

Higher-Order Differentials of Word-Oriented SPN Schemes with Low-Degree S-Boxes

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Abstract. Higher-order differential attacks are among the most powerful attacks against iterated ciphers and hash functions over binary extension fields. Predicting the evolution of the algebraic degree (as a function of the number of iterations) is the main obstacle in assessing the feasibility of these attacks. For ciphers with round functions of algebraic degree δ , it is a common belief that the degree of the cipher grows almost exponentially with δ . However, for an iterated Even–Mansour cipher whose round function can be described as an invertible low-degree polynomial it has recently been shown that the algebraic degree grows linearly with the number of rounds, and not exponentially.

In this paper we generalize these results to SPN schemes and show that the growth of the algebraic degree is often linear for schemes with low-degree S-Boxes as well. Most notably, our analysis applies to SPN ciphers designed to be competitive for recent applications like MPC, FHE, SNARKs, and STARKs, e.g. HadesMiMC. Our findings have been practically verified on small-scale ciphers.

Keywords: Higher-Order Differential Cryptanalysis · SPN · Algebraic Degree

1 Introduction

One of the most powerful cryptanalytic methods for symmetric primitives over \mathbb{F}_2^n with low-degree building blocks is higher-order differential cryptanalysis. In essence, this method allows to distinguish a given Boolean function from a random one. More precisely, given an instance of a block cipher $E_k : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ under a fixed but unknown secret key k , higher-order differential cryptanalysis exploits the fact that for any vector subspace $\mathcal{V} \subseteq \mathbb{F}_2^n$ with dimension strictly greater than the algebraic degree of E_k and for any $c \in \mathbb{F}_2^n$

$$\bigoplus_{x \in \mathcal{V}} E_k(x \oplus c) = 0.$$

An analogous result holds when working with a fixed permutation $P : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$. Since the same property does not, in general, hold for a permutation drawn at random, it can be exploited to distinguish a given (keyed or keyless) permutation from a random permutation and/or to set up key-recovery attacks. The idea was first introduced by Lai [Lai94], albeit without a concrete application. Knudsen [Knu94] then used higher-order differentials to break low-degree ciphers which were deemed secure against standard differential cryptanalysis. Several variants and generalisations of higher-order differential attacks have since been proposed in the literature, including *cube attacks* [DS09] and the *division property* [Tod15].

1.1 Preventing Higher-Order Differential Attacks – State of the Art

We focus on the case of *iterated* block ciphers, that is, ciphers consisting of several iterations of the same round function parametrized by different round keys. To prevent higher-order differential attacks on ciphers over \mathbb{F}_2^n , ideally one would like to make a statement such as:

“After r rounds, there is no output bit with algebraic degree strictly smaller than $n - 1$.”

To achieve this goal, one needs to estimate the growth of the algebraic degree, which is in general a difficult task. In other words, predicting the evolution of the algebraic degree of the cipher when the number of rounds varies is the main challenge in higher-order differential cryptanalysis. A naive bound for the algebraic degree of the composition of two functions $F, G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ is given by

$$\deg(G \circ F) \leq \deg(G) \cdot \deg(F). \quad (1)$$

This bound leads to a first estimate about the number of rounds *necessary* to reach maximum algebraic degree in SPN ciphers. For an SPN cipher defined over $(\mathbb{F}_{2^n})^t$ with S-Box layer of algebraic degree δ it follows that at least $\lceil \log_\delta(n \cdot t - 1) \rceil$ rounds are necessary to prevent higher-order differential attacks (the affine layer does not increase the algebraic degree). We refer to Definition 1 for a clarification on what the terms ‘necessary’ and ‘sufficient’ number of rounds mean.

A Better Estimation for $\deg(G \circ F)$. In general, the naive upper bound (1) does not reflect the real growth of the algebraic degree when considering iterated ciphers, and the problem of estimating the growth of the algebraic degree has therefore been studied in the literature. After the initial work of Canteaut and Videau [CV02], a tighter upper bound was presented by Boura, Canteaut, and De Cannière in [BCD11]. In there, the authors show how to deduce a new bound for the algebraic degree of iterated permutations for a special category of SP networks over $(\mathbb{F}_{2^n})^t$, which includes functions that have a number of $t \geq 1$ balanced S-Boxes as their non-linear layer. As a consequence, the number of rounds necessary to prevent higher-order differential attacks is in general higher than the one obtained using the naive bound in (1). Apart from the bound of Boura, Canteaut and De Cannière, in a follow-up work Boura and Canteaut

studied the influence of F^{-1} on the algebraic degree of $\deg(G \circ F)$ [BC13]. As main result, they discuss how the algebraic degrees of F^{-1} and F affect each other, which subsequently allows them to bound the algebraic degree of $G \circ F$ by means of the degrees of G and F^{-1} .

Degree Growth in MiMC-Like Ciphers. MiMC [AGR⁺16,GRR⁺16] is an *iterated Even-Mansour cipher*, i.e., a cipher natively defined over \mathbb{F}_{2^n} , where the S-Box is given by the cube function $x \mapsto x^3$. Only recently a new upper bound on the algebraic degree growth of MiMC-like ciphers has been proposed in [EGL⁺20]. More precisely, the authors show that when the round function can be described as a low-degree polynomial function over \mathbb{F}_{2^n} of degree at most d , the algebraic degree $\delta(r)$ of r iterations of the round function grows linearly with the number of rounds, and not (almost) exponentially as previously assumed, i.e.

$$\delta(r) \leq \min\{\log_2(d^r), n - 1\}.$$

Consequently, this observation implies that roughly $(n - 1) \cdot \log_d(2)$ rounds are necessary for security against higher-order differential distinguishers. As a concrete application, the authors in [EGL⁺20] were able to exploit this result for setting up the first higher-order differential attack on MiMC better than exhaustive search.

1.2 Our Contribution

Results for SPN Schemes. In this work, we generalise the result presented in [EGL⁺20] and extend it to SPN schemes, where one round consists of the parallel application of $t \geq 2$ invertible low-degree functions over \mathbb{F}_{2^n} . Since we do not make any assumption about the round-keys (with the only exception that they are fixed), our results apply to both keyed and keyless permutations built from word-oriented SPN constructions. In the following, we refer to both of them by using the term “SPN schemes”.

As such, we work with SPN schemes over $(\mathbb{F}_{2^n})^t$. In particular, we improve on the best currently known estimation for the growth of the algebraic degree published in [BCD11]. We show in Section 4 that for all SPN schemes which consist of the parallel application of t copies of the same low-degree S-Box, the algebraic degree grows almost linearly with the number of rounds, and not exponentially as previously believed. More precisely, if d denotes the degree of the S-Boxes as a polynomial over \mathbb{F}_{2^n} and δ denominates the corresponding algebraic degree, we prove that

$$n \cdot \log_d(2) + \log_\delta(t)$$

rounds are necessary to provide security against higher-order differential attacks. In particular, our result improves on the belief that around

$$\log_\delta(n \cdot t - 1) \approx \log_\delta(n) + \log_\delta(t)$$

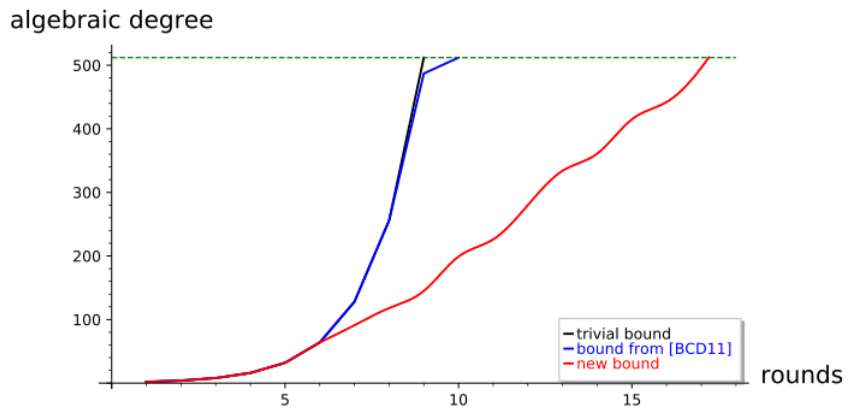


Fig. 1. Growth of upper bounds of the algebraic degree for an SPN cipher/permutation with S-Box $x \mapsto x^3$ over $(\mathbb{F}_{219})^{27}$.

rounds are necessary, based on the assumption of an exponentially growing algebraic degree. A concrete comparison of our bound and other upper bounds of the algebraic degree of a SHARK-like cipher [RDP⁺96] over $(\mathbb{F}_{219})^{27}$ (with the cubing function $x \mapsto x^3$ as S-Boxes and an MDS matrix as mixing layer) is depicted in Fig. 1.

Observations on Partial SPN Schemes. Our results apply to the case of partial SPN schemes as well (that is, schemes with partial non-linear layers). Also in this case, it is possible to show that for low-degree polynomial functions the algebraic degree of the cipher grows mostly linearly. More details are given in Section 6.

2 Preliminaries

In this section, we recall the most important results about polynomial representations of Boolean functions and summarize the currently best known results regarding bounding the algebraic degree in the context of SP networks. We start with a clarification about what we mean by *necessary* and *sufficient* number of rounds to provide security against an attack. Informally, ‘necessary’ means *at least* and ‘sufficient’ can be read as *at most*.

