Sublattice Attacks on Ring-LWE with Wide Error Distributions I

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Abstract

Since the Lyubashevsky-Peikert-Regev Eurocrypt 2010 paper the Ring-LWE has been the hard computational problem for lattice cryptographic constructions. The fundamental problem is its hardness which has been based on the conjectured hardness of approximating ideal-SIVP or ideal-SVP. Though it is now widely conjectured both are hard in classical and quantum computation model there are no sufficient attacks proposed and considered. In this paper we propose sublattice attacks on Ring-LWE over an arbitrary number field from *sublattice pairs with ideals*. We give a sequence of number fields $\mathbf{K}_n$ of degree $d_n \longrightarrow \infty$, such that the decision Ring-LWE with very wide error distributions over integer rings of $\mathbf{K}_n$ can be solved by a polynomial (in d_n) time algorithm from our sublattice attack. The widths of error distributions in our attack is in the range of Peikert-Regev-Stephens-Davidowitz hardness reduction results in their STOC 2017 paper. Hence we also prove that approximating ideal-$SIVP_{poly(d)}$ with some polynomial factor for ideal lattices in these number fields can be solved by a polynomial time quantum algorithm.

Keywords: Ring-LWE, Width of error distribution, Sublattice attack, Sublattice pair with an ideal

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1 Introduction

1.1 Algebraic number fields

An algebraic number field is a finite degree d extension of the rational number field $\mathbf{Q}$. Let $\mathbf{K}$ be an algebraic number field and $\mathbf{R_K}$ be its ring of integers in $\mathbf{K}$. From the primitive element theorem there exists an element $\theta \in \mathbf{K}$ such that $\mathbf{K} = \mathbf{Q}[x]/(f) = \mathbf{Q}[\theta]$, where $f(x) \in \mathbf{Z}[x]$ is an irreducible monic polynomial of degree d satisfying $f(\theta) = 0$ (see [13, 5]). It is well-known there is a positive definite inner product on $\mathbf{K} \otimes \mathbf{C}$ defined by $< u, v >= \Sigma_{i=1}^{d} \sigma_i(u)\sigma_i(\tilde{v})$, where σ_i, $i = 1, \ldots, d$, are d embeddings of $\mathbf{K}$ in $\mathbf{C}$, and $\tilde{v}$ is complex conjugate. Sometimes we use $||u||_{tr}$ to represent the norm $< u, u >^{1/2}$. This is the norm with respect to the canonical embedding (see [26]). An ideal in $\mathbf{R_K}$ is a subset of $\mathbf{R_K}$ which is closed under ring addition and multiplication by an arbitrary element in $\mathbf{R_K}$. An ideal is a sub-lattice in $\mathbf{R_K}$ of dimension $deg(\mathbf{K}/\mathbf{Q})$. For an ideal $\mathbf{I} \subset \mathbf{R_K}$, the (algebraic) norm of ideal $\mathbf{I}$ is defined by the cardinality $N(\mathbf{I}) = |\mathbf{R_K}/\mathbf{I}|$, we have $N(\mathbf{I} \cdot \mathbf{J}) = N(\mathbf{I})N(\mathbf{J})$. For a principal ideal $\mathbf{xR_K}$ generated by an element $\mathbf{x}$, then $N(\mathbf{x}) = N(\mathbf{xR_K})$, we refer to [5, 12] for the detail. The dual of a lattice $\mathbf{L} \subset \mathbf{K}$ of rank $\deg(\mathbf{K}/\mathbf{Q})$ is defined by $\mathbf{L}^\vee = \{\mathbf{x} \in \mathbf{K}, tr_{K/Q}(\mathbf{ax}) \in \mathbf{Z}, \forall \mathbf{a} \in \mathbf{L}\}$. An order $\mathbf{O} \subset \mathbf{K}$ in a number field $\mathbf{K}$ is a subring of $\mathbf{K}$ which is a lattice with rank equal to $\deg(\mathbf{K}/\mathbf{Q})$. We refer to [12, 13, 5] for number theoretic properties of orders in number fields.

Let ξ_n be a primitive n-th root of unity, the n-th cyclotomic polynomial Φ_n is defined as $\Phi_n(x) = \prod_{j=1,\gcd(j,n)=1}^{n}(x - \xi_n^j)$. This is a monic irreducible polynomial in $\mathbf{Z}[x]$ of degree $\phi(n)$, where ϕ is the Euler function. The n-th cyclotomic field is $\mathbf{Q}(\xi_n) = \mathbf{Q}[x]/(\Phi_n(x))$. When $n = p$ is an odd prime $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ and when $n = p^m$, $\Phi_{p^m}(x) = \Phi_p(x^{p^{m-1}}) = (x^{p^{m-1}})^{p-1} + \cdots + x^{p^{m-1}} + 1$. The ring of integers in $\mathbf{Q}(\xi_n)$ is exactly $\mathbf{Z}[\xi_n] = \mathbf{Z}[x]/(\Phi_n(x))$ (see Theorem 2.6 in [46]). Hence the cyclotomic number field $\mathbf{Q}[\xi_n]$ is a monogenic field. The discriminant of the cyclotomic field (also the discriminant of the cyclotomic polynomial Φ_n) is

$$(-1)^{\frac{\phi(n)}{2}} \frac{n^{\phi(n)}}{\prod_{p|n} p^{\frac{\phi(n)}{p-1}}}.$$

A polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \in \mathbf{Z}[X]$ satisfies the condition of the Eisenstein criterion at a prime p, if $p|a_i$ for $0 \leq i \leq n-1$

and p^2 not dividing a_0. A polynomial satisfying this condition is irreducible in $\mathbf{Z}[x]$ from the Eisenstein criterion (see [5, 13]).

1.2 Gaussian and discrete Gaussian

Set $\rho_{s,\mathbf{c}}(\mathbf{x}) = e^{-\pi||\mathbf{x}-\mathbf{c}||^2/s^2}$ for any vector $\mathbf{c}$ in $\mathbf{R}^n$ and any $s > 0$, $\rho_s = \rho_{s,\mathbf{0}}$, $\rho = \rho_1$. The Gaussian distribution around $\mathbf{c}$ with width s is defined by its probability density function $D_{s,\mathbf{c}} = \frac{\rho_{s,\mathbf{c}}(\mathbf{x})}{s^n}, \forall \mathbf{x} \in \mathbf{R}^n$.

Discretization. For any discrete subset $\mathbf{A} \subset \mathbf{R}^n$ we set $\rho_{s,\mathbf{c}}(\mathbf{A}) = \Sigma_{\mathbf{x}\in\mathbf{A}}\rho_{s,\mathbf{c}}(\mathbf{x})$ and $D_{s,\mathbf{c}}(\mathbf{A}) = \Sigma_{\mathbf{x}\in\mathbf{A}}D_{s,\mathbf{c}}(\mathbf{x})$. Let $\mathbf{L} \subset \mathbf{R}^n$ be a dimension n lattice, the discrete Gaussian distribution over $\mathbf{L}$ is the probability distribution over $\mathbf{L}$ defined by

$$\forall \mathbf{x} \in \mathbf{L}, D_{\mathbf{L},s,\mathbf{c}} = \frac{D_{s,\mathbf{c}}(\mathbf{x})}{D_{s,\mathbf{c}}(\mathbf{L})} = \frac{\rho_{s,\mathbf{c}}(\mathbf{x})}{\rho_{s,\mathbf{c}}(\mathbf{L})}.$$

When $\mathbf{c} = \mathbf{0}$, the discrete Gaussian distribution is denoted by $\mathbf{D}_{\mathbf{L},\mathbf{s}}$. We refer to [31] for the following properties of discrete Gaussian distributions.
1) If $\mathbf{x}$ is distributed according to $\mathbf{D}_{s,\mathbf{c}}$ and conditioned on $\mathbf{x} \in \mathbf{L}$, the conditional distribution of $\mathbf{x}$ is $D_{\mathbf{L},s,\mathbf{c}}$.
2) For any lattice $\mathbf{L}$ and any vector $\mathbf{c} \in \mathbf{R}^n$ we have $\rho_{s,\mathbf{c}}(\mathbf{L}) \leq \rho_s(\mathbf{L})$.
3) Set $C = c\sqrt{2\pi}e^{-\pi c^2} < 1$ for any $c > \frac{1}{\sqrt{2\pi}}$, and n dimensional lattice $\mathbf{L}$ and $\mathbf{v} \in \mathbf{R}^n$, $\rho(\mathbf{L} - c\sqrt{n}\mathbf{B}_n) \leq C^n\rho(\mathbf{L})$, $\rho((\mathbf{L} + \mathbf{v}) - c\sqrt{n}\mathbf{B}_n) \leq C^n\rho(\mathbf{L})$, where $\mathbf{B}_n$ is the unit-ball centered at the origin.
4 If a $\mathbf{e} \in \mathbf{R}^n$ is sampled according to a Gaussian distribution with width σ, then the Euclid norm $||\mathbf{e}||$ of $\mathbf{e}$ satisfies $||\mathbf{e}|| \leq \sqrt{3n}\sigma$ with an overwhelming probability.

Width with the canonical embedding

The Gaussian distribution depends on coordinates and the norm. We need to pay special attention to coordinates (or the basis with which coordinates are obtained) and the norm used when we say the "width" of a Gaussian distribution. The "canonical embedding' was used to define the Gaussian distribution on $\mathbf{K} \otimes \mathbf{C}$ (see [26, 27, 38, 7]). We refer the further analysis to [7, 40].

