Cracking Matrix Modes of Operation with Goodness-of-Fit Statistics

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Abstract. The Hill cipher is a classical poly-alphabetical cipher based on matrices. Although known plaintext attacks for the Hill cipher have been known for almost a century, feasible ciphertext only attacks have been developed only about ten years ago and for small matrix dimensions. In this paper we extend the ciphertext only attacks for the Hill cipher in two ways. First, we present two attacks for the affine version of the Hill cipher. Secondly, we show that the presented attacks can be extended to several modes of operations. We also provide the reader with several experimental results and show how the message's language can influence the presented attacks.

1 Introduction

Two classical ciphers based on linear algebra are the Hill cipher [11] and its affine version [12]. Both use invertible matrices over integers modulo a to encipher messages, where a is the size of the language alphabet $\mathcal{A}$. The first step of the encryption process is the encoding of each plaintext letter into a numerical equivalent. The simplest encoding is `"a"` $= 0$, `"b"` $= 1$ and so on. After encoding, the plaintext is divided into blocks of size k and, then, each block is multiplied with an invertible matrix of size k. In the affine case, a second matrix is added to the result. After each block is transformed, the result is converted back into letters. To decipher messages, one must perform the above steps in reverse.

Although both ciphers are vulnerable to known plaintext attacks[3], efficient ciphertext only attacks have been developed only a decade ago [6] and only for the Hill cipher[4] with small ks. Note that as k increases simple brute force attacks fail. For example, in the case of the Hill cipher with $a = 26$, we have around 2^{17} keys for $k = 2$, 2^{40} keys for $k = 3$ and 2^{73} keys for $k = 4$ [6]. According to [7, 19], given a and k the exact number of invertible matrices can be computed. Note that in the case of the affine Hill cipher the computational effort made to brute force the Hill cipher is multiplied with a^k.

In 2007, Bauer and Millward [6] introduced a ciphertext only attack for the Hill cipher[5], that was later improved in [15, 17, 23]. The attack was independently published by Khazaei and Ahmadi [13]. The main idea of these attacks is to do a brute force attack on the key rows, instead of the whole matrix, and then recover the decryption matrix.

In [14], Kiele suggests the usage of block-chaining procedures to complicate the algebraic cryptanalytic techniques developed for the Hill cipher. We will show in this paper how to adapt the attacks described in [6, 13, 23] to different modes of operation (not only the block-chaining one) for both the Hill cipher and its affine version. Note that some modes do not require the key to be invertible, thus the attack presented in [15] does not work for all Hill based modes. For uniformity, we will only extend Yum and Lee's attack and leave as future work the extension of [15] to modes requiring invertible matrices. We stress that out of the three attacks [6, 13, 23] Yum and Lee's attack has the best performance to message recovery ratio.

[3] *i.e.* after a number of known messages are encrypted, one can easily recover the encryption key(s) if he has access to the corresponding ciphertexts.

[4] To the authors' knowledge no attack against the affine Hill cipher has been published.

[5] Bauer and Millward's attack for $k = 3$ was previously and independently described online by Wutka [22].

Another paper that motivated this study is [5]. The authors of [5] conjecture that the fourth cryptogram of the Kryptos sculpture [3] is either encrypted using the affine Hill cipher or some other sort of cipher mode of operation. We provide the reader with a preliminary study of these conjectures. To prove or disprove these conjectures, one has to find a way to adapt all the presented ciphertext attacks to the secret encoding versions of the (affine) Hill cipher and their corresponding modes of operation. Various partial answers for the secret encoding Hill cipher are provided in [23].

Structure of the paper. Notations and definitions are presented in Section 2. The core of the paper consists of two parts, Sections 3 and 4, that contain several key ranking functions and ciphertext only attacks. Experimental results are provided in Section 5. We conclude in Section 6. The letter frequencies and the Vigenère attack used in Section 5 are given in Appendices A and B.

2 Preliminaries

Notations. Throughout the paper, k will denote a security parameter. We use the notation $x \overset{\$}{\leftarrow} X$ when selecting a random element x from a sample space X. We also denote by $x \leftarrow y$ the assignment of value y to variable x. The subset $\{0, \ldots, q-1\} \in \mathbb{N}$ will be referred to as $[0, q]$. The set of matrices with α rows, β columns and entries from $\mathbb{G}$ is denoted by $M(\alpha, \beta, \mathbb{G})$, the set of invertible matrices by $GL(\alpha, \mathbb{G})$ and the transpose of matrix A by A^T. The number of letters in a string m is represented by $|m|$ and the set of all strings by $\mathcal{A}^\times$.

In this paper we use some C++ language operators (*i*.e. $==$ for equality testing, $+=$, $*=$ as compound assignment operators, $++$ for incrementing a variable and & as reference to a variable) as well as some native function (*i*.e. $size()$ for returning the size of the object, $substring(pos, npos)$ for returning a substring starting from pos and containing $npos$ characters, $push\_back(val)$ to add val at the end of a vector and $sort$ to sort a vector in descending order). For initializing all the entries of a vector vec with a value val we use the notation $vec \leftarrow \{val\}$. When presenting algorithms we consider only lower case messages represented by ASCII codes (*i*.e. `"c"` $-$ `"a"` $= 99 - 97 = 2$).

Conventions. To minimize repetitions, we employ the following system. When reading the attacks against the Hill based modes of operation we invite the reader to ignore red colored text, while in the case of the affine Hill based modes ignore the blue text. Also, when describing algorithms we prefer using verbose names for variables, while for mathematical descriptions we prefer notations. The last convention used is to store constants in look-up tables when their size is small (*e.g.* letter frequencies) and in maps, otherwise (*e.g.* quadgraph frequencies).

2.1 Ciphers

A cipher consists of three probabilistic polynomial-time algorithms: *Setup*, *Encrypt* and *Decrypt*. The first one takes as input a security parameter and outputs the secret key. The secret key together with the *Encrypt* algorithm are used to encrypt a message m. The last algorithm decrypts any message encrypted using the known secret key.

Hill cipher. The Hill cipher is a poly-alphabetical cipher based on linear algebra introduced by Lester S. Hill in [11]. We briefly provide the algorithms for the Hill cipher. Note that before encrypting/decrypting a text, the corresponding letters are encoded/decoded as follows: `"a"` to/from 0, `"b"` to/from 1 and so on.

Setup(λ): Set an integer $k \geq \lambda$ and choose $K_1 \overset{\$}{\leftarrow} GL(k, \mathbb{Z}_a)$. Output the secret key $sk = K_1$.
Encrypt(sk, m): Pad message m until $|m| \equiv 0 \bmod k$[6]. Divide m into blocks $m = m_1 \| \ldots \| m_\ell$, where $|m_i| = k$. Compute $c_i^T \leftarrow K_1 \cdot m_i^T$. Output the ciphertext $c = c_1 \| \ldots \| c_\ell$.

[6] Usually a rarely used letter, such as `"x"`, is appended to m until we get the desired length.

Decrypt(sk, c): Divide c into ℓ blocks $c = c_1 \| \ldots \| c_\ell$ and compute $m_i^T \leftarrow K_1^{-1} \cdot c_i^T$. Recover m by removing the padding.

Example 1. For clarity, we further provide the reader with an example from [6]. The message `"matrixencryptioniseasy"` is mapped into

$$12, 0, 19, 17, 8, 23, 4, 13, 2, 17, 24, 15, 19, 8, 14, 13, 8, 18, 4, 0, 18, 24.$$

If $K_1 \leftarrow \begin{pmatrix} 1 & 3 \\ 4 & 11 \end{pmatrix}$, then the first block is encrypted into $\begin{pmatrix} 1 & 3 \\ 4 & 11 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 22 \end{pmatrix}$. Therefore, we obtain the ciphertext `"mwsdzzrdbnrbribrkweqmy"`.

