Efficient Perfectly Sound One-message Zero-Knowledge Proofs via Oracle-aided Simulation

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Abstract. In this paper we put forth new *one-message* proof systems for several practical applications, like proving that an El Gamal ciphertext (over a multiplicative group) decrypts to a given value and correctness of a shuffle. Our proof systems are not based on any setup/trust assumption like the RO or the common reference string model and are *perfectly sound*, that is they are written proofs in the sense of mathematics.

Our proof systems satisfy a generalization of zero-knowledge (ZK) that we call *harmless* zero-knowledge (HZK). The simulator of an O-HZK proof for a relation over a language L is given the additional capability of invoking an oracle O relative to which L is hard to decide. That is, the proof does not leak any knowledge that an adversary might not compute by itself interacting with an oracle O that does not help to decide the language.

Unlike ZK, non-interactivity and perfect soundness do not contradict HZK and HZK can replace ZK in any application in which, basically, the computational assumptions used in the application hold even against adversaries with access to O. An O-HZK proof is witness hiding, assuming some computational problem to be hard for adversaries with access to O, and strong-WI when quantifying over distributions that are indistinguishable by adversaries with access to O.

We provide a specific oracle DHInvO that is enough powerful to make our main proof systems DHInvO-HZK but not trivial: indeed, we show concrete and practical cryptographic protocols that can be proven secure employing a DHInvO-HZK proof in the reduction and that are instead not achievable using traditional ZK (unless assuming a CRS/RO).

Efficient one-message proof systems with perfect soundness were only known for relations over bilinear groups and were proven only witness indistinguishable.

As byproduct, we also obtain a perfectly sound non-interactive ZAP and HZK proof for NP from a number-theoretic assumption over multiplicative groups of hidden order.

Keywords: zero-knowledge, NIZK, RSA, ZAP.

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1 Introduction

Mathematical proofs vs interactive proofs. Zero-knowledge (ZK) proofs [GMR85,GMR89,GMW91] represent one of the most important concepts in computer science and turned out to be one of the key ingredients in modern cryptographic protocol design. Unlike traditional mathematical proofs, ZK proofs require interaction and allow the prover to cheat the verifier with a non-zero probability of error.

Non-interactive proofs. Since the discovery of ZK proofs, the importance of removing interaction led to the introduction of non-interactive (NI) zero-knowledge (NIZK) systems [DMP88]. NIZKs have been extensively studied for about 30 years [BFM88,DMP88,FLS90,RS92,Gol01,DDO+01]. Indeed, the concept of proving a statement in just one round without leaking any information has been intriguing for theoreticians and extremely useful as building block for designers of cryptographic protocols.

Truly non-interactive ZK proofs for non-BPP languages are provably impossible to achieve [Gol01], so the initial constructions for NIZKs worked in the common reference string (CRS) model [DMP88] and because of various limitations (e.g., the need of NP-reductions, the non-reusability of the CRS, the expensive computations) their impact was mainly in the theoretical foundations of cryptography. In the CRS model, the party who generates the CRS has to be trusted. Indeed, such party might either setup the CRS in a malicious way for which the soundness does not hold or hand a trapdoor to an adversary who can exploit the trapdoor to prove false theorems.[1]

Proofs vs arguments. The gap between NIZK proof (NIZKP) systems and NIZK argument (NIZKA) systems consists in a different soundness requirement. The soundness property aims to prevent a dishonest prover from convincing the verifier about the veracity of a false statement. The powerful concept of a NIZK proof requires the soundness guarantee to be unconditional, therefore the adversarial prover can be unbounded. Instead, the notion of a NIZK argument [BCC88] has a significantly weaker soundness guarantee since it applies to efficient adversarial provers only.[2].

The bridge between theory and practice: the Fiat-Shamir (FS) transform. The traditional power of the simulator in a NIZK proof/argument system consists in programming the common reference string (CRS). A popular alternative to the CRS model is the Random Oracle (RO) model [BR93]. The RO model assumes

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[1] In the rest of this work, when we say that an adversary "proves a false theorem" we actually mean that it produces a string π such that the verifier accepts (x, π) for a false statement x. Such string is an alleged proof of x, not a proof in the mathematical sense.

[2] In literature this difference is often overlooked. Despite this subtle difference, for simplicity we will call *proof* the string generated by the prover, irrespective of whether the prover be part of a proof or an argument system. We will however be precise on using the words "proof system" and "argument system".

the availability of a perfect random function to all parties. One of the most successful applications of the RO model in cryptography is the FS transform that allows to obtain very efficient NIZK arguments [FS87]. The simulator of such a NIZK argument programs the RO (i.e., the simulator replaces at least in part the RO in answering to RO queries of the adversary).

In concrete implementations of this transform, prover and verifier replace the RO by some "secure" hash function (e.g., SHA-3 [BDPA11]). NIZK arguments in the RO model obtained via the FS transform are orders of magnitude more efficient than the most efficient NIZK arguments in the CRS model [GS08].

Even if the RO methodology has been shown to be controversial already in [CGH98] and further negative results were published next [DNRS99,Bar01,GK03,Kal06] [BLV03,DRV12,BDSG+13,GOSV14,KRR16], NIZK arguments via the FS transform are widely used in concrete cryptographic protocols (e.g., in the popular Helios voting system [Adi08]). We remark that one could also consider a hybrid notion where the adversarial prover can be unbounded except that it can query the random oracle a polynomial number of times only. We stress that in this paper, when analyzing FS-derived NIZKs, we consider a truly unbounded adversarial prover. This difference can be crucial in applications.

The importance of unconditional soundness. In e-voting privacy cannot be achieved unconditionally unless losing universal verifiability or unless all the voters actually vote [CMFP+10]. There is instead no barrier to attain unconditional universal verifiability and the soundness guarantee of a ZK proof/argument used in an e-voting scheme impinges on the quality of the universal verifiability: an adversary that can break the soundness condition can subvert the result of the election.

In the context of proofs of solvency for cryptocurrencies [DBB+15,BBB+18] and similar applications, the soundness of the proof/argument system is significantly more important than the ZK property: the privacy of the transaction is expendable but breaking the soundness of the proof system gives the possibility of generating arbitrary coins blasting away the whole digital economy. The proof systems we will propose are unconditionally sound but are not short, so cannot be used in most of the applications of confidential transactions. Moreover, under widely accepted complexity assumptions, no proof system with short communication may exist [GH98]. Despite this fact, the new ideas we will introduce may open up the possibility of removing CRSs/ROs from the design of argument systems for this kind of applications.

Problem statement. The FS transform induces a significant soundness loss. Indeed it receives as input a constant-round public-coin honest-verifier zero-knowledge (HVZK) *proof* system and outputs a NIZK *argument* system. This is a step back compared to the known NIZK *proofs* in the CRS model [BFM88,FLS90] [GOS12,GS08].

Of course if one is interested in a NIZK proof system in the RO model there is a trivial approach: just evaluate the RO on input the instance x to get a random string that can be used to compute a NIZK proof in the common reference

string model (e.g., [FLS90]). However the trivial approach is very unsatisfying for the following two reasons: 1) it requires expensive computations (sometimes including an NP reduction) that make the NIZK proof completely impractical, and 2) it requires some complexity assumptions (e.g., trapdoor permutations in [FLS90]) therefore additionally incurring a significant security loss in the zero-knowledge guarantee.

For languages relative to bilinear groups the situation is better and we have NIZK proof systems and non-interactive ZAPs [DN00] with perfect soundness [GOS06b,GS08]. However, bilinear groups are less efficient and their security is less studied than other number-theoretic problems and the security of ZAPs is limited to witness indistinguishability (WI) that, for instance, is a vacuous guarantee for single witness relations as it is the case for the relation of valid DH tuples. Furthermore, even for non-single witness relations WI poses several threats and issues in several scenarios (see, e.g., discussion on verifiable functional encryption in Section 1.3.8).

These limitations of the FS-transform, of the above trivial approach, of ZK proofs in general (non-zero soundness error and non-interactivity inherently based on trust assumptions), and of weaker WI proofs, motivate the main questions of this work.

Practical open question: *is there any efficient non-interactive proof system (i.e., soundness is guaranteed also against unbounded adversarial provers) for practical languages not related to bilinear groups that can be used in relevant cryptographic applications and satisfies a meaningful notion of privacy?*

Theoretical open question: *is there any achievable, non-trivial, useful and usable variant of ZK that is compatible with perfect soundness and true non-interactivity?*

Later, we will answer positively these questions in a very strong sense by presenting completely non-interactive proof systems both for practical applications and general statements satisfying perfect soundness, not based on any trusted parameter and enjoying a close variant of ZK. Our proofs are in particular proper *mathematical proofs*.

The FS transform internals. Before digging in our results, we will first discuss the limitations of the FS transform, the most known technique to construct efficient NIZKAs, and review its internals.

Formal definitions of NIZK proofs and arguments of knowledge in the RO model through the FS transform have been investigated in several papers [FKMV12] [BPW12,BFW15]. For simplicity here we will now discuss the specific case of a 3-round public-coin HVZK proof system $\mathsf{3HVZK} = (\mathcal{P}, \mathcal{V})$ where the decision of the verifier is deterministic. However our discussion can be generalized to any constant-round public-coin HVZK argument system.

$\mathcal{P}$ sends a first message a to $\mathcal{V}$, also called the commitment. Then $\mathcal{V}$ sends back a random challenge c. Finally $\mathcal{P}$ outputs the final message z, the answer to c. The triple (a, c, z) is called the transcript of an execution of $\mathsf{3HVZK}$ for

an instance x and $\mathcal{V}$ takes deterministically the decision of accepting or not the transcript.

The FS transform constructs $\mathsf{NIZK} = (\mathsf{NIZK.Prove}, \mathsf{NIZK.Verify})$ as follows. $\mathsf{NIZK.Prove}$ computes a precisely as $\mathcal{P}$, but then the challenge c of $\mathcal{V}$ is replaced by the output of the RO on input the statement x and a, i.e., $c = H(x, a)$.[3] Finally $\mathsf{NIZK.Prove}$ computes z precisely as $\mathcal{P}$ would compute it.

NIZK is only computationally sound (i.e., it is an argument system) in the random oracle model. Indeed one can easily see that computing with non-negligible probability an accepting transcript for a false statement when the adversarial prover runs in polynomial time, implies that the challenge is the output of one out of a polynomially bounded number of evaluations of the RO, and this can be translated to proving with non-negligible probability a false statement to $\mathcal{V}$. Soundness can not be claimed when instead the adversarial prover is unbounded and can therefore make an unbounded number of queries to the RO.

If $\mathsf{3HVZK}$ is also HVZK, then the resulting NIZK argument system is additionally a computational ZK argument system. Indeed the ZK simulator can program the queries therefore being able to produce a simulated proof using the HVZK simulator that is computationally indistinguishable from the a real proof.

If $\mathsf{3HVZK}$ satisfies special soundness (i.e., there is a deterministic efficient extractor that from 2 different accepting transcripts for the same statement with the same first message outputs a witness), then the resulting NIZK argument system additionally enjoys witness extraction but limited to PPT adversarial provers. Known variations [Pas03a,Fis05,FKMV12] of the FS transform produce NIZK *argument* systems that suffer of the same limitation of witness extraction with respect to PPT provers. We also stress that, to our knowledge, all previous variants of the FS transform (e.g., the ones of Pass [Pas03a] and Fischlin [Fis05]) only attain *computational* soundness (i.e., there is no security guarantee against an unbounded adversarial prover that as such can have unlimited access to the random oracle). Moreover, to our knowledge, previous works on NIZK argument systems in the RO model only attained extraction (i.e., the proof of knowledge property) against bounded adversaries.

The soundness degradation of the FS transform. Suppose that the underlying interactive protocol has the following properties. The space of prover commitments has cardinality $\geq 2^{b(\lambda)}$, the verifier's challenges have length $k(\lambda)$, the soundness error is $2^{-k(\lambda)}$, with $k(\lambda) \in \omega(\log(\lambda)), b(\lambda) \geq \lambda + k(\lambda)$ and λ being the security parameter, and the prover computes the answer z deterministically based on (a, c). Suppose further that for each $x \notin L$ and each commitment a, there exists at least one challenge c such that (a, c, z) is an accepted transcript. (A natural Σ-protocol satisfying the above requirements will be shown soon. The latter hypothesis can be also seen to hold assuming that for each $x \notin L$, the soundness error is non-zero.)

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[3] When the challenge c is computed as $H(a)$, the FS transform offers weaker security guarantees (see [BPW12,CPS$^+$16]). In this overview of the FS transform, we will consider the *strong* FS transform.

Fix an $x \notin L$ and consider the following unbounded prover $\mathsf{NIZK.Prove}^\star$ that aims to compute an accepting proof for x. $\mathsf{NIZK.Prove}^\star$ searches over all pairs of commitments and challenges (a_c, c) such that the above property holds (i.e., (a_c, c, z) is an accepting tuple, where z is the deterministic answer of the prover to (a_c, c)) *and* RO maps (x, a_c) into c; if $\mathsf{NIZK.Prove}^\star$ can find a pair (a_c, c) that verifies such conditions, it outputs (a_c, c, z) as its proof, otherwise outputs some error $\perp$.

For each commitment and challenge pair (a_c, c) the probability that the RO maps (x, a_c) into c such that (a_c, c, z) is an accepted transcript is, by hypothesis, $\geq 2^{-k(\lambda)}$. Thus, since there are $2^{b(\lambda)} \geq 2^{\lambda + k(\lambda)}$ commitments, $\mathsf{NIZK.Prove}^\star$ fails in proving the false statement x with probability $< (1 - \frac{1}{2^{k(\lambda)}})^{2^{\lambda + k(\lambda)}}$. Therefore, $\mathsf{NIZK.Prove}^\star$ succeeds with probability $\geq 1 - (1 - \frac{1}{2^{k(\lambda)}})^{2^{k(\lambda)} \cdot 2^\lambda} \approx 1 - (\frac{1}{e})^{2^\lambda}$.[4]

This example shows that an unbounded prover can break the soundness of the FS transform applied to some particular proof system satisfying the above requirements. This is not an artificial counter-example as such requirements are satisfied by very natural proof systems like the ones of [CP93,CDS94].

Example. Consider for instance the protocol of Chaum and Pedersen [CP93] for proving that a tuple (g, h, u, v) of 4 group elements, in a group of prime order q, is a Diffie-Hellman (DH) tuple.

The prover chooses a random $r \leftarrow \mathbb{Z}_q$, where q is the order of the group, and sends the commitment $a \triangleq g^r, b \triangleq h^r$. The verifier sends a random challenge $c \leftarrow \mathbb{Z}_q$. The prover sends back deterministically $z \triangleq r + c \cdot w \mod q$ and the verifier accepts iff $g^z = a \cdot u^c$ and $h^z = b \cdot v^c$.

Let λ be the security parameter and $k(\lambda) \triangleq \lambda$ equal the length of the group elements. Then, challenges have length $k(\lambda)$, commitments have length $2 \cdot k(\lambda)$ and $k(\lambda)$ is also the soundness error. By using the simulator (of the special HVZK), it is easy to see that for each false statement $x \notin L$ and for each challenge c, there exists (a, z) such that (a, c, z) is an accepted transcript for x. Thus, the Chaum and Pedersen's protocol satisfies the above requirements and the soundness can be broken in time $\approx 2^{k(\lambda)}$.

Ineffectiveness of parallel repetition. A natural approach to adjust the FS transform in order to circumventing the above attack would be to execute p instances of the protocol in parallel and computing each challenge c_i, for $i = 1, \ldots, p$, as $\mathcal{RO}(x \| a_i \| i)$. Unluckily, this strategy does not improve the situation. In fact, while the number of possible challenges increases (each challenge now consists of $k \cdot p$ bits) the number of possible commitments also increases. A simple analysis shows that an attack similar to the previous one can be applied to such variant of the FS transform as well. Observe also that the previous attack can be viewed as a special case for $p(\lambda) = 1$.

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[4] This follows from the fact that $\lim_{\lambda \to \infty} 2^{k(\lambda)} = \infty$ and thus $\lim_{\lambda \to \infty} (1 - \frac{1}{2^{k(\lambda)}})^{-2^{k(\lambda)}} = e$.

In fact, consider a false statement x and an unbounded prover $\mathsf{NIZK.Prove^\star}$ similar to before aiming at computing an accepting proof for x. By the previous analysis on the protocol without repetitions (that can be seen as a special case for $p(\lambda) = 1$) and since the $p(\lambda)$ executions are independent, $\mathsf{NIZK.Prove^\star}$ succeeds with probability $\left(1 - (\frac{1}{e})2^\lambda\right)^{p(\lambda)}$ that is overwhelming in λ.

It is fundamental for the previous analysis to hold that the space of commitments be much bigger than the challenge space, as it is indeed the case in general for natural Σ-protocols for relations in which deciding membership is non-trivial. In fact, if for instance the challenge and commitment spaces had the same cardinality, the lower-bound on the winning probability of the previous prover would just be $\left(1 - \frac{1}{e}\right)^{p(\lambda)}$, a negligible function.

Moreover, it is very easy to observe that any ZK proof system, even for the RO model, cannot satisfy *perfect* soundness.

1.1 Our Results and Roadmap

The main result of this work is a *positive* answer to the above open questions: we construct new efficient perfectly sound *one-message* proof systems for practical languages satisfying a variant of zero-knowledge that we call *harmless zero-knowledge* (HZK). We provide an overview of HZK in Section 1.3 and in Section 2 we recall standard definitions and provide formal definitions for harmless ZK proof of knowledge systems. In Sections 2.2.4, 2.2.5 and 2.2.6 (sketch in Section 1.3.4) we present relations of our new notion to related ones.

In Section 3 (overview in Section 1.2) we present our new proof system for proving that an El Gamal ciphertext decrypts to a given value, for an El Gamal encryption scheme instantiated over a group of hidden order (more details in Sections 1.2 and 2). In Section 1.4.1 we show a variant of our proof system for proving correctness of a shuffle of El Gamal ciphertexts (instantiated over the same group).

In Section 1.4.2 we build OR proofs from proofs of correct shuffle, and in Section 1.4.3 we construct direct proofs for polynomial statements.

All our proof systems enjoy perfect soundness, are non-interactive and do not assume any trust assumptions like the Common Reference String (CRS) model or the RO model and do not assume any bound on the space of the verifier.

If the group parameter on which DH tuples are based is seen as a common parameter and is made public (though we do not require it to be setup in a trusted way), our proof for DH tuples also satisfies the standard definition of perfect extraction [GOS12] and additionally enjoys what we call harmless proof of knowledge (see Sections 1.3 and 2).

In Section 1.4.4 we construct a one-message perfectly sound WI proof (ZAP) for Boolean circuit satisfiability from a number-theoretic assumption related to DH over groups of hidden order. Our ZAP is also *computational* HZK and we sketch that, using know complexity leveraging arguments and techniques, it can be tweaked to be quasi-polynomial time simulatable.

Our proof systems are sufficient to construct an e-voting scheme in which authorities have zero probability of subverting the result of the election. This application and an application to verifiable functional encryption are presented in Sections 1.2.3 and 1.3.8.

In Section 1.5 we survey the known literature in the field and compare our results to it.

1.2 Overview of our main proof system

Before describing our new proof system, we recall the standard NIZK argument in the programmable RO model for proving correct decryption of an El Gamal ciphertext due to Chaum and Pedersen [CP93].

1.2.1 Standard NIZKA for correctness of El Gamal decryption Consider an (exponential) El Gamal ciphertext for public key $\mathsf{pk} = g^w$ and message m: $(a = g^r, b = \mathsf{pk}^r \cdot g^m)$, with g known generator of a group of prime order p (e.g., the group of quadratic residues modulo a prime q such that $q = 2p+1$ for a prime p). To prove that this ciphertext decrypts to m without revealing any information on the secret-key w, one can prove that the following tuple is DH: $(g, h \stackrel{\triangle}{=} a, u \stackrel{\triangle}{=} \mathsf{pk} = g^w, v \stackrel{\triangle}{=} b/g^m = h^w)$. Therefore, the problem of proving that an El Gamal ciphertext decrypts to some message boils down to proving that a tuple (g, h, u, v) is DH.

The standard Chaum-Pedersen's interactive proof for DH tuples [CP93] is the following. Let (g, h, u, v) be a DH tuple with witness w, i.e., $u = g^w, v = h^w$ for some $w \in \mathbb{Z}_p$. The prover knows the witness but the verifier does not and both the prover and the verifier share the common input (g, h, u, v).

- The prover sends $a = g^r, b = h^r$ to the verifier.
- The verifier sends a challenge $e \leftarrow \mathbb{Z}_p$.
- The prover sends back $z = r + e \cdot w \mod p$.
- The verifier accepts iff $g^z = a \cdot u^e, h^z = b \cdot v^e$.

Let us analyze the soundness. Let g, h, u, v be a non-DH tuple, that is $g, h, u = g^{w_1}, v = h^{w_2}$ for $w_1 \neq w_2 \mod p$. Let $a = g^{r_1}, b = h^{r_2}$ for some $r_1, r_2 \in \mathbb{Z}_p$ (the first message of the prover can be possibly ill-formed when $r_1 \neq r_2 \mod p$).

If the verifier accepts, the equations checked by the verifier imply that 1) $z = r_1 + e \cdot w_1 \mod p$ and $z = r_2 + e \cdot w_2 \mod p$. Subtracting the equations together, it holds that $r_1 - r_2 = e \cdot (w_1 - w_2) \mod p$. This means that for each w_1, w_2, r_1, r_2 there is exactly one value e (i.e., $e = (w_1 - w_2)/(e_1 - e_2) \mod p$) that satisfies the equations, and thus if e is randomly chosen the probability that the verifier accepts the false statement is $\frac{1}{p}$, a quantity negligible in $|p|$.

The proof is made non-interactive via the FS transform, i.e., computing $e = \mathcal{RO}((g, h, u, v), a, b)$. As we analyzed before, applying the FS transform we lose statistical soundness. For completeness, we sketch again the argument. Fix a false statement (g, h, u, v). The unbounded prover can search over all values $a = g^{r_1}$ and $b = g^{r_2}$ until it finds a pair (a, b) such that $\mathcal{RO}(g, h, u, v, a, b)$ equals the only

value e that satisfies the equation $e = (w_1 - w_2)/(e_1 - e_2) \mod p$. As there are 2^{2k} possible pairs (a, b) and 2^k possible values of e, with $k \triangleq |p|$, an unbounded prover succeeds with overwhelming probability in proving any false statement. Moreover, in practice one chooses a fixed hash function and in this case nothing can be said about the security: it might be that the hash function maps a false statement and a pair (a, b) into the only one "bad" e that satisfies the equations checked by the verifier.

In the following, we will propose a new proof system that both removes the use of the hash function (and thus it is not based on the RO heuristic or any other trust assumption or limitation) and achieves perfect soundness. Recall that perfect soundness cannot be attained even in the interactive case for ZK proofs. This system will also serve as base to build a proof system for general NP statements (see Section 1.4.4).

1.2.2 Our new non-interactive proof system Observe that in the above proof the prover can cheat only when $r_1 \neq r_2 \mod p$. To prevent such possibility of cheating, one could require the prover to send the value r in the clear in the first round but this is insecure as the prover also sends the value $z \triangleq r + e \cdot w \mod p$ in the last round. Sending instead $a \triangleq g^z$ and $b \triangleq h^z$ would not reveal z and would still allow the verifier to check the equations. In this case the verifier would need to additionally verify the well-formedness of the pair (a, b), i.e., that $a = g^r, b = g^r$, for some $r \in \mathbb{Z}_p$. Apparently, a way of proving (a, b) to be well-formed seems as difficult as proving a tuple to be DH. However, we will show that this can be done working in groups of hidden order.

Switching to groups of hidden order. Let us analyze the following completely non-interactive proof system (in particular, the verifier does not longer need to send any challenge to the prover). Let N be a Blum integer and consider the group of quadratic residues modulo N. In this group, we can construct an El Gamal-like encryption scheme (see Def. 25). Both the prover and the verifier share the tuple (g, h, u, v) and the modulus N. The order of the group of quadratic residues modulo N is hidden and equals $m \triangleq \phi(N)/4$.

The aim of the prover is to convince the verifier that the tuple (g, h, u, v) is DH for witness $w \in \mathbb{Z}_{\phi(N)}$. In Section 1.2.3, we will show that our proof can be used to prove that a ciphertext $\mathsf{ct} = (\mathsf{ct}_1, \mathsf{ct}_2)$ for public key pk decrypts to g^0.

Our first version of the proof system NIDDH assumes u (resp. v) to be in the same subgroup generated by g (resp. h) and we will subsequently show another proof system NISG that will be used in the final version of NIDDH to remove this restriction. However, we stress that in the analysis of NIDDH and NISG we will never assume N to be well-formed.

To the aim of highlighting potential attacks, we will first present a NI proof system subject to an attack and we will later show how to patch it.

A first attempt. The prover of NIDDH (in its first insecure version), on input a DH tuple (g, h, u, v) and a factorization of N, sends the following (non-interactive) proof:

$$r, X \stackrel{\triangle}{=} g^z, Y \stackrel{\triangle}{=} h^z, z' \stackrel{\triangle}{=} z^{-1} \mod \phi(N),$$

with $r \leftarrow \mathbb{Z}^\star_{\phi(N)}$ and $z \stackrel{\triangle}{=} r + w \mod \phi(N)$ subject to the following constraints: $z \in \mathbb{Z}^\star_{\phi(N)}$ and z' prime number. (The reason for z' to be prime will be explained later when we will also propose a change in the proof. We stress that we *require* z to be in $\mathbb{Z}^\star_{\phi(N)}$, that is the prover has to find randomness r such that $r + w \mod \phi(N)$ satisfies the constraint; it is easy to design an algorithm that computes values satisfying such constraints w.v.h.p.) Note that, notwithstanding the group is of "hidden order", the prover can compute $z^{-1} \mod \phi(N)$ from the factorization of N, that is the order of the group is hidden to the verifier but not to the prover.

The verifier of NIDDH is given N and the tuple (but not the factorization). The verifier accepts the proof if and only if z' is a prime number and all the following equalities hold:

$$X^{z'} = g, Y^{z'} = h, X = g^r \cdot u, Y = h^r \cdot v.$$

The idea is that z' should allow to verify that X and Y are such that $\mathbf{dlog}_g X = \mathbf{dlog}_h X$, i.e., $X = g^z, Y = h^z$ for some $z < \phi(N)$. Then, the soundness would follow from the observations highlighted above. However, the checks are not sufficient to guarantee soundness and we actually need some modifications as explained next.

A potential issue, how to fix it and soundness analysis of our first attempt. Notice that the soundness of the previous proof relies on the fact that z' should allow to check the well-formedness of the pair (X, Y), i.e., check that $X = g^t, Y = h^t$ for some non-negative integer $t < \phi(N)$. However, it might be that $X^{z'} = g, Y^{z'} = h$ but $\mathbf{dlog}_g X \neq \mathbf{dlog}_h Y$.[5] Thus, the previous checks are not sufficient. We guarantee this case cannot occur as follows.

Observe that if z' has no common factors with $\phi(N)$ and $X^{z'} = g, Y^{z'} = h$, then $X = g^t, Y = h^t$ for some integer $t < \phi(N)$. This can be seen by setting $t \stackrel{\triangle}{=} z'^{-1} \mod \phi(N)$. So, let us analyze the soundness supposing z' to be co-prime with $\phi(N)$. Recall that we are assuming u (resp. v) to be in the same subgroup generated by g (resp. h). We will later show how to remove this restriction. Let us assume for simplicity $\mathrm{ord}(g) = \mathrm{ord}(h)$ (see the general case in Theorem 14) and let $k \stackrel{\triangle}{=} \mathrm{ord}(g) = \mathrm{ord}(h)$. The verifier of NIDDH checks that $X = g^r \cdot u$ and $Y = h^r \cdot v$. By hypothesis, $u = g^{w_1}$ and $v = h^{w_2}$ for some $w_1, w_2 < k$. Letting $t \stackrel{\triangle}{=} z'^{-1} \mod \phi(N)$ and taking the discrete logs, resp. in base g and h, we have that $t \mod k = r + w_1 \mod k$ and $t \mod k = r + w_2 \mod k$. So, we have

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[5] For example, if $N = 35, g = 8, X = 2$, we have that $X^3 = g$ but $\mathbf{dlog}_g X$ does not exist, that is there is no x such that $8^x = 2 \mod 35$.

$w_1 = w_2 \mod k$, for some $k < \phi(N)$. Therefore, there exists $w < \phi(N)$ such that $u = g^w$ and $v = h^w$, as it was to prove.

Therefore, what is left to guarantee soundness is to enforce z' to be co-prime with $\phi(N)$. This can be done as follows. The prover repeats the basic NIDDH protocol in parallel a sufficient number of times s setting in the i-th execution, for $i = 1, \ldots, s$, the value z_i' to be prime and setting all z_i's to be *different*. If a dishonest prover could set for each $i \in [s]$ the value z_i' to have a common factor with $\phi(N)$, we would have a contradiction. Indeed, it is not possible for all z_i''s to have a factor in common with the order of the group $\mathbb{Z}_N^\star$ assuming s to be, e.g., greater or equal than the maximum possible number of factors of $\phi(N)$. See details in Theorem 14.

Observe that the soundness of NIDDH, as described so far, does not to rely on the well-formedness of the modulus N: whatever the modulus N is, the prover cannot cheat.

Proof of knowledge with perfect extraction. Our proof system NIDDH (in its first insecure version) has perfect extraction according to the standard definition [GOS12]. The common reference string (that has not to be trusted in our case) can be set to the modulus N. The extractor computes the modulo N with knowledge of the factorization, and thus of the hidden order $\phi(N)$. Given a proof accepted by the verifier and $\phi(N)$, the prover inverts $z' = z^{-1} = (r+w)^{-1} \mod \phi(N)$ to compute $z = r+w \mod \phi(N)$ and subtracts from it $r \mod \phi(N)$ to compute w, the witness.

So the NI satisfies perfect extraction assuming a common parameter (the modulus N) is set and made public at the beginning of the protocol. On the other hand, NIZKAs obtained via FS transform suffers annoying rewinding issues. We will later show that our proofs systems additionally enjoy a generalization of proof of knowledge to a purely non-interactive setting.

Guaranteeing that u belong to the subgroup generated by g. The previous proof of knowledge system NIDDH (in its first insecure version) can be simplified to a Schnorr-like non-interactive proof of knowledge system NISG to prove that an element u belongs to the subgroup generated by g, i.e., that $u = g^w$ for some $w < \phi(N)$. (This proof of knowledge system is still subject to the linear attacks we will describe next but later we will show a patch against them that applies both to NIDDH and NISG.)

Consider the following proof of knowledge system NISG. The prover sends $r, z' \triangleq z^{-1} \mod \phi(N)$, with $z \triangleq r+w \mod \phi(N) \in \mathbb{Z}_{\phi(N)}^\star$ and z' prime number. Let $H \triangleq g^r \cdot u$. The verifier checks that 1) $H^{z'} = g$.

The soundness should follow from the fact that, for each Y and X both $\neq 1$ and for each number z' co-prime with $\phi(N)$, if $Y^{z'} = X$ then Y belongs to the subgroup generated by X. This can be proven as follows. Let $t \triangleq z'^{-1} \mod \phi(N)$, then $X^t = Y$ and thus Y belongs to the subgroup generated by X (note that the inverse of z' exists as z' if is co-prime with $\phi(N)$). Guaranteeing

z' to be co-prime with $\phi(N)$ can be done as shown above with the trick of the repetitions. For simplicity, henceforth we assume z' to be co-prime with $\phi(N)$.