1.3 SVP and SIVP

A lattice $\mathbf{L}$ is a discrete subgroup in $\mathbf{R}^n$ generated by several linear independent vectors $\mathbf{b}_1, \ldots, \mathbf{b}_m$ over the ring of integers, where $m \leq n$, $\mathbf{L} := \{a_1\mathbf{b}_1 + \cdots + a_m\mathbf{b}_m : a_1 \in \mathbf{Z}, \ldots, a_m \in \mathbf{Z}\}$. The volume $vol(\mathbf{L})$ of this lattice is $\sqrt{det(\mathbf{B} \cdot \mathbf{B}^\tau)}$, where $\mathbf{B} := (b_{ij})$ is the $m \times n$ generator matrix of this lattice, $\mathbf{b}_i = (b_{i1}, \ldots, b_{in}) \in \mathbf{R}^n$, $i = 1, \cdots, m$, are base vectors of this lattice. The length of the shortest non-zero lattice vectors is denoted by $\lambda_1(\mathbf{L})$. The well-known shortest vector problem (SVP) is defined as follows. Given an arbitrary $\mathbf{Z}$ basis of an arbitrary lattice $\mathbf{L}$ to find a lattice vector with length $\lambda_1(\mathbf{L})$ (see [32]). The approximating shortest vector problem $SVP_{f(m)}$ is to find some lattice vectors of length within $f(m)\lambda_1(\mathbf{L})$ where $f(m)$ is an approximating factor as a function of the lattice dimension m (see [32]). The Shortest Independent Vectors Problem ($SIVP_{\gamma(m)}$) is defined as follows. Given an arbitrary $\mathbf{Z}$ basis of an arbitrary lattice $\mathbf{L}$ of dimension m, to find m independent lattice vectors such that the maximum length of these m lattice vectors is upper bounded by $\gamma(m)\lambda_m(\mathbf{L})$, where $\lambda_m(\mathbf{L})$ is the m-th Minkowski's successive minima of lattice $\mathbf{L}$ (see [32]). A breakthrough result of M. Ajtai [3] showed that SVP is NP-hard under the randomized reduction. Another breakthrough proved by Micciancio asserts that approximating SVP within a constant factor is NP-hard under the randomized reduction (see [32]). For the latest development we refer to Khot [20]. It was proved that approximating SVP within a quasi-polynomial factor is NP-hard under the randomized reduction. For the hardness results about SVP and $SIVP$ we refer to [20, 21, 44], we refer to [19] for Minkowski's first and second theorems on successive minima of lattices.

1.4 Plain LWE, Ring-LWE and LWE over number field lattices

Plain LWE

Plain LWE and its lattice-based cryptographic construction was originated from [42]. We refer to [43] for a survey. Let n be the security parameter, q be an integer modulus and χ be an error distribution over $\mathbf{Z}_q$. Let $\mathbf{s} \in \mathbf{Z}_q^n$ be a secret chosen uniformly at random. Given access to d samples of the form

$$(\mathbf{a}, [\mathbf{a} \cdot \mathbf{s} + e]_q) \in \mathbf{Z}_q^n \times \mathbf{Z}_q,$$

where $\mathbf{a} \in \mathbf{Z}_q^n$ are chosen uniformly at random and $\mathbf{e}$ are sampled from the

error distribution χ, the search LWE is to recover the secret $\mathbf{s}$. In general χ is the discrete Gaussian distribution with the width σ. Here $\mathbf{a} \cdot \mathbf{s} = \Sigma a_i s_i$ is the inner product of two vectors in $\mathbf{Z}_q^n$.

Write the d coefficient vectors $\mathbf{a}_1, \ldots, \mathbf{a}_d$ as columns of a matrix $\mathbf{A} \in \mathbf{Z}_q^{n \times d}$, Then the search LWE problem $LWE_{n,q,d,\chi}$ is to recover the secret from $\mathbf{A}^\tau \cdot \mathbf{s} + \mathbf{e} = \mathbf{b}\ mod\ q$ from public $(\mathbf{A}, \mathbf{b})$. Here τ is the transposition of a matrix and $(\mathbf{s}, \mathbf{e})$ is an unknown vector.

Solving decision $LWE_{n,q,d,\chi}$ is to distinguish with non-negligible probability whether $(\mathbf{A}, \mathbf{b}) \in \mathbf{Z}_q^{n \times d} \times \mathbf{Z}_q^d$ is sampled uniformly at random, or if it is of the form $(\mathbf{A}, \mathbf{A}^\tau \cdot \mathbf{s} + \mathbf{e})$ where $\mathbf{e}$ is sampled from the distribution χ.

Here $[\mathbf{a} \cdot \mathbf{s} + e]_q$ is the residue class in the interval $(-\frac{q}{2}, \frac{q}{2}]$. We refer to [43] for the detail and the background. When q is prime and polynomial bounded by $poly(n)$, there is a polynomial-time reduction between the search and decision LWE (see [43]). For plain LWE without the ring structure the reduction results from approximating SIVP to plain LWE were given in [43, 35, 6].

Ring-LWE

The algebraic structure of ring was first introduced to the hardness of computational problems of lattices in [29] (also in [24, 25]) for the consideration of efficiency. This is Ring-SIS (Short Integer Solution over Ring, see [29]) and it is the analogue of Ajtai's SIS problem. The one-wayness of some function was proved in [29] by assuming the hardness of some computational problems of cyclic lattices (ideal lattices). Ring-LWE was originated from 2010 paper [26] and then extended in [27]. We refer to [37] for a survey of the history of development, the theory and cryptographic constructions based on Ring-LWE and Ring-SIS.

If the $\mathbf{Z}_q^n$ in plain LWE is replaced by $\mathbf{P}_q = \mathbf{P}/q\mathbf{P}$ where $\mathbf{P} = \mathbf{Z}[x]/(f)$, $f(x)$ is a monic irreducible polynomial of degree n in $\mathbf{Z}[x]$, this is the polynomial learning with errors (PLWE). The inner product $\mathbf{a} \cdot \mathbf{s} = \Sigma a_i s_i$ is replaced by the multiplication $\mathbf{a} \cdot \mathbf{s}$ in the ring $\mathbf{P}_q$. The error distribution χ is defined as the discrete Gaussian distributions with respect to the basis $1, x, x^2, \ldots, x^{n-1}$ (see [18, 7]). We refer to [45] for relations and reductions between Ring-LWE and PLWE.

If the $\mathbf{Z}_q^n$ is replaced by $(\mathbf{R}_\mathbf{K})_q = \mathbf{R}_\mathbf{K}/q\mathbf{R}_\mathbf{K}$ where $\mathbf{R}_\mathbf{K}$ is the ring

of integers in an algebraic number field $\mathbf{K}$ of degree n, this is the Ring-LWE, learning with errors over the ring $\mathbf{R_K}$. The secret $\mathbf{s}$ is in the dual $(\mathbf{R_K}^\vee)_q = \mathbf{R_K}^\vee/q\mathbf{R_K}^\vee$ and $\mathbf{a} \in \mathbf{R_K}_q$ is chosen uniformly at random. The inner product $\mathbf{a}\cdot\mathbf{s} = \Sigma a_i s_i$ is replaced by the multiplication $\mathbf{a}\cdot\mathbf{s}$ in $(\mathbf{R_K}^\vee)_q$. The error $\mathbf{e}$ is in $(\mathbf{R_K}^\vee)_q = \mathbf{R_K}^\vee/q\mathbf{R_K}^\vee$. In this case the width of error distribution is defined by the trace norm on $\mathbf{K}\otimes\mathbf{R}$ via the canonical embedding (see [26, 7]). This is called the dual form of Ring-LWE problem . When $\mathbf{s} \in (\mathbf{R_K})_q$ and $\mathbf{e} \in (\mathbf{R_K})_q$ are assumed it is called the non-dual form of Ring LWE problem. As indicated in [38] page 10 in monogenic case a "tweak factor" $f'(\theta)$ can be used to make two versions equivalent.

LWE over number field lattice

Learning with errors over a number field lattice was introduced in [39]. Let $\mathbf{L} \subset \mathbf{K}$ be a rank $\deg(\mathbf{K})$ lattice and

$$\mathbf{O^L} = \{x \in \mathbf{K} : x \cdot \mathbf{L} \subset \mathbf{L}\}.$$

Then $\mathbf{O^L}$ is an order. Set $\mathbf{O^L}_q = \mathbf{O^L}/q\mathbf{O^L}$, $\mathbf{L^\vee}_q = \mathbf{L}^\vee/q\mathbf{L}^\vee$. The secret vector $\mathbf{s}$ is in $\mathbf{L^\vee}_q$ and $\mathbf{a}$ is in $\mathbf{O^L}_q$. Here we notice that $\mathbf{O}\cdot\mathbf{L}^\vee \subset \mathbf{L}^\vee$. Then the error $\mathbf{e} \in \mathbf{L^\vee}_q$. For the detail and hardness reduction we refer to [39].

1.5 Hardness reduction

The reduction results from approximating ideal-$SIVP_{poly(d)}$ (or approximating ideal-$SVP_{poly(d)}$) to Ring-LWE were first given in [26, 27] for search version and then a general form to decision version was proved for arbitrary number fields in [40]. We refer to [40] Corollary 6.3 for the following hardness reduction result.