Affine Hill cipher. An affine variation of the Hill cipher was introduced in [12]. We shortly provide the algorithms for the affine Hill cipher.

Setup(λ): Set an integer $k \geq \lambda$ and choose $K_1 \xleftarrow{\$} GL(k, \mathbb{Z}_a)$ and $K_2 \xleftarrow{\$} M(k, 1, \mathbb{Z}_a)$. Output the secret key $sk = (K_1, K_2)$.

Encrypt(sk, m): Pad message m until $|m| \equiv 0 \bmod k$. Divide m into blocks $m = m_1 \| \ldots \| m_\ell$, where $|m_i| = k$. Compute $c_i^T \leftarrow K_1 \cdot m_i^T + K_2$. Output the ciphertext $c = c_1 \| \ldots \| c_\ell$.

Decrypt(sk, c): Divide c into ℓ blocks $c = c_1 \| \ldots \| c_\ell$ and compute $m_i^T \leftarrow K_1^{-1} \cdot (c_i^T - K_2)$. Recover m by removing the padding.

Other affine variations of the Hill cipher. In Table 1 we present all the possible affine variations of the Hill cipher. Note that $K_3 \xleftarrow{\$} M(k, 1, \mathbb{Z}_a)$. After performing some computations, we can see that for all variations we can recover m_i^T using $f(c_i) = K_1' \cdot c_i^T + K_2'$. Since we are interested only in recovering the encrypted messages and not the initial secret keys, all the presented attacks try to recover K_1' and K_2'. Thus, for the affine Hill cipher we only consider f for recovering m_i^T.

Encrypt	*Decrypt*	K_1'	K_2'
$c_i^T \leftarrow K_1 \cdot m_i^T + K_2$	$m_i^T \leftarrow K_1^{-1} \cdot (c_i^T - K_2)$	K_1^{-1}	$-K_1^{-1} K_2$
$c_i^T \leftarrow K_1 \cdot (m_i^T + K_2)$	$m_i^T \leftarrow K_1^{-1} \cdot c_i^T - K_2$	K_1^{-1}	$-K_2$
$c_i^T \leftarrow K_1 \cdot (m_i^T + K_2) + K_3$	$m_i^T \leftarrow K_1^{-1} \cdot (c_i^T - K_3) - K_2$	K_1^{-1}	$-K_1^{-1} K_3 - K_2$

Table 1. Affine variations of the Hill cipher.

2.2 Cipher Modes of Operation

When we encrypt messages block by block[7], identical blocks are mapped into identical ciphertexts. Thus, block patterns are preserved. This is an information leakage that can lead to security breaches. To address this issue several modes of operation where introduced in [8], such as: CBC, CTR, CFB and OFB.

In [4], the authors introduce a generalization of the CBC-MAC construction[8]. Based on Alagic *e*t al.'s generalization, we present a possible adaptation of the CBC, CTR and CFB modes of operation to the (affine) Hill cipher.

Let $E_k, D_k : M(k, k, \mathbb{Z}_a) \to M(k, k, \mathbb{Z}_a)$ be the matrix transformations of the (affine) Hill cipher's encryption and decryption. We further describe the encryption and decryption algorithms for CBC and CFB.

Encrypt(sk, m): Choose $iv \xleftarrow{\$} M(1, k, \mathbb{Z}_a)$ and pad message m until $|m| \equiv 0 \bmod k$. Divide m into blocks $m = m_1 \| \ldots \| m_\ell$, where $|m_i| = k$. Let $m_0 \leftarrow iv$. For CBC compute $c_i \leftarrow E_k(c_{i-1} + m_i)$, while for CFB compute $c_i \leftarrow E_k(c_{i-1}) + m_i$. Let $c = c_1 \| \ldots \| c_\ell$. The output is ciphertext (iv, c).

[7] usually called the ECB mode of operation

[8] the XOR operation is replaced with a generic group operation

Decrypt(sk, iv, c): Divide c into ℓ blocks $c = c_1\| \dots \|c_\ell$. For CBC compute $m_i \leftarrow D_k(c_i) - c_{i-1}$ and for CFB compute $m_i \leftarrow c_i - E_k(c_{i-1})$. Recover m by removing the padding.

In the case of CTR, the sender and the receiver each keep a state *ctr*. The initial value is chosen at random $ctr \overset{\$}{\leftarrow} M(1, k, \mathbb{Z}_a)$. Before each encryption *ctr* is updated as follows:

Update(ctr): Let $ctr = (\alpha_0, \dots, \alpha_{k-1})$ and $i \leftarrow k - 1$. Compute the following
1. $\alpha_i \leftarrow (\alpha_i + 1) \bmod a$,
2. If $\alpha_i == 0$, then $i \leftarrow (i - 1) \bmod k$ and go to step 1.

Now, the encryption and decryption algorithm for this mode of operation are:

Encrypt(sk, m): Pad message m until $|m| \equiv 0 \bmod k$. Divide m into blocks $m = m_1\| \dots \|m_\ell$, where $|m_i| = k$. Compute $ctr \leftarrow Update(ctr)$ and $c_i \leftarrow E_k(ctr) + m_i$. The output is ciphertext $c = c_1\| \dots \|c_\ell$.
Decrypt(sk, iv, c): Divide c into ℓ blocks $c = c_1\| \dots \|c_\ell$. Compute $ctr \leftarrow Update(ctr)$ and $m_i \leftarrow c_i - E_k(ctr)$. Recover m by removing the padding.

Example 2. For clarity, we provide the reader with some examples for the *Update* function. Let $a = 26$ and $k = 2$. Then $Update((1, 2)) = (1, 3)$, $Update((1, 25)) = (2, 0)$ and $Update((25, 25)) = (0, 1)$.

Although our attacks do not apply to the OFB mode, for completeness we provide its description.

Encrypt(sk, m): Choose $iv \overset{\$}{\leftarrow} M(1, k, \mathbb{Z}_a)$ and pad message m until $|m| \equiv 0 \bmod k$. Divide m into blocks $m = m_1\| \dots \|m_\ell$, where $|m_i| = k$. Let $x_0 \leftarrow iv$. Compute $x_i \leftarrow E_k(x_{i-1})$ and $c_i \leftarrow m_i + x_i$. Let $c = c_1\| \dots \|c_\ell$. The output is ciphertext (iv, c).
Decrypt(sk, iv, c): Divide c into ℓ blocks $c = c_1\| \dots \|c_\ell$. Let $x_0 \leftarrow iv$. Compute $x_i \leftarrow E_k(x_{i-1})$ and $m_i \leftarrow c_i - x_i$. Recover m by removing the padding.

Remark 1. Note that the CFB, CTR and OFB modes do not require K_1 to be invertible.

2.3 Statistical Models

When brute forcing rows, we cannot tell immediately if the decrypted text is correct or not. But we can statistically analyze the letters of the resulting text and check if they are reasonable enough. Using the frequencies of the recovered letters and the frequencies of the characters in original language, we can rank the rows according to their relevance to the ciphertext.

In order to rank[9] all possible rows for the decryption key, Yum and Lee [23] introduce a goodness-of-fit score function. Compared to the score functions presented in [6, 13], Yum and Lee's function describes the exact probability of the recovered plaintext. We briefly describe the goodness-of-fit score function in Algorithm 1.