The check 1) implies that $H = g^t$ for some $t \in \mathbb{Z}_\phi(N)$ and thus $u = H \cdot g^{-r} = g^{t-r}$, that is u is in the subgroup generated by g as well.

Previously, we assumed NIDDH to work under the hypothesis of u (resp. v) being in the same subgroup generated by g (resp. h). To remove such a restriction, we require the prover of NIDDH to first invoke the prover of NISG to prove u (resp. v) to be in the same subgroup generated by g (resp. h).

A linear attack against our first attempts of NIDDH *and* NISG. Hereafter, for simplicity we are not considering the aforementioned modification to NIDDH that introduces parallel repetitions and invokes NISG as sub-protocol. Recall that in NIDDH, $r \stackrel{\triangle}{=} z - w \mod \phi(N)$. Let $s \stackrel{\triangle}{=} z^{-1} \mod \phi(N)$. The verifier can multiply (over the integers) s by r to get $1 + s \cdot w \mod \phi(N)$ and in turn, subtracting 1 (over the integers), can get $s \cdot w \mod \phi(N)$. Given another pair of group elements g^t and h^t, an attacker can power g^t to $s \cdot w \mod \phi(N)$ and h^t to s to check that the tuple (g, h, g^t, h^t) is DH for witness w, a destructive attack.

Potential attacks on multiple proofs. Given two proofs, it is possible to get a multiple of the order of the group, from which it is possible to factorize. Indeed, suppose to have two proofs for the same witness. That is, suppose we have $z_1 = r_1 + w \mod \phi(N)$ and $z_2 = r_2 + w \mod \phi(N)$. Combining them together we obtain $z_1 - z_2 = r_1 - r_2 \mod \phi(N)$. Multiplying over the integers this value by $z_1' = (r_1 + w)^{-1} \mod \phi(N)$, we have $z_1' \cdot (r_1 - r_2) = 1 - z_2 \cdot z_1' \mod \phi(N)$. Subtracting over the integers by 1 and multiplying over the integers by z_2' we finally obtain that $z_2'(z_1' \cdot (r_1 - r_2) - 1) = z_1' \mod \phi(N)$.

Therefore, $z_2' \cdot (z_1' \cdot (r_1 - r_2) - 1) - z_1'$ is a multiple of $\phi(N)$ and, by standard techniques, the hidden order $\phi(N)$ can be computed.

Defense against linear attacks and how to patch NIDDH *and* NISG. We now show how to counter the previous attacks. Yet, for simplicity we are not considering the modification to NIDDH and NISG that introduces parallel repetitions and the need for NIDDH to invoke NISG as sub-protocol. The insecurity of the NI proof of knowledge systems NIDDH and NISG, as presented so far, comes from the fact that the proof contains the value r in the clear. Such value can be multiplied by z' to get a value of the form $z' \cdot w \mod \phi(N)$, a fatal attack.

To overcome this attack, we require the prover to send the pair $a \stackrel{\triangle}{=} g^r, b \stackrel{\triangle}{=} h^r$ as in the original Chaum-Pedersen's proof. In addition the prover has to send the value $r^{-1} \mod \phi(N)$ that can be used by the verifier to check the well-formedness of the pair (a, b) (all previous considerations and the need for parallel repetition apply in this case as well). In this case the prover of NIDDH (similar consideration holds for NISG) does not need to transmit g^z and h^z as they can be derived from g^r, h^r and u, v. The previous soundness analysis stays roughly unchanged. The perfect extraction still holds: from the factorization of N and $r^{-1} \mod \phi(N)$, the value r and thus w can be computed.

Why is working in a group of hidden order not a trust assumption? One could naively think that working in a group of hidden order is a trust assumption. A trust assumption for proof systems requires a parameter to be chosen correctly and that the generator of the parameter is a trusted party who cannot collude with the adversaries against the proof system. If this is not the case (if the parameters are not correctly chosen or the generator colludes with the adversary), the security may not hold.

In our proof, *whatever* parameter is adversarially chosen, no proof for a false statement can be generated. That is, even if N and the group is setup in an incorrect way, no proof for a false statement can be produced. Notice that the prover must *not* convince the verifier that the group has the right form. Indeed, there exists *no* proof for a false statement at all, so the possibility of cheating is null.

What about privacy? If a parameter is ill-formed, can a proof leak information about the witness to the statement to prove? The answer is positive but this is *inherent* in proof systems. For any proof system, the prover could compute the proof in an ill-formed way (e.g., choosing the randomness not uniformly) so that the proof leaks knowledge. Moreover, the prover can ever collude with an adversary to give the adversary a witness to the statement to prove. Therefore, for any reasonable notion of privacy for proof systems, the randomness used to compute a proof has to be computed in a trusted way, and our proofs are not an exception.

What privacy? The so modified proof systems seems to withstand the afore-mentioned linear attacks: given $r^{-1} \mod \phi(N)$ and $(r + w)^{-1} \mod \phi(N)$, the attacker cannot seemingly form a multiple of the witness. This patch also appears to protect against the attacks on multiple proofs. Does the overall proof systems NIDDH and NISG satisfy a reasonable notion of "privacy" and what kind of security is it attained? This question will be discussed in depth in Section 1.3.

The overall detailed constructions and analysis for NIDDH and NISG are presented in Section 3.

1.2.3   Proof of correct decryption and its applications In Section 1.3.8 we will show that (actually, we will analyze a more general case) no PPT adversary can win with non-negligible probability in the following game against a challenger $\mathcal{C}$. The challenger $\mathcal{C}$ selects a random bit b, computes a well-formed DH tuple $T \stackrel{\triangle}{=} (, g, h, u, v)$ over $\mathbb{Z}_N^\star$ and gives the adversary two tuples $(T_0^b.T_1^b)$, with $T_0^b \stackrel{\triangle}{=} (g, h, u.v \cdot g^1)$ and $T_1^b \stackrel{\triangle}{=} (g, h, u, v \cdot g^{-1})$, and additionally a NIDDH proof that T is a well-formed DH tuple. The adversary outputs a bit b' and wins iff $b = b'$. Note that this is a non-trivial problem. To our knowledge, it was not know how to prove that no PPT adversary can win with non-negligible probability in the previous game instantiated with any other (completely) non-interactive proof (unless assuming trusted parameters.

The proof system NIDDH for DH tuples can be also used to prove that a ciphertext $\mathsf{ct} = (\mathsf{ct}_1, \mathsf{ct}_2)$ for public key pk decrypts to g^0 as follows. Observe first

that if ct_1 belongs to the subgroup generated by g, then the tuple (g, ct_1, pk, ct_2) is DH if and only if ct decrypts to $g^0 = 1$. However, ct_1 might not be in the group generated by g and in this case it might occur that $ct_2 = ct_1^{w_1} \cdot g^{m_1} = ct_1^{w_2} \cdot g^{m_2}$ for integers w_1, w_2, m_1, m_2 such that $w_1 \neq w_2 \mod \text{ord}(ct_1)$, $w_1 = w_2 \mod \text{ord}(g)$ and $m_1 \neq m_2 \mod \text{ord}(g)$. However, if N is generated properly as described in Section 2.1, QR_N is cyclic and g can be set to be a generator of QR_N. In this case (i.e, assuming (N, g) to be generated correctly) the issue can be overcome by having the decryption performed on (ct_1^2, ct_2^2) with respect to public key pk^2; indeed, ct_1^2 belongs to QR_N, so in this case it is generated by g (we assume g to be generated correctly as a generator of QR_N).

An alternative that does not require introducing trusted parameters would be for the encryptor to provide another ciphertext (beyond (ct_1, ct_2)) encrypting the randomness used in ct_1 so as to allow the authority to recover such randomness needed to compute the proof. We conjecture the resulting cryptosystem to be IND-CPA secure. As the change in the encryption scheme makes the analysis slightly more complicated, we skip the details.

Therefore, for the application to e-voting we have either to assume the parameters (N, g) to be generated in a trusted way or we have to use our proof system for NP of Section 1.4.4 to prove ct_1 to belong to the subgroup generated by g^6 or we have to use the above trick to recover the randomness.

Notice that if we use our proof system for NP or we use NIDDH with correctly generated parameters (or we use the alternative solution we sketched above), no unbounded authority may have a non-zero probability of subverting the election result when using our proofs. In contrast, if an authority were aware of trapdoor in a hash function used to instantiate known RO-based NIZKs, it could easily subvert the result of an election in, e.g., the Helios e-voting system [Adi08]. This comes at the cost of basing the privacy of the e-voting system on stronger oracle-based computational problems.

We also point out that, it was not known any *efficient* proof system, even in the CRS model and with statistical soundness, satisfying a non-trivial notion of privacy for proving that an El Gamal-like ciphertext decrypts correctly; in the bilinear group setting instead it was known how to prove efficiently with perfect soundness correct decryption of ciphertexts (for encryption schemes defined in the bilinear setting) with just WI security.

Applications of proof of correct decryption to e-voting. In an universally verifiable e-voting, proofs of correct decryption are a necessary component used to compute the tally in a privacy-preserving way and guarantee universal verifiability, that is that every party, not just who took part in the election, may verify the result of the election. Proofs of correct *encryption* are also used to enforce that the

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[6] Observe that this cannot be directly done using NIDDH since the prover of NIDDH needs the factorization of the modulus that the encryptor does not have. Instead, the proof system for NP does not suffer this limitation, as the prover for the NP system generates the modulus and all the group elements in the proof with knowledge of the corresponding factorization and discrete logs.

encrypted plaintext belong to a valid message space. Our techniques cannot be used to construct efficient (if we do not consider efficiency, we can instead use our proofs for NP relations of Section 1.4.4) proofs of correct encryption as the prover crucially needs the factorization of the modulus. Notwithstanding, we show that proofs of correct decryption can profitably replace proofs of correct encryption in e-voting under the very basic assumption that for each candidate c there exists at least one ciphertext encrypting a vote for c.

We would like to stress that in the following we are considering a setting with a single authority. The single authority has the secret key so is ever able to obtain the preference of any voter. This is inherent in e-voting. Even in a multi-authority setting, the authorities can ever collude together to decrypt the votes. Extending our results to a multi-authority setting is possible but beyond the scope of the work. Note that, unlike privacy, our results show that verifiability can be instead guaranteed even if the authorities (or multiple authorities) are completely malicious. In our analysis, we are also ignoring issues of malleability, so ours is far from and does not asire to being a complete e-voting solution. (Preventing malleability attacks can be done, e.g., encrypting in an onion way the voters' ciphertexts along with their corresponding signatures under the public key of another non-malleable cryptosystem. We skip the details.)

For simplicity, consider a referendum (voters should encrypt 0 or 1). We propose the following. The authority selects a random bit b and groups the ciphertexts in two classes Z_0, Z_1 putting in Z_b (resp. Z_{1-b}) the ciphertexts encrypting 0 (resp. 1). The authority proves that each pair of ciphertexts c and d in the same class encrypt the same bit by showing that the product of c and d^{-1} (where here we mean the usual operations between El Gamal ciphertexts; cf. Def. 26) decrypts to 0. Notice that this proves that each ciphertext in the same class decrypts to the same value. This fact, combined with the hypothesis that there is at least one ciphertext encrypting 0 and one ciphertext encrypting 1, implies that all ciphertexts in Z_0 encrypt a bit b and all ciphertexts in Z_1 encrypt $1 - b$.

Additionally, for each ciphertext that does not encrypt either 0 or 1, the authority provides a corresponding proof of the fact that the ciphertext decrypts to an invalid plaintext. In this way, the authority is able to prove that it tallies all and only the ciphertexts encrypting plaintexts that are 0 or 1. This can be extended to a larger space of voting options with the proof size growing linearly in the number of options.

Our El Gamal encryption scheme over the group of quadratic residues modulo N and our proofs of correct decryption for it can be used to construct an e-voting scheme satisfying the following indistinguishability-based security notion. No PPT adversary can win in the following game with non-negligible advantage. The adversary is given the public key of the e-voting system and selects two tuples of n votes $\boldsymbol{v}_0, \boldsymbol{v}_1$ such that $\sum_{i \in [n]} v_{0,i} = \sum_{i \in [n]} v_{1,i}$. A challenge bit b is chosen at random and the adversary is given n ciphertexts $\mathsf{ct}_i, i \in [n]$ encrypting resp. $v_{b,i}$ along with the tally v $(= \sum_{i \in [n]} v_{b,i})$ and a proof of correct computation of the tally. The goal of the adversary is to guess the bit b. The notion can be

extended to allow the adversary to choose ill-formed votes. See also Section 1.3 for more discussion about the e-voting application and on how to use our proofs to argue security.

1.3 Harmless Zero-Knowledge Proof of Knowledge

Let us recall a proof of the ZK property for the (interactive) Chaum-Pedersen's proof system. The simulator chooses random values $e, z \in \mathbb{Z}_p$ and sets $a \stackrel{\triangle}{=} g^z \cdot u^{-e}, b \stackrel{\triangle}{=} h^z \cdot v^{-e}$. It is easy to see that the transcript of the simulator is distributed identically to the output of the prover, thus the ZK property is *perfect*.

In our proof system, the simulator could likewise generate a, b but cannot compute $r^{-1} \mod \phi(N)$ because it does not know r and $\phi(N)$. Any approach to design an efficient simulator is doomed to fail because a ZK proof system for a non-trivial language cannot be perfectly sound. The reason is that if the proof system satisfied perfect soundness, the simulator might be used to decide the language, a contradiction. Moreover, no ZK proof, even with statistical soundness, can be completely non-interactive (without trusted parameters).

1.3.1 Harmless ZK According to the ZK paradigm, a proof carries no additional information (i.e., it is zero-knowledge) if whatever you can compute after seeing the proof, you could compute by yourself by means of a simulator. The power of the simulator has to be restricted. Indeed, if the simulator were allowed to have unbounded time, proofs that leak the witness would be declared ZK just because there exists a simulator that can simulate the proof in unbounded time. The obvious restriction is to limit the simulator to run in polynomial-time. We contend that this definition can be generalized to achieve more properties and enable more applications while still sticking to the the ZK paradigm.

Let us analyze the ZK paradigm in more detail. Suppose there exists a language L that is hard to decide for adversaries of time $\mathcal{O}(n^2)$ but easy for adversaries of time $\omega(n^2)$. Then, a proof leaking part or all of the witness might be declared ZK only because there might exist a simulator running in time $\mathcal{O}(n^3)$. This example suggests that simulators running in arbitrary polynomial-time should not be allowed and that the running-time of the simulator should *depend* on the hardness of the language.

One can abstract this line of reasoning: if the language is not decidable by adversaries using a given set S of "resources", the class of admissible simulators should include all algorithms having access to S. The resources comprise the time of execution of the algorithm but other resources can be considered as well. Indeed, if the language is not decidable by PPT adversaries interacting with some oracle O, the simulator should be allowed to both run in PPT and have oracle access to O. The oracle can be seen as an external entity handing some auxiliary information to the parties in the system.

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This leads to our notion of *harmless ZK* (HZK).[7] HZK is a generalization of the traditional ZK formulation in that it allows the simulator to have access to an oracle relative to which the language is still hard to decide. We stress that we do *not* allow the simulator to program the oracle. We will denote an algorithm/simulator with access to an oracle O an *O-aided* algorithm/simulator. Considering oracle machines in the analysis of the complexity of computational problems is as old as computer science and dates back to Turing himself. In Section 1.5 we compare our use of oracles to related notions of ZK and secure computation in general.

Our main proof system NIDDH is for the language of DH tuples (over some group of hidden order) and, conjecturing this language to be hard to decide even for adversaries given access to an oracle O (to be defined later), we will show that the proof system is HZK. To avoid trivial attacks, the adversary has to be restricted to not query the oracle on inputs not belonging to the language (see Remark 1 for a careful discussion on this point).

As we will show later, ZK essentially implies that the probability for a PPT adversary of computing a function of the witness given as input a statement for some language L and a ZK proof for it is bounded by the probability that any PPT adversary can compute the same function of the witness given only the statement. The latter probability is negligible if L is hard to decide for PPT adversaries. HZK makes this property more fine-grained by quantifying over PPT adversaries that attempt to compute a function of the witness given the statement but with additional access to some oracle O. Such probability may still be negligible for adversaries with oracle access to O.

Another way of comparing HZK to the standard ZK formulation is to look at an application in which a ZK proof system is employed. ZK proofs are used to enforce correctness in various applications like e-voting while preserving privacy. The privacy of an e-voting system that uses a ZK proof system can be based, e.g., on the (decisional) DH assumption. The running time of the simulator implicitly affects the assumption: if the simulator runs in time $\Theta(m)$, we have to assume DH to hold against adversaries running in time $\Omega(m)$. If the latter were not true, a simulator of time $\Theta(m)$ could not be used in a reduction to the assumption. This, again, is to reiterate the dependency of the resources of the simulator on the language for which the proof system is designed. Analogously, to make use of a simulator with access to an oracle O in a security reduction, the computational problem to which the security is reduced has to hold with respect to adversaries with access to O.

1.3.2 HZK of our main proof system To prove NIDDH to be HZK, we provide a simulator with access to an oracle relative to which we conjecture the language of DH tuples to be hard (under the constraint that the adversary cannot query the oracle on invalid tuples). The simulator needs to invoke the following oracle DHInvO (cf. Def. 29 for more details). The oracle DHInvO takes as input a tuple (N, g, h, u, v), checks whether $u = g^w$ and $v = h^w$ for some

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[7] The name harmless zero-knowledge was suggested to us by Geoffroy Couteau.

$w \in \mathbb{Z}^\star_{\phi(N)}$; if such value does not exist, it outputs error and outputs $(g^r, h^r, r^{-1}$ mod $\phi(N), (r+w)^{-1}$ mod $\phi(N))$ for a random $r \in \mathbb{Z}^\star_{\phi(N)}$, otherwise. (The oracle has to compute the randomness so as to guarantee that the inverses modulo $\phi(N)$ exist.) Our NI NIDDH can be proven O-HZK with respect to this oracle (see Section 3 for the details). We conjecture the language L of DH tuples (over our group of hidden order) to be hard to decide with respect to DHInvO and thus our simulator belongs to the class of legal oracles for a HZK proof system for L (cf. Def. 5). Precisely, we need to restrict the adversary to not query the oracle on invalid statements (i.e., non-DH tuples). See also Remark 1. Similar considerations hold for NISG that is DHInvO-HZK with respect to a simulator with access to the same oracle DHInvO.

Languages vs relations. Having a proof system for a NP language L means having a proof system for a polynomial-time relation $\mathcal{R}$ such that $x \in L$ iff there exists w such $(x, w) \in \mathcal{R}$; in this case we say that $\mathcal{R}$ is a relation over L. However, for each NP language L there are *different* relations over L and a proof system for a relation $\mathcal{R}$ over L does not necessarily imply a proof system for any another relation $\mathcal{R}'$ over L.

Indeed, a subtle point is that in our proof system for DH tuples (as well as the ones for correctness of a shuffle and polynomial statements) the prover cannot be run on input just the statement and the natural witness w (i.e., the exponent for the DH tuple) but additionally needs the factorization of the modulus N. Formally, our proof system is for the relation $\mathcal{R}(x, (w, [p_i, m_i]^l_{i=1}))$ whose witness also includes the factorization of the modulus N and checks if the tuple $x \stackrel{\triangle}{=} (N, g, h, u, v)$ is a DH tuple with exponent w, with the group operation being the multiplication modulo N, and $N = \prod_{i=1}^l p_i^{m_i}$. Notice that $\mathcal{R}$ is still a relation over the language of valid DH tuples but is different from the "natural" relation whose witness consists of just the exponent (and nothing else). This is reflected in applications: NIDDH can be used to provide proofs of correct *decryption* but not proofs of correct encryption because our prover needs to be run with an input that includes the factorization that is not given to an encryptor.

Notwithstanding, we will see that our techniques can be used to construct an e-voting scheme. Moreover, our proof system for Boolean circuit satisfiability of Section 1.4.4 does not share this limitation (its prover does not need any trapdoor) but this comes at the cost of trading perfect simulatability for computational simulatability.

1.3.3 Harmless proof of knowledge

We extend the above concepts to the proof of knowledge property that is challenging in a completely non-interactive setting (no trusted parameters, no RO...). First, we introduce the notion of a hard relation (cf. Def 13). A hard relation is one that is coupled with a distribution $\mathcal{D}$ over pairs in the relation and is such that no PPT adversary, with possibly access to some oracle, on input an instance x sampled from $\mathcal{D}$, can output (with non-negligible probability) the witness w sampled by $\mathcal{D}$. In the case of the relation

of valid DH tuples, we conjecture the hardness of the relation of valid DH tuples associated with the following distribution $\mathcal{D}$: $\mathcal{D}$ outputs pairs (x, w), with x being an uniformly distributed DH tuple (over $\mathbb{Z}_N^\star$, for a Blum integer N) and w being the corresponding witness. For the rest of this discussion, we sometimes omit the dependency on the distribution.

Consider now the following motivating protocol. Parties A and B interact in the following way. A chooses a random DH tuple x in the group of the quadratic residues modulo N with knowledge of its witness w and sends x to B. Party B has access to an oracle $W(\cdot)$ that, given as input x, outputs w. Party B sends a proof π for the validity of x to A. If the proof is accepted, A sends back to B the witness w and B outputs it. The security property P we require is that a malicious $B^\star$ that does not invoke the oracle W should not be able to output a valid witness w for the instance x chosen by A.

We can reduce the security of P to the following assumption: a PPT adversary cannot compute a witness for a randomly chosen DH tuple (over the group of quadratic residues modulo N) even if it has access to a factorization oracle. The reduction algorithm runs B on input the random tuple x, gets from B the accepted proof π and from it, using the factorization oracle, can get a witness to x. The factorization oracle is likely a legitimate resource since the relation of valid DH tuples over the group of the quadratic residues modulo N is probably hard even for adversaries with access to the factorization oracle.

Our definition of harmless proof of knowledge (HPoK) (cf. Def. 15) for a hard relation R postulates that there exists a PPT extractor algorithm Ext with access to an oracle O relative to which R is hard such that Ext, on input any x and any accepted proof for x, can extract a witness to x with probability 1.

As mentioned above, our NI NIDDH has also perfect extraction according to the standard definition [GOS12] when the modulus N is set to a common parameter (that does not have to be trusted). We call HZKPoK a proof system that is both HZK and HPoK.

HPoK does not contradict HZK. It is worth observing why HPoK does not contradict HZK. Consider the concrete example of NIDDH. Intuitively, one could think at an incompatibility of the two properties as it seems that there exists an efficient oracle adversary that can extract a witness to a randomly chosen DH tuple x as follows (and we conjectured such an adversary to not exist). The adversary could generate a proof for x using the DHInvO-aided simulator and then run the O-aided extractor on the proof to get a witness to x. The reasoning is mistaken as for the extraction the adversary needs to invoke an oracle O *different* from the oracle DHInvO for the simulation and it might not be longer hard, for adversaries with access to *both* DHInvO and O, to extract a witness to a randomly chosen instance. In the case of our HZKPoK for DH tuples, the oracle associated with the simulator used in combination with the oracle associated with the extractor allows indeed to compute a witness to a randomly chosen DH tuple.

Precisely, it is true that the existence of an O-HPoK proof is in contradiction with the existence of an O_2-HZK when the two oracles $O(\cdot)$ and $O_2(\cdot)$ are identical but it is not true in general.

HPoK implies hardness of obliviously computing accepted proofs. The immediate corollary of harmless proof of knowledge is that no PPT algorithm can output with non-negligible probability an accepted proof for a randomly chosen statement x (received from a challenger) for a hard relation without knowing a witness for x. Indeed, if there existed such an algorithm, it could be used by the extractor with oracle access to O to break the hardness of the relation, contradicting the hypothesis that no PPT algorithm even with access to O can break the hardness of the relation.

1.3.4 O-HZK $\iff$ O-function hiding $\rightarrow$ witness hiding

HZK implies WH. Our proofs are additionally 1-message harmless witness hiding proofs, under some computational assumptions. Harmless witness hiding (HWH) is a natural *strengthening* of witness hiding [FS90]. Witness hiding (WH) requires that no efficient adversary (playing the role of the verifier) can extract a witness with non-negligible probability after interacting with the prover on a randomly chosen instance for a hard relation. O-HWH requires the same property to hold but quantifying over adversaries with access to O; see Def. 20.

If DHInvO is the oracle associated with the simulator of our NI NIDDH, then NIDDH is DHInvO-HWH for the hard relation of DH tuples. This holds under the assumption that no PPT adversary, with access to DHInvO, can extract a witness from a randomly selected DH tuple over the multiplicative group $\mathbb{Z}_N^\star$, for a Blum integer N. In the following, let us call T the previous hardness assumption. Towards a contradiction, suppose there exist an adversary $\mathcal{A}$, with oracle access to DHInvO, that can extract with probability p a witness from a proof for a randomly selected DH tuple over this group. Then there exists a DHInvO-aided algorithm $\mathcal{B}$ that receives as input a randomly selected instance x, computes a proof π using the DHInvO-aided simulator, and returns the output $\mathcal{A}$ on (x, π). By definition of $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{B}$ outputs a witness to x with probability p, contradicting the hardness of T. See Lemma 4 for more generality and details. (A version of T in which the adversary is restricted to query the oracle only a certain bounded number of times is also equivalent to the DHInvO-HWH of NIDDH (that is, the reduction goes both ways.)

Notice that, for any oracle $O(\cdot)$, O-HWH is *stronger* than WH and indeed, under assumption T, NIDDH is WH (cf. Cor. 16). It is worth observing why this fact does not contradict the impossibility results of [HRS09,Pas11] of the existence of black-box reductions of "standard assumptions" to WH. The standard assumptions in [HRS09] can be naturally extended to the case of assumptions against oracle-aided adversaries but even in this setting the black-box impossibility results of [HRS09,Pas11] would break down.

The core of the previously cited black-box impossibility results is that if a reduction R using an adversary Adv, guaranteed to break the WH of a given proof

system (in our case NIDDH), breaks a computational problem P (in our case Adv is DHInvO-aided and P is the computational problem H stated above), then the reduction can be rewound to extract a witness. This would imply the existence of an adversary $\mathcal{B}$ that breaks T, in our case would imply the existence a DHInvO-aided adversary that breaks T. In the case of our proof NIDDH, an extractor is guaranteed to extract a witness only with access to the factoring oracle. So, for the previous reasoning to hold, we would need to give the reduction access to the factoring oracle but then the black-box impossibility results would imply the existence of an adversary $\mathcal{B}$ that breaks T but with access to *both* DHInvO and the factoring oracle. This is not a contradiction as T is not hard against adversaries with access to both DHInvO and the factoring oracle.

Finally, note that the assumption "the NI proof NIDDH is WH" (cf. Def. 21 and Assumption 3) is a falsifiable assumption according to the classification of Gentry and Wichs [GW11], which is a more liberal classification that Naor's [Nao03]. Indeed, the assumption of the WH of NIDDH can be stated by means of the following game (skipping minor details) between an efficient interactive challenger and an adversary. The challenger chooses N with knowledge of the corresponding factorization, a random DH tuple X over $\mathbb{Z}_N^\star$, computes (by means of the factorization) a NIDDH's proof π of the fact that X is DH and runs the adversary on input (N, X, π). The adversary outputs an alleged witness w and the challenger outputs "win" to indicate that the adversary broke the assumption iff w is a witness to X (this fact can be efficiently checked by the challenger).

An alternative formulation of HZK. An alternative definition of privacy for NI systems, that we call O-function (or feature) hiding (O-FH) (cf. Def. 23) that is implied by O-HZK and turns out to be equivalent to HZK in the case of single witness relations is the following.[8] Assume L to be an NP language that is worst-case hard to decide relative to O. For any possibly randomized function f, for any PPT algorithm Adv, for any pair $(x, w) \in \mathcal{R}_L$, let $P_{x,f,\mathsf{Adv}}$ the distance between the random variable $\mathsf{Adv}(x, \pi)$ and $f(x)$, where π is a proof for x computed by running the prover on input (x, w) and an uniformly distributed random string, and Adv and f are executed/evaluated on uniformly random strings. Then, for any randomized function f, for any PPT algorithm Adv, there exists a PPT algorithm Adv' with oracle access to O such that, for any $(x, w) \in \mathcal{R}_L$, the distance between $\mathsf{Adv}'^{O(\cdot)}(x)$ and $f(x)$ is equal to $P_{x,f,\mathsf{Adv}}$.

Simplifying, the definition essentially says: for a given function, if no O-aided PPT adversary, on input $x \in L$, can compute the function of x with probability $\geq p$, then no PPT adversary (without access to O), given x and a proof for $x \in L$, can compute the function of x with probability $\geq p$. That is, whatever you can compute from the proof, you could compute without the proof with

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[8] We would like to remark that, despite the name, our definition of O-FH is conceptually different from what is usually denoted as "feature hiding" as generalization of witness hiding [Pas06a]. Ours is a worst-case, non-distributional notion whereas feature hiding meant as generalization of witness hiding assumes inputs chosen from a distribution.

the same probability using "non-trivial" resources that do not help decide the language.

The notion of O-FH shares similarities with the original notion of semantic security for public key encryption of Goldwasser and Micali [GM84]. In the modern treatment of cryptography, semantic security is usually presented according to the simulation paradigm whereas the original pioneering work of Goldwasser and Micali adopted a function-based notion similar to the previous one. Moreover, the original seminal work of Goldwasser, Micali and Rackoff (GMR) first appeared in ACM STOC '85 [GMR85] was slightly syntactically different from what later appeared in the journal version of the same authors [GMR89]. To our knowledge, the first formal definition of ZK expliciting the role of the simulator was in Goldwasser, Micali and Wigderson (GMW) [GMW91], though the definition of ZK via approximability of random variables in the journal version of GMR [GMR89] implicitly defines a simulator and is equivalent to the simulation-based one of GMW.

An O-HZK proof system satisfies O-FH. Indeed, for any possibly randomized function f, for any PPT algorithm Adv, for any pair $(x, w) \in \mathcal{R}_L$, consider the following algorithm Adv' with access to O. The algorithm Adv' uses the simulator Sim with oracle access to O guaranteed by the definition of HZK to simulate a proof π that is identically distributed to a real proof (i.e., computed by the prover) and runs Adv on (x, π). By hypothesis the probability that Adv' outputs $f(x, w)$ equals $P_{x,f,\mathsf{Adv}}$.

The reverse also holds for a single witness relation (O-FH implies O-HZK). Indeed, consider the randomized function f that implements the prover algorithm that, on input x, outputs a proof for (x, w) computed with the NI system, where w is the unique witness to x. Consider an adversary Adv that, on input (x, π), just outputs π. The previous definition guarantees the existence of an O-aided algorithm Adv' that, on input x, is identical to the distribution $f(x)$, that is a proof for (x, w), except with statistical distance $P_{x,f,\mathsf{Adv}}$. By definition of Adv, the latter statistical distance equals 0. Such an adversary Adv' is thus a simulator with oracle access to O that generates a proof distributed identically to a proof for (x, w), as it was to show. See Lemma 6 for a more detailed proof of the previous implications.

It is also easy to see that ZK implies 1-FH[9] and, in the case of single witness relations, is equivalent to 1-FH, with $1(\cdot)$ being the oracle implementing the identity function. Therefore, as there exists no NI ZK proof for non-BPP languages, then there exists no NI FH proof for single witness relations associated to non-BPP languages.

In Remark 7 we discuss whether and why O-HZK and O-FH are not equivalent for relations with multiple witnesses.

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[9] Since there is no NI ZK proof, to give more sense to this implication, we should think more generally about generalizations of the definitions of O-HZK and O-FH for interactive proof systems.