Hardness reduction for decision Ring-LWE. *Let $\mathbf{K}$ be an arbitrary number field of degree n and $\mathbf{R} = \mathbf{R_K}$. Let $\alpha = \alpha(n) \in (0,1)$, and let $q = q(n)$ be an integer such that $\alpha q \geq 2\omega(1)$. Then there exists a polynomial-time quantum reduction from $\mathbf{K} - SIVP_\gamma$ to average-case, decision $\mathbf{R} - LWE_{q,\Upsilon_\alpha}$, for any $\gamma = \max\{\frac{\eta(\mathbf{I})\cdot 2}{\alpha\cdot\omega(1)}, \frac{\sqrt{2n}}{\lambda_1(\mathbf{I})}\} \leq \max\{\omega(\sqrt{nlogn}/\alpha), \sqrt{2}n\}$. Here $\mathbf{K} - SIVP_\gamma$ is the Shortest Independent Vector Problems for any fractional ideal lattice in $\mathbf{K}$. $\mathbf{I}$ is any ideal lattice and $\eta(\mathbf{I})$ is the smoothing parameter of $\mathbf{I}$.*

1.6 Known attacks

1.6.1 Attacks on LWE

The famous Blum-Kalai-Wasserman (BKW) algorithm in [4] was improved in [1, 22]. On the other hand some provable weak instances of Ring-LWE was given in [17, 18, 11] and analysed in [7, 38]. As showed in [38, 7] these instances of Ring-LWE can be solved by polynomial time algorithms mainly because the widths of Gaussian distributions of errors are too small or Gaussian distributions of errors are too skew. In [8] these attacks were improved for these modulus parameters which are factors of $f(u)$, where f is the defining equation of the number field and u is an arbitrary integer. However the Gaussian distribution is still required to be narrow such that this type of attack can be succeed. We refer to [2] for the dual lattice attack to LWE with small secrets.

1.6.2 Approximating ideal-SVP

In [14] it was proved approximating SVP with factor $2^{O(\sqrt{nlogn})}$ for principal ideals in cyclotomic integer rings $\mathbf{Z}[\xi_n]$ with $n = p^m$ can be found from an arbitrary generator within polynomial time by an efficient bounded distance decoding algorithm for the log-unit lattice. This work was extended in [15] and [41] such that sub-exponential complexity algorithms with some pre-processing for approx-SVP with some sub-exponential factor for ideal lattices can been achieved. The analysis of the approximating factor was recently published in [16]. For the recent developments we refer to [23, 34].

1.7 The ideal-attack is very restricted

In previous attacks on Ring-LWE in [18] (then analysed in [7, 38]) the Ring-LWE equation $\mathbf{a} \cdot \mathbf{s} + \mathbf{e} \equiv \mathbf{b}$ *mod* q was transformed to consider $\mathbf{a} \cdot \mathbf{s} + \mathbf{e} \equiv \mathbf{b}$ *mod* $\mathbf{P}$, where $\mathbf{P}$ is a prime ideal factor of the modulus parameter q with a polynomially bounded algebraic norm $N(\mathbf{P})$. This kind of attack initiated in [18] and then analysed in [7, 38] can be called ideal-attack on Ring-LWE. In ideal-attack on Ring-LWE $\lambda_1(\mathbf{P}^\vee)$ satisfies

$$\lambda_1(\mathbf{P}^\vee) \geq \sqrt{d} N(\mathbf{P}^\vee)^{1/d} \geq d^{1/2-c/d} \frac{1}{|\Delta_{\mathbf{K}}|^{1/d}}.$$

Since $\mathbf{P}$ has a polynomially bounded algebraic norm, the width has a small upper bound for solvable instances for some fixed positive integer c . In our

sublattice attack we propose to consider the equation $\mathbf{a} \cdot \mathbf{s} + \mathbf{e} \equiv \mathbf{b}\ mod\ \mathbf{L}$, where $\mathbf{L}$ is a sublattice with polynomially bounded index $|\mathbf{R_K}/\mathbf{L}|$ and satisfying $q\mathbf{R_K} \subset \mathbf{L}$. Then we find subtle sublattice $\mathbf{L}$ such that $\lambda_1(\mathbf{L}^\vee)$ is very small and there are many very short lattice vectors in $\mathbf{L}^\vee$. From Theorem 4.1 the above equation can be solved for very large widths of error distributions. Our main results indicate that asymptotically our sublattice attack on Ring-LWE is essentially much better than ideal-attack on Ring-LWE at least for certain number fields.

2 Sublattice attack

Sublattice attack was proposed in [9]. It works for arbitrary algebraically structured learning with errors problems. We restricted to LWE over number field lattices in [9]. In this paper we introduce the sublattice attacks with sublattice pairs. It achieves attacks on Ring-LWE with wide Gaussian error distributions. However it works even for other wide error distributions provided suitable sublattice pairs can be constructed. From this point of view we believe that learning with errors problem over algebraically-structured objects perhaps can be solved within polynomial times in many settings.

2.1 The motivation of sublattice attacks

In previous attacks on Ring-LWE, when polynomially bounded many samples $(\mathbf{a}, \mathbf{b}) \in \mathbf{R_K}/q\mathbf{R_K} \times \mathbf{R_K}/q\mathbf{R_K}$ are given, only the distributions of these samples over $\mathbf{R_K}/\mathbf{I}$ for some **ideals** satisfying $q\mathbf{R_K} \subset \mathbf{I} \subset \mathbf{R_K}$ and $|\mathbf{R_K}/\mathbf{I}| \leq poly(d)$ have been checked. This is not natural and not sufficient. We need to check the distributions of samples in $\mathbf{R_K}/\mathbf{L}$ where $\mathbf{L}$ takes over **all sublattices** satisfying

$$q\mathbf{R_K} \subset \mathbf{L} \subset \mathbf{R_K}$$

and

$$|\mathbf{R_K}/\mathbf{L}| \leq poly(d).$$

This is the motivation our previous paper [8]. In general when the learning with error problems with algebraic structures are used to improve the efficiency, sublattice attacks as above to analysis the distributions of samples over $\mathbf{M}/\mathbf{L}$ should be considered, where $\mathbf{M}$ is module over which the

module-LWE is defined and $\mathbf{L}$ takes over all sublattices of $\mathbf{M}$ satisfying

$$q\mathbf{M} \subset \mathbf{L} \subset \mathbf{M}$$

and

$$|\mathbf{M}/\mathbf{L}| \leq poly(d).$$

The previous attacks where $\mathbf{L}$ is restricted to ideals or sub-modules are not natural, special and not sufficient to guarantee the security, we refer to our next paper [10].

The basic point here is as follows. When we want to use the algebraic structure to improve the efficiency of lattice-based cryptographic constructions. The adversary is not so restricted to only check the distributions of samples over algebraic-structured object, the adversary can attack the problem by using lattices without any algebraic structure.

2.2 Sublattice pairs with ideals are needed

When $\mathbf{L}$ is just a sublattice of $\mathbf{R_K}$ satisfying $|\mathbf{R_K}/\mathbf{L}| \leq poly(d)$, for an uniformly distributed $\mathbf{a} \in \mathbf{R_K}$, $\mathbf{a} \cdot \mathbf{L} \subset \mathbf{L}$ is not valid. We even proved in Theorem 5.2 if $\mathbf{L}_1 \cdot \mathbf{L}_2 \subset \mathbf{L}_3$ where

$$|\mathbf{R_K}/\mathbf{L}_i| \leq poly(d),$$

where $i = 1, 2, 3$, then the length $\lambda_1(\mathbf{L}_3^{\vee})$ of the shortest nonzero lattice vectors in the dual $\mathbf{L}_3^{\vee}$ can not be very small. Hence we need to find an ideal $\mathbf{Q} \subset \mathbf{L}$ satisfying $|\mathbf{R_K}/\mathbf{Q}| \leq poly(d)$, then $\mathbf{s}$ satisfies $\mathbf{s} \in \mathbf{Q}$ with a probability $\frac{1}{poly(d)}$. Therefore we have $\mathbf{a} \cdot \mathbf{s} \in \mathbf{Q} \subset \mathbf{L}$ with a probability $\frac{1}{poly(d)}$ of secrets for uniformly distributed $\mathbf{a} \in \mathbf{R_K}$. Hence if $\mathbf{e} \in \mathbf{L}$ is satisfied with a probability

$$\mathbf{P_L} \geq \frac{d^c}{|\mathbf{R_K}/\mathbf{L}|}$$

where c is a fixed positive integer, we can distinguish samples $\mathbf{b}$ from Ring-LWE equations and uniformly distributed samples.

The condition

$$\mathbf{P_L} \geq \mathbf{P_{L_1}} \geq \frac{d^c}{|\mathbf{R_K}/\mathbf{L}|}$$

is achieved by an auxiliary sublattice $\mathbf{L}_1 \subset \mathbf{L}$ from the probability computation about $\mathbf{P}_{\mathbf{L}_1}$ in Theorem 4.1. Hence we need sublattice pairs $(\mathbf{L}_1, \mathbf{L})$ with ideals $\mathbf{Q}$ to mount our sublattice attack.

3 Our contribution

3.1 Sublattice pair attack

We consider the decision non-dual Ring-LWE over the integer ring $\mathbf{R_K}$ of a number field $\mathbf{K}$ of degree d. Let q be a modulus parameter. $\mathbf{a}$ and $\mathbf{s}$ are chosen uniformly at random in $(\mathbf{R_K})_q = \mathbf{R_K}/q\mathbf{R_K}$. The error $\mathbf{e}$ is sampled in $(\mathbf{R_K})_q$ according to a discrete Gaussian distribution (with respect to the canonical embedding) with the width σ. We define sublattice pair $(\mathbf{L}_1, \mathbf{L}_2)$ with an ideal $\mathbf{Q}$ as follows.

