

Attack Beyond-Birthday-Bound MACs in Quantum Setting

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Abstract. The security in the quantum setting of a series of message authentication codes (MACs) with provable beyond-birthday-bound (BBB) security is analyzed in this paper, including SUM-ECBC, PolyMAC, PMAC_Plus, 3kf9 and some variants (2K-ECBC_Plus, GCM-SIV2, 1k-PMAC_Plus, 2K-PMAC_Plus, PMAC_TBC3k and 2kf9). All these MACs have a security proof up to $2^{2n/3}$ (even $2^{3n/4}$) queries assuming the block size of the underlying (tweakable) block cipher is n bits. Given that the adversary can make quantum queries, we consider secret state recovery and partial key recovery attacks against these MACs. Both attacks lead to successful forgeries. For the first one, we apply Grover-meet-Simon algorithm to recover some secret states of SUM-ECBC, PolyMAC, PMAC_Plus, 3kf9 and so on. Our research shows this forgery attack costs at most $\mathcal{O}(2^{n/2}n)$ quantum queries using at most $\mathcal{O}(n^2)$ qubits. For the second one, we apply Grover’s algorithm to recover partial keys of PMAC_Plus, 3kf9, PMAC_TBC3k and so on. Our research shows this forgery attack costs $\mathcal{O}(2^{m/2})$ quantum queries and $\mathcal{O}(m + n^2)$ qubits assuming the size of one key is m bits. As far as we know, these are the first quantum attacks against BBB MACs. Our results show that in the quantum setting their securities go back to birthday bounds.

Keywords: Beyond-Birthday-Bound · Message Authentication Codes · Quantum Attacks · Grover’s Algorithm · Simon’s Algorithm · Grover-meet-Simon Algorithm

1 Introduction

Grover’s and Simon’s algorithms. Quantum computing poses an impending threat to many widely used cryptographic schemes. Other than Shor’s algorithm [29] which speeds up factoring integers and computing discrete logarithms from classic exponential time to polynomial time, breaking many asymmetric cryptography schemes, such as RSA, ECDSA and ECDH, the existing quantum attacks to symmetric schemes are mainly related to Grover’s or Simon’s algorithms. Grover’s algorithm [14] speeds up the search problem quadratically and Simon’s algorithm [30] finds the period of a periodic function in polynomial time. For a long time, Grover search was regarded as the only threat to

symmetric schemes, which can be thwarted by doubling the key length. However, from 2010, plenty of works applied Simon’s algorithm to attack symmetric schemes, directly reducing the complexity from exponent to polynomial. The broken schemes include 3-round Feistel cipher [22], Even-Mansour construction [23], message authentication code (MAC) modes [18] CBC-MAC, PMAC, GMA, etc., authenticated encryption modes [18] GCM, OCB, AEZ [5], etc., and tweakable enciphering schemes [13] CMC, EME, XCB, TET, etc.

Simon’s algorithm and birthday attacks. The procedure of the attack using Simon’s algorithm is as follows: first construct a periodic function $f(x)$ based on the scheme, where the period is a hidden value s such that $f(x) = f(x \oplus s)$ for all x ; second use Simon’s algorithm to find the period s ; third use the period s to carry out forgery etc. attacks. The period s also can be retrieved from a collision of $f(x) = f(y)$ for $x \neq y$, so that $s = x \oplus y$ with high probability. Therefore $\mathcal{O}(2^{n/2})$ classic queries is enough to find the period and break the scheme, where n is usually the block size of the underlying (tweakable) block ciphers. Therefore the schemes broken by using Simon’s algorithm are destined to suffer from birthday attacks.

Beyond-Birthday-Bound MACs. When $n = 64$ especially for lightweight (tweakable) block ciphers, birthday attacks become practical security problems [1, 3]. In recent years, crypto community made great efforts to enhance the security strength, by constructing beyond-birthday-bound (BBB) schemes, particularly MAC modes, which are secure for above $2^{n/2}$ queries. In 2010, Yasuda firstly proposed a BBB MAC: SUM-ECBC [32]. Later on, other BBB MACs, such as PMAC.Plus [33], 3kf9 [34], LightMAC [28], 1k-PMAC.Plus [12], PloyMAC [9] and so on were proposed. All these BBB MACs follow a generic design paradigm called Double-block Hash-then-Sum (in short DbHtS) [10], which generates double hash blocks on the message and then sum the two encrypted blocks as the output. So The computation of DbHtS consists of two chains, denoted as G and H :

$$DbHtS(M) = G(M) \oplus H(M),$$

for the message M . The double block brings $2n$ -bit internal state, making the classic birthday attack no longer applicable. The primary proofs show that they are secure up to $2^{2n/3}$ queries (ignoring the maximum message length). The best classical attacks against DbHtS MACs proposed by Leurent [25] in 2018 need $2^{3n/4}$ queries, including SUM-ECBC, PMAC.Plus, 1k-PMAC.Plus, 3kf9, GCM-SIV2 and so on. Recently Kim et al. [21] further proved that some of them are secure up to $2^{3n/4}$ queries, so the security bounds are tight.

Motivations. What about the security of BBB MACs in the quantum setting? From the above analyses, we see that the effort of BBB designs to strengthen security in the classic setting actually coincides with the designs against Simon’s algorithm in the quantum setting. *Are there any quantum attacks better than classic attacks against these BBB MACs? Are they still BBB in the quantum setting?* There is still no answer.

Grover-meet-Simon algorithm. For BBB MACs, Simon’s algorithm is invalid. We need new techniques. In 2017 Leander and May [24] combines Grover’s algorithm with Simon’s algorithm to attack FX construction [19, 20]. The main idea is to construct a function with two inputs based on FX, say $f(u, x)$. When the first input u equals to a special value k , the function has a hidden period s such that $f(k, x) = f(k, x \oplus s)$ for all x . Their combined algorithm use Grover’s algorithm to search k , by running many independent Simon’s algorithms to check whether the function is periodic or not, and recover both k and s in the end. The attack only costs $\mathcal{O}(2^{m/2}(m+n))$ quantum queries to FX, which is much less than the proved security bellow $2^{\frac{m+n}{2}}$ queries [19], where $|u| = m$ and $|s| = n$. Their heuristic work provides a new tool to study the security of symmetric schemes. We mainly utilize such Grover-meet-Simon algorithm to explore the security of BBB MACs.

Attacking strategies. According to the method of double block hash generation, these DbHtS MACs can be categorized into SUM-ECBC-like MACs or PMAC_Plus-like MACs. SUM-ECBC-like MACs handle the message twice, generating two hash blocks independently using the same function with different keys, so the chains G and H are the same except for their keys. PMAC_Plus-like MACs handle the message once by a common keyed permutation and then linearly combine the result into two hash blocks, so the chains G and H are two different functions.

Strategy 1: For SUM-ECBC-like MACs, G and H are not secure under the quantum attack using Simon’s algorithm. We can use the same method C , based on G (resp. H), to construct a periodic function denoted as $g(b, x) = C^G(b, x)$ (resp. $h(b, x) = C^H(b, x)$) where $b \in \{0, 1\}$ and $x \in \{0, 1\}^n$. The periods of g and h are denoted as $1||s_1$ and $1||s_2$ respectively. Then use the same method C based on $DbHtS = G \oplus H$, we get $C^{DbHtS}(b, x) = C^G(b, x) \oplus C^H(b, x) = g(b, x) \oplus h(b, x)$. Unfortunately s_1 is equal to s_2 usually with negligible probability, so $C^{DbHtS}(b, x)$ is not a periodic function. We construct

$$f(u, x) = C^{DbHtS}(0, x) \oplus C^{DbHtS}(1, x \oplus u) = g(0, x) \oplus h(0, x) \oplus g(1, x \oplus u) \oplus h(1, x \oplus u).$$

We can verify that when $u = s_1$ or s_2 , $f(u, x)$ is a periodic function: the period is $s_1 \oplus s_2$. Thus we can use Grove-meet-Simon algorithm to recover both s_1 and s_2 .

Strategy 2: For PMAC_Plus-like MACs, G and H are different in linear combination processes, Strategy 1 is not applicable. But we can use the same method based on G (resp. H), to construct a function denoted as $g(u, b, x)$ (resp. $h(u, b, x)$). When u equals a special value say u^* , both $g(u^*, b, x)$ and $h(u^*, b, x)$ are periodic functions with the same period $1||s$. If the method is applied to $DbHtS$, we get

$$f(u, b, x) = g(u, b, x) \oplus h(u, b, x).$$

For $u = u^*$, $f(u^*, b, x)$ is a periodic function. So Grove-meet-Simon algorithm can recover the special value u^* and the period $1||s$.

With these two strategies, we can utilize Grover-meet-Simon algorithm to recover some secret states of BBB MACs, which leads to successful forgery attacks.

Strategy of Grover search: We notice that most BBB MACs have more than one key. For example, PMAC_Plus, PMAC_TBC3k and 3kf9, are respectively keyed by three independent m -bit keys. For perfect crypto primitive, there should be no better way to recover the key than the exhaustive search, whose complexity is $\mathcal{O}(2^{3m})$ for $3m$ -bit keys in the classic setting. If we directly Grover search the whole $3m$ -bit keys, the number of queries is $\mathcal{O}(2^{3m/2})$. Our strategy is to Grover search only one of the three keys, so that we need only $\mathcal{O}(2^{m/2})$ queries. Once the key is known, a forgery attack is straightforward.

Table 1. Summary of the main results. q is the number of queries, ℓ is the number of maximum blocks of a query, σ is total number of processed blocks, n is the length of the message block, m is the length of the key of a block cipher. The expected lower bound and attack complexity is in number of constant length queries ($\ell = O(1)$).

Scheme	Key Space	Provable security bounds		Classical attacks		Quantum secret state recovery attack		Quantum key recovery attack	
		Advantage	Queries	Queries	Queries	Qubits	Queries	Qubits	
SUM-ECBC [32]	2^{4m}	$\ell^{\sigma(1)} q^{4/3} / 2^n + \ell^4 q^{4/3} / 2^{2n}$ [21]	$\Omega(2^{3n/4})$	$\mathcal{O}(2^{3n/4})$ [25]	$\mathcal{O}(2^{n/2}n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(2^{m}n)$	$\mathcal{O}(m+n^2)$	
2K-ECBC.Plus [10]	2^{3m}	$q\ell^2/2^n + q^3\ell^2/2^{2n}$ [10]	$\Omega(2^{2n/3})$	-	$\mathcal{O}(2^{n/2}n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(2^{m}n)$	$\mathcal{O}(m+n^2)$	
PolyMAC [9]	2^{2m+2n}	$\ell q^{4/3} / 2^n$ [21]	$\Omega(2^{3n/4})$	-	$\mathcal{O}(2^{n/2}n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(2^{(n+m)/2}n)$	$\mathcal{O}(m+n^2)$	
GCM-SIV2 [17]	2^{4m+2n}	$7\sigma^3/2^{2n} + 6\ell^2 q^3/2^{2n} + q/2^{2n}$ [17]	$\Omega(2^{2n/3})$	$\mathcal{O}(2^{3n/4})$ [25]	$\mathcal{O}(2^{n/2}n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(2^{(n+m)/2}n)$	$\mathcal{O}(m+n^2)$	
PMAC.Plus [33]	2^{3m}	$\ell/2^{n/2} + \ell^{2/3} q^{4/3} / 2^n + \ell^2 q / 2^n$ [21]	$\Omega(2^{3n/4})$	$\mathcal{O}(2^{3n/4})$ [25]	$\mathcal{O}(2^{n/2}n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(2^{m/2})$	$\mathcal{O}(m+n)$	
1k-PMAC.Plus [12]	2^m	$21\sigma/2^n + 224q\sigma^2/2^{2n}$ [12]	$\Omega(2^{2n/3})$	$\mathcal{O}(2^{3n/4})$ [25]	$\mathcal{O}(2^{n/2}n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(2^{m/2})$	$\mathcal{O}(m+n)$	
2K-PMAC.Plus [10]	2^{2m}	$q^3\ell/2^{2n} + q^2\ell^2/2^{2n}$ [10]	$\Omega(2^{2n/3})$	-	$\mathcal{O}(2^{n/2}n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(2^{m/2})$	$\mathcal{O}(m+n)$	
3kf9 [34]	2^{3m}	$\ell^{4/3} q^{4/3} / 2^n + \ell^2 q^2 / 2^{2n} + \ell^6 q^4 / 2^{3n}$ [21]	$\Omega(2^{3n/4})$	$\mathcal{O}(\sqrt[3]{n} 2^{3n/4})$ [25]	$\mathcal{O}(2^{n/2})$	$\mathcal{O}(n)$	$\mathcal{O}(2^{m/2})$	$\mathcal{O}(m+n)$	
2kf9 [10]	2^{2m}	$q^3\ell^4/2^{2n}$ [10]	$\Omega(2^{2n/3})$	-	$\mathcal{O}(2^{n/2})$	$\mathcal{O}(n)$	$\mathcal{O}(2^{m/2})$	$\mathcal{O}(m+n)$	
PMAC.TBC3k [27]	2^{3m}	$0.5q^2/(2^n - q)^2$ [27]	$\Omega(2^n)$	-	-	-	$\mathcal{O}(2^{m/2})$	$\mathcal{O}(m+n)$	

Our contributions. Table 1 summarizes our main results and comparisons with provable security claims and best classical attack results.