1.3.5 The impact of the oracle leakage in applications. As byproduct, HZK implies that the hardness of computing functions of a witness to some statement, given the statement and a proof for it (and with no access to any oracle), is bounded by the hardness of computing functions of the witness (given only the statement) for adversaries with access to the oracle (associated with the simulator). Therefore, the hardness of computing functions of the witness is bounded by the amount of leakage the oracle provides.

Such leakage may be harmful in some applications in which adversaries are given some auxiliary input. For instance, suppose that a proof of membership of some string x in some worst-case hard NP language L contain the value $f(x, w)$, for some one-way function f and suppose it be hard to decide L given access to an oracle that on input x returns $f(x, w)$.[10] If f is a trapdoor one-way function, the proof might help an adversary with access to the trapdoor to carry out some task it could not perform without having the proof. So, such proof might be risky for applications in which potential attackers do have access to the trapdoor but might be suitable in other applications. Different oracles give different leakage and may be harmless or harmful depending on the application. In the case of our main HZK proof for DH tuples, for example, the proof cannot be composed with sub-protocols in which the factorization of the modulus is made public.

As a general example, consider the following 3-party protocol between PPT parties $\mathcal{P}$, $\mathcal{V}$ and an unbounded party Q. $\mathcal{P}$ sends to $\mathcal{V}$ a HZK proof π for the membership of x in some worst-case hard language L. Suppose that the proof leaks the value $O(x)$. The verifier can send to Q some pair (x, y) and if $y = O(x)$ the party O sends back to $\mathcal{V}$ a witness w for x if any, or $\bot$ otherwise. It is clear that this protocol is not "zero-knowledge" since the information leaked by the HZK proof allows $\mathcal{V}$ to get non-trivial knowledge from Q. Hence, in this specific protocol the HZK proof turns out to be indeed harmful. It is easy to design other counter-examples in which the leakage given by the oracle is deleterious. However, a proof should be designed to provide "harmless" leakage in natural and practical applications; see Section 1.3.8.

It is interesting noticing why the previous "attacks" do not contradict O-FH. Fix a given function and consider for simplicity an NP language L associated with a single witness relation. If a PPT adversary, with access to O and on input $x \in L$, can compute the function of the witness to x with probability $> p$, then we *cannot* conclude that no PPT adversary (without access to O), given x and a proof for $x \in L$, can compute the function of the witness with probability $> p$. Indeed, in the case of the oracle O associated to our NI NIDDH, an adversary that has embedded the factorization of a modulus N (that corresponds in the real world to receiving the factorization from a colluding party) can compute the witness of a DH tuple over $\mathbb{Z}_N^\star$ with just one invocation to the oracle.

Contrast HZK with ZK that is equivalent to 1-FH. The following considerations apply more generally to interactive proof systems. On the one hand, it may be that there is an PPT oracle adversary that can use an oracle O to compute the function of the witness with probability $> p$, but no such non-oracle

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[10] For simplicity, we are glossing over issues of non-uniformity.

adversary exists. So, there may exist an O-FH (and thus O-HZK) proof system that is not 1-FH (and thus not ZK). On the other hand, not considering oracles (or equivalently restricting the oracle to be $1(\cdot)$) in the analysis of an proof system, we are potentially excluding "attacks" that may be instead harmless in a specific application. Viceversa, considering an oracle O in the analysis of an proof system, we are implicitly excluding as harmful all kind of attacks in which the adversary may gain knowledge using the oracle. For instance, if a particular application like e-voting assumes an adversary to not have access to the factorization, then a DHInvO-HZH proof system NIDDH is harmless in that particular application.

More in detail, fix an oracle O and consider a NI proof system for a single witness (this is for simplicity) NP relation. Let (f, Adv) be a pair consisting of a randomized function and a PPT (non-oracle) adversary (Adv is *not* given access to O or any other oracle). We call the pair (f, Adv) *bad* if there exists no PPT (non-oracle) adversary Adv' such that, for any $(x, w) \in \mathcal{R}_L$, the distance between $\mathsf{Adv}'(x)$ and $f(x, w; r)$ is equal to $P_{x,f,\mathsf{Adv}}$ (cf. Def. of O-FH) but there exists such an oracle adversary $\mathsf{Adv}'^{O(\cdot)}_2$ with access to O. In other words (simplifying), the pair is bad if (1) Adv, with input a proof for a statement x, can compute $f(x, w)$ with some probability $\geq p$ and (2) no PPT adversary can solve the same task without the proof and (3) instead a PPT oracle adversary with access to O (and without the proof) can. So, an oracle O induces a set of bad pairs. If such bad pairs are not harmful in the application in which the proof has to be utilized, then the proof is harmless for that particular application. For instance, the pair consisting of an adversary that has embedded the factorization and the identity function (meaning that the adversary can compute the entire witness) is bad but this may be not harmful in applications, like in e-voting, in which the verifier is not given the secret key. When analyzing the security via ZK (that is, 1-FH), the set of bad pairs (relative to the trivial oracle $1(\cdot)$) is ever empty but this comes at the cost of being unable to demonstrate useful security properties of the proof system in a particular application.

In other words, ZK is a powerful notion of privacy but protects against *any* possible leakage, even the ones that might not be harmful to the applications in which ZK protocols are usually employed. O-HZK instead exploits the fact that in most real-world protocols the corresponding ideal world harmlessly grants to the adversary an O-leakage. Consider the relation between ZK and WH, and more generally simulatability and WH. The existence of a simulator implies WH, under the assumption that no adversary, *as powerful as* the simulator, can compute a witness from a randomly selected instance for a hard relation. If we restrict the simulator to be PPT, we are not taking advantage that the fact that it is likely hard even for adversaries with oracle access to DHInvO to compute a witness from a randomly selected instance for a hard relation. So, in some case we might not able to reduce simulatability to WH. So, granting the simulator more resources may allow, as our results demonstrate, to deduce the WH of a specific proof system, fact that might not be provable without this leveraging of the simulator power.

Composition of HZK proofs. HZK proofs can raise issues of composition in a protocol I when the oracle associated with the simulator can make one of the hard problems, on which the security of I is based, easy. This is already implicit but less visible in the case of standard ZK proofs.

Consider a possible world (not ruled by our current knowledge on computational complexity) in which a problem P used in cryptographic constructions is provably hard for adversaries of time $\mathcal{O}(n^7)$ but it is provably breakable by an adversary of time $\mathcal{O}(n^{10})$. In this world, it would still make sense to design cryptographic protocols choosing appropriately the security parameter though one would have to carefully take in account the running-time of simulators for ZK proofs. Indeed, a simulator of time $\mathcal{O}(n^{10})$ should not be considered legitimate. Moreover, if a larger protocol were based both on P and on another problem P_2 with stronger hardness requirements (e.g., hard only for adversaries of time $\mathcal{O}(n^4)$), even a simulator of time n^6 would raise compositional issues. This scenario is however unlikely as we currently believe that PPT algorithms can be composed together without having "too much" effects on the computational assumptions used in the design of protocols.

Yet, the running-time of the simulator of a ZK proof affects the quality of a reduction of some protocol I that uses the ZK proof to some hard problem P on which the security of I is based: more time the simulator needs to carry out the simulation, more time the reduction needs to break P and thus, for a given security parameter and for a given bound on the computational power of adversaries, a stronger hypothesis on the hardness of P is required. Then, composing a sub-protocol for which a tight reduction is known with a ZK proof may imply a significant worsening of the tightness of the reduction.

Such compositional issues are more evident when using oracles. In particular, it may be easier to check whether an oracle help in breaking some computational problem and hence issues with oracle-aided simulation are made manifest.

In view of the above considerations, we advocate a pragmatic approach. ZK and HZK are tools for proving the security of larger protocols. HZK proofs can own additional properties provably impossible for ZK proofs, like non-interactivity and perfect soundness, but this comes at the cost of stronger "oracle-based" assumptions and potential composability issues with other protocols whose security relies on assumptions that do not hold relative to the oracle associated with the HZK simulator. In this paper we show real-world applications that crisply benefit from replacing ZK with HZK proofs.

Relation to computational ZK or formulations without auxiliary inputs. HZK, even in its perfect HZK formulation, is somehow qualitatively comparable to computational ZK without auxiliary input (cZK) in that the computational leakage gained by an adversary is harmless unless the adversary holds a trapdoor as it is illustrated by the following example.

Given an arbitrary NP-relation $\mathcal{R}$, define the relation $\mathcal{R}'$ so that, for each string pk, $\mathcal{R}'(x||\mathsf{pk}, w)$ holds iff $\mathcal{R}(x, w)$ holds (pk is artificially ignored).

Consider the following ZK proof for $\mathcal{R}'$. The prover, on input $x||\mathsf{pk}$ and w, for a valid El Gamal public key pk, interacts with the verifier using a ZK proof

for $\mathcal{R}$ but additionally in the last message sends an encryption of w computed under pk. The verifier ignores the ciphertext and accepts iff the ZK argument for $\mathcal{R}$ is accepting. This protocol is still simulatable (the simulator can set the ciphertext to be an encryption of an arbitrary string of the same length) but the output of the simulator is only computationally indistinguishable under some computational assumption[11] from the output of the honest prover, that is the protocol is cZK. Yet, an adversary having the secret key corresponding to the public key can compute the entire witness. So, similarly to HZK, a cZK protocol does not necessarily guarantee security against adversaries that may gain some trapdoor information.

1.3.6   On trivial and efficient oracles

Trivial oracles. A HZK simulator is coupled with an oracle. The choice of the oracle leads to qualitatively different assumptions. In the extreme case, for a proof system for a relation $\mathcal{R}_L$, one could set the oracle to be the trivial "prover oracle". The prover oracle gets as input a statement $x \in L$, finds a witness w such that $\mathcal{R}(x, w) = 1$ and returns the output of the prover on input (x, w) (and uniformly chosen random bits). Then the assumption on the hardness of the language would boil down to require L to be hard even for adversaries seeing proofs for statements in L. That, in turn, accounts to say that the protocol is "secure because it is secure".

One can object that the oracle associated with our main proof system is somehow trivial as well. We embrace a pragmatic approach. First of all, computational assumptions are such because of our current limitations on their provability or refusal. So, the real challenge is to reduce the security of protocols to assumptions that are at least enough simple to cryptanalyze. Simulators are a practical tool employed to prove the security of protocols and can be used in a neater way than witness indistinguishability that, not only introduces direct inefficiency in the security proofs due to a complicated and unnatural use of OR statements, but also results in severe limitations (see discussion on verifiable functional encryption in Section 1.3.8).

In Section 1.3.8 we show that our main proof system can be profitably used to prove the security of a concrete non-trivial simplified e-voting protocol and the overall security of such protocol can be reduced, via a reduction that uses the oracle-aided simulator, to Assumption 7 that is well-defined and seemingly cryptanalyzable. We stress that it was not know before how to construct an efficient protocol for the same task satisfying perfect verifiability (i.e., with proofs of correct decryption satisfying perfect soundness) without bilinear groups.

It appeared not obvious to us to come up with non-trivial oracles to design oracle-aided simulators for other known proof systems like the non-interactive ZAPs of Groth *et al.* [GOS12].

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[11] For each PPT adversary $\mathcal{D}$, for all sufficiently long valid El Gamal public keys pk, $\mathcal{D}$ has negligible advantage in distinguishing an encryption of 0 under pk from an encryption of 1 under pk.

Note that the trivial proof system for a single witness DH relation in which the prover outputs the witness in the clear satisfies WI but does not satisfy (under some reasonable computational assumption) DHInvO-HZK with respect to the oracle DHInvO previously defined in Section 1.3.2. Indeed, the DHInvO-aided simulator could be used to extract a witness from a DH tuple (given oracle access to DHInvO) and this seems a hard problem.

Oracles with auxiliary input. Our oracle DHInvO for our proof system for DH tuples cannot be implemented in polynomial-time, not even whether the oracle is given the trapdoor as hint. It is a natural question to ask whether there may exist a one-message HZK proof system associated with an oracle that runs in polynomial-time given a trapdoor as auxiliary input. We speculate this situation to be unlikely. The intuition is that if this were the case, the HZK proof system might be converted into a traditional ZK proof system by somehow "obfuscating" the oracle program (with the trapdoor embedded in its representation).

The question might be related to the problem of removing the oracle and achieving super-polynomial (but sub-exponential) simulation [Pas03b] via complexity leveraging arguments [CGGM00]. Along this direction, we mention that our HZK proof for Boolean circuit satisfiability is computational HZK (cf. Def. 6). Our proof for NP can be tweaked, using known techniques, to achieve one-message arguments with uniform soundness and quasi-polynomial time simulation or alternatively two message-arguments with standard soundness. In Section 1.4.4 we briefly discuss this point.

1.3.7 Undeniability of our proofs Though our proofs are non-interactive, they are *undeniable* as well. Indeed, you cannot claim to have generated the proof by yourself using the simulator. This is because you would need access to the oracle to run the simulator. In the case of interactive ZK proofs (as well as for NIZK proofs in the CRS and programmable RO model), the simulator can be also used to simulate proofs for false statements and thus having a transcript for a given statement does not represent evidence that you know the veracity of the statement. In contrast, if you have a perfectly sound HZK proof for a given statement, you *can* always check its validity and so you cannot deny that you *could* have learnt the validity of the statement since proofs for false statements cannot be simulated.

1.3.8 Applications and how to use HZK proofs

E-voting. Consider the following e-voting application (see also Section 1.2.3). (We would like to stress that we are in a setting with a single authority. The single authority has the secret key so is ever able to obtain the preference of any voter. This is inherent in e-voting. Even in a multi-authority setting, the authorities can collude together to decrypt the votes.)

The authority adds a proof that the the product ciphertext $\mathsf{ct} \stackrel{\triangle}{=} \mathsf{ct}_1 * \mathsf{ct}_2$ (where "$*$" has the usual meaning) decrypts to v, with v being $v_1 + v_2$. We would

like such a protocol to satisfy the following privacy requirement. The requirement (details in Assumption 6) states that a PPT adversary cannot distinguish whether either ct_1 encrypts 1 and ct_2 encrypts -1 or ct_1 encrypts -1 and ct_2 encrypts 1, given additionally a proof of the fact that the product ciphertext $ct_1 * ct_2$ decrypts to 0.

If the proof of correct decryption were ZK, we could reduce the security to the standard Decisional Diffie-Hellman (DDH) assumption, that is the problem of distinguishing a DH tuple from a random tuple. When using a HZK proof, one can naively think to reduce the privacy to a variant of DDH in which the adversary has access to the oracle DHInvO needed by the simulator of our DHInvO-HZK proof. Unfortunately, there is an issue.

Let us first analyze in more detail how the security reduction would work if the proof were ZK. Consider the following hybrid experiments (we do not include all the hybrid experiments necessary to argue the security and we do not take in consideration the CRS model).

- H_0. Hybrid experiment H_0 is identical to the real game except that ct_1 encrypts 1 and ct_2 encrypts -1.
- H_1. Hybrid experiment H_1 is identical to H_0 except that the proof is simulated (instead of being honestly generated).
- H_2. Hybrid experiment H_2 is identical to H_1 except that ct_1 encrypts a random group element (the proof is still simulated like in H_1).

One can construct an adversary against DDH from an adversary that distinguishes H_2 from H_1, that is an El Gamal encryption of 1 from an El Gamal encryption of a random element, given additionally a simulated proof. The point is that in H_2, the proof can be simulated even though the statement given as input to the simulator is *false*: ct_2 encrypts a random plaintext and hence the product ciphertext $ct \overset{\triangle}{=} ct_1 * ct_2$ is not a valid statement (it does not decrypt to 0).

A ZK simulator *can* be run on false statements and indeed one can prove that simulated ZK proofs for false statements are always accepted by the verifier. In contrast, a HZK simulator cannot be generally run on false statements. Precisely, we cannot attempt to reduce the privacy to the variant of DDH in which the adversary has access to the oracle DHInvO because, by definition, DHInvO returns an error when the input is not a valid DH tuple (cf. Def. 29); the assumption would be trivially false since the oracle might be used to distinguish a DH tuple from a random one. Therefore, when using HZK proofs in reductions one has to guarantee that the simulator is invoked only on valid statements. This can be done considering the following alternative series of hybrid experiments.

- H_0'. Hybrid experiment H_0' is identical to the real game except that ct_1 encrypts 1 and ct_2 encrypts -1.
- H_1'. Hybrid experiment H_1' is identical to H_0' except that the proof is simulated (instead of being honestly generated).
- H_2'. Hybrid experiment H_2' is identical to H_1' except that ct_1 encrypts -1 and ct_2 encrypts 1 (the proof is still simulated like in H_1).

– H_3'. Hybrid experiment H_3' is identical to H_2' except that the proof is honestly computed (instead of being simulated).

Observe that the simulator is always invoked on valid statements. The perfect indistinguishability of H_1' from H_0' (resp. H_3' from H_2') follows from the perfect HZK simulatability. The indistinguishability of H_2' from H_1' follows from Assumption 6 that may be easily seen to be equivalent to the following "simpler" assumption: a PPT adversary cannot distinguish a DH tuple (g, h, u, v) for witness w (i.e., $u = g^w, v = h^w$) from a random tuple, given additionally oracle access to DHInvO and values h', v' such that (g, h', u, v') is another DH tuple for the same witness w. This is stated in Assumption 7 and the reduction of Assumption 7 to Assumption 6 is proven in Lemma 10.

Verifiable Functional Encryption. Badrinarayanan [BGJS16] *et al.* put forth the powerful primitive of verifiable functional encryption that extends functional encryption in that malicious behavior of the central authority and encryptors can be detected. The difficulty in the construction of verifiable functional encryption of Badrinarayanan *et al.* stems from the fact that, in order to not rely on trusted parameters, NIZKs in the CRS model have to avoided and replaced by non-interactive WI proofs. Unfortunately, a complicated and unnatural use of WI limits the security to be selective (the adversary has to announce the challenge before seeing the public key). Essentially, to engineer multiple witnesses, the public key has to contain a commitment that in a hybrid experiment is set to be a commitment to the challenge message. This shows that often the use of WI is not only unnatural and causes an efficiency loss but also suffers inherent limitations: it is not know how to construct, from a fully secure functional encryption scheme and non-interactive WI proofs, a fully secure verifiable functional encryption scheme.

A computational O-HZK for NP, as the one we construct in Section 1.4.4, makes the construction of a fully secure verifiable functional encryption from an arbitrary fully secure functional encryption scheme straightforward: add to ciphertexts and tokens of a fully secure functional encryption scheme proofs of their correct computation. The perfect verifiability follows from the perfect soundness of the proof and the IND-CPA security for the verifiable functional encryption scheme follows assuming the underlying functional encryption scheme to be IND-CPA secure against adversaries with access to the oracle used by the proof system for NP (restricting the adversaries to never query the oracle on inputs returning error).

1.3.9 O-strong witness indistinguishability Strong witness indistinguishability (strong-WI) [Gol01,Gol04] requires that for two computationally indistinguishable statement distributions X_1 and X_2, a proof for statement $x_1 \leftarrow X_1$ must be computationally indistinguishable from a proof for statement $x_2 \leftarrow X_2$. The previous application to e-voting makes manifest a close relation to strong-WI. Indeed, consider the following two distributions $X_b, b \in \{0, 1\}$. Distribution X_b contains a randomly selected public key pk for our (variant of) El Gamal

encryption scheme and a ciphertext ct computed as follows. Compute two ciphertexts ct_0, ct_1 encrypting resp. b and $1 - b$ under pk and set ct to be $ct_0 * ct_1$. The statement output by the distribution states that ct is a ciphertext for public key pk that decrypts to 1 and $ct = ct_0 * ct_1$. Assumption 6 (that is in turn equivalent to Assumption 7) may be then seen to be equivalent to conjecturing our main proof system to be strong-WI with respect to these two specific distributions.

More generally, an O-HZK proof system is O-strong-WI, that is strong-WI when quantifying distributions that are computationally indistinguishable even by adversaries with access to O; see Def. 22 and Corollary 5. We are not aware of any work analyzing whether known non-interactive ZAPs are also strong-WI.

1.4 Extensions

In this section we sketch several extensions of our main proof system for DH tuples. While we provide a comprehensive overview, we sometimes skip technicalities and details, in particular we mostly opt for presenting only the details needed to construct our HZK proof for NP relations of Section 1.4.4.

1.4.1 Verifiable shuffle Our techniques can be extended to construct a non-interactive perfectly sound proof of correctness of a shuffle of (our variant of) El Gamal ciphertexts. Our starting point is the verifiable shuffle of Neff [Nef01].

The Iterated Logarithm Multiplication Problem. As in Neff, to construct a verifiable shuffle we first build proofs for the iterated logarithm multiplication problem (ILMP). In the ILMP for parameter k, one wants to prove that two tuples $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_k)$ are such that $\prod_{i=1}^{k} \mathbf{dlog}_g X_i = \prod_{i=1}^{k} \mathbf{dlog}_g Y_i$. For simplicity, hereafter we consider tuples of 3 elements, that is we consider the ILMP for $k = 3$. (For our construction of a HZK NI proof for NP relations of Section 1.4.4, we would actually need to just consider the case $k = 4$, but we believe that the core ideas can be presented more clearly with a slight loss of generality and we limit the presentation to the case $k = 3$.)

In our setting, the tuples are over $\mathbb{Z}_N^\star$, so it might be that for some $i \in [3]$, $\mathbf{dlog}_g X_i$ or $\mathbf{dlog}_g Y_i$ does not exist. Therefore, in the following we will often assume all the group elements in the tuples to belong to the subgroup generated by a public group element g (that, if computed honestly, is a generator of the group of quadratic residues modulo N). This limitation can be removed using our HZK NI proof NISG to prove each group element in the tuples to belong to the subgroup generated by g. Observe that ILMP for $k = 2$ corresponds to the problem of proving that a tuple of 4 group elements is DH, so ILMP can be seen as a generalization of the DH problem.

We assume the prover knows the discrete logs in base g of all the elements X_i's and Y_i's. In our proof system for Boolean circuit satisfiability of Section 1.4.4 the prover does know the discrete logs of all the group elements (because it is the prover to create such elements with knowledge of the corresponding exponents) and so this assumption is not a limitation for that application.

A HZK NI proof for the ILMP. Consider the following non-interactive proof (we will next show that, in order to make it sound, it has to be modified).

The prover sends values:

$$A_1 \triangleq Y_1^{\Theta_1}, A_2 \triangleq X_2^{\Theta_1} \cdot Y_2^{-\Theta_2}, A_3 \triangleq X_3^{\Theta_2},$$

and values r_1, r_2 satisfying the following three equations:

$$Y_1^{r_1} = A_1 \cdot X_1, X_2^{r_1} \cdot Y_2^{-r_2} = A_2, X_3^{r_2} = A_3 \cdot Y_3.$$

Observe that the prover can efficiently do that as it knows the exponents $x_1, x_2, x_3, y_1, y_2, y_3$.

The verifier accepts iff all the last equations are satisfied.

Let us set $\bar{r}_i = r_i - \Theta_i$. Assuming that the values A_1, A_2, A_3 are generated correctly, the first equation implies (1) $\bar{r}_i \cdot y_1 = x_1$, the second implies (2) $\bar{r}_1 \cdot x_2 - \bar{r}_2 \cdot y_2 = 0$ and the third (3) $\bar{r}_2 \cdot x_3 = y_3$. Replacing (1) and (3) in (2) we have $\frac{x_1 \cdot x_2}{y_1} + \frac{y_3 \cdot y_2}{x_3} = 0$ that implies $x_1 x_2 x_3 = y_1 y_2 y_3$.

Therefore, (perfect) soundness would hold if the values A_1, A_2, A_3 were ever computed correctly (if, e.g., $A_3 = X_3^{\Theta_2'}$ for $\Theta_2' \neq \Theta_2$ the above analysis would break down). For this reason we have to modify the above system to prove correctness of the computation of the values A_i's.

To that purpose, the prover also sends $z_1 = \Theta_1^{-1} \mod \phi(N), z_2 = \Theta_2^{-1} \mod \phi(N)$. It is easy to see that such values can be used to check the validity of the A_i's: the prover first checks that $A_1^{z_2} = Y_1$ then compute $C = A_2 \cdot X_2^{z_1}$, checks that $C^{z_2} = Y_2^{-1}$ and that $A_3^{z_2} = X_3$. However, it might be that the value z_i's are not co-prime with $\phi(N)$ and in such case the check would not be sufficient to guarantee soundness (see Section 1.2). As for our NI proof NIDDH, we have thus to employ the trick of the parallel repetitions (see Section 1.2). In the following, for simplicity we will skip this detail and assume the above version of the proof without parallel repetitions.

Let DHInvO be the oracle associated to the simulator for the NI NIDDH. The DHInvO-HZK can be proved as follows. The DHInvO-aided simulator SimILMP computes random values $r_1, r_2 \leftarrow \mathbb{Z}_{\phi(N)}$ and computes elements A_1, A_2, A_3 satisfying the equations

$$Y_1^{r_1} = A_1 \cdot X_1, X_2^{r_1} \cdot Y_2^{-r_2} = A_2, X_3^{r_2} = A_3 \cdot Y_3.$$

The distribution of the elements $(r_1, r_2, A_1, A_2, A_3)$ computed by the simulator is identical to the distribution of the same elements in the distribution of real proofs. Finally, the simulator invokes the oracle on (N, Y_1, Y_1, A_1, A_1) to get $z_i = \Theta_1^{-1} \mod \phi(N)$ and invokes the oracle on (N, X_3, X_3, A_3, A_3) to get $z_2 = \Theta_2^{-1} \mod \phi(N)$ and outputs $r_1, r_2, z_1, z_2, A_1, A_2, A_3$. By definition of the oracle, the so computed values z_1, z_2 are distributed identically to the values z_1, z_2 in the distribution of real proofs.

Note that, as mentioned above, to prove that all the group elements in the tuples belong to the subgroup generated by g, we have to invoke NISG that is also DHInvO-HZK, for the same oracle DHInvO$(\cdot)$. Furthermore, recall that we

are skipping the fact that, for the soundness to hold, the proof has to be repeated in parallel a certain number of times (in this case, the previous simulator would have to run the same computation several times).

The previous protocol can be generalized to deal with any tuple of $k > 3$ elements. For simplicity, we do not present the details. We remark that for our main application to proofs of Circuit Satisfiability of Section 1.4.4, it is sufficient to consider proofs for the ILMP for parameter $k = 4$. Hereafter, we assume to have a proof for the ILMP for any $k > 3$.

A HZK interactive proof for proving the correctness of a shuffle. From ILMP we move to a simple n-shuffle problem. Constants $c, d \in \mathbb{Z}^{\star}_{\phi(N)}$ are known to the prover and commitments $C = g^c$ and $D = g^d$ are published. The prover has to convince the verifier that there is some permutation $p : [n] \to [n]$ such that:

$$Y_i^d = X_{p(i)}^c,$$

for $i \in [n]$. We assume the prover to know the discrete logs in base g of all the elements X_i's and Y_i's; in the application to the proof for NP of Section 1.4.4, this assumption will not constitute a limitation as it will be the prover of such system to create the group elements with knowledge of the corresponding exponents.

We firstly consider an interactive protocol. The protocol proceeds as follows. The verifier sends to the prover a random $t \in \mathbb{Z}_{\phi(N)}$.

Prover and verifier publicly compute $U \stackrel{\triangle}{=} D^t = g^{dt}$ and $W \stackrel{\triangle}{=} C^t = g^{ct}$ and

$$\bar{X} \stackrel{\triangle}{=} (\bar{X}_1, \ldots, \bar{X}_n) = (X_1/U, \ldots, X_n/U)$$

and

$$\bar{Y} \stackrel{\triangle}{=} (\bar{Y}_1, \ldots, \bar{Y}_n) = (Y_1/W, \ldots, Y_n/W).$$

Prover sends a ILMP proof for the vectors $(\bar{X}, [C]^n)$ and $(\bar{Y}, [D]^n)$, where $[C]^n$ (resp. $[D]^n$) denotes a list containing the value C (resp. D) repeated n times. (Here, we implicitly assuming to have a generalization of the ILMP proof to the case $k = 2n$. For our proof for NP relations of Section 1.4.4 we will need to consider only a proof of correct shuffle for parameter $n = 2$ and thus in turn a proof for the ILMP problem for parameter $k = 4$.)

By the perfect soundness of the ILMP proof, if the proof is accepted then it holds that:

$$c^n \cdot \prod_{i=1}^{n}(x_i - dt) = d^n \cdot \prod_{i=1}^{n}(y_i - ct)$$

$$\Longleftrightarrow$$

$$\prod_{i=1}^{n}(x_i - dt) = d^n \cdot \prod_{i=1}^{n}(y_i/c - t)$$

$$\Longleftrightarrow$$

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$$\prod_{i=1}^{n}(x_i/d - t) = \prod_{i=1}^{n}(y_i/c - t).$$
$$\Longleftrightarrow$$
$$\prod_{i=1}^{n}(t - x_i/d) = \prod_{i=1}^{n}(t - y_i/c).$$

Denote by P (resp. Q) the left (resp. right) hand of the last equation, that is $P \stackrel{\triangle}{=} \prod_{i=1}^{n}(t - x_i/d)$ and $Q \stackrel{\triangle}{=} \prod_{i=1}^{n}(t - y_i/c)$. P and Q can be seen as polynomials $P[t]$ and $Q[t]$ over $\mathbb{Z}_{\phi(N)}$ in the variable t. If $P[t]$ and $Q[t]$ are identical polynomials (the lists of the coefficients are identical), then the statement (i.e., that there exists a permutation p over $[n]$ such that for any $i \in [n], Y_i^d = X_{p(i)}^c$), is true (the viceversa holds as well). Indeed, suppose $P[t], Q[t]$ to be identical polynomials (the list of coefficients of $P[t]$ is equal to the list of coefficients of $Q[t]$). By definition of $P[t]$ and $Q[t]$, and from the fact that a polynomial of degree n, as $P[t]$ and $Q[t]$ are, has at most n roots, it follows that there exists a permutation p over $[n]$ such that for every $i \in [n], x_i/d = y_{p(i)}/c \iff x_i \cdot c = y_{p(i)} \cdot d \iff X_i^c = Y_{p(i)}^d$, as it was to show. Hence, if the statement does not hold, the two polynomials have to be different.

The soundness error is the probability, over the random choices of $t \in \mathbb{Z}_{\phi(N)}$ of the verifier, that the proof is accepted but the statement does not hold. Therefore, the soundness error is equal to the probability that the last equation holds but the two polynomials $P[t]$ and $Q[t]$ are different. The probability that the last equation holds is equal to the probability, over a random $r \in \mathbb{Z}_{\phi(N)}$, that $P[t](r) = Q[t](r)$ (that is, that the evaluation of $P[t]$ at the point r equals the evaluation of $Q[t]$ at the point r). As a consequence, the probability that the last equation holds is equal to the probability, over the choices of $r \in \mathbb{Z}_{\phi(N)}$, that r is a zero of the polynomial $(P - Q)[t]$. Since $P[t]$ and $Q[t]$ are monic polynomials of degree n and they are different, the polynomial $(P - Q)[t]$ has at most $n - 1$ roots, so the soundness error is $\leq (n-1)/\phi(N)$.