Definition. *We assume that the modulus parameter q satisfies $d^{C_1} \leq q < d^{C_2}$ where C_1 and C_2 are two fixed positive integers. Let $\mathbf{L}_i \subset \mathbf{R_K}$ be a sublattice in $\mathbf{R_K}$ for $i = 1, 2$ and $\mathbf{Q} \subset \mathbf{R_K}$ be an ideal. They satisfy the following properties.*
1) $q\mathbf{R_K} \subset \mathbf{L}_i \subset \mathbf{R_K}$ for $i = 1, 2$ and $q\mathbf{R_K} \subset \mathbf{Q} \subset \mathbf{R_K}$;
2) $\mathbf{L}_1, \mathbf{L}_2, \mathbf{Q}$ are polynomially indexed in $\mathbf{R_K}$, that is, there exists a fixed positive integer C_3 such that $|\mathbf{R_K}/\mathbf{L}_i| \leq d^{C_3}$ for $i = 1, 2$ and $|\mathbf{R_K}/\mathbf{Q}| \leq d^{C_3}$;
3) $\mathbf{Q} \subset \mathbf{L}_2$ and $\mathbf{L}_1 \subset \mathbf{L}_2$;
4) The probability $\mathbf{P}_{\mathbf{L}_1}$ that $\mathbf{e} \in \mathbf{L}_1$ satisfies the inequality $\mathbf{P}_{\mathbf{L}_1} \geq \frac{d^{C_4}}{|\mathbf{R_K}/\mathbf{L}_2|}$ where C_4 is a fixed positive integer. We call $(\mathbf{L}_1, \mathbf{L}_2)$ a sublattice pair with the ideal $\mathbf{Q}$ for the Ring-LWE over $\mathbf{K}$ with the modulus parameter q.

Notice that the condition 4)

$$P_{\mathbf{L}_1} \geq \frac{d^{C_4}}{|\mathbf{R_K}/\mathbf{L}_2|}$$

and the condition 3) are the main points that the samples from Ring-LWE equations can be distinguished from uniformly ones within polynomial time. We refer to the proof of Theorem 2.1. The sublattice $\mathbf{L}_2$ in our sublattice attacks should be carefully constructed such that there is a sublattice $\mathbf{L}_1$ satisfying 3) and 4).

The following result is to transform the LWE equation $\mathbf{a} \cdot \mathbf{s} + \mathbf{e} \equiv \mathbf{b}$ *mod* q to a weaker equation $\mathbf{a} \cdot \mathbf{s} + \mathbf{e} \equiv \mathbf{b}$ *mod* $\mathbf{L}_2$ when $\mathbf{L}_2$ has a sublattice pair

with an ideal. In previous works [18, 7, 38] only the case $\mathbf{L}_2$ is an ideal was considered.

Theorem 3.1. *We consider the decision non-dual Ring-LWE over* $\mathbf{R_K}$ *with a general error distribution and a modulus parameter* q *satisfying* $d^{C_1} \leq q < d^{C_2}$ *where* C_1 *and* C_2 *are two fixed positive integers. Suppose that there exists a sublattice pair with an ideal as above. Then the decision Ring-LWE over* $\mathbf{R_K}$ *with the modulus parameter* q *can be solved within time complexity* $O(d^{4C_2C_3})$.

We consider the following sequence of irreducible polynomials $f_d = x^d - u_d x + u_d(u_d - 1) \in \mathbf{Z}[x]$, where $u_d = 2p_d$ is a polynomially bounded positive integer which contain only prime factors of exponent 1. From the Eisenstein criterion f_d is an irreducible polynomial. Let $\mathbf{K}_d = \mathbf{Q}[x]/(f_d(x))$ be a degree d extension field of $\mathbf{Q}$ with the integer ring $\mathbf{R_{K_d}}$. We take $W_d = u_d(u_d - 1)^3$ and $u_d = 2p_d$, where p_d is a prime number satisfying the conditions in the following Theorem.

Theorem 3.2. *Let* C *be an arbitrary fixed positive integer. There exist a sequence of polynomially bounded positive integers* $u_d = 2p_d$ *where* p_d *is a suitable polynomially bounded prime number satisfying*
1)All prime factors of $2p_d - 1$ *are distinct,* $\gcd(d - 1, 2p_d - 1) = 1$*;*
2)$p_d \geq d^{5C+2}$*;*
and a sequence of modulus parameters $W_d = p_d(2p_d - 1)^3$*, such that if the width of the error distributions satisfies* $\frac{\sqrt{d}}{\lambda_1(\mathbf{R_{K_d}}^\vee)} \leq \sigma \leq \frac{u_d(u_d-1)d^{C+1}}{4}$*, we can construct a sequence of sublattice pairs* $(\mathbf{L}_1^d, \mathbf{L}_2^d)$ *with an ideal* $\mathbf{Q}$ *satisfying*
1)$|\mathbf{R_{K_d}}/\mathbf{L}_2^d|$ *is polynomially bounded and at least* $\frac{u_d^2(u_d-1)^3}{4}$*;*
2)The probability $\mathbf{e} \in \mathbf{L}_1^d$ *is lower bounded by* $\frac{1}{64d^{3C+1}u_d^2(u_d-1)^2}$*;*
3)The index $|\mathbf{R_K}/\mathbf{Q}| \leq W_d^2$.

The construction in Theorem 2.2 implies that for Ring-LWE over $\mathbf{R_{K_d}}$, we can always have an effective sublattice attack if the upper bound on widths (wider than the range in hardness reduction results in [40]) in Theorem 2.2 is satisfied. We should notice that from the proof of Theorem 2.1 the partial information of the private key $\mathbf{s}$ *mod* $\mathbf{L}_2$ can be found within a polynomial time when polynomially bounded many samples are given, we refer to Section 5.

Corollary 3.1. *Let C be an arbitrary fixed positive integer. We consider the decision non-dual Ring-LWE over $\mathbf{R_{K_d}}$. There exist a sequence of polynomially bounded positive integers $u_d = 2p_d$ where p_d is a suitable polynomially bounded prime number, and a sequence of modulus parameters $q_d = p_d(2p_d - 1)^3$ only depending on d and C, such that if with the width of the error distributions satisfying $\frac{\sqrt{d}}{\lambda_1(\mathbf{R_{K_d}}^\vee)} \leq \sigma \leq \frac{u_d(u_d-1)d^{C+1}}{4}$, the decision non-dual Ring-LWE with modulus parameter q_d can be solved in polynomial time (in d and C).*

We can consider the attack on the dual form of decision Ring-LWE from Corollary 2.1 since we estimates the size $|f'(\theta)|$ of "tweak factors" in Corollary 4.1. We have the following result.

Corollary 3.2. *Let C be an arbitrary fixed positive integer. We consider the decision dual Ring-LWE over $\mathbf{R_K}^\vee$ where $\mathbf{K}$ is the number field as above. Suppose that the width of the error distributions satisfies $\frac{\sqrt{d}}{\lambda_1(\mathbf{R_{K_d}}^\vee)} \leq \sigma \leq d^C$. Then there exist a sequence of polynomially bounded modulus parameters q_d only depending on d and C, such that the decision dual Ring-LWE with above modulus parameter q_d can be solved in polynomial time (in d and C).*

From the hardness reduction result Corollary 6.3 in [40] (refer to Subsection 1.5) we have the following result. We refer to [9] for another proof of similar result without using the reduction to Ring-LWE.

Corollary 3.3. *Let $\mathbf{K}_d$ be a sequence of number field sequence with their degrees $d \longrightarrow \infty$ as in Theorem 2.2. Then approximating $SIVP_{d^{18}}$ with approximating factor d^{18} for ideal lattices in $\mathbf{K}_d$ can be solved by a polynomial (in d) time quantum algorithm.*

3.2 Cryptographic and algorithmic implications

From Corollary 3.1 the decision Ring-LWE over certain number fields can be solved by a polynomial time algorithm in classical computation model even for error distributions with the widths in recommended range of [40]. Sublattice attacks on the decision Ring-LWE over the two-to-power cyclotomic integer rings with wide error distributions will be presented in [10]. For the complexity theory of computational problems for ideal lattices, our main result Corollary 3.2 indicates that approximating ideal-SIVP with a polyno-

mial factor for certain number fields can be solved in quantum polynomial time. It is interesting to know for other number field sequences whether the approximating ideal-$SIVP_{poly(d)}$ can be solved in quantum polynomial time.

4 Probability computation

We need the following computation of probability in Theorem 3.2.

Theorem 4.1. *Let* $\mathbf{L}$ *be a rank* d *number field lattice in a degree* d *number field* $\mathbf{K}$*. Let* $\mathbf{L}_1$ *be rank* d *sublattice of* $\mathbf{L}^\vee$ *satisfying that* $q\mathbf{L}^\vee \subset \mathbf{L}_1 \subset \mathbf{L}^\vee$ *and the cardinality* $|\mathbf{L}^\vee/\mathbf{L}_1|$ *is polynomially bounded. Suppose that the width of the Gaussian distribution (with respect to the canonical embedding) of errors* $\mathbf{e}$ *satisfying* $\frac{\sqrt{d}}{\lambda_1(\mathbf{L})} \leq \sigma \leq \frac{\sqrt{c_1}}{\sqrt{\pi}\lambda_1(\mathbf{L}_1^\vee)}$ *and moreover there are at least* $\frac{|\mathbf{L}^\vee/\mathbf{L}_1|}{q^{c_2}}$ *lattice vectors in* $\mathbf{L}_1^\vee$ *satisfying* $||\mathbf{x}||_{tr} \leq \frac{\sqrt{c_1}}{\sqrt{\pi}\sigma}$*, where* c_1 *and* c_2 *are fixed positive real numbers. Then the probability* $\mathbf{e} \in \mathbf{L}_1$ *is*

$$\mathbf{P}_{\mathbf{L}_1} = \frac{\Sigma_{\mathbf{x}\in\mathbf{L}_1} e^{-\pi(\frac{||\mathbf{x}||_{tr}}{\sigma})^2}}{\Sigma_{\mathbf{x}\in\mathbf{L}^\vee} e^{-\pi(\frac{||\mathbf{x}||_{tr}}{\sigma})^2}}.$$