- 1) Reduce the query complexity from practical $\mathcal{O}(2^{3n/4})$ to at most $\mathcal{O}(2^{n/2}n)$ by our secret state recovery attacks. We use Strategy 1 to analyze SUM-ECBC-like MACs, including SUM-ECBC, 2K-ECBC-Plus, PolyMAC and GCM-SIV2, and Strategy 2 to PMAC_Plus-like MACs, including PMAC_Plus, 1k-PMAC_Plus, 2K-PMAC_Plus, 3kf9 and 2kf9.
- 2) Reduce the key search complexity from $\mathcal{O}(2^{3m})$ (for perfect MAC) to $\mathcal{O}(2^{m/2})$ by our key recovery attacks. We use the strategy of Grover search to analyze PMAC_Plus-like MACs, including PMAC_Plus, 1k-PMAC_Plus, 2K-PMAC_Plus, 3kf9, 2kf9 and PMAC_TBC3k.

Therefore our results show that the security of BBB DbHtS MACs goes back to the birthday bound in the quantum setting.

Organization of the paper. Section 2 introduces quantum algorithms, the quantum security of MAC and previous attack for MAC by Simon's algorithm. Section 3 applies Strategy 1 and 2 to make secret state attacks for SUM-ECBC-like and PMAC_Plus-like MACs respectively. Section 4 applies the strategy of Grover search to make key recovery attacks for PMAC_Plus-like MACs. Section

5 gives conclusions. The quantum basis, the concrete introductions of quantum algorithms is in Appendix A to D.

2 Preliminaries

For a positive integer m , let $\{0, 1\}^m$ is the set of all m -bit binary string. For two bit strings x and y , the concatenation is $x||y$, the *bitwise exclusive-or* is $x \oplus y$. Let $|\mathcal{U}|$ be the number of the elements in set \mathcal{U} .

2.1 Quantum Algorithms

In this section, we introduce some useful quantum techniques which will be involved in the following sections. We put quantum basis in Appendix A.

1) Grover's Algorithm Grover's algorithm [14] finds with high probability the input to a function that produces a particular output value. The specific problem is as follows:

Definition 1. (Grover Problem) *Let m be a positive integer, $test : \{0, 1\}^m \rightarrow \{0, 1\}$ be a boolean function ($|\{u : test(u) = 1\}| = e$). Find a u who satisfies $test(u) = 1$.*

Classically, we can search an element who satisfies $test(u) = 1$ with $\mathcal{O}(\frac{2^m}{e})$ queries to $test(\cdot)$. However, the Grover's algorithm [14] can find such an elements with only $\mathcal{O}(\sqrt{\frac{2^m}{e}})$ quantum queries [8]. More generally, we can't construct such an accurate $test$ function that coincides with U . We define the Grover problem with error in the following.

Definition 2. (Grover Problem with Error) *Let m be a positive integers, $\mathcal{U}(|\mathcal{U}| = e)$ be a subset in $\{0, 1\}^m$, $test : \{0, 1\}^m \rightarrow \{0, 1\}$ be a boolean function who satisfies*

$$\begin{cases} \Pr[test(u) = 1] = 1, & u \in \mathcal{U}, \\ \Pr[test(u) = 1] \leq p_1, & u \notin \mathcal{U}. \end{cases}$$

Find a $u \in \mathcal{U}$.

Grover's algorithm can solve the problem as well with some biases. In fact, Grover's algorithm do as follows: first there is an initial probability to get a u who satisfies $test(u) = 1$; second amplify the initial probability iteration by iteration; third measure the quantum state and get a u who satisfies $test(u) = 1$ with high probability. From definition 1, we obtain the initial probability to get a u who satisfy $test(u) = 1$ is between $[p_0, p_0 + p_1]$, where $p_0 = \frac{e}{2^m}$ and it is the initial probability to get a $u \in \mathcal{U}$. Bonnetain [6] has proved when the initial success probability to get a u where $test(u) = 1$ is between an interval $[p_0, p_0 + p_1]$, then after $t = \lceil \frac{\pi}{4 \arcsin \sqrt{p_0}} \rceil$ quantum queries to $test(\cdot)$, the final probability to get

a u who satisfies $test(u) = 1$ is $[1 - (\frac{p_1}{p_0} + \sqrt{p_0 + p_1} + 2\sqrt{1 + \frac{p_1}{p_0} p_0})^3]$. Among all elements satisfying $test(u) = 1$, the proportion of $u \in \mathcal{U}$ is at least $\frac{p_0}{p_0 + p_1}$. Multiple them and we get the following theorem.

Theorem 1. (Adapted from [6]) *Let $m, e, \mathcal{U}, test$ as defined in definition 2, $p_0 := \frac{e}{2^m}$. Assume the quantum implement of $test(\cdot)$ costs $\mathcal{O}(n)$ qubits. Then Grover's algorithm with $t = \lceil \frac{\pi}{4 \arcsin \sqrt{p_0}} \rceil$ quantum queries to $test(\cdot)$ and $\mathcal{O}(m + n)$ qubits will output a $u \in \mathcal{U}$ with probability at least $\frac{p_0}{p_0 + p_1} [1 - (\frac{p_1}{p_0} + \sqrt{p_0 + p_1} + 2\sqrt{1 + \frac{p_1}{p_0} p_0})^3]$.*

We put Grover's algorithm and the concrete proof of theorem 1 in Appendix B. When apply this algorithm to concrete attack for MACs, if $e = 1, p_1 \leq \frac{1}{2^{2m}}$, then for sufficient large m the Grover's algorithm with $\mathcal{O}(2^{m/2})$ quantum queries to $test(\cdot)$ and $\mathcal{O}(m + n)$ qubits will output a $u \in \mathcal{U}$ with probability almost 1 by theorem 1.

2) Simon's Algorithm Simon's algorithm [30] finds the period of a periodic function in polynomial time.

Definition 3. (Periodic/Aperiodic Function) *Let n, d be two positive integers, $f : \{0, 1\}^n \rightarrow \{0, 1\}^d$ be a boolean function. We call f is a periodic (resp. aperiodic) function if there is a unique (resp. no) $s \in \{0, 1\}^n \setminus \{0^n\}$ such that $f(x) = f(x \oplus s)$ for all $x \in \{0, 1\}^n$.*

Definition 4. (Simon Problem) *Let n, d be two positive integers, $f : \{0, 1\}^n \rightarrow \{0, 1\}^d$ be a periodic function with a period s . Find s .*

Classically, if f is a periodic function, we can find out the period by searching a collision with birthday bound queries $\mathcal{O}(2^{d/2})$. However, if f is given as a quantum oracle, Simon's algorithm [30] can solve it with only $\mathcal{O}(n)$ quantum queries. Let

$$\varepsilon(f) := \max_{t \in \{0, 1\}^n \setminus \{0^n, s\}} \Pr_x[f(x) = f(x \oplus t)]. \quad (1)$$

This parameter quantifies the disturbance of other partial periods. Kaplan et al. [18] have proved the following theorem.

Theorem 2. [18] *Let n, d, f, s as defined in definition 4, $\varepsilon(f)$ as defined in equation (1), c be an positive integer. Then Simon's algorithm with cn quantum queries to f and $\mathcal{O}(n + d)$ qubits will recover s with probability at least $1 - (2(\frac{1 + \varepsilon(f)}{2})^c)^n$.*

We put Simon's algorithm and the proof of theorem 2 in Appendix C.

3) Grover-meet-Simon Algorithm In 2017 Leander and May [24] combines Grover’s algorithm with Simon’s algorithm to analyze FX construction. A general problem is described as follows:

Definition 5. (Grover-meet-Simon Problem) *Let m, n, d be three positive integers, set $\mathcal{U} \subseteq \{0, 1\}^m$ ($|\mathcal{U}| = e$) and $f : \{0, 1\}^m \times \{0, 1\}^n \rightarrow \{0, 1\}^d$ be a function who satisfies*

$$\begin{cases} f(u, \cdot) \text{ is a period function with period } s_u, & u \in \mathcal{U}, \\ f(u, \cdot) \text{ is an aperiodic function,} & u \notin \mathcal{U}. \end{cases}$$

Set $\mathcal{U}_s := \{(u, s_u) : u \in \mathcal{U}, s_u \text{ is the period of } f(u, \cdot)\}$. Find any tuple $(u, s_u) \in \mathcal{U}_s$.

The problem is composed of the Grover problem as a whole and the Simon problem partially. The main idea is to search $u \in \mathcal{U}$ by Grover’s algorithm and check whether or not $u \in \mathcal{U}$ by whether $f(u, \cdot)$ is periodic or not, which can be implemented by Simon’s algorithm. Bonnetain [6] has generalized a algorithm. He gave the success probability of Grover-meet-Simon algorithm for problem in definition 5 when $|\mathcal{U}| = 1$. Further on, our paper generalizes the success probability of the algorithm for $|\mathcal{U}| \geq 1$. Let

$$\varepsilon(f) := \max_{(u,t) \in \{0,1\}^m \times \{0,1\}^n \setminus (\mathcal{U}_s \cup \{0,1\}^m \times \{0^n\})} \Pr_x[f(u, x) = f(u, x \oplus t)], \quad (2)$$

to quantify the disturbance of $u \notin \mathcal{U}$ and other partial periods ts for $u \in \mathcal{U}$. We show the conclusion in theorem 3.

Theorem 3. *Let $m, n, d, f, \mathcal{U}, \mathcal{U}_s, e$ as defined in definition 5, $\varepsilon(f)$ as defined in equation (2), c be a positive integer, $p_0 := \frac{e}{2^m}, p_1 := [2 \cdot (\frac{1+\varepsilon(f)}{2})^c]^n$. Then Grover-meet-Simon algorithm with $\lceil \frac{\pi}{4 \arcsin \sqrt{p_0}} \rceil \cdot cn$ quantum queries to f and $\mathcal{O}(m + cn^2 + cdn)$ qubits outputs a tuple $(u, s_u) \in \mathcal{U}_s$ with probability at least $\frac{(1-p_1)p_0}{p_0+p_1} [1 - (\frac{p_1}{p_0} + \sqrt{p_0 + p_1} + 2\sqrt{1 + \frac{p_1}{p_0}})^3 p_0^2]$.*

We put Grover-meet-Simon algorithm and the proof of theorem 3 in Appendix D. When apply this algorithm to concrete attack for MACs, if $\varepsilon(f) \leq 3/4, e \leq 2, d = m = n$ and n is sufficient large, then we let $c = 16$ and Grover-meet-Simon algorithm after $\mathcal{O}(2^{n/2}n)$ quantum queries to f using $\mathcal{O}(n^2)$ qubits will output a tuple $(u, s_u) \in \mathcal{U}_s$ with probability almost 1 by theorem 3.

2.2 Quantum Security of MACs

Message authentication code (MAC) generates a tag for any input message: $T = \text{MAC}_K(M)$, where T is the tag, K is the key and M is the message. The classic security of MAC requires that the attacker can not produce (M', T') such that $T' = \text{MAC}_K(M')$ and M' is a message that has never been queried to the MAC.

In the quantum setting, the attacker has access to the quantum oracle of $\text{MAC}_K(\cdot)$. Boneh and Zhandry [4] firstly define the Existential unforgeability against quantum chosen message attack (EUF-qCMA). One MAC is EUF-qCMA if no quantum attacker can output $q + 1$ distinct message-tag pairs with non-negligible probability after q quantum queries to $\text{MAC}_K(M)$. Notice that we can regard any classical query as a special quantum query. So the q quantum queries contain q quantum and classical queries in all.