It is easy to see that the protocol is DHInvO-HZK, with DHInvO$(\cdot)$ being the same oracle associated to the simulator of NIDDH. The DHInvO-oracle aided simulator SimShfl for the above protocol works as follows. The simulator takes as input $(C, D, X_1, \ldots, X_n, Y_1, \ldots, Y_n)$, chooses a random $t \in \mathbb{Z}_{\phi(N)}$ and computes $U \stackrel{\triangle}{=} D^t = g^{dt}, W \stackrel{\triangle}{=} C^t = g^{ct}$ and

$$\bar{X} \stackrel{\triangle}{=} (\bar{X}_1, \ldots, \bar{X}_n) = (X_1/U, \ldots, X_n/U)$$

and

$$\bar{Y} \stackrel{\triangle}{=} (\bar{Y}_1, \ldots, \bar{Y}_n) = (Y_1/W, \ldots, Y_n/W).$$

Such values are distributed identically as in a real proof. Finally, SimShfl returns the output of SimILMP on input the vectors $(\bar{X}, [C]^n)$ and $(\bar{Y}, [D]^n)$. As the distribution of the SimILMP's output for the ILMP problem is identical to a real proof for the ILMP problem, so the distribution of the SimShfl output is identical to the distribution of the real transcripts for the above protocol.

Removing interaction and de-randomizing the protocol. The protocol can be made non-interactive using a non-programmable RO by computing the verifier message as output of the RO (note that the verifier message is the first message in the interaction, not the second like in a sigma protocol).

To remove the RO and the negligible soundness error, we recall that, as we proved above, there are at most $n-1$ values of t that can make the verifier to err. Therefore, the protocol can be repeated n times with *different* values of t. It is straightforward to see that the above simulator SimShfl can be adapted to this modification as well.

(Note that de-randomizing in this way the first message in the interactive protocol of Neff is possible but does not offer any advantage. Indeed, the Neff's interactive proof consists in a first random message sent by the verifier followed by a sigma protocol between prover and verifier. De-randomizing the first message would remove the need of the RO in the first verifier message but not for the verifier's message sent in the next three-round protocol.)

From a shuffle of group elements to a shuffle of El Gamal ciphertexts. The previous protocol could be improved, as done in Neff [Nef01] and following our ideas, to make it non-interactive and perfectly sound, to remove the limitation that the shuffler needs to know the discrete logs of all the X_i's and Y_i's; the resulting protocol would have computational HZK, that is the simulator with access to the oracle $\mathsf{DHInvO}(\cdot)$ outputs a transcript computationally indistinguishable from the one of the prover (cf. Def. 6). Moreover, as in Neff, the protocol can be generalized to any tuple of $k \geq 2$ elements (we have been implicitly assuming that in the previous analysis) and to a shuffle of El Gamal ciphertexts (i.e., pairs of group elements). We skip further details.

We instead show how our previous shuffle (in its limited version in which the prover needs to know the discrete logs of all the elements in the statement) can be adapted in a simpler way to a shuffle of 2 El Gamal ciphertexts. This is sufficient for our application to proofs for Circuit Satisfiability of Section 1.4.4.

For a given ciphertext $\mathsf{ct} = (c_1, c_2)$, let us denote by ct^l the left part c_1 (resp. by ct^r the right part c_2). Let ct_1 and ct_2 be two El Gamal ciphertexts and let $P_1 \triangleq \mathsf{ct}_1^l \cdot \mathsf{ct}_1^r$ and $P_2 \triangleq \mathsf{ct}_2^l \cdot \mathsf{ct}_2^r$.

The shuffler computes random commitments $C = g^c, D = g^d$, selects a random permutation p over $[2]$ and sets $\mathsf{ct}_1' \triangleq ((\mathsf{ct}_{p(1)}^l)^{d/c}, (\mathsf{ct}_{p(1)}^r)^{d/c})$ and $\mathsf{ct}_2' \triangleq ((\mathsf{ct}_{p(2)}^l)^{d/c}, (\mathsf{ct}_{p(2)}^r)^{d/c})$, $P_1' \triangleq \mathsf{ct}_1'^l \cdot \mathsf{ct}_1'^r$ and $P_2' \triangleq \mathsf{ct}_2'^l \cdot \mathsf{ct}_2'^r$. Note that P_1' (resp. P_2') is the re-randomized permutation of P_1 (resp. P_2).

The shuffler uses our previous proof of correct shuffle to prove that all the following statements hold:

- (1). $(\mathsf{ct}_1'^l, \mathsf{ct}_2'^l)$ is a shuffle with respect to the commitments C, D of $(\mathsf{ct}_1^l, \mathsf{ct}_2^l)$.
- (2). $(\mathsf{ct}_1'^r, \mathsf{ct}_2'^r)$ is a shuffle with respect to the commitments C, D of $(\mathsf{ct}_1^r, \mathsf{ct}_2^r)$.
- (3). (P_1', P_2') is a shuffle with respect to the commitments C, D of (P_1, P_2).

The verifier checks that $P_1 = \mathsf{ct}_1^l \cdot \mathsf{ct}_1^r$, $P_1' = \mathsf{ct}_1''^l \cdot \mathsf{ct}_1''^r$, $P_2 = \mathsf{ct}_2^l \cdot \mathsf{ct}_2^r$, $P_2' = \mathsf{ct}_2''^l \cdot \mathsf{ct}_2''^r$ and that the three proofs above are correct. Furthermore, the verifier checks that $\mathsf{ct}_1^l \neq \mathsf{ct}_2^l$ and $\mathsf{ct}_1^r \neq \mathsf{ct}_2^r$.

Roughly speaking, the shuffler computes three shuffles, one for the left part of the two ciphertexts, one for the right part, and one for the products, and proves that each shuffle individually is correct, and additionally performs some tests to enforce the prover to use the same permutation in all the shuffles.

Let us analyze the soundness. By (1), $(\mathsf{ct}_1''^l, \mathsf{ct}_2''^l) = ((\mathsf{ct}_{p(1)}^l)^s, (\mathsf{ct}_{p(2)}^l)^s)$, for some permutation p and $s = d/c$. By (2), $(\mathsf{ct}_1''^r, \mathsf{ct}_2''^r) = ((\mathsf{ct}_{q(1)}^r)^s, (\mathsf{ct}_{q(2)}^r)^s)$, for some permutation q and $s = d/c$. By (3), $(\mathsf{ct}_1''^l \cdot \mathsf{ct}_1''^r) = (\mathsf{ct}_{v(1)}^l \cdot \mathsf{ct}_{v(1)}^r)^s$ and $(\mathsf{ct}_2''^l \cdot \mathsf{ct}_2''^r) = (\mathsf{ct}_{v(2)}^l \cdot \mathsf{ct}_{v(2)}^r)^s$, for some permutation v and $s = d/c$.

So, $P_1' =$ (by the verifier's checks) $= \mathsf{ct}_1''^l \cdot \mathsf{ct}_1''^r =$ (by (1) and (2)) $= (\mathsf{ct}_{p(1)}^l \cdot \mathsf{ct}_{q(1)}^r)^s =$ (by (3)) $= (\mathsf{ct}_{v(1)}^l \cdot \mathsf{ct}_{v(1)}^r)^s$, and similarly for P_2'. Suppose towards a contradiction that the proof is accepted (all the verifier's checks pass) but the ciphertexts are not permuted correctly, that is that $p \neq q$. We analyze four mutually exclusive cases and we reach a contradiction.

- p and v are the identity permutations. In this case, $P_1' = (\mathsf{ct}_1^l \cdot \mathsf{ct}_2^r)^s = (\mathsf{ct}_1^l \cdot \mathsf{ct}_1^r)^s$. This contradicts the fact that $\mathsf{ct}_1^r \neq \mathsf{ct}_2^r$.
- p is the identity and v is not the identity permutation. In this case, $P_1' = (\mathsf{ct}_1^l \cdot \mathsf{ct}_2^r)^s = (\mathsf{ct}_2^l \cdot \mathsf{ct}_2^r)^s$. This contradicts the fact that $\mathsf{ct}_1^l \neq \mathsf{ct}_2^l$.
- p is not the identity and v is the identity permutation. In this case, $P_2' = (\mathsf{ct}_1^l \cdot \mathsf{ct}_2^r)^s = (\mathsf{ct}_2^l \cdot \mathsf{ct}_2^r)^s$. This contradicts the fact that $\mathsf{ct}_1^l \neq \mathsf{ct}_2^l$.
- p and v are both different from the identity permutation. In this case, $P_2' = (\mathsf{ct}_1^l \cdot \mathsf{ct}_2^r)^s = (\mathsf{ct}_1^l \cdot \mathsf{ct}_1^r)^s$. This contradicts the fact that $\mathsf{ct}_1^r \neq \mathsf{ct}_2^r$.

To demonstrate the DHInvO-HZK of this modified proof, the previously described DHInvO-aided simulator SimShfl has to be modified as follows. The simulator SimShfl on input the two commitments C, D, two pairs of ciphertext ct_1 and ct_2 and the re-encrypted (according to C and D) and permuted resulting ciphertexts ct_1' and ct_2', computes what follows. It simulates a proof (using the previously described version of the simulator for the proof of correct shuffle of group elements) of the fact that $(\mathsf{ct}_1''^l, \mathsf{ct}_2''^l)$ is a shuffle with respect to the commitments C, D of $(\mathsf{ct}_1^l, \mathsf{ct}_2^l)$, of the fact that $(\mathsf{ct}_1''^r, \mathsf{ct}_2''^r)$ is a shuffle with respect to the commitments C, D of $(\mathsf{ct}_1^r, \mathsf{ct}_2^r)$ and of the fact that (P_1', P_2') is a shuffle with respect to the commitments C, D of (P_1, P_2), where, by their definitions, P_1, P_2, P_1', P_2' are publicly computable from the statement. By the previous analysis, it is straightforward to see that this modified simulator carries out a perfect DHInvO-aided simulation.

We stress that in our proof system for Boolean circuit satisfiability of Section 1.4.4 the prover does know the discrete logs of all the group elements. Furthermore, in our proof for Circuit satisfiability, the prover only needs to compute shuffles of 2 El Gamal ciphertexts. So the previous limited version of the shuffle (i.e., in which the prover needs to know the discrete logs) extended with the above modification to a shuffle of El Gamal ciphertexts suffices for that application.

1.4.2   OR proofs from verifiable shuffle In an OR proof, the prover needs to convince the verifier that a compound statement like $x \in L \vee x_2 \in L$ holds. OR proofs [CDS94] for sigma protocols are achievable using the simulator constructively in the proof. The idea is that the prover uses the simulator to generate a proof for the part of the OR statement for which it does not know the witness. Proving that ciphertext c decrypts to 0 or 1 is easy using such a technique. This trick cannot be applied to NIDDH as its prover cannot execute the simulator without the oracle and in general cannot be applied to a HZK proof as the simulator for it can be only invoked on valid statements.

We can construct an OR proof for proving that an El Gamal ciphertext $\mathsf{ct} \stackrel{\triangle}{=} (c_1, c_2)$ (over $\mathbb{Z}_N^\star$) for public key pk decrypts to 0 or 1 from the previous proof of correctness of a shuffle of El Gamal ciphertexts.

We assume the prover to know the discrete logs in base g of c_1 and c_2. In our application to proofs for Circuit Satisfiability of Section 1.4.4 this is not a limitation as the prover of the NI for Circuit Satisfiability applies OR proofs from ciphertexts created by itself with knowledge of all the corresponding exponents.

The prover holds an El Gamal public key pk (over $\mathbb{Z}_N^\star$) and a ciphertext ct_1 encrypting a bit b and computes another ciphertext ct_2 encrypting $1 - b$. Then the prover computes a shuffle $(\mathsf{ct}_{s,1}, \mathsf{ct}_{s,2})$ of $(\mathsf{ct}_1, \mathsf{ct}_2)$, that is it computes random commitments $C \stackrel{\triangle}{=} g^c, D \stackrel{\triangle}{=} g^d$ and a random permutation p over [2] and sets $\mathsf{ct}_{s,1}$ and $\mathsf{ct}_{s,2}$ such that $\mathsf{ct}_{s,i}^d = \mathsf{ct}_{s,p(i)}^c$, for $i = 1, 2$. (Given a ciphertext $\mathsf{ct} \stackrel{\triangle}{=} (c_1, c_2)$, we denote by ct^d the ciphertext (c_1^d, c_2^d).) The prover computes a proof of correctness of the previous shuffle of the two previous pairs of ciphertexts with respect to the commitments C, D. Furthermore, the prover uses NIDDH to show what values $\mathsf{ct}_{s,1}$ and $\mathsf{ct}_{s,2}$ decrypt to. The verifier checks that the shuffle is correct and that ct_s and ct_s' decrypt to two bits that sum up to 1 so it is convinced that ct_1 encrypts a bit (0 or 1).

It is easy to see that the proof inherits the same soundness guarantee of the underlying proof of correct shuffle, so it is perfectly sound. The DHInvO-aided simulation is straightforward given a DHInvO-aided simulator for the proof of correct shuffle. More in detail, consider the following DHInvO-aided simulator SimOR. The simulator SimOR, on input a public key pk and a ciphertext ct_1, computes what follows. Let b the bit encrypted in ct_1. SimOR computes a ciphertext encrypting 1 and uses the homomorphic properties of El Gamal to compute a ciphertext ct_2 encrypting $1 - b$. Note that the simulator does not know the bit b but is able to obliviously compute a ciphertext encrypting the bit complement. Then, as the prover, the simulator computes a shuffle $(\mathsf{ct}_{s,1}, \mathsf{ct}_{s,2})$ of $(\mathsf{ct}_1, \mathsf{ct}_2)$, that is it computes random commitments $C \stackrel{\triangle}{=} g^c, D \stackrel{\triangle}{=} g^d$ and a random permutation p over [2] and sets $\mathsf{ct}_{s,1}$ and $\mathsf{ct}_{s,2}$ such that $\mathsf{ct}_{s,i}^d = \mathsf{ct}_{s,p(i)}^c$, for $i = 1, 2$. The simulator invokes SimShfl to simulate a proof of correctness of the previous shuffle. It is straightforward to see that the simulation of SimOR is perfect as the simulation of SimShfl is.

1.4.3 Polynomial statements The ideas and construction in this section are due to Geoffroy Couteau. An alternative approach to prove OR statements and more general polynomial statements is the following. We first present a proof subject to an issue and then we will show how to fix it. Let (g, h) be the El Gamal public key (our our group of quadratic residues modulo N), and let w be the secret key (hence $h = g^w$). Let (u, v) be a ciphertext that decrypts to a message m: $(u, v) = (g^r, h^r \cdot g^m)$. To prove that m is a bit, it suffices to prove that there exist values (w, m, x) such that the following four equations are satisfied:

1. $h = g^w$.
2. $v = u^w \cdot g^m$.
3. $1 = g^x \cdot h^m$. This equation ensures that $x = -w \cdot m$.
4. $1 = u^x \cdot (v/g)^m$. This equation ensures that $u^{-w \cdot m} \cdot (v/g)^m = 1$, which reduces to $g^{m \cdot (m-1)} = 1$, hence m is a bit.

The proof that these equations are satisfied works as follows.

Pick random masks w', m', and x' in $\mathbb{Z}_m^\star$, where m is the order of the group. Compute $c_0 = g^{w'}, c_1 = u^{w'} \cdot g^{m'}, c_2 = g^{x'} \cdot h^{m'}$, and $c_3 = u^{x'} \cdot (v/g)^{m'}$. The proof is:

$$(c_0, c_1, c_2, c_3, s_1, s_2, s_3, s_4, s_5, s_6),$$

with

$$s_1 = w'^{-1} \mod \phi(N), s_2 = m'^{-1} \mod \phi(N), s_3 = x'^{-1} \mod \phi(N),$$

$$s_4 = (w+w')^{-1} \mod \phi(N), s_5 = (m+m')^{-1} \mod \phi(N), s_6 = (x+x')^{-1} \mod \phi(N).$$

Then, similarly to the proof for DH tuples, the verifier uses s_1, s_2, s_3 to check that c_0, c_1, c_2, c_3 are well formed. To illustrate the check, suppose you want to check that $c_1 = u^{w'} \cdot g^{m'}$. This check is in fact equivalent to verifying:

$$(c_1^{s_2}/g)^{s_1} = u^{s_2},$$

which can be done using s_1 and s_2. Then, analogously the verifier uses s_4, s_5, s_6 to check that $(h \cdot c_0, v \cdot c_1, v \cdot c_2, v \cdot c_3)$ are well-formed.

There is a problem with the previous proof system: if we reveal $(m + m')^{-1}$ and m'^{-1}, then we leak whether m is equal to 0 or not. However, there is a simple fix: simply add 1 to m homomorphically and prove that m is equal to 1 or 2; the previous proof system has to be slightly modified in an obvious way[12] but we skip the details.

Again, we implicitly assumed all group elements to belong to the same subgroup. This limitation can be removed as detailed for the proof for DH tuples. To simulate a proof for polynomial statements, the simulator needs oracle access to DHInvO. We skip further details.

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[12] The essential modification is to adapt the equations so to ensure that $g^{(m-2)(m-1)} = 1$.

1.4.4 ZAP and computational HZK proof for NP relations The previous OR proofs or proofs for polynomial statements can be used to construct a one-message perfectly sound WI proof (non-interactive ZAP) for Boolean circuit satisfiability as follows. Our solution is inspired by [GOS12]. Assume the circuits consist only of NAND gates. If w_0, w_1 are the values corresponding to the input wires of a gate and w_2 is the value corresponding to its output wire, it is easy to see that w_0, w_1, w_2 are a valid assignment (i.e., $w_2 = \neg(w_0 \wedge w_1)$) iff $w_0 + w_1 + 2w_2 - 2 \in \{0, 1\}$ and $w_0, w_1, w_2 \in \{0, 1\}$.

The prover creates a public key (N, g, h) for our variant of El Gamal and associates a ciphertext to each wire of the circuit in the following way. To each input wire corresponding to a bit b of the witness, the prover associates a ciphertext encrypting b. The prover evaluates the circuit at each gate and associates to each output wire of a gate the encryption of the corresponding bit (computed homomorphically).

To each output wire of a gate and to each input wire of the circuit, the prover adds a proof of the fact that the associated ciphertext decrypts to 0 or 1.[13] Let t be a ciphertext encrypting the integer -2 (this can be done with trivial randomness, so that the verifier can verify that it is correctly computed). For each gate with ciphertexts $\mathsf{ct}_0, \mathsf{ct}_1$ associated to its input wires and ciphertext ct_2 associated to its output wire, the prover computes the ciphertext $G \stackrel{\triangle}{=} \mathsf{ct}_0 * \mathsf{ct}_1 * \mathsf{ct}_2^2 * t$ and adds a proof that G decrypts to 0 or 1. (Here, by "$*$" and exponentiation we mean the usual operations on El Gamal ciphertexts; cf. Def. 26.) Finally, the prover shows that the output gate decrypts to 1.

Using the homomorphic property of El Gamal and the above fact, it is easy to see that if all the proofs are accepted, the computation is consistent. Therefore, since the ciphertexts associated to the input wires decrypt to 0 or 1 and the output wire of the circuit decrypts to 1, the circuit is satisfiable. Notice that we can easily deal with the issue mentioned in Section 1.2.3. Indeed, for each ciphertext $\mathsf{ct} = (\mathsf{ct}_1, \mathsf{ct}_2)$, the prover can add a proof that ct_1 belongs to the subgroup generated by g; this can be done by the prover since the prover can generate N with knowledge of the factorization.

It is easy to see that the computational WI property follows from the assumption that a PPT adversary cannot distinguish the encryption of one of two bits having in addition a proof (computed as described previously) of the fact that the ciphertext encrypts a bit (0 or 1) and access to an oracle (that can be invoked only on valid tuples) for computing proofs of valid statements of this sort. This can be seen to be equivalent to state that a PPT adversary cannot distinguish the encryption of one of two bits having the possibility of invoking the oracle DHInvO of the HZK simulator of our main proof system only on valid

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[13] Observe that the prover for the NP proof can use our OR proof from proof of correct shuffle in the simplified version in which the input to the prover for the proof of shuffle includes the discrete logs of the group elements in the statement. Indeed, in this case the prover for the NP proof can generate such group elements with knowledge of the corresponding discrete logs.

DH tuples. The precise statement is given in Assumption 5. We skip further details.

cHZK proof for Boolean circuit satisfiability. The previous non-interactive ZAP is also computational DHInvO-HZK: the simulator computes the public key and simulates an OR proof for each ciphertext associated with an internal wire and a proof of the fact that the ciphertext associated with the output gate decrypts to 1.

Therefore, for the simulator to carry out the simulation is sufficient to provide the simulator with access to the same oracle DHInvO associated with the HZK proof for DH tuples, and in addition another oracle O_{NP} (the two oracles can be seen as a single oracle). The oracle O_{NP}, if invoked on a satisfiable Boolean circuit, computes the public key, and a bit-by-bit encryption of a witness, specifically the lexicographically first witness satisfying the statement, or returns error on input a non-satisfiable Boolean circuit.

Let DHInvO$'$ be the oracle resulting from joining DHInvO and O_{NP}. The simulation is not perfect as the ciphertexts leak information (the witness chosen by the oracle may differ from the one in a real proof). However, our NI for Boolean circuit satisfiability can be conjectured to be computational HZK (cf. Def. 6) with respect to the oracle DHInvO$'$. We state this as an assumption in itself and skip further details.[14] We did not investigate whether the proof be simulatable via more "concise" oracles.

As discussed in Section 1.3.4 the existence of an O-aided simulator for a proof for a relation $\mathcal{R}$ implies that the proof is witness hiding with respect to distributions D (over pairs (x, w) such that $\mathcal{R}(x, w) = 1$) with the following property: there exists no PPT oracle adversary $\mathcal{B}$ with oracle access to O such that if $(x, w) \leftarrow D$, then $\mathcal{B}$, on input x, outputs w with non-negligible probability. With respect to any such distribution over Boolean circuits and satisfying assignments for them, our proof is witness hiding.

Complexity leveraging and quasi-polynomial time simulation. Pass [Pas03b] showed how to use complexity leveraging arguments [CGGM00] to construct quasi-polynomial time simulatable two-message arguments extended in Barak and Pass [BP04] to quasi-polynomial time simulatable one-message arguments with uniform soundness.

The idea of Pass [Pas03b] is (simplifying) the following. The verifier sends some random challenge $c = f(r)$ for some one-way function f and the prover proves, using an argument of knowledge, that $x \in L$ or there exists a pre-image for c. The soundness follows from the one-wayness of f and the security of the argument of knowledge. The quasi-polynomial time simulatability holds setting the length of c so that the simulator can extract a pre-image in quasi-polynomial time. The protocol as described results in a four round argument (due to the

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[14] The cHZK property might be reduced to "simpler" assumptions similar to the one used to prove the computational WI of our non-interactive ZAP, that is Assumption 5.

use of the argument of knowledge), the rounds can be reduced to two using non-interactive ZAP and extractable commitments but we skip the details.

Barak and Pass [BP04] builds on this idea to remove the interaction. In essence, the prover sends a commitment c and a proof, using for instance a non-interactive ZAP, that $x \in L$ or there exists a pair of collisions for a hash function. Assuming uniform adversarial prover strategies only, the soundness holds because it is a difficult for a uniform adversary to come up with a pair of collisions for a hash function. The quasi-polynomial time simulatability follows setting the parameters appropriately so that the simulator can find a pair of collisions in quasi-polynomial time.

Our proof for Boolean circuit satisfiability is also harmless proof of knowledge, so it can be used in both constructions to achieve quasi-polynomial time simulation.

1.5 Related Work and Comparison

Zero-knowledge proofs and arguments. Zero-knowledge proofs, introduced in the seminal work of Goldreich, Micali and Rackoff [GMR85,GMR89] and further refined in [GMW86,GMW91], have been one of the main building blocks of modern cryptography [GMW87,CCD88], have provided useful in complexity theory to establish that certain languages are unlikely to be NP-complete [BHZ87,For87] and represent a fascinating concept in itself. Argument systems have been introduced by Brassard *et al.* [BCC88] as a natural relaxation of the notion of proof system in which the dishonest prover is restricted to be efficient.

Zero-knowledge Proof of knowledge systems. Proof of knowledge systems [BM88,GMR89] [BG93,Gol01] extend proof systems adding the property that if a prover succeeds in convincing the verifier that, e.g., some NP statement holds, then the prover "knows" a witness for the statement; this is formalized requiring the existence of an efficient extractor that can extract the witness given oracle access to the prover with probability higher than the success probability of the prover in convincing the verifier (this is a simplification, the actual definition has to take in account a knowledge error).

Limitations of zero-knowledge. Strong limitations for ZK proofs and arguments have been studied since its discovery. The impossibility of non-interactive ZK (in the plain model) is straightforward (see Goldreich [Gol01]). Goldreich and Oren [GO94] proved the impossibility of non-trivial 2-round ZK arguments. Regarding black-box simulation, the possibility of non-trivial 3-round ZK arguments has been ruled out by Goldreich and Krawczyk [GK90] while Canetti *et al.* [CKPR01] ruled out the existence of non-trivial constant-round concurrent ZK arguments. The limitations of strict polynomial-time simulators have been studied by Barak and Lindell [BL02]; the oracle-aided simulators in our proof systems run in strict polynomial-time.

Fortnow [For87], and Aiello and Håstad [AH87] study the limits of perfect and statistical ZK (that guarantee that the ensemble output by the simulator is,

resp., distributed identically or statistically indistinguishable from the ensemble output by the prover) showing that is unlikely the existence of a perfect or statistical ZK proof system for an NP-complete language.

CRS-based NIZKs. CRS-based NIZK proof and argument systems have been intensively studied in the last 30 years in a sequel of works [BFM88,DMP88,FLS90,RS92] [BY93,BY96,Gol01,DDO$^+$01,Can01,Pas03a,BCNP04,Ps05,GOS06b,GOS06a,AF07] [GS08,GK08,GOS12,Pas13,BFS16,PS19].

One of the initial motivations for CRS-based NIZK proof was CCA-security [NY90,CS98,Sah99,DDO$^+$01,ES02,CS03,Lin06]. In this setting, the CRS is computed by the receiver, while the NIZK proofs are computed by the sender of ciphertexts. Thus, for CCA-security the CRS model does not pose any issue.

In contrast, in e-voting [Cha81,CGS97,Adi08,RS06,JCJ10] the authority cannot be the same party to generate the CRS. Indeed, the authority must compute proofs of correctness of the tally and thus the CRS has to be setup by a trusted party. Our proofs for NP of Section 1.4.4 are not based on any trusted parameter and may be used in several existing e-voting schemes (e.g., [Adi08]) to overcome this limitation.

NIZKs in the CRS model can be obtained from any trapdoor permutation [FLS90] (and thus from factoring), from bilinear groups [GOS06a], and recently from the learning with error problem [CLW18,PS19], only to cite the most notable constructions.

The RO model and NIZKs derived via FS-like transformations. An alternative to the CRS model is the RO model that assumes the availability of a perfect random function that in practice is implemented with a hash function. The RO model does not solve the issues of the CRS model but often leads to the design of more efficient protocols. The RO methodology has been introduced in the groundbreaking work of Bellare and Rogaway [BR93]. Canetti *et al.* [CGH98] show that the RO methodology is unsound in general and several works [DNRS99,Bar01,GK03,BLV03,BDSG$^+$13,GOSV14,KRR16] studied the security of the FS methodology. A first rigorous analysis of the FS transform (applied to the case of signature schemes) appeared in Pointcheval and Stern [PS00]. Since the introduction of the FS transform [FS87], there has been a lot of work in investigating alternative transformations that achieve further properties or mitigate some issues of FS.

Pass [Pas03a] and Fischlin [Fis05] introduce new transformations with straight-line extractors to address some problems that arise when using the NIZK argument systems resulting from the FS transform in larger protocols [SG02]. The NIZK systems resulting from the Pass' and Fischlin's transforms share the same limitation of FS of being *arguments*, i.e., sound only against computationally bounded adversaries. Furthermore, Fischlin's transform also results in a completeness error.

(Note that the definition of online extractability of Fischlin implicitly assumes that the list of RO queries given to the extractor has polynomial size and thus only withstands adversaries that are possibly computationally unbounded *but*

limited to a polynomial number of RO queries; according to our terminology, this limitation brings to an argument system with computational extractability.[15])

Damgård *et al.* [DFN06] propose a new transformation for the standard model but it results in NIZK argument systems that are only *designated verifier*, rests on computational assumptions and has soundness limited to a logarithmic number of theorems. Designated verifier NIZK proofs are sufficient for some applications (e.g., non-malleable encryption [PsV06]) but not for others like e-voting in which public verifiability is a wished property. The limitation on the soundness of the Damgård's transformation has been improved in the works of Ventre and Visconti [VV09] and Chaidos and Groth [CG15].

Lindell [Lin15] (see also the improvement of Ciampi *et al.* [CPSV16]) puts forward a new transformation that requires both a *non-programmable* RO and a CRS and has computational complexity only slightly higher than FS. The transformations of Lindell and Ciampi *et al.* are based on computational assumptions.

Mittelbach and Venturi [MV16] investigate alternative classes of interactive protocols where the FS transform does have standard-model instantiations but their result yields NIZK argument systems and is based on strong assumptions like indistinguishability obfuscation [GGH+13], and as such is far from being practical. Moreover the result of Mittelbach and Venturi seems to apply only to the weak FS transform in which the statement is not hashed along with the commitment. The weak FS transform is known to be insecure in some applications [BPW12].

The work of Mittelbach and Venturi has been improved by Kalai *et al.* [KRR16] that, building on [BLV03,DRV12], have shown how to transform any public-coin interactive proof system into a *two-round* argument system using strong computational assumptions. The latter work does *not* yield non-interactive argument systems.

Ishai *et al.*[IMS12] and Mahmoody and Xiao [MX13] construct unconditional sub-linear ZK *arguments* in the RO model using the FS transform.

Faust *et al.* [FKMV12] and Bernhard *et al.* [BFW15] provide a careful study of the definitions and security properties of the NIZK argument systems resulting from the FS transform but they do not investigate the possibility of achieving *statistically* sound proofs. Both works make use of the general forking lemma of Bellare and Neven [BN06] that extends the forking lemma of Pointcheval and Stern [PS00]. Wee [Wee09] presents a hierarchy of RO models and some separations among them.

Recently, Canetti *et al.* [CLW18] showed how to instantiate correlation-intractable functions [CGH98] and turned a particular sigma protocol into a NIZK in the CRS model based on a variant of fully homomorphic encryption. The result of Canetti *et al.* has been exploited by Peikert and Shiehian [PS19] to construct the first NIZK in the CRS model based on learning with error (LWE), a widely studied post-quantum safe problem.

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[15] The FS transform leads to statistically sound proof systems against computationally unbounded provers if constrained to a polynomial number of RO queries.

The FS and NIZKs in the quantum setting. In the quantum setting, the Fiat-Shamir and NIZK proof/argument systems in the RO model has been studied in several works [Unr12,ARU14,Unr15] but this goes beyond the scope of our work.

Sigma protocols. Sigma protocols, which efficient NIZK arguments in the RO model are based on, have been intensively studied [GQ88,CP93,CDS94,FKI06,BR08] [Dam10,ABB+10,YZ12,Mau15,GMO16] and incorporate properties both of interactive proof systems and proofs of knowledge systems.

Chaum and Pedersen [CP93] construct sigma protocols for the well-formedness of DH tuples that can be FS-transformed to non-interactive arguments in the programmable RO model. Our proof system of Section 3 (overview in Section 1.2) for the well-formedness of (a variant of) DH tuples is non-interactive and perfectly sound and does not rely on any trusted parameter.

Cramer *et al.* [CDS94] present techniques to achieve sigma protocol for compound statements and Maurer [Mau15] show that many known sigma protocols can be seen as a special case of an abstract protocol. Our techniques extend as well to OR proofs and polynomial statements; see Sections 1.4.2 and 1.4.3.