It satisfies

$$\mathbf{P}_{\mathbf{L}_1} \geq \frac{1}{e^{c_1} q^{c_2}}$$

when q *is sufficiently large.*

Proof. We calculate the probability $\mathbf{P}_{\mathbf{L}_1}$ of the condition $\mathbf{e} \equiv 0 \ mod \ \mathbf{L}_1$. It is clear

$$\mathbf{P}_{\mathbf{L}_1} = \frac{\Sigma_{\mathbf{x}\in\mathbf{L}_1} e^{-\pi(\frac{||\mathbf{x}||_{tr}}{\sigma})^2}}{\Sigma_{\mathbf{x}\in\mathbf{L}^\vee} e^{-\pi(\frac{||\mathbf{x}||_{tr}}{\sigma})^2}}.$$

Set $Y_3(0) = \frac{\Sigma_{\mathbf{x}\in\mathbf{L}^\vee} e^{-\pi(\frac{||\mathbf{x}||_{tr}}{\sigma})^2}}{\sigma^n}$ and $Y_4(0) = \frac{\Sigma_{\mathbf{x}\in\mathbf{L}_1} e^{-\pi(\frac{||\mathbf{x}||_{tr}}{\sigma})^2}}{\sigma^n}$. From the Poisson summation formula (see [31]) we have

$$Y_3(0) = \frac{1}{\det(\mathbf{L}^\vee)} \Sigma_{\mathbf{x}\in\mathbf{L}} e^{-\pi(||\mathbf{x}||_{tr}\sigma)^2}.$$

and

$$Y_4(0) = \frac{1}{\det(\mathbf{L}_1)} \Sigma_{\mathbf{x}\in(\mathbf{L}_1)^\vee} e^{-\pi(||\mathbf{x}||_{tr}\sigma)^2}.$$

Since $\sigma \geq \frac{\sqrt{d}}{\lambda_1(\mathbf{L})}$ then $\Sigma_{\mathbf{x}\in\mathbf{L}-\mathbf{0}} e^{-\pi(||\mathbf{x}||_{tr}\sigma)^2} \leq 1 + \frac{1}{2^d}$ from Lemma 3.2 in [31]. For lattice vectors $\mathbf{x} \in \mathbf{L}_1^\vee$ satisfying

$$||\mathbf{x}||_{tr} \leq \frac{\sqrt{c_1}}{\sqrt{\pi}\sigma}$$

we have

$$e^{-\pi(||\mathbf{x}||_{tr}\sigma)^2} \geq e^{-c_1}.$$

Hence $\mathbf{P}_{\mathbf{L}_1} \geq \frac{1}{|\mathbf{L}^\vee/\mathbf{L}_1|}(1 + \frac{1}{e^{c_1}} \cdot \frac{|\mathbf{L}^\vee/\mathbf{L}_1|}{q^{c_2}})$. The conclusion follows directly.

5 Number theory

The following proposition is useful in this paper. Please refer to [13, 5] for the proof.

Proposition 5.1. *Let $\mathbf{K} = \mathbf{Q}[\alpha]$ be a number field of degree n and $f(T) \in \mathbf{Q}[T] = a_n T^n + a_{n-1}T^{n-1} + \cdots + a_T + a_0$ be the minimal polynomial of α. Write*

$$f(T) = (T - \alpha)(c_{n-1}T^{n-1} + \cdots + c_1(\alpha)T + c_0(\alpha))$$

where $c_j(\alpha) = \Sigma_{i=j+1}^n a_i \alpha^{i-j-1}$. The dual base of $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ relative to the trace product is

$$\{\frac{c_0(\alpha)}{f'(\alpha)}, \frac{c_1(\alpha)}{f'(\alpha)}, \ldots, \frac{c_{n-1}(\alpha)}{f'(\alpha)}\}$$

.

Let p be a positive integer and $p\mathbf{R_K} = \mathbf{P}^{e_1} \cdots \mathbf{P}_t^{e_t}$ where $\mathbf{P}_i$ are prime ideals and $e_i \geq 1$ are positive integers, is the factorization of the ideal $p\mathbf{R_K}$ to the product of prime ideals.

Proposition 5.2. *If $\mathbf{I} \subset \mathbf{R_K}$ is an ideal containing the positive integer p, then $\mathbf{I}$ is of the form*

$$\mathbf{P}_{j_1}^{e'_1} \cdots \mathbf{P}_{j'_t}^{e'_{t'}}$$

where $t' \leq t$ $e'_i \leq e_{j_i}$.

Proof. Set $\mathbf{I} = \prod_j \mathbf{Q}_j$ the factorization of $\mathbf{I}$ to the product of prime ideals. Then $p \in \mathbf{Q}_j$ and $\mathbf{Q}_j$ is a prime ideal over p. The conclusion follows

directly.

From Proposition 5.2 only few ideals $\mathbf{I}$ satisfy the condition $q\mathbf{R}_K \subset \mathbf{I}$. Hence in sublattice attack it is not natural to require a sublattice $\mathbf{L}$ satisfying $q\mathbf{R_K} \subset \mathbf{L} \subset \mathbf{R_K}$ to be an ideal.

We refer to [33] Theorem 1 for the following result, which is useful to estimate the trace norm of an algebraic integer.

Proposition 5.3. *For any positive integer n and $1 \leq k \leq n-1$, let $P(x) = x^n + a_{n-k-1}x^{n-k-1} + \cdots + a_0$ be a complex polynomial such that $a_0 \neq 0$. For any root α of P, we have*

$$|\alpha| \leq (n-k)^{\frac{1}{k+1}} \max_{1 \leq j \leq n} |a_{n-j}|^{\frac{1}{j}}.$$

Here $|\alpha|$ is the absolute value of the complex number α.

Set $f = x^n - ux + u(u-1)$ where u is an positive integer with only prime factors of exponent 1. Then $f(x) \in \mathbf{Z}[x]$ is an irreducible polynomial from the Eisenstein criterion. Let $\theta \in \mathbf{C}$ be a root of f, we have the following result from Proposition 4.3.

Corollary 5.1. *Let u be a positive integer satisfying $n < u \leq n^c$ for some fixed positive integer c. Then $1 < |\theta| \leq \frac{9}{8}$ when n is sufficiently large. We have $||\theta||_{tr} \leq \sqrt{2n}(u(u-1))^{\frac{1}{n}}$ and then*

$$||\theta^{n-1}||_{tr} \leq \sqrt{2n}(u(u-1))$$

and

$$||\theta^{n-2}||_{tr} \leq \sqrt{2n}(u(u-1)).$$

Moreover $f'(\theta) = n\theta^{n-1} - u$ satisfies

$$\frac{nu(u-1)}{2} \leq |f'(\theta)| \leq \frac{3nu(u-1)}{2}.$$

Proof. From Proposition 4.3 the inequality $|\theta| \leq \frac{9}{8}$ holds when n is sufficiently large. If $|\theta| < 1$, $|\theta^{n-1}| = |u - \frac{u(u-1)}{\theta}| > \frac{8|u(u-1)|}{9} - u \geq \frac{|u(u-1)|}{2}$. This is a contradiction. Hence $1 < |\theta|$. The other conclusions follows from

Proposition 4.3 directly.

The following Kummer lemma (see [12, 5]) is useful for the decomposition of prime numbers to the product of prime ideals in number fields.

Proposition 5.4. *Let $\mathbf{K} = \mathbf{Q}[\theta]$ be a number field, where θ is an algebraic integer whose monic minimal polynomial is denoted by $f(X)$. Then for any prime p not dividing $|\mathbf{R_K}/\mathbf{Z}[\theta]|$ one can obtain the prime decomposition of $p\mathbf{R_K}$ as follows. Let $f(X) \equiv \prod_{i=1}^{g} f_i(X)^{e_i}$ mod p be the decomposition of $f(X)$ module p into irreducible factors in $\mathbf{F}_p[X]$ where f_i are taken to be monic. Then*

$$p\mathbf{R_K} = \prod_{i=1}^{g} \mathbf{P}_i^{e_i},$$

where

$$\mathbf{P}_i = (p, f_i(\theta)) = p\mathbf{R_K} + f_i(\theta)\mathbf{R_K}.$$

Furthermore the residual index of $\mathbf{P}_i$ is equal to the degree of f_i.

We refer to [12] Theorem 6.1.4 for the following Dedekind criterion which is helpful to decide $f = |\mathbf{R_K}/\mathbf{Z}[\theta]|$.

Proposition 5.5 (Dedekind Criterion) *Let $\mathbf{K} = \mathbf{Q}[\theta]$ be a number field, $T(x) \in \mathbf{Z}[x]$ the monic minimal polynomial of θ and p be a prime number. Denote by $\bar{a}$ the reduction module p (in $\mathbf{Z}$ or $\mathbf{Z}[\theta]$). Let*

$$\bar{T(x)} = \prod_{i=1}^{k} \bar{t_i}^{e_i}$$

be the factorization of $T(x)$ module p in $\mathbf{F}_p[x]$, and set

$$g(x) = \prod_{i=1}^{k} t_i(x),$$

where $t_i \in \mathbf{Z}[x]$ are arbitrary lifts of $\bar{t_i}$. Let $h(x)$ be a monic lift of $\frac{\bar{T(x)}}{\bar{g(x)}}$ and set $f(x) = \frac{g(x)h(x)-T(x)}{p}$. Then $|\mathbf{R_K}/\mathbf{Z}[\theta]|$ is not divisible by p if and only if $\gcd(\bar{f}, \bar{g}, \bar{h}) = 1$ in $\mathbf{F}_p[x]$.

Let $f = x^n - ux + u(u-1)$ where u is a positive integer which has only prime factors of exponent 1. Let $\theta \in \mathbf{C}$ be a root of f, $\mathbf{K} = \mathbf{Q}[\theta] = \mathbf{Q}[x]/(f)$

is the number field. Let $\mathbf{R_K}$ is the ring of integers in $\mathbf{K}$, $\mathbf{Z}[\theta] \subset \mathbf{R_K}$ is an order.