For all concrete MACs in these paper, we assume the message block size is n -bit and the message M has been padded with 10^* padding and divided into n -bit blocks. Also, we assume the underlying keyed (tweakable) block cipher of MACs is a (tweakable) random permutation.

2.3 Attacking ECBC-MAC

MACs of single-chain like ECBC-MAC are broken by using Simon's algorithm [18], with only $\mathcal{O}(n)$ quantum queries. We write the MAC as a function G . The attack in [18] is to construct a periodic function g based on G using a method C . We denote it as $g(b, x) = C^G(b, x)$ with a period $1\|s$.

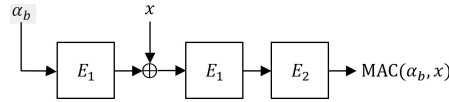


Fig. 1. ECBC-MAC of two inputs $M = (\alpha_b, x)$.

In the following, we demonstrate how they give the construction g for the ECBC-MAC variant [2], the estimation of $\varepsilon(g)$ and the forgery attack after recovery of s . ECBC-MAC uses a block cipher keyed by two independent keys, denote as E_1, E_2 .

Construction of function g . Let $b \in \{0, 1\}$, $x \in \{0, 1\}^n$, α_0, α_1 are two arbitrary fixed number in $\{0, 1\}^n$. ECBC-MAC with message $M = (M[1], M[2]) = (\alpha_b, x)$ are showed in figure 1, which can be wrote as

$$\text{MAC}(\alpha_b, x) = g(b, x),$$

where

$$g(b, x) = E_2(E_1(E_1(\alpha_b) \oplus x)).$$

Obviously, g has a period of $1\|s$ where $s = E_1(\alpha_0) \oplus E_1(\alpha_1)$:

$$\begin{aligned} g(b', x') = g(b, x) &\Leftrightarrow E_2(E_1(E_1(\alpha_{b'}) \oplus x')) = E_2(E_1(E_1(\alpha_b) \oplus x)) \\ &\Leftrightarrow \begin{cases} x' \oplus x = 0^n & \text{if } b' \oplus b = 0, \\ x' \oplus x = E_1(\alpha_0) \oplus E_1(\alpha_1) & \text{if } b' \oplus b = 1. \end{cases} \end{aligned}$$

Therefore, $\varepsilon(f) = 0$ and s can be recovered with $\mathcal{O}(n)$ quantum queries to f using $\mathcal{O}(n)$ qubits by theorem 2.

Forgery attack. After recovering s , by using the property of $g(b, x) = g(b, x \oplus s)$, they make a successful forgery after one classic queries as follows:

- 1) Query $M_1 = (\alpha_0, x)$ and get tag T ;
- 2) Forge $M_2 = (\alpha_1, x \oplus s)$ and its tag T .

To break the notion of EUF-qCMA security, they produce $q + 1$ valid tags with only q queries to the quantum oracle of MAC. Let $q' = O(n)$ denote the number of quantum queries made to find s . The attacker just repeats the above classic forgery step $q' + 1$ times. So that $2q' + 2$ messages with valid tags were produced, using a total of $2q' + 1$ classical and quantum queries. Therefore, ECBC-MAC is broken by a quantum existential forgery attack.

3 Secret State Recovery Attack for BBB MACs

We focus on DbHtS MAC [10], the sum of two chains is denoted as: $DbHtS(M) = G(M) \oplus H(M)$.

3.1 Secret State Recovery Attack for SUM-ECBC-like MACs

Both chains of G and H in SUM-ECBC-like MACs use the same MAC with different keys. Each chain is vulnerable to Simon's algorithm as showed in section 2.3. So it is easy to construct a periodic function g (resp. h) based on G (resp. H) using a method C . We denote it as $g(b, x) = C^G(b, x)$ (resp. $h(b, x) = C^H(b, x)$) with a period $1||s_1$ (resp. $1||s_2$). Therefore we get $C^{DbHtS}(b, x) = g(b, x) \oplus h(b, x)$.

If $s_1 = s_2$, it is a periodic function with the same period, which we can apply Simon's algorithm to recover. However, it happens with negligible probability for SUM-ECBC-like MACs. Strategy 1 constructs

$$f(u, x) = g(0, x) \oplus h(0, x) \oplus g(1, x \oplus u) \oplus h(1, x \oplus u).$$

When $u = s_1$ or s_2 , $f(u, x)$ has a period of $s_1 \oplus s_2$. If $\varepsilon(f) \leq 3/4$, then by theorem 3, Simon-meet-Grover algorithm can find both s_1 and s_2 with at most $O(2^{n/2}n)$ quantum queries and $O(n^2)$ qubits.

In the following, for any concrete SUM-ECBC-like MAC, we only give the construction of function f , the estimation of $\varepsilon(f)$, and the forgery attack after recovery of s_1 and s_2 .

The method applies to SUM-ECBC [32], PolyMAC [21], the authentication part of GCM-SIV2 [17] and 2K-ECBC.Plus [10]. In the following, we only give SUM-ECBC and PolyMAC as examples.

1) Secret State Recovery Attack for SUM-ECBC. SUM-ECBC was designed by Yasuda in 2010 [32], which is the sum of two independent ECBC-MACs. The scheme uses a block cipher keyed with four independent keys, denoted as E_1, E_2, E_3, E_4 . Assume the message has been padded and divided into n -bit blocks.

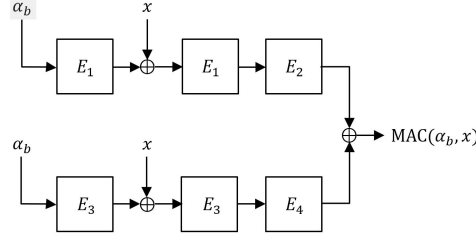


Fig. 2. SUM-ECBC of two inputs $M = (\alpha_b, x)$.

Construction of function f . Let $b \in \{0, 1\}$, $x \in \{0, 1\}^n$, α_0, α_1 are two arbitrary fixed number in $\{0, 1\}^n$. SUM-ECBC with message $M = (M[1], M[2]) = (\alpha_b, x)$ is showed in figure 2, which can be wrote as

$$\text{MAC}(\alpha_b, x) = g(b, x) \oplus h(b, x),$$

where

$$\begin{aligned} g(b, x) &= E_2(E_1(E_1(\alpha_b) \oplus x)), \\ h(b, x) &= E_4(E_3(E_3(\alpha_b) \oplus x)). \end{aligned}$$

Obviously, g (resp. h) has a period of $1 \parallel s_1$ where $s_1 = E_1(\alpha_0) \oplus E_1(\alpha_1)$ (resp. $1 \parallel s_2$ where $s_2 = E_3(\alpha_0) \oplus E_3(\alpha_1)$). Given that E_1, E_3 are two independent pseudorandom permutations, the probability of $s_1 = s_2$ is at most $1 - 1/2^n$. So in the following we assume $s_1 \neq s_2$. Let

$$f(u, x) = \text{MAC}(\alpha_0, x) \oplus \text{MAC}(\alpha_1, x \oplus u).$$

Estimation of $\varepsilon(f)$. Here, $\mathcal{U}_s = \{(s_1, s_1 \oplus s_2), (s_2, s_1 \oplus s_2)\}$,

$$\varepsilon(f) = \max_{(u,t) \in \{0,1\}^n \times \{0,1\}^n \setminus (\mathcal{U}_s \cup \{0,1\}^n \times \{0^n\})} \Pr_x[f(u, x) = f(u, x \oplus t)].$$

We consider $u = s_1$ as an example and the other situation is similar. In this case $f(u, x) = f(s_1, x) = E_4(E_3(x \oplus E_3(\alpha_0)) \oplus E_4(E_3(x \oplus s_1 \oplus E_3(\alpha_1))))$. We will prove $\varepsilon(f(s_1, \cdot)) \leq \frac{1}{2}$ with overwhelming probability. Otherwise, there is $t \notin \{0^n, s_1 \oplus s_2\}$ such that $\Pr_x[f(s_1, x) = f(s_1, x \oplus t)] > 1/2$, i.e.,

$$\Pr_x \left[\begin{array}{c} E_4(E_3(x \oplus E_3(\alpha_0))) \oplus E_4(E_3(x \oplus s_1 \oplus E_3(\alpha_1))) \oplus \\ E_4(E_3(x \oplus t \oplus E_3(\alpha_0))) \oplus E_4(E_3(x \oplus t \oplus s_1 \oplus E_3(\alpha_1))) \end{array} \right] > 1/2. \quad (3)$$

By $t \notin \{0^n, s_1 \oplus s_2\}$ and $s_1 \neq s_2$, we know the four inputs of $E_4(E_3(\cdot))$ are different from each other. By E_4 is a pseudorandom function and E_3 is a permutation, then the equation (3) happens with negligible probability.

Forgery attack. After recovering s_1 and s_2 , by using the property of $f(s_1, x) = f(s_1, x \oplus s_1 \oplus s_2)$, we can make a successful forgery after 3 classic queries as follows:

- 1) Query $M_1 = (\alpha_0, x)$ and get tag T_1 ;

- 2) Query $M_2 = (\alpha_1, x \oplus s_1)$ and get tag T_2 ;
- 3) Query $M_3 = (\alpha_0, x \oplus s_1 \oplus s_2)$ and get tag T_3 ;
- 4) Forge $M_4 = (\alpha_1, x \oplus s_2)$ and its tag $T_1 \oplus T_2 \oplus T_3$.

To break the notion of EUF-qCMA security, we must produce $q+1$ valid tags with only q queries to the quantum oracle of MAC. If $q' = \mathcal{O}(2^{n/2})$ denote the number of quantum queries made to find s_1 and s_2 . The attacker just repeats the above classic forgery step $q'+1$ times. So that $4q'+4$ messages with valid tags were produced, using a total of $4q'+3$ classical and quantum queries. Therefore, SUM-ECBC is broken by a quantum existential forgery attack. Generally, the EUF-qCMA attack is straightforward after we find the hidden periods. So we omit it in the following examples.

2) Secret State Recovery Attack for PolyMAC. Replace the block cipher E_1 and E_3 in SUM-ECBC with multiplication functions $H_{k_1}(x) = k_1 \cdot x$ and $H_{k_3}(x) = k_3 \cdot x$, we get PolyMAC [21]. k_1, k_3 are two independent keys in $\{0, 1\}^n$ and they are independent with the keys of E_2, E_4 . The chain of MAC is actually PolyHash, which is used in the authentication of associated data in GCM-SIV2 [17], GCM [26] and HCTR [31].

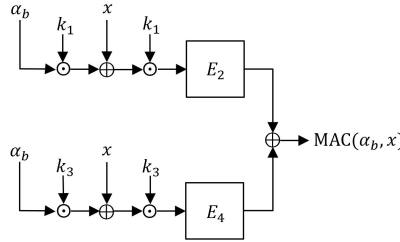


Fig. 3. The left part of PolyMAC with two inputs $M = (M[1], M[2]) = (\alpha_b, x)$.

Construction of function f . Let $b \in \{0, 1\}$ and $x \in \{0, 1\}^n$, α_0, α_1 are two arbitrary fixed number in $\{0, 1\}^n$. PolyMAC with message $M = (M[1], M[2]) = (\alpha_b, x)$ is showed in figure 3, which can be wrote as

$$\text{MAC}(\alpha_b, x) = g(b, x) \oplus h(b, x)$$

where

$$\begin{aligned} g(b, x) &= E_2(k_1^2 \alpha_b \oplus k_1 x), \\ h(b, x) &= E_4(k_3^2 \alpha_b \oplus k_3 x). \end{aligned}$$

Obviously, g (resp. h) has a period of $1 \parallel s_1$ where $s_1 = k_1 \alpha_0 \oplus k_1 \alpha_1$ (resp. $1 \parallel s_2$ where $s_2 = k_3 \alpha_0 \oplus k_3 \alpha_1$). Given that k_1, k_3 are two independent keys, the probability of $s_1 \neq s_2$ is at most $1 - 1/2^n$. So in the following we assume $s_1 \neq s_2$. Let

$$f(u, x) = \text{MAC}(\alpha_0, x) \oplus \text{MAC}(\alpha_1, x \oplus u).$$

Similar as SUM-ECBC, we can prove $\varepsilon(f) \leq 3/4$.

