p_1+p_2+n-32} + 2^{p_2}. \end{aligned}$$

As will be shown in the differential characteristics, $p_1 \approx 2$, $\text{MIN}(2^{p_3}, 2^{p_4}) \leq 2^5$ and $n \leq 3$. Thus, we have

$$T = \text{MIN}(2^{p_3}, 2^{p_4}) \times 2^{p_1+p_2+n-32} + 2^{p_2} \leq 2^{5+2+p_2+3-32} + 2^{p_2} \approx 2^{p_2}. \quad (1)$$

In other words, under our semi-free-start collision attack framework, the time complexity to find a semi-free-start collision for t steps of RIPEMD-160 is fully dominated by the probabilistic right branch.

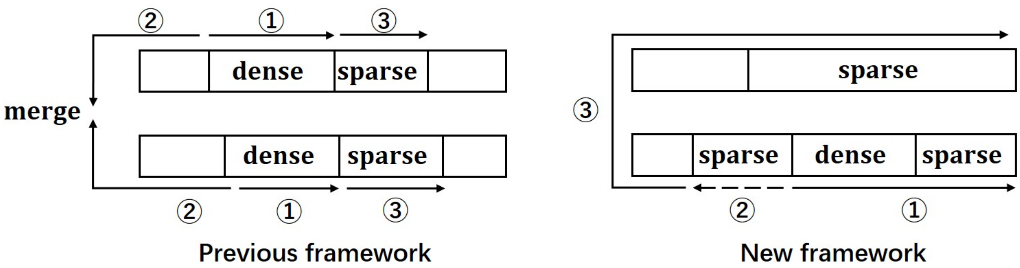

Figure 4: Comparison between our framework and previous frameworks [LMW17, MPS+13]

3.4 Advantage

As can be observed, our new semi-free-start collision attack framework is completely different from previous ones presented at Asiacrypt 2013 [MPS+13] and Asiacrypt 2017 [LMW17], as depicted in Figure 4. Compared with the semi-free-start collision attack frameworks for 36 steps of RIPEMD-160 [LMW17, MPS+13], our new semi-free-start collision attack framework can bring the following three advantages.

- The memory complexity is negligible for our new framework, while it is 2^{32} in previous work [LMW17, MPS+13] after an optimization based on [LMW17].
- Our new framework is extendable and allows us to mount semi-free-start collision attack on several reduced versions of RIPEMD-160 when inserting message difference at message word m_{12}. However, it seems impossible to attack more steps when adopting the framework [LMW17, MPS+13] by inserting difference at m_7. As will be shown, the new framework can be used to mount semi-free-start collision attack on 36/37/38/40 steps of RIPEMD-160.
- The framework can provide significantly improved results for the semi-free-start collision attack on reduced RIPEMD-160.

3.4.1 Remark

With the start-from-the-middle structure, while it is hard to turn a semi-free-start collision attack into a collision attack due to the match with the predefined initial value, it is easy to turn a collision attack into a semi-free-start collision attack. For the dense-left-and-sparse-right (DLSR) collision attack framework in [LDM+18], an intuitive idea to convert it into a semi-free-start collision attack framework is to remove the connecting phase. Specifically, the starting point is a solution for X_i $(11 \leq i \leq 23)$ in the DLSR framework. Then, the attacker can always keep the conditions on X_i $(24 \leq i \leq 32)$ hold with single-step message modification. Finally, for each valid value for X_i $(24 \leq i \leq 32)$, all message words become fixed and therefore the attacker can compute the remaining internal states on both branches and verify their conditions. For the 34-step differential characteristic, the probability that the conditions on the remaining internal states hold is not too low, i.e. greater than 2^{-40}, one can repeat choosing valid values for X_i $(24 \leq i \leq 32)$ with single-step message modification and verify these conditions. Once they are satisfied, a semi-free-start collision is found. Obviously, the time complexity to find a semi-free-start collision for 34 steps of RIPEMD-160 will not exceed 2^{40} and is practical. However, if it is directly applied to a longer differential characteristic, one has to deal with the conditions in the third round on the left branch.

Compared with the above naive semi-free-start collision attack framework derived from the DLSR collision attack framework [LDM+18], our new framework adopts an optimal freedom degree utilization strategy, thus performing better for a longer differential characteristic. In our new framework, the starting point is a solution for X_i $(13 \leq i \leq t)$ while it is a solution for X_i

($11 \leq i \leq 23$) in [LDM+18]. Especially, we also prove that the total time complexity is fully dominated by right branch under our new semi-free-start collision attack framework if obtaining a suitable differential characteristic. Determining such an optimal freedom degree utilization strategy is obviously non-trivial.

4 Differential Characteristics

Our semi-free-start collision attack procedure to find a semi-free-collision for reduced RIPEMD-160 has been explained in details in Section 3. Thus, the next task is to find a suitable differential characteristic to make the framework work efficiently. Thanks to the use of automated techniques [MNS11, MNS13, EMS14], this task can be finished efficiently. Thus, the remaining work is to add some constraints on the differential characteristic before the search in order to find a desirable one.

As explained in Section 3, once a solution for the starting point is found, we can immediately utilize the freedom degrees provided by X_{12}. Besides, there will be a filtering phase to filter invalid X_{12}. We hope there are sufficient valid X_{12} left after filtering. Thus, the desirable differential characteristic have the following properties.

- There should be only one active bit in X_{12} so that there is only one bit condition on X_{11}.
- The probability that $LQ_i = (X_i \boxminus X_{i-4}^{\lll 10})^{\ggg s_i^l}$ ($13 \leq i \leq 16$) satisfy their corresponding equations should be as high as possible.
- The total number of active bits in X_{13} and X_{14} should be as small as possible so that there are a few bit conditions on X_{12}. This is to ensure X_{12} can take as many possible values as possible before filtering.

When taking the generation of new starting points into account, we can expect that the cost is as small as possible. Besides, there should be sufficient freedom degrees provided by X_{14} and X_{15} and m_4. Thus, the desirable differential characteristic should have the following extra properties.

- The total number of active bits in X_{15}, X_{16}, X_{17} should be as small as possible. Besides, the probability that $LQ_i = (X_i \boxminus X_{i-4}^{\lll 10})^{\ggg s_i^l}$($17 \leq i \leq 19$) satisfy their corresponding equations should be as high as possible. In this way, it is expected that X_{14} and X_{15} can provide sufficient freedom degrees.
- The probability that the conditions on X_i ($36 \leq i \leq t \leq 39$) hold should not be too small. Then, we can also utilized the freedom degrees provided by m_4.