Theorem 5.1. *Let u be a positive integer satisfying*
1) u is square-free; and
2) All prime factors of $u-1$ are distinct and $\gcd(n-1,p)=1$ for any prime factor p of $u-1$.
Let $f=x^n-ux+u(u-1)$ as above, p be a prime factor of u or $u-1$. Then $|\mathbf{R_K}/\mathbf{Z}[\theta]|$ is not divisible by p.

Proof. First of all we have $f \equiv (x^2-ux+u^2-u)(x^{n-2}+ux^{n-3}+ux^{n-4}+\cdots+ux+u)$ *mod* W if W is a factor of $u(u-1)^2$. When p is a factor of u, we can take $g(x)=x$, $h(x)=x^{n-1}$. Then $f(x)=\frac{x^n-(x^n-ux+u(u-1))}{p}=\frac{u}{p}x-\frac{u}{p}(u-1)$. It is easy to verify $\gcd(\bar{f},\bar{g},\bar{h})=1$ in $\mathbf{F}_p[x]$. The conclusion follows from the Dedekind criterion.

When p is a prime factor of $u-1$ we have $f(x) \equiv x(x-1)(x^{n-2}+x^{n-3}+x^{n-4}+\cdots+x+1)$ *mod* p. Since $\gcd(n-1,p)=1$, then $x^{n-1}-1$ has $n-1$ distinct roots in the algebraic closure of $\mathbf{F}_p$. We can take $g(x)=x(x-1)(x^{n-2}+\cdots+x+1)$, $h(x)=1$. Then $f(x)=\frac{x^n-x-(x^n-ux+u(u-1))}{p}=\frac{u-1}{p}x-u\frac{u-1}{p}$. It is easy to verify that $\gcd(\bar{h},\bar{f},\bar{g})=1$ in $\mathbf{F}_p$ from the condition 2). The conclusion follows from the Dedekind criterion.

The main construction in Theorem 3.2 is as follows. There should be many very short lattice vectors in the dual $\mathbf{L}_1^\vee$ of the number field lattice $\mathbf{L}_1$ satisfying $q\mathbf{R_{K_d}} \subset \mathbf{L}_1 \subset \mathbf{R_{K_d}}$. For given $\mathbf{x}_1,\ldots,\mathbf{x}_t$, t elements in $\mathbf{R_K}^\vee/q\mathbf{R_K}^\vee$, we define a number field lattice $\mathbf{L}(\mathbf{x}_1,\ldots,\mathbf{x}_t)$ by the equations $Tr(\mathbf{x}_i\mathbf{y}) \equiv 0$ *mod* q, where $\mathbf{y} \in \mathbf{R_K}$, $i=1,\ldots,t$. It is obvious $q\mathbf{R_K} \subset \mathbf{L} \subset \mathbf{R_K}$. Moreover it is clear the definition of $\mathbf{L}(\mathbf{x}_1,\ldots,\mathbf{x}_t)$ only depends on the residue classes of $\mathbf{x}_i$'s in $\mathbf{R_K}^\vee/q\mathbf{R_K}^\vee$.

Proposition 5.6. *The vectors $\frac{\mathbf{x}_1}{q},\ldots,\frac{\mathbf{x}_t}{q}$ are in the dual lattice*

$$\mathbf{L}(\mathbf{x}_1,\ldots,\mathbf{x}_t)^\vee \subset \frac{\mathbf{R_K}^\vee}{q}.$$

If $\mathbf{a} \in \mathbf{R_K}$ is an invertible element in $\mathbf{R_K}/q\mathbf{R_K}$, then there is a $\mathbf{Z}/q\mathbf{Z}$ linear isomorphism from $\mathbf{L}(\mathbf{x}_1,\ldots,\mathbf{x}_t)$ to $\mathbf{L}(\mathbf{a}^{-1}\mathbf{x}_1,\ldots,\mathbf{a}^{-1}\mathbf{x}_t)$ defined by $\mathbf{y} \longrightarrow \mathbf{ay}$. In particular the cardinalities of

$$\mathbf{R_K}/\mathbf{L}(\mathbf{x}_1,\ldots,\mathbf{x}_t)$$

and

$$\mathbf{R_K}/\mathbf{L}(\mathbf{a}^{-1}\mathbf{x}_1, \ldots, \mathbf{a}^{-1}\mathbf{x}_t)$$

are the same.

Proof. The first conclusion is direct from the definition. The second conclusion is a simple computation.

The following result gives a restriction on the $\lambda_1(\mathbf{L}^\vee)$ of number field lattice $\mathbf{L}$ if $\mathbf{L}$ containing the product of two number field lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ satisfying $|\mathbf{R_K}/\mathbf{L}_i| \leq poly(n)$.

Theorem 5.2. *Let $\mathbf{L}_1, \mathbf{L}_2$ and $\mathbf{L}_3$ be three polynomially bounded index sublattices of rank d in the integer ring $\mathbf{R_K}$ of a degree d number field $\mathbf{K}$. That is $|\mathbf{R_K}/\mathbf{L}_i| \leq d^c$ holds for a fixed positive integer c and $i = 1, 2, 3$. We assume $\mathbf{L}_2 \cdot \mathbf{L}_3 \subset \mathbf{L}_1$. Then $\lambda_1(\mathbf{L}_1^\vee) \geq \Omega(\frac{1}{|\Delta_\mathbf{K}|^{\frac{3}{2d}} d^{\frac{2c}{d}}})$.*

Proof. For $\mathbf{x} \in \mathbf{L}_1^\vee$, let $\mathbf{X}$ be the matrix representation of the multiplication of $\mathbf{x}$ with respect to a fixed $\mathbf{Z}$-base of $\mathbf{R_K}$. For a number field lattice $\mathbf{L}$ set $\mathbf{B}(\mathbf{L})$ to be the matrix representation of $\mathbf{L}^\vee$ with respect to this fixed base of $\mathbf{R_K}$. Then

$$|\det(\mathbf{B}(\mathbf{L}_2^\vee))| = |\Delta_\mathbf{K}|^{-1} \cdot |(\det(\mathbf{B}(\mathbf{L}_2)))^{-1}| \geq \frac{1}{|\Delta_\mathbf{K}|^{3/2} d^c}$$

from the definition of dual lattice. Since $\mathbf{x} \in (\mathbf{L}_2 \cdot \mathbf{L}_3)^\vee$, $\mathbf{xy} \in \mathbf{L}_2^\vee$ for each $\mathbf{y} \in \mathbf{L}_3$. Then

$$\mathbf{B}(\mathbf{L}_3) \cdot \mathbf{X} = \mathbf{M} \cdot \mathbf{B}(\mathbf{L}_2^\vee)$$

for some non-singular integer matrix $\mathbf{M}$. We have

$$|\det(\mathbf{X})| \geq |\det(\mathbf{M})| \cdot \frac{1}{|\Delta_\mathbf{K}|^{3/2} d^{2c}} \geq \frac{1}{|\Delta_\mathbf{K}|^{3/2} d^{2c}}$$

since $|\det(\mathbf{M})| \geq 1$. It is clear

$$||\mathbf{x}||_{tr} = (\Sigma_{i=1}|\sigma_i(\mathbf{x})|^2)^{1/2} \geq \sqrt{d}(\prod_{i=1}^{d} \sigma_i(\mathbf{x}))^{1/d} = \sqrt{d}(N(x\mathbf{R_K}))^{1/d} = \sqrt{d}|\det(\mathbf{X})|^{1/d}.$$

The conclusion follows directly.

From Theorem 5.2 if a sublattice $\mathbf{L}$ in $\mathbf{R_K}$ contains the product of two polynomially bounded cardinality sublattices, the $\lambda_1(\mathbf{L}^\vee)$ is lower bounded

by $\Omega(\frac{1}{|\Delta_{\mathbf{K}}|^{\frac{3}{2d}} d^{\frac{2c}{d}}})$ when d is sufficiently large. In particular if both $\mathbf{L}$ and $\mathbf{O^L}$ are with polynomially bounded cardinalities, $\lambda_1(\mathbf{L}^\vee)$ can not be very small. The sublattice attack with non-negligible $\mathbf{O^L}$ suggested in [8] has a strong restriction on the bound of width as the attack when $\mathbf{L}_1$ is required to be an ideal as in [18, 7, 38].

6 Proofs of main results

Proof of Theorem 3.1. First of all the probability that uniformly chosen $\mathbf{a} \in \mathbf{R_K}/q\mathbf{R_K}$ is in the ideal $\mathbf{Q}/q\mathbf{R_K}$ is at least $\frac{1}{d^{C_3}}$ from the condition 2 of the sublattice pair with the ideal $\mathbf{Q}$. We take the d^C samples $(\mathbf{a}, \mathbf{b})$'s from the Ring-LWE equation satisfy $\mathbf{a} \in \mathbf{Q}/q\mathbf{R_K}$. This can be achieved within time complexity $O(d^{2(C+C_3)})$ by checking the algebraic condition $\mathbf{a} \in \mathbf{Q}/q\mathbf{R_K}$ when $\mathbf{Q}$ is explicitly given.