Definition 1. *Given an iterated cipher and a certain number of rounds $\mathcal{R} \geq 1$, we say \mathcal{R} rounds are necessary to prevent a certain attack if this attack can be set up for each number of rounds r with $r < \mathcal{R}$. Similarly, a certain number of rounds $\mathcal{R} \geq 1$ is sufficient to prevent an attack if the attack cannot be set up for all r with $r \geq \mathcal{R}$.*

While upper-bounding the algebraic degree (i.e., lower-bounding the number of rounds) is more important from an attacker’s point of view, lower-bounding

the algebraic degree (i.e., upper-bounding the number of rounds) is more relevant when arguing about the security from a designer’s viewpoint. However, at the current state of the art and to the best of our knowledge, it seems very hard to find such a lower bound for a given cipher without investigating concrete instances experimentally. An experimental approach, of course, limits the scope of any analysis. We emphasize that in general it is only possible to provide a *necessary* number of rounds for security against higher-order differential attacks.

2.1 Polynomial Representations over Binary Extension Fields

We denote addition (and subtraction) in binary extension fields and polynomial rings over binary extension fields by the symbol \oplus . For $n, t \in \mathbb{N}$, every function $F : (\mathbb{F}_{2^n})^t \rightarrow \mathbb{F}_{2^n}$ can be uniquely represented by a polynomial over \mathbb{F}_{2^n} in t variables with maximum degree $2^n - 1$ in each variable, i.e., as

$$F(X_1, \dots, X_t) = \bigoplus_{u=(u_1, \dots, u_t) \in \{0, 1, \dots, 2^n - 1\}^t} \varphi(u) \cdot X_1^{u_1} \cdot \dots \cdot X_t^{u_t}, \quad (2)$$

for certain $\varphi(u) \in \mathbb{F}_{2^n}$. We refer to this representation as the *word-level representation*. At the same time, the function F admits a unique representation as an n -tuple (F_1, \dots, F_n) of polynomials over \mathbb{F}_2 in $N := n \cdot t$ variables with maximum degree 1 in each variable. Here, F_i takes the form

$$F_i(X_1, \dots, X_N) = \bigoplus_{u=(u_1, \dots, u_N) \in \{0, 1\}^N} \varphi_i(u) \cdot X_1^{u_1} \cdot \dots \cdot X_N^{u_N}, \quad (3)$$

where the coefficients $\varphi_i(u) \in \mathbb{F}_2$ can be computed by the *Moebius transform* with time complexity $\mathcal{O}(n \cdot 2^n)$. We call this alternative description the *bit-level representation* of F . Combining Eq. (3) into a single polynomial representation leads to a description of F as a single polynomial in $n \cdot t$ variables, but now with coefficients in \mathbb{F}_2^n , instead of \mathbb{F}_2 .

Whenever we refer to the degree of a single variable in F (or F_i), we shall speak of the *univariate degree*. In contrast, the degree of F (or F_i) as a multivariate polynomial shall be called its *multivariate degree*, or just its *degree*. We denote functions $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ as *Boolean functions* and hence functions of the form $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$, for $n, m \in \mathbb{N}$, as *vectorial Boolean functions*. If not explicitly stated otherwise, we work with vectorial Boolean functions where $m = n$. The unique polynomial representation of a Boolean function is called its *algebraic normal form* (ANF), which we emphasize with the following definition.

Definition 2. *Let $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ be a Boolean function. The algebraic normal form (ANF) of F is the unique representation as a polynomial over \mathbb{F}_2 in n variables and with maximum univariate degree 1, i.e., the representation*

$$F(X_1, \dots, X_n) = \bigoplus_{u=(u_1, \dots, u_n) \in \{0, 1\}^n} \varphi(u) \cdot X_1^{u_1} \cdot \dots \cdot X_n^{u_n}.$$

The algebraic degree $\delta(F)$ of F is the degree of the above representation of F as a multivariate polynomial over \mathbb{F}_2 . When the function F is clear from the context, we also write δ instead of $\delta(F)$. If $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ is a vectorial Boolean function and (G_1, \dots, G_n) is its representation as an n -tuple of multivariate polynomials over \mathbb{F}_2 , then its algebraic degree $\delta(G)$ is defined as the maximal algebraic degree of its coordinate functions G_i , i.e. as $\delta(G) := \max_{1 \leq i \leq n} \delta(G_i)$.

The link between the algebraic degree and the univariate degree of a vectorial Boolean function is well-known, e.g. it is established in [CCZ98, Sect. 2.2]: due to the isomorphism of \mathbb{F}_2 -vector spaces $\mathbb{F}_{2^n} \cong \mathbb{F}_2^n$, every function over \mathbb{F}_{2^n} can be considered as a function over \mathbb{F}_2^n and thus admits a representation as an univariate polynomial over \mathbb{F}_{2^n} . Thus, the algebraic degree of a vectorial Boolean function can be computed from its univariate representation. Lemma 1 makes this link explicit.

Lemma 1. *Let $F : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ be a function over \mathbb{F}_{2^n} and let $F(X) = \sum_{i=0}^{2^n-1} \varphi_i \cdot X^i$ denote the corresponding univariate polynomial description over \mathbb{F}_{2^n} . The algebraic degree $\delta(F)$ of F as a vectorial Boolean function is the maximum over all Hamming weights⁵ of exponents of non-vanishing monomials, that is*

$$\delta(F) = \max_{0 \leq i \leq 2^n-1} \{\text{hw}(i) \mid \varphi_i \neq 0\}.$$

2.2 Higher-Order Differentials and SPN Schemes – State of the Art

The currently best-known generic upper bound for the algebraic degree of the composition of two functions is given by Boura, Canteaut, and De Cannière.

Proposition 1 ([BCD11]). *Let F be a function from \mathbb{F}_2^N to \mathbb{F}_2^N corresponding to the concatenation of t smaller balanced⁶ S-Boxes S_1, \dots, S_t defined over \mathbb{F}_2^n . Then, for any function G from \mathbb{F}_2^N to \mathbb{F}_2^N , it holds*

$$\deg(G \circ F) \leq \min \left\{ \deg(F) \cdot \deg(G), N - \frac{N - \deg(G)}{\gamma} \right\}, \quad (4)$$

where

$$\gamma := \max_{i=1, \dots, n-1} \frac{n-i}{n-\delta_i} \leq n-1, \quad (5)$$

and δ_i is defined as the maximal algebraic degree of the product of any i coordinates of any of the smaller S-Boxes.

We emphasize that γ and δ_i depend on the details of the S-Box. Namely, two S-Boxes with the same algebraic degree have in general different γ . In Proposition 2, we present a *direct* upper bound of the algebraic degree of SPN schemes in the simple but most common case where all S-Boxes are equal. With ‘direct’

⁵ Given $x = \sum_{i=0}^s x_i \cdot 2^i \in \mathbb{Z}$, for $x_i \in \{0, 1\}$, then $\text{hw}(x) = \sum_{i=0}^s x_i$.

⁶ A function $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ is said to be *balanced* if each element in \mathbb{F}_2^m has exactly 2^{n-m} preimages.

upper bound we mean that we iteratively apply (4) to the round functions of an SPN scheme and thus obtain a statement about the algebraic degree after a certain number of rounds.

Proposition 2. *Let F be a function from \mathbb{F}_2^N to \mathbb{F}_2^N corresponding to the concatenation of t copies of a balanced S -Box S over \mathbb{F}_{2^n} with algebraic degree $\delta \geq 2$. For any affine functions L_1, L_2, \dots, L_r from \mathbb{F}_2^N to \mathbb{F}_2^N and any integer $r \geq 1$ consider the SPN scheme E_r from \mathbb{F}_2^N to \mathbb{F}_2^N defined as*

$$E_r := L_r \circ F \circ L_{r-1} \circ F \circ \dots \circ L_2 \circ F \circ L_1 \circ F.$$

Then the algebraic degree $\delta(r)$ of E after r rounds is upper-bounded by

$$\delta(r) \leq \begin{cases} \delta^r & \text{if } r \leq R_0 = \left\lceil \log_\delta \left(N \cdot \frac{\gamma-1}{\gamma \cdot \delta - 1} \right) \right\rceil, \\ \frac{\delta^{R_0}}{\gamma^{R_0}} + N \cdot \frac{\gamma^{r-1}}{\gamma^r} & \text{if } R_0 < r \leq \mathcal{R}_{\text{[BCD11]}}, \end{cases} \quad (6)$$

where $\mathcal{R}_{\text{[BCD11]}}$ is the number of rounds necessary to prevent secret-key higher-order differential distinguishers defined by

$$\mathcal{R}_{\text{[BCD11]}} := \underbrace{\left\lceil \log_\delta \left(N \cdot \frac{\gamma-1}{\gamma \cdot \delta - 1} \right) \right\rceil}_{=R_0} + \lceil \log_\gamma (N - \delta^{R_0}) \rceil \quad (7)$$

and where γ is defined as in Eq. (5).

Proof. According to [BCD11] (see Proposition 1) the algebraic degree of E_1 is upper bounded by

$$\deg(L_1 \circ F) \leq \min \left\{ \delta, N \cdot \left(1 - \frac{1}{\gamma} \right) + \frac{1}{\gamma} \right\} = \delta,$$

since $t, \gamma \geq 2$ and therefore

$$N \cdot \left(1 - \frac{1}{\gamma} \right) + \frac{1}{\gamma} \geq \frac{N}{2} = \frac{n \cdot t}{2} > n - 1 \geq \delta.$$

In other words, for at least one round the naive bound (1) for the growth of the algebraic degree is better than the bound in [BCD11]. Therefore, we now look for the maximum number of rounds R_0 with this behaviour. This corresponds to solving the following equation for R_0

$$\delta^{R_0} = N \cdot \left(1 - \frac{1}{\gamma} \right) + \frac{\delta^{R_0-1}}{\gamma}.$$

A straightforward computation shows

$$R_0 = \log_\delta \left(N \cdot \frac{\gamma-1}{\gamma \cdot \delta - 1} \right) + 1.$$

To put it another way, for any number of rounds $r \leq R_0$, the degree of E_r is upper-bounded by δ^r . As a next step, we find the minimum additional number of rounds to prevent higher-order differential attacks, i.e., the minimum additional number of rounds R_1 such that the algebraic degree after $R_0 + R_1$ rounds is $N - 2$ (the biggest non-trivial subspace of \mathbb{F}_2^N has dimension $N - 1$).