In a word, the differential characteristic located at X_i ($13 \leq i \leq 17$) and X_i ($36 \leq i \leq t$) should be as sparse as possible. Then, we can solve the nonlinear differential characteristic located at X_i ($18 \leq i \leq 35$) with the use of automated techniques [MNS11, MNS13, EMS14]. The desirable discovered differential characteristics are displayed at Table 11, Table 12, Table 13 and Table 14 in Appendix A respectively. In addition, we also provide the solution for X_i ($16 \leq i \leq 35$) in Table 4.

As will be shown, the semi-free-start collision attack on 36 steps of RIPEMD-160 will become practical. For the attack on 37 and 38 steps of RIPEMD-160, the time complexity is below 2^{54}, which may become practical with more powerful computing resources. However, such powerful computing resources are out of our reach. For the attack on 39 steps under our framework, the right branch will hold with probability about 2^{-59}, suggesting the time complexity will be about 2^{59} if a suitable differential characteristic can be found. Thus, for the attack with relatively high time complexity, we focus on more steps. In other words, we will concentrate on the theoretical semi-free-start collision attack on 40 steps of RIPEMD-160.

Table 4: Solution for X_i ($16 \leq i \leq 35$)

36 steps		37 steps	
m_0 = 0x6c2c8526, m_2 = 0x16188d15, m_3 = 0xc6c5da57, m_5 = 0xf7a7a97a, m_6 = 0xa7cbbf38, m_8 = 0xb6477677, m_9 = 0x47f24a3e, m_{10} = 0xb1bdf3b5, m_{11} = 0x78aaa252, m_{12} = 0x69a579f0, m_{14} = 0xbb877480, m_{15} = 0x5caa647e.		m_0 = 0x2a3e3e5d, m_2 = 0xc5ab4a9c, m_3 = 0xdc1f16ce, m_5 = 0x848cc0fe, m_6 = 0xf11aa5a3, m_8 = 0x9e6914b7, m_9 = 0xfe96a9cf, m_{10} = 0xda48b5c6, m_{11} = 0x59b4296f, m_{12} = 0x14a47a10, m_{14} = 0x3b3e4837, m_{15} = 0x7f4d5b3f.	
X_{16}	10100111011110011u10011110n1100	X_{16}	10101000001000110 1n1000101uu0110
X_{17}	n0010101101100111100110111101111	X_{17}	u1000100111000111001111001000100
X_{18}	0100010111101110011111 01nu001001	X_{18}	000000000100 11111u0001001u111000
X_{19}	11001101110111 11u010100n00100001	X_{19}	n1111110100011000110111u10000001
X_{20}	1101000011000110 1n1000101101 0000	X_{20}	0010n100n0110111u1uu110010100110
X_{21}	nnnn1nn1010111110 1011nu110100u10	X_{21}	001110u00010110100010 0u100011011
X_{22}	1001001nu1101u0n1001n1nuununnu11	X_{22}	1101100u10u000un0100001uuuuu1000
X_{23}	nn110u11n10nu100n001nnn10u11nnu1	X_{23}	1un11nuu11100u0u11u1uuun1001010u
X_{24}	1101nu0010uun01nu1n0000nn1101u10	X_{24}	110n11nn1001uu110u0n10u1u0nu001u
X_{25}	11uu1nn10n011001n0u01uuu1n101uuu	X_{25}	00n0010n0nn0un0000nnn01u11u10100
X_{26}	110101 1u1un0u100u01uuuuuu0010010	X_{26}	0nnnnn001101011110nnnnnn011u0u10
X_{27}	0u11u0010011n111uuun001111000111	X_{27}	01000011nn0110u1100n01uu00n101u1
X_{28}	11000100n11nnn0n11100n010n0n0nn1	X_{28}	1nu11111uu1un0u01n0n1nnnn1100n00
X_{29}	01n00nu000u01nnu101uuu0unununn11n	X_{29}	u001uuuu00u0101un010011u0001n0uu
X_{30}	n1u01n10u010011n000110100000un1u	X_{30}	111000010 1110u10u100uuu0010uuu01
X_{31}	01n1000nnn0110101011 0111nnn0110u	X_{31}	1101111001010 1nnn0110nnn10110111
X_{32}	10n010100000001111100100100 0010u	X_{32}	00001101101 11u01n00101110111110 1
X_{33}	01n0001101101000100100001000110u	X_{33}	10110110000011110111111011011001
X_{34}	01010101010010111110101000111111	X_{34}	01011101011010100010101100001110
X_{35}	10110000101111011111111011111 01101	X_{35}	00001111000000101011111001100010 0

38 steps		40 steps	
m_0 = 0xc32cb8b2, m_2 = 0xdcebf941, m_3 = 0x473889d3, m_5 = 0x38875789, m_6 = 0xcc53e680, m_8 = 0xb8dce09a, m_9 = 0xcb87a927, m_{10} = 0x67c766e5, m_{11} = 0x9c0866a4, m_{12} = 0x6dcd4ef1, m_{14} = 0x74e28f11, m_{15} = 0x898b12aa.		m_0 = 0x31973617, m_2 = 0x3f5a3668, m_3 = 0xfc3ffea2, m_5 = 0x97ccd10f, m_6 = 0x41688e61, m_8 = 0x69a1d2a2, m_9 = 0x5b2331f3, m_{10} = 0x1c7c9435, m_{11} = 0x9d41f6eb, m_{12} = 0x1f5f0b1b, m_{14} = 0x19ec64c9, m_{15} = 0x86c9080.	
X_{16}	0001000101011111111u1000101uu1100	X_{16}	0011100100110011 10u111101n0u1001
X_{17}	u1111101010000011100001110001110	X_{17}	u1110110001101101100001110100001
X_{18}	01010111110010001111000 01u101000	X_{18}	00000010101011010111 11000u011010
X_{19}	01010110011010 0nu111101n10101010	X_{19}	10000111000101 10u001111u10000001
X_{20}	0000001u1101101uu11110011010011	X_{20}	11110011u111011n0u10010001001011
X_{21}	11011101011000001100 01u100011000	X_{21}	01011111000001000 01001u001000000
X_{22}	1100n0unu10un1nn01unnnnnnnnnnn10	X_{22}	101001010101100u11000101 10011110
X_{23}	n0uuu1100010nuuu0uuuu1nun101nu11	X_{23}	100010u011001u10001110000 1010111
X_{24}	1011111101un01u10011000001101000	X_{24}	10011n0u0101101111 0unnnn1001un11
X_{25}	uu1110n001000100010nuu01u11101n0	X_{25}	11110001 0n0u0nnn010011un10011100
X_{26}	01nu101u11u00010100u000011u10111	X_{26}	unu001uu001u01001u101101111u11u1
X_{27}	01000uu0010u1n0100nnn11011n11101	X_{27}	011un011nnuuuuuuuu10000u00u100u0
X_{28}	0u10u00011000001110000100000 0101	X_{28}	00111nun0111u1nn0nnnnnn10n00nnn1
X_{29}	10101010111n001000001n0n1001n100	X_{29}	nn111010n010101010 0010u1un10101n
X_{30}	n100010u01101100u010100110n10101	X_{30}	0111u11010000n01101100uuu11101u0
X_{31}	00uuuuuu0110nu1111000101011111 10	X_{31}	n001u110n10u1110nn11u0100111110n
X_{32}	u11111n100110110n01u0101nuu10000	X_{32}	1000uun010n1n011u011110100n0nu01
X_{33}	00011111110un0100010101100001100	X_{33}	0u1110101100u01u1111n01u000111n0
X_{34}	u11n1001101110101101000100000111	X_{34}	011n11n0010u10n01000000010000101 1
X_{35}	1001001001001111011010111010 1110	X_{35}	0u00000010011011011111111110001n1