3.2 Secret State Recovery Attack for PMAC_Plus-like MACs

The two chains G and H of PMAC_Plus-like MAC are different in the linear combination processes, so Strategy 1 is on longer applicable. Strategy 2 uses the same method based on G (resp. H), to construct a function denoted as $g(u, b, x)$ (resp. $h(u, b, x)$). When u equals a special value, say u^* , both $g(u^*, b, x)$ and $h(u^*, b, x)$ are periodic functions with the same period $1 \parallel s$. So we get

$$f(u, b, x) = g(u, b, x) \oplus h(u, b, x).$$

For $u = u^*$, $f(u^*, b, x)$ is a periodic function. Then that we can use Grover-meet-Simon algorithm to recover u^* and s .

If $s \neq 0^n$, $\varepsilon(f) \leq 3/4$, apply Grover-meet-Simon algorithm (theorem 3) to recover u^* , s with at most $\mathcal{O}(2^{n/2}n)$ quantum queries and $\mathcal{O}(n^2)$ qubits.

If $s = 0^n$, apply Grover algorithm (theorem 1) to recover u^* with at most $\mathcal{O}(2^{n/2})$ quantum queries and $\mathcal{O}(n)$ qubits.

In the following, for any concrete PMAC_Plus-like MAC, we only give the construction of function f , the estimation of $\varepsilon(f)$.

The method applies to PMAC_Plus [33], 1k-PMAC_Plus [11, 12], 3kf9 [34], 2K-PMAC_Plus [10] and 2kf9 [10]. In the following, we only give PMAC_Plus [33] and 3kf9 [34] as examples.

1) Secret State Recovery Attack for PMAC_Plus. PMAC_Plus was designed by Yasuda in 2011 [33]. The scheme uses block cipher E_1, E_2, E_3 , who are keyed by three independent keys. The block size is n bits.

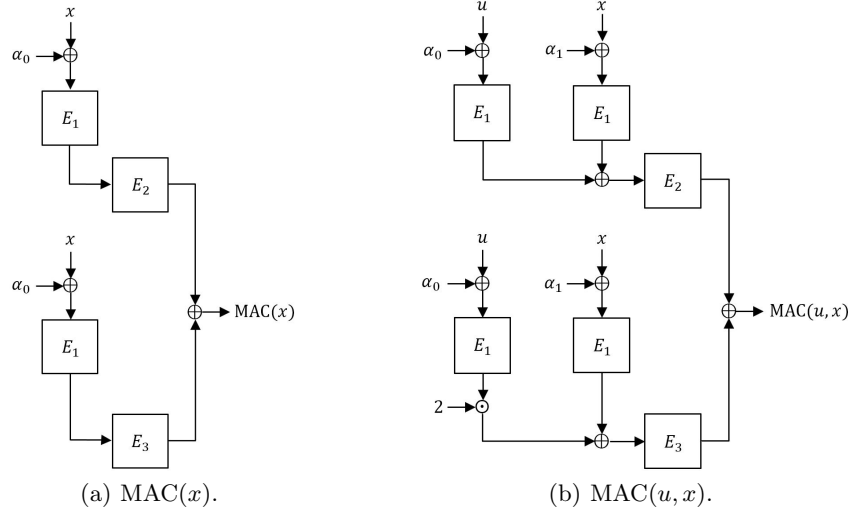


Fig. 4. PMAC.Plus with message $M = (x)$ and $M = (u, x)$.

Construction of function f . Let $b \in \{0, 1\}, u, x \in \{0, 1\}^n$ and

$$\alpha_b := \begin{cases} 2E_1(0) \oplus 2^2E_1(1), & \text{if } b = 0, \\ 2^2E_1(0) \oplus 2^4E_1(1), & \text{if } b = 1. \end{cases}$$

PMAC_Plus with message $M = (M[1]) = (x)$ and message $M = (M[1], M[2]) = (u, x)$ are shown as figure 4, which can be wrote as

$$\text{MAC}(M) = \begin{cases} g(u, 0, x) \oplus h(u, 0, x), & \text{if } M = (x), \\ g(u, 1, x) \oplus h(u, 1, x), & \text{if } M = (u, x), \end{cases}$$

where

$$g(u, b, x) = \begin{cases} E_2(E_1(x \oplus \alpha_0)), & \text{if } b = 0, \\ E_2(E_1(x \oplus \alpha_1) \oplus E_1(u \oplus \alpha_0)), & \text{if } b = 1, \end{cases}$$

$$h(u, b, x) = \begin{cases} E_3(E_1(x \oplus \alpha_0)), & \text{if } b = 0, \\ E_3(E_1(x \oplus \alpha_1) \oplus 2E_1(u \oplus \alpha_0)), & \text{if } b = 1. \end{cases}$$

We define

$$f(u, b, x) = \begin{cases} \text{MAC}(x), & \text{if } b = 0, \\ \text{MAC}(u, x), & \text{if } b = 1. \end{cases}$$

Let $u^* \in \{0, 1\}^n$ such that $E_1(u^* \oplus \alpha_0) = 0^n$. When $u = u^*$, $f(u, b, x)$ has a period $1 \parallel \alpha_0 \oplus \alpha_1$. So if $\varepsilon(f) \leq 3/4$, then we can apply Grover-meet-Simon algorithm to make forgery attack.

Estimation of $\varepsilon(f)$. Here $\mathcal{U}_s = \{(u^*, 1 \parallel \alpha_0 \oplus \alpha_1)\}$. Let $\mathcal{U}_t := \{0, 1\}^n \times \{0, 1\} \times \{0, 1\}^n \setminus (\mathcal{U}_s \cup \{0, 1\}^n \times \{0^{n+1}\})$, then

$$\varepsilon(f) = \max_{(u, t_1, t_2) \in \mathcal{U}_t} \Pr_{b, x} [f(u, b, x) = f(u, b \oplus t_1, x \oplus t_2)].$$

We consider $u = u^*$ as example and the other is similar. Firstly, we divide the scope $t_1 \parallel t_2 \in \{0, 1\}^{n+1} \setminus \{0^{n+1}, 1 \parallel \alpha_0 \oplus \alpha_1\}$ into two parts $t_1 = 0, t_2 \neq 0^n$ and $t_1 = 1, t_2 \neq \alpha_0 \oplus \alpha_1$. We take the former as example. In fact, when $u = u^*, t_1 = 0, t_2 \neq 0^n$, the equation $f(u, b, x) = f(u, b \oplus t_1, x \oplus t_2)$ equals

$$E_2(E_1(x \oplus \alpha_b)) \oplus E_2(E_1(x \oplus t_2 \oplus \alpha_b)) \oplus E_3(E_1(x \oplus \alpha_b)) \oplus E_3(E_1(x \oplus t_2 \oplus \alpha_b)) = 0^n. \quad (4)$$

By $t_2 \neq 0^n$ and E_1 is a permutation, we obtain both the two inputs of E_2 and the two inputs of E_3 are different respectively. Therefore, by the pseudo-randomness of E_2, E_3 , the equation (4) holds with probability at most $1/2$ with overwhelming probability.

2) Secret State Recovery Attack for 3kf9. 3kf9 was designed by Zhang et.al. [34]. The scheme uses block cipher E_1, E_2, E_3 , who are keyed by three independent keys. The block size is n bits.

Construction of function f . Let $b \in \{0, 1\}, u, x \in \{0, 1\}^n$. The 3kf9 with message $M = (M[1]) = (x)$ and message $M = (M[1], M[2]) = (u, x)$ are showed in figure 5, which can be wrote as

$$\text{MAC}(M) = \begin{cases} g(u, 0, x) \oplus h(u, 0, x), & \text{if } M = (x), \\ g(u, 1, x) \oplus h(u, 1, x), & \text{if } M = (u, x), \end{cases}$$

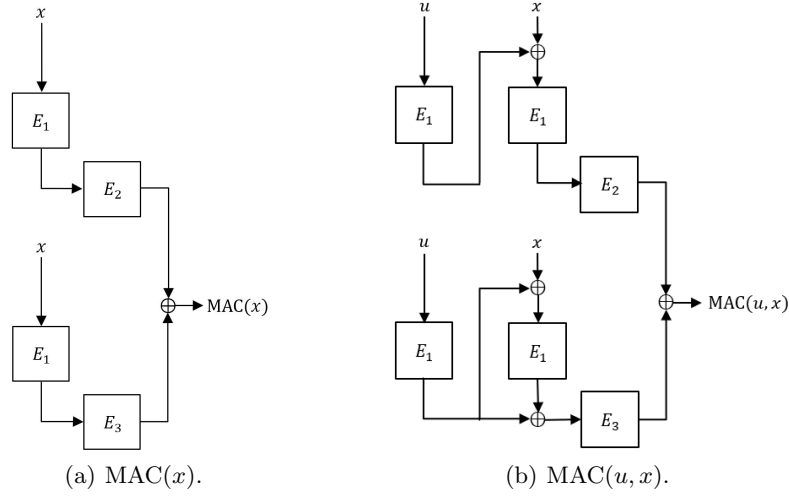


Fig. 5. 3kf9 with message $M = (x)$ and $M = (u, x)$.

where

$$g(u, b, x) = \begin{cases} E_2(E_1(x)), & \text{if } b = 0, \\ E_2(E_1(x \oplus E_1(u))), & \text{if } b = 1, \end{cases}$$

$$h(u, b, x) = \begin{cases} E_3(E_1(x)), & \text{if } b = 0, \\ E_3(E_1(x \oplus E_1(u)) \oplus E_1(u)), & \text{if } b = 1. \end{cases}$$

Let

$$f(u, b, x) = \begin{cases} \text{MAC}(x), & \text{if } b = 0, \\ \text{MAC}(u, x), & \text{if } b = 1. \end{cases}$$

Let $u^* \in \{0, 1\}^n$ such that $E_1(u^*) = 0^n$. It is easy to obtain u^* is unique by permutation E_1 . Then when $u = u^*$, $f(u^*, 0, x) = f(u^*, 1, x)$ holds for all $x \in \{0, 1\}^n$. It means the period is $1||0^n$, which is trivial. So we apply Grover algorithm to recover u^* directly. Let $test : \{0, 1\}^n \rightarrow \{0, 1\}$ be defined as

$$test(u) = \begin{cases} 1, & \text{if } f(u, 0, x_i) = f(u, 1, x_i), \text{ where } i = 1, \dots, q, \\ 0, & \text{otherwise.} \end{cases}$$

If $\max_{u \in \{0, 1\}^n \setminus \{u^*\}} \Pr[test(u) = 1] \leq 2^{-2n}$ when $q = \mathcal{O}(1)$, then Grover's algorithm (theorem 1) will recover u^* using $\mathcal{O}(n)$ qubits and at most than $\mathcal{O}(2^{n/2})$ quantum queries.

Estimation of $\max_{u \in \{0, 1\}^n \setminus \{u^*\}} \Pr[test(u) = 1] \leq 2^{-2n}$. The deviation

$$\begin{aligned} & \max_{u \in \{0, 1\}^n \setminus \{u^*\}} \Pr[test(u) = 1] \leq 2^{-2n} \\ & = \max_{u \in \{0, 1\}^n \setminus \{u^*\}} \Pr[f(u, 0, x_1) = f(u, 1, x_1), \dots, f(u, 0, x_q) = f(u, 1, x_q)]. \end{aligned}$$

Here, the equation system

$$f(u, 0, x_i) = f(u, 1, x_i), i = 1, 2, \dots, q,$$

equals

$$E_2(y_i^1) \oplus E_2(y_i^2) \oplus E_3(y_i^3) \oplus E_3(y_i^4) = 0^n, i = 1, 2, \dots, q,$$

where $y_i^1 = E_1(x_i), y_i^2 = E_1(x_i \oplus E_1(u)), y_i^3 = E_1(x_i), y_i^4 = E_1(x_i \oplus E_1(u)) \oplus E_1(u)$. To calculate the probability of these q equations, we consider sampling about E_2 . If y_i^1, y_i^2 , who are the inputs of E_2 in i th equation, have all appeared in the other $q - 1$ equations, then we don't sample in the i th equation. In fact, if $x_i \oplus x_j = E_1(u)$ then $y_i^1 = y_j^2, y_i^2 = y_j^1$. Therefore, we have to sample E_2 in at least $\lfloor \frac{q+1}{2} \rfloor$ equations among q . For every equation, by the randomness of E_2 , it holds with probability at most $\frac{1}{2^{2n-2q}}$. Therefore, for any $u \in \{0, 1\}^n \setminus \{u^*\}$, we have $\Pr[\text{test}(u) = 1] \leq (\frac{1}{2^{2n-2q}})^{\frac{q-1}{2}}$. When $q = 7$, we have $\Pr[\text{test}(u) = 1] \leq 2^{-2n}$ for sufficient large n .