For $r > R_0$, the bound in [BCD11] is better than the naive bound, hence, the algebraic degree of E_r after $r = R_0 + 1$ rounds is upper-bounded by

$$\deg(E_{R_0+1}) \leq \underbrace{N \cdot \left(1 - \frac{1}{\gamma}\right)}_{=:C} + \frac{\delta^{R_0}}{\gamma} = C + \frac{\delta^{R_0}}{\gamma},$$

and after $r = R_0 + 2$ rounds by

$$\deg(E_{R_0+2}) \leq C + \frac{1}{\gamma} \cdot \left(C + \frac{\delta^{R_0}}{\gamma}\right) = C + \frac{C}{\gamma} + \frac{\delta^{R_0}}{\gamma^2}.$$

Continuing this way, we conclude that after $r = R_0 + s$ rounds, for an integer $s \geq 1$, the algebraic degree is upper bounded by

$$\deg(E_{R_0+s}) \leq \frac{\delta^{R_0}}{\gamma^s} + C \cdot \sum_{i=0}^{s-1} \frac{1}{\gamma^i} = \frac{\delta^{R_0}}{\gamma^s} + \frac{1 - \frac{1}{\gamma^s}}{1 - \frac{1}{\gamma}} = \frac{\delta^{R_0}}{\gamma^s} + N \cdot \frac{\gamma^s - 1}{\gamma^s}.$$

This means, the minimum additional number of rounds R_1 to prevent higher-order differential distinguishers is given by

$$\frac{N \cdot (\gamma^{R_1} - 1)}{\gamma^{R_1}} + \frac{\delta^{R_0}}{\gamma^{R_1}} = N - 1,$$

or equivalently, by

$$R_1 = \log_{\gamma} (N - \delta^{R_0}).$$

We conclude, the number of rounds $\mathcal{R}_{\text{[BCD11]}}$ necessary to prevent higher-order differential distinguishers is given by

$$\mathcal{R}_{\text{[BCD11]}} = \left\lceil \log_{\delta} \left(N \cdot \frac{\gamma - 1}{\gamma \cdot \delta - 1} \right) \right\rceil + \lceil \log_{\gamma} (N - \delta^{R_0}) \rceil.$$

Cost of Computing γ . Before going on, we emphasize that the growth of the degree predicted in (4) depends on the value of γ . Computing γ can be very expensive for large S-boxes. Indeed, one has to consider all possible combinations of the product of any i coordinates of the given S-boxes, which implies a cost of order

$$\mathcal{O} \left(\sum_{i=1}^n \binom{n}{i} \right) \approx \mathcal{O}(2^n).$$

This means that for large S-Boxes (e.g., $n \geq 64$) it is infeasible to compute the real value of γ , and an approximation is needed. Our results in Section 4 do not have this limitation. They depend on known parameters of the cipher and can be computed straight away.

3 Higher-Order Differential Analysis of Iterated Even–Mansour Schemes

As a main result, in [EGL⁺20] the authors show that the algebraic degree of iterated Even–Mansour ciphers may grow much slower than what is commonly suggested in the literature. More precisely, they show that in some cases the algebraic degree grows linearly with the number of rounds and not exponentially. In this section⁷, we briefly recall the results published in [EGL⁺20].

Let’s start by considering an *iterated Even–Mansour cipher* $EM_k^r : \mathbb{F}_{2^N} \rightarrow \mathbb{F}_{2^N}$ defined as

$$EM_k^r(x) := k_r \oplus (\cdots R(k_1 \oplus R(k_0 \oplus x)) \cdots) \quad (8)$$

for $r \geq 1$ rounds, where $k_0, \dots, k_r \in \mathbb{F}_{2^N}$ are derived from a master key $k \in \mathbb{F}_{2^N}$ using a certain key schedule, and where each round function $R : \mathbb{F}_{2^N} \rightarrow \mathbb{F}_{2^N}$ is simply defined as some invertible polynomial function

$$R(x) := \rho_0 \oplus \bigoplus_{i=1}^d \rho_i \cdot x^i \quad (9)$$

of degree $d \geq 3$ and with $\rho_i \in \mathbb{F}_{2^N}$, $\rho_d \neq 0$. A cipher in the literature that falls into this category is e.g. MiMC. In [EGL⁺20], the authors derive a necessary condition on the number of rounds to prevent higher-order differential attacks against iterated Even–Mansour ciphers, see Proposition 3. In the remarks following Proposition 3 we discuss the scope of this result in more detail.

Proposition 3 ([EGL⁺20]). *Let R be the round function of an iterated Even–Mansour cipher EM_k^r with degree d defined as in Eq. (9). The number of rounds necessary to prevent secret-key higher-order differential distinguishers is given by*

$$\mathcal{R}_{EM} := \lceil \log_d(2^{N-1} - 1) \rceil \approx (N - 1) \cdot \log_d(2). \quad (10)$$

The idea of the proof is simple: to prevent higher-order differential attacks, the algebraic degree of EM_k^r must reach its maximum value $N - 1$. Due to the relation between the word-level degree and the algebraic degree (see Section 2.1), EM_k^r has algebraic degree $N - 1$ if at least one monomial with exponent $2^N - 2^j - 1$ (for $0 \leq j < N$) is present in the univariate polynomial representation. Since the smallest exponent of this form is $2^{N-1} - 1$, the number of rounds r must satisfy $r \geq \lceil \log_d(2^{N-1} - 1) \rceil$.

Forward versus Backward Direction. As originally proved in Corollary 3 of [BC13], given a fixed key k , the algebraic degrees of EM_k^r and its compositional inverse EM_k^{-r} are related in a particular way: the algebraic degree of

⁷ Since the authors in [EGL⁺20] focus on Even–Mansour ciphers, in this section we work specifically with ciphers. However, all these results apply as well to the case of permutations.

EM_k^r is maximal (i.e. $N - 1$) if and only if the algebraic degree of EM_k^{-r} is maximal. As an immediate consequence we state the following observation for iterated Even–Mansour ciphers:

“The number of rounds to reach maximal algebraic degree in encryption and decryption direction is the same.”

This fact is particularly surprising if one direction of an iterated Even–Mansour cipher has a low-degree round function, while the inverse direction is built from a round function of high degree. For example, for the round function $R(x) = x^3$, the inverse round function is given by $R^{-1}(x) = x^{(2^{N+1}-1)/3}$. Here, R has algebraic degree 2, while R^{-1} has algebraic degree $(n + 1)/2$.

Only a Necessary Condition. We stress once more, the condition on the number of rounds in Proposition 3 is only a *necessary* condition, not a sufficient one. This comes from the (complicated) cancellation behaviour of powers of polynomials over finite fields: even if the univariate degree satisfies $d^r \geq 2^{N-1} - 1$ after a certain number of rounds r , it is not guaranteed that monomials of algebraic degree $N - 1$ will be present in the encryption polynomial. A deeper analysis and concrete example of this behaviour is given in Section 5.2.

Full versus Partial Zero-Sums. Another point to keep in mind is that the result of Proposition 3 only focuses on security against *full* zero-sums, i.e. zero-sums over all outputs bits. We mention that this does not provide security against *partial* zero sums, i.e. zero-sums only in some particular output bits. For example, consider the extreme case of a function over \mathbb{F}_2^N with ANF $y_0 = x_0 + \prod_{i=1}^{n-1} x_i$ and $y_i = x_i$ for $1 \leq i \leq N - 1$. Even if a zero-sum cannot be set up for output bit y_0 , it is straightforward to set up a zero-sum over the remaining $N - 1$ bits, since the degree of y_i , for $1 \leq i \leq N - 1$, is just 1.

4 Higher-Order Differential Analysis of SPN Schemes

In this section we prove a new upper bound on the growth of the algebraic degree in SPN schemes with low-degree polynomial S-Boxes. Focusing on the ciphers case (analogous for permutations, by assuming fixed public known constants instead of fixed secret keys⁸), in the following let $E_k^r : (\mathbb{F}_{2^n})^t \rightarrow (\mathbb{F}_{2^n})^t$ denote the application of r rounds of an SPN cipher under a fixed but unknown secret key $k \in (\mathbb{F}_{2^n})^t$ with $n \geq 3$, $t \geq 2$, and $N := n \cdot t$. For every $x = (x_1, \dots, x_t) \in (\mathbb{F}_{2^n})^t$ we write

$$E_k^r(x) := (F_r \circ \dots \circ F_1)(x \oplus k_0), \quad (11)$$

where $F_i : (\mathbb{F}_{2^n})^t \rightarrow (\mathbb{F}_{2^n})^t$ is defined as $F_i(x) := R(x) \oplus k_i$, for $1 \leq i \leq r$. The subkeys $k_0, \dots, k_r \in (\mathbb{F}_{2^n})^t$ may be derived from the master key $k \in (\mathbb{F}_{2^n})^t$ by means of a key schedule, or they may just as well be randomly chosen elements.

⁸ Note that our results are independent of the value of the (secret or public) keys.

