Private Identity Agreement for Private Set Functionalities

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Abstract. Private set intersection and related functionalities are among the most prominent real-world applications of secure multiparty computation. While such protocols have attracted significant attention from the research community, other functionalities are often required to support a PSI application in practice. For example, in order for two parties to run a PSI over the unique users contained in their databases, they might first invoke on a support functionality to agree on the primary keys to represent their users.

This paper studies a secure approach to agreeing on primary keys. We introduce and realize a functionality that computes a common set of identifiers based on incomplete information held by two parties, which we refer to as *private identity agreement*. We explain the subtleties in designing such a functionality that arise from privacy requirements when intending to compose securely with PSI protocols. We also argue that the cost of invoking this functionality can be amortized over a large number of PSI sessions, and that for applications that require many repeated PSI executions, this represents an improvement over a PSI protocol that directly uses incomplete or fuzzy matches.

Keywords: private set intersection, private identity agreement, garbled circuits

1 Introduction

In recent years *Private Set Intersection* (PSI) and related two-party protocols have been deployed in real-world applications [IKN+17]. In the simplest setting of PSI, each party has a set X_i as its input, and the output will be the intersection $\bigcap X_i$. More generally the parties may wish to compute some function f over the intersection and obtain output $f(\bigcap X_i)$ [IKN+17,CO18,PSWW18,FNO18,PSTY19].

Owing to its importance in real-world applications, PSI has been the topic of a significant body of research. Common PSI paradigms include DDH-style protocols [HFH99,AES03,DCKT10,Lam16,SFF14], approaches based on oblivious transfer [PSZ14,PSSZ15,DCW13,RR16] or oblivious polynomial evaluation [FNP04,DSMRY09], and approaches based on garbled circuits [HEK12,PSSZ15,PSTY19]. Performance improvements have been dramatic, especially the computational overhead of PSI.

State-of-the-art PSI protocols require *exact* matches to compute the intersection; in other words, the intersection is based on bitwise equality. In real-world application scenarios the parties may not have inputs that match exactly. As an example, consider the case of two centralized electronic medical record (EMR) providers, which supply and aggregate medical records for medical practitioners, who wish to conduct a study about the number of patients who develop a particular disease after their recent medical histories indicate at-risk status. The EMR providers could use a PSI protocol to count the total number of unique diagnoses among their collective patients. Unfortunately, the EMR providers may not have the same set of information about each patient in their databases; for example, one might identify Alice by her street address and phone number, while the other might use her phone number and email address. Further complicating matters, Bob could use “bob@email.com” for one provider, but “BobDoe123@university.edu” for another.

It may appear that naively applying PSI to each column in two parties’ databases would allow them to realize their desired functionality, but such an approach has many flaws. For example, in the case that individuals use different identifying information for the different services, this approach could incur false negatives. To remedy this issue, there has been previous research on the private record linkage problem, in which “fuzzy matches” between records are permitted [WD14,HMFS17]. In this problem, two rows from different parties’ databases can be said to match if they satisfy some closeness relation, for example by matching approximately on t out of n columns. However, fuzzy matching PSI protocols are not as performant as exact-matching protocols.

As a design goal, we consider applications in which two parties would like to run PSI many times over respective databases. In our EMR example, the rows comprising users change slowly as new patients enter the system and some are expunged. However, auxiliary medical data could change frequently, at least daily. If the EMR providers wish to continuously update their medical models or run multiple analyses, they may run many PSI instances with auxiliary data [IKN+17].

In general, for many applications it is desirable for two parties to run PSI-style protocols many times over their respective data sets, and in this work we assume the parties will perform many joint queries. It is therefore advantageous for the parties to first to establish a new column for their databases, containing a single key for each row that can be used for the most performant exact-match PSI protocols.

As a second design goal, we relax an assumption that is standard for the private record linkage problem. We believe that it is not always realistic in practice to assume or to ensure that each participant’s database uniquely maps its rows to identities. For example, one EMR provider may unknowingly have multiple records about the same person in its database, as a result of that person providing different identifying information to different medical providers. As part of a correct protocol, some preprocessing phase must identify records that belong to the same individual – using both parties’ records – and group them accordingly. This is especially important for PSI applications that compute aggregate statistics.

This correctness requirement introduces an additional privacy concern. Consider the case in which party A has a single row in its database that matches more than one row in party B's database. Naively running a protocol to produce primary keys which link records would inevitably reveal some information to one of the parties. Either party A would learn that the party B is unaware that its rows could be merged, or party B would learn that it has several rows that correspond to a single person. Either way, one party will learn more about the other party's input than it should.

This work focuses on resolving the apparent trade-offs in privacy and performance between state-of-the-art exact-matching and fuzzy-matching PSI protocols. Our approach is to design a new two-party protocol that computes a new identifier for every row in both databases that will give exact matches. To avoid the additional leakage problem described above, our protocol outputs either (a) shares of the new identifiers, or (b) encryptions of the new identifiers for a generic CPA-secure encryption scheme with XOR homomorphism, which can be decrypted with a key held by the other party. (Our protocol can also output both share and encryptions, and we in fact prove security in the case that it outputs both.) The regime of PSI protocols that can be composed with our protocol is limited to those that can *combine shares or decrypt online* without revealing the plaintext to either party. However, the flexibility we provide in producing outputs offers flexibility to the design of PSI protocols which can be composed with ours. Additionally, although our identifier-agreement protocol is computationally intensive compared to the subsequent PSI protocol, we argue that this is a one-time cost that can be amortized over many PSI computations.

1.1 Our Contributions

This work addresses two problems: (1) The performance and accuracy tradeoffs between exact matching PSI and fuzzy matching PSI protocols. (2) The correctness and privacy problems introduced to PSI by the possibility of poorly defined rows. We address both of these problems in one shot by defining a functionality that computes shared primary keys for two parties' databases, such that the keys can be used multiple times as inputs to successive efficient PSI protocols, *without revealing the keys to the parties.* We refer to our stated problem as the *private identity agreement* functionality, and define it formally. We additionally discuss the security implications of composing our identity agreement functionality and subsequent PSI functionalities. We note that identity agreement is substantially more complex than private set intersection and private record linkage because of the concerns introduced by producing an intermediate output of a larger functionality.

After defining the identity agreement problem, we present a novel two-party protocol that solves the problem. We additionally describe a modification to our generic protocol that allows the outputs to naturally compose with DDH-style PSI protocols. Finally we present performance of our prototype implementation.

2 Problem Definition

Our setting assumes two parties, each holding some database, that wish to engage in inner-join style queries on their two databases, which we refer to as the *private joint-database query* functionality $\mathcal{F}^{\mathsf{Query}}$. The join will be over some subset of columns, and will be a disjunction i.e. two rows are matched if any of the columns in the join match. In Figure 1 we present the ideal private joint-database query functionality.

We consider a scenario in which it is advantageous for the parties to first establish a new database column containing keys for each record, so that this key can be used for many exact-match PSI protocols. We refer to this as the *private identity agreement* functionality, denoted $\mathcal{F}^{\mathsf{ID}}$ and described in Figure 2. As we have explained, establishing these keys is a setup phase in a general protocol that realizes $\mathcal{F}^{\mathsf{Query}}$. Importantly, the newly established identities should not be revealed to either party, as this could also reveal information about the other party's input. This makes it impossible to separate the protocol for $\mathcal{F}^{\mathsf{ID}}$ from the subsequence PSI-style protocols that the parties will use for their joint queries. We must instead modify the PSI-style protocols as well to ensure a secure composition with $\mathcal{F}^{\mathsf{ID}}$. Our security analysis must therefore consider the entire $\mathcal{F}^{\mathsf{Query}}$ as a single system.

2.1 The Identity Agreement Functionality

We denote the set of possible identifiers that either party may hold by $I = \bigotimes I_i$, with each set I_i being one column and having $\perp \in I_i$. To define a "match" we define an equivalence relation $S_1 \overset{\mathsf{user}}{\sim} S_2$ as follows: if there exists component of $s_{1,j} \neq \perp$ of S_1 and a component $s_{2,k} \neq \perp$ of S_2 such that $s_{1,j} = s_{2,k}$ then $S_1 \overset{\mathsf{user}}{\sim} S_2$. In other words, we consider two rows to be equivalent if any of their non-empty columns are equal. [3]

For each party p_i with database $\mathcal{D}_{p_i}$, we assume for simplicity of exposition that every pair of rows S_1 and S_2 satisfies $S_1 \overset{\mathsf{user}}{\not\sim} S_2$. (This means that the party does not have sufficient information to conclude that the two rows represent the same element.) Note, however, that it is possible for p_1 to have rows $S_{1,1}, S_{1,2}$, and for p_2 to have a row S_2 such that $S_{1,1} \overset{\mathsf{user}}{\sim} S_2 \overset{\mathsf{user}}{\sim} S_{1,2}$. In such a situation, p_1 is not aware that its database contains two rows that represent the same element.

The goal of the identity agreement functionality $\mathcal{F}^{\mathsf{ID}}$ is to compute a map $\Lambda \colon \mathcal{D}_{p_1} \cup \mathcal{D}_{p_2} \to \mathcal{U}$ such that for any $S_1 \overset{\mathsf{user}}{\sim} S_2$, $\Lambda(S_1) = \Lambda(S_2)$, and for all $S_1 \overset{\mathsf{user}}{\not\sim} S_2$, $\Lambda(S_1) \neq \Lambda(S_2)$. As we explain below, the parties will not learn $\Lambda(S_i)$ for their respective databases; they will only see encryptions of the map.

We define the privacy goals of $\mathcal{F}^{\mathsf{ID}}$ in relation to the overall query functionality $\mathcal{F}^{\mathsf{Query}}$: to compute some PSI functionality where the intersection is determined by the $\overset{\mathsf{user}}{\sim}$ relation. Importantly, if $\mathcal{F}^{\mathsf{ID}}$ is composed with other protocols to realize $\mathcal{F}^{\mathsf{Query}}$, then $\mathcal{F}^{\mathsf{ID}}$ may not reveal any information about $\overset{\mathsf{user}}{\sim}$ to either party. Consider, for example, a situation where p_1 has in its input S_1 and S_2, and in p_2's input there is a S_* such that $S_1 \overset{\mathsf{user}}{\sim} S_* \overset{\mathsf{user}}{\sim} S_2$. It should not be the case that p_1 will learn $S_1 \overset{\mathsf{user}}{\sim} S_2$, beyond what can be infered from the output of the PSI functionality. Likewise, p_2 should not learn that p_1 has such elements in its input. If $\mathcal{F}^{\mathsf{ID}}$ revealed such information, then some party could learn more from the composition of $\mathcal{F}^{\mathsf{ID}}$ with another functionality than it would learn from querying only $\mathcal{F}^{\mathsf{Query}}$.

Private Query Functionality $\mathcal{F}^{\mathsf{Query}}$

1. **Setup:** Upon receiving messages of the form $(\mathsf{setup}, \mathsf{sid}, \mathcal{D}_{p_i})$ from p_i, store $(\mathsf{sid}, \mathcal{D}_{p_i}, i)$. After receiving setup from both parties with matching sid, ignore all future setup messages with sid.
2. **Queries:** Upon receiving messages of the form $(\mathsf{query}, \mathsf{sid}, f)$ from party p_i, where $f = (f_1, f_2)$ is the description of a two-party functionality over two databases:
   (a) If both parties have not sent setup messages for sid, ignore the message. Otherwise proceed as follows.
   (b) Record the message $(\mathsf{query}, \mathsf{sid}, (f_1, f_2), i)$. If have already recorded $(\mathsf{query}, \mathsf{sid}, (f_1, f_2), 3 - i)$, then send $(\mathsf{response}, \mathsf{sid}, f_1(\mathcal{D}_{p_1}, \mathcal{D}_{p_2}))$ to p_1 and $(\mathsf{response}, \mathsf{sid}, f_2(\mathcal{D}_{p_1}, \mathcal{D}_{p_2}))$ to p_2.

Fig. 1: Query functionality $\mathcal{F}^{\mathsf{Query}}$, which receives two parties' databases and responds to queries over functions of the databases.

[3] It is possible to establish $\overset{\mathsf{user}}{\sim}$ for S_1 and S_2 for any binary relation that $s_{1,j}$ and $s_{2,k}$ may satisfy; however, we feel that equality is the most natural, and consider only equality in this work. We remark later to indicate when one could substitute another relation for equality in the construction.