Let E_K and D_K be the encryption and, respectively, decryption function of a cipher. Also, let $c \leftarrow E_K(m)$ be the given cryptogram and K' the key we want to rank. The goodness-of-fit function takes as input the letter frequency table *letter_freq* associated with the language m is written in (see Appendix A for some examples) and the letter frequency table *occurence* observed in $D_{K'}(c)$.

To automatically separate meaningful messages from random texts, we use an approach similar with the ones described in [10, 16]. When testing a list of strings for meaning, we first score each of them using Algorithm 2 and then output the highest scoring message.

The first and second inputs of the score function are a string *in* and the block frequency map (in our case either a digraph *di_freq* or a quadgraph *quad_freq* frequency map) associated with the language we are interested in. The fourth variable *num_of_letters* controls if we are observing digraphs (*i.e.* *num_of_letters* = 2) or quadgraph (*i.e.* *num_of_letters* = 4). When computing block frequency maps, some blocks may be missing entirely from the training corpus. To avoid assigning a likelihood of zero to these blocks, we use the *ad hoc* method found in [16][10].

[9] according to their relevance to a given cryptogram

[10] *i.e.* *block_default* $\leftarrow \log_{10}(0.01/\,num\_of\_blocks)$, where the total number of blocks found in the training corpus is denoted by *num_of_blocks*

Algorithm 1. The goodness-of-fit score function.

```
Input: A vector of letter occurrences $occurrence$.
Output: The vector's goodness-of-fit score $score$.
Function $goodness\_of\_fit(letter\_freq, occurrence)$:
    $score \leftarrow 1$;
    for $i \in [0, alphabet\_size]$ do
        $score \mathrel{*}= letter\_freq[i]^{occurrence[i]} / occurrence[i]!$
    end
    return $score$;
```

Algorithm 2. The score function.

```
Input: A string $in$, the bound $number\_of\_rows$.
Output: The string's score $score$.
Function $score\_function(in, block\_freq, block\_freq, num\_of\_letters)$:
    $score \leftarrow 0$;
    for $i \in [0, in.\,size() - num\_of\_letters]$ do
        $temp \leftarrow in.substr(i, num\_of\_letters)$;
        if $temp \in block\_freq$ then
            $score \mathrel{+}= block\_freq[temp]$;
        end
        else
            $score \mathrel{+}= block\_default$;
        end
    end
    return $score$;
```

Remark 2. To ease description, all frequency tables/maps will be implicit when presenting algorithms, unless otherwise specified.

3 Ranking Functions

The first step in attacking the (affine) Hill cipher and the associated modes of operation is to rank all possible rows according to their relevance to a given cryptogram. In this section we describe the ranking functions latter used in the attacks presented in Section 4.

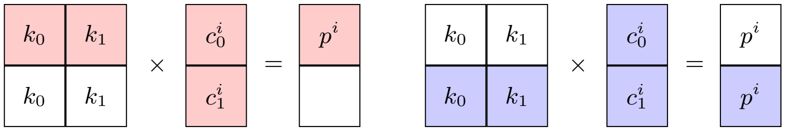
3.1 (Affine) ECB

In [23], the authors describe a ranking algorithm for the Hill cipher. We choose to present it in this section (Algorithm 3, red text) because it is tightly linked with the affine version that we introduce (Algorithm 3, blue text).

Let $matrix\_size = k = 2$ and let $enc = c$ be a Hill cipher cryptogram. We illustrate the influence of a given row on the decrypted plaintext p in Figure 1. We observe that if the first and second rows are equal we obtain the same letter p^i after decryption. Thus, is enough to decrypt the ciphertext using only the first row ($hill\_line\_decrypt$). Since we do not have duplicates, the resulting text msg is k times shorter than c. After decryption we compute the letter frequency observed in msg and use the $goodness\_of\_fit$ function to obtain the row's score. After all the rows have been ranked, we sort them in descending order according to their score. In the case of the affine Hill cipher the ranking algorithm is similar. The main difference is that instead of having to brute force k_0 and k_1, we also have to do an exhaustive search on k_2 (Figure 2). The algorithm for the generic case is given in Algorithm 3.

In some cases storing a vector of size a^k[11] might be troublesome. Thus, we further consider that $fit.size() = B$, where B is dependent on the available memory. Note that in this case fit must be sorted and when an element is inserted we first check if its score is higher than the lowest score from fit and if it is, the element replaces the lowest scoring element from fit.

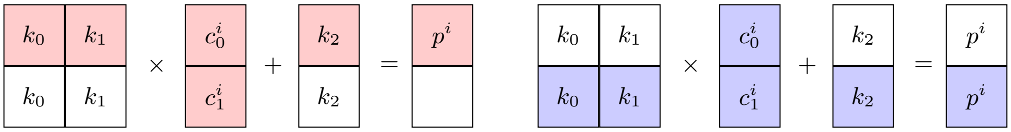
We usually work with small values of $alphabet\_size$ and the $msg.size()$ and thus we consider the complexity of the $goodness\_of\_fit$ and of multiplication as $\mathcal{O}(1)$. Hence, the Hill version of Algorithm 3 performs $\mathcal{O}(a^k)$ $hill\_line\_decryptions$ and sorts a vector of size B. So, it has a complexity of $\mathcal{O}(ka^k + B\log B)$. In the case of the affine Hill cipher, the only change is that we perform $\mathcal{O}(a^{k+1})$ $affine\_hill\_line\_decryptions$. So, the complexity becomes $\mathcal{O}(ka^{k+1} + B\log B)$.



(a) Line 1.

(b) Line 2.

Fig. 1. Line propagation in ECB.



(a) Line 1.

(b) Line 2.

Fig. 2. Line propagation in affine ECB.

3.2 (Affine) CBC, CTR, CFB

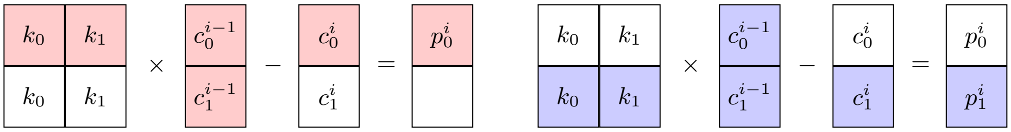
Again, let $matrix\_size = 2$ and let enc be a Hill cipher cryptogram. The effect of a given row on the decrypted plaintext is shown in Figure 3 for CBC, in Figure 4 for CTR and in Figure 5 for CFB. Compared to ECB, we can easily see that if the first and second row are identical the resulting letters are different. Thus, we need the full decryption of the Hill cipher to rank rows. After decryption, we break the resulting msg in two parts msg_0 and msg_1. The first part contains the letters in even positions and the second one the letters in odd positions. After we score each part, we store them in $fit[0]$ and, respectively, $fit[1]$. The last step is to sort the two vectors in descending order by score. The case of the affine Hill cipher is similar.

For the Hill modes attack, we perform $\mathcal{O}(a^k)$ decryptions, while for the affine version the number of decryptions is $\mathcal{O}(a^{k+1})$. Both algorithms sort k vectors of size B. Thus, the complexities are $\mathcal{O}(k^2a^k + kB\log B)$ and $\mathcal{O}(k^2a^{k+1} + kB\log B)$ for the Hill attack and, respectively, for the affine attack.

[11] a^{k+1} for the affine version

Algorithm 3. The algorithm for ranking all possible rows for (affine) ECB.