5 Application

In this section, we present the results of the new semi-free-start collision attack on 36/37/38/40 steps of RIPEMD-160. As our attack framework requires, we have to focus on the conditions on the following part:

1. The conditions on the right branch, which will influence the whole time complexity.
2. The conditions on X_{12}, which will influence the total number of possible values for X_{12} before filtering.
3. The conditions on X_{11} and LQ_i ($13 \leq i \leq 16$), which will influence the filtering phase.
4. The conditions on X_{13} and LQ_{17}, which will influence the time to generate a new starting point.
5. The conditions on X_i ($14 \leq i \leq 15$), which will influence the freedom degrees to generate a new starting point. We stress here that we have added extra conditions on X_{14} and X_{15} to make LQ_i ($18 \leq i \leq 19$) satisfy their equations. Therefore, even if X_{14} and X_{15} are changed, LQ_i ($18 \leq i \leq 19$) will always satisfy their equations.
6. The conditions on X_i ($36 \leq i \leq t$), which will influence the total number of valid m_4. In other words, it will also influence the freedom degrees to generate a starting point.

Therefore, when describing the semi-free-start collision attack in next sections, we will firstly list the above conditions.

5.1 Practical Semi-Free-Start Collision on 36 Steps of RIPEMD-160

As discussed above, we first list in Table 6 some conditions influencing the performance of the semi-free-start collision attack, which are not presented in Table 11. As Table 6 shows, the probability that LQ_{36} satisfies its corresponding equations is close to 1 (there is no need to consider the bit conditions on X_{36} when we attack 36 steps of RIPEMD-160), while the conditions on X_{13} holds with probability 2^{-5}. Therefore, we use **Strategy 1** to generate a new starting point, whose cost can be neglected.

Moreover, based on Table 6 and Table 11, there will be $2^{32-3} = 2^{29}$ possible values for X_{12} for a given starting point. After filtering, about $2^{29-1.7} = 2^{27.3}$ valid values for X_{12} are left. Since the right branch holds with probability 2^{-41}, we need to generate about $2^{41-27.3} = 2^{13.7}$ starting points. It should be noticed in Table 6 that there are sufficient free bits in X_i ($13 \leq i \leq 15$). Therefore, we can only use one solution for X_i ($16 \leq i \leq 35$). Thus, the time complexity to mount semi-free-start collision attack on 36 steps of RIPEMD-160 can be evaluated with the formula 1 in Section 3.3, where

$$(p_1, p_2, p_3, p_4, n) = (1.7, 41, 5, 0, 3).$$

Therefore, the time complexity to find a semi-free-start collision for 36 steps is 2^{41}. Due to the low time complexity, we can give the first practical semi-free-start collision for 36 steps of RIPEMD-160, as shown in Table 5. Such an instance is found in about 5.5 hours with 25 CPUs on a Linux Server.

Table 5: Semi-free-start collision for 36 steps of RIPEMD-160

$h_0 \sim h_4$	809825f7 d2a55861 6bd86be7 fc58a6cb 11f6a005
M	6c2c8526 dc3084cc 16188d15 c6c5da57 73f15b99 f7a7a97a a7cbbf38 53a4b30 b6477677 47f24a3e b1bdf3b5 78aaa252 69a579f0 72b32f35 bb877480 5caa647e
M'	6c2c8526 dc3084cc 16188d15 c6c5da57 73f15b99 f7a7a97a a7cbbf38 53a4b30 b6477677 47f24a3e b1bdf3b5 78aaa252 69a5f9f0 72b32f35 bb877480 5caa647e
hash value	88f79fa4 c9973719 dcf0ff7f 15cef816 a9d702a5

Table 6: Other conditions influencing the attack for the 36-step differential characteristic

	Conditions	Probability
Y_{18}	$Y_{18,31} = Y_{17,31}$	2^{-1}
Y_{22}	$Y_{22,9} = Y_{21,9}$	2^{-1}
Y_{26}	$Y_{26,20} = Y_{25,20}$, $Y_{26,19} = Y_{25,19}$	2^{-2}
Y_{30}	$Y_{30,0} = Y_{29,0}$, $Y_{30,29} = Y_{29,29}$, $Y_{30,30} = Y_{29,30}$	2^{-3}
Y_{34}	$Y_{34,7} \vee \neg Y_{33,7} = 1$, $Y_{34,10} \vee \neg Y_{33,10} = 1$	2^{-1}
RQ_{16}	$(RQ_{16} \boxplus \texttt{0x8000})^{\lll 6} = RQ_{16}^{\lll 6} \boxplus \texttt{0x200000}$	Negligible
RQ_{28}	$(RQ_{28} \boxplus \texttt{0x8000})^{\lll 7} = RQ_{28}^{\lll 7} \boxplus \texttt{0x400000}$	Negligible
RQ_{35}	$(RQ_{35} \boxplus \texttt{0xffffc80})^{\lll 15} = RQ_{35}^{\lll 15} \boxplus \texttt{0xfe400000}$	Negligible
Right Branch		$2^{-33-8} = 2^{-41}$
X_{12}	$X_{12,19} \neq X_{13,29}$, $X_{12,18} \neq X_{13,28}$, $X_{12,11} = X_{14,21}$	2^{-3}
X_{11}	$X_{11,11} = X_{12,21}$	2^{-1}
LQ_{13}	$(LQ_{13} \boxplus \texttt{0x8000})^{\lll 6} = LQ_{13}^{\lll 6} \boxplus \texttt{0x200000}$	Negligible
LQ_{14}	$(LQ_{14} \boxplus \texttt{0x200000})^{\lll 7} = LQ_{14}^{\lll 7} \boxplus \texttt{0x10000000}$	$2^{-0.1}$
LQ_{15}	$(LQ_{15} \boxplus \texttt{0xf0200000})^{\lll 9} = LQ_{15}^{\lll 9} \boxplus \texttt{0x3ffffe0}$	$2^{-0.6}$
LQ_{16}	$(LQ_{16} \boxplus \texttt{0xffffe0})^{\lll 8} = LQ_{16}^{\lll 8} \boxplus \texttt{0xffffe010}$	Negligible
Filtering		$2^{-1.7}$
X_{13}	$X_{13,27} = X_{14,5}$, $X_{13,20} \neq X_{14,30}$, $X_{13,19} \neq X_{15,29}$, $X_{13,18} = X_{15,28}$	2^{-4}
X_{15}	$X_{15,13} = X_{14,3}$, $X_{15,4} = X_{14,26}$, $X_{15,21} \neq X_{16,31}$	2^{-3}
The number of free bits in X_{14} and X_{15}: $64 - 3 - 4 = 57$		
LQ_{36}	$(LQ_{36} \boxplus \texttt{0x38007})^{\lll 7} = LQ_{36}^{\lll 7} \boxplus \texttt{0x1c00380}$	Negligible
The expected number of valid m_4: $2^{32-0} = 2^{32}$		
The expected number of starting points for a fixed X_i $(16 \leq i \leq 35)$: $2^{57+32-5} = 2^{84}$		

