Two-Round Oblivious Linear Evaluation from Learning with Errors

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Abstract

Oblivious Linear Evaluation (OLE) is a simple yet powerful cryptographic primitive which allows a sender, holding an affine function $f(x) = a + bx$ over a finite field, to let a receiver learn $f(w)$ for a w of the receiver's choice. In terms of security, the sender remains oblivious of the receiver's input w, whereas the receiver learns nothing beyond $f(w)$ about f. In recent years, OLE has emerged as an essential building block to construct efficient, reusable and maliciously-secure two-party computation.

In this work, we present efficient two-round protocols for OLE based on the Learning with Errors (LWE) assumption. Our first protocol for OLE is secure against malicious unbounded receivers and semi-honest senders. The receiver's first message is reusable, meaning that it can be reused over several executions of the protocol, and it may carry information about a batch of inputs, and not just a single input. We then show how we can extend the above protocol to provide malicious security for both parties, albeit at the cost of reusability.

1 Introduction

Oblivious Linear Evaluation (OLE) is a cryptographic primitive between a sender and a receiver, where the sender inputs an affine function $f(x) = a + bx$ over a field $\mathbb{F}$, the receiver inputs an element $w \in \mathbb{F}$, and in the end the receiver learns $f(w)$. The sender remains oblivious of the receiver's input w and the receiver learns nothing beyond $f(w)$ about f. OLE can be seen as a generalization of the well-known Oblivious Transfer (OT) primitive.[1] In fact, just as secure computation of *Boolean* circuits can be based on OT, secure computation of *arithmetic* circuits can be based on OLE [GMW87, IPS09].

In recent years, OLE has emerged as one of the most promising avenues to realize efficient two-party secure computation in different settings [IPS09,

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[1] It is easy to see that, if we consider the affine function $f : \{0,1\} \to \{0,1\}$ such that $f(b) = m_0 + b(m_1 - m_0)$, OLE trivially implements OT.

ADI+17, DGN+17, BCGI18, HIMV19, CDI+19]. Interestingly, OLE has found applications, not just in the secure computation of generic functions, but also in specific tasks such as Private Set Intersection [GN19, GS19] or Machine Learning related tasks [MZ17, JVC18].

One aspect which sets OLE apart from OT is *reusability*, meaning that the first message of a protocol is reusable across multiple executions. While two-party *reusable* non-interactive secure computation (NISC) is impossible in the OT-hybrid model [CDI+19], reusable NISC for general Boolean circuits is known to be possible in the (reusable) OLE-hybrid model assuming one-way functions [CDI+19]. The result stated above is meaningful only if we have access to a reusable two-round OLE protocol. However, the only efficient realizations of this primitive are based on the Decisional Composite Residuosity (DCR) and the Quadratic Residuosity assumptions [CDI+19] and thus insecure in the presence of a quantum computer.

Given this state of affairs and the importance of OLE in constructing two -party secure computation protocols, we ask the following question:

Can we build efficient and reusable two-round OLE protocols from (presumed) post-quantum hardness assumptions?

1.1 Our Results

In this work, we partially answer the question above. Specifically, we present two simple, efficient and round-optimal protocols for OLE based on the hardness of the Learning with Errors (LWE) assumption [Reg05] with super-polynomial modulus-to-noise ratio, which is conjectured to be post-quantum secure.

Before we start we clarify what type of OLE we obtain. OLE comes in many flavors, one of the most useful being *vector OLE* where the sender inputs two vectors $a = \mathbf{a}, b = \mathbf{b} \in \mathbb{F}_q^\ell$ and the receiver obtains a linear combination of them $\mathbf{z} = \mathbf{a} + w\mathbf{b} \in \mathbb{F}_q^\ell$ [BCGI18]. For simplicity, we just refer to this variant as OLE.

Both of our protocols implement the functionality in just two-rounds and have the following properties:

- Our first protocol (Section 4) for OLE achieves statistical security against a corrupted receiver and computational semi-honest security against a corrupted sender. Additionally, the receiver's first message is reusable in the sense of [CDI+19]. This means that the first message sent by the receiver can be reused across several runs of the protocol. Additionally, we show how we can extend this protocol to implement *batch reusable OLE*, a functionality similar to OLE where the receiver can input a batch of values $\{x_i\}_{i\in[k']}$, instead of just one value.

- We then show how to extend the above protocol to provide malicious security for both parties, at the cost of reusability (Section 5). The protocol makes λ invocations of a two-round Oblivious Transfer protocol (which exists under LWE [PVW08, DGH+20]), where λ is the security param-

eter. Instantiate the OT with the LWE-based protocol of [PVW08], we preserve statistical security against a malicious receiver.