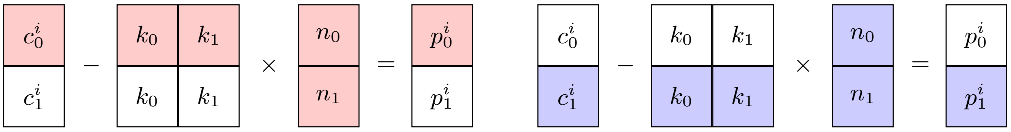
```
Input: The ciphertext $enc$.
Output: A vector $fit$ containing all possible rows sorted by the goodness-of-fit score.
Function $affine\_hill\_line\_decrypt(conv, key_1, key_2)$:
    $msg\_int[enc.size()/matrix\_size] \leftarrow \{0\}$;
    for $i \in [0, conv.size()/matrix\_size]$ do
        for $j \in [0, matrix\_size]$ do
            $msg\_int[i] \leftarrow (msg\_int[i] + key_1[j] \cdot conv[i \cdot matrix\_size + j]) \bmod alphabet\_size$;
        end
        $msg\_int[i] \leftarrow (msg\_int[i] + key_2[i \bmod matrix\_size]) \bmod alphabet\_size$;
    end
    return $msg\_int$;
Function $affine\_ecb\_rank(enc)$:
    for $key_1[0], \ldots, key_1[matrix\_size - 1] \in [0, alphabet\_size]$ do
        for $key_2 \in [0, alphabet\_size]$ do
            $letter\_freq[alphabet\_size] \leftarrow \{0\}$;
            $conv \leftarrow encode(enc)$;
            $msg\_int \leftarrow hill\_line\_decrypt(enc, key_1)$;
            $msg\_int \leftarrow affine\_hill\_line\_decrypt(enc, key_1, key_2)$;
            $msg \leftarrow decode(msg\_int)$
            for $i \in [0, msg.size()]$ do
                $letter\_freq[msg[i] -$ "a"$]{+}{+}$;
            end
            $letter\_freq.sort()$; \\only for Algorithm 6;
            $letter\_freq.sort()$; \\only for Algorithm 6;
            $score \leftarrow goodness\_of\_fit(letter\_freq, letter\_freq)$;
            $fit.push\_back((key_1, score))$;
            $fit.push\_back((key_1, key_2, score))$;
        end
    end
    $fit.sort()$;
    return $fit$;
```



(a) Line 1.

(b) Line 2.

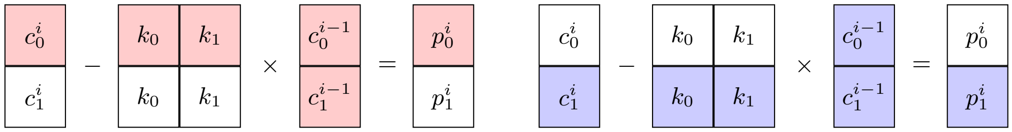
Fig. 3. Line propagation in CBC.



(a) Line 1.

(b) Line 2.

Fig. 4. Line propagation in CTR.



(a) Line 1.

(b) Line 2.

Fig. 5. Line propagation in CFB.

Algorithm 4. The algorithm for ranking all possible rows for (affine) CBC, CTR, CFB.

```
Input: The ciphertext $enc$ and the initialization vector $iv$.
Output: A family of vectors $fit$ containing all possible rows sorted by the goodness-of-fit score.
Function $affine\_mode\_rank(enc, iv)$:
    for $a[0], \ldots, a[matrix\_size-1] \in [0, alphabet\_size]$ do
        for $b \in [0, alphabet\_size]$ do
            $letter\_freq[matrix\_size][alphabet\_size] \leftarrow \{0\}$;
            for $i \in [0, matrix\_size]$ do
                for $j \in [0, matrix\_size]$ do
                    $key_1[i][j] \leftarrow a[j]$;
                end
                $key_2[i] \leftarrow b$;
            end
            $conv \leftarrow encode(enc)$;
            $msg\_int \leftarrow mode\_decrypt(enc, iv, key_1)$;
            $msg\_int \leftarrow affine\_mode\_decrypt(enc, iv, key_1, key_2)$;
            $msg \leftarrow decode(msg\_int)$
            for $i \in [0, msg.size()/ matrix\_size]$ do
                for $j \in [0, matrix\_size]$ do
                    $letter\_freq[j][msg[i \cdot matrix\_size + j] -$ "a"$]{+}{+}$;
                end
            end
            $letter\_freq.sort()$; \\only for Algorithm 8;
            for $i \in [0, matrix\_size]$ do
                $letter\_freq[i].sort()$; \\only for Algorithm 8;
                $score \leftarrow goodness\_of\_fit(letter\_freq, letter\_freq[i])$;
                $fit[i].push\_back((a, score))$;
                $fit[i].push\_back((a, b, score))$;
            end
        end
    end
    for $i \in [0, matrix\_size]$ do
        $fit[i].sort()$;
    end
    return $fit$;
```

4 Message Recovering Attacks

After the ranking step is over, we can proceed to the recovering step. When searching for the original message a lot of random text is produced. To filter random messages from ones with meaning we use the *score_function* to score each message and we always output the highest scoring one.

4.1 (Affine) ECB

The authors of [6, 23] describe the message recovering algorithm for the Hill cipher, but they do not provide an automatic detection method for the original message. On the other hand, the authors of [13] trade-off success probability for an unique output. The gap is filled in [15]. We present the algorithm in this section (Algorithm 5, red text), instead of Section 2, because of its link to the affine version we introduce (Algorithm 5, blue text). Due to better results in practice, in Algorithm 5 we use a different scoring function[12] than the one from [15][13]. Also, compared to [15], we only output the highest scoring message without lowering the success probability.

After ranking all possible rows, we need to find the decryption key's rows (*check_variants*) and their order (*check_variant*). Hence, Algorithm 5 checks all possible row combinations with index less than $number\_of\_rows = B$. Note that the success probability is dependent on $number\_of\_rows$[14]. After selecting k rows from fit, we test all possible row permutations[15], decrypt *enc* and rank the result. If one of the decrypted texts has a higher score than the stored message *global_msg*, we overwrite *global_msg* and update *global_score*. The main differences between the Hill cipher attack and the affine Hill cipher attack are: the call to the affine ranking algorithm, the creation of k_2 and the call to the affine decryption algorithm.

For the same reasons as in Section 3.1, we further consider the complexity of the *score_function* as $\mathcal{O}(1)$. After the row ranking step, both message recovering algorithms perform $\mathcal{O}(B!/(B-k)!)$ decryptions. Thus, the complexities for the Hill attack and for the affine attack are $\mathcal{O}(ka^k + B\log B + k^2B!/(B-k)!)$ and, respectively, $\mathcal{O}(ka^{k+1} + B\log B + k^2B!/(B-k)!)$.

4.2 Affine ECB (Second Approach)

In [23], the authors propose a ranking method for the Hill cipher with unknown encoding and decoding functions. The basic idea is that encoding functions act as substitution ciphers and thus leave letter distributions intact. According to their method, to score a row one needs to sort in ascending order both *letter_freq* and *occurrence* and then use Algorithm 1 to obtain the row's score. Note that Yum and Lee do not provide a message recovering algorithm.

The affine Hill cipher can be seen as the composition of a Hill cipher and a Vigenère cipher. Thus, we use Yum and Lee's ranking method to find K_1''s rows (*ecb_rank*), decrypt the cryptogram using the trial K_1' (*hill_decrypt*) and then use a Vigenère message recovery algorithm (*break_vigenere*) to find K_2'. This method is formally described in Algorithm 6. Note that *break_vigenere*[16] returns the score of the *trial_msg*. Unfortunately, we can not use only this score to filter messages. For example, when $k = 2$ the texts `easy` and `aeys` have the same *trial_score*. Hence, we use a second scoring system based on quadgrams to differentiate between trial messages with the same score. Note that the only difference between *check_variants*_2 and *check_variants* is that the latter is using the *check_variant*_2 function.