Here, R denotes the composition of the S-Box and the linear layer, i.e., we have $R : (\mathbb{F}_{2^n})^t \rightarrow (\mathbb{F}_{2^n})^t$ with

$$R(x) := (M \circ S)(x) := M(S_1(x_1), \dots, S_t(x_t)),$$

where all $S_i : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ are assumed to be non-linear polynomial S-Boxes of degree $d \geq 3$ defined as

$$S_i(x) := \bigoplus_{j=0}^d \sigma_j^{(i)} \cdot x^j, \quad (12)$$

for $\sigma_j^{(i)} \in \mathbb{F}_{2^n}$ and $\sigma_d^{(i)} \neq 0$. Finally, M denotes an invertible *non-trivial* linear layer $M : (\mathbb{F}_{2^n})^t \rightarrow (\mathbb{F}_{2^n})^t$ defined by the multiplication with a matrix

$$M(x) := \begin{pmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,t} \\ M_{2,1} & M_{2,2} & \dots & M_{2,t} \\ \vdots & & \ddots & \vdots \\ M_{t,1} & M_{t,2} & \dots & M_{t,t} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{pmatrix}, \quad (13)$$

where $M_{i,j} \in \mathbb{F}_{2^n}$ for $i, j = 1, \dots, t$. The notion of a non-trivial linear layer is made precise in the following definition.

Definition 3. *A linear layer M is non-trivial if it ensures full diffusion (in the sense that each word of the output depends on each word of the input and vice versa) after a finite number of rounds.*

We remark that all SPN ciphers over $(\mathbb{F}_{2^n})^t$ can be written as described above. Just to give some examples, if the linear layer is defined by an MDS matrix⁹, the cipher is similar to SHARK [RDP⁺96]. For AES [DR02] or AES-like ciphers, where the linear layer is obtained by a combination of the ShiftRows and the MixColumns operations, many elements of the matrix M are equal to 0 (see e.g. [BB02]).

4.1 New Bound on the Necessary Number of Rounds to Prevent Higher-Order Differential Attacks

The main result in this section is the following.

Theorem 1. *Let $n \geq 3$ and $t \geq 2$. Consider r rounds of an SPN scheme E^r over $(\mathbb{F}_{2^n})^t$ as defined in Eq. (11), with the additional assumption that all S-Boxes S_1, \dots, S_t are defined via the same function S of univariate degree $d \geq 3$ with algebraic degree δ . Then the algebraic degree of E^r after r rounds, denoted by $\delta(r)$, is upper-bounded by*

$$\delta(r) \leq \begin{cases} \delta^r & \text{if } r \leq R_{exp} := 1 + \lceil \log_\delta(t) \rceil, \\ t \cdot \delta + \lceil t \cdot \log_2(d^{r-R_{exp}}) \rceil & \text{if } R_{exp} < r \leq \mathcal{R}_{SPN}, \end{cases} \quad (14)$$

⁹ A matrix $M \in \mathbb{F}^{t \times t}$ is called a maximum distance separable (MDS) matrix iff every $u \times u$ submatrix of M is invertible, where $u \leq t$.

where

$$\mathcal{R}_{\text{SPN}} := \lceil n \cdot \log_d(2) \rceil + \lceil \log_\delta(t) \rceil \quad (15)$$

is the number of rounds necessary to prevent higher-order differential distinguishers.

We emphasize that we are speaking of a *necessary* number of rounds to prevent higher-order differential distinguishers. While we do *not* claim the above number of rounds to be *sufficient* for E^r to have maximum algebraic degree, the finesse of our new bound is that it is considerably closer to a sufficient bound than the currently best known results in the literature.

Idea of the Proof Before we prove Theorem 1, we give a brief overview of the proof itself and explain our terminology and underlying assumptions.

Idea of the Proof. The roadmap for the proof of Theorem 1 reads as follows:

1. Lemma 2 makes a statement about which monomials can occur in the polynomial representation of the encryption function.
2. In Lemma 3 we tightly upper-bound the number of rounds with exponentially growing algebraic degree.
3. Finally, we use this observation to prove Proposition 4, which is a slightly reformulated equivalent to Theorem 1.

General Assumptions. For simplicity, and since this is the most common case in the literature, we assume that all S-Boxes S_1, \dots, S_t are defined via the same *monomial* function $S : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ with $S(x) := S_1(x) = \dots = S_t(x) = x^d$. We remark that our results can be extended to the case where the S-Boxes are defined via more general *polynomial* mappings as described in Eq. (12).

According to the remark about the connection of forward and backward direction in Section 3, it suffices to focus only on one direction of the cipher when attempting to reach the maximal algebraic degree. We focus on the forward direction.

Furthermore, our analysis is independent of the concrete instantiation of the linear layer, besides the fact that we assume the matrix M to be non-trivial (see Definition 3). However, depending on the instantiation of the linear layer the algebraic degree might grow slower than we predict, but never faster because a linear function does not increase the algebraic degree. Therefore our analysis focuses on the polynomial representation of a single output word E_j for a certain $1 \leq j \leq t$. Hence, when we refer to *the* encryption polynomial, it is one of the polynomials E_j^r .

Details of the Proof

Terminology. We recall part of our terminology relevant for this section. For an unknown but fixed secret key $k = (k_1, \dots, k_t) \in (\mathbb{F}_{2^n})^t$ let $E_k^r = (E_{k,1}^r, \dots, E_{k,t}^r)$ denote the representation of an SPN cipher $E_k^r : (\mathbb{F}_{2^n})^t \rightarrow (\mathbb{F}_{2^n})^t$ after r rounds as a t -tuple of polynomials over \mathbb{F}_{2^n} with maximum univariate degree $2^n - 1$. The variables X_1, \dots, X_t shall represent n -bit words. Furthermore, let $m_\alpha := X_1^{\alpha_0} \dots X_t^{\alpha_t}$ denote the monomial with exponent vector $\alpha = (\alpha_1, \dots, \alpha_t) \in \{0, \dots, 2^n - 1\}^t$. The word-level degree of the S-Box $S(X) = X^d$ is denominated by d , while the algebraic degree of S is denoted by $\delta := \text{hw}(d)$. The base-2 expansion of d is written as $d = \sum_{i=1}^{\delta} 2^{d_i}$, for appropriate $d_i \in \mathbb{N}$. When we speak of monomials to be *expected*¹⁰ in the encryption polynomial we allude to the fact that the actual number of non-zero coefficients in the polynomial representation of E_k^r also depends on the secret key k , and as a result, the coefficient of a monomial may be zero under a specific key. We presume to be sloppy on this point and occasionally just speak of monomials that appear.

Lemma 2. *Let $r \in \mathbb{N}$ with $r \geq 1$ and $d = \sum_{i=1}^{\delta} 2^{d_i}$ be the base-2 expansion of d . Let $m_{\alpha_1}, m_{\alpha_2}, \dots, m_{\alpha_\delta}$ be monomials in the encryption polynomial after r rounds. Only those monomials given by*

$$m_{\alpha_1}^{2^{d_1}} \cdot m_{\alpha_2}^{2^{d_2}} \cdot \dots \cdot m_{\alpha_\delta}^{2^{d_\delta}}$$

are to be expected in the encryption polynomial after $r + 1$ rounds.

Proof. For each $j = 1, \dots, t$, the expression

$$E_j^{r+1} = M_{j,1} \cdot (E_1^r \oplus k_1)^d \oplus M_{j,2} \cdot (E_2^r \oplus k_2)^d \oplus \dots \oplus M_{j,t} \cdot (E_t^r \oplus k_t)^d$$

describes the encryption polynomial of word j after $r + 1$ rounds. If we write

$$E_j^r \oplus k_j = \bigoplus_{\alpha \in (0,1,\dots,2^n-1)^t} \varphi_\alpha \cdot m_\alpha,$$

and use the fact that $\text{char}(\mathbb{F}_{2^n}) = 2$, we obtain

$$\begin{aligned} (E_j^r \oplus k_j)^d &= \left(\bigoplus_{\alpha \in (0,1,\dots,2^n-1)^t} \varphi_\alpha \cdot m_\alpha \right)^{2^{d_1} + \dots + 2^{d_\delta}} \\ &= \prod_{i=1}^{\delta} \left(\bigoplus_{\alpha \in (0,1,\dots,2^n-1)^t} \varphi_\alpha^{2^{d_i}} \cdot m_\alpha^{2^{d_i}} \right) = \bigoplus_{\alpha_1, \dots, \alpha_\delta \in (0,1,\dots,2^n-1)^t} \left(\prod_{i=1}^{\delta} \varphi_{\alpha_i}^{2^{d_i}} \cdot m_{\alpha_i}^{2^{d_i}} \right). \end{aligned}$$

Hence, we conclude that only monomial products of the form

$$m_{\alpha_1}^{2^{d_1}} \cdot m_{\alpha_2}^{2^{d_2}} \cdot \dots \cdot m_{\alpha_\delta}^{2^{d_\delta}} \tag{16}$$

are expected to occur in the encryption polynomial after $r + 1$ rounds. The monomials $m_{\alpha_1}, \dots, m_{\alpha_\delta}$ are not necessarily different, therefore the exponents in Eq. (16) are either powers of 2 or sums of powers of 2.

¹⁰ This notion of expectation has nothing to do with statistical expectation.