ID Agreement Functionality $\mathcal{F}^{\mathsf{ID}}$

1. **Inputs:** Each party p_i has a tuple $\mathsf{input}_{p_i} = (\mathcal{D}_{p_i}, \mathsf{k}^{\mathsf{enc}}_{p_i})$, where $\mathcal{D}_{p_i}$ is the party's database and $\mathsf{k}^{\mathsf{enc}}_{p_i}$ is an encryption key for a CPA-secure encryption scheme (referred to as CPA).
2. **Generate Identifiers:** Upon receiving $(\mathsf{submit}, \mathsf{sid}, \mathsf{input}_{p_i})$ from each p_i for session sid, $\mathcal{F}^{\mathsf{ID}}$ computes a map $\Lambda\colon \mathcal{D}_{p_1} \cup \mathcal{D}_{p_2} \to \mathcal{U}$, where $\mathcal{U} = \{0,1\}^{\ell}$ is a universe of user identities, such that for any two rows S_1 and S_2 for which $S_1 \overset{\mathsf{user}}{\sim} S_2$, $\Lambda(S_1) = \Lambda(S_2)$, and for all two rows for which $S_1 \overset{\mathsf{user}}{\not\sim} S_2$, $\Lambda(S_1) \neq \Lambda(S_2)$.
   Let $\Lambda(S)$ be the identifier assigned to a row S. For each row $S \in \mathcal{D}_{p_i}$, $\mathcal{F}^{\mathsf{ID}}$ returns $(\mathsf{sid}, S, \mathsf{CPA.enc}(\mathsf{k}^{\mathsf{enc}}_{p_{3-i}}, \Lambda(S)))$ to p_i.

Fig. 2: ID Agreement functionality $\mathcal{F}^{\mathsf{ID}}$

Two-Party Component Labeling Functionality $\mathcal{F}^{\mathsf{lbl}}$

1. **Inputs:** Each party p_i has a tuple $\mathsf{input}_{p_i} = (G_{p_i}, \mathsf{k}^{\mathsf{enc}}_{p_i}, \rho_{p_i})$, where $G_{p_i} = (V_{p_i}, E_{p_i})$ is a graph, $\mathsf{k}^{\mathsf{enc}}_{p_i}$ is an encryption key for a CPA-secure encryption scheme (referred to as CPA), and $\rho_{p_i} = \{\rho_{p_i,j}\}_{j\in[N]}$ is a list of N ℓ-bit one-time pads.
2. **Generate Labels:** Upon receiving $(\mathsf{submit}, \mathsf{sid}, \mathsf{input}_{p_i})$ from each p_i for session sid, $\mathcal{F}^{\mathsf{lbl}}$ samples random elements from a universe $\mathcal{U} = \{0,1\}^{\ell}$ and computes a map $\Lambda\colon V_{p_1} \cup V_{p_2} \to \{0,1\}^{\ell}$ such that for any two vertices $v_1, v_2 \in V_{p_1} \cup V_{p_2}$, $\Lambda(v_1) = \Lambda(v_2)$ if and only if v_1 and v_2 are in the same connected component in $G_1 \cup G_2$.
   For $i \in \{1,2\}$ and $j \in [|V_{p_i}|]$, let $v_{p_i,j}$ be the jth vertex in V_{p_i}. $\mathcal{F}^{\mathsf{lbl}}$ computes $f^{\mathsf{enc}}_{p_i} = \{\mathsf{CPA.enc}(\mathsf{k}^{\mathsf{enc}}_{p_i}, \Lambda(v_{p_i,j})\}_{j\in[|V_{p_i}|]}$ and $f^{\mathsf{mask}}_{p_i} = \{\rho_{p_i,j} \oplus \Lambda(v_{p_i,j})\}_{j\in[|V_{p_i}|]}$
   $\mathcal{F}^{\mathsf{lbl}}$ sends $(\mathsf{labeling}, \mathsf{sid}, f^{\mathsf{enc}}_{p_1}, f^{\mathsf{mask}}_{p_2})$ to p_1 and $(\mathsf{labeling}, \mathsf{sid}, f^{\mathsf{enc}}_{p_2}, f^{\mathsf{mask}}_{p_1})$ to p_2.

Fig. 3: Two-Party Component Labeling Functionality $\mathcal{F}^{\mathsf{lbl}}$

3 Security Primitives and Cryptographic Assumptions

3.1 Garbled Circuits

Garbled circuits were proposed by Andrew Yao [Yao82] as a means to a generic two-party computation protocol. Yao's protocol consists of two subprotocols: a garbling scheme [BHR12] and an oblivious transfer. Garbled circuit constructions have promising performance characteristics. Most of the CPU work of a garbled circuit protocol involves symmetric primitives, and as Bellare et al. show, garbling schemes can use a block cipher with a fixed key, further improving performance [BHR12]. A drawback of garbled circuits is that they require as much communication as computation, but this can be mitigated by using garbled circuits to implement efficient subprotocols of a larger protocol.

3.2 ElGamal Encryption

The ElGamal encryption scheme is CPA-secure under the DDH assumption, and supports homomorphic group operations (denoted $\mathsf{ElGl}.\mathit{Mul}$ below). If the plaintext space is small, addition in the exponent can also be supported, but decryption in this case requires computing a discrete logarithm. Using the identity element of the group, $\mathsf{ElGl}.\mathit{Mul}$ can be used to re-randomize a ciphertext.

Definition 1 (ElGamal Encryption) *The ElGamal encryption scheme [ElG85] is an additively homomorphic encryption scheme, consisting of the following probabilistic polynomial-time algorithms:*

ElGl.**Gen** *Given a security parameter λ, $\mathsf{ElGl}.\mathit{Gen}(\lambda)$ returns a public-private key pair (pk, sk), and specifies a message space $\mathcal{M}$.*

ElGl.**Enc** *Given the public key pk and a plaintext message* $m \in \mathcal{M}$*, one can compute a ciphertext* $\mathsf{ElGl}.Enc(pk,m)$*, an El Gamal encryption of m under pk.*

ElGl.**Dec** *Given the secret key sk and a ciphertext* $\mathsf{ElGl}.Enc(pk,m)$*, one can run* $\mathsf{ElGl}.Dec$ *to recover the plaintext m.*

ElGl.**Mul** *Given the public key pk and a set of ciphertexts* $\{\mathsf{ElGl}.Enc(pk,m_i)\}$ *encrypting messages* $\{m_i\}$*, one can homomorphically compute a ciphertext encrypting the product of the underlying messages:*

$$\mathsf{ElGl}.Enc(pk,\prod_i m_i) = \mathsf{ElGl}.Mul(\{\mathsf{ElGl}.Enc(pk,m_i)\}_i)$$

As described in Section 4.4, we will use $\mathsf{ElGl}.Mul$ to induce an XOR homomorphism by encrypting bit-by-bit. This allows us to avoid performing group operations in the garbled circuit part of our protocol and reduces our communication cost, at the cost of CPU effort.

4 Secure ID Agreement as Secure Two-Party Component Labeling

In this section, we define a graph problem that we call Two-Party Component Labeling, and provide a reduction between ID agreement and Two-Party Component Labeling. We then describe an algorithm to compute component labeling and a two-party protocol that securely implements it.

4.1 Two-Party Component Labeling

In the two-party component labeling problem, each party $p_i \in \{p_1, p_2\}$ has a graph $G_{p_i} = (V_{p_i}, E_{p_i})$, where there exists some universe of vertices V for which $V_{p_1} \subset V$ and $V_{p_2} \subset V$. Each party's graph contains at most N vertices, which are distributed among connected components of size at most m. Both N and m are parameters of the problem. As shorthand, we refer to a connected component as a component. As output, each party assigns a label to every component in its graph. If there are two components $C_1 \in G_{p_1}$ and $C_2 \in G_{p_2}$ for which C_1 and C_2 have a non-empty intersection, then p_1 and p_2 must assign the same label to C_1 and C_2. Just as we explained with ID Agreement, this property induces a transitive relation. If two vertices $v \in G_{p_1}$ and $u \in G_{p_2}$ are in the same component of $G = G_{p_1} \cup G_{p_2}$, then their components in G_{p_1} and G_{p_2}, respectively, must be assigned the same label.

More precisely, consider parties p_1 and p_2 with graphs G_{p_1} and G_{p_2}, respectively, and let $\mathcal{C}_{p_i}$ represent the set of components that constitute G_{p_i}. Moreover, assume the vertices in G_{p_1} and G_{p_2} are drawn from some universe of vertices V. The two-party component labeling problem is to construct a map $\Lambda\colon \mathcal{C}_{p_1} \cup \mathcal{C}_{p_2} \to \mathcal{U}$, where $\mathcal{U}$ is a universe of labels. For any two components $C_1, C_n \in \mathcal{C}_{p_1} \cup \mathcal{C}_{p_2}$, $\Lambda(C_1) = \Lambda(C_n)$ if and only if there is some series of components $C_2, \ldots, C_{n-1} \in \mathcal{C}_{p_1} \cup \mathcal{C}_{p_2}$ such that $C_i \cap C_{i+1} \neq \emptyset$ for $i \in \{1 \ldots n-1\}$.

4.2 Reducing ID Agreement to Two-Party Component Labeling

We reduce the two-party identity agreement problem to two-party component labeling. Each party p represents its database as a graph $G_p = (V_p, E_p)$ as follows. Each piece of identifying information in party p's database is represented by a vertex in p's graph. (Empty entries in a database are simply left out of the graph.) Edges in the graph connect vertices which represent identifying information of the same user. Therefore, each set in a party's database is represented as a component in the party's graph.

The component labeling of two graphs G_{p_1} and G_{p_2} can be trivially used to assign user identities. The identifier of a user represented by a component C in G_p is directly copied from the label assigned to C during component labeling.

Intuitively, the reduction works because the two parties compute ID agreement over their databases by computing a union of their graphs. If the two parties' graphs contain the same vertex v (meaning both databases contain the same piece of identifying information), then the components containing v in G_{p_1} and G_{p_2} are the same component in the union graph $G = G_{p_1} \cup G_{p_2}$.

4.3 An Algorithm for Two-Party Component Labeling

In this section, we present a component labeling algorithm without explicitly addressing privacy concerns. In Section 4.4, we present a protocol to implement the algorithm while preserving privacy. We illustrate an execution of our percolate-and-match algorithm in Figure 5 and provide pseudo-code in Figure 7.

Our component labeling algorithm is an iterative procedure in which the two parties assign labels to every vertex in their respective graphs and then progressively update their vertices' labels. To initialize the procedure, each party p constructs an initial labeling for its graph by assiging a unique label to every vertex in its graph G_p. Specifically, every vertex v in G_p is assigned the label v, the encoding of the vertex itself. [4] Notice that in the initial labeling, no two vertices within a party's graph are assigned the same label. However, any vertex that is included in both graphs is assigned the same label by both parties.

By the end of the iterative procedure, two properties of the labelings must be met. First, within each component of a party's graph, all vertices must have the same label. Second, if any vertex is in both parties' graphs, then the vertex has the same label in both parties' labelings. Together, these two requirements enforce that every two vertices within a component of $C \subset G$ have the same label. This common label can then be taken as the component's label.

Each step in our iterative procedure is a two phase process. The first phase operates on each party's graph independently. It enforces the property that for each component in a party's graph, every vertex in the component has the same label. In this phase, the algorithm assigns to every vertex $v \in G_p$ the (lexicographic) minimum of all labels in its component in G_p. We call this a *percolation phase* because common labels are percolated to every vertex in a component.

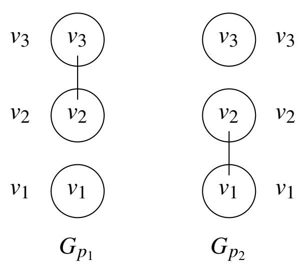
In the second phase, the algorithm operates on the vertices which are common to both parties' graphs. It ensures that every vertex v which is common to both parties' graphs has been assigned the same label in the two parties' labelings. If one party's label for v differs from the other party's label for v, then both labels are updated to the minimum of two labels that have been assigned to v. We call this a *matching phase* because vertices which are common to both graphs are assigned matching labels in the two labelings.

If some vertex's label is updated in a matching phase, then its label may differ from the labels of the other vertices in its component. Therefore, the iterative procedure repeats until labelings stabilize. During percolation, each vertex's label is set to the minimum label of all vertices in its component. If some vertex's label changes during a matching phase, its new label must be "smaller" than its previous label. During the next percolation phase, the change is propagated by again updating the label of each vertex in the component is to the minimum label in the component. In Appendix B, we prove that if m denotes the maximum size of a component in $G_{p_1} \cup G_{p_2}$, then at most $m-1$ iterations are necessary for vertex labels to stabilize.