We consider the complexity of *break_vigenere* as being $\mathcal{O}(1)$, since it is linear in the cryptogram's size. Then, the complexity of the second algorithm is $\mathcal{O}(ka^k + B\log B + k^2B!/(B-k)!)$.

4.3 (Affine) CBC, CTR, CFB

The main difference between ECB and the other the modes is that after the ranking step is over, in the former case we know the exact position of the key rows. Thus, in Algorithm 7 we iterate over all rows (*check_variants_mode*), decrypt the cryptogram and then score the result (*check_variant_mode*).

The *check_variants_mode* function performs $\mathcal{O}(B^k)$ decryptions. Thus, Algorithm 7's complexity for the Hill based modes attack and for the affine versions is $\mathcal{O}(k^2a^k + kB\log B + k^2B^k)$ and, respectively, $\mathcal{O}(k^2a^{k+1} + kB\log B + k^2B^k)$.

[12] based on quadgraphs

[13] based on the index of coincidence

[14] see Section 5 for the experimental results

[15] σ_i denotes the ith permutation of length *mat_size*

[16] see Appendix B for a concrete algorithm

Algorithm 5. The algorithm for breaking (affine) ECB.

```
Input: The ciphertext $enc$, the bound $number\_of\_rows$.
Output: The best possible message $global\_msg$ and its associated score $global\_score$.
Function $check\_variant(enc, rows, \& global\_score, \& global\_msg)$:
    $best\_score \leftarrow -\infty$;
    for $i \in [0, matrix\_size!]$ do
        for $s \in [0, matrix\_size]$ do
            for $t \in [0, matrix\_size]$ do
                $key_1[s][t] \leftarrow rows[\sigma_i[s]].key_1[t]$;
            end
            $key_2[s] \leftarrow rows[\sigma_i[s]].key_2$;
        end
        $trial\_msg \leftarrow hill\_decrypt(enc, key_1)$;
        $trial\_msg \leftarrow affine\_hill\_decrypt(enc, key_1, key_2)$;
        $trial\_score \leftarrow score\_function(trial\_msg, quad\_freq, quad\_default, 4)$;
        if $trial\_score > best\_score$ then
            $best\_score \leftarrow trial\_score$;
            $best\_msg \leftarrow trial\_msg$;
        end
    end
    if $best\_score > global\_score$ then
        $global\_score \leftarrow best\_score$;
        $global\_msg \leftarrow best\_msg$;
    end
Function $check\_variants(enc, fit, number\_of\_rows)$:
    $global\_score \leftarrow -\infty$;
    $global\_msg \leftarrow$ "";
    for $i_0 \in [0, number\_of\_rows]$ do
        for $i_1 \in [i_0 + 1, number\_of\_rows]$ do
            ...
            for $i_{matrix\_size-1} \in [i_{matrix\_size-2} + 1, number\_of\_rows]$ do
                $trial\_rows \leftarrow \varnothing$;
                for $j \in [0, matrix\_size]$ do
                    $trial\_rows.push\_back(fit[i_j])$;
                end
                $check\_variant(enc, trial\_rows, global\_score, global\_msg)$;
            end
        end
    end
    return $(global\_score, global\_msg)$;
Function $affine\_ecb\_attack(enc, number\_of\_rows)$:
    $fit \leftarrow affine\_ecb\_rank(enc)$;
    return $check\_variants(enc, fit, number\_of\_rows)$;
```

4.4 Affine CBC, CTR, CFB (Second Approach)

As in the case of the affine Hill cipher, attacking a affine based mode can be interpreted as attacking a Hill-Vigenère cipher mode of operation. We present this complementary attack in Algorithm 8. Note that the only difference between *check_variants_mode* and *check_variants_mode_2* is that the former uses the *check_variant_mode_2* function. The time complexity of Algorithm 8 is $\mathcal{O}(k^2 a^k + kB \log B + k^2 B^k)$.

Algorithm 6. The algorithm for breaking affine ECB (second approach).

```
Input: The ciphertext $enc$, the bound $number\_of\_rows$.
Output: The best possible message $global\_msg$ and its associated score $global\_score$.
Function $check\_variant\_2(enc, rows, \& global\_score, \& global\_msg)$:
    $best\_score \leftarrow -\infty$;
    for $i \in [0, matrix\_size!]$ do
        for $s \in [0, matrix\_size]$ do
            for $t \in [0, matrix\_size]$ do
                $key_1[s][t] \leftarrow rows[\sigma_i[s]].key_1[t]$;
            end
        end
        $hill\_msg \leftarrow hill\_decrypt(enc, key_1)$;
        $(trial\_score, trial\_msg) \leftarrow break\_vigenere(hill\_msg)$;
        if $trial\_score > best\_score$ then
            $best\_score \leftarrow trial\_score$;
            $best\_msg \leftarrow trial\_msg$;
        end
        if $trial\_score == best\_score$ then
            $first\_quad\_score \leftarrow score\_function(best\_msg, quad\_freq, quad\_default, 4)$;
            $second\_quad\_score \leftarrow score\_function(trial\_msg, quad\_freq, quad\_default, 4)$;
            if $second\_quad\_score > first\_quad\_score$ then
                $best\_msg \leftarrow trial\_msg$;
            end
        end
    end
    if $best\_score > global\_score$ then
        $global\_score \leftarrow best\_score$;
        $global\_msg \leftarrow best\_msg$;
    end
Function $affine\_ecb\_attack\_2(enc, number\_of\_rows)$:
    $fit \leftarrow ecb\_rank(enc)$;
    return $check\_variants\_2(enc, fit, number\_of\_rows)$;
```

5 Experimental Results

We implemented Algorithms 5 to 8 in order to see the relation between B and the algorithms' success probability[17]. To see the influence of the message's native language on the attack algorithms' recovery rate, we tested this type of relation for eight languages: Danish (DN), English (EN), Finnish (FN), French (FR), German (GE), Polish (PL), Spanish (SP) and Swedish (SW). We also computed the running time of Algorithms 5 to 8 for the English language and $k = 2$ (Section 5.5). Besides providing the reader with some benchmarks, we also wanted to have a precise comparison[18] between the two affine attack approaches.

In our implementations, frequency tables have $a = 26$ values and are derived from the frequencies provided in [16]. For completeness, we describe the tables in Appendix A. The quadgrams for the English language are downloaded from [16], while the digraph[19] frequencies are computed from the quadgraph map. The algorithm for breaking Vigenère is given in Appendix B.

For computing the success probability we used 100 texts with 100 letters (without diacritical marks) for each language. Each text was encrypted with a different key(s)/initialization vector/counter. The texts are taken from news items found in the Leipzig Corpora Collection [9]. The keys, initialization vectors and

[17] We refer the reader to Sections 5.2 to 5.4 for the results.

[18] that takes into account the hidden constants found in asymptotic notations

[19] If *abcd* is a quadgraph, we consider *ac* as a digraph.