Lemma 3. *Let $d = \sum_{i=1}^{\delta} 2^{d_i}$ for appropriate $d_i \in \mathbb{N}$. Only for the first $1 + \lceil \log_{\delta}(t) \rceil$ rounds the algebraic degree of the encryption polynomial grows as fast as δ^r .*

Proof. The idea of the proof is to observe the growth of the algebraic degree with the help of Lemma 2. After the first round, all monomials of the form X_1^d, \dots, X_t^d appear in the encryption polynomial. The algebraic degree of each of these monomials is δ . According to Lemma 2, after one more round the monomial

$$(X_1^d)^{2^{d_1}} \cdot (X_2^d)^{2^{d_2}} \cdot \dots \cdot (X_{\delta}^d)^{2^{d_{\delta}}}$$

appears in the encryption polynomial, and it has algebraic degree δ^2 . To see why it has algebraic degree δ^2 , we note that: (a) raising a (word-level) monomial m to the power of 2^k , $k \in \mathbb{N}$, does not change its algebraic degree, and (b) if two (word-level) monomials m_1, m_2 do not contain any shared variable, the algebraic degree of the product $m_1 \cdot m_2$ is the sum of the algebraic degrees. In the same way as before, after another round, the monomial

$$\underbrace{(X_1^{d 2^{d_1}} \dots X_{\delta}^{d 2^{d_{\delta}}})^{2^{d_1}}}_{\text{algebr. degree } \delta^2} \underbrace{(X_{\delta+1}^{d 2^{d_1}} \dots X_{2\delta}^{d 2^{d_{\delta}}})^{2^{d_2}}}_{\text{algebr. degree } \delta^2} \dots \underbrace{(X_{(\delta-1)\delta+1}^{d 2^{d_1}} \dots X_{\delta\delta}^{d 2^{d_{\delta}}})^{2^{d_{\delta}}}}_{\text{algebr. degree } \delta^2}$$

has algebraic degree δ^3 and, again, appears in the encryption polynomial. Continuing this way, we conclude that the algebraic degree grows as fast as δ^r until all t variables are exhausted, i.e., until $\delta^r = \delta \cdot t$.

It remains to show that $\lceil \log_{\delta}(\delta \cdot t) \rceil = 1 + \lceil \log_{\delta}(t) \rceil$ is the maximum number of rounds with exponential growth. To see this, we argue with the connection between the word-level degree and the bit-level degree (which is the algebraic degree). After $1 + \lceil \log_{\delta}(t) \rceil$ rounds, we have a monomial that uses all t variables X_1, \dots, X_t . Therefore, the algebraic degree cannot be increased any further by multiplying bit variables that belong to different words, but only by multiplying bit variables that belong to the same word. As we have discussed in Section 3, in this case the algebraic degree in each variable X_i grows at most as fast as $\log_2(d) \cdot r$, and hence in total at most by $t \cdot \log_2(d) \cdot r$.

Proposition 4. *Only after at least*

$$\lceil n \cdot \log_d(2) \rceil + \lceil \log_{\delta}(t) \rceil$$

rounds is the encryption polynomial expected to have maximum algebraic degree $n \cdot t - 1$.

Proof. As shown in Lemma 3, only for the first $1 + \lceil \log_{\delta}(t) \rceil$ rounds the algebraic degree grows exponentially with the number of rounds and, eventually, adds up to $t \cdot \delta$. The idea of the proof is to give the minimum additional number of rounds until the maximum algebraic degree $n \cdot t - 1$ is reached.

As discussed at the end of the proof of Lemma 3, from now on it suffices to observe the algebraic degree in a single variable X_i . Hence, to reach algebraic

degree n in a single variable and maximum overall algebraic degree $n \cdot t - 1$ respectively, it takes at least

$$\lceil n \cdot \log_d(2) \rceil - 1$$

more rounds, because the algebraic degree in each variable is already δ and we need degree n in $t-1$ variables and degree $n-1$ in one variable to reach maximum overall degree.

Remark. Our result in Theorem 1 and Proposition 4 respectively, highlights the different phases of the growth of the algebraic degree, namely a phase of exponential growth and a phase of linear growth. Once more, we emphasize the new insight that for SPN ciphers with low-degree polynomial S-Boxes the phase of linear growth may dominate the overall degree growth. For example, if $n \cdot \log_d(2)$ is large compared to $\log_\delta(t)$, namely in the case of *large* S-Boxes (i.e., $n \geq t$) described by *low-degree* polynomials (i.e., $d \ll n$) after some initial rounds for which the growth is exponential, the growth of the degree predicted by our formula is mostly linear in r .

4.2 Comparison with Related Work

For a better insight when the bound \mathcal{R}_{SPN} improves upon the one given by $\mathcal{R}_{[\text{BCD11}]}$ we ask the following question: *For which values of n, t, d and δ is*

$$\mathcal{R}_{\text{SPN}} \geq \mathcal{R}_{[\text{BCD11}]}$$

satisfied? Substituting the corresponding expression we obtain the following inequality

$$\lceil \log_d(2^n - 1) \rceil + \lceil \log_\delta(t) \rceil \geq \left\lceil \log_\delta \left(N \cdot \frac{\gamma - 1}{\gamma \cdot \delta - 1} \right) \right\rceil + \left\lceil \log_\gamma \left(N \cdot \frac{\gamma \cdot (\delta - 1)}{\gamma \cdot \delta - 1} \right) \right\rceil.$$

Using the relations $\gamma \cdot \delta - 1 \geq \gamma - 1$ and $\gamma \cdot \delta - 1 \geq \delta - 1$ (note that $\gamma, \delta \geq 2$), an upper bound for $\mathcal{R}_{[\text{BCD11}]}$ is given by

$$\mathcal{R}_{[\text{BCD11}]} \leq 1 + \lceil \log_\delta(N) \rceil + \lceil \log_\gamma(N) \rceil \leq 1 + \lceil \log_\delta(N) \rceil + \lceil \log_2(N) \rceil.$$

Thus, the condition $\mathcal{R}_{\text{SPN}} \geq \mathcal{R}_{[\text{BCD11}]}$ is satisfied if

$$\mathcal{R}_{\text{SPN}} = \lceil \log_d(2^n - 1) \rceil + \lceil \log_\delta(t) \rceil \geq 1 + \log_\delta(n \cdot t) + \log_2(n \cdot t),$$

or to put it another way, if

$$\log_d(2^n - 1) \approx \underbrace{n \cdot \log_d(2)}_{\in \mathcal{O}(n)} \geq \underbrace{\log_2(n) \cdot \left(1 + \frac{1}{\log_2(\delta)} \right)}_{\in \mathcal{O}(\log_2(n))} + \log_2(t) + 1.$$

It is easy to see that for any fixed values of d, δ , and t , the previous inequality can be satisfied if n is large enough. For a concrete example we refer to Fig. 1. There we compare the *upper* bound predicted by our formula and the one proposed in [BCD11] in the case of a SHARK-like SPN scheme with the cube S-Box $S(x) = x^3$ over $(\mathbb{F}_{2^n})^t$ for $n = 19$ and $t = 27$.

5 Practical Results

In this section, we present our practical results on SPN ciphers over $(\mathbb{F}_{2^n})^t$ (defined as in Section 4) with low-degree S-Boxes. The practical tests have been performed in the following way: instead of computing the ANF of a keyed permutation (which is quite expensive already for small field sizes), we evaluate the zero-sum property for a specific input vector space. Namely, for random keys and constants, given an input subspace of dimension $N - 1$, where $N = n \cdot t$, we look for the minimum number of rounds r for which the corresponding sum of the ciphertexts is different from zero. Such a number corresponds to the number of rounds necessary to prevent higher-order differential distinguishers. To avoid a bias by weak keys or “bad” round constants, we have repeated the tests multiple times (with new random keys and round constants). The code we used for the practical tests can be found on GitHub:

<https://github.com/IAIK/higher-order-differential>.

The practical number of rounds we report is *the smallest number of rounds among all tested keys and round constants* to prevent higher-order differential distinguishers. This means that potentially a higher number of rounds can be attacked by choosing the keys and round constants in a particular way.

Practical Results on SPN Schemes with $S(x) = x^3$. For our practical results, we focus on a SHARK-like cipher [RDP⁺96], namely a SPN cipher over $(\mathbb{F}_{2^n})^t$, where the S-Box is $S(x) = x^3$ and the mixing layer is defined by the multiplication with an invertible $t \times t$ matrix. Our practical results are reported in Table 1. The theoretical values for $\mathcal{R}_{[\text{BCD11}]}$ are computed assuming $\gamma = (n + 1)/2$ (we refer to [EGL⁺20, Lemma 2] for a detailed argument supporting this point). We observe that the number of rounds that can be covered by a higher-order differential distinguisher is (almost) always equal to the one predicted by our formula (in some cases a little higher, but never smaller). Moreover, especially when the size of the S-Box is not too small, the round number \mathcal{R}_{SPN} predicted by our formula is significantly larger than $\mathcal{R}_{[\text{BCD11}]}$.

Test Methodology. In order to derive the results shown in Table 1, we wrote a custom script in C. We then searched for zero sums with various round numbers, using different random input subspaces of dimension $N - 1$, together with random round constants and random round keys for each test. The round numbers we found for full zero sums were then verified again using additional test runs.

5.1 Influence of the Linear Layer

In order to understand how the linear layer influences the number of rounds *necessary* to provide security against higher-order differential distinguishers, in our practical tests we consider two extreme cases:

Table 1. Theoretical and practical round numbers *necessary* to guarantee security against secret-key higher-order differential distinguishers on SPN schemes over $(\mathbb{F}_{2^n})^t$ for several values of n and $t \geq 2$ (where $N = n \cdot t$). The chosen S-Box is the cube function $S(x) = x^3$. To better understand the influence of the linear layer, we consider both the case of a matrix that provides full diffusion after one round (e.g., an MDS matrix, if possible) – denoted by “Practical \mathcal{R} (MDS)” – and the case of a matrix that provides the “worst” possible diffusion (e.g., a sparse matrix as in Eq. (17)) – denoted by “Practical \mathcal{R} (Sparse)”. $\mathcal{R}_{[\text{BCD11}]}$ are computed assuming $\gamma = (n + 1)/2$.