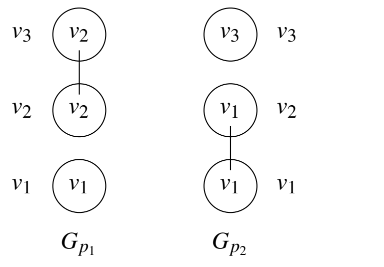
4.4 Private Component Labeling

We now provide a protocol which implements the component labeling algorithm described in Section 4.3 while preserving privacy. The ideal functionality $\mathcal{F}^{\mathsf{lbl}}$ for private two-party component labeling is given in Figure 3.

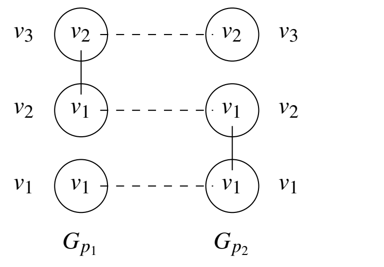
[4] In an application, the label would be the data that the vertex represents. Additionally, if the parties agree on an encoding scheme beforehand, types (address, zip, phone) can be encoded as part of a label at the cost of only a few bits.



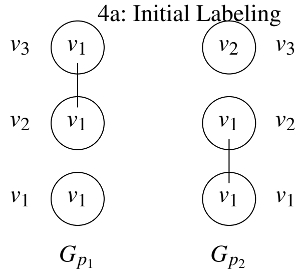
4a: Initial Labeling



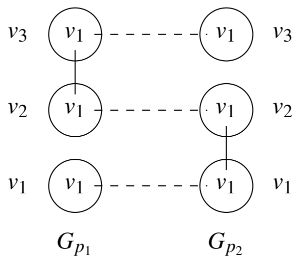
4b: Labelings after first percolation phase.



4c: Labelings after first matching phase.



4d: Labelings after second percolation phase.



4e: Labelings after second matching phase.

Fig. 5: Depiction of the percolate-and-match algorithm for component labeling. v_1, v_2, and v_3 are common to both G_{p_1} an G_{p_2}. Each vertex's label is drawn inside the vertex, and its identity is on the side. p_1's graph has an edge between v_2 and v_3, while p_2's graph has an edge between v_1 and v_2. Solid lines depict edges in each party's graph. Dotted lines depict matches that occur during matching phases. The figures show the evolution of the algorithm over 2 iterations.

Approach The challenge of securely implementing the percolate-and-match algorithm arises from the fact that percolate-and-match performs two operations: (1) comparisons on vertices, and (2) updates on vertex labels. However, if either party knew the output of any such operation on its vertices, then it would learn information about the other party's graph. Consider that if a participant learns that its vertex's label changed during the any matching phase or during a percolation phase, it learns that one of the vertices in its graph has a matching vertex in the other party's graph. Similarly, it a party learns that its vertex's label *isn't* updated during the first matching or percolation phase, it learns its vertex isn't in the other party's graph.

Our approach is to perform both vertex comparisons and label updates without revealing the output of any comparison or update, and to encrypt all intermediate and output labels so that no information is leaked about the computation. Naively adapting state-of-the-art PSI protocols in order to perform our matching phase does not work for this approach, because in addition to finding the common vertices in the two parties' graphs, we also must perform updates on matching labels, and state of the art PSI protocols do not provide easy ways to modify auxiliary data without revealing additional information.

To implement comparisons and updates, we use garbled circuits. Importantly, garbled circuits must implement oblivious algorithms, whose operation is independent of the input data. Notably, for any branch in the execution tree of an oblivious algorithm, we must perform operations for both possible paths, replacing the untaken path with dummy operations. Additionally, random accesses to an array (i.e. those for which the index being read is input-dependent) must either scan over the entire array, incurring a $O(N)$ cost, or use *Oblivious RAM* techniques, which incurs $\log(N)$ communication overhead per access [LN18,AKL$^+$18].

Matching via Garbled Circuits To perform our matching phase obliviously, we adapt a technique described by Huang, Evans, and Katz for PSI [HEK12]. In their scheme, called Sort-Compare-Shuffle, each party sorts its elements, then provides its sorted list to a garbled circuit which merges the two lists. If two parties submit the same element, then the two copies of the element land in adjacent indices in the merged list. The circuit

```
procedure ComponentLabeling($G_1, G_2, m$)
    $\mathcal{L}_1 \leftarrow$ Setup($G_1$)
    $\mathcal{L}_2 \leftarrow$ Setup($G_2$)
    for $m-1$ times do
        $\mathcal{L}_1 \leftarrow$Percolate($G_1, \mathcal{L}_1$)
        $\mathcal{L}_2 \leftarrow$Percolate($G_2, \mathcal{L}_2$)
        $\mathcal{L}_1, \mathcal{L}_2 \leftarrow$Match($\mathcal{L}_1, \mathcal{L}_2$)
    end for
    return $\mathcal{L}_1, \mathcal{L}_2$
end procedure
function Setup($G$)
    $\mathcal{L} \leftarrow []_N$
    for all vertex $v \in G$ do
        $\mathcal{L}$.append((vrt : $v$, lbl : $v$) )
    end for
    return $\mathcal{L}$
end function
function UpdateLabel($\mathcal{L}, v$, lbl$^*$)
    FindVert($\mathcal{L}, v$).lbl $\leftarrow$ lbl$^*$          ▷ updates $v$'s label in $\mathcal{L}$
end function
function Percolate($G, \mathcal{L}$)
    for all component $C \in G$ do
        lbl$^* \leftarrow \min_{u \in C}$ FindVert($\mathcal{L}, u$).lbl
        for all vertex $v \in C$ do
            UpdateLabel($\mathcal{L}, v$, lbl$^*$)
        end for
    end for
    return $\mathcal{L}$
end function
function Match($\mathcal{L}_1, \mathcal{L}_2$)
    for all vertex $v$ in $\mathcal{L}_1$ and in $\mathcal{L}_2$ do
        $a \leftarrow$ FindVert($\mathcal{L}_1, v$).lbl
        $b \leftarrow$ FindVert($\mathcal{L}_2, v$).lbl
        UpdateLabel($\mathcal{L}_1, v, \min(a,b)$ )
        UpdateLabel($\mathcal{L}_2, v, \min(a,b)$ )
    end for
    return $\mathcal{L}_1, \mathcal{L}_2$
end function
function FindVert($\mathcal{L}, v$)
    return (vrt, lbl) $\in \mathcal{L}$ for which vrt $= v$
end function
```

Fig. 7: Pseudocode for Component Labeling algorithm.

iterates through the list, comparing elements at adjacent indices in order to identify common elements. After comparing, their circuit shuffles thesorted list before revealing elements in the intersection.

Our construction adapts the sort-compare-shuffle technique to efficiently perform our matching phase with label updates as follows. Each party submits a sorted list of vertices to a garbled circuit, including auxiliary information for each vertex that represents the vertex's currently assigned label. To perform matching, we merge the two parties' lists of vertices into one sorted list $\mathcal{L}$ and iterate through $\mathcal{L}$. At each pair of adjacent indices in $\mathcal{L}$, we conditionally assign both elements' current labels to the minimum of the two labels only if the vertices match. Matching in this way via garbled circuit hides from both parties all of the matches that are made between the two parties' graphs and their respective label updates.

Percolation via Garbled Circuits To percolate labels within a component, a party can submit all of the vertices in one of its components to a garbled circuit along with each vertex's current label. The circuit computes the minimum of the labels and assigns the minimum label to each vertex in the component.

Stitching Percolation and Matching Together The remaining question is how to efficiently stitch together percolation phases and matching phases without revealing intermediate labels of any vertices.We perform percolation and matching in the same circuit. To transition between percolation and matching phases, we permute the list of vertices. We define a permutation π which is hidden from both parties, and apply π and π^{-1} to $\mathcal{L}$ to transition from matching phase to percolation phase and back.

Our garbled circuit begins by merging the two parties' sorted lists into one large list $\mathcal{L}$. We then apply π to the list to shuffle the list, hiding all information about the sorted order of $\mathcal{L}$. Next, we reveal indices of each party's components in $\pi(\mathcal{L})$. For the graph G_{p_i} of each party i, and for each component $C \subset G_{p_i}$, both parties learn the indices of C's vertices in $\pi(\mathcal{L})$. We use these indices to hard-wire the min-circuit that percolates labels within C. After percolating, we apply the inverse permutation π^{-1} to $\pi(\mathcal{L})$ and can again iterate through $\mathcal{L}$ to merge. The circuit repeatedly applies π and π^{-1} to $\mathcal{L}$ to transition from matching phase to percolation phase and back. After $m-1$ iterations, the circuit outputs encrypted labels to the two parties.

We remark that permuting $\mathcal{L}$ and revealing indices of each party's components avoids in-circuit random access to look up current vertex labels for each percolation circuit, and hence the overhead of ORAM.

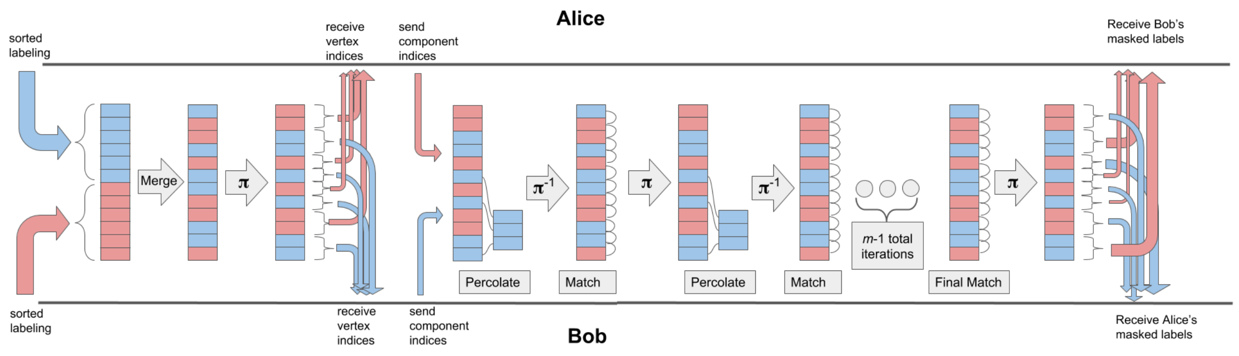


Fig. 8: Illustration of the percolate-and-match garbled circuit approach.

Revealing indices allows our circuit to hard-wire the indices of the min-circuits that perform percolation, achieving $O(1)$ cost per label lookup at the expense of an $O(n\log n)$ shuffle between phases. We can consider that this technique allows us to amortize the expense of two permutations with cost $n\log n$ over the n memory accesses we do for each iteration, which yields the same asymptotic complexity as ORAM [LN18,AKL$^+$18].

Graph Structure There is one additional caveat. Revealing the indices of the vertices of each component in $\mathcal{L}$ reveals the structure of the parties' graphs. To prevent this, we require that both parties pad their graphs to uniform structure. If the two parties do not have uniformly structured graphs, then they must agree on a graph structure, and then pad their graphs using randomly selected vertices. To simplify the presentation, we set a number of components C and a maximum size of each component m, and then have each party pad its graphs using randomly selected vertices until each graph contains $N = Cm$ vertices.

Outputs We remark that in the protocol we present, each party receives as output XOR-shares of both parties' labels and their own labels encrypted under a key held by the other party.

Protocol in Depth We now describe how to privately implement our percolate-and-match algorithm. Figure 8 illustrates our approach, Figure 9 contains the full protocol, and Figure 10 describes our garbled circuits.

Protocol Inputs As input, each party p has a graph $G_p = (V_p, E_p)$. If some party has fewer than m vertices in some component(s) of its graph, it pads its graph by randomly sampling vertices to add to its component(s). The two parties must agree on a number of components C and the maximum size of a component m, and each must pad its graph until it has $N = Cm$ vertices.

In addition, each party p_i has a key $\mathsf{k}^{\mathsf{enc}}_{p_i}$ for a CPA-Secure encryption scheme with XOR homomorphism. Third, it has a list of N ℓ-bit random strings ρ_p. The key and random strings are used for the output; the key will encrypt the other party's output labels, and random strings will hide p's own labels from the other party.

Initial Labelings Each party represents the current labeling of its graph as a list of *vertex descriptors*. A descriptor $\mathcal{L}(v)$ of a vertex v is a triple $(\mathsf{img}_v, \mathsf{lbl}_v, \mathsf{party}_v)$, where img_v is the image of vertex v under a shared function $H\colon \{0,1\}^* \to \{0,1\}^\ell$, lbl_v is the label assigned to v, and party_v is the party's identifier (1 or 2). We note that we use H to hash each vertex descriptor to a uniform length. In the description of the protocol and the proof, we treat H as a random oracle.