4 Partial Key Recovery Attack for PMAC_Plus-like MACs

We notice that most BBB MACs have several keys. So we consider a partial key recovery attack by applying Grover's algorithm. We observe that PMAC_Plus-like MACs such as PMAC_Plus [33], 3kf9 [34] etc., [27, 12, 10] with message $M = (M[1], M[2], M[3])$ share a common structure as in figure 6.

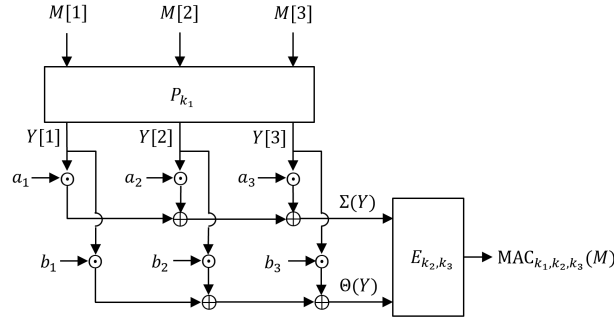


Fig. 6. PMAC_Plus-like MACs with message $M = (M[1], M[2], M[3])$.

Let message $M = (M[1], M[2], M[3]) \in (\{0, 1\}^n)^3$, the tag $\text{MAC}_{k_1, k_2, k_3}(M) \in \{0, 1\}^n$, the independent keys $(k_1, k_2, k_3) \in (\{0, 1\}^m)^3$, P_{k_1} be a permutation from $3n$ bit to $3n$ bit keyed by k_1 and E_{k_2, k_3} be a function from $2n$ bit to n bit keyed by k_2, k_3 , $Y = (Y[1], Y[2], Y[3]) \in (\{0, 1\}^n)^3$, public constants $A = (a_1, a_2, a_3) \in (\{0, 1\}^n)^3, B = (b_1, b_2, b_3) \in (\{0, 1\}^n)^3$. Then the procedure of $\text{MAC}_{k_1, k_2, k_3}(M)$ is as follows.

- 1) Given message M , compute $Y = P_{k_1}(M)$;
- 2) Compute $\Sigma(Y) := a_1Y[1] \oplus a_2Y[2] \oplus a_3Y[3]$ and $\Theta(Y) := b_1Y[1] \oplus b_2Y[2] \oplus b_3Y[3]$;

3) Compute $E_{k_2, k_3}(\Sigma(Y), \Theta(Y))$ and output it.

We notice the middle part of the above structure is a little special. We extract it in figure 7 and call it *linear combination processes*. In fact, it is a equa-

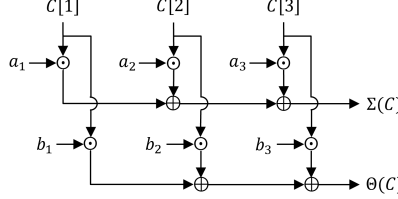


Fig. 7. *Linear combination processes* with input $C = (C[1], C[2], C[3])$ and output $(\Sigma(C), \Theta(C))$.

tion system with two equations about variable $C = (C[1], C[2], C[3])$, where $C[1], C[2], C[3]$ are three independent components. We write it as

$$\begin{cases} a_1 C[1] \oplus a_2 C[2] \oplus a_3 C[3] = \Sigma(C), \\ a_1 C[1] \oplus a_2 C[2] \oplus a_3 C[3] = \Theta(C). \end{cases}$$

Obviously, for any fixed outputs $(\Sigma(C), \Theta(C)) = (z_0, z_1)$, such equation system has more than one solution. In the following attack, we will choose two different solutions $C_0 = (C_0[1], C_0[2], C_0[3])$ and $C_1 = (C_1[1], C_1[2], C_1[3])$ with a same output to construct the test function of Grover's algorithm.

Construction of function *test*. Let set

$$\mathcal{C} := \left\{ (C_0, C_1) \mid \begin{array}{l} \Sigma(C_0) = \Sigma(C_1), \Theta(C_0) = \Theta(C_1), \text{ where} \\ C_j = (C_j[1], C_j[2], C_j[3]) \in (\{0, 1\}^n)^3, j = 0, 1. \end{array} \right\}$$

and function $f : \{0, 1\}^m \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ as

$$f(k, C) = \text{MAC}_{k_1, k_2, k_3}(P_k^{-1}(C)).$$

Then we define $\text{test} : \{0, 1\}^m \rightarrow \{0, 1\}$ as

$$\text{test}(k) = \begin{cases} 1, & \text{if } f(k, C_0^i) = f(k, C_1^i), \text{ for } i = 1, \dots, q, \\ 0, & \text{otherwise,} \end{cases}$$

where $(C_0^i, C_1^i) \in \mathcal{C}$. We notice when $k = k_1$, $\text{test}(k) = 1$. Given quantum oracle of $\text{MAC}_{k_1, k_2, k_3}(\cdot)$, if the deviation $\max_{k \in \{0, 1\}^m \setminus \{k_1\}} \Pr[\text{test}(k) = 1] \leq 2^{-2m}$ when $q = \mathcal{O}(1)$, then we can recover k_1 by Grover's algorithm (theorem 1) with at most $\mathcal{O}(2^{m/2})$ quantum queries and $\mathcal{O}(m + n)$ qubits.

Forgery attack. After recovering k_1 , we can make a successful forgery after a classical query as following.

- 1) Choose an arbitrarily pair $(C_0, C_1) \in \mathcal{C}$.
- 2) Compute $M_0 = (P_{k_1})^{-1}(C_0)$ and $M_1 = (P_{k_1})^{-1}(C_1)$;

- 3) Query M_0 to $\text{MAC}_{k_1, k_2, k_3}(\cdot)$ and get T ;
- 4) Forge M_1 and its tag T .

Generally, the EUF-qCMA attack is straight forward the same as in section 2.3. So we omit it.

The method apply to PMAC_Plus [33], PMAC_TBC3k [27], 1k-PMAC_Plus [11, 12], 3kf9 [34], 2K-PMAC_Plus [10] and 2kf9 [10]. In the following, we only give PMAC_Plus [33] and 3kf9 [34] as examples. And for them, we only give the estimation of the deviation $\max_{k \in \{0,1\}^m \setminus \{k_1\}} \Pr[\text{test}(k) = 1]$.

4.1 Partial Key Recovery Attack for PMAC_Plus

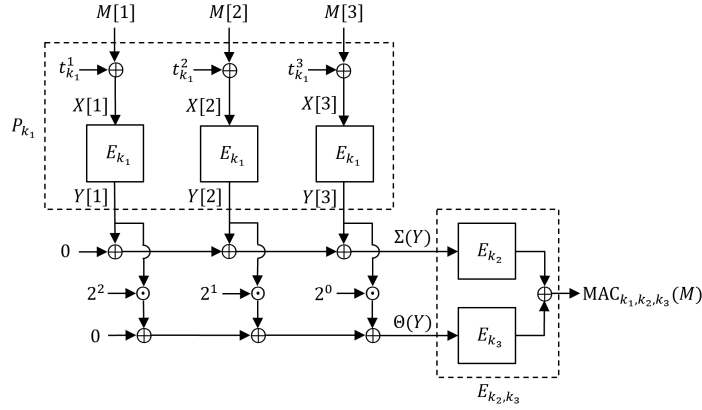


Fig. 8. PMAC_Plus with message $M = (M[1], M[2], M[3])$.

We have introduced PMAC_Plus in section 3.2. Assume the three independent keys are $(k_1, k_2, k_3) \in (\{0, 1\}^m)^3$. The construction with three-block message $M = (M[1], M[2], M[3])$ is showed in figure 8, where $t_{k_1}^j = 2^j E_{k_1}(0^n) \oplus 2^{2j} E_{k_1}(0^{n-1} \parallel 1)$, $j = 1, 2, 3$.

Estimation of $\max_{k \in \{0,1\}^m \setminus \{k_1\}} \Pr[\text{test}(k) = 1]$. The deviation is equals to

$$\max_{k \in \{0,1\}^m \setminus \{k_1\}} \Pr[f(k, C_0^1) = f(k, C_1^1), \dots, f(k, C_0^q) = f(k, C_1^q)].$$

Here, the equation system

$$f(k, C_0^i) = f(k, C_1^i), i = 1, 2, \dots, q, \quad (5)$$

equals

$$E_{k_2}(\Sigma(Y_0^i)) \oplus E_{k_3}(\Theta(Y_0^i)) = E_{k_2}(\Sigma(Y_1^i)) \oplus E_{k_3}(\Theta(Y_1^i)), i = 1, 2, \dots, q,$$

where

$$\begin{aligned}\Sigma(Y_b^i) &= E_{k_1}(X_b^i[1]) \oplus E_{k_1}(X_b^i[2]) \oplus E_{k_1}(X_b^i[3]), b = 0, 1, \\ \Theta(Y_b^i) &= 2^2 E_{k_1}(X_b^i[1]) \oplus 2E_{k_1}(X_b^i[2]) \oplus E_{k_1}(X_b^i[3]), b = 0, 1,\end{aligned}$$

and

$$\begin{aligned}X_b^i[1] &= E_k^{-1}(C_b^i[1]) \oplus t_k^1 \oplus t_{k_1}^1, \\ X_b^i[2] &= E_k^{-1}(C_b^i[2]) \oplus t_k^2 \oplus t_{k_1}^2, \\ X_b^i[3] &= E_k^{-1}(C_b^i[3]) \oplus t_k^3 \oplus t_{k_1}^3.\end{aligned}$$

We assume all $C_b^i[a], i = 1, \dots, q, b = 0, 1, a = 1, 2, 3$ are different. This can be realized easily. Then all $X_b^i[1], i = 1, \dots, q, b = 0, 1$ are different, all $X_b^i[2], i = 1, \dots, q, b = 0, 1$ are different and all $X_b^i[3], i = 1, \dots, q, b = 0, 1$ are different as well.

In the following, we only consider the equations which have new sample of E_{k_1} among the q equations in (5). If $X_b^i[a], b = 0, 1, j = 1, 2, 3$, who are the inputs of E_{k_1} in i th equation, have all appeared in the other $q - 1$ equations, then we don't sample in the i th equation. In fact, there may be $X_b^i[a_1] = X_{b'}^{i'}[a_2] = X_{b''}^{i''}[a_3]$, where a_1, a_2, a_3 are three different values belong to $\{1, 2, 3\}, b, b', b'' \in \{0, 1\}, i', i'' \in \{1, \dots, q\}$. Take $X_0^i[1]$ as example, there may be $b', b'' \in \{0, 1\}, i', i'' \in \{1, \dots, q\}$ such that $X_0^i[1] = X_{b'}^{i'}[2] = X_{b''}^{i''}[3]$. Therefore, it is easily to obtain that we have to sample E_{k_1} in at least $\lfloor \frac{q+2}{3} \rfloor$ equations among q . Then we consider the probability of the i th equation $f(k, C_0^i) = f(k, C_1^i)$ where we have new sample of E_{k_1} .

1) If

$$\Sigma(Y_0^i) = \Sigma(Y_1^i), \Theta(Y_0^i) = \Theta(Y_1^i), \quad (6)$$

then the i th equation holds. We want to know the upper bound of the probability of this case. So we only consider $\Sigma(Y_0^i) = \Sigma(Y_1^i)$. It means

$$E_{k_1}(X_0^i[1]) \oplus E_{k_1}(X_0^i[2]) \oplus E_{k_1}(X_0^i[3]) = E_{k_1}(X_1^i[1]) \oplus E_{k_1}(X_1^i[2]) \oplus E_{k_1}(X_1^i[3]).$$

By the randomness of E_{k_1} , the probability to make the above equation holds by sampling E_{k_1} is at most $\frac{1}{2^n - 6q}$.