Parameters			Theoretical # of rounds		Practical # of rounds	
N	n	t	\mathcal{R}_{SPN}	$\mathcal{R}_{[\text{BCD11}]}$	MDS matrix	Sparse matrix
35	5	7	7	6	8	15
35	7	5	8	5	8	12
36	9	4	8	6	9	11
33	11	3	9	5	10	10
39	13	3	11	6	11	12
34	17	2	12	6	12	12
38	19	2	13	6	14	14
66	11	6	9	6	-	-
65	13	5	11	6	-	-
60	15	4	12	6	-	-
66	17	4	13	7	-	-
63	21	3	15	6	-	-
66	33	2	22	7	-	-
132	11	12	10	8	-	-
135	15	9	14	8	-	-
133	19	7	14	7	-	-
132	33	4	23	8	-	-
129	43	3	29	7	-	-
130	65	2	43	8	-	-

1. We evaluate the case in which the linear layer is defined as the multiplication with an MDS matrix (for parameters n and t that allow us to do so¹¹), which corresponds to the case of the “strongest” linear layer from a diffusion point of view.
2. We also evaluate the case in which the linear layer is “weak”, which could happen if it is defined by the multiplication with a matrix containing a large number of zero coefficients. For this second case, we used a $t \times t$ matrix M with coefficients $M_{r,c}$, for $r, c = 0, \dots, t - 1$, given by

$$M_{r,c} = \begin{cases} 1 & \text{if } r = 0 \text{ OR if } c \equiv r + 1 \pmod{t}, \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

We note that by using M we need t rounds to have full diffusion (at word level), instead of just one round as for the MDS case. Hence, especially for large

¹¹ We recall that a $t \times t$ MDS matrix with elements in $GF(2^n)$ exists if the condition $\log_2(2t + 1) \leq n$ is satisfied.

t , we expect that more rounds than previously predicted may be necessary to guarantee security against higher-order differential distinguishers. In Table 1 we report empirical evidence for this expectation: the gap between the number of rounds predicted by our formula and the one found by practical tests in the case of a sparse matrix is close to zero for “small” t , and grows for “large” t . An *open problem* for future research would be to take into account the details of the linear layer for a more accurate prediction of the number of rounds that can be covered by a higher-order differential distinguisher.

5.2 Practical Results for Specific S-Boxes

Besides the effect of the linear layer, we also tried to better understand the impact of the details of the S-Box. As already mentioned in [EGL⁺20, Sect. 3.1], whenever the output of the S-Box is sparse, or full, this fact will affect the gap between the practical number of rounds needed for security and the one predicted by our formula (which is just a lower bound since it is only a necessary condition). In particular, focusing on an S-Box defined by a power map $x \mapsto x^d$ there are two “extreme” cases, namely $d = 2^{d'} \pm 1$ for some $d' \in \mathbb{N}$. For these cases it is possible to observe that for $d = 2^{d'} + 1$ the output of a single round

$$(x \oplus y)^{2^{d'}+1} = x^{2^{d'}+1} \oplus x^{2^{d'}} \cdot y \oplus y^{2^{d'}} \cdot x \oplus y^{2^{d'}+1}$$

is sparse. We note that it contains only 4 terms instead of $d+1 = 2^{d'} + 2$. While for $d = 2^{d'} - 1$ the output of a single round

$$(x \oplus y)^{2^{d'}-1} = \bigoplus_{i=0}^{2^{d'}-1} x^i \cdot y^{2^{d'}-1-i}$$

contains much more monomials. In order to better understand this fact, we performed practical tests on a MiMC-like cipher with different S-Boxes $x \mapsto x^d$ of the form $d = 2^{d'} + 1$ and $d = 2^{d'} - 1$. We emphasize that the choice of working on MiMC-like ciphers and not on SPN ciphers has been made to prevent the influence of the linear layer. Our practical results are presented in Appendix A, see Table 3 and Table 4 for more details. As expected, we found that the polynomial that describes the encryption function is in general dense in the case $d = 2^{d'} - 1$. As a result, for $d = 2^{d'} - 1$ the sufficient number of rounds that provides security against secret-key higher-order differential distinguishers is close to \mathcal{R}_{SPN} , while the gap between these two figures is in general larger for the case $d = 2^{d'} + 1$. These observations match the theoretical expectation expressed before, and the results concerning “Variants of MiMC” presented in [AGR⁺16].

6 Observations on Partial SPN Schemes

The results on the growth of the algebraic degree in Even–Mansour and SPN schemes can be used to derive preliminary results for partial SPN schemes over

$(\mathbb{F}_{2^n})^t$ in the case in which the round function is described by a low-degree polynomial over \mathbb{F}_{2^n} .

6.1 Partial SPN Schemes

In partial SPN schemes over $(\mathbb{F}_{2^n})^t$, the round function is defined as

$$R = M \circ \underbrace{(S_1, \dots, S_s, I, \dots, I)}_{t \text{ words}}, \quad (18)$$

for $1 \leq s < t$, where the $S_i : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ denote some non-linear functions on \mathbb{F}_{2^n} and where I denotes the identity function on \mathbb{F}_{2^n} . The function $M : (\mathbb{F}_{2^n})^t \rightarrow (\mathbb{F}_{2^n})^t$ is defined as the multiplication of the state vector with a $t \times t$ -matrix over \mathbb{F}_{2^n} . The distinctive feature of a partial SPN schemes is that the non-linear functions S_i are only applied to part of the state, while the rest of the state remains unchanged. To set up higher-order differential distinguishers on partial SPN schemes, the attacker can employ several strategies. The two most extreme ones are:

1. use a proper initial subspace with “maximum” dimension $n \cdot t - 1$ bits;
2. use a proper initial subspace of dimension $n \cdot r$ bits for a certain $1 \leq r \leq t - 1$.
If the choice of such an initial subspace is made in a clever way (e.g., as described in the following), it is possible to “skip” at most r initial rounds (in the sense that the input of the S-Box in the first r rounds is always constant, hence the algebraic degree does not change in the first r rounds).

Depending on the details of the cipher, each of the previous strategies can be the best one, in the sense that it can cover the highest number of rounds with a higher-order differential distinguisher. For SPN schemes, the first strategy is in general the only possible, since it is not possible to “skip” rounds for free without activating any S-Box.

For simplicity, we limit ourselves to consider only the case in which $s = 1$, that is, the case in which one single S-Box is applied in each round (which e.g. corresponds to the instantiation of LowMC used in variants of the Picnic signature scheme [CDG⁺17, CDG⁺19] and to the partial rounds in the recently proposed HadesMiMC permutation [GLR⁺20, GKK⁺19]).

Proposition 5. *Let $E : (\mathbb{F}_{2^n})^t \rightarrow (\mathbb{F}_{2^n})^t$ be a partial SPN scheme in which the S-Box layer is composed of one S-Box and $t - 1$ identity functions. Let us assume that the S-Box is defined as some invertible (low degree) polynomial function over \mathbb{F}_{2^n} of the form $S(x) := \rho_0 + \sum_{i=1}^d \rho_i \cdot x^i$ of degree $d \geq 3$ and with $\rho_i \in \mathbb{F}_{2^n}$, $\rho_d \neq 0$. Then at least*

$$\mathcal{R}_{\text{PSPN}} := t - 1 + \lceil \log_d(2^n - 1) \rceil$$

rounds are necessary to prevent higher-order differential distinguishers. In particular, to prevent higher-order differential distinguishers on $(t - 1) \cdot n$ bits (namely, all bits except the n ones corresponding to the position of the S-Box), one more round is necessary.

Proof. Without loss of generality, we assume that the S-Box is applied to the first word¹². In such a case, it is possible to “skip” $t - 1$ rounds (namely, to impose that no S-Box is active in the first $t - 1$ rounds) if the initial input $x = (x_0, x_1, \dots, x_{t-1})$ satisfies the condition

$$\forall i = 0, \dots, t - 2 : \quad [M^i \cdot x]_0 = \text{constant}$$

where M^i is just the i -fold product of M , with M^0 being the identity matrix I , and where for $y = (y_0, y_1, \dots, y_{t-1})$ the expression $[y]_0 := y_0$ denotes the word at position 0. Hence the first $t - 1$ rounds do not increase the degree.

After at least t rounds (we remark that depending on the details of the linear layer it is potentially possible to skip more rounds), the S-Box is active. Since only one word is active, we can simply reuse the results for the Even–Mansour case: a necessary condition to guarantee security is that the algebraic degree is at least n , which happens only in the case where a monomial of the form $x^{2^n - 1}$ appears in the encryption polynomial. Since the degree of the monomial over \mathbb{F}_{2^n} grows as fast as $d^{r-(t-1)}$, it follows that the number of rounds r must satisfy $d^{r-(t-1)} \geq 2^n - 1$ to prevent higher-order differential distinguishers.

Remark. We note, that a *full* zero sum on N bits after the S-Box layer implies always (at least) a *partial* zero sum on $n \cdot (t - 1)$ bits after the next S-Box layer. Indeed, a full zero sum on N bits after the S-Box layer implies a full zero-sum on N bits after the mixing layer (since it is linear). The result follows from the fact that the non-linear layer is partial (in our case, just 1 S-Box and $t - 1$ identity functions are applied).

6.2 Related Work

We note that the result just presented is less surprising than the corresponding one given for SPN schemes. Indeed, the linear growth of the algebraic degree in case of partial SPN ciphers had already been observed in the literature when considering the security of LowMC [ARS⁺15] and Bison [CLL⁺19]. For example, we quote from Section 6.2 of [CLL⁺19]:

“[...] the degree of any NLFSR increases linearly with the number of rounds. To the best of our knowledge, this is the first time this has been observed in this generality. We like to add that this is in sharp contrast to how the degree increases for SPN ciphers. For SPN ciphers the degree usually increases exponentially until a certain threshold is reached.”

A formal analysis of the growth of the algebraic degree in a partial SPN cipher has been given e.g. for LowMC in [ARS⁺15].

¹² If this is not the case, it is always possible to find an equivalent representation – via a different but equivalent linear layer – for which this is the case.