Since $\mathbf{a} \in \mathbf{Q}/q\mathbf{R_K}$ then $\mathbf{a}\cdot\mathbf{s} \in \mathbf{Q}/q\mathbf{R_K} \subset \mathbf{L}_2/q\mathbf{R_K}$ for arbitrary unknown secret $\mathbf{s}$ from the condition 3) of the sublattice pair. Hence for the d^C samples $(\mathbf{a}, \mathbf{b})$'s from the Ring-LWE equation satisfy $\mathbf{a} \in \mathbf{Q}/q\mathbf{R_K}$, the probability that $\mathbf{b} \in \mathbf{L}_2$ is bigger than the probability $\mathbf{P}_{\mathbf{L}_1}$ that $\mathbf{e} \in \mathbf{L}_1$ since $\mathbf{L}_1 \subset \mathbf{L}_2$ and $\mathbf{a} \cdot \mathbf{s} + \mathbf{e} \in \mathbf{L}_2$ from the fact $\mathbf{a} \cdot \mathbf{s} \in \mathbf{L}_2$. Then for these d^C samples $(\mathbf{a}, \mathbf{b})$'s from the Ring-LWE equation satisfy $\mathbf{a} \in \mathbf{Q}/q\mathbf{R_K}$, the probability that $\mathbf{b} \in \mathbf{L}_2$ is bigger than

$$\mathbf{P}_{\mathbf{L}_1} \geq \frac{d^{C_4}}{|\mathbf{R_K}/\mathbf{L}_2|} \geq \frac{2}{|\mathbf{R_K}/\mathbf{L}_2|}.$$

One the other hand we look at $(\mathbf{a}, \mathbf{b})$ uniformly distributed samples in

$$\mathbf{R_K}/\mathbf{Q} \times \mathbf{R_K}/\mathbf{L}_2.$$

For any fixed first coset, the second components $\mathbf{b}$'s have to be equally distributed in $\mathbf{R_K}/\mathbf{L}_2$. Therefore by checking the algebraic condition $\mathbf{b} \in \mathbf{L}_2$ for these d^C samples $(\mathbf{a}, \mathbf{b})$'s from the Ring-LWE equation satisfy $\mathbf{a} \in \mathbf{Q}/q\mathbf{R_K}$, we can distinguish from these uniformly chosen $(\mathbf{a}, \mathbf{b})$ within time complexity $O(d^{4C_3})$.

Another proof can be given as follows. We consider the secret $\mathbf{s} \in \mathbf{Q}$, this happens with a probability at least $\frac{1}{d^{C_3}}$. For such a secret, given d^C samples $(\mathbf{a}, \mathbf{b})$, we have

$$\mathbf{a} \cdot \mathbf{s} \in \mathbf{Q} \subset \mathbf{L}_2.$$

The probability that $\mathbf{b} \in \mathbf{L}_2$ is bigger than the probability $\mathbf{P}_{\mathbf{L}_1}$ that $\mathbf{e} \in \mathbf{L}_1$ since $\mathbf{L}_1 \subset \mathbf{L}_2$. Then for these d^C samples $(\mathbf{a}, \mathbf{b})$'s, the probability that $\mathbf{b} \in \mathbf{L}_2$ is bigger than

$$\mathbf{P}_{\mathbf{L}_1} \geq \frac{d^{C_4}}{|\mathbf{R_K}/\mathbf{L}_2|} \geq \frac{2}{|\mathbf{R_K}/\mathbf{L}_2|}.$$

This can be distinguished from the uniformly distributed d^C samples.

Proof of Theorem 3.2. First of all we have $x^d - u_d x + u_d(u_d - 1) = (x^2 - u_d x + u_d(u_d - 1))(x^{d-2} + u_d x^{d-3} + u_d x^{d-4} + \cdots + u_d x + u_d) - u_d(u_d - 1)^2(x^{d-3} + \cdots + x^2 + x + 1)$. Then $x^d - u_d x + u_d(u_d - 1) \equiv (x^2 - u_d x + u_d(u_d - 1) + u_d(u_d - 1)^2(\frac{1}{d-1}x - 1))(x^{d-2} + u_d x^{d-3} + \cdots + u_d + u_d(u_d - 1)^2(G_3(x) - G_2(x) - \frac{1}{d-1}G_1(x)))$ mod $u_d(u_d - 1)^3$, where

$$G_1(x) = \frac{x^{d-1} - (d-1)x + d - 2}{(x-1)^2},$$

$$G_2(x) = \frac{x^{d-3} - (d-3)x + d - 4}{(x-1)^2},$$

and

$$G_3(x) = \frac{x^{d-2} - (d-2)x + d - 3}{(x-1)^2}.$$

Notice that $\gcd(d-1, u_d - 1) = 1$ and then $d - 1$ is invertible in the ring $\mathbf{Z}/(u_d - 1)\mathbf{Z}$, $G_1(x), G_2(x), G_3(x)$ are polynomials in $\mathbf{Z}/(u_d - 1)\mathbf{Z}[x]$. Here we have $x^{d-3} + \cdots + x^2 + x + 1 \equiv (1 - \frac{1}{d-1}x)(x^{d-2} + u_d(x^{d-3} + \cdots + 1)) + (x^2 - u_d x + u_d(u_d - 1))(G_2(x) + \frac{1}{d-1}G_1(x) - G_3(x))$ mod $u_d - 1$, where

$$G_3(x) - G_2(x) - \frac{1}{d-1}G_1(x) = \frac{\frac{-1}{d-1}x^{d-1} + x^{d-2} - x^{d-3} + \frac{1}{d-1}}{(x-1)^2}.$$

We take $W_d = u_d(u_d - 1)^3$ and $u_d = 2p_d$, where p_d is a prime number such that $2p_d - 1$ satisfies the following conditions.
1) All prime factors of $2p_d - 1$ are distinct and not smaller than $d - 1$, $\gcd(d-1, 2p_d - 1) = 1$;
2) p_d is polynomially bounded and $p_d \geq d^{5C+2}$.
From Theorem 4.1 we have

$$\mathbf{R}_{\mathbf{K_d}}/W_d\mathbf{R}_{\mathbf{K_d}} = \mathbf{Z}[\theta]/W_d\mathbf{Z}[\theta],$$

Hence

$$\mathbf{R_{K_d}}^{\vee}/W_d\mathbf{R_{K_d}}^{\vee} = \mathbf{Z}[\theta]^{\vee}/W_d\mathbf{Z}[\theta]^{\vee}.$$

The sublattice pair $(\mathbf{L}_1^d, \mathbf{L}_2^d)$ with an ideal $\mathbf{Q}$ are defined by six vectors in

$$\mathbf{R_{K_d}}^{\vee}/W_d\mathbf{R_{K_d}}^{\vee} = \mathbf{Z}[\theta]^{\vee}/W_d\mathbf{Z}[\theta]^{\vee}$$

as follows. We set

$$\mathbf{x}_1 = \frac{\theta^{d-1}}{f_d'(\theta)},$$

$$\mathbf{x}_2 = \frac{\theta^{d-2}}{f_d'(\theta)},$$

and

$$\mathbf{x}_3 = \frac{1}{f_d'(\theta)}.$$

Set

$$\mathbf{L}_1^d = \mathbf{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3).$$

We consider the following three elements in

$$\mathbf{R_{K_d}}^{\vee}/W_d\mathbf{R_{K_d}}^{\vee} = \mathbf{Z}[\theta]^{\vee}/W_d\mathbf{Z}[\theta]^{\vee}.$$

$$\mathbf{x}_4 = \frac{(u_d-1)(\theta^{d-1} + (u_d-1)\theta^{d-2} - u_d)}{f_d'(\theta)},$$

$$\mathbf{x}_5 = \frac{(u_d-1)^2(\theta^{d-1} - u_d)}{f_d'(\theta)},$$

$$\mathbf{x}_6 = \frac{(u_d-1)((u_d^2-u_d-1)\theta^{d-1} - u_d(u_d-1)\theta^{d-2} + u_d(u_d-2))}{f_d'(\theta)}.$$

Notice that $(u_d-1)(\theta-1)(\theta^{d-2}+u_d\theta^{d-3}+\cdots+u_d+u_d(u_d-1)^2(G_3(\theta)-G_2(\theta)-\frac{2}{d-1}G_1(\theta))) \equiv (u_d-1)(\theta^{d-1}+(u_d-1)\theta^{d-2}-u_d)$ *mod* W_d,
$(u_d-1)^2(\theta-u_d)(\theta^{d-2}+u_d\theta^{d-3}+\cdots+u_d+u_d(u_d-1)^2(G_3(\theta)-G_2(\theta)-\frac{2}{d-1}G_1(\theta))) \equiv (u_d-1)^2(\theta^{d-1}-u_d)$ *mod* W_d,
and $\theta(\theta-1)^2(\theta^{d-2}+u_d\theta^{d-3}+\cdots+u_d+u_d(u_d-1)^2(G_3(\theta)-G_2(\theta)-\frac{1}{d-1}G_1(\theta))) \equiv (u_d-1)((u_d^2-u_d-1)\theta^{d-1}-u_d(u_d-1)\theta^{d-2}+u_d(u_d-2))$ *mod* W_d.

Hence $\mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6$ are in the ideal generated by

$$\frac{\theta^{d-2} + u_d(\theta^{d-3} + \cdots + \theta + 1) + u_d(u_d - 1)^2(G_3(\theta) - \frac{1}{d-1}G_1(\theta) - G_2(\theta))}{f'(\theta))}.$$

Set

$$\mathbf{L}_2^d = \mathbf{L}(\mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6).$$

It is obvious $\mathbf{L}_1^d = \mathbf{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \subset \mathbf{L}_3^d$ since $\mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6$ are linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ with integer coefficients. We can take $\mathbf{Q}$ to be the ideal in $\mathbf{R}_{\mathbf{K_d}}/W\mathbf{R}_{\mathbf{K_d}} = \mathbf{Z}[\theta]/W_d\mathbf{Z}[\theta]$ generated by the element

$$\theta^2 - u_d\theta + u_d^2 - u_d + u_d(u_d - 1)^2(\frac{1}{d-1}\theta - 1).$$

Actually this ideal $\mathbf{Q}$ has index in $\mathbf{R}_{\mathbf{K}}/W_d\mathbf{R}_{\mathbf{K}} = \mathbf{Z}[\theta]/W_d\mathbf{Z}[\theta]$ at most W_d^2. Then $\mathbf{Q} \subset \mathbf{L}_2^d$ since $\mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6$ are in the ideal generated by the element

$$\frac{\theta^{d-2} + u_d\theta^{d-3} + u_d\theta^{d-4} + \cdots + u_d\theta + u_d + u_d(u_d - 1)^2(G_3(\theta) - \frac{1}{d-1}G_1(\theta) - G_2(\theta))}{f'(\theta)}$$

and $(\theta^2 - u_d\theta + u_d(u_d - 1) + u_d(u_d - 1)^2(\frac{1}{d-1}\theta - 1))(\theta^{d-2} + u_d\theta^{d-3} + u_d\theta^{d-4} + \cdots + u_d\theta + u_d + u_d(u_d - 1)^2(G_3(\theta) - \frac{1}{d-1}G_1(\theta) - G_2(\theta))) \equiv \theta^d - u_d\theta + u_d(u_d - 1)$ *mod* W_d.