5.2 Semi-Free-Start Collision for 37/38/40 Steps of RIPEMD-160

Similarly, for the semi-free-start collision attack on 37/38/40 steps of RIPEMD-160 under our attack framework, we first list some conditions influencing the performance of the semi-free-start collision attack in Table 7, Table 8, Table 9, which are not presented in Table 12, Table 13, Table 14.

5.2.1 Attack on 37 steps of RIPEMD-160

Based on Table 7 and Table 12, we can know that there will be $2^{32-2} = 2^{30}$ possible values for X_{12} for a given starting point. After filtering, about $2^{30-2} = 2^{28}$ are left. Since the conditions on X_i $(36 \leq i \leq 37)$ holds with probability $2^{-2.3}$ and the condition on X_{13} holds with probability 2^{-4}, we use **Strategy 1** to generate a new starting point, whose cost is about $2^{2.3}$ computations. Since the right branch holds with probability 2^{-49}, it is expected to generate $2^{49-28} = 2^{21}$ starting points. As Table 7 shows, X_i $(13 \leq i \leq 15)$ can provide sufficient freedom degree to generate so many starting points for a fixed solution for X_i $(16 \leq i \leq 35)$. Thus, the time complexity to mount semi-free-start collision attack on 37 steps of RIPEMD-160 can be evaluated with the formula 1, where

$$(p_1, p_2, p_3, p_4, n) = (2, 49, 4, 2.3, 2).$$

Therefore, the time complexity to find a semi-free-start collision for 37 steps of RIPEMD-160 is 2^{49}.

5.2.2 Attack on 38 steps of RIPEMD-160

Based on Table 8 and Table 13, we can know that there will be $2^{32-2} = 2^{30}$ possible values for X_{12} for a given starting point. After filtering, about $2^{30-2} = 2^{28}$ are left. Since the conditions on X_i $(36 \leq i \leq 38)$ holds with probability $2^{-13.3}$ and the condition on X_{13} holds with probability 2^{-4}, we use **Strategy 2** to generate a new starting point, whose cost is about 2^4 computations. Since the right branch holds with probability 2^{-53}, it is expected to generate $2^{53-28} = 2^{25}$ starting points. As Table 7 shows, X_i $(14 \leq i \leq 15)$ can provide sufficient freedom degree to generate so many starting points for a fixed solution for X_i $(16 \leq i \leq 35)$. Specifically, for a valid m_4, there are 57 free bits in X_{14} and X_{15}, while the conditions on X_{13} hold with probability 2^{-4}. Therefore, for a fixed solution for X_i $(16 \leq i \leq 35)$ and a valid m_4, we can expect to generate $2^{57-4} = 2^{53}$ starting points in total.

Table 7: Other conditions influencing the attack for the 37-step differential characteristic

	Conditions	Probability
Y_{18}	$Y_{18,31} = Y_{17,31}$	2^{-1}
Y_{22}	$Y_{22,9} = Y_{21,9}$	2^{-1}
Y_{26}	$Y_{26,20} = Y_{25,20}, Y_{26,19} = Y_{25,19}$	2^{-2}
Y_{30}	$Y_{30,0} = Y_{29,0}, Y_{30,29} = Y_{29,29}, Y_{30,30} = Y_{29,30}$	2^{-3}
Y_{34}	$Y_{34,7} \vee \neg Y_{33,7} = 1, Y_{34,10} \vee \neg Y_{33,10} = 1$	2^{-1}
RQ_{16}	$(RQ_{16} \boxplus \texttt{0x8000})^{\lll 6} = RQ_{16}^{\lll 6} \boxplus \texttt{0x200000}$	Negligible
RQ_{28}	$(RQ_{28} \boxplus \texttt{0x8000})^{\lll 7} = RQ_{28}^{\lll 7} \boxplus \texttt{0x400000}$	Negligible
RQ_{35}	$(RQ_{35} \boxplus \texttt{0xffffc80})^{\lll 15} = RQ_{35}^{\lll 15} \boxplus \texttt{0xfe400000}$	Negligible
Right Branch		$2^{-41-8} = 2^{-49}$
X_{12}	$X_{12,18} \neq X_{13,28}, X_{12,11} \neq X_{14,21}$	2^{-2}
X_{11}	$X_{11,11} \neq X_{12,21}$	2^{-1}
LQ_{13}	$(LQ_{13} \boxplus \texttt{0x8000})^{\lll 6} = LQ_{13}^{\lll 6} \boxplus \texttt{0x200000}$	Negligible
LQ_{14}	$(LQ_{14} \boxplus \texttt{0xffe00000})^{\lll 7} = LQ_{14}^{\lll 7} \boxplus \texttt{0xf0000000}$	$2^{-0.1}$
LQ_{15}	$(LQ_{15} \boxplus \texttt{0xfe00000})^{\lll 9} = LQ_{15}^{\lll 9} \boxplus \texttt{0xc0000020}$	$2^{-0.6}$
LQ_{16}	$(LQ_{16} \boxplus \texttt{0xd0000020})^{\lll 8} = LQ_{16}^{\lll 8} \boxplus \texttt{0x1fd0}$	$2^{-0.3}$
Filtering		2^{-2}
X_{13}	$X_{13,20} \neq X_{14,30}, X_{13,18} \neq X_{15,28}$	2^{-2}
X_{15}	$X_{15,13} = X_{14,3}, X_{15,4} = X_{14,26}$	2^{-2}
The number of free bits in X_{14} and X_{15}: $64 - 2 - 7 = 55$		
LQ_{36}	$(LQ_{36} \boxplus \texttt{0xe1c0000})^{\lll 7} = LQ_{36}^{\lll 7} \boxplus \texttt{0xe000007}$	$2^{-0.2}$
LQ_{37}	$(LQ_{37} \boxplus \texttt{0xf2000000})^{\lll 14} = LQ_{37}^{\lll 14} \boxplus \texttt{0xffffc80}$	$2^{-0.1}$
The expected number of valid m_4: $2^{32-0.2-0.1-2} = 2^{29.7}$		
The expected number of starting points for a fixed X_i $(16 \leq i \leq 35)$: $2^{55+29.7-4} = 2^{80.7}$		