Table 2. Theoretical and practical number of rounds necessary to prevent full higher-order differential distinguishers for partial SPN schemes with a single ($s = 1$) S-Box of the form $x \mapsto x^3$. The mixing layer is defined as the multiplication with a matrix that maximizes the branch number, e.g., an MDS matrix. We present practical results with two different initial subspaces, one of dimension $N - 1$, and one of dimension n (chosen to skip as many initial rounds as possible).

Parameters			Theoretical # of rounds	Practical # of rounds	
N	n	t	$\mathcal{R}_{\text{PSPN}}$	dimension n	dimension $N - 1$
35	5	7	10	10	19
35	7	5	9	9	15
36	9	4	10	9	13
33	11	3	9	9	12
39	13	3	11	11	14
34	17	2	12	12	14
38	19	2	12	13	15
65	5	13	16	16	-
65	13	5	13	13	-
63	7	9	13	13	-
63	9	7	12	12	-
68	17	4	14	14	-
133	7	19	23	23	-
133	19	7	18	18	-
135	9	15	20	20	-
135	15	9	18	18	-

Proposition 6 ([ARS⁺15]). *Let F be a function that corresponds to the parallel application of s balanced n -bit S-Boxes and an identity function of width $l = N - s \cdot n$. Thus, F is a mapping from $\mathbb{F}_2^{n \cdot s + l}$ to $\mathbb{F}_2^{n \cdot s + l}$. Let δ_k be the maximum algebraic degree of the product of any k output bits of the S-Box. Then for any function G from $\mathbb{F}_2^{n \cdot s + l}$ to \mathbb{F}_2^N , we have*

$$\deg(G \circ F) \leq \min\{\deg(G) \cdot \deg(F), \beta \cdot s + \deg(G)\},$$

where $\beta = \max_{1 \leq i \leq n} (\delta_i - i)$.

Focusing on the case where only one S-Box is applied in each round, in Appendix B we present a closed formula that describes the growth of the degree in partial SPN schemes (as previously done for the case of SPN schemes).

6.3 Practical Results

Our practical results for the cubing S-Box are shown in Table 2, and they are obtained using the strategy described before. We considered the two extreme cases, namely an initial subspace of maximum dimension $N - 1$ and an initial subspace of dimension n chosen in such a way to skip as many initial rounds as possible without increasing the degree. As can be seen, the number of rounds obtained in practice in this last case matches the theoretical ones.

On the other hand, the gap for the number of rounds obtained in the case of an initial subspace of maximum dimension $N - 1$ is in many cases non-trivial. This choice of the initial subspace seems to allow an attacker to break more rounds than by using an initial subspace of dimension n . As a result, in many cases it seems possible to break many more rounds than the ones predicted by \mathcal{R}_{SPN} . We conjecture that this result is due to the fact that several cancellations of monomials in the encryption polynomial occur. Thus, our understanding of the degree growth for P-SPN ciphers is far from complete and we leave this as an open problem. As future work, the goal would be to estimate the growth of the degree in the case in which $N - 1$ bits are active.

7 Summary and Open Problems

Summary. Our main result on the security of SPN schemes against higher-order differential attacks can be summarized as follows: The number of rounds necessary to prevent *full* secret-key higher-order differential distinguishers on SPN schemes over $(\mathbb{F}_{2^n})^t$ as defined in Eq. (11) is given by

$$\mathcal{R}_{\text{SPN}} = \lceil \log_d(2^n - 1) \rceil + \lceil \log_\delta(t) \rceil,$$

where all S-Boxes are defined via the same low-degree polynomial function with univariate degree d and algebraic degree δ . We refer to the remark about full versus partial zero sums in Section 3 for a clarification of our terminology in this regard. Most importantly, our result implies that the necessary number of rounds for security against higher-order differential distinguishers grows (for the most part) linearly with n and not logarithmically, as previously presented in the literature.

Open Problems. Below we list the main open problems we identified during our research:

1. One may take into account the details of the linear layer for improving the bounds on the growth of the degree presented in this paper.
2. HadesMiMC [GLR⁺20] is probably the most suitable candidate in order to apply our results. A natural question arises immediately: by combining the results for SPN ciphers and partial SPN ciphers presented here, how many rounds of this scheme is it possible to break by higher-order differential attacks? Moreover, is it possible to break the full scheme by exploiting the slow growth of the algebraic degree in the case of a weak linear layer [KR20,BCD⁺20] (namely, a linear layer M for which $M^2 = \mu I$ for a certain $\mu \in \mathbb{F}$ where I is the identity matrix)?
3. The analysis of the growth of the degree for partial SPN ciphers is far from being complete. Our current results are similar to the ones obtained for the \mathbb{F}_p -case in [BCD⁺20]. However, we expect that better results can be obtained for the Boolean cases, due to the larger number of subspaces that exist in $(\mathbb{F}_{2^n})^t$.

4. Finally, a next step would be to extend the higher-order differential distinguishers presented in this paper to e.g. key-recovery attacks, as was already done for MiMC in [EGL⁺20].

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Table 3. Theoretical and practical round numbers *necessary* to prevent higher-order differential distinguishers for MiMC-like schemes over \mathbb{F}_{2^N} , in the case of a round function of the form $R(x) = x^d$ for several values of $d = 2^{d'} + 1$.

Parameters		Theoretical # of rounds		Practical # of rounds
d	N	\mathcal{R}_{EM}	$\mathcal{R}_{[\text{BCD11}]}$	dimension $N - 1$
5	13	6	5	7
	17	8	5	8
	33	15	6	15
	65	28	7	-
	129	56	8	-
9	13	5	5	6
	17	6	5	7
	33	11	6	12
	65	21	7	-
	129	41	8	-
17	13	4	5	6
	17	5	5	7
	33	9	6	10
	65	16	7	-
	129	32	8	-
33	13	3	5	6
	17	4	5	7
	33	7	6	9
	65	13	7	-
	129	26	8	-
65	13	4	5	7
	17	3	5	7
	33	6	6	9
	65	11	7	-
	129	22	8	-

A Practical Tests on Iterated Even–Mansour Schemes

In this section, we present our practical results for Even–Mansour schemes instantiated with S-Boxes $x \mapsto x^d$ for which the exponent is of the form $d = 2^{d'} + 1$ and $d = 2^{d'} - 1$ (see Table 3 and Table 4 for more details). We recall that $x \mapsto x^d$ is a permutation in \mathbb{F}_{2^N} if and only if $\gcd(d, 2^N - 1) = 1$.

Estimation for $\mathcal{R}_{[\text{BCD11}]}$ and Bound for γ . In the case in which $d = 2^{d'} + 1$, the algebraic degree of the round function $x \mapsto x^d$ is always 2. This means that we can reuse the bound for γ given in [EGL⁺20, Sect. 3.1], namely $\gamma = (n+1)/2$. In the case in which $d = 2^{d'} - 1$, the algebraic degree of the round function $x \mapsto x^d$ is equal to d' . Hence, it is not hard to show that

$$\gamma \leq \max \left\{ \frac{N - \lfloor (N-1)/d' \rfloor}{N - \delta \cdot \lfloor (N-1)/d' \rfloor}, n - \left\lfloor \frac{N-1}{d'} \right\rfloor - 1 \right\}. \quad (19)$$

Table 4. Theoretical and practical round numbers *necessary* to prevent higher-order differential distinguishers for the case of MiMC-like schemes over \mathbb{F}_{2^N} , in the case of a round function of the form $R(x) = x^d$ for several values of $d = 2^{d'} - 1$.

Parameters		Theoretical # of rounds		Practical # of rounds
d	N	\mathcal{R}_{EM}	$\mathcal{R}_{[\text{BCD11}]}$	Practical \mathcal{R}
7	13	5	3	5
	17	7	3	7
	33	12	4	14
	65	24	4	-
	128	46	5	-
15	13	4	2	4
	17	5	3	5
	33	9	3	9
	65	17	4	-
	129	34	4	-
31	13	3	2	4
	17	4	2	5
	33	7	3	8
	64	13	3	-
	129	27	3	-
63	13	3	2	3
	17	3	2	6
	35	6	2	8
	65	11	3	-
	129	22	3	-

Practical Results. As we have already explained in Section 3, we expect that the gap between the theoretical and the practical results can be larger for $d = 2^{d'} + 1$ than for the case $d = 2^{d'} - 1$. Again, this is due to the fact that in the case $d = 2^{d'} + 1$, the encryption polynomial is in general relatively sparse. Our practical results confirm this expectation.

As a result, for $d = 2^{d'} - 1$ the real number of rounds that provides security against (secret-key) higher-order differential distinguishers is close to \mathcal{R}_{EM} previously given, while the gap between these two numbers is in general bigger for the case $d = 2^{d'} + 1$. This fits with the theoretical prediction made before (and with the results regarding “Variants of MiMC” presented in [AGR⁺16]). For completeness, we point out that \mathcal{R}_{EM} is similar for $d = 2^{d'} + 1$ and $d = 2^{d'} - 1$ when both are close to the maximum value of $2^N - 1$, since

$$\log_{2^{d'}+1}(2^N - 1) \approx \log_{2^{d'}-1}(2^N - 1) \approx \frac{N}{d'}.$$

However, we remark one more time that the gap between \mathcal{R}_{EM} and the (practical/real) number of rounds necessary to provide security is in general bigger for $d = 2^{d'} + 1$ than for $d = 2^{d'} - 1$.

B Closed Formula for Partial SPN Schemes

Focusing on the case where only one S-Box is applied in each round, it is possible to obtain a closed formula that describes the growth of the degree in partial SPN schemes. We compare our results on the round necessary number of rounds in to prevent higher-order differential distinguishers in partial SPN ciphers with the numbers derived in Proposition 7. For this comparison, we present a slightly extended variant of Table 2.