We refer to p's labeling as $\mathcal{L}_p = \{\mathcal{L}(v)\}_{v\in G_p}$. In the initial labeling that each party constructs, each vertex's label is initially set to its image under H (meaning $\mathsf{img}_v = \mathsf{lbl}_v$). After constructing its labeling $\mathcal{L}_{p_i}$, each party sorts its labeling $\mathcal{L}_{p_i}$ on the images of its vertices under H.

Garbled Circuit Part 1: Merge, Permute, and Reveal Order After the parties set up their inputs, they invoke a garbled circuit that merges their descriptors into a combined labeling $\mathcal{L}$ and reveals the indices of their descriptors under a random permutation. First, each party p_i submits its sorted labeling $\mathcal{L}_{p_i}$ to a garbled circuit. The garbled circuit merges $\mathcal{L}_{p_1}$ and $\mathcal{L}_{p}$ into one list $\mathcal{L}$ using a Batcher Merge [Bat68]. Second, the circuit shuffles $\mathcal{L}$ using a permutation π unknown to either party using Waksman networks [Wak68,BD02]. Each party p_i randomly samples a permutation π_{p_i} on $2N$ elements and inputs its selection bits s_{p_i} to the circuit. We define $\pi = \pi_{p_1} \circ \pi_{p_2}$; the circuit permutes $\mathcal{L}$ as $\pi(\mathcal{L})$.

Next, the circuit reveals to each party the indices of its own vertices in $\pi(\mathcal{L})$. First, each party samples $2N$ one-time pads σ_{p_i} and submits them to the circuit. The garbled circuit then iterates through $\pi(\mathcal{L})$, and at each index, the descriptor's third element (which is the party identity) determines what to output to each party. If the vertex at index j was submitted by p_i, then p_i receives from the garbled circuit $\pi(\mathcal{L})[j].\mathsf{img}$, the image of the vertex at index j in the permuted list, and p_{3-i} receives $\pi(\mathcal{L})[j].\mathsf{img} \oplus \sigma_{p_i,j}$, which is the same image but masked by p_1's jth one-time pad.

Given the indices of each of its vertices in $\pi(\mathcal{L})$, each party computes the indices composing each of its components. Let $\mathsf{idx}(v)$ denote the index of vertex v in $\pi(\mathcal{L})$. For each component $C \subset G_{p_i}$, p_i computes $\mathsf{idx}(C) = \{\mathsf{idx}(v)\}_{v \in C}$. For each component C in G_{p_i}, p_i shares $\mathsf{idx}(C)$ with p_{3-i}. Note that each party p_i learns the indices of its own vertices in $\pi(\mathcal{L})$ *and* it learns the indices corresponding to each of p_{3-i}'s components in $\pi(\mathcal{L})$. However, neither party learns the original positions of either party's vertices in $\mathcal{L}$. In the proof, we show that revealing these indices in $\pi(\mathcal{L})$ reveals no information about the other party's inputs.

Garbled Circuit Part 2: Percolate and Match In the second subcircuit, the parties perform percolation and matching. The parties use the information revealed about their components' indices in $\pi(\mathcal{L})$ to hard-wire the indices of each component in order to perform percolation. Percolation happens via independent subcircuit for each component in both parties' graphs. For each component C, let $\mathsf{idx}(C)$ be the indices of the component's vertices in $\pi(\mathcal{L})$. The circuit computes $\mathsf{lbl}^* \leftarrow \min_{j \in \mathsf{idx}(C)} \pi(\mathcal{L})[j].\mathsf{lbl}$, and then assigns $\pi(\mathcal{L})[j].\mathsf{lbl} \leftarrow \mathsf{lbl}^*$ for each $j \in \mathsf{idx}(C)$.

Given a circuit with the parties' descriptors arranged as $\pi(\mathcal{L})$, the circuit applies $\pi^{-1} = \pi_{p_2}^{-1} \circ \pi_{p_1}^{-1}$ to $\pi(\mathcal{L})$ to retrieve $\mathcal{L}$. It then performs matching by iterating through $\mathcal{L}$ and obliviously comparing the descriptors at each pair of adjacent indices in $\mathcal{L}$. [5] Let $\mathcal{L}[i] = (\mathsf{img}_i, \mathsf{lbl}_i, \mathsf{party}_i)$ be the descriptor at index i in $\mathcal{L}$, and let $\mathcal{L}[i+1] = (\mathsf{img}_{i+1}, \mathsf{lbl}_{i+1}, \mathsf{party}_{i+1})$ be the descriptor at index $i+1$. If $\mathsf{img}_i = \mathsf{img}_{i+1}$, then both lbl_i and lbl_{i+1} are set to be the minimum among lbl_i and lbl_{i+1}.

The circuit iterates between percolation and matching $m-1$ times, applying π to transition from matching to percolation, and π^{-1} to transition from percolation to matching. After the final matching phase, the circuit again applies π to transition to the output phase.

Encrypting Vertex Labels At the end of the protocol, each party must receive its vertex labels encrypted under a key known only to the other party. We show how to move encryption outside of the garbled circuit in order to save the cost of online encryption at the expense of a few extra rounds of communication.

For a generic CPA-secure encryption scheme, we use the following technique. Consider a message m, computed within a circuit, that needs to be encrypted under a key k known only to p_1 without either party learning m. At the end, p_2 should learn $c = \mathsf{enc}(k, m)$. We can encrypt m under k as follows. p_2 samples a one-time pad υ and submits it to the circuit. The circuit outputs $m \oplus \upsilon$ to p_1. p_1 computes $c' = \mathsf{enc}(\mathsf{key}, m \oplus \upsilon)$. Then, p_1 sends c' to p_2, and p_2 computes $c = c' \oplus \upsilon = \mathsf{enc}(k, m)$.

[5] If not using equality to compare the descriptors, then one could substitute any other comparison circuit to evaluate matching between two elements.

The garbled circuit produces outputs to the parties as follows. Let out_i be the set of indices of p_i's vertices in $\pi(\mathcal{L})$. For the jth index in out_i, p_{3-i} receives $\pi(\mathcal{L})[\mathsf{out}_i[j]].\mathsf{lbl} \oplus \rho_{p_i}[j]$, which is p_i's jth label masked by p_i's jth random pad. The parties use the technique above to recover their encrypted labels.

Interfacing with DDH-Style PSI Protocols We present our technique for ElGamal encrypting vertex labels. As we show below, DDH-style PSI protocols can be modified to accept ElGamal-encrypted inputs.

First, the parties agree on ℓ group elements $\{h_{j,0}\}_{j\in[\ell]}$. They then compute $h_{j,1} = h_{i,0}^{-1}$ as the inverse of each element. Neither party should know the discrete logarithm of these group elements; this can be accomplished, for example, by using the Diffie-Hellman key exchange to agree on $h_{j,0}$ for each j.

We represent a label m using these group elements by letting each pair of group elements $(h_{j,0}, h_{j,1})$ corresponds to the possible values of the bit at position j of the label (the same group elements are used for all labels). Let m_j be the jth bit of m. m can be represented as $\{h_{j,m_j}\}_{j\in[|m|]}$. Notice that each bit can be inverted by computing $h_{i,b}^{q-2}$ where q is the order of the group. Therefore, to compute $m \oplus p$, it is sufficient to invert the elements representing m where the bits of p are 1.

Suppose m is the label of one of $p_i's$ vertices. As we described above, p_i submits a mask υ to the circuit, and the circuit outputs $m' = m \oplus \upsilon$ to p_{3-i}. For each bit m'_j in m', p_{3-i} will encrypt h_{j,m'_j} under its public key, and will send the ℓ ciphertexts for m' to p_i. When p_i receives its ciphertexts, it removes υ as follows. For each bit υ_j of υ where $\upsilon_j = 1$, p_i uses $\mathsf{ElGl}.\mathit{Mul}$ to invert the plaintext of the corresponding bit ciphertext. Finally, p_i uses $\mathsf{ElGl}.\mathit{Mul}$ to combine the bit-ciphertexts into a single label.

Recall that DDH-style PSI protocols proceed as follows:

1. p_1 chooses a random exponent R_1 and, for each element $S_{1,i}$ in its set, sends $S_{1,i}^{R_1}$ to p_2.
2. p_2 chooses a random exponent R_2 and, for each element $S_{2,j}$ in its set, computes $S_{2,j}^{R_2}$. It then computes $(S_{1,i}^{R_1})^{R_2}$, and sends $\{S_{1,i}^{R_1 R_2}\}$ and $\{S_{2,j}^{R_2}\}$ to p_1.
3. p_1 computes $(S_{2,j}^{R_2})^{R_1}$ and the intersection.

We note that the exponentiations of each party's input elements can actually be performed using $\mathsf{ElGl}.\mathit{Mul}$. Since the other party has the key to decrypt these ciphertexts, the protocol proceeds as follows:

1. p_1 samples a random exponent R_1 and, for each element $\mathsf{enc}_{pk_2}(S_{1,i})$ in its set, sends $\mathsf{enc}_{pk_2}(S_{1,i}^{R_1})$ to p_2.
2. p_2 samples a random exponent R_2 and, for each element $\mathsf{enc}_{pk_1}(S_{2,j})$ in its set, computes $\mathsf{enc}_{pk_1}(S_{2,j}^{R_2})$. p_2 computes $S_{1,i}^{R_1} \leftarrow \mathsf{dec}_{sk_2}(\mathsf{enc}_{pk_2}(S_{1,i}^{R_1}))$ and then $(S_{1,i}^{R_1})^{R_2}$, and sends $\{S_{1,i}^{R_1 R_2}\}$ and $\{\mathsf{enc}_{pk_1}(S_{2,j}^{R_2})\}$ to p_1.
3. p_1 decrypts, computes $(S_{2,j}^{R_2})^{R_1}$, and computes the intersection.

Protocol Outputs Each party outputs two sets of encrypted labels. First, each party outputs the other party's masked labels, which it receives from the garbled circuit. Second it output its own vertices' encrypted labels.

Parties associate their encrypted labels with their vertices based on the order in which they receive their encrypted labels. In the earlier reveal phase, the parties learn the indices of their own vertices in the permuted list. They sort their vertices based on their indices in that list, and then associate the sorted vertices in order with the encrypted labels they receive.

To choose a component label, a party arbitrarily selects any label assigned to a vertex in the component.