Verifiable shuffle. The original concept of mix-nets has been introduced by Chaum [Cha81]. A variant of Chaum's mix-net called shuffle, based on the re-encrypt and permute paradigm, has been introduced by Park *et al.* [PIK94] and made verifiable by Sako and Kilian [SK94]. To our knowledge, all known efficient (without passing through NP reductions) non-interactive "proofs" of correct shuffle of ciphertexts of any semantically secure cryptosystem, both in the CRS and RO model, and even the non-short ones, only achieve soundness against computationally bounded provers. Examples include [SK94,FS01,Nef01,Gro03,Gro05b] [Wik05,GL07a,GL07b,TW10]. Our verifiable shuffle of Section 1.4.1 extends the one of Neff [Nef01] making it non-interactive and perfectly sound and is not based on any trust assumption like the CRS or the RO model.

Witness indistinguishable systems. Dwork and Naor [DN00] constructed 2-round witness indistinguishable proofs, called ZAPs, for any NP relation (assuming trapdoor permutations exist). The construction of Dwork and Naor allows for the first message (from verifier to prover) to be reused, thus only one message is required even if many statements have to be proven. Barak *et al.* [BOV03] constructed the first non-interactive ZAPs for any NP relation. Groth *et al.* [GOS12] construct non-interactive ZAPs for any NP relation from number-theoretic assumptions over bilinear groups. Using the same techniques of Groth *et al.* [GOS12], the NIZK proofs of Groth and Sahai [GS08] can be used to build non-interactive ZAPs. The latter are the only known non-interactive ZAPs for practical statements that do not employ NP-reductions. Bitansky and Paneth [BP15] construct non-interactive ZAPs from indistinguishability obfuscation [GGH+13] and one-way functions. In Section 1.4.4 we sketch the construction of a non-interactive ZAP with perfect soundness for Boolean circuit satisfiability. Our non-interactive ZAP is also computational HZK.

ZAPs only offer witness indistinguishability (WI) [FS90], a security guarantee strictly weaker than ZK. Indeed, for relations with single witness, a trivial proof system that outputs the witness as proof satisfies WI. On the other hand, HZK proofs can be applied to relations with single witness as well, as it is the case for the relation of well-formed DH tuples.[16] WI makes the construction of security protocols more complex; to make use of WI proofs one has to introduce relations with artificial witnesses to be able to switch witnesses in hybrid experiments. This problem does not only affect security reductions. In practice, the prover invoked by the actual protocol has to compute a proof for a more complex relation (that usually consists in an OR of different statements, some of which are not related to the statement to prove, but are artificially needed to make the reduction go through), and this comes with an *additional* efficiency loss. As for ZK, HZK proofs instead do not pose this kind of efficiency loss.

As a reference, in the verifiable functional encryption construction of Badrinarayanan [BGJS16], four functional encryption instances have to be used in parallel along with non-interactive WI proofs for very complex relations. Moreover, due to the use of WI the construction of Badrinarayanan *et al.* is inherently selectively secure (i.e., the adversary has to choose the challenge messages before seeing the public key). On the other hand, our non-interactive HZK proof with perfect soundness for NP relations implies a fully (i.e., non-selective) secure verifiable functional encryption scheme based on just a single instance of a fully secure functional encryption in conjunction with HZK proofs.

To our knowledge, no efficient non-interactive WI statistically sound proof for DH-like languages was known (unless going through expensive NP-reductions). Hash proof systems [CS02] provide a different avenue to obtain variants of designated verifier proof systems for several practical relations including the DH one.

Other formulations of privacy for proof and argument systems. Pass [Pas03b] proposed a relaxation of ZK by allowing the simulator to run in sub-exponential time and in this setting presented two-message straight-line concurrent ZK arguments that are composable without requiring trusted parameters bypassing the impossibility results of [CKPR01,GK90]. The work of Pass is similar in spirit to ours in that the simulator is allowed more than PPT resources. HZK can be seen to be more general as a sub-exponential time computation can be simulated by an oracle.[17] In [Pas03b], the motivation was to extend the simulation techniques to enable advanced composition and reduce the round complexity of protocols. Pass [Pas03b] only provides computationally sound and interactive protocols, whereas our protocols are completely non-interactive and perfectly sound.

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[16] For simplicity, our formal definition of DH relation over groups of non-prime order is not single witness but can be done so.

[17] Let D be the oracle that gets as input the representation of a program P and an input x, and returns the output of the execution of P on x. For each sub-exponential simulator Sim, consider the D-aided PPT simulator Sim$'$ that, on input x, invokes the oracle D on the bit representation of Sim concatenated to x, and returns the output of the oracle.

Barak and Pass [BP04] de-randomize the two-message protocol of Pass [Pas03b] constructing a one-message quasi-polynomial time simulatable argument system for NP, in which the soundness condition holds only against uniform machines, under general non-standard complexity theoretic assumptions. Chung *et al.*[CLMP12] present barriers to using black-box reductions for proving soundness of quasi-polynomial time simulatable arguments, essentially showing that the results in the aforementioned works are optimal.

Our proofs can be tweaked using complexity leveraging arguments [CGGM00] along with the techniques in [Pas03b,BP04] to achieve one-message arguments with uniform soundness and quasi-polynomial time simulation or two-message arguments with standard soundness and quasi-polynomial simulation; we briefly discuss this point in Section 1.4.4.

Prabhakaran and Sahai [PS04] realized tasks in the Universal Composability framework [Can01], known to be impossible without trust assumptions, by allowing the adversary and the environment to have super-polynomial computational power. The work of Prabhakaran and Sahai also introduces the notion of imaginary angels that are oracles available to the environment and the adversary and shows this to be an useful abstraction at the aim of defining and analyzing the security of complex protocols. Our oracles are very similar in spirit to imaginary angels. Observe that, like for imaginary angels in the work of Prabhakaran and Sahai, our oracles are used to define and analyze the security but are not used by the actual parties in proof systems.

Strong-WI [Gol01,Gol04] is a stronger security guarantee than WI. Strong-WI requires that for two computationally indistinguishable statement distributions X_1 and X_2, a pair (x_1, π_1), in which π_1 is a proof for statement $x_1 \leftarrow X_1$, must be computationally indistinguishable from a pair (x_2, π_2), in which π_2 is a proof for statement $x_2 \leftarrow X_2$. Jain *et al.* [JKKR17] observe that there is no evidence of whether ZK is actually necessary to enforce honest behavior of protocols with indistinguishability-based security and, borrowing ideas from Aiello *et al.* [ABOR00], design variant of argument systems that can be used to recover several applications of ZK. Their argument systems satisfy strong-WI and a weaker simulation strategy, called distinguisher-dependent simulation, in which the simulator can depend on the distinguisher. This is done in a setting, called the delayed-input distributional setting, in which the instance is only determined by the prover in the last round of the interaction.

We are not aware of any work analyzing whether known non-interactive ZAPs are also strong-WI. Pass [Pas06a] show that it is not possible to reduce the strong-WI of a constant-round public-coin proof systems (with negligible soundness error) for Boolean satisfiability to one-way functions under black-box reductions. An *O*-HZK proof is *O*-strong-WI, that is strong-WI when quantifying distributions that are computationally indistinguishable even by adversaries with access to *O*; see Section 1.3.9, Def. 22 and Corollary 5.

Khurana and Sahai [KS17] constructed the first two-message arguments for NP achieving quasi-polynomial time simulation in which the simulated proofs are indistinguishable from real proofs by distinguishers running in time significantly

larger than that of the simulator (improving on Pass [Pas03b], in which the simulated proofs were instead indistinguishable by distinguishers running in time significantly smaller than that of the simulator). Kalai *et al.* [KKS18] construct two-message arguments satisfying the above property but with simulated proofs statistically indistinguishable from real proofs by distinguishers running in time significantly smaller than that of the simulator. In the delayed-input distributional setting, with distinguisher-dependent (polynomial time) simulation, Kalai *et al.* [KKS18] improve the work of Jain *et al.* [JKKR17] achieving statistical privacy.

Deshpande and Kalai [DK18] construct the first 2-message adaptively sound witness hiding argument for NP (in the non-delayed input setting). Our proofs are additionally 1-message harmless witness hiding proofs, under seemingly reasonable computational assumptions. Harmless witness hiding is a natural strengthening of witness hiding. Witness hiding requires that no efficient verifier can extract a witness after interacting with the prover on a randomly chosen instance. Harmless witness hiding quantifies over efficient adversaries with access to some oracle. For instance, our NI NIDDH is harmless witness hiding, under the assumption that no PPT adversary, with access to the oracle DHInvO (cf. Def. 29), can extract a witness from a randomly selected DH tuple over the multiplicative group $\mathbb{Z}_N^\star$, for a Blum integer N. For more details, see Section 1.3.4, Def. 20 and Lemma 4. Haitner *et al.* [HRS09] and Pass [Pas11] present impossibility results for black-box reductions of standard assumptions to WH. In Section 1.3.4 we elaborate on why the fact that NIDDH is WH (under some computational assumption) does not contradict the aforementioned black-box impossibility results. In Section 1.3.4, we argue that the assumption "the NI proof system NIDDH is WH" is a falsifiable assumption according to the classification of Gentry and Wichs [GW11] (see also Naor [Nao03]).

Dwork and Stockmeyer [DS02] present two-round zero-knowledge proofs for provers that are resource-bounded (e.g., running in time n^4) and simulators allowed to run in longer time than the prover (long enough to break the soundness of the system); they also observe that one-message proofs cannot be obtained in their setting.

De Santis *et al.* [DPY92] showed that when a bound on the space of the verifier is known, then there exists a one-message zero-knowledge proof system for NP with statistical soundness.

Bellare *et al.* [BFS16] study the problem of subversion resistance, that is they consider the case of an adversary colluding with the CRS generator. Our proofs are not based on any trusted parameter, hence are additionally subversion resistant.

Pass, Halpern and Raman [HPR09] present an epistemic formula that holds iff a proof system is ZK. It would be interesting to find a similar characterization for O-HZK proof systems.

Precise ZK (see Pass [Pas06b]) aims at bounding the knowledge gained by a verifier in an interaction in terms of the actual computation rather than the potential computation. The notion does not apply to NI systems. However, one-

message HZK proofs are not precise in that the simulator is allowed access to a resource (the oracle) that a potential verifier *might* use in an attempt to attack the system. A more "precise" definition of O-HZK, meaningful also for NI systems, might be formulated as a stricter variant of O-FH (see Sections 1.3.4 and 2.2.6) requiring that for every adversary of time t computing a function of the witness from a proof with probability p, there exists a related O-aided adversary of time t that can compute the same function of the witness (without the proof) with the same probability p.

Cryptography in groups of hidden order. The power of groups of hidden order has been established in the seminal work of Rivest, Shamir and Adleman [RSA78] on the homonymous RSA cryptosystem. El Gamal-like encryption schemes based on trapdoors like ours have been already proposed in several works. Paillier [Pai99] and its elliptic curve variant [Gal02], Bresson *et al.* [BCP03] and [CL15] constructed linearly homomorphic encryption schemes based on variants of the discrete log problem with trapdoors. McCurley [McC88] and Schmuely [Shm85] observed that the computational Diffie-Hellman assumption in multiplicative groups modulo a composite number is equivalent to factoring. Hofheinz and Kiltz [HK09] proposed a cryptographic group in which variants of the El Gamal encryption scheme can be proven CCA-secure under the factoring assumption. Groth [Gro05a] show the potential usefulness of working in small subgroups of $\mathbb{Z}_N^\star$.

Summary of our improvements of the state of the art. Summing up, our work improves the state of the art as follows.

- We construct the first *non-interactive* proofs not relying on trusted parameters for non-trivial relations. Known non-interactive proofs without trusted parameters were only proven WI, whereas ours satisfy the stronger HZK property, a close variant of ZK rooted in the simulation paradigm. Our use of oracles to define and analyze security of proof systems is novel but shares similarities with imaginary angels introduced by Prabhakaran and Sahai [PS04] in the context of multi-party computation and universal composability. In this work, we show that HZK is useful and usable to prove security of larger protocols by presenting concrete examples (see Section 1.3.8).
- We construct proof systems with *perfect* soundness enjoying HZK. Interactive ZK proof systems cannot be perfectly sound.
- For the relation of well-formed DH tuples and correctness of shuffles, only NIZK arguments in the RO model resulting from FS-transforming sigma protocols for the same relations were known. We construct proofs for these relations enjoying perfect soundness and not based on any trusted parameter whereas previously known efficient constructions only achieved computational soundness in the programmable RO model. Our techniques extend to compound and polynomial statements.
- Our proof systems are additionally proofs of knowledge. In a completely non-interactive setting like ours, a proof of knowledge guarantees that an

adversary that gets a randomly chosen statement cannot output an accepted proof for it (to come up with an accepted proof, it would need to know a witness for the statement). If the group parameter for the DH tuples is seen as a common parameter made public (that, however, does not have to be trusted), our proof for DH tuples also satisfies the standard definition of perfect extraction [GOS12].

– We construct a one-message perfectly sound WI proof (i.e., non-interactive ZAP) for Boolean circuit satisfiability from a number-theoretic assumption related to multiplicative groups of hidden order. Previous non-interactive ZAPs for Boolean circuit satisfiability were based on bilinear group assumptions [GOS12]. Our non-interactive ZAP is also (computational) HZK.

– Our proofs, and HZK proofs in general, satisfy O-strong-WI. While strong-WI requires that for two computationally indistinguishable statement distributions X_1 and X_2, a proof for statement $x_1 \leftarrow X_1$ must be computationally indistinguishable from a proof for statement $x_2 \leftarrow X_2$, O-strong-WI has a similar requirement but quantifies over statement distributions that are computationally indistinguishable even by adversaries with access to O. It has been observed that strong-WI can recover several applications of ZK. Similarly, this is true for O-strong-WI at the cost of basing the security on assumptions that hold even with respect to adversaries with access to O. In this paper we present concrete examples in which this is the case; see Section 1.3.8.

– Witness hiding requires that no efficient verifier can extract a witness after interacting with the prover on a randomly chosen instance. Harmless witness hiding quantifies over efficient adversaries with access to some oracle and thus is stronger than witness hiding. Our NIs are 1-message harmless witness hiding proof systems under seemingly reasonable oracle-based computational assumptions.

2 Definitions

Notation. We use $\mathbb{N}$ to denote the set of all natural numbers. For any natural number $m > 0$, we let U_m stand for the uniform distribution over binary strings of length m. A *negligible* function $\mathsf{negl}(\cdot)$ is a function that is smaller than the inverse of any polynomial in λ (starting from a certain point). We denote by $[n]$ the set of numbers $\{1, \ldots, n\}$, by $|x|$ the bit length of $x \in \{0,1\}^\star$ and by $x||y$ the concatenation of any two strings x and y in $\{0,1\}^\star$. A function $\epsilon(\cdot)$ is non-negligible if ϵ is not a negligible function.

If S is a finite set, we denote by $a \leftarrow S$ the process of setting a equal to a uniformly chosen element of S. We denote by $\perp$ a special symbol not in $\{0,1\}^\star$. An oracle O is a, possibly computationally unbounded and stateful, randomized algorithm that takes as input strings in $\{0,1\}^\star$ and outputs strings in $\{0,1\}^\star \cup \{\perp\}$.

We let PPT stand for probabilistic polynomial-time and nuPPT for non-uniform PPT (by Adleman's theorem [MR95] we could consider only non-uniform

deterministic algorithms without loss of generality). Unless otherwise specified, all our adversaries are modeled as nuPPT algorithms. Precisely, a non-uniform algorithm Adv is a family of probabilistic algorithms $\{\mathsf{Adv}_\lambda\}_{\lambda>0}$ parameterized by λ such that there exists a polynomial $p(\cdot)$ such that for all $\lambda > 0$, Adv_λ takes as inputs strings of length $p(\lambda)$, runs in time at most $p(\lambda)$ and its description has length at most $p(\lambda)$. We do not advocate that the non-uniform is the right modeling of adversarial behavior, in particular when proving implications of the form $A \to B$ via non-uniform reductions and A, B are hardness problems postulated in the non-uniform model, we are assuming the stronger hypothesis that B is breakable by non-uniform algorithms, hypothesis that might be unrealistic. The choice of the non-uniform model is for simplicity and our assumptions and results can be often translated to the uniform model as well. We will point out results and conclusions that present issues in the non-uniform model. In particular, at the end of Lemma 6 we remark that, because of the non-uniform model adopted both in the definition of O-HZK and O-HWH, the statement of the Lemma is not precise.

For a probabilistic algorithm A, $A(x)$ denotes the probability distribution of the output of A when run with x as input. We use $A(x; r)$ instead to denote the output of A when run on input x and coin tosses r. Analogously, we denote by $f(x; r)$ the output of a (possibly uncomputable) function on input x and random coins r.

An *oracle* $O(\cdot)$ is a possibly unbounded algorithm. An algorithm that can invoke an oracle and gets its output during its execution is called an oracle algorithm. We assume that, when an oracle algorithm can invoke an oracle to get an output in 1 step. We denote by $A^{O(\cdot)}$ the execution of A with access to an oracle $O(\cdot)$. The oracle is possibly *stateful* when it can store internal information between two executions and is possibly randomized when can choose random coins in its execution. When it is clear from the context, we denote by 1 the trivial identity oracle $1(\cdot)$ that, on input an arbitrary string x, outputs x. We call an oracle algorithm with access to an oracle O an *O-aided* algorithm.

Given two families of random variables $X_0 \triangleq \{X_{0,\lambda}\}_\lambda$ and $X_1 \triangleq \{X_{1,\lambda}\}$, a function $\epsilon(\cdot)$ and a nuPPT algorithm $\mathcal{D} = \{D_\lambda\}_{\lambda>0}$, we say that $\mathcal{D}$ cannot *distinguish* X_0 from X_1 with *advantage* more than $\epsilon(\cdot)$ whether $|\mathrm{Prob}[\mathcal{D}_\lambda(X_0) = 1 | \boldsymbol{x} \leftarrow X_{0,\lambda}] - \mathrm{Prob}[\mathcal{D}_\lambda(X_1) = 1 | \ x \leftarrow X_{1,\lambda}]| \leq \epsilon(\lambda)$. If in addition $\mathcal{D}$ is given access to an oracle $O(\cdot)$ we say that $\mathcal{D}^{O(\cdot)}$ cannot distinguish X_0 from X_1 with advantage more than $\epsilon(\lambda)$. We say that two families of random variables $X_0 = \{X_{0,\lambda}\}_\lambda$ and $X_1 = \{X_{1,\lambda}\}$ are computationally indistinguishable if there exists no nuPPT algorithm $\mathcal{D}$ and there exists no non-negligible function $\epsilon(\cdot)$ such $\mathcal{D}$ distinguishes X_0 from X_1 with advantage more than $\epsilon(\cdot)$.

A *polynomial-time* relation $\mathcal{R}$ is a relation for which membership of (x, w) in $\mathcal{R}$ can be decided in time polynomial in $|x|$. If $(x, w) \in \mathcal{R}$ then we say that w is a *witness* for *instance* x. A polynomial-time relation $\mathcal{R}$ is naturally associated with the NP language $L_\mathcal{R}$ defined as $L_\mathcal{R} = \{x| \ \exists w : \ (x, w) \in \mathcal{R}\}$. Similarly, an NP language is naturally associated with a polynomial-time relation. Given an NP language L, for any natural number $k > 0$, we denote by L_k the language

$L \cap \{0,1\}^{\leq k}$. A relation $\mathcal{R}$ is a polynomial-time *single witness* (or unique witness) relation if $\mathcal{R}$ is a polynomial-time relation and for any two pairs $(x, w_1), (x, w_2)$, if $(x, w_1) \in \mathcal{R}, (x, w_2) \in \mathcal{R}$ then $w_1 = w_2$.

We assume familiarity with interactive algorithms (see [Gol01] for more details). Given two interactive algorithms M_0 and M_1, we denote by $\langle M_0(x_0), M_1(x_1) \rangle(x)$ the output of M_1 when running on input x_1 and interacting with M_0 running on input x_0 and common input x and by $\mathsf{view}_A \langle A(x_A), B(x_B) \rangle(x)$ the *view* of A during the interaction with B when both are executed on common input x and A (resp. B) is executed on input x_A (resp. x_B).

For any $k > 0$, any distribution X over inputs of length k, and any Boolean predicate $f : \{0,1\}^k \to \{0,1\}$, we denote by $\mathrm{Prob}[f(x) | \; x \leftarrow X]$ the probability that $f(x) = 1$ for $x \leftarrow X$.

With a slight abuse of notation, in the description of algorithms we sometimes use the symbols "$\triangleq$" and "$=$" interchangeably to denote a copy of a memory area (corresponding to a variable in pseudo-code) into another.

2.1 Number-theoretic facts and definitions.

We call a λ-bit integer $N = p \cdot q$ an RSA *modulus* [RSA78] for security parameter λ if p and q are two distinct $\lambda/2$-bit odd primes and p and q are both congruent to 3 modulo 4; such a modulus N is also called a *Blum integer*. The group $\mathbb{Z}_N^\star$ consists of all the integers of $\mathbb{Z}_N$ that have an inverse modulo N with group operation being the multiplication modulo N and has order $\phi(N) = (p-1)(q-1)$, where $\phi(N)$ is the Euler's totient function. We denote by QR_N the set of all the elements y of $\mathbb{Z}_N$ such that there exists an integer $x \in \mathbb{Z}_N$ satisfying $y = x^2$ mod N. QR_N can be shown to be a group under multiplication modulo N and is called the *group of quadratic residues modulo N*. If N is a Blum integer, it can be seen that $-1 \notin \mathsf{QR}_N$.

The Legendre symbol $\left(\frac{x}{p}\right)$ of an integer $x \in \mathbb{Z}_p$ for a prime p is defined to be 1 if $x \in \mathsf{QR}_p, x \neq 0$, -1 if $x \notin \mathsf{QR}_p$ and 0 if $x = 0$. It can be proven that for p prime $\left(\frac{x}{p}\right) = x^{(p-1)/2} \mod p$. For an RSA modulus $N = p \cdot q$, we denote by $\left(\frac{x}{N}\right)$ the Jacobi symbol of x modulo N and we define it as $\left(\frac{x}{N}\right) \triangleq \left(\frac{x}{p}\right) \cdot \left(\frac{x}{q}\right)$. There is a PPT algorithm to compute the Jacobi symbol of x modulo N for any RSA modulo N [BSJS96].

By the Chinese remainder theorem (CRT) [BSJS96], the group $\mathbb{Z}_N^\star$, for an RSA modulus $N = p \cdot q$, is isomorphic to the product group $(\mathbb{Z}_p^\star \times \mathbb{Z}_q^\star)$. $\mathbb{Z}_p^\star$ (resp. $\mathbb{Z}_q^\star$) is cyclic of order $p - 1$ (resp. $q - 1$). Therefore QR_p (resp. QR_q) is cyclic of order $(p-1)/2$ (resp. $(q-1)/2$). By the Chinese remainder theorem, the group QR_N, for an RSA modulus $N = p \cdot q$, has order $(p-1)/2 \cdot (q-1)/2 = \phi(N)/4$ and is isomorphic to the product group $(\mathsf{QR}_p \times \mathsf{QR}_q)$, and hence is cyclic if $\gcd((p-1)/2, (q-1)/2) = 1$. Therefore, with respect to an RSA modulus N for primes $p \triangleq 2p' + 1$ and $q \triangleq 2q' + 1$, for primes p', q', the group QR_N is cyclic.

We let $\mathsf{GenRSA}(1^\lambda)$ be a PPT algorithm that generates elements (N, p, q, g) such that $N = p \cdot q$ is an RSA modulus for security parameter λ, $\gcd((p -$

$1)/2, (q-1)/2) = 1$ and g is a generator of QR_N. From the previous facts, it is easy to see that such an algorithm exists.

If g is an element of some group, we denote by $\mathrm{ord}(g)$ its order.

By a generalization of the CRT, a system of equations over the integers

$$x = a \mod m_1,$$

$$x = b \mod m_2,$$

has a unique integer solution modulo $m_1 m_2 / \gcd(m_1, m_2)$ if $a = b \mod \gcd(m_1, m_2)$, otherwise it has no solution.

We will make use of a PPT algorithm to recognize prime numbers [AKS04].[18]

2.2 Proof systems.

2.2.1 Interactive and NI proof systems
We now recall notions related to interactive proof systems and put forth our definitions for non-interactive harmless zero-knowledge proof of knowledge systems.

Definition 1 [Interactive proof system [BM88,GMR89]] A pair $(\mathcal{P}, \mathcal{V})$ of PPT interactive algorithms is a *interactive proof system* for polynomial-time relation $\mathcal{R}$ associated with a language L if the following properties of completeness and soundness hold:

– *Completeness.* For every $(x, w) \in \mathcal{R}$, it holds that:

$$\mathrm{Prob}[\langle \mathcal{P}(w), \mathcal{V} \rangle(x) = 1] = 1.$$

The property may be relaxed to hold statistically by requiring the probability to be negligible in $|x|$. For the sake of our results, we are satisfied with statistical completeness. Actually, our proof systems, as they are described, satisfy statistical completeness because, e.g., the prover has to find random values that are invertible under some constraint. If the provers in our proof systems select the randomness properly, the required property holds w.v.h.p., so the statistical completeness follows. We believe that the provers can be changed so as to enjoy perfect completeness but we did not investigate the details. See also Remark 9.

– *Soundness.* For every non-uniform (possibly computationally unbounded) algorithm $\mathcal{P}^{\star} \triangleq \{\mathcal{P}^{\star}_{\lambda}\}_{\lambda > 0}$, it holds that for every polynomial $p(\cdot)$, there exists a constant n such that for every $\lambda \geq n$, for every $x \in \bar{L} \cap \{0, 1\}^{\lambda}$, it holds that:

$$\mathrm{Prob}[\langle \mathcal{P}^{\star}_{\lambda}, \mathcal{V} \rangle(x) = 1] \leq 1/p(\lambda).$$

---

[18] If we replace this algorithm with an algorithm that errs with negligible probability like the Miller-Rabin algorithm [Rab80], our proof systems would incur a negligible soundness error in which the probability of error is over the random choices of the verifier, that is if the verifier fails (with some probability over its random coins) in detecting that an integer sent from the prover is *not* a prime, the verifier may accept a false statement.

$\triangle$

The soundness can be generalized to $s(\cdot)$-soundness as follows.

Definition 2 [$s(\cdot)$-soundness] Let $s(\cdot)$ be a function. An interactive proof system $(\mathcal{P}, \mathcal{V})$ for polynomial-time relation $\mathcal{R}$ associated with a language L satisfies $s(\cdot)$-soundness if the following holds. For every non-uniform (possibly computationally unbounded) algorithm $\mathcal{P}^\star \triangleq \{\mathcal{P}_\lambda^\star\}_{\lambda > 0}$, it holds that there exists a constant n such that for every $\lambda \geq n$, for every $x \notin L \cap \{0,1\}^\lambda$, it holds that:

$$\mathrm{Prob}[\langle \mathcal{P}_\lambda^\star, \mathcal{V}\rangle(x) = 1] \leq 1/s(\lambda).$$

$\triangle$

Definition 3 [Perfect soundness] An interactive proof system $(\mathcal{P}, \mathcal{V})$ for polynomial-time relation $\mathcal{R}$ associated with a language L satisfies perfect soundness (or it is perfectly sound) if it satisfies 0-soundness. $\triangle$

Definition 4 [Non-interactive proof system] A pair of PPT (non-interactive) algorithms $(\mathcal{P}, \mathcal{V})$ is a non-interactive proof (NI) system for polynomial-time relation $\mathcal{R}$ associated with a language L if $(\mathcal{P}, \mathcal{V})$ is a perfectly sound proof system and $\mathcal{V}$ is deterministic. For any $(x, w) \in \mathcal{R}$, we call any string π in the range of $\mathcal{P}(x, w)$ a proof for x. $\triangle$

Note that in the above definition we require $(\mathcal{P}, \mathcal{V})$ to be non-interactive algorithms and thus the prover, on input (x, w) (and the random coins), outputs a proof π and the verifier $\mathcal{V}$, on input two strings x and π, outputs its decision on whether $x \in L$.

According to the previous definition, the term NI is synonymous of a non-interactive perfectly sound proof system. Sometimes, in this paper we denote by NI just a pair of non-interactive algorithms prover and verifier, and we use instead the term "NI proof" to stress the perfect soundness condition.

2.2.2 *O*-HZK

Definition 5 [Legitimate oracle] We definite a legitimate oracle by first defining an illegitimate (non-legitimate) oracle. A (possibly stateful and randomized) oracle $O(\cdot)$ is illegitimate for polynomial-time relation $\mathcal{R}$ associated with a language L if there exists a nuPPT O-aided algorithm $\mathsf{Adv}^{O(\cdot)} \triangleq \{\mathsf{Adv}_\lambda^{O(\cdot)}\}_{\lambda > 0}$ such that the following conditions hold:

- For every $x \in L$, $\mathsf{Adv}_{|x|}^{O(\cdot)}$ outputs 1 with probability $\geq 2/3$ and $\mathsf{Adv}_{|x|}^{O(\cdot)}$ never queries $O(\cdot)$ on a point y such that $O(y) = \bot$.
- For every $x \notin L$, $\mathsf{Adv}_{|x|}^{O(\cdot)}$ outputs 1 with probability $\leq 1/3$ and $\mathsf{Adv}_{|x|}^{O(\cdot)}$ never queries $O(\cdot)$ on a point y such that $O(y) = \bot$.

That is, an illegitimate oracle for a relation $\mathcal{R}$ over a language L makes L easy to decide against O-aided nuPPT adversaries.

An oracle $O(\cdot)$ for a polynomial-relation $\mathcal{R}$ is *legitimate* if $O(\cdot)$ is *not* illegitimate for $\mathcal{R}$, in such case we write $O(\cdot) \in \mathsf{LegOr}^{\mathcal{R}}$. $\triangle$

Remark 1 The requirement on restricting the adversary to not query the oracle on points on which the oracle returns $\bot$ (to indicate error) is necessary to prevent the adversary to decide the language in a trivial way, e.g., querying the oracle on instances not belonging to the language.

At first sight, this condition may appear "weird" for the following reason. If the oracle cannot be invoked on invalid instances, how could an adversary ever benefit from the access to the oracle? Does not an adversary that invokes the oracle already "know" that the instance is in the language? It would seem so that the oracle can be never invoked. A careful reflection shows instead that the oracle might be invoked on any other valid instance that is not necessarily related to the input of the adversary or might be invoked on a string that is just a part of the whole input of the adversary; we will later further discuss this point.

What does the definition of legitimate oracle model? Later on, we will conjecture some language to be hard with respect to an adversary with access to a legitimate oracle for that language. We will now argue that this conjecture is necessary (it is a minimal assumption) to make any kind of HZK proof useful in cryptographic protocols.

Let us briefly recall the setting for traditional ZK proofs. A verifier needs a ZK proof of the fact that $x \in L$ because it cannot decide whether $x \in L$ or $x \notin L$ by itself. If, for instance, it were easy to distinguish DH tuples from non-DH tuples, we would not need any ZK proof for the well-formedness of DH tuples. So, in this sense the hypothesis that deciding the language of DH tuples is *worst-case* hard is a minimal assumption to make ZK proofs for this language useful. Observe that in practice one uses such ZK proofs for the well-formedness of DH tuples in protocols whose overall security is based on a related average-case assumption, e.g., the problem of distinguishing a random DH tuple from a random tuple. However, if the worst-case assumption does not hold, the average-case assumption does not hold as well, so the former is ever a necessary assumption.

Consider now the possibility of a world in which deciding the language of DH tuples (in the worst-case) is hard but it is easy if the adversary is given access to some oracle O that is only invoked on valid DH tuples. Then, a verifier that is given access to O would not need any proof of the fact that $x \in L$. Then, assuming every party to have access to O (an implicit assumption in the setting of HZK), requiring O to be a legitimate oracle is a necessary condition.