Thus, the time complexity to mount semi-free-start collision attack on 38 steps of RIPEMD-160 can be evaluated with the formula 1, where

$$(p_1, p_2, p_3, p_4, n) = (2, 54, 4, 13.3, 2).$$

Therefore, the time complexity to find a semi-free-start collision for 38 steps of RIPEMD-160 is 2^{53}.

Table 8: Other conditions influencing the attack for the 38-step differential characteristic

	Conditions	Probability
Y_{18}	$Y_{18,31} = Y_{17,31}$	2^{-1}
Y_{22}	$Y_{22,9} = Y_{21,9}$	2^{-1}
Y_{26}	$Y_{26,20} = Y_{25,20}, Y_{26,19} = Y_{25,19}$	2^{-2}
Y_{30}	$Y_{30,0} = Y_{29,0}, Y_{30,29} = Y_{29,29}, Y_{30,30} = Y_{29,30}$	2^{-3}
Y_{34}	$Y_{34,7} \vee \neg Y_{33,7} = 1, Y_{34,10} \vee \neg Y_{33,10} = 1$	2^{-1}
RQ_{16}	$(RQ_{16} \boxplus \texttt{0x8000})^{\lll 6} = RQ_{16}^{\lll 6} \boxplus \texttt{0x200000}$	Negligible
RQ_{28}	$(RQ_{28} \boxplus \texttt{0x8000})^{\lll 7} = RQ_{28}^{\lll 7} \boxplus \texttt{0x400000}$	Negligible
RQ_{35}	$(RQ_{35} \boxplus \texttt{0xffffc80})^{\lll 15} = RQ_{35}^{\lll 15} \boxplus \texttt{0xfe400000}$	Negligible
RQ_{38}	$(RQ_{38} \boxplus \texttt{0x7})^{\lll 6} = RQ_{38}^{\lll 6} \boxplus \texttt{0x1c0}$	Negligible
Right Branch		$2^{-45-8} = 2^{-53}$
X_{12}	$X_{12,18} \neq X_{13,28}, X_{12,11} = X_{14,21}$	2^{-2}
X_{11}	$X_{11,11} \neq X_{12,21}$	2^{-1}
LQ_{13}	$(LQ_{13} \boxplus \texttt{0x8000})^{\lll 6} = LQ_{13}^{\lll 6} \boxplus \texttt{0x200000}$	Negligible
LQ_{14}	$(LQ_{14} \boxplus \texttt{0xffe00000})^{\lll 7} = LQ_{14}^{\lll 7} \boxplus \texttt{0xf0000000}$	$2^{-0.1}$
LQ_{15}	$(LQ_{15} \boxplus \texttt{0xfe00000})^{\lll 9} = LQ_{15}^{\lll 9} \boxplus \texttt{0xc0000020}$	$2^{-0.6}$
LQ_{16}	$(LQ_{16} \boxplus \texttt{0xd0000020})^{\lll 8} = LQ_{16}^{\lll 8} \boxplus \texttt{0x1fd0}$	$2^{-0.3}$
Filtering		2^{-2}
X_{13}	$X_{13,20} = X_{14,30}, X_{13,18} \neq X_{15,28}, X_{13,27} \neq X_{14,5}$	2^{-3}
X_{15}	$X_{15,13} = X_{14,3}, X_{15,4} = X_{14,26}, X_{15,21} \neq X_{16,31}$	2^{-3}
The number of free bits in X_{14} and X_{15}: $64 - 3 - 4 = 57$		
LQ_{36}	$(LQ_{36} \boxplus \texttt{0xeffff04})^{\lll 7} = LQ_{36}^{\lll 7} \boxplus \texttt{0xffff81f8}$	$2^{-0.1}$
LQ_{37}	$(LQ_{37} \boxplus \texttt{0x7fc8})^{\lll 14} = LQ_{37}^{\lll 14} \boxplus \texttt{0x1ff20000}$	$2^{-0.2}$
LQ_{38}	$(LQ_{38} \boxplus \texttt{0xe0000000})^{\lll 9} = LQ_{38}^{\lll 9} \boxplus \texttt{0x1c0}$	2^{-3}
The expected number of valid m_4: $2^{32-0.1-0.2-3-10} = 2^{18.7}$		
The expected number of starting points for a fixed X_i $(16 \leq i \leq 35)$: $2^{57+18.7-4} = 2^{71.7}$		

5.2.3 Attack on 40 steps of RIPEMD-160

Based on Table 9 and Table 14, we can know that there will be $2^{32-2} = 2^{30}$ possible values for X_{12} for a given starting point. After filtering, about $2^{30-2} = 2^{28}$ are left. Since the conditions on

X_i ($36 \leq i \leq 39$) holds with probability $2^{-21.8}$, the conditions on X_{13} hold with probability 2^{-4}, and LQ_{40} satisfies its equation with a probability close to 1, we use **Strategy 2** to generate a new starting point, whose cost is about 2^4 computations. Since the right branch holds with probability $2^{-74.6}$, it is expected to generate $2^{74.6-28} = 2^{47.6}$ starting points. As Table 7 shows, X_i ($14 \leq i \leq 15$) can provide sufficient freedom degree to generate so many starting points for a fixed solution for X_i ($16 \leq i \leq 35$). Specifically, for a valid m_4, there are 57 free bits in X_{14} and X_{15}, while the conditions on X_{13} hold with probability 2^{-4}. Therefore, for a fixed solution for X_i ($16 \leq i \leq 35$) and a valid m_4, we can expect to generate $2^{57-4} = 2^{53}$ starting points in total. Indeed, we can also store some solutions for m_4 in an array, whose memory requirement can be neglected. Then, as stated previously, we not only can choose valid values for X_{14} and X_{15}, but also can randomly choose valid values for m_4 from this array. In this way, the freedom degree of m_4 can be utilized as well. Thus, the time complexity to mount semi-free-start collision attack on 40 steps of RIPEMD-160 can be evaluated with the formula 1, where

$$(p_1, p_2, p_3, p_4, n) = (2, 74.6, 4, 21.8, 2).$$

Therefore, the time complexity to find a semi-free-start collision for 40 steps of RIPEMD-160 is $2^{74.6}$.