1.2 Related Work and Comparison

In the following, we briefly review some proposals from prior work and compare them with our proposal. We only consider schemes that are provable UC-secure as our protocols. OLE can be trivially implemented using Fully/Somewhat Homomorphic Encryption (e.g., [JVC18]) or using generic solutions for two-party secure computation (such as [GS18, BL18]). However, these solutions fall short in achieving a *decent* level of security and/or efficiency.

The work of Döttling et al. [DKM12, DKMQ12] proposed an OLE protocol with unconditional security, in the stateful tamper-proof hardware model. The protocol takes only two rounds, however further interaction with the token is needed by the parties.

In [IPS09], a semi-honest protocol for oblivious multiplication was proposed, which can be easily extended to a OLE protocol. The protocol is based on noisy encodings. Based on the same assumption, [GNN17] proposed a maliciously-secure OLE protocol, which extends the techniques of [IPS09]. However, their protocol takes eight rounds of interaction.

Chase et al. [CDI+19] presented a round-optimal reusable OLE protocol based on the Decisional Composite Residuosity (DCR) and the Quadratic Residuosity (QR) assumptions. The protocol is maliciously-secure and, to the best of our knowledge, it is the most efficient protocol for OLE proposed so far. However, it is well-known that both the DCR and the QR problems are quantumly insecure.

We also remark that our protocols implement vector OLE where the sender's input are vectors over a field, as in [GNN17].

In Table 1, a brief comparison between several UC-secure OLE protocols is presented.

	Hardness Assumption	Setup Assumption	Rounds	Reusability	Security
[IPS09]	Noisy Encodings	OT	3	-	semi-honest
[DKM12]	2	Stateful tamper proof hardware	-	-	malicious
[GNN17]	Noisy Encodings	OT	8	-	malicious
[CDI+19]	DCR & QR	CRS	2	✓	malicious
This work	LWE	CRS	2	✓	malicious receiver
	LWE	CRS & OT	2	✗	malicious

Table 1: Comparison between different OLE schemes.

1.3 Open Problems

Our fully maliciously-secure protocol (in Section 5) does not have reusability of the first message. Hence, the main open problem left in our work is the following: Can we construct a reusable maliciously-secure two-round OLE protocol based on the LWE assumption?

Also, remark that our protocols are secure assuming the hardness of LWE with *super-polynomial modulus-to-noise ratio*. Can we build an efficient and reusable OLE protocol whose hardness is based on the LWE with *polynomial modulus-to-noise ratio*?

2 Technical Outline

In this section, we give a brief overview of our protocols.

2.1 A Two-Round Semi-Honest Protocol

In our protocol, both the sender S and the receiver R have access to a common reference string $\mathsf{crs} = \mathbf{A} \leftarrow\!\!\$\, \mathbb{Z}_q^{k\times n}$. The core idea of our protocol is the following:

1. R computes $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$ where $\mathbf{e}$ is a short noise vector and the i-th coordinate $x_i \in \mathbb{Z}_q$ of $\mathbf{x}$ corresponds to its input.
2. S samples a *short* matrix $\mathbf{R}$ and computes the pair $(\mathbf{s}_0 = \mathbf{a}'\mathbf{R}, \mathbf{s}_1 = \mathbf{a}_i\mathbf{R})$ where $\mathbf{a}_i$ is the i-th row of $\mathbf{A}$. It sends $\mathbf{A}_{-i}\mathbf{R}$ to the receiver, where $\mathbf{A}_{-i}$ is the matrix $\mathbf{A}$ with the i-th row removed.
3. R computes $\tilde{\mathbf{y}} = \mathbf{x}_{-i}\mathbf{C}$ where $\mathbf{x}_{-i}$ is the vector $\mathbf{x}$ with the i-th coordinate removed.

Observe that

$$\begin{aligned}\tilde{\mathbf{y}} &= \mathbf{x}_{-i}\mathbf{C} = \mathbf{x}_{-i}\mathbf{A}_{-i}\mathbf{R} = (\mathbf{x}\mathbf{A} - x_i\mathbf{a}_i)\mathbf{R} \\ &= (\mathbf{x}\mathbf{A} + \mathbf{e})\mathbf{R} - (x_i\mathbf{a}_i)\mathbf{R} - \mathbf{e}\mathbf{R} = \mathbf{s}_0 + x_i\mathbf{s}_1 + \mathbf{e}'\end{aligned}$$

for some short vector $\tilde{\mathbf{e}}$. Hence, the vector obtained by R is a linear combination of $(\mathbf{s}_0, \mathbf{s}_1)$ up to some *noise*, which can be corrected using an error-correcting code (ECC) as we will see below.

We can extend this idea into a fully functional protocol as follows, by making additional use of a linear error correcting code (ECC) against short errors in the euclidean norm.