As for the case of traditional ZK proofs for DH tuples, in practice one would use our HZK proof for the well-formedness of DH tuples in protocols whose overall security is based on a related average-case assumption, e.g., Assumption 7 (see Section 1.3.8 on how to use HZK proofs to argue security, and in particular reducing the security of an e-voting application that uses our HZK proofs to such assumption). The latter assumption essentially states that it is difficult to distinguish a random DH tuple for witness w from a random tuple given additionally

another random DH tuple for the same witness w and access to an oracle O that can be only invoked on valid DH tuples. If the oracle were not legitimate, it is easy to see that the latter assumption would not hold; this again to stress that requiring the oracle to be legitimate is a necessary assumption.

Assumption 7 also exemplifies why restricting the adversary to not query the oracle on invalid tuples is natural and useful. It turns out that if an adversary can break Assumption 6, there exists an adversary breaking Assumption 7 (see Lemma 10). The latter assumption essentially states that it is difficult to distinguish whether two El Gamal ciphertexts encrypt resp. $(m, -m)$ or $(-m, m)$ given additionally a proof that their "product" decrypts to 0. When proving that Assumption 7 implies Assumption 6 via hybrid experiments, we will have that the adversary, in one case, has to simulate one hybrid with input $Z = A^w$, where w is the witness for the DH tuple, and in another case, has to simulate another hybrid but with input a random Z. However, the adversary also gets as input a related DH tuple for the same witness w and such tuple, that is by definition always well-formed, is the input on which the adversary invokes the oracle (the HZK proof that that the adversary has to simulate in both hybrids is set to be the output of the oracle on such input). So, notwithstanding the restriction on the oracle queries, the adversary makes a non-trivial use of the oracle.

Assumption 7 and more generally our oracle-based assumptions have somehow the flavor of "one-more" discrete log assumptions [BNPS02] in that an adversary is asked to break some problem given access to an oracle that breaks other instances. Indeed, access to oracles in our assumptions allows the check the membership in some language.

One can argue that the restriction on the oracle queries in Assumption 7 is more natural than the restriction on the oracle queries in, e.g., the following assumption (that we will state later): the oracle DHInvO (cf. Def. 29), that returns error on input an invalid DH tuple, is legitimate for the relation of valid DH tuple (that we will define in Section 2.3). It seems that, unlike Assumption 7, the adversary cannot benefit from the oracle access. We remark that if in this case an adversary cannot make any non-trivial use of the oracle, all the more we are justified to allow the simulator access to the oracle. Indeed, we stress again that we would like to give the simulator access to all resources that cannot help the adversary to decide the language (without seeing proofs); if one could prove that, for the case of the relation of valid DH tuples, the oracle does not add any power to an adversary attempting to break the worst-case decisional hardness of the corresponding language, all the more so the oracle is a harmless resource and as such we grant the simulator access to it. Note that the assumption that the oracle DHInvO be legitimate for the relation of valid DH tuples basically accounts to say that an efficient adversary cannot distinguish a DH tuple from a non-DH tuple seeing additionally proofs of well-formedness of valid DH tuples of its choice.

We also point out that the oracle we will consider in this paper for the case of extraction (i.e., the factoring oracle) never outputs error. $\triangle$

Definition 6 [Computational harmless zero-knowledge] A NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ for a polynomial-time relation $\mathcal{R}$ is computational harmless zero-knowledge (cHZK) if there exists a, possibly stateful and randomized, oracle $O(\cdot) \in \mathsf{LegOr}^{\mathcal{R}}$ and a PPT algorithm $\mathsf{Sim}^{O(\cdot)}$ (called the simulator) with oracle access to $O(\cdot)$ such that, for for every sequence $\{(x_\lambda, w_\lambda)\}_{\lambda > 0}$ such that $\forall \lambda > 0$ $(x_\lambda, w_\lambda) \in \mathcal{R}$ and $|x_\lambda| \geq \lambda$, for every nuPPT O-aided distinguisher $\mathcal{D}^{O(\cdot)} = \{\mathcal{D}_\lambda^{O(\cdot)}\}_{\lambda > 0}$, for every polynomial $p(\cdot)$, there exists a number $n > 0$ such that for every $\lambda \geq n$, $\mathcal{D}_{|x_\lambda|}^{O(\cdot)}$ can distinguish the following two random variables with advantage $< 1/p(|x_\lambda|)$:

- $(x_\lambda, \mathcal{P}(x_\lambda, w_\lambda; U_{m(|x_\lambda|)}))$. (Where $m(\lambda)$ is the number of random coins $\mathcal{P}$ uses on an input x of length λ.)
- $(x_\lambda, \mathsf{Sim}^{O(\cdot; U_{d(|\cdot|)})}(x_\lambda; U_{s(|x_\lambda|)}))$. (Where $d(\lambda)$ is the total number of random coins O uses in a single invocation when running on an input x of length λ, and $s(\lambda)$ is the number of random coins $\mathsf{Sim}^{O(\cdot)}$ uses on an input x of length λ.)

$\triangle$

Definition 7 [Statistical harmless zero-knowledge] A NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ for a polynomial-time relation $\mathcal{R}$ is statistical harmless zero-knowledge (sHZK) if there exists a, possibly stateful and randomized, oracle $O(\cdot) \in \mathsf{LegOr}^{\mathcal{R}}$ and a PPT algorithm $\mathsf{Sim}^{O(\cdot)}$ (called the simulator) with oracle access to $O(\cdot)$ such that, for for every sequence $\{(x_\lambda, w_\lambda)\}_{\lambda > 0}$ such that $\forall \lambda > 0$ $(x_\lambda, w_\lambda) \in \mathcal{R}$ and $|x_\lambda| \geq \lambda$, for every non-uniform (possibly unbounded) distinguisher algorithm $\mathcal{D} = \{\mathcal{D}_\lambda\}_{\lambda > 0}$, for every polynomial $p(\cdot)$, there exists a number $n > 0$ such that for every $\lambda \geq n$, $\mathcal{D}_{|x_\lambda|}$ can distinguish the following two random variables with advantage $< 1/p(|x_\lambda|)$:

- $(x_\lambda, \mathcal{P}(x_\lambda, w_\lambda; U_{m(|x_\lambda|)}))$. (Where $m(\lambda)$ is the number of random coins $\mathcal{P}$ uses on an input x of length λ.)
- $(x_\lambda, \mathsf{Sim}^{O(\cdot; U_{d(|\cdot|)})}(x_\lambda; U_{s(|x_\lambda|)}))$. (Where $d(\lambda)$ is the number of random coins O uses in a single invocation when running on an input x of length λ, and $s(\lambda)$ is the number of random coins $\mathsf{Sim}^{O(\cdot)}$ uses on an input x of length λ.)

$\triangle$

Definition 8 [Perfect harmless zero-knowledge] A NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ for a polynomial-time relation $\mathcal{R}$ is perfect harmless zero-knowledge (HZK) or simply harmless zero-knowledge if there exists a, possibly stateful and randomized, oracle $O(\cdot) \in \mathsf{LegOr}^{\mathcal{R}}$ and a PPT algorithm $\mathsf{Sim}^{O(\cdot)}$ (called the simulator) with oracle access to $O(\cdot)$ such that, for for every sequence $\{(x_\lambda, w_\lambda)\}_{\lambda > 0}$ such that $\forall \lambda > 0$ $(x_\lambda, w_\lambda) \in \mathcal{R}$ and $|x_\lambda| \geq \lambda$, there exists a number $n > 0$ such that for every $\lambda \geq n$, the following two random variables are identically distributed:

- $(x_\lambda, \mathcal{P}(x_\lambda, w_\lambda; U_{m(|x_\lambda|)}))$. (Where $m(\lambda)$ is the number of random coins $\mathcal{P}$ uses on an input x of length λ.)

– $(x_\lambda, \mathsf{Sim}^{O(\cdot; U_{d(|\cdot|)})}(x_\lambda; U_{s(|x_\lambda|)}))$. (Where $d(\lambda)$ is the number of random coins O uses in a single invocation when running on an input x of length λ, and $s(\lambda)$ is the total number of random coins $\mathsf{Sim}^{O(\cdot)}$ uses on an input x of length λ.)

$\triangle$

Definition 9 [O-HZK] We say that a NI NI is O-HZK for polynomial-time relation $\mathcal{R}$, for some oracle $O(\cdot)$, if NI satisfies definition 8 with respect to $\mathcal{R}$ and a simulator with oracle access to $O(\cdot)$. Analogously, we say that a NI is O-cHZK or O-sHZK. $\triangle$

In this work, sometimes we informally talk about HZK even when we are actually referring to cHZK and sometimes.

Definition 10 [ZK] We say that a NI NI is zero-knowledge (ZK) for polynomial-time relation $\mathcal{R}$ if NI satisfies definition 8 with respect to $\mathcal{R}$ and a simulator with oracle access to the trivial identity oracle $1(\cdot)$. Analogously, we say that a NI is cZK or sZK. $\triangle$

Remark 2 Observe that the distinguisher for the real and simulated random variables in the definition of cHZK is given access to the oracle. This is necessary to prove sequential composition (see next). Also, compare it with the definition of ZK in the non-programmable and explicitly programmable RO model given in Wee [Wee09] in which the distinguisher is likewise given access to the RO.

Note that our definitions are conditional in that we require the oracle to be legal with respect to the relation. That is, a NI for a relation $\mathcal{R}$ over a language L is not O-HZK if $O \notin \mathsf{LegOr}^{\mathcal{R}}$, even if there exists an O-aided simulator satisfying the definition. This constraint is just to stress that, e.g., in that case a proof for $\mathcal{R}$ would not be meaningful as L is decidable (with access to O). The condition can be removed without any effect on our results.

Our definition of ZK is syntactically different from the standard definition of ZK but is equivalent regarding language recognition. $\triangle$

Definition 11 [Unbounded computational harmless zero-knowledge] A NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ for a polynomial-time relation $\mathcal{R}$ is unbounded computational harmless zero-knowledge (ucHZK) if there exists a, possibly stateful and randomized, oracle $O(\cdot) \in \mathsf{LegOr}^{\mathcal{R}}$ and a PPT algorithm $\mathsf{Sim}^{O(\cdot)}$ (called the simulator) with oracle access to $O(\cdot)$ such that, for every $m > 0$, every sequence $\{X_\lambda \stackrel{\triangle}{=} [(x_\lambda^i, w_\lambda^i)]_{i \in [m]}\}_{\lambda > 0}$ such that $\forall \lambda > 0, i \in [m]$, $(x_\lambda^i, w_\lambda^i) \in \mathcal{R}$ and $|x_\lambda^i| \geq \lambda$, for every nuPPT O-aided distinguisher $\mathcal{D}^{O(\cdot)} = \{\mathcal{D}_\lambda^{O(\cdot)}\}_{\lambda > 0}$, for every polynomial $p(\cdot)$, there exists a number $n > 0$ such that for every $\lambda \geq n$, $\mathcal{D}_{|X_\lambda|}^{O(\cdot)}$ can distinguish the following two random variables with advantage $< 1/p(|X_\lambda|)$:

– $R_{\lambda, m} \stackrel{\triangle}{=} [(x_\lambda^i, \mathcal{P}(x_\lambda^i, w_\lambda^i; U_{m(|x_\lambda^i|)}))]_{i \in [m]}$. (Where $m(\lambda)$ is the number of random coins $\mathcal{P}$ uses on an input x of length λ.)

- $S_{\lambda,m} \overset{\triangle}{=} [(x_\lambda^i, \mathsf{Sim}^{O(\cdot; U_{d(|\cdot|)})}(x_\lambda^i; U_{s(|x_\lambda^i|)}))]_{i\in[m]}$. (Where $d(\lambda)$ is the total number of random coins O uses in a single invocation when running on an input x of length λ, and $s(\lambda)$ is the total number of random coins $\mathsf{Sim}^{O(\cdot)}$ uses on an input x of length λ.)

A NI NI is a O-ucHZK for polynomial-relation $\mathcal{R}$ if NI is ucHZK with respect to an oracle $O(\cdot)$. $\triangle$

Lemma 1 [Composition of cZK proofs] If $\mathsf{NI} \overset{\triangle}{=} (\mathcal{P}, \mathcal{V})$ is a O-cZK NI for a polynomial-time relation $\mathcal{R}$, then NI is a O-ucHZK NI for $\mathcal{R}$. $\triangle$

Proof. (Sketch) This follows by a standard hybrid argument observing that the distinguisher against a cHZK proof is given access to the oracle, by means of which, the distinguisher can compute simulated proofs. $\triangle$

Lemma 2 [Impossibility of non-interactive ZK for non-trivial languages] If $\mathsf{NI} \overset{\triangle}{=} (\mathcal{P}, \mathcal{V})$ is a statistically sound cZK proof for a polynomial-time relation $\mathcal{R}$ associated with a language L, then L is trivial in the following sense: there exists a nuPPT adversary $\mathsf{Adv} \overset{\triangle}{=} \{\mathsf{Adv}_\lambda\}_{\lambda>0}$ such that for every sequence $\{x_\lambda\}_{\lambda>0}$ such that for every $\lambda > 0, |x_\lambda| \geq \lambda$, there exists a value $k > 0$ such that for every $\lambda \geq k$, if $x_\lambda \in L$, $\mathsf{Adv}_{|x|}(x) = 1$ with probability $\geq 2/3$ and if $x_\lambda \notin L$, $\mathsf{Adv}_{|x|}(x) = 1$ with probability $\leq 1/3$.

$\triangle$

Proof. Let Sim be the 1-aided oracle guaranteed by the cZK property. In the following, without loss of generality, we assume Sim to not be oracle-aided (any invocation to the trivial identity oracle $1(\cdot)$ can be simulated with a constant overhead). Let $\mathsf{Adv} \overset{\triangle}{=} \{\mathsf{Adv}_\lambda\}_{\lambda>0}$ be the following nuPPT adversary against the worst-case membership hardness of L. Algorithm $\mathsf{Adv}_{|x|}$ receives as input a string x, invokes Sim on x to get an output π and returns $\mathcal{V}(x,\pi)$ as its decision on whether $x \in L$ or $x \notin L$. (Recall that, according to our definition of NI (cf. Def. 4), the verifier is deterministic. The impossibility can be easily extended to the case of probabilistic verifiers.)

Let us analyze the behavior of Adv. Consider an arbitrary sequence $\{(x_\lambda, w_\lambda)\}_{\lambda>0}$ such that $\forall \lambda > 0 \ (x_\lambda, w_\lambda) \in \mathcal{R}, |x_\lambda| \geq \lambda$.

By the cZK property, for every nuPPT distinguisher $\mathcal{D} = \{\mathcal{D}_\lambda\}_{\lambda>0}$, for every polynomial $p(\cdot)$, there exists a number $n > 0$ such that for every $\lambda \geq n$, $\mathcal{D}_{|x_\lambda|}$ can distinguish the following two random variables R_λ and S_λ with advantage $< 1/p(|x_\lambda|)$:

- $R_\lambda \overset{\triangle}{=} (x_\lambda, \mathcal{P}(x_\lambda, w_\lambda; U_{m(|x_\lambda|)}))$. (Where $m(\lambda)$ is the number of random coins $\mathcal{P}$ uses on an input x of length λ.)
- $S_\lambda \overset{\triangle}{=} (x_\lambda, \mathsf{Sim}(x_\lambda; U_{s(|x_\lambda|)}))$. (Where $s(\lambda)$ is the number of random coins Sim uses on an input x of length λ.)

By statistical completeness, there is a negligible function $\nu(\cdot)$ such that (1) for any $\lambda > 0$, $\mathcal{V}$ accepts (x_λ, π), with $\pi \leftarrow \mathcal{P}(x'_\lambda, w_\lambda)$, with probability $\geq 1 - \nu(\lambda)$ over the random coins of $\mathcal{P}$ and $\mathcal{V}$.

Suppose towards a contradiction that (2) there exists a polynomial $p(\cdot)$ such that for every $n > 0$ there exists $\lambda \geq n$ such that $\mathcal{V}$ accepts (x_λ, π), with $\pi \leftarrow \mathsf{Sim}(x_\lambda)$, with probability $< 1 - \nu(\lambda) - \frac{1}{p(|x_\lambda|)}$ over the random coins of Sim and $\mathcal{V}$. Consider the following nuPPT distinguisher $\mathcal{D}' \triangleq \{\mathcal{D}'_\lambda\}_{\lambda > 0}$ against the families of ensembles $\{R_\lambda\}_{\lambda > 0}$ and $\{S_\lambda\}_{\lambda > 0}$. $\mathcal{D}'_{|x_\lambda|}$ receives a string (x_λ, π) and outputs $\mathcal{V}(x_\lambda, \pi)$.

For every $\lambda > 0$, if $(x_\lambda, \pi) \leftarrow R_\lambda$, then, by (1) and definitions of R_λ and $\mathcal{D}'_{|x_\lambda|}$, $\mathcal{D}'_{|x_\lambda|}(x_\lambda, \pi) = 1$ with probability $\geq 1 - \nu(\lambda)$. By (2), for every $n > 0$, there exists $\lambda \geq n$ such that if $(x_\lambda, \pi) \leftarrow S_\lambda$, then, by definitions of S_λ and $\mathcal{D}'_{|x_\lambda|}$, $\mathcal{D}'_{|x_\lambda|}(x_\lambda, \pi) = 1$ with probability $< 1 - \nu(\lambda) - 1/p(|x_\lambda|)$. Therefore, there exists a polynomial $p(\cdot)$ such that for every $n > 0$ there exists $\lambda \geq n$ such that $\mathcal{D}'_{|x_\lambda|}$ can distinguish the distributions R_λ and S_λ with advantage $\geq \frac{1}{p(|x_\lambda|)}$, a contradiction to the cZK property.

Since (2) does not hold, then there exists a negligible function $\nu'(\cdot)$ such that (3) for any $\lambda > 0$, $\mathcal{V}$ accepts (x_λ, π), with $\pi \leftarrow \mathsf{Sim}(x_\lambda)$, with probability $\geq 1 - \nu'(\lambda)$ over the random coins of Sim and $\mathcal{V}$. Since ν' is a negligible function, there exists a value $k_1 \geq 0$ such that for every $\lambda > k_1$, $\nu'(\lambda) \leq 1/3$. Hence, by definition of Adv, we have that (4) there exists a value $k_1 \geq 0$ such that for every $\lambda > k_1$, $\mathsf{Adv}_\lambda(x_\lambda) = 1$ with probability $\geq 2/3$.

By statistical soundness there exists a negligible function $\mu(\cdot)$ such that for every string π, every $x \notin L$, $\mathcal{V}(x, \pi) = 1$ with probability $\leq \mu(|x|)$ over its random coins. Since μ is a negligible function, there exists a value $k_2 \geq 0$ such that for every $\lambda > k_2$, $\mu(\lambda) \leq 1/3$. Hence, there exists a value $k_2 \geq 0$ such that for every $\lambda > k_2$, string π, $x_\lambda \notin L \cap \{0, 1\}^{\geq \lambda}$, $\mathcal{V}(x_\lambda, \pi) = 1$ with probability $\leq \mu(\lambda) \leq 1/3$ over its random coins. Therefore, (5) for every sequence $\{x_\lambda\}_\lambda$ such that for every $\lambda > 0, |x_\lambda| \geq \lambda$, there exists a value $k_2 \geq 0$ such that for every $\lambda > k_2$, string π, if $x_\lambda \notin L$, $\mathcal{V}(x_\lambda, \pi) = 1$ with probability $\leq \mu(\lambda) \leq 1/3$ over its random coins. By definition of Adv and (5), we have that (6) for every sequence $\{x_\lambda\}_\lambda$ such that for every $\lambda > 0, |x_\lambda| \geq \lambda$, there exists a value $k_2 \geq 0$ such that for every $\lambda > k_2$, string π, if $x_\lambda \notin L$, $\mathsf{Adv}(x_\lambda) = 1$ with probability $\leq \mu(\lambda) \leq 1/3$ over its random coins.

Let k be the maximum of k_1 and k_2. Then, by (4) and (6) we have that for every sequence $\{x_\lambda\}_{\lambda > 0}$ such that for every $\lambda > 0, |x_\lambda| \geq \lambda$, there exists a value $k > 0$ such that for every $\lambda \geq k$, if $x_\lambda \in L$, $\mathsf{Adv}(x) = 1$ with probability $\geq 2/3$ and if $x_\lambda \notin L$, $\mathsf{Adv}(x) = 1$ with probability $\leq 1/3$. This concludes the proof. $\triangle$

Remark 3 Note that considering in the above lemma only cZK, rather than perfect ZK, and statistical soundness, rather than perfect soundness, makes the impossibility result stronger. $\triangle$

2.2.3 Hard relations and O-HPoK

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Definition 12 [Hard relation] A polynomial-time relation $\mathcal{R}$ associated with a language L is said to be hard with respect to an algorithm Gen (called the generator) if:

- Gen, on input 1^λ, outputs a pair $(x, w) \in \mathcal{R}$ where $|x| = \lambda$.
- For all nuPPT algorithms $\mathcal{A} = \{A_\lambda\}_{\lambda > 0}$, the quantity $\epsilon(\lambda) \triangleq \mathrm{Prob}[(x, w) \in \mathcal{R}|\ x \leftarrow \mathsf{Gen}(1^\lambda);\ \mathsf{w} \leftarrow \mathcal{A}_\lambda(\mathsf{x})]$ is a negligible function in λ.

A polynomial-time relation $\mathcal{R}$ associated with a language L is said to be hard if it is hard with respect to some PPT algorithm Gen. $\triangle$

Definition 13 [Hard relation with respect to an oracle] Let $O(\cdot)$ be a, possibly stateful and randomized, oracle. A polynomial-time relation $\mathcal{R}$ associated with a language L is said to be hard with respect to an algorithm Gen and $O(\cdot)$ if:

- Gen, on input 1^λ, outputs a pair $(x, w) \in \mathcal{R}$ where $|x| = \lambda$.
- For all nuPPT algorithms $\mathcal{A}^{O(\cdot)} = \{A_\lambda^{O(\cdot)}\}_{\lambda > 0}$ with access to $O(\cdot)$, the quantity $\epsilon(\lambda) \triangleq \mathrm{Prob}[(x, w') \in \mathcal{R}|\ (x, w) \leftarrow \mathsf{Gen}(1^\lambda);\ \mathsf{w}' \leftarrow \mathcal{A}_\lambda^{\mathsf{O}(\cdot)}(\mathsf{x})]$ is a negligible function in λ.

Let $O(\cdot)$ be a, possibly stateful and randomized, oracle. A polynomial-time relation $\mathcal{R}$ associated with a language L is said to be hard with respect to $O(\cdot)$ if it is hard with respect to some PPT algorithm Gen and to $O(\cdot)$. $\triangle$

Definition 14 [Legitimate oracle for extraction] A, possibly stateful and randomized, oracle $O(\cdot)$ is said to be a legitimate oracle for extraction for a polynomial-relation $\mathcal{R}$ if $\mathcal{R}$ is a hard relation with respect to $O(\cdot)$. In such case, we write $O(\cdot) \in \mathsf{LegOrHR}^{\mathcal{R}}$. $\triangle$

Definition 15 [Harmless proof of knowledge] A NI system $\mathsf{NI} \triangleq (\mathcal{P}, \mathcal{V})$ for a polynomial-time relation $\mathcal{R}$ is said to be harmless proof of knowledge (HPoK) if there exists a, possibly stateful and randomized, oracle $O(\cdot) \in \mathsf{LegOrHR}^{\mathcal{R}}$ and a PPT algorithm $\mathsf{Ext}^{O(\cdot)}$ (called the extractor) with oracle access to $O(\cdot)$ such that the following holds:

- For any strings $x, \pi \in \{0, 1\}^\star$, if $\mathcal{V}(x, \pi) = 1$ then $\mathrm{Prob}[(x, w) \in \mathcal{R}(x, w)|\ w \leftarrow \mathsf{Ext}^{O(\cdot)}(x)] = 1$.

$\triangle$

Definition 16 [O-HPoK] We say that a NI NI is O-HPoK for polynomial-time relation $\mathcal{R}$, for some oracle $O(\cdot)$, if NI satisfies definition 15 with respect to $\mathcal{R}$ and an extractor with oracle access to $O(\cdot)$. $\triangle$

HPoK implies the following corollary.

Corollary 3 If $(\mathcal{P}, \mathcal{V})$ is a HPoK NI system for some polynomial-time relation $\mathcal{R}$, then the following holds:

– Let Gen be a PPT algorithm such that $\mathcal{R}$ is hard with respect to Gen. For any nuPPT algorithm $\mathsf{Adv} = \{\mathsf{Adv}_\lambda\}_{\lambda>0}$, $\mathrm{Prob}[\mathcal{V}(x,\pi) = 1|\ x \leftarrow \mathsf{Gen}(1^\lambda);\ \pi \leftarrow \mathsf{Adv}_\lambda(\mathsf{x})] = 0$.

$\triangle$

We call a NI system HZKPoK if it satisfies both HZK and HPoK.

Definition 17 [NIHZKPoK] A NIHZK is a NI system satisfying HZK and a NIHZKPoK is a NI system satisfying both HZK and HPoK. $\triangle$

2.2.4 O-WI and O-WH

Definition 18 [Witness indistinguishable NI system] A NI system for a polynomial-time relation $\mathcal{R}$ associated with a language L, consisting of a pair $(\mathcal{P}, \mathcal{V})$ of PPT algorithms, is called witness indistinguishable (WI) if it satisfies the following property.

– *Witness indistinguishability (WI):*
For every two sequences $\{(x_\lambda, w_\lambda^0)\}_{\lambda>0}$, $\{(x_\lambda, w_\lambda^1)\}_{\lambda>0}$, such that $\forall \lambda > 0$ $(x_\lambda, w_\lambda^0) \in \mathcal{R}$ and $(x_\lambda, w_\lambda^1) \in \mathcal{R}$ and $|x_\lambda| \geq \lambda$, for every nuPPT distinguisher $\mathcal{D} = \{\mathcal{D}_\lambda\}_\lambda$, for every polynomial $p(\cdot)$, there exists a number $n > 0$ such that for every $\lambda \geq n$, $\mathcal{D}_{|x_\lambda|}$ can distinguish the following two random variables with advantage $< 1/p(|x_\lambda|)$:
  - $\mathcal{P}(x_\lambda, w_\lambda^0; U_{m(|x_\lambda|)})$.
  - $\mathcal{P}(x_\lambda, w_\lambda^1; U_{m(|x_\lambda|)})$.
   (Where $m(\lambda)$ is the number of random coins $\mathcal{P}$ uses on an input x of length λ.)

The above definition can be naturally extended to nuPPT distinguishers with access to an oracle O and in this case we talk about O-WI. $\triangle$

A NI system that satisfies WI is also called one-message (or non-interactive) *ZAP*.

Definition 19 [Harmless witness hiding] A NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathcal{P}, \mathcal{V})$ for a hard polynomial-time relation $\mathcal{R}$ associated to generator Gen is said to be harmless witness hiding (HWH) if there exists a, possibly stateful and randomized, oracle $O(\cdot) \in \mathsf{LegOrHR}^\mathcal{R}$ such that the following holds:

– For all nuPPT algorithms $\mathcal{A}^{O(\cdot)} = \{A_\lambda^{O(\cdot)}\}_{\lambda>0}$ with access to $O(\cdot)$, the quantity $\epsilon(\lambda) \stackrel{\triangle}{=} \mathrm{Prob}[(x, w') \in \mathcal{R}|\ (x, w) \leftarrow \mathsf{Gen}(1^\lambda);\ \pi \leftarrow \mathcal{P}(\mathsf{x}, \mathsf{w});\ \mathsf{w}' \leftarrow \mathcal{A}_\lambda^{O(\cdot)}(\mathsf{x}, \pi)]$ is a negligible function in λ.

$\triangle$

Definition 20 [O-HWH] We say that a NI NI is O-HWH, or simply O-WH, for a hard polynomial-time relation $\mathcal{R}$, for some oracle $O(\cdot)$, if NI satisfies definition 19 with respect to $\mathcal{R}$ and oracle $O(\cdot)$. $\triangle$

Definition 21 [WH] We say that a NI NI is witness hiding (WH) for a hard polynomial-time relation $\mathcal{R}$ if NI satisfies definition 19 with respect to $\mathcal{R}$ and the trivial identity oracle $1(\cdot)$. $\triangle$

Remark 4 [O-WH $\to$ WH] It is easy to see that, for any oracle O, O-WH implies WH as defined above that is equivalent to the traditional notion of witness hiding [FS90]. $\triangle$

Lemma 4 [O-HZK $\to$ O-WH] If a polynomial-time relation $\mathcal{R}$ is hard, then an O-HZK NI $\mathsf{NI} \stackrel{\triangle}{=} (\mathcal{P}, \mathcal{V})$ for $\mathcal{R}$ is O-WH. $\triangle$

Proof. Let $\mathcal{R}$ be hard with respect to generator Gen. Suppose towards a contradiction NI to not be O-WH. Then, there exists a nuPPT algorithm $\mathcal{A}^{O(\cdot)} = \{A_\lambda^{O(\cdot)}\}_{\lambda > 0}$ with access to $O(\cdot)$ such that the quantity $\epsilon(\lambda) \stackrel{\triangle}{=} \mathrm{Prob}[(x, w') \in \mathcal{R}| \ (x, w) \leftarrow \mathsf{Gen}(1^\lambda); \ \pi \leftarrow \mathcal{P}(\mathsf{x}, \mathsf{w}); \ \mathsf{w}' \leftarrow \mathcal{A}_\lambda^{\mathsf{O}(\cdot)}(\mathsf{x}, \pi)]$ is a non-negligible function in λ. Consider the adversary $\mathsf{Adv}'^{O(\cdot)}$ with access to O that, on input x, runs the simulator $\mathsf{Sim}^{O(\cdot)}$ guaranteed by the O-HZK property, to compute an identically (to the proof computed by the prover) distributed proof π and outputs $\mathsf{Adv}^{O(\cdot)}(x, \pi)$. Then, $\epsilon'(\lambda) \stackrel{\triangle}{=} \mathrm{Prob}[(x, w') \in \mathcal{R}| \ (x, w) \leftarrow \mathsf{Gen}(1^\lambda); \ \mathsf{w}' \leftarrow \mathsf{Adv}'^{\mathsf{O}(\cdot)}_\lambda(\mathsf{x})] = \epsilon(\lambda)$ is a non-negligible function in λ, contradicting the fact that $\mathcal{R}$ is hard with respect to Gen. $\triangle$

2.2.5 O-strong-WI

Definition 22 [O-strong-WI] Let O be a, possibly stateful and randomized, oracle and $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ be a proof system for a polynomial-time relation $\mathcal{R}$. We say that NI is O-strong-WI if the following holds.