Secure Two-Party Component Labeling Protocol

- **Shared Inputs** The two parties share a security parameter λ and a random oracle $H\colon \{0,1\}^* \to \{0,1\}^\ell$
- **Local Input** Each party $p_i \in \{p_1, p_2\}$ has the following inputs:
  - $G_{p_i} = (V_{p_i}, E_{p_i})$ is a party's graph. For each party, V_{p_i} is a subset of a universal set V. Moreover, $|V_{p_i}| = N$. Each graph G_{p_i} is composed of components containing at most m vertices.
  - $\mathsf{k}^{\mathsf{enc}}_{p_i}$ is an encryption key for a CPA-secure encryption scheme with XOR homomorphism
  - $\rho_{p_i} = \{\rho_{p_i,j}\}_{j\in N}$ is a set of N ℓ-bit strings.
- **Setup**: Each party p_i
  - Randomly selects a permutation on $2N$ elements π_{p_i} and generates its Waksman select bits s_{p_i}.
  - Samples $2N$ ℓ-bit masks $\sigma_{p_i} = \{\sigma_{p_i,j}\}_{j\in[2N]}$ uniformly at random.
  - Computes $V^H_{p_i} \leftarrow \mathsf{sort}(\mathsf{map}(V_{p_i}, \lambda(x)\colon H(x)), \lambda(x,y)\colon x < y)$, which applies H to each of p_i's vertices and then sorts the images lexicographically.
  - Computes $\mathcal{L}_{p_i} = \{(\mathsf{img} : V^H_{p_i,j}, \mathsf{lbl} : V^H_{p_i,j}, \mathsf{party} : i)\}_{j\in[N]}$
- **Revealing order under permutation**. Each party p_i:
  - Submits $\mathcal{L}_{p_i}$, σ_{p_i}, and s_i to the garbled circuit $\mathsf{GC}^{\mathsf{RevealOrder}}$, which merges $\mathcal{L}_{p_1}$ and $\mathcal{L}_{p_2}$ as $\mathcal{L}$, and permutes $\mathcal{L}$ as $\pi(\mathcal{L})$. Each party receives $2N$ ℓ-bit outputs from $\mathsf{GC}^{\mathsf{RevealOrder}}$. Let $o^{\mathsf{order}}_{p_i} = \{o^{\mathsf{order}}_{p_i,j}\}_{j\in[2N]}$ be the set of outputs received by party p_i.
  - Computes the indices of both parties' labels in the list $\pi(\mathcal{L})$ computed by $\mathsf{GC}^{\mathsf{RevealOrder}}$. p_i sets $\mathsf{out}_{p_i} \leftarrow \{j\colon o^{\mathsf{order}}_{p_i,j} \in V^H_{p_i}\}$, and sets $\mathsf{out}_{p_{3-i}} \leftarrow \{j\colon o^{\mathsf{order}}_{p_i,j} \notin V^H_{p_i}\}$, where out_{p_i} is the set of indices of p_i's vertices in $\pi(\mathcal{L})$.
  - Records the index of each of its vertices in $\pi(\mathcal{L})$. For all $v \in V_{p_i}$, assigns $\mathsf{idx}(v) \leftarrow j\colon o^{\mathsf{order}}_{p_i,j} = H(v)$.
  - Groups the indices in $o^{\mathsf{order}}_{p_i}$ by component. For each component $C \in G_{p_i}$, assign $\mathsf{idx}(C) \leftarrow \{\mathsf{idx}(v)\}_{v\in C}$.
  - Sends $I_{p_i} = \{\mathsf{idx}(C)\}_{C\in G_{p_i}}$ to p_{3-i}.
- **Percolate and Match**
  - The parties use and I_{p_1}, I_{p_2}, out_{p_1}, and out_{p_2} to hard-wire $\mathsf{GC}^{\mathsf{Perc\&Match}}_{I_{p_1}, I_{p_2}, \mathsf{out}_{p_1}, \mathsf{out}_{p_2}}$. Each party p_i submits ρ_{p_i} to the circuit and receives the other party's masked labels. Let $\{o^{\mathsf{pm}}_{p_i,j}\}_{j\in[N]}$ be p_i's final output from the garbled circuit.
- **Encrypting Labels**. Each party p_i:
  - Encrypts the other party's masked labels using $\mathsf{k}^{\mathsf{enc}}_{p_i}$ and sends them to p_{3-i}. For all $j \in [N]$, p_i computes $\phi_{p_{3-i},j} = \mathsf{CPA.enc}(\mathsf{k}^{\mathsf{enc}}_{p_i,1}, o^{\mathsf{pm}}_{p_i,j})$ and sends $\{\phi_{p_{3-i},j}\}_{j\in[N]}$ to p_{3-i}.
- **PostProcessing**. Each party p_i:
  - Removes the masks from the encrypted labels it receives. p_i receives $\{\phi_{p_i,j}\}_{j\in[N]}$ from p_{3-i}, and computes $\gamma_{p_i,j} \leftarrow \phi_{p_i,j} \oplus \rho_{p_i,j}$.
  - Maps its vertices to its encrypted output labels. p_i computes $Y \leftarrow \mathsf{sort}(V_{p_i}, \lambda(x,y)\colon \mathsf{idx}(x) < \mathsf{idx}(y))$. Then p_i constructs $\Lambda_{p_i} = \{(Y_j, \gamma_{p_i,j})\}_{j\in[N]}$.
- **Outputs** Each party p_i outputs $\{o^{\mathsf{pm}}_{p_i,j}\}_{j\in[N]}$ and Λ_{p_i}.

Fig. 9: Full Protocol for Secure Component Labeling.

5 Evaluation

5.1 Asymptotic Analysis

The offline cost of the protocol is dominated by setup and encryption phases. In the setup, sorting a list of N of vertices offline requires $O(N \log N)$ offline comparisons. During the encryption phase, each party encrypts the other party's N labels and performs N XOR operations to retrieve its own encrypted labels.

The garbled circuit performs the following computations for the percolate-and-match algorithm. Merging two sorted lists of size N requires $O(N \log N)$ oblivious comparisons using a Batcher merge. For each iteration of the loop, it computes C (the number of components) min-circuits over m-sized lists to perform the percolation phase. We can find the minimum of a list with m elements using m comparisons; therefore, in total the min circuits require $N = Cm$ comparisons per iteration. To perform matching in each iteration, we require $O(N)$ pairwise comparisons. In addition, each Waksman network requires $O(N \log N)$ oblivious swaps, and two permutation networks are computed per iteration. Therefore, each iteration of the loop

```
procedure RevealOrder($\mathcal{L}_{p_1}, \mathcal{L}_{p_2}, \sigma_{p_1}, \sigma_{p_2}, s_{p_1}, s_{p_2}$)
    // Merge, Permute, and Reveal the permuted index of each vertex to the vertex's owner
    $\mathcal{L} \leftarrow$ BatcherMerge($\mathcal{L}_{p_1}, \mathcal{L}_{p_2}$, CmpImg )
    $\mathcal{L} \leftarrow$ Permute($\mathcal{L}, s_{p_1}, s_{p_2}$)
    $o_{p_1} \leftarrow []_{2N}$, $o_{p_2} \leftarrow []_{2N}$
    for all $j \in [2N]$ do
        if $\mathcal{L}[j]$.party $= p_1$ then
            $o_{p_1}[j] \leftarrow \mathcal{L}[j]$.img
            $o_{p_2}[j] \leftarrow \mathcal{L}[j]$.img $\oplus \sigma_{p_1,j}$
        else
            $o_{p_1}[j] \leftarrow \mathcal{L}[j]$.img $\oplus \sigma_{p_2,j}$
            $o_{p_2}[j] \leftarrow \mathcal{L}[j]$.img
        end if
    end for
    Output $o_{p_1}$ to $p_1$ and $o_{p_2}$ to $p_2$
end procedure
procedure Perc&Match( $\mathcal{L}, \rho_{p_1}, \rho_{p_2}, s_{p_1}, s_{p_2}$ )
    // $I_{p_1}, I_{p_2}$, out$_{p_1}$, out$_{p_2}$ must be hard-coded
    // $\mathcal{L}$ is passed to this subcircuit already permuted
    for all $i \in [m-1]$ do
        $\mathcal{L} \leftarrow$ Percolate($\mathcal{L}, I_{p_1}$)
        $\mathcal{L} \leftarrow$ Percolate($\mathcal{L}, I_{p_2}$)
        $\mathcal{L} \leftarrow$ InvertPermute($\mathcal{L}, s_{p_1}, s_{p_2}$)
        $\mathcal{L} \leftarrow$ Match($\mathcal{L}$)
        $\mathcal{L} \leftarrow$ Permute($\mathcal{L}, s_{p_1}, s_{p_2}$)
    end for
    $o_{p_1} \leftarrow$MaskFixedOutputs($\mathcal{L}, \rho_{p_2}$, out$_{p_2}$)
    $o_{p_2} \leftarrow$MaskFixedOutputs($\mathcal{L}, \rho_{p_1}$, out$_{p_1}$)
    Output $o_{p_1}$ to $p_1$ and $o_{p_2}$ to $p_2$
end procedure
function Permute($L, s_1, s_2$)
    return Waksman(Waksman($L, s_1$), $s_2$)
end function
function InvertPermute($L, s_1, s_2$)
    return Waksman$^{-1}$(Waksman$^{-1}$($L, s_2$), $s_1$)
end function
function Percolate($\mathcal{L}, I$)
    // $I$ provides the indices in $\mathcal{L}$ that compose each component $C$ in some party's graph.
    for all idx($C$) $\in I$ do
        $\psi \leftarrow []_m$
        for all $j \in [m]$ do
            $\psi[j] \leftarrow \mathcal{L}[$idx$(C)[j]]$.lbl
        end for
        minlbl $\leftarrow \min_{j \in [m]} \psi[j]$
        for all $j \in [m]$ do
            $\mathcal{L}[$idx$(C)[j]]$.lbl $\leftarrow$ minlbl
        end for
    end for
    return $\mathcal{L}$
end function
function Match($\mathcal{L}$)
    for all $i \in [2N-1]$ do
        lbl$^* \leftarrow$ min($\mathcal{L}[i]$.lbl, $\mathcal{L}[i+1]$.lbl)
        if $\mathcal{L}[i]$.img $= \mathcal{L}[i+1]$.img then
            $\mathcal{L}[i]$.lbl $\leftarrow$ lbl$^*$
            $\mathcal{L}[i+1]$.lbl $\leftarrow$ lbl$^*$
        end if
    end for
    return $\mathcal{L}$
end function
function MaskFixedOutputs($\mathcal{L}, \rho$, out)
    $j \leftarrow 0$
    $o \leftarrow []_N$
    for all $i \in$ out do
        $o[j] \leftarrow \mathcal{L}[i] \oplus \rho[j]$
        $j \leftarrow j+1$
    end for
    return $o$
end function
function CmpImg($l_1, l_2$)
    return $l_1$.img $\leq l_2$.img
end function
```

Fig. 10: Garbled Circuit for Secure Component Labeling. The subcircuits $\mathsf{GC}^{\mathsf{RevealOrder}}$ and $\mathsf{GC}^{\mathsf{Perc\&Match}}$ are defined by the procedures RevealOrder and PercAndMatch. In $\mathsf{GC}^{\mathsf{Perc\&Match}}$, variables out$_{p_i}$ and I_{p_i} are public and must be hard-coded.

requires $O(N \log N)$ operations, and the iterative procedure loops $m-1$ times. In total, the garbled circuit performs $O(Nm \log N)$ comparisons and swaps. The circuit must also compute $2N$ conditional XOR operations for the first output to the two parties, and an addition N XORs for the final output.

The total cost of the protocol is dominated by the garbled circuit implementation of the percolate-and-match algorithm. The number of Yao gates depends on the output length ℓ of the hash function H because each comparison is performed over ℓ-bit values. The total cost of the circuit is therefore $O(Nm\ell \log N)$ gates. In Appendix D, we show how ℓ is set as a function of the input size N and the tolerable correctness error ε. Specifically, we set show that $\ell \geq \lceil 2\log(2N) - \log(\varepsilon) - 1 \rceil$, making the total size of the circuit $O(Nm \log(N)(\log(N) + \log(\frac{1}{\varepsilon})))$ gates.

5.2 Experiments

We implemented our protocol using Obliv-C [ZE15] and Absentminded Crypto Kit [Doe17], adding our own optimizations to Obliv-C. Our tests were performed in parallel on Google Compute Platform (GCP) on n1-highmem-32 (32 vCPUs with 208GB memory) machines between pairs of machines in the same datacenter. El-Gamal operations were performed over elliptic curve secp256r1. We present our experimental results in Appendix A.

5.3 Security Analysis

We prove our protocol secure in the honest-but-curious model. The proof is presented in Appendix C. Although the honest-but-curious model is admittedly weak, we believe it is a realistic model for our desired application. Adaptation to malicious security is future work.

6 Related Work

Private set intersection with "fuzzy" matches has been considered in previous research. An early work by Freedman, Nissim, and Pinkas on PSI included a proposed fuzzy matching protocol based on oblivious polynomial evaluation [FNP04]. Unfortunately that protocol had a subtle flaw identified by Chmielewski and Hoepman, who proposed solutions based on OPE, secret sharing, and private Hamming distance [CH08].

Wen and Dong presented a protocol solving the *private record linkage* problem, which is similar to the common identifiers problem in this work [WD14]. In that setting the goal is to determine which records in several databases correspond to the same person, and to then reveal those records. Wen and Dong present two approaches, one for exact matches using the *garbled bloom filter* technique from previous work on PSI [DCW13] and one for fuzzy matches that uses locality-sensitive hash functions [IM98] to build the bloom filter. One important difference between the PRL setting and ours is that our privacy goal requires the matches and non-matches to remain secret from both parties. We also assign a label to each record, with the property that when two records match they are assigned the same label.

Huang, Evans, and Katz compared the performance of custom PSI protocols to approaches based on garbled circuits [HEK12]. One of their constructions, which they call *sort-compare-shuffle*, is familiar to our approach. Our protocol uses the sort-compare-shuffle technique as a repeated subroutine. Unlike their constructions, our output is not a set intersection.

7 Conclusion and Future Work

We have presented a two-party protocol that can be used as a setup for subsequent PSI-style computations. Although we work in the honest-but-curious model, only small modifications would be needed to adapt the ID agreement protocol to the malicious model. That said, modifying a PSI protocol to securely compose with the ID agreement protocol in the malicious model appears challenging, and we leave it for future work.

Our ID-agreement protocol was designed for use with DDH-style PSI protocols. In particular, we rely on the fact that in DDH-style protocols it is straightforward to work with ElGamal encryptions by taking advantage of the homomorphism over the group operation. We believe similar techniques can be applied to other PSI paradigms, which we leave for future work.

In a real-world application it is possible that the parties will update their respective databases and require new encrypted labels for their modified rows. One approach to computing the updated labels would be to run the entire protocol again, but this would be expensive if the updates occur frequently. More efficiently updating labels without scaning over bother parties' entire inputs is an interesting future direction.