2) When the equation set (6) doesn't holds but

$$E_{k_2}(\Sigma(Y_0^i)) \oplus E_{k_3}(\Theta(Y_0^i)) = E_{k_2}(\Sigma(Y_1^i)) \oplus E_{k_3}(\Theta(Y_1^i)), \quad (7)$$

then the i th equation holds as well. Firstly, we exclude the case that $\Sigma(Y_0^i), \Theta(Y_0^i), \Sigma(Y_1^i), \Theta(Y_1^i)$ in i th equation have all appeared in other $q - 1$ equations, whose probability is at most $(\frac{2q}{2^n - 6q})^4$. Then we assume that in i th equation that at least $\Sigma(Y_0^i)$ hasn't been appeared in other $q - 1$ equations, which means $E_{k_2}(\Sigma(Y_0^i))$ is a new sample. Thus the i th equation holds with probability at most $\frac{1}{2^n - 2q}$. Overall, this case happens with probability at most $(\frac{2q}{2^n - 6q})^4 + \frac{1}{2^n - 2q}$.

Sum of case 1) and 2), the i th equation holds with probability at most $\frac{1}{2^{n-6q}} + (\frac{2q}{2^n-6q})^4 + \frac{1}{2^{n-2q}} \leq \frac{q}{2^{n-3}}$ assuming $6q \leq 2^{n-1}$. Therefore, the q equations happens with probability at most $(\frac{q}{2^{n-3}})^{\frac{q-1}{3}}$. For PMAC.Plus, the key length $m \leq 2n$. Then when $q = 16$, we have $\Pr[\text{test}(k) = 1] \leq 2^{-2m}$ for sufficient large m and any $k \in \{0, 1\}^m \setminus \{k_1\}$.

4.2 Partial Key Recovery Attack for 3kf9

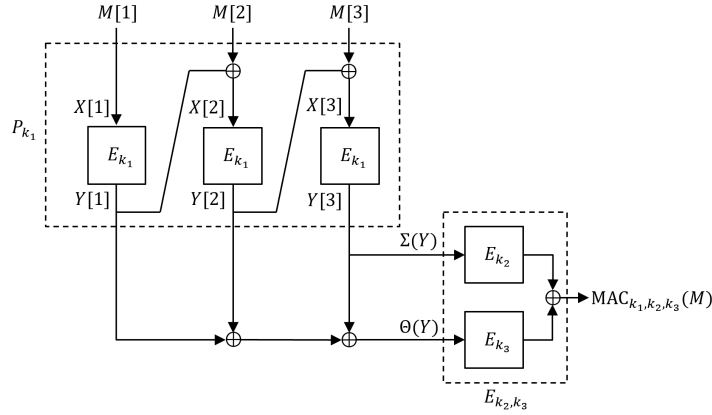


Fig. 9. 3kf9 with a 3-block message $M = (M[1], M[2], M[3])$.

We have introduced 3kf9 in section 3.2. Assume the three keys are $(k_1, k_2, k_3) \in (\{0, 1\}^m)^3$. The construction with message $M = (M[1], M[2], M[3])$ is defined as in figure 9.

Estimation of $\max_{k \in \{0, 1\}^m \setminus \{k_1\}} \Pr[\text{test}(k) = 1]$. The deviation is equals to

$$\max_{k \in \{0, 1\}^m \setminus \{k_1\}} \Pr[f(k, C_0^1) = f(k, C_1^1), \dots, f(k, C_0^q) = f(k, C_1^q)].$$

Here, the equation system

$$f(k, C_0^i) = f(k, C_1^i), i = 1, 2, \dots, q, \quad (8)$$

equals

$$E_{k_2}(\Sigma(Y_0^i)) \oplus E_{k_3}(\Theta(Y_0^i)) = E_{k_2}(\Sigma(Y_1^i)) \oplus E_{k_3}(\Theta(Y_1^i)), i = 1, 2, \dots, q,$$

where

$$\Sigma(Y_b^i) = E_{k_1}(X_b^i[3]), b = 0, 1,$$

$$\Theta(Y_b^i) = E_{k_1}(X_b^i[1]) \oplus E_{k_1}(X_b^i[2]) \oplus E_{k_1}(X_b^i[3]), b = 0, 1,$$

and

$$\begin{aligned} X_b^i[1] &= E_k^{-1}(C_b^i[1]), \\ X_b^i[2] &= E_{k_1}(X_b^i[1]) \oplus C_b^i[1] \oplus E_k^{-1}(C_b^i[2]), \\ X_b^i[3] &= E_{k_1}(X_b^i[2]) \oplus C_b^i[2] \oplus E_k^{-1}(C_b^i[3]). \end{aligned}$$

We assume all $C_b^i[1], i = 1, \dots, q, b = 0, 1$ are different. This can be realized easily. Then all $X_b^i[1], i = 1, \dots, q, b = 0, 1$ are different from each other, which means we have to sample for $E_{k_1}(X_b^i[1])$ in every equation in (8). Similar as the PMAC_Plus in section 4.1, every equation $f(k, C_0^i) = f(k, C_1^i)$ holds with probability at most $\frac{q}{2^{n-3}}$. Therefore, the q equations happens with probability at most $(\frac{q}{2^{n-3}})^q$. For 3kf9, the key length $m \leq 2n$. Then when $q = 5$, we have $\Pr[\text{test}(k) = 1] \leq 2^{-2m}$ for sufficient large m and any $k \in \{0, 1\}^m \setminus \{k_1\}$.

5 Conclusions

In this paper, we introduce secret state recovery attacks and key recovery attacks for a series of BBB MACs, leading to forgery attacks and drawing the security back to birthday bound.

Notice that we didn't apply secret state attack to PMAC_TBC3k in section 3.2 as other PMAC_Plus-like MACs. The reason is that for PMAC_Plus-like MACs based on block cipher, we can set an independent variable x that is xored with another value α_i before a block cipher $E_k: E_k(x \oplus \alpha_i)$, which is also called separability property in [15]. However, PMAC_TBC3k is based on tweakable block cipher and doesn't have such separability property.

Another notice is that we didn't apply key recover attack to SUM-ECBC-like MACs as PMAC_Plus-like MACs in section 4. Because SUM-ECBC-like MACs don't have *linear combination processes*, which means we can't construct a valid *test* function. However, we can still make another key recovery attack. Take SUM-ECBC as example. The complexity of the attack is $\mathcal{O}(2^m n)$ quantum queries assuming the size of block message is n bits and the size of keys is $4m$ bits. It is still more than birthday bound but better than Grover search for keys, whose complexity is $\mathcal{O}(2^{2m})$. So we just mention it as follows. Let $b \in \{0, 1\}, x \in \{0, 1\}^n$. Similar as in introduction (section 1, strategy 1), we can construct a function $C^{\text{MAC}_{k_1, k_2, k_3, k_4}}(b, x) = g_{k_1, k_2}(b, x) \oplus h_{k_3, k_4}(b, x)$ from SUM-ECBC through method C , where $g_{k_1, k_2}(b, x)$ and $h_{k_3, k_4}(b, x)$ have periods $1||s_1$ and $1||s_2$ respectively and k_1, k_2, k_3, k_4 are keys. Then we can construct a function $f: \{0, 1\}^m \times \{0, 1\}^m \times \{0, 1\} \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ as

$$\begin{aligned} f_{k_1, k_2, k_3, k_4}(k'_3, k'_4, b, x) &= C^{\text{MAC}_{k_1, k_2, k_3, k_4}}(b, x) \oplus h_{k'_3, k'_4}(b, x) \\ &= g_{k_1, k_2}(b, x) \oplus h_{k_3, k_4}(b, x) \oplus h_{k'_3, k'_4}(b, x). \end{aligned}$$

When $(k'_3, k'_4) = (k_3, k_4)$, then f equals $g_{k_1, k_2}(b, x)$ and have a period $1||s_1$. By applying Grover-meet-Simon algorithm we can recover k_3, k_4, s_1 , which leads to a forgery attack.

The further question is whether our attacks to DbHtS MACs are optimal or whether there are quantum security proofs to show the tightness of the bounds. We leave it as an open problem for future researches.

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A Quantum Basics

For two n -bit strings $x = x_1x_2\dots x_n$ and $y = y_1y_2\dots y_n$ where $x_i, y_i \in \{0, 1\}$, the *inner product* of them is $x \cdot y = x_1y_1 \oplus \dots \oplus x_ny_n$.

Qubits. We call quantum bits as *qubits*. Let notation *ket* " $|\cdot\rangle$ " represent a column vector. The n -qubit system is associated with the 2^n -dimension Hilbert space in complex field. Let the unit orthogonal basis of the Hilbert space be $\{|x\rangle\}$ where $x \in \{0, 1\}^n$, which also is the basis of the n -qubit system. If we let $|x\rangle$ be an unit column vector whose x -th component is 1 and other components are 0. Then any n -qubit state can be represented as the linear combination of the basis:

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle = \begin{bmatrix} \alpha_{00\dots 0} \\ \alpha_{00\dots 1} \\ \dots \\ \alpha_{11\dots 1} \end{bmatrix}.$$

where $\alpha_x \in \mathbb{C}$ and $\sum_{x \in \{0,1\}^n} |\alpha_x|^2 = 1$. It means any n -qubit state $|\psi\rangle$ is a unit length complex vector in the Hilbert space. If we measure $|\psi\rangle$, the superposition state will collapse into a basis state $|x\rangle$ with probability $|\alpha_x|^2$. Let notation *bra* " $\langle\cdot|$ " represent a row vector. Then $\langle\psi| = (|\psi\rangle)^\dagger = [\alpha_{00\dots 0}^\dagger, \dots, \alpha_{11\dots 1}^\dagger]$. We call $\langle\psi_1|\psi_2\rangle$ as *inner product* and $|\psi_1\rangle\langle\psi_2|$ as *outer product*. The *orthogonal basis* means the inner product of any two different vectors in the basis is equal to 0. For two independent quantum system $|\psi_1\rangle = \sum_{x_1 \in \{0,1\}^n} \alpha_{x_1} |x_1\rangle$ and $|\psi_2\rangle = \sum_{x_2 \in \{0,1\}^m} \alpha_{x_2} |x_2\rangle$, the joint state can be represented by *tensor product*:

$|\psi_1\rangle \otimes |\psi_2\rangle = \sum_{x_1 \in \{0,1\}^n} \sum_{x_2 \in \{0,1\}^m} \alpha_{x_1} \alpha_{x_2} (|x_1\rangle \otimes |x_2\rangle)$, where $|x_1\rangle \otimes |x_2\rangle$ can be represented as $|x_1 x_2\rangle$ as well.

Quantum Operations. Unitary operation (unitary matrix, unitary gate) U can transform a quantum state $|\psi_1\rangle$ to another quantum state $|\psi_2\rangle = U|\psi_1\rangle$. For a joint system of two independent quantum system $|\psi_1\rangle$ and $|\psi_2\rangle$, a joint quantum unit operation on the system can be represented as by tensor product $U \otimes V$, where $U \otimes V(|\psi_1\rangle \otimes |\psi_2\rangle) = (U|\psi_1\rangle) \otimes (V|\psi_2\rangle)$. There are some useful unitary operations. The first is Hadamard transform

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

on a single qubit. For example, when we apply it to state $|1\rangle$, we get $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ by

$$H|1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle.$$

If we apply H on $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ again, it is easy to know we will get $|1\rangle$ again. So H is the inverse of itself. Let $H^{\otimes n}$ be the operation that apply H to every qubit of n -qubit quantum state. Then for n -qubit basis state $|x\rangle$, we get $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$. Assume $|\psi_1\rangle, |\psi_2\rangle$ are n qubits state. The second unitary operation is $D_{|\psi_1\rangle} = 2|\psi_1\rangle\langle\psi_1| - I_{2^n}$. The transform $D_{|\psi_1\rangle}|\psi_2\rangle$ implements flipping the vector $|\psi_2\rangle$ with $|\psi_1\rangle$ as the symmetry axis, which is the core operation of Grover's algorithm.

Quantum Queries. Let $O_f : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle$ be a quantum oracle for implementing function $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$, where $|y\rangle$ is ancilla m qubits and $|x\rangle, |y\rangle$ are basis states. For $f : \{0, 1\}^n \rightarrow \{0, 1\}$, there is another available quan-

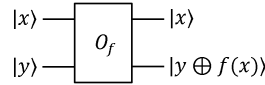


Fig. 10. The oracle $O_f : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle$.

tum oracle O'_f , which is constructed from O_f by the following quantum circuit. The input of the quantum circuit is $|x\rangle|1\rangle$ and the output is $(-1)^{f(x)}|x\rangle|1\rangle$. If we neglect the last qubit $|1\rangle$, then we get the quantum oracle $O'_f : |x\rangle \rightarrow (-1)^{f(x)}|x\rangle$.