Now we prove the conclusions 1) and 2) in Theorem 2.2. First of all it is easy to verify the cardinality $|\mathbf{R}_{\mathbf{K_d}}/\mathbf{L}_1^d| = W_d^3$ from Theorem 4.1 and Proposition 4.1. We have

$$||\mathbf{x}_1||_{tr} \leq \frac{2||\theta^{d-1}||_{tr}}{d(u_d(u_d - 1))} \leq \sqrt{\frac{8}{d}},$$

$$||\mathbf{x}_2||_{tr} \leq \frac{2||\theta^{d-2}||_{tr}}{d(u_d(u_d - 1))} \leq \sqrt{\frac{8}{d}}$$

from Corollary 4.1. It is obvious

$$||\mathbf{x}_3||_{tr} \leq \frac{2}{\sqrt{d}u_d(u_d - 1)}$$

from Proposition 4.3. Then the probability that $\mathbf{e} \in \mathbf{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is at least

$$\frac{1}{64d^{3C+1}u_d^2(u_d - 1)^2}$$

by counting the number of lattice vectors $m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + m_3\mathbf{x}_3 \leq \frac{W_d}{\sigma}$ for $m_1, m_2, m_3 \in \mathbf{Z}$ and Theorem 3.1. Actually the number of integers of m_i satisfying $||m_i\mathbf{x}_i||_{tr} \leq \frac{W_d}{3\sigma}$ is at least $\frac{W_d}{4\sigma||\mathbf{x}_i||_{tr}}$, when $\frac{W_d}{\sigma}$ is sufficiently large. Here we can always chose a polynomially bounded u_d satisfying $\frac{W_d}{\sigma} \geq d^{10}$. Hence there are at least $\frac{W_d 3}{64\sigma^3||\mathbf{x}_1||_{tr}||\mathbf{x}_2||_{tr}||\mathbf{x}_3||_{tr}}$ lattice vectors in the dual lattice $\mathbf{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)^\vee$. The lower bound 2) in Theorem 2.2 follows from Theorem 3.1.

We now prove the conclusion 1) of Theorem 2.2, we need to calculate $\mathbf{R_{K_d}}/\mathbf{L}(\mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6)$.

The two conditions $Tr(\mathbf{x}_i \cdot \mathbf{y}) \equiv 0 \ mod \ p_d(u_d - 1)^3$ for $i = 4, 5$ and $\mathbf{y} \in \mathbf{R_{K_d}}/W_d\mathbf{R_{K_d}} = \mathbf{Z}[\theta]/W_d\mathbf{Z}[\theta]$ are equivalent to the two conditions $Tr(\frac{\theta^{d-1}+(u_d-1)\theta^{d-2}-u_d}{f_d'(\theta)} \cdot \mathbf{y}) \equiv 0 \ mod \ p_d(u_d - 1)^2$ and $Tr(\frac{\theta^{d-1}-u_d}{f_d'(\theta)} \cdot \mathbf{y}) \equiv 0 \ mod \ p_d(u_d - 1)$. These two conditions implies that there are $p_d^2(u_d - 1)^2$ residual classes in $\mathbf{R_{K_d}}/\mathbf{L}(\mathbf{x}_4, \mathbf{x}_5)$.

On the other hand from the third condition $Tr(\mathbf{x}_6 \cdot \mathbf{y}) \equiv 0 \ mod \ p_d(u_d - 1)^3$ it is equivalent to $Tr(\frac{(u_d^2-u_d-1)\theta^{d-1}-u_d(u_d-1)\theta^{d-2}+u_d(u_d-2)}{f_d'(\theta)} \cdot \mathbf{y}) \equiv 0 \ mod \ p_d(u_d-1)^2$, then there are at least $p_d^2(u_d-1)^3$ residual classes in $\mathbf{R_{K_d}}/\mathbf{L}(\mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6)$. The conclusion 1) is proved.

Proof of Corollary 3.1. For a given arbitrary positive integer C, we can take polynomially bounded $p_d \geq d^{5C+2}$ then the constructed sublattices $\mathbf{L}_1^d$, $\mathbf{L}_2^d$ and the ideal $|bfQ$ satisfy the requirement as in Definition 2.1 for $q_d = u_d(u_d - 1)^3$ and suitable constants.

Proof of Corollary 3.2. Since we have

$$\mathbf{R_{K_d}}^\vee/W\mathbf{R_{K_d}}^\vee = \mathbf{Z}[\theta]^\vee/W_d\mathbf{Z}[\theta]^\vee$$

when $W = u_d(u_d-1)^3$ and u_d satisfies the conditions of Theorem 4.1. Then

$$\mathbf{R_{K_d}}^\vee/q_d\mathbf{R_{K_d}}^\vee = \mathbf{Z}[\theta]^\vee/q_d\mathbf{Z}[\theta]^\vee.$$

The size $|f'(\theta)|$ was estimated in Corollary 4.1 the conclusion follows from the conversion of dual form Ring-LWE to non-dual form Ring-LWE by by "tweak factors" on the widths.

Proof of Corollary 3.3. We take an arbitrary large positive integer C in Corollary 2.3 then we get a polynomial factor d^{9C} from the Hardness reduction result in Subsection 1.5. To satisfy the main condition in the Hardness reduction result we take $C = 2$ then an approximation factor d^{18} can be achieved.

7 The algorithm

In this algorithm we take $W_d = u_d(u_d - 1)^3$ and $u_d = 2p_d$, where p_d is a prime number such that $2p_d - 1$ satisfies the following conditions.
1) All prime factors of $2p_d - 1$ are distinct, $\gcd(d - 1, 2p_d - 1) = 1$;
2) p_d is polynomially bounded and $p_d \geq d^{5C+2}$.
The conditions in Theorem 5.1 are always satisfied. The ideal $\mathbf{Q}$ in $\mathbf{R_{K_d}}/W\mathbf{R_{K_d}} = \mathbf{Z}[\theta]/W_d\mathbf{Z}[\theta]$ is generated by the element $\theta^2 - u_d\theta + u_d^2 - u_d + u_d(u_d-1)^2(\frac{1}{d-1}\theta - 1)$. $\mathbf{x}_1, \ldots, \mathbf{x}_6$ are defined as in the proof of Theorem 2.2. The sublattice $\mathbf{L}_2^d = \mathbf{L}(\mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6)$ and the auxiliary sublattice $\mathbf{L}_1^d = \mathbf{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

Step 1. For given polynomially bounded many samples $(\mathbf{a}_1, \mathbf{b}_i)$, $i = 1, \ldots, W_d^6$, we find at least $\frac{W_d^6}{W_d^2}$ samples $\mathbf{a}_i$'s which are in the ideal $\mathbf{Q}$ generated by the element $\theta^2 - u_d\theta + u_d^2 - u_d + u_d(u_d-1)^2(\frac{1}{d-1}\theta - 1)$. It is within the time $O(W_d^8)$.

Step 2. For samples $(\mathbf{a}, \mathbf{b})$'s with the first component $\mathbf{a} \in \mathbf{Q}$ we check the probability $\mathbf{b} \in \mathbf{L}_3^d$. If these samples are not from the Ring-LWE equation this probability is $\frac{1}{|\mathbf{R_K}/\mathbf{L}_2^d|}$. If it is from the Ring-LWE equation, this probability is bigger than the probability $\mathbf{P}_{\mathbf{L}_1^d}$ that $\mathbf{e} \in \mathbf{L}_1^d$. Since $\mathbf{P}_{\mathbf{L}_1^d} \geq \frac{2}{|\mathbf{R_K}/\mathbf{L}_2^d|}$, we can distinguish within time complexity $O(W_d^8)$.

8 Conclusion

The essence of sublattice attack on Ring-LWE is that the error distributions of sublattices in $\mathbf{R_K}$ should be checked for these polynomially bounded index sublattices $\mathbf{L}$. This gives new large bounds on widths of solvable instances of Ring-LWE, which are closely related to the $\lambda_1(\mathbf{L}^\vee)$ and the number of very short lattice vectors in $\mathbf{L}^\vee$. In this paper we construct a sequence of number fields such that that decision Ring-LWE can be solved within a polynomial

time complexity for error distributions with the widths in the range of hardness reduction results in [40]. This is the first sequence of number fields with degrees going to the infinity such that Ring-LWE with large width error distributions can be solved by a polynomial time algorithm. From the hardness reduction results in [40] the approximating $SIVP_{poly(d)}$ for ideal lattices in these number fields can be solved within quantum polynomial time. This is also the first sequence of number fields with degrees going to the infinity such that their approximating ideal-$SIVP_{poly(d)}$ can be solved by a polynomial time quantum algorithm. The sublattice attack on Ring-LWE with wide errors over two-to-power cyclotomic integer rings will be presented in [10].

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