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A Experimental Results

For each problem size, we ran our tests for three label lengths, each length corresponding to a parameterization of the correctness error, as explained in Appendix D; specifically, we ran tests for parameters that bound the correctness error probability at 2^{-40}, 2^{-60}, and 2^{-80}. For each problem size N and correctness parameter ε, we computed the requisite bit length $\ell = \ell(N, \varepsilon)$ and then rounded up the label length to the next full byte. We then instantiated the function H by truncating SHA256 outputs to the first ℓ bits.

We summarize our experimental results in Figures 11a, 11b, 11c and 11d. We evaluated the performance of our generic the garbled circuit protocol (including outputting of masked labels), and present the results in Figures 11a, Figure 11b, and Figure 11c. In the generic protocol, encryptions of output labels were computed using AES in CTR mode. Each test was performed with $m = 4$ (maximum component size); Figure 11c contains the time per iteration of each percolate-and-match subcomponent, which can be used to roughly extrapolate to other values of m. All circuit sizes were run 5 times. In Figure 11d, we evaluate our El Gamal interface for composing with DDH-style PSI protocols. We benchmarked El-Gamal encryption for only three (smaller) problem sizes.

Importantly, our prototype implementation was not parallelized. We believe performance could be improved substantially by parallelizing, in particular because parts of the garbled circuit and the vast majority of the El-Gamal phase are embarrassingly parallel.

B Termination of Percolate-and-Match

Theorem 1 (Termination of Percolate-and-Match). *Let V be a set of vertices, let $G_{p_1} = (V_1, E_1)$, $G_{p_2} = (V_2, E_2)$ such that $V_1 \subset V$ and $V_2 \subset V$, and let $G = G_1 \cup G_2$. If the maximum size (by number of vertices) of a connected component in G is m, then in the worst case, $m-1$ iterations of the percolate-and-match algorithm are both necessary and sufficient for vertex labels to stabilize.*

Proof. We show that vertex labels stabilize after $m-1$ iterations, where m is a parameter denoting the largest component (by number of vertices) in $G = G_{p_1} \cup G_{p_2}$. We do so by analyzing how labels percolate between vertices during each iteration. If in the beginning an iteration, the label of some vertex $u \in V_p$ is lbl, and at the end of the iteration, the label of u is lbl', then we say lbl' *reaches u* during the iteration.

[6] We measured the iteration time twice per experiment and take the average as the iteration time for that experiment.

Input Size	Label Length (bits)	Non-Free Gates (billions)	Time (minutes)
4,096	72	0.21	1.53 ± 0.2
	88	0.26	1.76 ± 0.2
	112	0.33	2.4 ± 0.4
16,384	72	0.98	6.7 ± 0.8
	96	1.30	8.5 ± 1.0
	112	1.51	10.1 ± 0.8
65,536	80	4.88	29.8 ± 0.5
	96	5.85	39.2 ± 3.1
	120	7.29	49.0 ± 1.5
262,144	80	21.74	123.9 ± 3.9
	104	28.18	161.5 ± 8.4
	120	32.48	173.9 ± 3.3
1,048,576	88	105.25	590.5 ± 31.2
	104	124.18	740.5 ± 22.7
	128	152.58	824.8 ± 7.1
2,097,152	88	220.20	1280.8 ± 21.1
	104	259.82	1563.2 ± 140.8
	128	319.25	1928.8 ± 43.4

11a: Runtime evaluation of our generic protocol. For each experiment, we provide the number of non-free gates and time to evaluate with 95% confidence interval.

	Proportion of Execution Time		
Input Size	4,096	65,536	1,048,576
Merge	$0.15{\pm}1.6E^{-2}$	$0.15{\pm}1.0E^{-2}$	$0.15{\pm}2.0E^{-3}$
Permute	$0.55{\pm}3.6E^{-2}$	$0.58{\pm}1.9E^{-2}$	$0.61{\pm}1.5E^{-3}$
Percolate	$0.01{\pm}5.1E^{-3}$	$0.01{\pm}1.2E^{-3}$	$0.01{\pm}1.0E^{-4}$
Match	$0.06{\pm}2.7E^{-3}$	$0.05{\pm}3.3E^{-3}$	$0.04{\pm}4.8E^{-4}$

11b: Proportion of total execution time spent performing each component of our garbled circuit, with 95% confidence interval.

	Per-Iteration Time (minutes)		
Input Size	4,096	65,536	1,048,576
Minutes	$0.60{\pm}5.30E^{-2}$	$12.3{\pm}6.00E^{-1}$	217.4±1.97

11c: Time spent [6]per iteration of percolate and match, with 95% confidence interval.

Input Size	Label Length (bits)	El Gamal Time (minutes)
4,096	112	32.7 ±7.7
16,384	112	121.4 ±26.2
65,536	120	448.2 ±13.8

11d: Runtime evaluation, with 95% confidence interval, of the El Gamal interface for Diffie-Hellman Style PSI Protocols.

Fig. 12: Evaluation of our prototype implementation. We tested three different label lengths per input size; with lengths corresponding to correctness parameters (which upperbound the error probability) of 2^{-40}, 2^{-60}, and 2^{-80}. In Figure 11a, we present results for all three lengths. In Figures 11b, 11c, and 11d, we present results only for error probability of 2^{-80}.

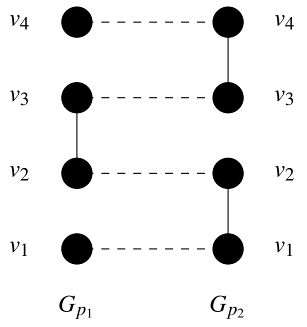

Fig. 13: Worst case example for the number of iterations until labels stabilize. v_1, v_2, v_3, and v_4 are in both G_{p_1} and G_{p_2}. Solid lines represent edges in G_{p_1} or G_{p_2}. Dotted lines represent matches during the matching phase. Three iterations are required to percolate a label from v_1 to v_4 in p_1's graph.

Consider any component $C \subset G$. Let v be the vertex in C with the minimum label in either party's initial labeling, and let its label be lbl^*. We show that in each iteration until labels stabilize, lbl^* must reach at least one new vertex in both G_{p_1} and G_{p_2}. As m denotes the maximum number of vertices in a component, lbl^* reaches every vertex in C in at most $m-1$ iterations.

If in any percolation phase, lbl^* does not reach a new vertex in either party's graph (meaning it does not reach a new vertex in G_{p_1} and it does not reach a new vertex in G_{p_2}), then the label has finished percolating because it cannot match to a new vertex in the following matching phase. Similarly, if during any matching phase, lbl^* does not match from a vertex in one party's graph to a vertex in the other party's graph which is not yet labeled lbl^*, then the label has finished percolating because it reaches no new components in the phase. Therefore, if there is ever an iteration of the algorithm in which lbl^* does not reach a new vertex in either party's graph, then iteration is complete. By contrapositive, while iteration is not complete, lbl^* must reach at least one new vertex in some party's graph during each percolation phase, and in each matching phase it must match to the same vertex in the other party's graph.

In Figure 13, we provide an example of a graph in which lbl^* reaches only one new vertex in each iteration. This completes the proof, as it shows that $m-1$ iterations are required for some graphs in which the maximum component size is m. One could analogously construct an example for any value of m. At each percolation phase, lbl^* reaches one new vertex, and in the following matching phase the label matches to a new sub-component in the other party's graph. The algorithm therefore takes $m-1$ iterations to percolate a label to all m vertices in C.

C Security Proof

Our protocol for component labeling achieves security in the honest-but-curious model with random oracles. We write the proof in a hybrid model in which the parties have access to a functionality $\mathcal{F}^{\mathsf{GC}}$ that takes the place of their garbled circuit evaluations. $\mathcal{F}^{\mathsf{GC}}$ takes the description of a circuit c and two parties' inputs and it returns the evaluation of c on those inputs to the parties, revealing the order of c's output gates. The parties invoke $\mathcal{F}^{\mathsf{GC}}$ to evaluate their garbled circuits.

We denote by $\mathcal{F}^{\mathsf{lbl}} = (\mathcal{F}_1^{\mathsf{lbl}}, \mathcal{F}_2^{\mathsf{lbl}})$ the two-party component labeling functionality. Recall that $\mathcal{F}_i^{\mathsf{lbl}}$ is a pair $(f_{p_{3-i}}^{\mathsf{mask}}(x,y), f_{p_i}^{\mathsf{enc}}(x,y))$. We denote by Π our component labeling protocol. Denote by $\mathsf{output}^{\Pi} = (\mathsf{output}_1^{\Pi}, \mathsf{output}_2^{\Pi})$ the pair of random variables describing each party's output of a real execution of Π.

Let $\mathsf{VIEW}_p^{\Pi}(x,y,\lambda)$ be the view of party p in a real execution of Π when p_1 has input x, p_2 has input y, and λ is the security parameter. Recall that $x = (G_{p_1}, \mathsf{k}_{p_1}^{\mathsf{enc}}, \rho_{p_1})$ and $y = (G_{p_2}, \mathsf{k}_{p_2}^{\mathsf{enc}}, \rho_{p_2})$. p_i's view in an execution of Π is composed of its input, its internal randomness r_{p_i}, and the messages it receives during the protocol.

Theorem 2 (Security with respect to honest-but-curious adversaries). *In the $\mathcal{F}^{\mathsf{GC}}$-hybrid model with random oracles, there exist PPT simulators $\mathcal{S}_1$ and $\mathcal{S}_2$ such that for all inputs x,y and security parameter λ:*

$$\{(\mathsf{VIEW}_{p_1}^{\Pi}(x,y,\lambda), \mathsf{output}^{\Pi}(x,y,\lambda))\}_{x,y,\lambda} \approx \\ \{(\mathcal{S}_1(1^{\lambda}, x, \mathcal{F}_1^{\mathsf{lbl}}(x,y)), \mathcal{F}^{\mathsf{lbl}}(x,y))\}_{x,y,\lambda}$$

$$\{(\mathsf{VIEW}_{p_2}^{\Pi}(x,y,\lambda), \mathsf{output}^{\Pi}(x,y,\lambda))\}_{x,y,\lambda} \approx \\ \{(\mathcal{S}_2(1^{\lambda}, y, \mathcal{F}_2^{\mathsf{lbl}}(x,y)), \mathcal{F}^{\mathsf{lbl}}(x,y))\}_{x,y,\lambda}$$

Proof. We describe how $\mathcal{S}_1$ simulates the view of p_1; $\mathcal{S}_2$ is analogous. At a high level, $\mathcal{S}_1$ generates a view by randomly sampling an input graph for p_2, and then faithfully simulating the interaction until the last step.

For the last step $\mathcal{S}_1$ uses its knowledge of p_1's ideal-functionality outputs to ensure that p_1's simulated view implies the correct outputs, and that the final messages sent by p_1 are consistent with the honest p_2's outputs.

Intuitively, the strategy works for $\mathcal{S}_1$ for the following reason. p_1 learns two kinds of information from the messages it receives. First, it learns encryptions of its own and the other party's vertex labels. For these messages, $\mathcal{S}_1$ generates encryptions of junk which are indistinguishable from p_1's real messages. Second, p_1 learns information about the ordering of the images of p_1 and p_2's vertices under a permutation π, which is defined by composing π_{p_1} and π_{p_2}, the permutations selected by the honest parties. Because each honest party randomly selects its permutation π_{p_i}, π is a random permutation as long as one participant is honest. Therefore, the permuted ordering is distributed identically to the ordering of the images of p_1's vertices and $\mathcal{S}_1$'s dummy vertices when $\mathcal{S}$ selects its own random permutation $\pi_{\mathcal{S}}$ and the images are permuted by $\pi' = \pi_{p_1} \circ \pi_{\mathcal{S}}$.