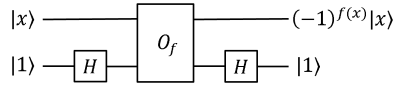


Fig. 11. The quantum circuit to construct oracle O'_f from oracle O_f .

In fact, for state

$$\begin{aligned}
 |\psi\rangle &= \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle \\
 &= \sum_{f(x)=0} \alpha_x |x\rangle + \sum_{f(x)=1} \alpha_x |x\rangle \\
 &= \sqrt{\sum_{f(x)=0} \alpha_x^2} \sum_{f(x)=0} \frac{\alpha_x}{\sqrt{\sum_{f(x)=0} \alpha_x^2}} |x\rangle + \sqrt{\sum_{f(x)=1} \alpha_x^2} \sum_{f(x)=1} \frac{\alpha_x}{\sqrt{\sum_{f(x)=1} \alpha_x^2}} |x\rangle \\
 &= \cos \theta |\psi_0\rangle + \sin \theta |\psi_1\rangle,
 \end{aligned}$$

$O'_f |\psi\rangle = O'_f (\cos \theta |\psi_0\rangle + \sin \theta |\psi_1\rangle) = \cos \theta |\psi_0\rangle - \sin \theta |\psi_1\rangle$. That is to say, O'_f flip the vector $|\psi\rangle$ with $|\psi_0\rangle$ as its symmetry axis.

In fact, in EUF-qCPA the quantum adversary maintains its state as follows. Let $|0^n\rangle$ be the initial state of adversary. Let O_i be the i -th quantum oracle query for adversary of MAC function and let U_0, U_1, \dots, U_q be the unitary operations applied by adversary between queries. Then after q quantum queries, the final state of adversary will be $U_q O_q \dots U_1 O_1 U_0 |0^n\rangle$. Finally, the adversary applies the final state to get some useful information and make forgeries.

Quantum Complexity. There are three dimensions to measure the complexities of a quantum algorithm: *query complexity*, *time complexity*, *memory complexity*. The *query complexity* counts the number of the superposition oracle queries O_f used for function f . Notice that the classical queries are specific cases of superposition queries. So we add the number of classical queries to query complexity in the quantum algorithm. The *time complexity* is the number of quantum operations (gates, unitaries). The *memory complexity* is the number of qubits in a quantum circuit. In our work, the time complexity of the quantum algorithm is close to query complexity, so we only consider query complexity and memory complexity.

B Grover's Algorithm and Proof of Theorem 1

B.1 Grover's Algorithm

The Grover's algorithm consists of a series of Grover's routines. Before all iterations, when we measure $|\psi\rangle$ the initial probability to get a u who satisfies $test(u) = 1$ is small. However, every routine of the algorithm will amplify the amplitude of such elements. When the amplitude of such elements is large enough,

then when we measure the state we will get a u who satisfies $test(u) = 1$. In our paper, we will apply the Grover algorithm to find some hidden useful information, such as the correct secret key. The quantum circuit of Grover's algorithm is showed figure 12 and the algorithm is showed in algorithm 1.

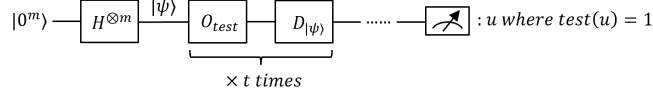


Fig. 12. The quantum circuit of Grover's algorithm.

Algorithm 1 Grover's Algorithm

Input: $m, t, test : \{0, 1\}^m \rightarrow \{0, 1\}$

Output: u who satisfies $test(u) = 1$

Let $O_{test}|u\rangle = (-1)^{test(x)}|u\rangle$, $D_{|\psi\rangle} = 2|\psi\rangle\langle\psi| - I_{2^m}$.

- 1: Initialize m qubits registers $|0^m\rangle$;
 - 2: Apply $H^{\otimes m}$ to obtain $|\psi\rangle = H^{\otimes m}|0^m\rangle$;
 - 3: Repeat Grover's routines with t times to get $|\phi\rangle = (D_{|\psi\rangle}O_{test})^t|\psi\rangle$;
 - 4: Measure $|\phi\rangle$ in order to get u who satisfies $test(u) = 1$;
 - 5: **return** u ;
-

Firstly, we divide the initial superposition state $|\psi\rangle = H^{\otimes m}|0^m\rangle = 2^{-\frac{m}{2}} \sum_{u \in \{0,1\}^m} |u\rangle$ as two parts by whether $test(u) = 0$ or not. Let $\theta = \arcsin \sqrt{\frac{e}{2^m}}$, $|\psi_0\rangle = \sum_{test(u)=0} \frac{1}{\sqrt{2^m-e}} |u\rangle$, $|\psi_1\rangle = \sum_{test(u)=1} \frac{1}{\sqrt{e}} |u\rangle$. Then the initial state $|\psi\rangle = \cos \theta |\psi_0\rangle + \sin \theta |\psi_1\rangle$. It is easy to know that $|\psi_1\rangle$ and $|\psi_0\rangle$ are two orthogonal unit vectors. Then we can establish a coordinate system with $|\psi_0\rangle$ and $|\psi_1\rangle$ as its orthogonal coordinate axis. In this coordinate system, the vector $|\psi\rangle$ is a unit-length vector with angle θ . In the first Grover routine, the query O_{test} flip the state $|\psi\rangle$ with $|\psi_0\rangle$ as the symmetry axis to a unit-length vector whose angle is $-\theta$, i.e., $\cos \theta |\psi_0\rangle - \sin \theta |\psi_1\rangle$. Then the flip operation $D_{|\psi\rangle}$ will flip vector $O_{test}|\psi\rangle$ with $|\psi\rangle$ as its symmetry axis to get a unit-length vector whose angle is 3θ , i.e., $\cos 3\theta |\psi_0\rangle + \sin 3\theta |\psi_1\rangle$. We show the above process in figure 13. It is easy to know that every iteration will add an angle 2θ . After t Grover iterations, we will get a quantum state $|\phi\rangle = \cos((2t+1)\theta) |\psi_0\rangle + \sin((2t+1)\theta) |\psi_1\rangle$. For $t = \lceil \frac{\pi}{4\theta} \rceil$, the final state $|\phi\rangle$ will be close to $|\psi_1\rangle$ and we will get a good elements with probability almost 1.

In the above Grover algorithm, we amplify some amplitudes of a uniform superposition state $|\psi\rangle = 2^{-m/2} \sum_{u \in \{0,1\}^m} |u\rangle$, which is produced by $H^{\otimes m}$ on $|0^m\rangle$. In the following, we will introduce a more general Grover algorithm: amplitude amplification algorithm (algorithm 2). It can amplify some amplitudes

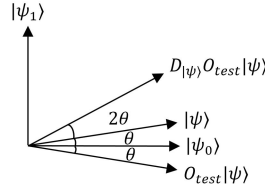


Fig. 13. The effect of the first Grover's routine.

of any quantum state $|\psi\rangle = \sum_{u \in \{0,1\}^m} \alpha_u |u\rangle$ as long as we can produce $|\psi\rangle$ by a unitary operation U on $|0^m\rangle$.

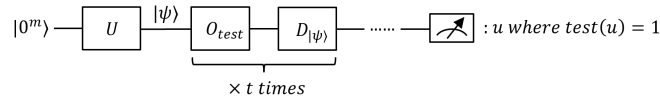


Fig. 14. The quantum circuit of amplitude amplification algorithm.

Algorithm 2 Amplitude Amplification Algorithm

Input: $m, t, test : \{0, 1\}^m \rightarrow \{0, 1\}$, unitary operation U

Output: u who satisfies $g(u) = 1$

Let $O_{test}|u\rangle = (-1)^{test(u)}|u\rangle$, $D_{|\psi\rangle} = 2|\psi\rangle\langle\psi| - I_{2^m}$.

1: Initialize m qubits registers $|0^m\rangle$;

2: Apply U to obtain $|\psi\rangle = U|0^m\rangle$;

3: Repeat Grover's routines with t times to get $|\phi\rangle = (D_{|\psi\rangle}O_{test})^t|\psi\rangle$;

4: Measure $|\phi\rangle$ in order to get a u who satisfies $test(u) = 1$;

5: **return** u ;

B.2 Proof of Theorem 1

Firstly, we prove the following lemma.

Lemma 1. (Adapted from [6]) Let $test : \{0, 1\}^m \rightarrow \{0, 1\}$, U as defined in definition 2. Assume in algorithm 2 the initial probability to get a $u \in \mathcal{U}$ after measuring $|\psi\rangle$ is p_0 and the quantum implement of $test(\cdot)$ costs j qubits. Then amplitude amplification algorithm with $t = \lceil \frac{\pi}{4 \arcsin \sqrt{p_0}} \rceil$ quantum queries to $test(\cdot)$ and $\mathcal{O}(m + j)$ qubits will output a $u \in \mathcal{U}$ with probability at least

$$\frac{p_0}{p_0 + p_1} [1 - (\frac{p_1}{p_0} + \sqrt{p_0 + p_1} + 2\sqrt{1 + \frac{p_1}{p_0} p_0})^2].$$

Proof. From lemma 1 we get the initial probability of measuring $|\psi\rangle$ to get a u who satisfies $test(u) = 1$ is between $[p_0, p_0 + p_1]$. Paper [6] has proofed when the initial probability is between $[p_0, p_0 + p_1]$, then after $t = \lceil \frac{\pi}{4 \arcsin \sqrt{p_0}} \rceil$ Grover's routines the probability to get a u who satisfies $test(u) = 1$ is at least $1 - (\frac{p_1}{p_0} + \sqrt{p_0 + p_1} + 2\sqrt{1 + \frac{p_1}{p_0}} p_0)^2$. Among all u who satisfies $test(u) = 1$, the proportion of $u \in \mathcal{U}$ is at least $\frac{p_0}{p_0 + p_1}$. Multiple them and then we can get the lower bound of the probability of getting a $u \in \mathcal{U}$.

By setting $U = H^{\otimes m}$ in amplitude amplification algorithm (algorithm 2), We obtain Grover's algorithm (algorithm 1) and $p_0 = \frac{\epsilon}{2^m}$. By lemma 1, we prove theorem 1.

C Simon's Algorithm and Proof of theorem 2

C.1 Simon's Algorithm

Simon's algorithm consist of many Simon's routines. The quantum circuit and the quantum algorithm of of a Simon's routine is showed in figure 16 and algorithm 3. If f is a periodic function with period s , Simon's routine outputs $v_i \in \{0, 1\}^n$ who is perpendicular to the period s . Assume l Simon's routines output v_1, v_2, \dots, v_l . If v_1, v_2, \dots, v_l span the whole space $\{0^n, s\}^\perp$, then we can get the nontrivial period s by solving the equation system $s \cdot v_i = 0, i = 1, \dots, l$. The whole Simon's algorithm is in algorithm 4.

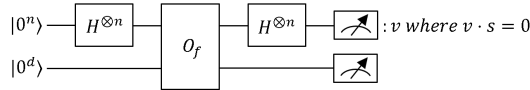


Fig. 15. The quantum circuit of Simon's routine.

In fact, we can parallel Simon's routines to construct Simon's algorithm as in algorithm 5. The quantum circuit of algorithm 5 is in figure 16.

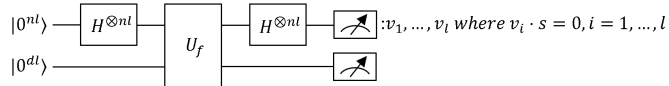


Fig. 16. The quantum circuit of Simon's algorithm.

Algorithm 3 Simon's routine

Input: $n, d, f : \{0, 1\}^n \rightarrow \{0, 1\}^d$ who has a hidden period s

Output: v who satisfies $v \cdot s = 0$

Let $O_f : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle$.

- 1: Initialize $n + d$ qubits registers $|0^n\rangle|0^d\rangle$;
- 2: Apply $U = (H^{\otimes n} \times I_{2^d})O_f(H^{\otimes n} \times I_{2^d})$ on $|0^n\rangle|0^d\rangle$ to get

$$|\psi\rangle = 2^{-n} \sum_{v \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot v} |v\rangle |f(x)\rangle;$$

- 3: Measure $|\psi\rangle$ and get the first n -bit v ;
 - 4: **return** v ;
-

Algorithm 4 Simon's algorithm

Input: $n, d, l, f : \{0, 1\}^n \rightarrow \{0, 1\}^d$ who has a hidden period s

Output: the period s

Let $O_f : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle$.