Let $\{X_\lambda^b\}_{\lambda > 0}, b \in \{0, 1\}$ be two ensembles of distributions such that for any $b \in \{0, 1\}$, X_λ^b outputs a pair $(x_\lambda, w_\lambda) \in \mathcal{R}$ such that $|x_\lambda| \geq \lambda$. Suppose that for every O-aided nuPPT adversary $\mathsf{Adv}^{O(\cdot)} = \{\mathsf{Adv}_\lambda^{O(\cdot)}\}_{\lambda > 0}$, for every polynomial $p(\cdot)$, there exists a number $n > 0$ such that for every $\lambda \geq n$,

$$\left| \mathrm{Prob}[\mathsf{Adv}_\lambda^{O(\cdot)}(x_\lambda^0) = 1| \ (x_\lambda^0, w_\lambda^0) \leftarrow X_\lambda^0] - \mathrm{Prob}[\mathsf{Adv}_\lambda^{O(\cdot)}(x_\lambda^1) = 1| \ (x_\lambda^1, w_\lambda^1) \leftarrow X_\lambda^1] \right| \leq 1/p(|x_\lambda|).$$

Then, for every nuPPT (non-oracle) adversary $\mathcal{B} = \{\mathcal{B}_\lambda\}_{\lambda > 0}$, for every polynomial $p(\cdot)$, there exists a number $n > 0$ such that for every $\lambda \geq n$,

$$\left| \mathrm{Prob}[\mathcal{B}_\lambda(x_\lambda^0, \mathcal{P}(x_\lambda^0, w_\lambda^0)) = 1| \ (x_\lambda^0, w_\lambda^0) \leftarrow X_\lambda^0] - \right.$$

$$\left. \mathrm{Prob}[\mathcal{B}_\lambda(x_\lambda^1, \mathcal{P}(x_\lambda^1, w_\lambda^1)) = 1| \ (x_\lambda^1, w_\lambda^1) \leftarrow X_\lambda^1] \right| \leq 1/p(|x_\lambda|).$$

That is, O-strong-WI relaxes strong-WI [Gol01] by quantifying over distributions $\{X_\lambda^b\}_{\lambda > 0}, b \in \{0, 1\}$ that are computationally indistinguishable by O-aided nuPPT adversaries. $\triangle$

Remark 5 It is easy to see that the standard definition of strong-WI is equivalent to 1-strong-WI for the trivial identity oracle $1(\cdot)$. $\triangle$

The following corollary follows straightforward from the definition of O-HZK.

Corollary 5 Let O be an oracle and $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ be an O-HZK system for a polynomial-time relation $\mathcal{R}$. Them, NI is O-strong-WI. $\triangle$

2.2.6 O-FH. An alternative definition of privacy for NI systems, that we call O-function (or feature) hiding (O-FH) and is implied by O-HZH and turns out to be equivalent to HZK in the case of single witness relations, is the following.

Definition 23 [O-FH] A NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ for a polynomial-time relation $\mathcal{R}$ is O-function (or feature) hiding (O-FH) if there exists a, possibly stateful and randomized, oracle $O(\cdot) \in \mathsf{LegOr}^{\mathcal{R}}$ such that the following holds.

For any possibly randomized function f with finite range R, for any nuPPT algorithm $\mathsf{Adv} \stackrel{\triangle}{=} \{\mathsf{Adv}_\lambda\}_{\lambda > 0}$, for any pair $(x, w) \in \mathcal{R}$, let $P_{x,w,f,\mathsf{Adv}}$ the following quantity:

$$\sum_{y \in R} \left| \mathrm{Prob}[\mathsf{Adv}_{|x|}(x, \pi; r) = y \mid r, s \leftarrow \{0,1\}^{d(|x|)}; \ \pi \leftarrow \mathcal{P}(x, w; s)] - \mathrm{Prob}[f(x; r) = y \mid r \leftarrow \{0,1\}^{d(|x|)}] \right|,$$

where $d(\lambda)$ is the maximum of the random coins used by $\mathcal{P}$ when invoked on an input of length λ and by Adv_λ.

Then, for any randomized function f with finite range R, for any nuPPT algorithm $\mathsf{Adv} \stackrel{\triangle}{=} \{\mathsf{Adv}_\lambda\}_{\lambda > 0}$ using at most $d(\lambda)$ coins on inputs of length λ, there exists a nuPPT O-aided algorithm $\mathsf{Adv}'^{O(\cdot)} \stackrel{\triangle}{=} \{\mathsf{Adv}_\lambda'^{O(\cdot)}\}_{\lambda > 0}$ such that, for any $(x, w) \in \mathcal{R}$, we have that:

$$\begin{aligned} P_{x,w,f,\mathsf{Adv}} = & \\ & \sum_{y \in R} |\mathrm{Prob}[\mathsf{Adv}_{|x|}'^{O(\cdot)}(x; r||s) = y \mid r \leftarrow \{0,1\}^{d(|x|)}; \ s \leftarrow \{0,1\}^{d'(|x|) - d(|x|)}] \\ & - \mathrm{Prob}[f(x; r) = y \mid r \leftarrow \{0,1\}^{d(|x|)}]|, \end{aligned} \tag{1}$$

where $d'(\lambda) \geq d(\lambda)$ is the maximum of the random coins used by $\mathsf{Adv}'^{O(\cdot)}$ when invoked on an input of length λ. $\triangle$

Definition 24 [FH] A NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ for a polynomial-time relation $\mathcal{R}$ is function (or feature) hiding (FH) if NI is 1-FH for the trivial identity oracle $1(\cdot)$. $\triangle$

Lemma 6 [O-HZH $\iff$ O-FH] A NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ for a polynomial-time relation $\mathcal{R}$ is O-FH if NI is O-HZK. A NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ for a single witness polynomial-time relation $\mathcal{R}$ is O-FH if and only if NI is O-HZK. (See note at the end of the proof.) $\triangle$

Proof. — **If.** Let $\mathsf{Sim}^{O(\cdot)}$ be the O-aided simulator guaranteed by the O-HZK of NI. Let $s(\lambda)$ be the maximum number of coins (including the coins used in all the invocations to its oracle) that $\mathsf{Sim}^{O(\cdot)}$ uses when executed on an input of length λ. For any possibly randomized function f with finite range R, for any nuPPT algorithm $\mathsf{Adv} \triangleq \{\mathsf{Adv}_\lambda\}_\lambda$, for any pair $(x,w) \in \mathcal{R}$, consider the following non-uniform oracle algorithm $\mathsf{Adv}'^{O(\cdot)} \triangleq \{\mathsf{Adv}_\lambda'^{O(\cdot)}\}_{\lambda>0}$ with access to O. The algorithm $\mathsf{Adv}_{|x|}'^{O(\cdot)}$, on input x, computes $\pi \leftarrow \mathsf{Sim}^{O(\cdot)}(x)$ simulating an invocation of Sim to O with its own oracle O, and outputs $\mathsf{Adv}_{|x|}(x,\pi)$. The algorithm $\mathsf{Adv}_{|x|}$ uses $d'(|x|)$ coins where $d'(\lambda)$ is the sum of $s(\lambda)$ and the maximum number of coins $d(\lambda)$ used by Adv_λ.

$$P_{x,w,f,\mathsf{Adv}} \triangleq$$

$$= \sum_{y \in R} |\mathrm{Prob}[\mathsf{Adv}_{|x|}(x,\pi;r) = y|\ r,s \leftarrow \{0,1\}^{d(|x|)};\ \pi \leftarrow \mathcal{P}(x,w;s)]$$

$$- \mathrm{Prob}[f(x;r) = y|\ r \leftarrow \{0,1\}^{d(|x|)}]| =$$

(by the perfect O-HZK of NI)

$$= \sum_{y \in R} |\mathrm{Prob}[\mathsf{Adv}_{|x|}(x,\pi;r) = y|\ r \leftarrow \{0,1\}^{d(|x|)};\ s \leftarrow \{0,1\}^{s(|x|)};\ \pi \leftarrow \mathsf{Sim}^{O(\cdot)}(x;s)]$$

$$- \mathrm{Prob}[f(x;r) = y|\ r \leftarrow \{0,1\}^{d(|x|)}]| =$$

(by the definition of Adv')

$$= \sum_{y \in R} |\mathrm{Prob}[\mathsf{Adv}_{|x|}'^{O(\cdot)}(x,\pi;r||s) = y|\ r \leftarrow \{0,1\}^{d(|x|)};\ s \leftarrow \{0,1\}^{s(|x|)}]$$

$$- \mathrm{Prob}[f(x;r) = y|\ r \leftarrow \{0,1\}^{d(|x|)}]|,$$

(2)

as it was to show.

– **Only if (in the case of single witness relations).** Let L the language associated with the single witness polynomial-time relation $\mathcal{R}$. Consider the randomized function $f(x;r)$ that, on input a string x, outputs $\perp$ if $x \notin L$, and $\mathcal{P}(x,w;r)$, where w is such that $(x,w) \in \mathcal{R}$, otherwise. That is:

$$f(x;r) \triangleq \left\{ \begin{array}{ll} \perp, & \text{if } x \notin L \\ \mathcal{P}(x,w;r), & \text{where } (x,w) \in \mathcal{R}, \text{ otherwise.} \end{array} \right\}$$

Note that the function is well-defined as $\mathcal{R}$ is a single witness relation, so for any $x \in L$, there is a *unique* witness w such that $(x,w) \in \mathcal{R}$. Let R be the range of f.

Consider an adversary Adv that, on input (x,π), just outputs π and uses at most $d(|x|)$ random coins. By O-FH, there exists a nuPPT oracle algorithm $\mathsf{Adv}'^{O(\cdot)} \triangleq \{\mathsf{Adv}_\lambda'^{O(\cdot)}\}_{\lambda>0}$ with oracle access to O such that, for any $(x,w) \in$

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$\mathcal{R}$, $P_{x,w,f,\mathsf{Adv}}$ is equal to the following quantity:

$$\sum_{y \in R} |\mathrm{Prob}[\mathsf{Adv}'^{O(\cdot)}_{|x|}(x;r||s) = y|\ r \leftarrow \{0,1\}^{d(|x|)};\ s \leftarrow \{0,1\}^{d'(|x|)-d(|x|)}] \tag{3}$$
$$- \mathrm{Prob}[f(x;r) = y|\ r \leftarrow \{0,1\}^{d(|x|)}]|.$$

We have that:

$$P_{x,w,f,\mathsf{Adv}} \stackrel{\triangle}{=}$$

$$\sum_{y \in R} |\mathrm{Prob}[\mathsf{Adv}_{|x|}(x,\pi;r) = y|\ r,s \leftarrow \{0,1\}^{d(|x|)};\ \pi \leftarrow \mathcal{P}(x,w;s)]$$

$$- \mathrm{Prob}[f(x;r) = y|\ r \leftarrow \{0,1\}^{d(|x|)}]| =$$
(by the definition of Adv)

$$= \sum_{y \in R} |\mathrm{Prob}[\pi = y|\ s \leftarrow \{0,1\}^{d(|x|)};\ \pi \leftarrow \mathcal{P}(x,w;s)]$$

$$- \mathrm{Prob}[f(x;r) = y|\ r \leftarrow \{0,1\}^{d(|x|)}]| =$$
(by the definition of π)

$$= \sum_{y \in R} |\mathrm{Prob}[\mathcal{P}(x,w;s) = y|\ s \leftarrow \{0,1\}^{d(|x|)}]$$

$$- \mathrm{Prob}[f(x;r) = y|\ r \leftarrow \{0,1\}^{d(|x|)}]| =$$
(by the definition of f)

$$= \sum_{y \in R} \left|\mathrm{Prob}[\mathcal{P}(x,w;s) = y|\ s \leftarrow \{0,1\}^{d(|x|)}] - \mathrm{Prob}[\mathcal{P}(x,w;s) = y|\ s \leftarrow \{0,1\}^{d(|x|)}]\right| =$$

$$= 0.$$
$$\tag{4}$$

By equations 3 and 4 and definition of f, we conclude that

$$\sum_{y \in R} |\mathrm{Prob}[\mathsf{Adv}'^{O(\cdot)}_{|x|}(x;r||s) = y|\ r \leftarrow \{0,1\}^{d(|x|)};\ s \leftarrow \{0,1\}^{d'(|x|)-d(|x|)}] \tag{5}$$
$$- \mathrm{Prob}[\mathcal{P}(x,w;s) = y|\ s \leftarrow \{0,1\}^{d(|x|)}]| = 0.$$

Therefore, adversary $\mathsf{Adv}'^{O(\cdot)}$ is an O-aided simulator such that, for any $(x,w) \in \mathcal{R}$, the distribution $\mathsf{Adv}'^{O(\cdot)}_{|x|}(x)$ is distributed identically to the distribution $\mathcal{P}(x,w)$, as it was to show (see next note).

Note. Note that in the proof for the "only if part" we actually constructed a nuPPT simulator whereas in the definition of O-HZK we required the simulator to be PPT. This is an artifact of the definition of O-FH. To formally prove the Lemma, we should change either the definition of O-HZK weakening the simulator to be nuPPT or considering PPT algorithms in the definition of O-FH. To not overburden the presentation, we skip these details.

$\triangle$

Remark 6 It is easy to observe that the equivalence between O-FH (resp. FH) and O-HZK (resp. ZK) holds even in the case of a NI $\mathsf{NI} \stackrel{\triangle}{=} (\mathcal{P}, \mathcal{V})$ for a general polynomial-time relation $\mathcal{R}$ when, for any x, w_1, w_2 such that $(x, w_1) \in \mathcal{R}$ and $(x, w_2) \in \mathcal{R}$, the random variable $\mathcal{P}(x, w_1)$ is identically distributed to $\mathcal{P}(x, w_2)$.

$\triangle$

Remark 7 [O-FH is weaker than O-HZK for relations with multiple witnesses] It is worth observing why the previous equivalence breaks down in the case of relations with multiple witnesses. For simplicity, let us neglect the oracle in the analysis and only consider the relation between FH and ZK for NI proofs. Recall that according to our definition, ZK requires perfect simulation. ZK implies that the simulated proof for a statement x has to be distributed identically to a real proof for *any* pair (x, w) in the relation. Therefore, ZK forces the distribution of proofs for any two pairs $(x, w_1), (x, w_2)$ in the relation to be identical.

In contrast, FH requires that for any randomized function of a statement x, if the output of an adversary, on input x, and a proof for a pair (x, w') in the relation, has distance p from $f(x)$, then there exists a simulator Adv' such that $\mathsf{Adv}'(x)$ is distributed at the same distance p from $f(x)$. Consider the function f_i such that $f_i(x; r) \stackrel{\triangle}{=} \mathcal{P}(x, w_i; r)$, where w_i is the i-th witness (in lexicographical order) to x. Let w' be an arbitrary witness to x. Suppose an adversary Adv, on input x, and a proof π for (x, w') is such that $\mathsf{Adv}(x, \pi)$ has distance p from the distribution $f_i(x) = \mathcal{P}(x, w_i)$. Then, FH implies the existence of a simulator Sim_i such that $\mathsf{Sim}_i(x)$ has distance p from $\mathcal{P}(x, w_i)$. Notice that the simulator guaranteed by FH can depend on the function and thus on the witness w_i!

As a consequence, unlike ZK, FH does *not* force the output distribution of proofs for different witnesses to be identical.

Similar considerations would apply to formulations of FH and ZK for interactive proof systems.

$\triangle$

Corollary 7 If $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ is a NI system for a single witness polynomial-time relation $\mathcal{R}$ associated with a non-easy language in the sense of Lemma 2, then NI is not FH.

$\triangle$

Proof. This follows from Lemma 2 and the fact that a NI system $\mathsf{NI} \stackrel{\triangle}{=} (\mathsf{Prove}, \mathsf{Verify})$ for a single witness polynomial-time relation $\mathcal{R}$ is FH if and only if NI ZK.

$\triangle$

2.3 Multiplicative groups of hidden order

2.3.1 El Gamal over groups of hidden order Let GenRSA be defined as in Section 2.1, i.e., on input the security parameter 1^λ generates elements (N, p, q, g) such that $N = p \cdot q$ is an RSA modulus for security parameter λ, $\gcd((p-1)/2, (q-1)/2) = 1$ and g is a generator of QR_N. Henceforth, we will often denote by m the order of QR_N that, as shown in Section 2.1, equals $\phi(N)/4$.

Assumption 1 [DDH over GenRSA] Let $X_{0,\lambda}$ be the random variable (N, g, h, u, v), with $(N, p, q, g) \leftarrow \mathsf{GenRSA}(1^\lambda)$, $w \leftarrow \mathbb{Z}_m^\star$, $h \stackrel{\triangle}{=} g^w$, $u \leftarrow \mathsf{QR}_N$, $v \stackrel{\triangle}{=} u^w$. Let $X_{1,\lambda}$ be the random variable (N, g, h, u, v), with $(N, p, q, g) \leftarrow \mathsf{GenRSA}(1^\lambda)$, $h, u, v \leftarrow \mathsf{QR}_N$. We say that the Decisional Diffie-Hellman assumption (DDH) holds for generator GenRSA if for every nuPPT algorithm $\mathsf{Adv} = \{\mathsf{Adv}_\lambda\}_{\lambda > 0}$, the following quantity is negligible in λ:

$$|\mathrm{Prob}[\mathsf{Adv}_\lambda(x) = 1|\ x \leftarrow X_{0,\lambda}] - \mathrm{Prob}[\mathsf{Adv}_\lambda(X_1) = 1|\ x \leftarrow X_{1,\lambda}]|\,.$$

DDH holds if it holds for generator GenRSA. $\triangle$

We assume the reader have familiarity with the notion of public key encryption scheme. We define an exponential El Gamal encryption scheme over the group QR_N and message space $\mathcal{M} \stackrel{\triangle}{=} \{0, 1, \ldots, d\}$ where d is an integer such that for any $M \in \mathcal{M}$, M can be computed by g^M in polynomial-time.

Definition 25 [El Gamal over groups of hidden order] Our El Gamal encryption scheme $\mathsf{ElGamal} = (\mathsf{Setup}, \mathsf{Enc}, \mathsf{Dec})$ over message space $\mathcal{M}$ is a tuple of 3 PPT algorithms.

$\mathsf{Setup}(1^\lambda)$: on input the security parameter λ it runs $(N, q, q, g) \leftarrow \mathsf{GenRSA}(1^\lambda)$, computes $h = g^w$ for randomly chosen integer in $\mathbb{Z}_{\phi(N)}^\star$ and outputs *public key* $\mathsf{pk} \stackrel{\triangle}{=} (N, g, h)$ and *secret key* $\mathsf{sk} \stackrel{\triangle}{=} (\mathsf{p}, \mathsf{q}, \mathsf{w})$.

$\mathsf{Enc}(\mathsf{pk}, M)$: on input public key $\mathsf{pk} \stackrel{\triangle}{=} (N, g, h)$ and a message $M \in \mathcal{M}$, outputs *ciphertext* $\mathsf{ct} \stackrel{\triangle}{=} (g^r, h \cdot g^M)$.

$\mathsf{Dec}(\mathsf{sk}, \mathsf{ct})$: on input secret key $\mathsf{sk} \stackrel{\triangle}{=} (\mathsf{p}, \mathsf{q}, \mathsf{w})$ and ciphertext $\mathsf{ct} \stackrel{\triangle}{=} (\mathsf{ct}_1, \mathsf{ct}_2)$, compute $y = \mathsf{ct}_2 \cdot \mathsf{ct}_1^{-w}$ and, by brute force search over all elements $M \in \mathcal{M}$ until an element M such that $y = g^M$ is found and in such case output M; if no such element can be found, output $\perp$.
When it is clear from the context, we sometimes assume Dec to just output y, that is Dec computes y as above and outputs it.

$\triangle$

It is easy to see that the following facts hold.

Fact 8 $\mathsf{ElGamal}$ satisfies *(perfect) correctness*, that is that for all $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda)$, all $M \in \mathcal{M}$, for all $\mathsf{ct} \leftarrow \mathsf{Enc}(\mathsf{pk}, M)$, $\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) = \mathsf{M}$. $\triangle$

Fact 9 If DDH holds, then $\mathsf{ElGamal}$ is *indistinguishable against chosen message attack* (IND-CPA), that is for every 2 messages $M, M' \in \mathcal{M}$, and every PPT adversary $\mathcal{A}$, the following quantity is negligible in λ:

$$\left|\mathrm{Prob}[\mathcal{A}(\mathsf{Enc}(\mathsf{pk}, M)) = 1 \mid \mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda))] - \mathrm{Prob}[\mathcal{A}(\mathsf{Enc}(\mathsf{pk}, M')) = 1 \mid \mathsf{pk} \leftarrow \mathsf{Setup}(1^\lambda))]\right|\,.$$

$\triangle$

Definition 26 [Operations on El Gamal ciphertexts]

- Multiplication of El Gamal ciphertexts. Let $\mathsf{ct}_1 \stackrel{\triangle}{=} (\mathsf{ct}_1^l, \mathsf{ct}_1^r), \mathsf{ct}_2 \stackrel{\triangle}{=} (\mathsf{ct}_2^l, \mathsf{ct}_2^r)$ be two El Gamal ciphertexts for the same public key pk. We denote by $\mathsf{ct}_1 * \mathsf{ct}_2$ the ciphertext $(\mathsf{ct}_1^l \cdot \mathsf{ct}_2^l, \mathsf{ct}_2^r \cdot \mathsf{ct}_2^r)$, that is we multiply the two ciphertexts entry by entry.
- Exponentiation of an El Gamal ciphertext to a constant. Let y be an integer and $\mathsf{ct} \stackrel{\triangle}{=} (\mathsf{ct}^l, \mathsf{ct}^r)$ an El Gamal ciphertext. We denote by ct^y the ciphertext $((\mathsf{ct}^l)^y, (\mathsf{ct}^r)^y))$, that is we exponentiate each entry of the ciphertext to y.

$\triangle$

2.3.2 Our relations $\mathcal{R}_{\mathsf{DDH}}$ and $\mathcal{R}_{\mathsf{SG}}$

Definition 27 [$\mathcal{R}_{\mathsf{DDH}}$] The relation of well-formedness of Diffie-Hellman (DH) tuples is defined as follows: $\mathcal{R}_{\mathsf{DDH}}((N, g, h, u, v), (w, [p_i, m_i]_{i=1}^l)) = 1$ iff $g, h, u, v \in \mathbb{Z}_N$ and $u = g^w, v = h^w$ for some non-negative integer $w < \phi(N)$ (with group operation being the multiplication modulo N) and $N = \prod_{i=1}^l p_i^{m_i}$. $\triangle$

Definition 28 [$\mathcal{R}_{\mathsf{SG}}$] The relation of *subgroup membership* between two group elements is defined as follows: $\mathcal{R}_{\mathsf{SG}}((N, g, u), (w, [p_i, m_i]_{i=1}^l)) = 1$ iff $g, u \in \mathbb{Z}_N$ and $u = g^w$ for some non-negative integer $w < \phi(N)$ (with group operation being the multiplication modulo N) and $N = \prod_{i=1}^l p_i^{m_i}$. $\triangle$

Remark 8 We refer the reader to Section 1.3 for a discussion about the need of the factorization of the modulus in the definition of our relations. Note also that the relations do not guarantee N to have the correct form or g to be in QR_N. If this is not the case, $\mathsf{ElGamal}$ might be not secure.

The definition of $\mathcal{R}_{\mathsf{DDH}}$ is very general, in particular it is trivially satisfied for particular choices of g and h (e.g., when g and h have co-prime orders). In relevant applications, the relation will be used for g and h belonging to the same subgroup or in conjunction with a proof for $\mathcal{R}_{\mathsf{SG}}$ to guarantee g and h to be in the same subgroup. $\triangle$

2.3.3 Our main oracle DHInvO

Definition 29 [Oracle DHInvO] The oracle DHInvO takes as input a tuple (N, g, h, u, v) and checks whether $u = g^w$ and $v = h^w$ for some $w \in \mathbb{Z}_{\phi(N)}^\star$; if such value w does not exist, it outputs $\perp$ to indicate an error; otherwise, it outputs $(g^r, h^r, r^{-1} \mod \phi(N), (r+w)^{-1} \mod \phi(N))$, with w being an integer $< \mathrm{ord}(g)$ such that $u = g^w, v = h^w$ and $r \leftarrow \mathbb{Z}_{\phi(N)}^\star$ under the constraint that r be a prime number and $(r+w) \mod \phi(N) \in \mathbb{Z}_{\phi(N)}^\star$. $\triangle$

Remark 9 Actually, our proof systems, as they are described, satisfy statistical completeness because, e.g., the prover has to find random values that are invertible under some constraint. As our provers might incur a statistical completeness

error, in order to get perfect simulation the simulator should err as the prover. Hence, we implicitly assume for all our NIs that if a prover or simulator fails in some bounded polynomial time to find values satisfying the constraints, it outputs an error symbol (different from $\bot$), e.g., $(0, 0, 0, 0)$. We believe that our provers (and thus the corresponding simulators) can be changed so as to enjoy perfect completeness but we did not investigate the details. $\triangle$

Assumption 2 The assumption states that $\mathsf{DHInvO} \in \mathsf{LegOr}^{\mathcal{R}_{\mathsf{DDH}}}$ and $\mathsf{DHInvO} \in \mathsf{LegOr}^{\mathcal{R}_{\mathsf{SG}}}$. That is, DHInvO is a legitimate oracle for both the polynomial-time relations $\mathcal{R}_{\mathsf{DDH}}$ and $\mathcal{R}_{\mathsf{SG}}$. $\triangle$

Note that the latter assumption basically accounts to say that an efficient adversary cannot distinguish a DH tuple from a non-DH tuple seeing additionally proofs of well-formedness of valid DH tuples of its choice. See also Remark 1.

2.3.4 Hardness assumptions

Assumption 3 Let $\mathsf{GenDDH}(1^\lambda)$ be the generator that, on input security parameter 1^λ, computes $(N, p, q, g) \leftarrow \mathsf{GenRSA}(1^\lambda)$ and outputs statement (N, g, g^r, g^w, g^{rw}) and witness (w, p, q), with $r, w \leftarrow \mathbb{Z}^\star_{\phi(N)}$. The assumption states that $\mathcal{R}_{\mathsf{DDH}}$ is hard with respect to GenDDH. $\triangle$

Definition 30 The factoring oracle FactO takes as input N and outputs the list of all prime factors of N. $\triangle$

Assumption 4 Let GenSG be the following PPT algorithm. The algorithm, GenSG on input the security parameter 1^λ computes $(N, p, q, g) \leftarrow \mathsf{GenRSA}(1^\lambda)$, and $u = g^w$, with $w \leftarrow \mathbb{Z}^\star_{\phi(N)}$, and outputs statement $x \stackrel{\triangle}{=} (N, g, u)$ and witness (w, p, q). The assumption states that $\mathcal{R}_{\mathsf{DDH}}$ and $\mathcal{R}_{\mathsf{SG}}$ are hard with respect to GenSG and FactO. Thus, FactO is a legitimate oracle for extraction for both the polynomial-time relations $\mathcal{R}_{\mathsf{DDH}}$ and $\mathcal{R}_{\mathsf{SG}}$, that is $\mathsf{FactO} \in \mathsf{LegOrHR}^{\mathcal{R}_{\mathsf{DDH}}}$ and $\mathsf{FactO} \in \mathsf{LegOrHR}^{\mathcal{R}_{\mathsf{SG}}}$. $\triangle$

We assume that the reader is familiar with the notion of cryptographic games.

Assumption 5 Let $\mathsf{ElGamal} = (\mathsf{Setup}, \mathsf{Enc}, \mathsf{Dec})$ be the encryption scheme as in definition 25 and DHInvO the oracle as in Definition 29. The assumption holds if no nuPPT adversary $\mathsf{Adv}^{\mathsf{DHInvO}} = \{\mathsf{Adv}^{\mathsf{DHInvO}}_\lambda\}_{\lambda > 0}$ with access to DHInvO can win with non-negligible advantage in λ in the following game between Adv and a challenge $\mathcal{C}$. For security parameter λ, the game is the following.

- **Setup Phase.** The challenger $\mathcal{C}$ picks a pair $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda)$ and gives pk to Adv_λ which outputs two pair of messages m_0 and m_1 under the constraint that $m_0, m_1 \in \{0, 1\}$.
- **Challenge Phase.** The challenger picks a random bit $b \leftarrow \{0, 1\}$ and gives to Adv_λ a ciphertext ct encrypting m_b.

- **Winning Condition.** Adv_λ outputs a bit b' and wins iff $b' = b$ and DHInvO is never invoked on a input y such that $\mathsf{DHInvO}(y) = \bot$.

$\triangle$

Assumption 6 Let $\mathsf{ElGamal} = (\mathsf{Setup}, \mathsf{Enc}, \mathsf{Dec})$ be the encryption scheme as in Def. 25 and let NIDDH be the NIHZK system for $\mathcal{R}_{\mathsf{DDH}}$ of Section 3. The assumption holds if no nuPPT adversary $\mathsf{Adv} = \{\mathsf{Adv}_\lambda\}_{\lambda>0}$ can win with non-negligible advantage in λ in the following game between Adv and a challenge $\mathcal{C}$. For security parameter λ, the game is the following.

- **Setup Phase.** The challenger $\mathcal{C}$ picks a pair $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Setup}(1^\lambda)$ and gives pk to Adv_λ which outputs two pair of messages $(v_{0,0}, v_{0.1})$ and $(v_{1,0}, v_{1,1})$ under the constraint that $v_{0,0} + v_{0,1} = v_{1,0} + v_{1,1} = 0$.
- **Challenge Phase.** The challenger picks a random bit $b \leftarrow \{0, 1\}$ and gives to Adv_λ two ciphertexts $\mathsf{ct}_0, \mathsf{ct}_1$ encrypting respectively $v_{b,0}, v_{b,1}$ along with a proof π of the fact that $\mathsf{ct} \stackrel{\triangle}{=} \mathsf{ct}_1 * \mathsf{ct}_2$ decrypts to 0 computed with NIDDH using statement ct and witness sk.
- **Winning Condition.** Adv_λ outputs a bit b' and wins iff $b' = b$.

$\triangle$

The following assumption and its relation with the previous one are due to Geoffroy Couteau; it basically states that it is difficult to distinguish a random DH tuple for witness w from a random tuple given additionally another random DH tuple for the same witness w and access to DHInvO that can be only invoked on valid DH tuples.

Assumption 7 The assumption holds if no nuPPT adversary $\mathsf{Adv}^{\mathsf{DHInvO}(\cdot)} = \{\mathsf{Adv}_\lambda^{\mathsf{DHInvO}(\cdot)}\}_{\lambda>0}$ can win with non-negligible advantage in λ in the following game between $\mathsf{Adv}^{\mathsf{DHInvO}(\cdot)}$ and a challenge $\mathcal{C}$. For security parameter λ, the game is the following.

The challenger $\mathcal{C}$ picks an instance $(N, p, q, g) \leftarrow \mathsf{GenRSA}(1^\lambda)$, random values $w, a \leftarrow \mathbb{Z}_{\phi(N)}^\star, h, h', Z_1 \leftarrow \mathsf{QR}_N$, a random bit $b \in \{0, 1\}$, sets $A = g^a$, $Z' = h'^w$, $Z_0 = A^w$ and runs $\mathsf{Adv}_\lambda^{O(\cdot)}$ on input $(N, g, h, A, Z_b, h', Z')$.

Adv_λ outputs a bit b' and wins iff $b' = b$ and Adv_λ never queried the oracle on an input y such that $\mathsf{DHInvO}(y) = \bot$. $\qquad \triangle$

Lemma 10 If there exists an adversary $\mathsf{Adv}^{\mathsf{DHInvO}(\cdot)}$ breaking Assumption 6 with advantage ϵ, there exists an adversary $\mathcal{B}^{\mathsf{DHInvO}(\cdot)}$ breaking Assumption 7 with advantage 2ϵ. $\qquad \triangle$

Proof. We consider the following series of hybrid experiments.

- H_0. Hybrid H_0 is identical to the game of Assumption 7 except that the bit $b = 0$ instead of being chosen randomly.
- H_1. Hybrid H_1 is identical to H_0 except that $v_{0,0}$ is a random message in $\mathbb{Z}_{\phi(N)}$ and $v_{0,1} = -v_{0,0}$.

– H_0. Hybrid H_0 is identical to the game of Assumption 7 except that the bit $b = 1$ instead of being chosen randomly.

W.l.o.g., we will show that if Adv can distinguish H_0 from H_1 with advantage ϵ, there exists an adversary $\mathcal{B}$ breaking Assumption 7 with advantage ϵ. This will prove the lemma.