The description of $\mathcal{S}_1$ follows:

1. **Random Tapes:** $\mathcal{S}_1$ uniformly samples r_{p_1} as its random tape for the simulation of p_1 and r_{p_2} as its random tape for the emulation of p_2. (Each party's tape must be long enough to provide randomness for every encryption that it must compute, all of the one-time pads it must generate.)
2. **Simulated inputs for p_2:** $\mathcal{S}_1$ randomly samples an input graph $G_{\mathcal{S}}$ that it uses as input for p_2. It samples $G_{\mathcal{S}}$ by sampling N vertices $V_{\mathcal{S}} \leftarrow V$, and then randomly partitioning $V_{\mathcal{S}}$ into components of size m subject to the constraint that in the graph $G = G_{p_1} \cup G_{\mathcal{S}}$, there are no components of size larger than m. $\mathcal{S}_1$ does not randomly sample an encryption key to serve as $\mathsf{k}^{\mathsf{enc}}_{p_2}$ and it does not sample one time pads to serve as replacements for ρ_{p_2}.
3. **Honest Execution:** For every step of the protocol except for those in which p_1 receives its protocol outputs, $\mathcal{S}_1$ faithfully emulates the behavior of p_1 and p_2 in interaction with each other using inputs x for p_1, $G_{\mathcal{S}}$ as p_2's graph, and r_{p_1} and r_{p_2} as the parties' internal random tapes.
4. **Fixing p_1's outputs:** $\mathcal{S}_1$ deviates from its strategy of faithfully emulating the execution of p_1 and p_2 order to ensure that the view constructed for p_1 is consistent with the ideal-functionality output of p_1. Recall that $\mathcal{F}_1^{\mathsf{lbl}}(x,y) = (f_{p_2}^{\mathsf{mask}}(x,y), f_{p_1}^{\mathsf{enc}}(x,y))$. $\mathcal{S}_1$ fixes the messages received by p_1 as follows:
   (a) $f_{p_2}^{\mathsf{mask}}(x,y)$: $\mathcal{S}_1$ provides p_1 with $f_{p_2}^{\mathsf{mask}}$, which is p_2's ideal-functionality labels masked with random strings in p_2's input y.
   (b) $f_{p_1}^{\mathsf{enc}}(x,y)$: $\mathcal{S}_1$ masks the encrypted labels given by $\mathcal{F}_1^{\mathsf{lbl}}$ with ρ_{p_1}, which are the random strings in p_1's input designated for masking its final labels. (Recall that in a real execution, p_1 submits these to the garbled circuit.) Let e_j for $j \in [N]$ be the jth encrypted label given by $f_{p_1}^{\mathsf{enc}}$. $\mathcal{S}_1$ provides $e_j \oplus \rho_{p_1,j}$ for each masked-and-encrypted label in p_1's output.

We proceed to compare the distributions of messages that p_1 receives in a real execution with the distributions of messages that $\mathcal{S}_1$ constructs for p_1.

1. **p_1's first message:** In a real execution, p_1 receives $\{o_{p_i,j}^{\mathsf{order}}\}_{j \in [2N]}$. We divide these $2N$ strings into two sets of N strings. In the first set, the garbled circuit returns the images of p_1's vertices, randomly permuted. In the other set, p_1 receives the images of p_2's vertices under H, randomly permuted and masked by one-time pads generated by p_2. The first set in $\{o_{p_i,j}^{\mathsf{order}}\}_{j \in [2N]}$ reveal the indices of p_1's vertex images when sorted with p_2 vertex images and permuted by π.
   In $\mathcal{S}$'s generated view, the message received by p_1 is different in two ways. First, its vertices are permuted by some other random permutation π'; second, the messages it receives for p_2's vertices are the images of $V_{\mathcal{S}}$, masked by one-time pads generated by $\mathcal{S}_1$, rather than the images of V_{p_2} masked by pads generated by p_2.

2. **p_1's second message:** In a real execution, for each component $C \in G_{p_2}$, p_1 receives the indices of C in the permuted image of $V_{p_1}||V_{p_2}$. (Recall that each component is size m, so for each $C \in G_{p_2}$, p_1 receives m unique indices in $[2N]$.) In the view generated by $\mathcal{S}$, for each component $C \in G_{\mathcal{S}}$, p_1 receives the indices of C in the permuted form of $V_{p_1}||V_{\mathcal{S}}$.
3. **p_1's third message:** In a real execution, p_1 invokes $\mathcal{F}^{\mathsf{GC}}$ to evaluate $\mathsf{GC}^{\mathsf{Perc\&Match}}$ and receives the set of p_2's output labels, masked by p_2's one-time pads. $\mathcal{S}_1$ does not invoke $\mathcal{F}^{\mathsf{GC}}$, but simply provides p_1 with $f^{\mathsf{mask}}_{p_2}(x,y)$, which it learns from p_1's ideal-functionality output. The difference between the messages in the real and simulated execution is that in a real execution, these labels are images of the vertices in V_{p_1} under H and masked with ρ_{p_2}, while in the view generated by $\mathcal{S}_1$, the labels are sampled randomly by $\mathcal{F}^{\mathsf{lbl}}$ and masked with ρ_{p_2}.
4. **p_1's fourth message:** In a real execution, p_1 receives $\{\phi_{p_1,j}\}_{j\in[N]}$ from p_2, where each $\phi_{p_1,j} = \mathsf{enc}(\mathsf{k}^{\mathsf{enc}}_{p_2}, o_j) = \mathsf{enc}(\mathsf{k}^{\mathsf{enc}}_{p_2}, \mathsf{lbl}_j \oplus \rho_{p_1,j}) = \mathsf{enc}(\mathsf{k}^{\mathsf{enc}}_{p_2}, \mathsf{lbl}_j) \oplus \rho_{p_1,j}$. p_1 removes its masks, after which it has N labels encrypted under $\mathsf{k}^{\mathsf{enc}}_{p_2}$.
   $\mathcal{S}_1$ fixes this message to ensure that p_1's simulated view is consistent with $f^{\mathsf{enc}}_{p_1}$, which it learns from $\mathcal{F}^{\mathsf{lbl}}$. For $j \in [N]$, $\mathcal{S}_1$ provides p_1 with $e_j \oplus \rho_{p_1,j}$. (In a real execution, an honest p_1 would remove the masks $\rho_{p_1,j}$ to derive its encrypted labels.)

To satisfy the definition of security, we must also show that the view generated by $\mathcal{S}_1$ is consistent with the ideal functionality outputs of p_2. Recall that $\mathcal{F}^{\mathsf{lbl}}_2(x,y) = (f^{\mathsf{mask}}_{p_1}(x,y), f^{\mathsf{enc}}_{p_2}(x,y))$.

1. $f^{\mathsf{enc}}_{p_2}(x,y)$: The consistency of this output of p_2 with the view output by $\mathcal{S}_1$ is implied by the fact that $\mathcal{S}_1$ fixes p_1's penultimate message to be exactly $f^{\mathsf{mask}}_{p_2}$, which it receives from the ideal functionality. Recall that in a real execution, p_1 encrypts the padded outputs it receives from the final garbled circuit using its encryption key $\mathsf{k}^{\mathsf{enc}}_{p_1}$, and then p_2 removes the pads ρ_{p_2} from these encryptions after receiving them from p_1. p_2's output $f^{\mathsf{enc}}_{p_2}$ is precisely the set of encryptions yielded by removing the pads ρ_{p_2} from the encryptions sent by p_1.
   Therefore, in the simulated view, $\mathcal{S}_1$ provides p_1 with precisely the masked pads that p_1 would then encrypt and send back to p_2; this is the message $f^{\mathsf{mask}}_{p_2}(x,y)$ which $\mathcal{S}$ receives from $\mathcal{F}^{\mathsf{lbl}}$ as part of p_1's output and forwards to p_1. The consistency of p_2's ideal functionality output with the message sent by p_1 follows directly from the fact that the output of p_2 is a set of encryptions of the messages that $\mathcal{S}$ forwards to p_1 from the ideal functionality.
2. $f^{\mathsf{mask}}_{p_1}(x,y)$: This is the set of labels of p_1's vertices, masked with p_1's one-time pads, which p_2 outputs. Recall that in a real execution, p_2 receives these masked labels from $\mathsf{GC}^{\mathsf{Perc\&Match}}$, encrypts them using its encryption key, and sends the encryptions to p_1, who removes the masks and outputs the encryptions. This output must be consistent with
   (a) the message that p_2 computes as a function of $f^{\mathsf{mask}}_{p_1}(x,y)$ and sends to p_1 as p_1's final message. (Specifically, this is the set of encryptions that p_2 computes and sends to p_1 in order to unmask and output.)
   (b) the garbled circuit inputs that are in p_1's view.

   Consistency with p_1's final message: In order for $f^{\mathsf{mask}}_{p_1}(x,y)$ to be consistent with p_1's final message in the view generated by $\mathcal{S}_1$, it must be the case that the message p_1 receives is composed of encryptions of $f^{\mathsf{mask}}_{p_1}(x,y)$ under p_2's encryption key.
   This is the case, since $\mathcal{S}_1$ fixes p_1's final message to be the pairwise XOR of $f^{\mathsf{enc}}_{p_1}$ with ρ_{p_1}, where $f^{\mathsf{enc}}_{p_1}$ is given to $\mathcal{S}$ by $\mathcal{F}^{\mathsf{lbl}}$, and $f^{\mathsf{enc}}_{p_1}$ is defined to be the result of encrypting $f^{\mathsf{mask}}_{p_1}$ and then homomorphically removing the pads ρ_{p_1}.

Consistency with GC Inputs: All of p_2's inputs to $\mathsf{GC}^{\mathsf{Perc\&Match}}$ are outside of p_1's view. Therefore, $f_{p_1}^{\mathsf{mask}}(x,y)$ must be consistent with the inputs to the garbled circuit which are in p_1's simulated view. p_1 has two inputs to $\mathsf{GC}^{\mathsf{Perc\&Match}}$: $\mathcal{L}_{p_1}$ and ρ_{p_1}. ρ_{p_1} is specified by p_1's input, and $\mathcal{L}_{p_1}$ is a function of p_1's input vertices and the random oracle H.
Consider that from p_1's inputs (specifically its one-time pads ρ_{p_1}) and $f_{p_1}^{\mathsf{mask}}(x,y)$, it is possible to learn p_1's true vertex labels and to map p_1's vertices to their final labels. Therefore, $f_{p_1}^{\mathsf{mask}}$ and the mapping of p_1's vertices to their final labels must be consistent with p_1's inputs. This follows directly from the correctness of $f_{p_1}^{\mathsf{mask}}$ provided by $\mathcal{F}^{\mathsf{lbl}}$, and the fact that $\mathcal{S}$ can program the random oracle so that queries on p_1's vertices give images that correspond to the output prescribed by $f_{p_1}^{\mathsf{mask}}$.

We proceed with the full proof of security in presence of a corrupt p_1; specifically, in the $\mathcal{F}^{\mathsf{GC}}$-hybrid model, there exists a PPT simulator $\mathcal{S}_1$ such that for all inputs x,y and security parameters λ:

$$\{(\mathsf{VIEW}_{p_1}^{\Pi}(x,y,\lambda), \mathsf{output}^{\Pi}(x,y,\lambda))\}_{x,y,\lambda} \approx$$
$$\{(\mathcal{S}_1(1^{\lambda}, x, \mathcal{F}_1^{\mathsf{lbl}}(x,y)), \mathcal{F}^{\mathsf{lbl}}(x,y))\}_{x,y,\lambda}$$

The proof follows from a hybrid argument. In each hybrid we give a simulator that produces a random variable describing the view of p_1 in either a real or simulated execution.