- 1: Initialize $n + d$ qubits registers $|0^n\rangle|0^d\rangle$;
- 2: **For** $i = 1$ to l **do**
- 3: Apply $U = (H^{\otimes n} \times I_{2^d})O_f(H^{\otimes n} \times I_{2^d})$ on $|0^n\rangle|0^d\rangle$ to get

$$|\psi_i\rangle = 2^{-n} \sum_{v_i \in \{0,1\}^n, x_i \in \{0,1\}^n} (-1)^{x_i \cdot v_i} |v_i\rangle |f(x_i)\rangle;$$

- 4: Measure $|\psi_i\rangle$ to get the first n -bits values v_i ;
 - 5: **end for**
 - 6: Compute the period s by solving the equation system $s \cdot v_i = 0, i = 1, 2, \dots, l$;
 - 7: **return** s ;
-

Algorithm 5 Simon's algorithm

Input: $n, d, l, f : \{0, 1\}^n \rightarrow \{0, 1\}^d$ who has a hidden period s

Output: the period s

Let $O_f : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle, U_f|x_1\rangle \dots |x_l\rangle|y_1\rangle \dots |y_l\rangle \rightarrow |x_1\rangle \dots |x_l\rangle|y_1 \oplus f(x_1)\rangle \dots |y_l \oplus f(x_l)\rangle$ with l calls to O_f .

- 1: Initialize $nl + dl$ qubits registers $|0^{nl}\rangle|0^{dl}\rangle$;
- 2: Apply $U = (H^{\otimes nl} \times I_{2^{dl}})U_f(H^{\otimes nl} \times I_{2^{dl}})$ on $|0^{nl}\rangle|0^{dl}\rangle$ to get

$$|\psi\rangle = 2^{-nl} \sum_{\substack{v_1, \dots, v_l \in \{0,1\}^n, \\ x_1, \dots, x_l \in \{0,1\}^n}} (-1)^{x_1 \cdot v_1} |v_1\rangle \dots (-1)^{x_l \cdot v_l} |v_l\rangle |f(x_1)\rangle \dots |f(x_l)\rangle;$$

- 3: Measure $|\psi\rangle$ to get the first nl -bits values v_1, v_2, \dots, v_l ;
 - 4: Compute the period s by solving the equation system $s \cdot v_i = 0, i = 1, 2, \dots, l$;
 - 5: **return** s ;
-

C.2 Proof of Theorem 2

Proof. Firstly, let us focus on Simon's routine. Kaplan et al (Appendix A in [18]) have proved for $t \in \{0, 1\}^n \setminus \{0^n\}$, there is a relationship between the probability of get a v who satisfies $v \cdot t = 0$ after measuring $|\psi\rangle$ and the proportion of x who satisfies $f(x) = f(x \oplus t)$ among $\{0, 1\}^n$. It is

$$\Pr_v[v \cdot t = 0] = \frac{1}{2}(1 + \Pr_x[f(x) = f(x \oplus t)]). \quad (9)$$

If $t = s$, we have $\Pr_x[f(x) = f(x \oplus s)] = 1$, which leads to $\Pr_v[v \cdot s = 0] = 1$ by equation (9). That is to say, for function f with period s , we can always get a v who satisfies $v \cdot s = 0$ after Simon's routine. By

$$\varepsilon(f) = \max_{t \in \{0, 1\}^n \setminus \{0^n, s\}} \Pr_x[f(x) = f(x \oplus t)],$$

we have $\Pr_v[v \cdot t = 0] \leq \frac{1}{2}(1 + \varepsilon(f))$ for $t \in \{0, 1\}^n \setminus \{0^n, s\}$. That is to say, the probability of getting a v who satisfies $v \cdot t = 0$ for $t \in \{0, 1\}^n \setminus \{0^n, s\}$ is at most $\frac{1}{2}(1 + \varepsilon(f))$.

Now, let us focus on Simon's algorithm. The line 1 to 3 are l parallel Simon routines. Then v_1, \dots, v_l are all satisfy $v_i \cdot s = 0, i \in \{1, \dots, l\}$. Therefore, the space spanned by v_1, \dots, v_l is the subspace of $\{0^n, s\}^\perp$. If the space spanned by v_1, \dots, v_l is equal to $\{0^n, s\}^\perp$, then we can get s by solving the equation system $v_i \cdot s = 0, i = 1, \dots, l$. However, Simon's algorithm may fail when there is at least one $t \in \{0, 1\}^n \setminus \{0^n, s\}$ such that $v_i \cdot t = 0, i = 1, \dots, l$. The probability of this bad case is at most $2^n \cdot (\frac{1+\varepsilon(f)}{2})^l$. Let $l = cn$ then we get theorem 2.

D Grover-meet-Simon Algorithm and Proof of Theorem 3

D.1 Grover-meet-Simon Algorithm

For Grover-meet-Simon problem in definition 5, Leader and May [24] firstly propose Grover-meet-Simon algorithm to solve it. The main idea is to search $u \in \mathcal{U}$ by Grover's algorithm and in every Grover's routine check whether or not each $u \in \mathcal{U}$ by whether $f(u, \cdot)$ is periodic or not, which can be implemented by Simon's algorithm. Assume the l parallel Simon routines in Simon's algorithm output v_1, \dots, v_l . For simplicity, we only check whether or not the rank of v_1, \dots, v_l is at most $n - 1$ instead of whether $f(u, \cdot)$ is periodic or not, this the first proposed in [16] and then combined with Grover's algorithm in [6]. The replacement is available for the following reason. For $u \in \mathcal{U}$, $f(u, x)$ is a periodic function. Thus the space spanned by v_1, \dots, v_l is the subspace of $\{0^n, s\}^\perp$. So the rank of such space is no more than $n - 1$. However, for $u \notin \mathcal{U}$, $f(u, x)$ is an aperiodic function. Thus the space spanned by v_1, \dots, v_l is the subspace of $\{0, 1\}^n$. For sufficient large l , the v_1, \dots, v_l can span the whole space $\{0, 1\}^n$. We let the the output of the test function be 1 when the rank of $\{v_1, \dots, v_l\}$ is at most $n - 1$. Otherwise,

it is 0. Therefore, Grover's routine will amplify the amplitude of $u \in \mathcal{U}$. At last, we can get a $u \in \mathcal{U}$ and its corresponding v_1, \dots, v_l . Like the Simon's algorithm, we can get s_u by solving the equation system $v_i \cdot s = 0, i = 1, \dots, l$ in the end. The whole Grover-meet-Simon is in algorithm 6 and the quantum circuit is in figure 17. More accurately, the whole algorithm is an amplitude amplification algorithm (algorithm 2) with Hadamard transform and the parallel Simon's routines without measurement as the unitary operation U in algorithm 2.

D.2 Proof of Theorem 3

Proof. When $u \in \mathcal{U}$, the classifier function $test$ will output 1. If we measure $|\psi\rangle$, it is easy to know the probability to get a $u \in \mathcal{U}$ is $\frac{\epsilon}{2^m}$. For $u \notin \mathcal{U}$, if there is at least one $t \in \{0, 1\}^n \setminus \{0^n\}$ who satisfies $t \cdot v_i = 0, i = 1, \dots, l$, then $test$ output 1 as well. By

$$\epsilon(f) = \max_{(u,t) \in \{0,1\}^m \times \{0,1\}^n \setminus (\mathcal{U}_s \cup \{0,1\}^m \times \{0^n\})} \Pr_x[f(u, x) = f(u, x \oplus t)],$$

this case happens with probability at most $2^n \cdot (\frac{1+\epsilon(f)}{2})^l$. By lemma 1, we will get the lower bound of the probability of get a $u \in \mathcal{U}$ after measuring $|\phi\rangle$. For $u \in \mathcal{U}$, Simon's algorithm with function $f(u, \cdot)$ output the period s_u with probability at least $1 - 2^n \cdot (\frac{1+\epsilon(f)}{2})^l$. Multiple them and then we can get lower bound of the probability of get a tuple $(u, s_u) \in \mathcal{U}_s$. Let $l = cn$. By paper [7], we get the qubits of these algorithm is $\mathcal{O}(m + cn^2 + cdn)$. Now, we have proved the theorem 3.

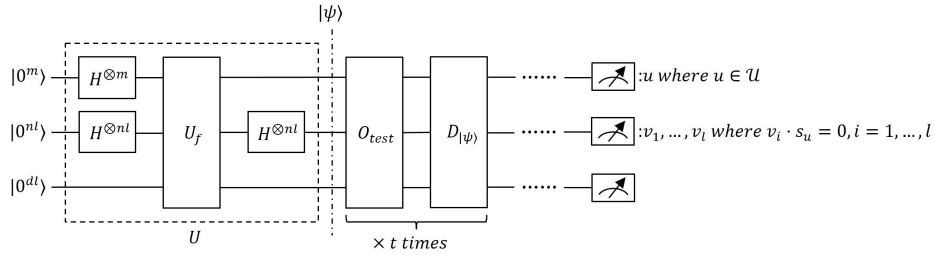


Fig. 17. The quantum circuit of Grover-meet-Simon algorithm.

Algorithm 6 Grover-meet-Simon Algorithm

Input: $m, n, r, l, t, f : \{0, 1\}^m \times \{0, 1\}^n \rightarrow \{0, 1\}^d$, for $u \in \mathcal{U} \subseteq \{0, 1\}^m$ that $f(u, \cdot)$ is a periodic function, otherwise it is an aperiodic function.

Output: a good element x

Let $O_f|u\rangle|x\rangle|y\rangle = |u\rangle|x\rangle|y \oplus f(u, x)\rangle, U_f|u\rangle|x_1\rangle \dots |x_l\rangle|y_1\rangle \dots |y_l\rangle = |u\rangle|x_1\rangle \dots |x_l\rangle|y_1 \oplus f(u, x_1)\rangle \dots |y_l \oplus f(u, x_l)\rangle$ with l calls to O_f , $test : \{0, 1\}^{m+nl+dl} \rightarrow \{0, 1\}$ with

$$test(u, v_1, \dots, v_l, y_1, \dots, y_l) = \begin{cases} 1, & \dim\{v_1, \dots, v_l\} \leq n-1 \\ 0, & \dim\{v_1, \dots, v_l\} = n \end{cases},$$

$O_{test}|u, v_1, \dots, v_l, y_1, \dots, y_l\rangle = (-1)^{test(u, v_1, \dots, v_l, y_1, \dots, y_l)}|u, v_1, \dots, v_l, y_1, \dots, y_l\rangle, D_{|\psi\rangle} = 2|\psi\rangle\langle\psi| - I_{2^m}$.

- 1: Initialize $m + nl + dl$ qubits registers $|0^m\rangle|0^{nl}\rangle|0^{dl}\rangle$;
- 2: Apply $U = (I_{2^m} \otimes H^{\otimes nl} \otimes I_{2^{dl}})U_f(H^{\otimes m} \otimes H^{\otimes nl} \otimes I_{2^{dl}})$ to $|0^m\rangle|0^{nl}\rangle|0^{dl}\rangle$ to obtain

$$|\psi\rangle = 2^{-\left(\frac{m}{2} + nl\right)} \sum_{\substack{u \in \{0, 1\}^m, \\ v_1, \dots, v_l \in \{0, 1\}^n, \\ x_1, \dots, x_l \in \{0, 1\}^n}} |u\rangle(-1)^{x_1 \cdot v_1} |v_1\rangle \dots (-1)^{x_l \cdot v_l} |v_l\rangle |f(u, x_1)\rangle \dots |f(u, x_l)\rangle;$$

- 3: Repeat Grover's routines with t times to get $|\phi\rangle = (D_{|\psi\rangle} O_{test})^t |\psi\rangle$;
 - 4: Measure $|\phi\rangle$ to in order to get the first $(m + nl)$ -bit values $u \in \mathcal{U}$ and v_1, \dots, v_l who satisfy $s_u \cdot v_i = 0, i = 1, \dots, l$;
 - 5: Compute the period s_u by solving the equation system $s_u \cdot v_i = 0, i = 1, 2, \dots, l$;
 - 6: **return** u, s_u ;
-