Table 9: Other conditions influencing the attack for the 40-step differential characteristic

	Conditions	Probability
Y_{18}	$Y_{18,31} = Y_{17,31}$	2^{-1}
Y_{22}	$Y_{22,9} = Y_{21,9}$	2^{-1}
Y_{26}	$Y_{26,20} = Y_{25,20}, Y_{26,19} = Y_{25,19}$	2^{-2}
Y_{30}	$Y_{30,0} = Y_{29,0}, Y_{30,29} = Y_{29,29}, Y_{30,30} = Y_{29,30}$	2^{-3}
Y_{37}	$Y_{37,3} \vee \neg Y_{36,3} = 1$	$2^{-0.5}$
Y_{38}	$Y_{38,17} \vee \neg Y_{37,17} = 1, Y_{38,20} \vee \neg Y_{37,20} = 1$	2^{-1}
RQ_{16}	$(RQ_{16} \boxplus \texttt{0x8000})^{\lll 6} = RQ_{16}^{\lll 6} \boxplus \texttt{0x200000}$	Negligible
RQ_{28}	$(RQ_{28} \boxplus \texttt{0x8000})^{\lll 7} = RQ_{28}^{\lll 7} \boxplus \texttt{0x400000}$	Negligible
RQ_{35}	$(RQ_{35} \boxplus \texttt{0x380})^{\lll 15} = RQ_{35}^{\lll 15} \boxplus \texttt{0x1c00000}$	Negligible
RQ_{38}	$(RQ_{38} \boxplus \texttt{0xfffffff7})^{\lll 6} = RQ_{38}^{\lll 6} \boxplus \texttt{0xffffdc0}$	Negligible
RQ_{39}	$(RQ_{39} \boxplus \texttt{0xfff20000})^{\lll 6} = RQ_{39}^{\lll 6} \boxplus \texttt{0xfc800000}$	$2^{-0.1}$
RQ_{40}	$(RQ_{40} \boxplus \texttt{0xffffffc8})^{\lll 14} = RQ_{40}^{\lll 14} \boxplus \texttt{0xfff20000}$	Negligible
Right Branch		$2^{-66-8.6} = 2^{-74.6}$
X_{12}	$X_{12,18} = X_{13,28}, X_{12,11} \neq X_{14,21}$	2^{-2}
X_{11}	$X_{11,11} \neq X_{12,21}$	2^{-1}
LQ_{13}	$(LQ_{13} \boxplus \texttt{0x8000})^{\lll 6} = LQ_{13}^{\lll 6} \boxplus \texttt{0x200000}$	Negligible
LQ_{14}	$(LQ_{14} \boxplus \texttt{0xffe00000})^{\lll 7} = LQ_{14}^{\lll 7} \boxplus \texttt{0xf0000000}$	$2^{-0.1}$
LQ_{15}	$(LQ_{15} \boxplus \texttt{0xefe00000})^{\lll 9} = LQ_{15}^{\lll 9} \boxplus \texttt{0xbfffffe0}$	$2^{-0.6}$
LQ_{16}	$(LQ_{16} \boxplus \texttt{0x2fffffe0})^{\lll 8} = LQ_{16}^{\lll 8} \boxplus \texttt{0xffffe030}$	$2^{-0.3}$
Filtering		2^{-2}
X_{13}	$X_{13,20} = X_{14,30}, X_{13,18} = X_{15,28}, X_{13,27} = X_{14,5}$	2^{-3}
X_{15}	$X_{15,13} = X_{14,3}, X_{15,4} = X_{14,26}, X_{15,21} \neq X_{16,31}$	2^{-3}
The number of free bits in X_{14} and X_{15}: $64 - 3 - 4 = 57$		
X_{36}	$X_{36,6} \vee \neg X_{35,6} = 1$	$2^{-0.5}$
LQ_{36}	$(LQ_{36} \boxplus \texttt{0x50c3fee0})^{\lll 7} = LQ_{36}^{\lll 7} \boxplus \texttt{0x61ff7028}$	$2^{-1.3}$
LQ_{37}	$(LQ_{37} \boxplus \texttt{0xd6008f90})^{\lll 14} = LQ_{37}^{\lll 14} \boxplus \texttt{0x23e3f580}$	$2^{-0.5}$
LQ_{38}	$(LQ_{38} \boxplus \texttt{0xdc1c0000})^{\lll 9} = LQ_{38}^{\lll 9} \boxplus \texttt{0x37ffffb8}$	$2^{-0.6}$
LQ_{39}	$(LQ_{39} \boxplus \texttt{0xc8000048})^{\lll 13} = LQ_{39}^{\lll 13} \boxplus \texttt{0x8f900}$	$2^{-0.4}$
LQ_{38}	$(LQ_{40} \boxplus \texttt{0xfff20700})^{\lll 15} = LQ_{40}^{\lll 15} \boxplus \texttt{0x37ffff9}$	Negligible
The expected number of valid m_4: $2^{32-1.3-0.5-0.6-0.4-19} = 2^{10.2}$		
The expected number of starting points for a fixed X_i ($16 \leq i \leq 35$): $2^{57+10.2-4} = 2^{63.2}$		

5.2.4 Experiments

To make the above theoretical analysis more convincing, we carried out the following experiments. For the t-step ($t \geq 37$) differential characteristic and its corresponding solution for X_i ($16 \leq i \leq 35$), we exhaust all possible values for X_{36} to verify the conditions on X_i ($37 \leq i \leq t \leq 39$) and record how many valid m_4 exists. Moreover, for a fixed valid m_4, we also randomly choose 2^{32} valid values for (X_{14}, X_{15}) and compute X_{13}. Then, we count the success times when the conditions on X_{13} hold (we will also check the conditions on X_{40} if analyzing 40 steps of RIPEMD-160). We list the experimental results in Table 10. Obviously, our theoretical analysis is reasonable.

Table 10: Experimental results

Steps	The number of valid m_4	Success times	Success probability
37	`0x36d40000`	`0x10001110`	2^{-4}
38	`0xe0000`	`0xffff6f3`	2^{-4}
40	`0x2d80`	`0xf1bd6ed`	2^{-4}

6 Conclusion

Relying on the specifics of RIPEMD-160's message expansion, a semi-free-start collision attack framework for reduced RIPEMD-160 is developed. Compared with previous semi-free-start collision attack framework, this new framework is extendable and allows us to attack as many steps of RIPEMD-160 as possible. One more advantage of this new framework is negligible requirement of memory. As a direct result, we present the first practical semi-free-start collision for 36 steps of RIPEMD-160 with time complexity 2^{41}. Moreover, benefiting from this framework, we can mount semi-free-start collision attack on 37/38/40 steps of RIPEMD-160 with time complexity $2^{49}/2^{53}/2^{74.6}$ respectively, thus extending the previously best known semi-free-start collision attack on RIPEMD-160 by four steps.