Hyb_0 This is the viewproduced by a simulator $\mathcal{S}$ that knows p_1's and p_2's inputs and faithfully executes the protocol on their behalves using their inputs. It is distributed identically to $(\mathsf{VIEW}_{p_1}^{\Pi}(x,y,\lambda)$

Hyb_1 This identical to Hyb_0, except for the order in which p_1 receives its outputs from $\mathcal{F}^{\mathsf{GC}}$ on the evaluation of $\mathsf{GC}^{\mathsf{RevealOrder}}$ and for the indices that it receives in its second message.
Order in first message: Recall that in a real execution, p_1 and p_2 each select $2N$ one-time pads σ_{p_i} and Waksman selection bits s_{p_i} and submit these along with their vertices to $\mathcal{F}^{\mathsf{GC}}$ in order to evalaute $\mathsf{GC}^{\mathsf{RevealOrder}}$. In this hybrid, instead of delivering p_1's output in the order corresponding to $\mathcal{F}^{\mathsf{GC}}(\mathsf{GC}^{\mathsf{RevealOrder}}, \mathcal{L}_{p_1}, \mathcal{L}_{p_2}, s_{p_1}, s_{p_2}, \sigma_{p_1}, \sigma_{p_2})$ (as in a real execution), $\mathcal{S}$ randomly samples a graph $G_{\mathcal{S}}$, Waksman select bits $s_{\mathcal{S}}$, and one-time pads $\sigma_{\mathcal{S}}$ that it uses as input for p_2, and additionally invokes $\mathcal{F}^{\mathsf{GC}}(\mathsf{GC}^{\mathsf{RevealOrder}}, \mathcal{L}_{p_1}, \mathcal{L}_{\mathcal{S}}, s_{p_1}, s_{\mathcal{S}}, \sigma_{p_1}, \sigma_{\mathcal{S}}))$. $\mathcal{S}$ then changes the indices in which p_1 receives its own labels as if receiving output from $\mathcal{F}^{\mathsf{GC}}(\mathsf{GC}^{\mathsf{RevealOrder}}, \mathcal{L}_{p_1}, \mathcal{L}_{\mathcal{S}}, s_{p_1}, s_{\mathcal{S}}, \sigma_{p_1}, \sigma_{\mathcal{S}})$. (Note that $\mathcal{S}$ does not actually use the outputs of $\mathcal{F}^{\mathsf{GC}}(\mathsf{GC}^{\mathsf{RevealOrder}}, \mathcal{L}_{p_1}, \mathcal{L}_{\mathcal{S}}, s_{p_1}, s_{\mathcal{S}}, \sigma_{p_1}, \sigma_{\mathcal{S}})$, but it delivers p_1's outputs from $\mathcal{F}^{\mathsf{GC}}(\mathsf{GC}^{\mathsf{RevealOrder}}, \mathcal{L}_{p_1}, \mathcal{L}_{p_2}, s_{p_1}, s_{p_2}, \sigma_{p_1}, \sigma_{p_2})$ in the order defined by $\mathcal{F}^{\mathsf{GC}}(\mathsf{GC}^{\mathsf{RevealOrder}}, \mathcal{L}_{p_1}, \mathcal{L}_{\mathcal{S}}, s_{p_1}, s_{\mathcal{S}}, \sigma_{p_1}, \sigma_{\mathcal{S}})$.)
To better discuss the order in which labels are output, we define an *n-ordering* to be a list containing an n-sized subset of $\{1, \ldots 2n\}$. We define a procedure $O(\pi, l_1, l_2)$ which takes a permutation π and two lists l_1 and l_2 such that $|l_1| = |l_2| = n$ and no list l_i contains duplicate elements (although the two lists have elements in common with each other). O constructs $l_3 = l_1 || l_2$, and then sorts l_3. Finally, it applies π to l_3. O returns an n-ordering which for each element of l_1 gives its index in l_3.
Observe that the indices of p_1's outputs of $\mathcal{F}^{\mathsf{GC}}$ in Hyb_0 correspond to $O(\pi, V_1, V_2)$ and that the indices of p_1's outputs of $\mathcal{F}^{\mathsf{GC}}$ in Hyb_1 correspond to $O(\pi', V_1, V_{\mathcal{S}})$, where $\pi = \pi_1 \circ \pi_2$ and $\pi' = \pi_1 \circ \pi_{\mathcal{S}}$. Because both π and π' are random, the n-orderings describing p_1's indices in Hyb_0 and Hyb_1 are independent of both the third argument to O and the contents of l_1 and l_2. It follows that p_1's first message is identically distributed in Hyb_0 and Hyb_1.
Indices in second message: p_1's second message contains random assignments of p_2's indices into k-sized components. As we argued for p_1's first message, the division of p_2's indices into components is independent of p_2's vertices because the permutation on $V_{p_1} || V_{p_2}$ (or $V_{p_1} || V_{\mathcal{S}}$) induced by π (or π') is

random. It follows that the message p_1 receives in Hyb_0 is distributed identically to the message that it receives in Hyb_1. Thus, p_2's second message is identically distributed in Hyb_0 and Hyb_1, and Hyb_0 is distributed identically to Hyb_1.

Hyb_2 This is identical to Hyb_1, except for the outputs that p_1 receives from invoking $\mathcal{F}^{\mathsf{GC}}$ on $\mathsf{GC}^{\mathsf{RevealOrder}}$. $\mathcal{S}$ invokes $\mathcal{F}^{\mathsf{GC}}$ on $\mathsf{GC}^{\mathsf{RevealOrder}}$ with p_1's inputs and the dummy inputs that it generates for p_2. In contrast to Hyb_1, $\mathcal{S}$ no longer discards the output of $\mathcal{F}^{\mathsf{GC}}(\mathsf{GC}^{\mathsf{RevealOrder}}, \mathcal{L}_{p_1}, \mathcal{L}_{\mathcal{S}}, s_{p_1}, s_{\mathcal{S}}, \sigma_{p_1}, \sigma_{\mathcal{S}})$ but uses it in place of the output of $\mathcal{F}^{\mathsf{GC}}(\mathsf{GC}^{\mathsf{RevealOrder}}, \mathcal{L}_{p_1}, \mathcal{L}_{p_2}, s_{p_1}, s_{p_2}, \sigma_{p_1}, \sigma_{p_2})$

We divide our analysis of p_1's first message in Hyb_1 and Hyb_2 into two parts. First we consider only those indices for which p_1 receives one of p_2's masked labels. Second we consider only those indices for which p_1 receives its own labels.

The set of strings corresponding to p_2's indices in Hyb_1 and in Hyb_2 are identically distributed, since both σ_{p_2} and $\sigma_{\mathcal{S}}$ are one-time pads sampled uniformly at random, and mask the labels output to p_1 which are not its own. The distributions of these masked strings that p_1 receives at p_2's indices in Hyb_1 and Hyb_2 are both identical to the uniform random distribution.

For all the indices for which p_1 receives one of its own labels, the set of strings it receives in Hyb_2 is distributed identically to the set that p_1 receives in Hyb_1. In both cases it receives precisely the images of its own vertex labels under the random oracle H. We conclude that Hyb_2 is distributed identically to Hyb_1.

Hyb_3 This is identical to Hyb_2, except that the third message received by p_1 is replaced by $f_{p_2}^{\mathsf{mask}}$, which $\mathcal{S}$ learns from p_1's ideal-functionality output. This takes the place of the output of $\mathcal{F}^{\mathsf{GC}}(\mathsf{GC}^{\mathsf{Perc\&Match}}, \cdots)$, which $\mathcal{S}$ does not invoke in this Hybrid. Because in both Hyb_2 and Hyb_3, p_1 receives the other party's vertex labels masked by ρ_{p_2}, this message in Hyb_3 is identically distributed to Hyb_2 if the labels generated by $\mathcal{F}^{\mathsf{lbl}}$ are identically distributed to the vertex labels constructed in a real execution. Because in a real execution, the parties invoke a random oracle to compute the images of their vertices, the images of their vertices in a real execution are uniformly randomly distributed. However, this is not identical to an interaction with $\mathcal{F}^{\mathsf{lbl}}$, since $\mathcal{F}^{\mathsf{lbl}}$ samples random labels subject to the constraint that if two vertices are not in the same component, they are assigned different labels, while the hash function is subject to the birthday bound. However, the output space of the random oracle can be chosen to be large enough that the probability of a collision is negligible. Therefore, for appropriately chosen parameters, Hyb_2 and Hyb_3 are computationally indistinguishable.

Hyb_4 This is identical to Hyb_3, except that the fourth message received by p_1 is replaced by the element-wise xor of $f_{p_1}^{\mathsf{enc}}$ and ρ_{p_1}. In both Hyb_3 and Hyb_4, p_1 receives encryptions of its output labels, masked with ρ_{p_1}. Remove the pad from each using ρ_{p_1}, and p_1 has N encryptions of plaintext labels under $\mathsf{k}_{p_2}^{\mathsf{enc}}$. By the semantic security of the encryption scheme, these are computationally indistinguishable.

Hyb_4 is identical to $\mathcal{S}_1(1^\lambda, x, \mathcal{F}_1^{\mathsf{lbl}}(x,y))$; therefore, as we have already shown that the view output by $\mathcal{S}_1$ is consistent with the ideal functionality output, we conclude that $\{(\mathcal{S}_1(1^\lambda, x, \mathcal{F}_1^{\mathsf{lbl}}(x,y)), \mathcal{F}^{\mathsf{lbl}}(x,y))\}_{x,y} \approx_c \{(\mathsf{VIEW}_{p_1}^{\Pi}(x,y,\lambda), \mathsf{output}^{\Pi}(x,y))\}_{x,y}$. The simulator and analysis for a corrupt p_2 is analogous. This concludes the proof.

D Fine-Grained Correctness

We observe that if an application does not require that correctness holds except with negligible probability, then it is possible to tune the protocol to allow error with some tolerable probability ε in order to improve efficiency. Specifically, we show that we can set the bit length ℓ of the labels assigned to vertices as a function

Modified Two-Party Component Labeling Functionality $\mathcal{F}_{\varepsilon}^{\mathsf{lbl}}$

1. **Inputs:** Each party p_i has a tuple $\mathsf{input}_{p_i} = (G_{p_i}, \mathsf{k}_{p_i}^{\mathsf{enc}}, \rho_{p_i})$, where $G_{p_i} = (V_{p_i}, E_{p_i})$ is a graph, $\mathsf{k}_{p_i}^{\mathsf{enc}}$ is an encryption key for a CPA-secure encryption scheme (referred to as CPA), and $\rho_{p_i} = \{\rho_{p_i,j}\}_{j\in[N]}$ is a list of N ℓ-bit one-time pads.
2. **Generate Labels:** Upon receiving $(\mathsf{submit}, \mathsf{sid}, \mathsf{input}_{p_i})$ from each p_i for session sid, $\mathcal{F}_{\varepsilon}^{\mathsf{lbl}}$ samples elements from a universe $\mathcal{U} = \{0,1\}^{\ell}$ and computes a map $\Lambda \colon V_{p_1} \cup V_{p_2} \to \mathcal{U}$ such that with probability $1-\varepsilon$, it is the case that for every two vertices $v_1, v_2 \in V_{p_1} \cup V_{p_2}$, $\Lambda(v_1) = \Lambda(v_2)$ if and only if v_1 and v_2 are in the same connected component in $G_1 \cup G_2$. For $i \in \{1,2\}$ and $j \in [|V_{p_i}|]$, let $v_{p_i,j}$ be the jth vertex in V_{p_i}. $\mathcal{F}^{\mathsf{lbl}}$ computes $f_{p_i}^{\mathsf{enc}} = \{\mathsf{CPA.enc}(\mathsf{k}_{p_i}^{\mathsf{enc}}, \Lambda(v_{p_i,j}))\}_{j\in[|V_{p_i}|]}$ and $f_{p_i}^{\mathsf{mask}} = \{\rho_{p_i,j} \oplus \Lambda(v_{p_i,j})\}_{j\in[|V_{p_i}|]}$
   $\mathcal{F}^{\mathsf{lbl}}$ sends $(\mathsf{labeling}, \mathsf{sid}, f_{p_1}^{\mathsf{enc}}, f_{p_2}^{\mathsf{mask}})$ to p_1 and $(\mathsf{labeling}, \mathsf{sid}, f_{p_2}^{\mathsf{enc}}, f_{p_1}^{\mathsf{mask}})$ to p_2.

Fig. 14: Modified Two-Party Component Labeling Functionality $\mathcal{F}_{\varepsilon}^{\mathsf{lbl}}$

of ε. Reducing bit length of the labels directly improves the cost of the protocol, as the number of gates in the garbled circuit is linear in the bit length of the labels.

Recall that correctness of our protocol requires that for every pair of vertices u, v in $G = G_{p_1} \cup G_{p_2}$, u and v assigned the same label if and only if they are in the same component in G. In our protocol, if u and v are in the same component in G, then they are assigned the same label by construction. If they are not in the same component, then they may be assigned the same label only if there is a spurious collision in the images of two unconnected vertices under H.

Our analysis is an application of the birthday bound. Recall that each party has a graph of size N, and consider that H is a random oracle. Let $\mathsf{collision}$ be the event that any two distinct vertices in G are mapped to the same image under H. By the birthday bound, it follows that

$$\mathsf{Pr}[\mathsf{collision}] \leq \frac{(2N)^2}{2 * 2^{\ell}} \tag{1}$$

where N is the number of vertices in each participant's graph, and ℓ is the output length of H.

Therefore, it is possible to achieve correctness with probability $1-\varepsilon$ by upperbounding the probability of a spurious collision by ε. To do so, we set $\ell = \lceil 2\log(2N) - \log(\varepsilon) - 1 \rceil$. In our experiments, we additionally round ℓ up to the next full byte, guaranteeing that the probability of collision is less than the desired parameter.

The Ideal Functionality

In Figure 14, we present a new ideal functionality parameterized by ε, which represents the probability of correctness error. The proof needs to be modified only in the transition from Hybrid 2 to Hybrid 3. In particular, the probability of a spurious collision in the real game is precisely the ε that we use to parameterize the error probability of the ideal functionality. In the real game, a spurious collision is the only way that vertices which are not members of the same component are assigned the same label; therefore, the probability of error in the real game and the consequences of such an error are identical to the probability of error by the ideal functionality and the consequences of such an error by the ideal functionality.