Algorithm 7. The algorithm for breaking (affine) CBC, CTR, CFB.

```
Input: The ciphertext $enc$, the initialization vector $iv$, the bound $number\_of\_rows$.
Output: The best possible message $global\_msg$ and its associated score $global\_score$.
Function $check\_variant\_mode(enc, iv, rows, \& global\_score, \& global\_msg)$:
    for $s \in [0, matrix\_size]$ do
        for $t \in [0, matrix\_size]$ do
            $key_1[s][t] \leftarrow rows[s].a[t]$;
        end
        $key_2[s] \leftarrow rows[s].b$;
    end
    $trial\_msg \leftarrow mode\_decrypt(enc, iv, key_1)$;
    $trial\_msg \leftarrow affine\_mode\_decrypt(enc, iv, key_1, key_2)$;
    $trial\_score \leftarrow score\_function(trial\_msg, quad\_freq, quad\_default, 4)$;
    if $trial\_score > global\_score$ then
        $global\_score \leftarrow trial\_score$;
        $global\_msg \leftarrow trial\_msg$;
    end
Function $check\_variants\_mode(enc, fit, number\_of\_rows)$:
    $global\_score \leftarrow -\infty$;
    $global\_msg \leftarrow$ "";
    for $i_0 \in [0, number\_of\_rows]$ do
        for $i_1 \in [0, number\_of\_rows]$ do
            ...
            for $i_{matrix\_size-1} \in [0, number\_of\_rows]$ do
                $trial\_rows \leftarrow \varnothing$;
                for $j \in [0, matrix\_size]$ do
                    $trial\_rows.push\_back(fit[j][i_j])$;
                end
                $check\_variant\_mode(enc, iv, trial\_rows, global\_score, global\_msg)$;
            end
        end
    end
    return $(global\_score, global\_msg)$;
Function $affine\_mode\_attack(enc, number\_of\_rows)$:
    $fit \leftarrow affine\_mode\_rank(enc, iv)$;
    return $check\_variants\_mode(enc, iv, fit, number\_of\_rows)$;
```

counters are generated using the default generator found in the GMP library [2]. When invertible keys were needed, we computed the inverse using the Armadillo library [20] and tested if the determinant is coprime with 26.

5.1 Unicity Distance of a Cipher

When analyzing the experimental results, the reader will observe different message recovery rates for different languages. These differences arise from distinct unicity distances[20] for distinct languages. The exact formula for the unicity distance when $a = 26$ is $\log_2 26^k / (\log_2 26 - H)$, where H is the language's entropy. Note that in our case the unicity distance is computed for one key row and we estimated the entropy from the frequency tables provided in Appendix A. The results for the unicity distance are provided in Table 2. We can see that in the case of the Polish language we need more letters per row than for the Finnish language. This gap will be more pronounced when determining the message recovery rates.

[20] The minimum ciphertext length required to determine the secret key almost uniquely.

Algorithm 8. The algorithm for breaking affine CBC, CTR, CFB (second approach).

```
Input: The ciphertext $enc$, the initialization vector $iv$, the bound $number\_of\_rows$.
Output: The best possible message $global\_msg$ and its associated score $global\_score$.
Function $check\_variant\_mode\_2(enc, iv, rows, \& global\_score, \& global\_msg)$:
    for $s \in [0, matrix\_size]$ do
        for $t \in [0, matrix\_size]$ do
            $key_1[s][t] \leftarrow rows[s].a[t]$;
        end
    end
    $hill\_msg \leftarrow mode\_decrypt(enc, iv, key_1)$;
    $(trial\_score, trial\_msg) \leftarrow break\_vigenere(hill\_msg)$;
    if $trial\_score > global\_score$ then
        $global\_score \leftarrow trial\_score$;
        $global\_msg \leftarrow trial\_msg$;
    end
Function $affine\_mode\_attack\_2(enc, number\_of\_rows)$:
    $fit \leftarrow mode\_rank(enc, iv)$;
    return $check\_variants\_mode\_2(enc, iv, fit, number\_of\_rows)$;
```

Language	$k = 2$	$k = 3$	$k = 4$
Danish	15.4323	23.1485	30.8647
English	18.2180	27.3270	36.4359
Finnish	12.0307	18.0460	24.0614
French	13.3713	20.0569	26.7425
German	15.6257	23.4386	31.2515
Polish	22.3918	33.5878	44.7837
Spanish	13.7891	20.6836	27.5781
Swedish	16.4837	24.7256	32.9674

Table 2. Unicity distance.

5.2 Hill Modes of Operation Message Recovery Rates

	B	DN	EN	FN	FR	GE	PL	SP	SW
ECB	2	94	93	100	96	95	84	96	95
	4	99	100	100	98	100	91	100	100
CBC	1	95	95	100	99	97	84	99	99
	2	99	99	100	100	100	90	100	100
CTR	1	96	93	100	96	98	87	100	98
	2	99	98	100	99	100	90	100	100
CFB	1	97	92	99	96	95	87	98	98
	2	100	99	100	100	99	91	100	100

Table 3. Number of recovered messages for the Hill modes of operation when $k = 2$.

	B	DN	EN	FN	FR	GE	PL	SP	SW
ECB	8	88	59	97	90	71	22	87	80
	16	95	77	100	95	86	45	96	94
	32	97	87	100	98	94	68	99	99
CBC	4	86	57	99	92	71	18	91	78
	8	93	68	99	96	80	34	96	86
	16	96	80	100	96	89	55	97	96
CTR	4	64	40	84	65	46	11	68	45
	8	80	59	94	87	67	19	83	66
	16	91	75	97	93	80	48	92	77
CFB	4	85	53	99	90	73	12	89	78
	8	93	66	99	94	81	36	94	87
	16	96	79	100	97	91	52	96	96

Table 4. Number of recovered messages for the Hill modes of operation when $k = 3$.

	B	DN	EN	FN	FR	GE	PL	SP	SW
ECB	512	78	48	97	89	72	10	85	74
	1024	88	65	98	91	89	19	94	86
	2048	95	80	99	95	94	39	95	93
CBC	32	78	50	97	89	69	13	88	72
	64	87	67	99	91	86	21	93	84
	128	93	78	99	95	94	45	95	93
CTR	32	71	37	91	77	55	6	80	64
	64	87	58	97	90	79	21	90	83
	128	93	75	100	95	94	40	99	88
CFB	32	78	48	97	88	69	14	86	73
	64	87	65	98	91	85	18	92	85
	128	93	75	99	95	95	45	94	95

Table 5. Number of recovered messages for the Hill modes of operation when $k = 4$.

5.3 Affine Hill Modes of Operation Message Recovery Rates (First Approach)

	B	DN	EN	FN	FR	GE	PL	SP	SW
ECB	2	89	80	100	90	88	54	93	92
	4	97	94	100	98	99	79	98	99
	8	99	99	100	99	99	87	99	100
CBC	1	93	85	100	99	85	57	96	93
	2	97	88	100	99	93	68	98	100
	4	99	95	100	99	99	78	100	100
CTR	1	92	72	100	93	90	48	96	95
	2	97	88	100	96	98	68	99	99
	4	98	97	100	99	99	78	100	100
CFB	1	89	80	100	95	91	54	98	93
	2	97	92	100	98	97	69	100	99
	4	99	97	100	99	99	83	100	100

Table 6. Number of recovered messages for the affine Hill modes of operation when $k = 2$.

	B	DN	EN	FN	FR	GE	PL	SP	SW
ECB	32	70	43	97	86	49	3	85	63
	64	84	50	99	91	62	11	87	75
	128	93	65	99	93	79	21	94	88
CBC	32	71	40	98	86	47	5	83	61
	64	82	50	99	93	65	11	90	74
	128	90	65	99	93	78	25	95	97
CTR	32	35	13	56	40	19	3	37	18
	64	58	28	85	63	36	6	60	45
	128	81	49	98	82	59	13	83	77
CFB	32	70	38	97	87	50	3	83	74
	64	84	49	99	93	64	8	89	86
	128	91	63	99	93	77	23	94	96

Table 7. Number of recovered messages for the affine Hill modes of operation when $k = 3$.