Proposition 7. *Let F be a function on $(\mathbb{F}_{2^n})^t$ corresponding to the concatenation of $s = 1$ balanced S-Box defined over \mathbb{F}_{2^n} and the identity function on $(\mathbb{F}_{2^n})^{t-1}$. Moreover, assume that the S-Box has algebraic degree $\delta \geq 2$, and let $\beta := \max_{1 \leq i \leq n}(\delta_i - i)$, where δ_i is the maximum algebraic degree of the product of any i output bits of the S-Box. For any affine functions L_1, L_2, \dots, L_r on $\mathbb{F}_{2^n}^t$, consider the permutation $E_r : (\mathbb{F}_{2^n})^t \rightarrow (\mathbb{F}_{2^n})^t$ defined as*

$$E_r = L_r \circ F \circ L_{r-1} \circ F \circ \dots \circ L_2 \circ F \circ L_1 \circ F$$

for a certain number of rounds $r \geq 1$. Then, the minimum number of rounds $\mathcal{R}_{[ARS+15]}$ necessary to prevent higher-order differential distinguishers on $N = n \cdot t$ bits is given by

$$\mathcal{R}_{[ARS+15]} := 1 + \left\lceil \log_\delta \left(\frac{\beta}{\delta - 1} \right) \right\rceil + \left\lfloor \frac{N}{\beta} - \frac{1}{\delta - 1} - \frac{\gamma}{\gamma - 1} \right\rfloor + \left\lceil \log_\gamma \left(\frac{\beta}{\gamma - 1} \right) \right\rceil, \quad (20)$$

where γ and β are defined as before. In particular, to prevent higher-order differential distinguishers on $(t - 1) \cdot n$ bits one more round is necessary.

Proof. 1st Part. We know that

$$\deg(G \circ F) \leq \min \left\{ \deg(G) \cdot \deg(F), \beta + \deg(G), \frac{N \cdot (\gamma - 1)}{\gamma} + \frac{\deg(G)}{\gamma} \right\},$$

where β is defined as before. First of all, note that we are not considering a generic function G , but a concrete encryption function E_r . Since the growth of the degree is independent of any affine/linear function, it follows that the degree of such an encryption function E_r is upper-bounded by

$$\deg(E_r) \leq \delta^R$$

To apply the bound from Eq. (4), the first step is to find the maximum number of rounds R_0 such that $\deg(G) \cdot \deg(F) \leq \beta + \deg(G)$. In our case, this corresponds to

$$\delta^{R_0+1} \leq \beta + \delta^{R_0},$$

which implies that

$$R_0 = \left\lfloor \log_\delta \left(\frac{\beta}{\delta - 1} \right) \right\rfloor.$$

Note that $\beta \geq \delta - 1$ by definition. As a result, $R_0 \geq 0$.

Next, we find the maximum number of rounds R_1 such that $\beta + \deg(G) \leq \frac{N \cdot (\gamma - 1)}{\gamma} + \frac{\deg(G)}{\gamma}$. This corresponds to

$$\beta \cdot (R_1 + 1) + \delta^{R_0} \leq \frac{N \cdot (\gamma - 1)}{\gamma} + \frac{\beta \cdot R_1 + \delta^{R_0}}{\gamma},$$

which implies

$$R_1 = \left\lfloor \frac{N}{\beta} - \frac{\delta^{R_0}}{\beta} - \frac{\gamma}{\gamma - 1} \right\rfloor.$$

Finally, we need to find the minimum number of rounds in order to prevent higher-order differential attacks. Remember that given a function $f(\cdot)$ of degree d , the sum over the outputs of the function applied to all elements of a vector space \mathcal{V} of dimension $\geq \delta + 1$ is zero. Since we only consider attacks that use less than the full code book, the biggest subspace of \mathbb{F}_2^N has dimension $N - 1$. As a result, a higher-order differential distinguisher can be set up for at most $\mathcal{R}^{[ARS+15]}$ rounds, where $\deg(E(\mathcal{R}^{[ARS+15]})) \leq N - 1$. For this last step, we can simply reuse the computation proposed in the previous proof. It follows that $\deg(E_r) \leq N - 1$ if

$$\frac{N \cdot (\gamma^{R_2} - 1)}{\gamma^{R_2}} + \frac{\beta \cdot R_1 + \delta^{R_0}}{\gamma^{R_2}} \leq N - 1,$$

which means that

$$R_2 = \lceil \log_\gamma (N - \beta \cdot R_1 - \delta^{R_0}) \rceil.$$

In conclusion, the minimum number of rounds $\mathcal{R}_{[ARS+15]}$ of $E(\mathcal{R}_{[ARS+15]})$ necessary to prevent (secret-key) higher-order differential distinguishers on N bits is given by

$$\begin{aligned} \mathcal{R}_{[ARS+15]} &\geq \underbrace{\left\lfloor \log_\delta \left(\frac{\beta}{\delta - 1} \right) \right\rfloor}_{\equiv R_0} + \underbrace{\left\lfloor \frac{N}{\beta} - \frac{\delta^{R_0}}{\beta} - \frac{\gamma}{\gamma - 1} \right\rfloor}_{\equiv R_1} + \lceil \log_\gamma (N - \beta \cdot R_1 - \delta^{R_0}) \rceil \\ &\approx 1 + \left\lfloor \log_\delta \left(\frac{\beta}{\delta - 1} \right) \right\rfloor + \left\lfloor \frac{N}{\beta} - \frac{1}{\delta - 1} - \frac{\gamma}{\gamma - 1} \right\rfloor + \left\lceil \log_\gamma \left(\frac{\beta}{\gamma - 1} \right) \right\rceil, \end{aligned}$$

where the last inequality is actually an equality if both $\log_\delta \left(\frac{\beta}{\delta - 1} \right)$ and $\frac{N}{\beta} - \frac{\delta^{R_0}}{\beta} - \frac{\gamma}{\gamma - 1}$ are integer numbers.

2nd Part. The previous formula is obtained by considering the case in which the attacker works with a subspace of size $N - 1$ bits. However, another strategy could allow the attacker to cover more rounds in general: Due to the partial S-Box layer, the attacker can cleverly choose the input subspace in order to not

active any S-Boxes in the first x rounds. In particular, by skipping x rounds, the subspace that the attacker can exploit has dimension at most $n \times (t - x)$. It follows that

$$\begin{aligned} \mathcal{R}_{[ARS+15]} &\geq \max_{0 \leq x \leq t-1} \left\{ x + \underbrace{\left\lceil \log_{\delta} \left(\frac{\beta}{\delta-1} \right) \right\rceil}_{\equiv R_0} + \underbrace{\left\lceil \frac{n \times (t-x)}{\beta} - \frac{\delta^{R_0}}{\beta} - \frac{\gamma}{\gamma-1} \right\rceil}_{\equiv R_1} + \right. \\ &\quad \left. + \left\lceil \log_{\gamma} \left(n \times (t-x) - \beta \cdot R_1 - \delta^{R_0} \right) \right\rceil \right\} \approx \\ &\approx \max_{0 \leq x \leq t-1} \left\{ x + 1 + \left\lceil \log_{\delta} \left(\frac{\beta}{\delta-1} \right) \right\rceil + \left\lceil \frac{n \times (t-x)}{\beta} - \frac{1}{\delta-1} - \frac{\gamma}{\gamma-1} \right\rceil + \left\lceil \log_{\gamma} \left(\frac{\beta}{\gamma-1} \right) \right\rceil \right\}. \end{aligned}$$

3rd Part: Conclusion. What is the best choice for x (equivalently, how many rounds x does it make sense to “skip”)? By simple computation, note that $\frac{d}{dx} \mathcal{R}_{[ARS+15]} \simeq 1 - \frac{n}{\beta}$. By definition of β , it follows that $1 \leq \beta = \max_{1 \leq i \leq n} (\delta_i - i) \leq n - 1$ since $-i \leq -1$ and since $\delta_i \leq n$ (the product of i coordinates of an S-Box over \mathbb{F}_2^n cannot be of degree bigger than n). As a result

$$\frac{d}{dx} \mathcal{R}_{[ARS+15]} \simeq 1 - \frac{n}{\beta} \leq 1 - \frac{n}{n-1} = -\frac{1}{n-1} < 0,$$

where remember that $n \geq 3$. This means that the maximum is taken for $x = 0$, that is, the best choice is $x = 0$.

Table 5. Theoretical and practical number of rounds necessary to prevent full higher-order differential distinguishers for P-SPN ciphers/permutations with a single ($s = 1$) S-Box of the form $x \mapsto x^3$. The mixing layer is defined as the multiplication with a matrix that maximizes the branch number, e.g., an MDS matrix. We present practical results with two different initial subspaces, one of dimension $N-1$, and one of dimension n (chosen to skip as many initial rounds as possible). The values for $\mathcal{R}_{[ARS+15]}$ are computed under the assumption $\beta = (n-1)/2$ and $\gamma = (n+1)/2$.

Parameters			Theoretical # of rounds		Practical # of rounds	
N	n	t	$\mathcal{R}_{\text{PSPN}}$	$\mathcal{R}_{[ARS+15]}$	dimension n	dimension $N-1$
35	5	7	10	18	10	19
35	7	5	9	12	9	15
36	9	4	10	10	9	13
33	11	3	9	8	9	12
39	13	3	11	8	11	14
34	17	2	12	7	12	14
38	19	2	12	7	13	15
65	5	13	16	33	16	-
65	13	5	13	12	13	-
63	7	9	13	22	13	-
63	9	7	12	17	12	-
68	17	4	14	11	14	-
133	7	19	23	45	23	-
133	19	7	18	17	18	-
135	9	15	20	35	20	-
135	15	9	18	21	18	-