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A Differential Characteristics

The 36-step, 37-step, 38-step and 40-step differential characteristic are displayed in Table 11, Table 12, Table 13 and Table 14 respectively.

Table 11: 36-step differential characteristic

$\Delta m_{12} = 2^{15}$				
i	X	$\pi_1(i)$	Y	$\pi_2(i)$
1	-------- -------- -------- --------	0	-------- -------- -------- --------	5
2	-------- -------- -------- --------	1	-------- -------- -------- --------	14
3	-------- -------- -------- --------	2	-------- -------- -------- --------	7
4	-------- -------- -------- --------	3	-------- -------- -------- --------	0
5	-------- -------- -------- --------	4	-------- -------- -------- --------	9
6	-------- -------- -------- --------	5	-------- -------- -------- --------	2
7	-------- -------- -------- --------	6	-------- -------- -------- --------	11
8	-------- -------- -------- --------	7	-------- -------- -------- --------	4
9	-------- -------- -------- --------	8	-------- -------- -------- --------	13
10	-------- -------- -------- --------	9	-------- -------- -------- --------	6
11	-------- -------- -------- --------	10	-------- -------- -------- --------	15
12	-------- -------- -------- --------	11	-------- -------- -------- --------	8
13	-------- --n----- -------- --------	12	-------- -------- ----0--- --------	1
14	--nu---- -------- -------- --------	13	-------- -------- ----1--- --------	10
15	-n------ -------- -------- --u-----	14	-------- --n----- -------- --------	3
16	-0-0---- -------- --u----- 110n----	15	-------- -------- -------- --------	12
17	n0------ -------1 1-0----1 -110----	7	-------- --1----- -------- --------	6
18	0------- 1------- 01-----1 nu-0--0-	4	-------- --1----- -------- --------	11
19	---01101 1-011111 u0----0n 001--0-1	13	u------- -------- -------- --------	3
20	11010000 ------10 1n1-00-0 11010000	1	-------- -------- -------- --------	7
21	nnnn1nn1 01011111 -1011nu1 10100u10	10	1------- -------- -------- -------0	0
22	1001-01n u1-01u0n 1--1n1nu ununnu1-	6	1------- -------- -------- -------1	13
23	nn110u11 n10nu100 n001nnn1 0u11nnu1	15	-------- -------- -----un- --------	5
24	1101nu0- 10uun01n u1n0000n n1101u10	3	-------- -------- -------- --------	10
25	-1uu1nn1 -n011001 n0u01uuu 1n101uuu	12	-------- -------- ---0-01- --------	14
26	1101011u 1un0u10- u01uuuuu u001-010	0	-------- -------- ---1-11- --------	15
27	0u11u001 0011n1-1 uuun0011 11000111	9	-------- -n-un--- -------- --------	8
28	11000100 n11nnn0n 11100n-1 0n0n0nn1	5	-------- -------- -------- --------	12
29	01n00nu0 00u01nnu -01uuu-u nunun11n	2	-------- -1--1--- -------- --------	4
30	n1u01n10 u010011n 000110-0 0000un1u	14	--1----- -------- -------- -------1	9
31	-1n100-n nn01-010 1011-11- nnn0-10u	11	--u----- -------- -------- -------n	1
32	10n010-0 00--0-11 1110010- 100--10u	8	--1----- -------- -------- -------1	2
33	--n----- ------00 1----0-- 1------u	3	------0- -0------ -------- --------	15
34	--0----- -------- -------- 0------1	10	------u- -n------ -------- --------	5
35	-------- -------- -------- --------	14	-------- -------- -----n-- u-------	1
36	------n- -u------ -------- --------	4	-------- -------- -------- --------	3

Table 12: 37-step differential characteristic

$\Delta m_{12} = 2^{15}$				
i	X	$\pi_1(i)$	Y	$\pi_2(i)$
1	-------- -------- -------- --------	0	-------- -------- -------- --------	5
2	-------- -------- -------- --------	1	-------- -------- -------- --------	14
3	-------- -------- -------- --------	2	-------- -------- -------- --------	7
4	-------- -------- -------- --------	3	-------- -------- -------- --------	0
5	-------- -------- -------- --------	4	-------- -------- -------- --------	9
6	-------- -------- -------- --------	5	-------- -------- -------- --------	2
7	-------- -------- -------- --------	6	-------- -------- -------- --------	11
8	-------- -------- -------- --------	7	-------- -------- -------- --------	4
9	-------- -------- -------- --------	8	-------- -------- -------- --------	13
10	-------- -------- -------- --------	9	-------- -------- -------- --------	6
11	-------- -------- -------- --------	10	-------- -------- -------- --------	15
12	-------- -------- -------- --------	11	-------- -------- -------- --------	8
13	----0--- ---n---- -------- --------	12	-------- -------- -------- --------	1
14	---u0--- -------- -------- --0-----	13	-------- -------- ----0--- --------	10
15	-u------ --00---- -------- --n-----	14	-------- -------- ----1--- --------	3
16	10-0---- -------- --n----- -1uu----	15	-------- --n----- -------- --------	12
17	u1------ --1----1 100----0 -100----	7	-------- -------- -------- --------	6
18	0------- 0------- 1u0----0 -u1-----	4	-------- --1----- -------- --------	11
19	n1------ 100----- 01----1u -0--00--	13	-------- --1----- -------- --------	3
20	0010n100 n0110-11 u1uu1-00 10--0---	1	u------- -------- -------- --------	7
21	001110u0 0-101101 00010-u1 00011011	10	-------- -------- -------- --------	0
22	110-100u 10u--0un 010-001u uuuu-00-	6	1------- -------- -------- -------0	13
23	1un11nuu 1-100u0u 11u1uuun 1001010u	15	1------- -------- -------- -------1	5
24	110n11nn 1001uu11 0u0n10u1 u0nu001u	3	-------- -------- -----un- --------	10
25	00n0010n 0nn0un00 00nnn01u 11u101-0	12	-------- -------- -------- --------	14
26	0nnnnn-0 1101011- 10nnnnnn 011u0u10	0	-------- -------- ---0-01- --------	15
27	010-0011 nn0110u1 100n01uu 00n101u1	9	-------- -------- ---1-11- --------	8
28	1nu11111 uu1un0u0 1n0n1nnn n1100n00	5	-------- -n-un--- -------- --------	12
29	u00-uuuu 00u0101u n0-0011u 0001n0uu	2	-------- -------- -------- --------	4
30	111--001 -1110u10 u--0uuu0 0--uuu--	14	-------- -1--1--- -------- --------	9
31	1--1-11- --01-1nn n011-nnn --1---11	11	--1----- -------- -------- -------1	1
32	--001101 1011-u01 n-01-111 -1111101	8	--u----- -------- -------- -------n	2
33	----011- ---0-1-- 0------- --------	3	--1----- -------- -------- -------1	15
34	------0- -1------ -------- --------	10	--1---0- -0------ -------- -------1	5
35	-------- -------- -------- ----0--0	14	------u- -n------ -----1-- 1-------	1
36	-------- -------- -------- ----n--u	4	------1- -1------ -----n-- u-------	3
37	-------- -------- -----u-- n-------	9	-------- -------- -------- --------	7