	B	DN	EN	FN	FR	GE	PL	SP	SW
ECB	16384	82	53	98	90	79	14	89	79
	32768	92	69	99	93	93	26	94	88
	65536	96	83	100	95	95	54	96	94
CBC	16384	80	53	98	89	76	14	88	78
	32768	89	69	99	93	92	27	94	87
	65536	96	80	100	95	95	61	96	93
CTR	16384	77	46	95	86	63	11	86	74
	32768	87	66	98	92	89	26	92	85
	65536	95	79	100	97	95	53	96	92
CFB	16384	81	53	98	89	76	15	88	77
	32768	90	68	99	93	92	27	94	87
	65536	96	81	100	95	95	59	96	93

Table 8. Number of recovered messages for the affine Hill modes of operation when $k = 4$.

5.4 Affine Hill Modes of Operation Message Recovery Rates (Second Approach)

	B	DN	EN	FN	FR	GE	PL	SP	SW
ECB	128	73	59	59	40	70	7	28	72
	256	92	83	98	97	90	32	98	89
	512	100	98	100	100	99	100	100	100
CBC	16	84	35	99	96	82	2	97	63
	32	95	57	100	97	92	4	100	87
	64	98	84	100	98	96	10	100	95
CTR	16	65	33	94	96	68	1	96	49
	32	92	58	100	97	87	5	100	82
	64	99	80	100	98	96	12	100	95
CFB	16	79	39	99	95	80	2	97	63
	32	94	60	100	98	91	6	100	86
	64	98	80	100	98	95	9	100	95

Table 9. Number of recovered messages for the secret coding affine Hill modes of operation when $k = 2$.

	B	DN	EN	FN	FR	GE	PL	SP	SW
ECB	4096	24	25	63	71	44	0	71	32
	8192	53	54	97	98	71	4	94	64
	16384	99	93	100	100	98	89	100	97
CBC	4096	34	34	96	91	55	0	92	45
	8192	68	62	100	99	79	3	100	76
	16384	100	96	100	100	98	47	100	96
CTR	4096	30	36	88	88	54	0	91	44
	8192	68	60	100	99	81	3	100	77
	16384	100	96	100	100	98	41	100	100
CFB	4096	34	35	96	92	56	0	92	47
	8192	65	62	100	99	82	3	100	74
	16384	100	95	100	100	98	47	100	100

Table 10. Number of recovered messages for the secret coding affine Hill modes of operation when $k = 3$.

	B	DN	EN	FN	FR	GE	PL	SP	SW
ECB	200000	46	32	86	76	30	0	82	53
	300000	73	64	96	98	71	2	95	78
	400000	93	86	100	100	91	16	100	94
CBC	200000	54	38	89	83	37	0	91	56
	300000	76	67	97	99	76	1	98	82
	400000	93	87	100	100	91	5	100	94
CTR	200000	54	41	90	84	38	0	85	56
	300000	76	67	96	95	73	2	98	82
	400000	95	86	100	100	94	4	100	95
CFB	200000	54	38	90	83	38	0	91	54
	300000	76	66	97	99	76	1	97	81
	400000	94	87	100	100	91	6	100	94

Table 11. Number of recovered messages for the secret coding affine Hill modes of operation when $k = 4$.

5.5 Running time

In this section we provide some benchmarks for Algorithms 5 to 8. The algorithms were run on a CPU Intel i7-4790 4.00 GHz and compiled with GCC with the O3 flag activated and the *omp_get_wtime()* function [1] was used to compute the running times. Due to resource constrains, we stopped the experiments at $k = 3$ for the Hill attacks and at $k = 2$ for the affine attacks. To obtain a fair comparison, when computing the running times, we used higher B values than the one presented in Sections 5.2 to 5.4. We present the exact margins in Table 12.

Mode	Hill	Afine Hill (1)	Afine Hill (2)	Hill
	($k = 2$)	($k = 2$)	($k = 2$)	($k = 3$)
ECB	4 (100%)	8 (99%)	512 (98%)	128 (97%)
CBC	2 (99%)	4 (95%)	256 (96%)	128 (95%)
CTR	2 (98%)	4 (97%)	256 (98%)	128 (96%)
CFB	2 (99%)	4 (97%)	256 (98%)	128 (96%)

Table 12. The threshold B and the corresponding success probability for the English language.

In Table 13, the second and third columns contain the total time necessary to recover 100 independent texts, the fourth and fifth columns the total time necessary to recover 8 texts. It is clear from the presented results that the first approach (Affine Hill (1)) has significantly lower running times than the second approach (Affine Hill (2)). Note that in the case of the second approach the difference between the ECB attack and the rest of the attacks is due to the extra *score_function* calls made when the *trial_score* is equal to the *best_score*.

Mode	Hill	Afine Hill (1)	Afine Hill (2)	Hill
	$(k = 2)$	$(k = 2)$	$(k = 2)$	$(k = 3)$
ECB	0.94057	23.1658	1805.98	1415.60
CBC	1.75324	45.4769	379.762	1502.20
CTR	1.75827	45.9883	374.439	1423.39
CFB	1.75271	48.5864	360.428	1509.62

Table 13. Running times of Algorithms 5 to 8.

Let $k = 2$. To see if the chosen bounds have the same success rate for other texts, we encrypted 1000 independent texts[21] and then we run Algorithms 5 and 7. The number of plaintexts recovered is presented in Table 14. We can see that for the Hill based modes the success probabilities are almost the same, while for the affine versions the probabilities are a little lower than the initial estimations.

Cipher	ECB	CBC	CTR	CFB
Hill	995	987	982	982
Afine Hill (1)	970	956	945	953

Table 14. Success rates for Algorithms 5 and 7 when $k = 2$.

6 Conclusions

In this paper we adapted Yum and Lee's attack to the affine Hill cipher. Also, we introduced new ranking and message recovery algorithms for the CBC, CTR and CFB modes of operation. We also conducted a series of experiments to determine and test the success rates of these algorithms.

Future Work. The row ranking algorithms perform the same instructions for disjoint rows. Thus, an interesting implementation direction is to parallelize Algorithms 3 and 4. The recovering algorithms also perform the same instructions, but for independent keys. Hence, Algorithms 5 and 7 can also be parallelized.

Another possible speed-up is to parallelize the algorithm presented [15] for the Hill cipher. Note that this speed-up can also be applied to the Hill CBC mode. From a theoretical point of view, it would be interesting to see if the Leap *et.al.*'s algorithm can be tweaked to work for the affine Hill cipher. If it can be tweaked we might obtain faster decryption times for the affine Hill and the corresponding CBC mode.

A time-memory trade-off attack for the Hill cipher is presented in [17]. Thus, it might be interesting to see if this attack can be adapted to the affine version and to the (affine) modes of operation versions. From an implementation point of view, it might worth seeing if McDevitt *et.al.*'s attack can be parallelized.

In [23], the authors provide a ranking algorithm when the the encoding and decoding functions are unknown, but they do not describe a message recovery algorithm. This cipher can be seen as a composition of a substitution cipher, a Hill cipher and a second substitution cipher. Note that the two substitution

[21] different from the 100 texts used for computing the bounds

ciphers do not necessarily have the same key. A generic version of the secret coding cipher can be obtained by combining a generic Vigenère cipher[22], a Hill cipher and a second generic Vigenère cipher. Note that in this case Yum and Lee's ranking algorithm still works. Hence, another possible research direction is to find message recovery algorithms[23] for this generic cipher.