The adversary $\mathcal{B}$ receives as input a tuple (g, h, A, Z, h', Z') and works as follows. The adversary $\mathcal{B}$ runs Adv on input the public key (N, g, h) and gets from Adv two pairs of messages $(v_{0,0}, v_{0.1})$ and $(v_{1,0}, v_{1,1})$ such that $v_{0,0} + v_{0,1} = v_{1,0} + v_{1,1} = 0$ (if this is not the case $\mathcal{B}$ aborts outputting an arbitrary bit).

The adversary $\mathcal{B}$ sets $\mathsf{ct}_0 = (A, Z \cdot g^{v_{0,0}})$, $\mathsf{ct}_1 = (h' \cdot A^{-1}, Z' \cdot Z^{-1} \cdot g^{-v_{0,0}})$. It is easy to verify that the "product" of ct_0 and ct_1 is the ciphertext (h', Z') encrypting 0. Therefore, $\mathcal{B}$ invokes the oracle DHInvO on input (N, g, h, h', Z') and returns its output to Adv as proof π of the fact that (h', Z') encrypts 0. By definition of the oracle, π has the right distribution in both the experiments.

Notice that if $Z = A^w$, ct_0 encrypts $v_{0,0}$ and ct_1 encrypts $-v_{0,0}$, that is $\mathcal{B}$ simulated to Adv the experiment H_0. On the other hand, if Z is a random element in QR_N, ct_0 encrypts a random message $v_{0,0}$ and ct_1 encrypts $-v_{0,0}$, that is $\mathcal{B}$ simulated to Adv the experiment H_1. This concludes the proof. $\triangle$

Remark 10 Assumption 6 is also equivalent (the reduction goes both ways) to a variant of Assumption 7 in which the adversary is restricted to call the oracle DHInvO only a certain bounded number of times. $\triangle$

3 Our HZKPoK proofs for subgroup membership and DH tuples

In this Section we present our main HZPoKs for the relations $\mathcal{R}_{\mathsf{SG}}$ (cf. Def. 28) and $\mathcal{R}_{\mathsf{DDH}}$ (cf. Def. 27).

3.1 HZKPoK for $\mathcal{R}_{\mathsf{SG}}$

We firstly construct a HZKPoK for polynomial-time relation $\mathcal{R}_{\mathsf{SG}}$ that will be used as building block in the HZKPoK for $\mathcal{R}_{\mathsf{DDH}}$.

Construction 1 The NI system $\mathsf{NISG} = (\mathsf{ProveSG}, \mathsf{VerifySG})$ for polynomial-time relation $\mathcal{R}_{\mathsf{SG}}$ (cf. Def. 28) consists of the following algorithms.

Note: In the following, we let $s \overset{\triangle}{=} |N|$. We will later show how to optimize the parameter s.

$\mathsf{ProveSG}$, on inputs statement (N, g, u) and witness $(w, [p_i, m_i]_{i=1}^l)$ for $\mathcal{R}_{\mathsf{SG}}$, computes the following proof. We assume w to be the integer $< \mathrm{ord}(g)$ such that $u = g^w$. If this is not the case, the prover finds the value w' with such property and executes the below protocol with witness w'. We skip this detail in the algorithm description.

Algorithm ProveSG:
Inputs: statement (N, g, u) and witness $(w, [p_i, m_i]_{i=1}^l)$ for $\mathcal{R}_{\mathsf{SG}}$.

- **For each** $i \in [s]$, **do**
  - $r_i \leftarrow \mathbb{Z}_{\phi(N)}^\star, z_i = r_i + w \mod \phi(N)$ under the constraint that r_i be a prime number and $(r_i + w) \mod \phi(N) \in \mathbb{Z}_{\phi(N)}^\star$.
  - $r_i' = r_i^{-1} \mod \phi(N), z_i' = z_i^{-1} \mod \phi(N)$.
  - $R_i = g^{r_i}$.
- **endFor**
- **Output** $(R_i, r_i', z_i')_{i \in [s]}$.

VerifySG, on inputs statement (N, g, u) and proof $(R_i, r_i', z_i')_{i \in [s]}$, outputs a binary decision, 0 to denote rejection and 1 to denote acceptance.

Algorithm VerifySG:
Inputs: statement (N, g, u) and proof $(R_i, r_i', z_i')_{i \in [s]}$.

1. **If** $g^{\prod_{i \in [s]} r_i'} = 1$ **then Return** 0.
2. **For each** $i \in [s]$, **do**
   (a) **If** r_i' is not a prime number, **then Return** 0.
   (b) **If** $R_i^{r_i'} \neq g$ **then Return** 0.
   (c) $H_i = R_i \cdot u$.
   (d) **If** $H_i^{z_i'} \neq g$ **then Return** 0.
3. **endFor**
4. **If** $\exists i, j \in [s], i \neq j, r_i' = r_j'$ **then Return** 0.
5. **Output** 1.

$\triangle$

Theorem 11 The NI system $\mathsf{NISG} = (\mathsf{ProveSG}, \mathsf{VerifySG})$ for polynomial-time relation $\mathcal{R}_{\mathsf{SG}}$ of Construction 1 is complete and perfectly sound. $\triangle$

Proof. The proof for completeness is trivial; see Remark 9.

Let us analyze soundness. Suppose that VerifySG, on inputs statement (N, g, u) and proof $(R_i, r_i', z_i')_{i \in [s]}$, outputs 1 (i.e., it accepts the proof). We will prove that $u = g^w$ for some $w < \phi(N)$ (the order of $\mathbb{Z}_N^\star$) and thus $\mathcal{R}_{\mathsf{SG}}((N, g, u), (w, [p_i, m_i]_{i=1}^l)) = 1$, with $N = \prod_{i=1}^l p_i^{m_i}$.

We firstly argue there exists at least one index $j \in [s]$ such that r_j' is co-prime with $\phi(N)$. Suppose towards a contradiction this to be false, that is, for each $i \in [s], r_i'$ has a common factor with $\phi(N)$. By check 2.a, for each $i \in [s], r_i$ is prime and, by check 4, the primes $r_1', \ldots, r_s'$ are all different. Then, since $s = |N|$, the product $t \stackrel{\triangle}{=} \prod_{i \in [s]} r_i'$ has to be a multiple of $\phi(N)$. Since $\phi(N)$ is the order of $\mathbb{Z}_N^\star$ and the order of each group element divides the order of the group, we

have that $g^t = 1$ and hence in check 1 VerifySG refuses the proof, contradicting the hypothesis.

Let $j \in [s]$ be such that r'_j is co-prime with $\phi(N)$. Check 2.b implies $R_j^{r'_j} = g$. Being r'_j co-prime with $\phi(N)$, it has an inverse modulo $\phi(N)$. Let $y \triangleq r_j^{-1}$ mod $\phi(N)$. Then, powering both sides of the equation $R_j^{r'_j} = g$ to y, we have $R_j = g^y$. Analogously, check 2.d implies $H_i = g^{y_2}$ for some $y_2 < \phi(N)$.

Then, by step 3.b, $H_i = R_i \cdot u$ and this implies $u = H_i \cdot R_i^{-1} = g^{y_2} \cdot g^{-y} = g^{y_2 - y}$, as we had to prove. $\triangle$

Theorem 12 Let DHInvO be the oracle of Def. 29. If Assumption 2 holds, then the NI system NISG $=$ (ProveSG, VerifySG) of Construction 1 is DHInvO-HZK for polynomial-time relation $\mathcal{R}_{\mathsf{SG}}$. $\triangle$

Proof. By Assumption 2, DHInvO is a legitimate oracle for polynomial-time relation $\mathcal{R}_{\mathsf{SG}}$. What is left to prove is to show a PPT simulator algorithm for NISG satisfying (perfect) HZK. Consider the following simulator $\mathsf{SimSG}^{\mathsf{DHInvO}(\cdot)}$ with oracle access to DHInvO$(\cdot)$. The simulator takes as input a statement (N, g, u) and computes a simulated proof as follows with the help of the oracle DHInvO.

Algorithm SimSG:
Inputs: statement (N, g, u).
Oracle: DHInvO.

– **For each** $i \in [s]$, **do**
  • $(R_i, Y_i, r'_i, z'_i) = \mathsf{DHInvO}(N, g, g, u, u)$.
– **endFor**
– **Output** $(R_i, r'_i, z'_i)_{i \in [s]}$.

By the definition of DHInvO, it is easy to see that for each $i \in [s]$, the tuple (R_i, r'_i, z'_i) in the output of the oracle on input (N, g, u, g, u) has the same distribution as a tuple (R_i, r'_i, z'_i) output by the prover; see Remark 9. Then, the output of $\mathsf{SimSG}^{\mathsf{DHInvO}(\cdot)}(N, g, u)$ is distributed identically to the output of $\mathsf{ProveSG}((N, g, u), (w, p, q))$, where $N = p \cdot q$ and $u = g^w$. $\triangle$

Theorem 13 Let FactO be the oracle of Def. 30. If Assumption 4 holds, then the NI system NISG $=$ (ProveSG, VerifySG) of Construction 1 is FactO-HPoK for polynomial-time relation $\mathcal{R}_{\mathsf{SG}}$. $\triangle$

Proof. By Assumption 4, $\mathcal{R}_{\mathsf{SG}}$ is hard with respect to FactO, that is FactO $\in$ LegOrHR$^{\mathcal{R}_{\mathsf{SG}}}$. What is left to prove is to show a PPT extractor $\mathsf{ExtSG}^{\mathsf{FactO}(\cdot)}$ with oracle access to FactO$(\cdot)$ such that the following holds: for any strings $x, \pi \in \{0, 1\}^\star$, if $\mathsf{VerifySG}(x, \pi) = 1$ then $\mathrm{Prob}[(x, w) \in \mathcal{R}_{\mathsf{SG}}(x, w) | w \leftarrow \mathsf{ExtSG}^{\mathsf{FactO}(\cdot)}(x)] = 1$.

Consider the following extractor ExtSG with oracle access to FactO. The extractor ExtSG takes as input a statement (N, g, u) and a proof $(R_i, r'_i, z'_i)_{i \in [s]}$

and computes what follows. ExtSG invokes the oracle to factorize N and gets its factorization $[p_i, m_i]_{i=1}^l$ from which it can compute $m \triangleq \phi(N)/4$, the order of $\mathbb{Z}_N^\star$. By construction $z_i' = (r_i'^{-1} \mod \phi(N) + w)^{-1} \mod \phi(N)$ with $z_i' \in \mathbb{Z}_{\phi(N)}^\star$. Using $\phi(N)$, one can invert r_i' to get $y_i \triangleq r_i'^{-1} \mod \phi(N)$, invert z_i' to get $(r_i'^{-1} \mod \phi(N)) + w \mod \phi(N) = y_i + w \mod \phi(N)$, and subtract from the latter value y_i to get $w \mod \phi(N)$ that, along with the factorization, forms a valid witness for $\mathcal{R}_{\mathsf{SG}}$. $\triangle$

3.2 HZKPoK for $\mathcal{R}_{\mathsf{DDH}}$

Construction 2 The NI system $\mathsf{NIDDH} = (\mathsf{ProveDDH}, \mathsf{VerifyDDH})$ for polynomial-time relation $\mathcal{R}_{\mathsf{DDH}}$ (cf. Def. 27) consists of the following algorithms.

Note: In the following, we let $s \triangleq |N|$. We will later show how to optimize the parameter s.

Let $\mathsf{NISG} = (\mathsf{ProveSG}, \mathsf{VerifySG})$ be the NI for polynomial-time relation $\mathcal{R}_{\mathsf{SG}}$ of Construction 1. $\mathsf{ProveDDH}$, on inputs statement (N, g, h, u, v) and witness $(w, [p_i, m_i]_{i=1}^l)$ for $\mathcal{R}_{\mathsf{DDH}}$, computes the following proof. We assume w to be the integer $< \mathrm{ord}(g)$ such that $u = g^w, v = h^w$. If this is not the case, the prover finds the value w' with such property and executes the below protocol with witness w'. We skip this detail in the algorithm description.

Algorithm $\mathsf{ProveDDH}$:
Inputs: statement (N, g, h, u, v) and witness $(w, [p_i, m_i]_{i=1}^l)$ for $\mathcal{R}_{\mathsf{DDH}}$.

- **For each** $i \in [s]$, **do**
    - $r_i \leftarrow \mathbb{Z}_{\phi(N)}^\star, z_i = r_i + w \mod \phi(N)$ under the constraint that r_i be a prime number and $(r_i + w) \mod \phi(N) \in \mathbb{Z}_{\phi(N)}^\star$.
    - $r_i' = r_i^{-1} \mod \phi(N), z_i' = z_i^{-1} \mod \phi(N)$.
    - $X_i = g^{r_i}, Y_i = h^{r_i}$.
- **endFor**
- $\pi_u \leftarrow \mathsf{ProveSG}((N, g, u), (w, p, q))$ and $\pi_v \leftarrow \mathsf{ProveSG}((N, h, v), (w, p, q))$.
- **Output** $(\pi_u, \pi_v, X_i, Y_i, r_i', z_i')_{i \in [s]}$.

$\mathsf{VerifyDDH}$, on inputs statement (N, g, h, u, v) and proof $(\pi_u, \pi_v, (X_i, Y_i, r_i', z_i')_{i \in [s]})$, outputs a binary decision, 0 to denote rejection and 1 to denote acceptance.

Algorithm VerifyDDH:
Inputs: statement (N, g, h, u, v) and proof $(\pi_u, \pi_v, (X_i, Y_i, r_i', z_i')_{i \in [s]})$.

1. **If** $g^{\prod_{i \in [s]} r_i'} = 1$ **then Return** 0.
2. **For each** $i \in [s]$, **do**
   (a) **If** r_i' is not a prime number, **then Return** 0.
   (b) **If** $X_i^{r_i'} \neq g \vee Y_i^{r_i'} \neq h$ **then Return** 0.
   (c) $H_i = X_i \cdot u, Z_i = Y_i \cdot v$.
   (d) **If** $H_i^{z_i'} \neq g \vee Z_i^{z_i'} \neq h$ **then Return** 0.
3. **endFor**
4. **If** $\mathsf{VerifySG}((N, g, u), \pi_u) = 0 \vee \mathsf{VerifySG}((N, h, v), \pi_v) = 0$ **then Return** 0.
5. **If** $\exists i, j \in [s], i \neq j, r_i' = r_j'$ **then Return** 0.
6. **Output** 1.

$\triangle$

Theorem 14 The NI system $\mathsf{NIDDH} = (\mathsf{ProveDDH}, \mathsf{VerifyDDH})$ for polynomial-time relation $\mathcal{R}_{\mathsf{DDH}}$ of Construction 2 is complete and perfectly sound. $\quad \triangle$

Proof. The proof for completeness is trivial; see Remark 9.

Let us analyze soundness. Suppose that $\mathsf{VerifyDDH}$, on inputs statement (N, g, h, u, v) and proof $(\pi_u, \pi_v, (X_i, Y_i, r_i', z_i')_{i \in [s]})$, outputs 1 (i.e., it accepts the proof). We will prove that $u = g^w, v = h^w$ for some $w < \phi(N)$ (the order of $\mathbb{Z}_N^\star$) and thus $\mathcal{R}_{\mathsf{DDH}}((N, g, h, u, v), (w, [p_i, m_i]_{i=1}^l)) = 1$, with $N = \prod_{i=1}^l p_i^{m_i}$.

We firstly argue there exists at least one index $j \in [s]$ such that r_j' is co-prime with $\phi(N)$. Suppose towards a contradiction this to be false, that is, for each $i \in [s]$, r_i' has a common factor with $\phi(N)$. By check 2.a, for each $i \in [s], r_i$ is prime and, by check 5, the primes $r_1', \ldots, r_s'$ are all different. Then, since $s = |N|$, the product $t \stackrel{\triangle}{=} \prod_{i \in [s]} r_i'$ has to be a multiple of $\phi(N)$. Since $\phi(N)$ is the order of $\mathbb{Z}_N^\star$ and the order of each group element divides the order of the group, we have that $g^t = 1$ and hence in check 1 $\mathsf{VerifyDDH}$ refuses the proof, contradicting the hypothesis.

Let $j \in [s]$ be such that r_j' is co-prime with $\phi(N)$. Check 2.b implies $X_j^{r_j'} = g$ (resp. $Y_j^{r_j'} = h$). Being r_j' co-prime with $\phi(N)$, it has an inverse modulo $\phi(N)$. Let $y \stackrel{\triangle}{=} r_j'^{-1} \mod \phi(N)$. Then, powering both sides of the equation $X_j^{r_j'} = g$ (resp. $Y_j^{r_j'} = h$) to y, we have $X_j = g^y$ (resp. $Y_j = h^y$). Analogously, check 2.d implies $H_i = g^{y_2}$ (resp. $Z_i = h^{y_2}$) for some $y_2 < \phi(N)$.

By perfect soundness of NISG, check 4 guarantees that $u = g^{w_1}$ for some $w_1 < \phi(N)$ and $v = h^{w_2}$ for some $w_2 < \phi(N)$. Let k_1 (resp. k_2) be the order of g (resp. h). Then, by step 2.c, $H_i = X_i \cdot u$ and $Z_i = Y_i \cdot v$ and this implies:

$$g^{w_1} = u = H_i \cdot X_i^{-1} = g^{y_2} \cdot g^{-y} = g^{y_2 - y},$$

and
$$h^{w_2} = v = Z_i \cdot Y_i^{-1} = h^{y_2} \cdot h^{-y} = h^{y_2-y}.$$

Taking the discrete logs, resp. in base g and h of the last two equations, we have $w_1 = y_2 - y \mod k_1$ and $w_2 = y_2 - y \mod k_2$. Recall that the system of equations

$$x = a \mod m_1,$$

$$x = b \mod m_2,$$

has a unique solution modulo $m_1 m_2 / \gcd(m_1, m_2)$ if $a = b \mod \gcd(m_1, m_2)$. Thus, there exists integer x such that $x = y_2 - y \mod k_1$ and $x = y_2 - y \mod k_2$ and setting $w \stackrel{\triangle}{=} x \mod \phi(N)$ we have that $u = g^{w_1 \mod k_1} = g^{w \mod k_1} = g^w$ and $v = h^{w_2 \mod k_2} = h^{w \mod k_2} = h^w$, as it was to prove. $\triangle$

Theorem 15 Let DHInvO be the oracle of Def. 29. If Assumption 2 holds, then the NI system $\mathsf{NIDDH} = (\mathsf{ProveDDH}, \mathsf{VerifyDDH})$ of Construction 2 is DHInvO-HZK for polynomial-time relation $\mathcal{R}_{\mathsf{DDH}}$. $\triangle$

Proof. By Assumption 2, DHInvO is a legitimate oracle for polynomial-time relation $\mathcal{R}_{\mathsf{DDH}}$. What is left to prove is to show a PPT simulator algorithm for NIDDH satisfying (perfect) HZK. Let $\mathsf{SimSG}^{\mathsf{DHInvO}(\cdot)}$ be the PPT simulator with oracle access to $\mathsf{DHInvO}(\cdot)$ of Theorem 12. Consider the following simulator $\mathsf{SimDDH}^{\mathsf{DHInvO}(\cdot)}$ with oracle access to $\mathsf{DHInvO}(\cdot)$. The simulator takes as input a statement (N, g, h, u, v) and computes a simulated proof as follows using SimSG and with the help of the oracle DHInvO.

Algorithm SimDDH:
Inputs: statement (N, g, h, u, v).
Oracle: DHInvO.

  – **For each** $i \in [s]$, **do**
    • $(X_i, Y_i, r_i', z_i') = \mathsf{DHInvO}(N, g, h, u, v)$.
  – **endFor**
  – $\pi_u \leftarrow \mathsf{SimSG}^{\mathsf{DHInvO}(\cdot)}((N, g, u)), \pi_v \leftarrow \mathsf{SimSG}^{\mathsf{DHInvO}(\cdot)}(N, h, v)$.
  – **Output** $(\pi_u, \pi_v, (X_i, Y_i, r_i', z_i')_{i \in [s]})$.

By the definition of DHInvO, it is easy to see that for each $i \in [s]$, the tuple (X_i, Y_i, r_i', z_i') output by the oracle on input (N, g, u, h, v) has the same distribution as a tuple (X_i, Y_i, r_i', z_i') output by the prover; see Remark 9. By Theorem 12, the output of $\mathsf{SimSG}^{\mathsf{DHInvO}(\cdot)}(N, g, u)$ (resp. $\mathsf{SimSG}^{\mathsf{DHInvO}(\cdot)}(N, h, v)$) is distributed identically to $\mathsf{ProveSG}((N, g, u), (w, [p_i, m_i]_{i=1}^l))$ (resp. $\mathsf{ProveSG}((N, h, v), (w, [p_i, m_i]_{i=1}^l)))$ for any witness $(w, [p_i, m_i]_{i=1}^l)$ for $\mathcal{R}_{\mathsf{SG}}$.

Then, the output of $\mathsf{SimDDH}^{\mathsf{DHInvO}(\cdot)}(N, g, h, u, v)$ is distributed identically to the output of $\mathsf{ProveDDH}((N, g, h, u, v), (w, [p_i, m_i]_{i=1}^l))$ for any witness $(w, [p_i, m_i]_{i=1}^l)$ for $\mathcal{R}_{\mathsf{DDH}}$. $\triangle$

Corollary 16 If Assumption 2 holds, then the NI system $\mathsf{NIDDH} = (\mathsf{ProveDDH}, \mathsf{VerifyDDH})$ of Construction 2 is WH (cf. Def. 21) for polynomial-time relation $\mathcal{R}_{\mathsf{DDH}}$. $\triangle$

Proof. By Assumption 3, $\mathcal{R}_{\mathsf{DDH}}$ is a hard relation and, by Theorem 15, NIDDH is DHInvO-HZK. By Lemma 4, if a polynomial-time relation $\mathcal{R}$ is hard, then for any oracle $O(\cdot)$, an O-HZK NI $\mathsf{NI} \triangleq (\mathcal{P}, \mathcal{V})$ for $\mathcal{R}$ is O-WH. Thus, NI is WH for $\mathcal{R}$. $\triangle$

Theorem 17 Let FactO be the oracle of Def. 30. If Assumption 4 holds, then the NI system $\mathsf{NISG} = (\mathsf{ProveDDH}, \mathsf{VerifyDDH})$ of Construction 2 is FactO-HPoK for polynomial-time relation $\mathcal{R}_{\mathsf{DDH}}$. $\triangle$

Proof. By Assumption 4, $\mathcal{R}_{\mathsf{DDH}}$ is hard with respect to FactO, that is $\mathsf{FactO} \in \mathsf{LegOrHR}^{\mathcal{R}_{\mathsf{DDH}}}$. What is left to prove is to show a PPT extractor $\mathsf{ExtDDH}^{\mathsf{FactO}(\cdot)}$ with oracle access to $\mathsf{FactO}(\cdot)$ such that the following holds: for any strings $x, \pi \in \{0,1\}^\star$, if $\mathsf{VerifyDDH}(x, \pi) = 1$ then $\mathrm{Prob}[(x, w) \in \mathcal{R}_{\mathsf{DDH}}(x, w) |\ w \leftarrow \mathsf{ExtDDH}^{\mathsf{FactO}(\cdot)}(x)] = 1$.

Let ExtDDH be the extractor with oracle access to FactO for NISG guaranteed by Theorem 13. Consider the following extractor ExtDDH with oracle access to FactO that uses ExtDDH. The extractor ExtDDH takes as input a statement (N, g, h, u, v) and a proof $(\pi_u, \pi_v, (X_i, Y_i, r'_i, z'_i)_{i \in [s]})$ and computes what follows. ExtDDH invokes ExtSG with input (N, g, u) and proof π_u simulating to ExtSG the oracle FactO. ExtDDH outputs the witness output by ExtSG. By Theorem 13, the witness $(w, [p_i, m_i]_{i=1}^l)$ output by ExtSG is such that $\mathcal{R}_{\mathsf{SG}}((N, g, u), (w, [p_i, m_i]_{i=1}^l)) = 1$, that is $u = g^w$. By perfect soundness of NIDDH, $h = g^w$ as well. Then, ExtDDH computes a valid witness. $\triangle$

3.3 Optimizations and a more efficient NIZK proof in the CRS model

Reducing the length of the proof. In the Constructions 1 and 2, the parameter s is set to $|N|$. What is actually needed for the perfect soundness to hold is just having s larger than the possible maximum number of prime factors of an integer N of λ bits.

There is a better upper bound on the maximum number s of prime factors of a composite number N of λ bits. Indeed, if the number of *different* prime factors of an integer N is $\geq s$, then $N \geq s!$ and thus, by Stirling's approximation, $N \geq \sqrt{2\pi} \cdot s^{s+1/2} \cdot e^{-s}$. Taking the discrete log in base 2 of both sides, we have that (1) $\lambda \geq 1.32 \cdot (\lfloor \log_2(s) \rfloor (s + 1/2) - \lfloor \log_2(e) \rfloor \cdot s) \geq 1.32 \cdot s \cdot (\lfloor \log_2(s) \rfloor - 2)$. This equation can be used to compute an integer s such that no integer of λ bits can have s different prime factors, that is to compute an integer s that represents a number of parallel repetitions sufficient to guarantee perfect soundness when the statement is with respect to a modulus N of λ bits. For instance, setting $\lambda = 1024$ (resp. $\lambda = 2048$) in (1), we see that $s = 156$ (resp. $s = 259$) repetitions are sufficient to guarantee perfect soundness.

We can also optimize the paramer s and reduce the length of the proof as follows. The verifier rejects if N has a prime factor of bit length $< k$. In this case, each factor of N has to be of bit length $\geq k$ and so s can be set to $\lceil |N|/k \rceil + 1$. As a consequence, we have a trade-off between the length of the proof and the computational power of the verifier.

NIZK proof in the CRS model. There are two sources of inefficiency in the NI NIDDH. Checking whether a prime number is co-prime with the group order induces the need for parallel repetitions, and additionally checking if an element belongs to the correct group. Can be NIDDH improved moving to the CRS model?

A possibility is to work in the group of signed quadratic residues modulo a Blum integer N [HK09] where one can efficiently check if a group element belongs to the group of signed quadratic residues. The properties of signed quadratic residues are guaranteed when -1 is not a quadratic residue and this is the case when the modulus N is setup honestly as a Blum integer. In the CRS model, the CRS can be set to be a pair consisting of a Blum integer N and a generator of the group of signed quadratic residues modulo N. Moreover, this can be done so that the generator have prime order. Therefore, having a CRS setup in that way, the need for parallel repetitions disappears and checking whether an element belong to the correct group is easy.

Even in the CRS model, to our knowledge there is no known efficient perfectly sound proof system for proving correct decryption of El Gamal ciphertexts over multiplicative groups.

The resulting proof system we obtain moving to the CRS model still seems to require oracle-aided simulation.

4   Conclusions

Since the introduction of zero-knowledge proofs [GMR89], the importance of removing coordination has been recognized as fundamental both from a theoretical point of view and for practical applications like e-voting that require universal verifiability. Unfortunately, one-message zero-knowledge proofs provably do not exist, so non-interactive zero-knowledge proof systems have been proposed subject to some limitations like the existence of a trusted player that setups a shared common reference string [DMP88] or in the so called random oracle model [BR93] or assuming a known bound on the space of the verifier [DPY92].

In this work we have put forth proofs for non-trivial and useful cryptographic relations that (1) can be communicated in one-message and enjoy zero soundness error, that is they are proofs in the mathematical sense, (2) are efficient and (3) satisfy a new privacy notion that we call harmless zero-knowledge. In addition, we presented proofs with the properties (1) and (3) for general NP relations.

Harmless zero-knowledge is rooted in the simulation paradigm and represents a generalization of zero-knowledge in that it allows the simulator to have access

to an oracle relative to which the language is still hard to decide. Essentially, we exploit the fact that in several real-world protocols that use cryptographic proofs, we can assume adversaries to *not* have access to some trapdoor information (e.g., a secret key); restricted to this class of adversaries, a harmless zero-knowledge proof does not leak knowledge that enables the adversary to attack a larger system in which the proof is employed.

For example, taking advantage of the fact that adversaries against the privacy of an encryption scheme do not have access to the secret-key, we can construct a perfectly sound one-message harmless zero-knowledge proof of correct decryption of El Gamal ciphertexts that is not based on any trust assumption.[19] Instead, the soundness of proofs of correct decryption obtained via the FS transform is completely breakable by adversaries discovering a trapdoor, e.g., in the hash function used to instantiate the random oracle. The drawback is that the privacy of the application obtained using our harmless zero-knowledge proof of correct decryption (for instance, the task of distinguishing whether two ciphertexts encrypt $(1, -1)$ or $(-1, 1)$ given a proof that the product ciphertext decrypts to 0) is based on a less-studied oracle-based assumption. Therefore, one has trade a qualitatively different assumption used for the privacy for removing trust and computational assumptions used for the verifiability.

Contrast this state of affairs with other "implementations" of the simulation paradigm. For instance, the variant of zero-knowledge secure in the Universal Composability model [Can01] is stronger and offers more general composability guarantees than standard zero-knowledge but this comes at the cost of limiting the achievability only to the CRS, RO or restricted models. Analogously, zero-knowledge is stronger and can be composed with a larger class of protocols than harmless zero-knowledge but this comes at the cost of sacrificing non-interactivity and zero soundness error. For specific applications, harmless zero-knowledge may be useful, usable and secure.

More in detail, an oracle O induces a set of bad pairs (f, Adv) (see Section 1.3.5) that, roughly speaking, represent the "non-simulatable" adversaries Adv that, given a proof of some statement x, can compute $f(x, w)$ for some witness w to x. Such an adversary may be, for instance, the one that is able to compute the witness to a DH tuple X over $\mathbb{Z}_N^\star$ given a NIDDH proof for X and additionally the factorization of N as auxiliary input. In some applications like e-voting, we can safely assume adversaries to not have access to the factorization, thus in those particular applications such an "attack" is harmless. Therefore, analyzing the security of an O-HZK proof deployed in a larger protocol accounts to studying whether the bad pairs induced by O correspond to real attacks in the protocol.

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[19] Due to the limitations highlighted in Section 1.2.3, we have to use either our proof for NP or to use our efficient proof but assuming the pair (N, g) to be correctly generated (we additionally sketch an alternative solution that requires a change in the encryption scheme). To our knowledge, it was not known how to construct a one-message perfectly sound proof for correct decryption satisfying a non-trivial notion of privacy beyond WI useful and usable in security proofs.

Another way of thinking about about the limitations and power of O-HZK proofs is to look at the fact that O-HZK implies O-strong-WI (cf. Def. 22 and discussion in Section 1.3.9): some distributions of statements X and Y are not computationally indistinguishable by distinguishers with access to the oracle and thus may not hold that (X, π) is computationally indistinguishable (by distinguishers without access to the oracle) from (Y, π). In the analysis of a security protocol, this limitation can be not necessary to prove the security, and so the limitation turns out to be harmless (moreover, without introducing the oracle in the analysis of the security, we could not be able to prove the security). That is, O-strong-WI punctures out a set of "bad" distributions, the ones that are indistinguishable by distinguishers without access to the oracle but are distinguishable by distinguishers with access to the oracle, that are non-simulatable (by simulators without access to the oracle). If such bad distributions never occur in the analysis of a protocol using an O-HZK proof, then O-HZK suffices to prove the security of that particular protocol (and in some applications it may be necessary to consider HZK as non-interactive ZK proofs do not exist). In view of these considerations, it would be interesting to characterize O-HZK in terms of epistemic logic [HPR09].

Our work is far from being a comprehensive or completely satisfactory study of alternative models to zero-knowledge proofs compatible with zero verification error and pure non-interactivity and introduces novel computational assumptions. We hope, however, it can shed light on this intriguing possibility.

Acknowledgments

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