Table 13: 38-step differential characteristic

$\Delta m_{12} = 2^{15}$				
i	X	$\pi_1(i)$	Y	$\pi_2(i)$
1	--------------------------------	0	--------------------------------	5
2	--------------------------------	1	--------------------------------	14
3	--------------------------------	2	--------------------------------	7
4	--------------------------------	3	--------------------------------	0
5	--------------------------------	4	--------------------------------	9
6	--------------------------------	5	--------------------------------	2
7	--------------------------------	6	--------------------------------	11
8	--------------------------------	7	--------------------------------	4
9	--------------------------------	8	--------------------------------	13
10	--------------------------------	9	--------------------------------	6
11	--------------------------------	10	--------------------------------	15
12	--------------------------------	11	--------------------------------	8
13	----------n---------------------	12	--------------------------------	1
14	---u0---------------------------	13	--------------------0-----------	10
15	-n------------------------n-----	14	--------------------1-----------	3
16	-0-1--------------u------1uu----	15	----------n---------------------	12
17	u---------------1-0----1-000----	7	--------------------------------	6
18	0-------1------0111-------u------	4	----------1---------------------	11
19	01-101100--0---nu------n-01---10	13	----------1---------------------	3
20	------01u1----01uu1--10011------	1	u-------------------------------	7
21	110111010110000011----u-00--1000	10	--------------------------------	0
22	1-00n-unu--un-nn0-unnnnnnnnnnn1-	6	1------------------------------0	13
23	n0uuu-100-10nuuu0uuuu1nun101nu11	15	1------------------------------1	5
24	1011111101un01u1001100000-1-1000	3	---------------------un---------	10
25	uu1110n0010-01000-0nuu01u11101n0	12	--------------------------------	14
26	-1nu101u--u0-01--00u00--11u-011-	0	-------------------0-01---------	15
27	-1000uu0-10u1n0--0nnn110--n-1--1	9	-------------------1-11---------	8
28	0u10u00--100-001110000-0000-01--	5	---------n-un-------------------	12
29	1--01--0111n-01-00---n0n--01n---	2	--------------------------------	4
30	n--0--0u-11011--u----001--n10---	14	---------1--1-------------------	9
31	00uuuuuu--10nu--1-00-10---1111--	11	--1----------------------------1	1
32	u-1111n1---101--n--u--0-nuu1-0--	8	--u----------------------------n	2
33	0-01--1--1-un---0--0--1-00001---	3	--1----------------------------1	15
34	u11n--0--0-11---1--1--0-----0---	10	--1---0--0---------------------1	5
35	1001--1--1--------------0------	14	------u--n-----------1--1-------	1
36	------n--u-0--0-------1--------1	4	------1--1-----------n--u-------	3
37	------1--1-u--n-----------------	9	---------------------1--1---1--1	7
38	--------------------------------	15	----------------------n--u------	14

Table 14: 40-step differential characteristic

		$\Delta m_{12} = 2^{15}$		
i	X	$\pi_1(i)$	Y	$\pi_2(i)$
1	--------------------------------	0	--------------------------------	5
2	--------------------------------	1	--------------------------------	14
3	--------------------------------	2	--------------------------------	7
4	--------------------------------	3	--------------------------------	0
5	--------------------------------	4	--------------------------------	9
6	--------------------------------	5	--------------------------------	2
7	--------------------------------	6	--------------------------------	11
8	--------------------------------	7	--------------------------------	4
9	--------------------------------	8	--------------------------------	13
10	--------------------------------	9	--------------------------------	6
11	--------------------------------	10	--------------------------------	15
12	--------------------------------	11	--------------------------------	8
13	----------n---------------------	12	--------------------------------	1
14	---u----------------------------	13	--------------------0-----------	10
15	-u-----------------------1u-----	14	--------------------1-----------	3
16	-0-1--------------u------n0u----	15	----------n---------------------	12
17	u1--------------1-0----1-010----	7	--------------------------------	6
18	0-------1------1011----0-u------	4	----------1---------------------	11
19	1-------0-------u-----1u-0------	13	----------1---------------------	3
20	--------u------n-u----00-1------	1	u-------------------------------	7
21	------1-0----1-0-0----u---------	10	--------------------------------	0
22	-0--01-10------u-1000-0-------1-	6	1------------------------------0	13
23	100010u0----1u-0----10-001---1--	15	1------------------------------1	5
24	---1-n0u0--1-011110unnnn--01un--	3	---------------------un---------	10
25	-1-100010n0u0nnn010011un1001-10-	12	--------------------------------	14
26	unu001uu001u01001u10--01111u11u1	0	-------------------0-01---------	15
27	011un011nnuuuuuuuu10000u00u100u0	9	-------------------1-11---------	8
28	00111nun0111u1nn0nnnnnn10n00nnn1	5	---------n-un-------------------	12
29	nn111010n0101010100--0u1un10101n	2	--------------------------------	4
30	011-u11010000n-1101100uuu11101u0	14	---------1--1-------------------	9
31	n001u11-n-0u1110nn--u0100111110n	11	--1-----------------------------1	1
32	1000uun01-n1n011u01111010-n0nu01	8	--u-----------------------------n	2
33	0u1110101-00u01u1-11n0-u--0111n0	3	--1------------------1--1------1	15
34	-11n-1n00-0u1-n---000001--00101-	10	--1---0--0-----------0--0------1	5
35	0u000000--01--1-----111111-000-n-	14	------n--u-----------1--1-------	1
36	-1-------------------0--0-----1-	4	------1--1-----------n--u------1	3
37	--------------------1u-1n-------	9	---1-----1-----0-----10-10--1--0	7
38	-----------10-10-----1--1-------	15	-----0--0--1--1-------u--u--0--1	14
39	-----------0n-0n----------------	8	-----u--n-------------1--0--n--u	6
40	-----n--u-------------------u--n	1	--------------------------------	9