In [12], Hill introduces a variation of the affine Hill cipher in which the elements of the key matrix are matrices. Thus, an interesting problem is to study the impact of the message recovering algorithms on the version presented in [12].

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[22] By a generic Vigenère cipher we understand a Vigenère cipher with random alphabets.

[23] that might use Yum and Lee's ranking algorithm

A Letter Frequencies

To have uniform letter frequency tables, we added the probability of letters with diacritical marks to the probability of their base letter. For example, in Danish, the letter O has a 0.0464 occurrence probability and the letter Ø one of 0.0094. We added the two and we recorded O's probability as 0.0558. Note that the frequency tables we used for computing our tables are from [16].

A, Å, Æ	0.0809	H	0.0162	O, Ø	0.0558	V	0.0233
B	0.0200	I	0.0600	P	0.0176	W	0.0007
C	0.0056	J	0.0073	Q	0.0001	X	0.0003
D	0.0586	K	0.0339	R	0.0896	Y	0.0070
E	0.1545	L	0.0523	S	0.0581	Z	0.0003
F	0.0241	M	0.0324	T	0.0686		
G	0.0408	N	0.0724	U	0.0198		

Table 15. Relative frequencies of Danish letters.

A	0.0855	H	0.0496	O	0.0747	V	0.0106
B	0.0160	I	0.0733	P	0.0207	W	0.0183
C	0.0316	J	0.0022	Q	0.0010	X	0.0019
D	0.0387	K	0.0081	R	0.0633	Y	0.0172
E	0.1210	L	0.0421	S	0.0673	Z	0.0011
F	0.0218	M	0.0253	T	0.0894		
G	0.0209	N	0.0717	U	0.0268		

Table 16. Relative frequencies of English letters.

A, Ä	0.1580	H	0.0185	O, Ö	0.0605	V	0.0225
B	0.0028	I	0.1082	P	0.0184	W	0.0009
C	0.0028	J	0.0204	Q	0.0001	X	0.0003
D	0.0104	K	0.0497	R	0.0287	Y	0.0174
E	0.0797	L	0.0576	S	0.0786	Z	0.0005
F	0.0019	M	0.0320	T	0.0875		
G	0.0039	N	0.0883	U	0.0501		

Table 17. Relative frequencies of Finnish letters.

A, À, Â	0.0808	H	0.0093	O, Ô, Œ	0.0546	V	0.0129
B	0.0096	I, Î, Ï	0.0726	P	0.0298	W	0.0008
C, Ç	0.0344	J	0.0030	Q	0.0085	X	0.0043
D	0.0408	K	0.0016	R	0.0686	Y	0.0034
E, È, É, Ê	0.1745	L	0.0586	S	0.0798	Z	0.0010
F	0.0112	M	0.0278	T	0.0711		
G	0.0118	N	0.0732	U, Ù, Û, Ü	0.0559		

Table 18. Relative frequencies of French letters.

A, Ä	0.0688	H	0.0411	O, Ö	0.0299	V	0.0094
B	0.0221	I	0.0760	P	0.0106	W	0.0140
C	0.0271	J	0.0027	Q	0.0004	X	0.0007
D	0.0492	K	0.0150	R	0.0771	Y	0.0013
E	0.1599	L	0.0372	S, ß	0.0656	Z	0.0122
F	0.0180	M	0.0275	T	0.0643		
G	0.0302	N	0.0959	U, Ü	0.0376		

Table 19. Relative frequencies of German letters.

A, Ą	0.0997	H	0.0125	O, Ó	0.0879	V	0.0000
B	0.0139	I	0.0809	P	0.0292	W	0.0478
C, Ć	0.0422	J	0.0226	Q	0.0000	X	0.0000
D	0.0323	K	0.0354	R	0.0506	Y	0.0370
E, Ę	0.0849	L, Ł	0.0418	S, Ś	0.0504	Z, Ź, Ż	0.0590
F	0.0041	M	0.0273	T	0.0394		
G	0.0154	N, Ń	0.0602	U	0.0259		

Table 20. Relative frequencies of Polish letters.

A	0.1250	H	0.0081	O	0.0898	V	0.0098
B	0.0127	I	0.0691	P	0.0275	W	0.0003
C	0.0443	J	0.0045	Q	0.0083	X	0.0019
D	0.0514	K	0.0008	R	0.0662	Y	0.0079
E	0.1324	L	0.0584	S	0.0744	Z	0.0042
F	0.0079	M	0.0261	T	0.0442		
G	0.0117	N, Ñ	0.0731	U	0.0400		

Table 21. Relative frequencies of Spanish letters.

A, Ä, Å	0.1252	H	0.0209	O, Ö	0.0579	V	0.0242
B	0.0154	I	0.0582	P	0.0184	W	0.0014
C	0.0149	J	0.0061	Q	0.0002	X	0.0016
D	0.0470	K	0.0314	R	0.0843	Y	0.0071
E	0.1015	L	0.0528	S	0.0659	Z	0.0007
F	0.0203	M	0.0347	T	0.0769		
G	0.0286	N	0.0854	U	0.0192		

Table 22. Relative frequencies of Swedish letters.

B Vigenère Cryptanalysis

In [21], the author describes an algorithm for breaking the Vigenère cipher. Because of better results in practice, we changed the scoring function from [21] with a scoring function based on digraphs. The result is presented in Algorithm 9.

Algorithm 9. The algorithm for breaking the Vigenère cipher with key length *matrix_size*.

```
Input: The ciphertext $enc$.
Output: The best possible message $best\_msg$ and its associated score $best\_score$.
Function decrypt_vigenere($enc, key$):
    $fragment[matrix\_size] \leftarrow \{\texttt{""}\}$;
    for $i \in [0, enc.size()/\, matrix\_size]$ do
        for $j \in [0, matrix\_size]$ do
            $fragment[j] \mathrel{+}= (enc[i \cdot matrix\_size + j] - \texttt{"a"} + key) \bmod alphabet\_size + \texttt{"a"}$;
        end
    end
    return $fragment$;
Function compute_score($best\_fragment\_score$):
    $best\_score \leftarrow 0$;
    for $i \in [0, matrix\_size]$ do
        $best\_score \mathrel{+}= best\_fragment\_score[i]$;
    end
    return $best\_score$;
Function recompose_msg($best\_fragment$):
    $best\_msg \leftarrow \texttt{""}$;
    for $i \in [0, enc\_size \,/\, matrix\_size]$ do
        for $j \in [0, matrix\_size]$ do
            $best\_msg \mathrel{+}= best\_fragment[j][i]$;
        end
    end
    return $best\_msg$;
Function break_vigenere($enc$):
    $best\_fragment\_score[matrix\_size] \leftarrow \{-\infty\}$;
    for $key \in [0, alphabet\_size]$ do
        $fragment \leftarrow decrypt\_vigenere(enc, key)$;
        for $i \in [0, matrix\_size]$ do
            $fragment\_score[i] \leftarrow score\_function(fragment[i], digraph\_freq, digraph\_default, 2)$;
            if $fragment\_score[i] > best\_fragment\_score[i]$ then
                $best\_fragment\_score[i] \leftarrow fragment\_score[i]$;
                $best\_fragment[i] \leftarrow fragment[i]$;
            end
        end
        $best\_score \leftarrow compute\_score(best\_fragment\_score)$;
        $best\_msg \leftarrow recompose\_msg(best\_fragment)$;
    end
    return $(best\_score, best\_msg)$;
```