1. R chooses a random $\mathbf{x} \leftarrow\!\!\$\, \mathbb{Z}_q^k$ such that x_i (i.e., the i-th coordinate of $\mathbf{x}$) corresponds to R's input and compute an LWE sample $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$ for an error vector $\mathbf{e}$.
2. Upon receiving $\mathbf{a}'$ from R, S chooses a *short* matrix $\mathbf{R}$ and computes $\mathbf{s}_0 = \mathbf{a}'\mathbf{R}$ and $\mathbf{s}_1 = \mathbf{a}_i\mathbf{R}$. Moreover, it sends $\mathbf{C} = \mathbf{A}_{-i}\mathbf{R}$, $\mathbf{t}_0 = \mathbf{s}_0 + \tilde{\mathbf{e}} + \hat{\mathbf{z}}_0$ and $\mathbf{t}_1 = \mathbf{s}_1 + \hat{\mathbf{z}}_1$ where $(\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_1)$ is an encoding of S's inputs $(\mathbf{z}_0, \mathbf{z}_1)$.

3. Upon receiving $\mathbf{C}$, R computes

$$\tilde{\mathbf{y}} = \mathbf{x}_{-i}\mathbf{C} - (\mathbf{t}_0 - x_i\mathbf{t}_1) = (\hat{\mathbf{z}}_0 + x_i\hat{\mathbf{z}}_1) + \mathbf{e}'.$$

Then, it decodes $\tilde{\mathbf{y}}$ to obtain $\mathbf{y}$.

Security against a semi-honest sender can be routinely established from the LWE assumption. To argue that the protocol is secure against a semi-honest receiver, we can show that conditioned on $\mathbf{y} = \mathbf{z}_0 + x_i\mathbf{z}_1$ the values $(\mathbf{z}_0, \mathbf{z}_1)$ are statistically hidden from the view of R. This argument relies on the fact that $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$ is *well-formed*, that $\mathbf{C} = \mathbf{A}_{-i}\mathbf{R}$ is statistically close to uniform (which can be established via the Smoothing Lemma [MR07]), and that the error term $\tilde{\mathbf{e}}$ *drowns* the residual term $\mathbf{e}\mathbf{R}$. But this means that $\tilde{\mathbf{e}}$ needs to be sampled from a super-polynomially wider distribution than $\mathbf{e}\mathbf{R}$, which is the reason why we need a super-polynomial modulus-to-noise ratio.

A closer inspection at the protocol reveals that, in fact, security holds even against an unbounded *malicious* receiver. In order to prove UC-security, we need to construct a simulator which extracts R's input [Can01]. By generating a matrix $\mathbf{A}$ in the CRS along with a lattice-trapdoor[2] $\mathsf{td}_{\mathbf{A}}$ in the sense of [GPV08, MP12], the simulator can extract $\mathbf{x}$ from $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$, provided that $\mathbf{e}$ is *short*. However, if $\mathbf{e}$ is *too large* and the extraction fails, we will be able to rely on the fact that the lattice $\Lambda_q(\mathbf{A}')$ generated by the row-span of $\mathbf{A}' = \begin{pmatrix}\mathbf{A}\\ \mathbf{a}'\end{pmatrix}$ has no short vectors. The *smoothing lemma* [MR07] guarantees that if $\mathbf{R}$ is sampled according to a discrete Gaussian distribution (with a properly chosen parameter), then the distribution of the product $\mathbf{A}'\mathbf{R}$ is statistically close to uniform. Thus, in this case, R just obtains random garbage and the simulation succeeds.

Reusability and batch reusable OLE. An important feature of the protocol described above is that it has *reusability* in the sense of [CDI+19]. Reusability (in a two-round protocol) means that the first message sent by the receiver can be reused in a polynomial number of executions of the protocol. This makes our protocol a perfect fit to be used as a building block in NISC protocols [CDI+19].

We also define a new OLE functionality that we called *batch reusable OLE*. In this functionality, the receiver *commits* to a batch of inputs $\{x_i\}_{i\in[k']}$. Later, the sender can send its inputs $(\mathbf{z}_0, \mathbf{z}_1)$ together with an index $j \in [k']$ corresponding to which input the receiver is using in this execution of the OLE. The receiver outputs the linear combination $\mathbf{z}_0 + x_j\mathbf{z}_1$.

A slight variant of the OLE protocol described above implements this functionality: In the first round, R chooses $\mathbf{x} \in \mathbb{Z}_q^k$ such that, say, the first k' coordinates correspond to its inputs $(x_1, \ldots, x_{k'})$, and computes $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$.

[2] Recall that a lattice trapdoor $\mathsf{td}_{\mathbf{A}}$ (as in [GPV08, MP12]) can be generated along with a matrix $\mathbf{A}$ such that it allows to invert LWE samples. That is, given $\mathbf{x}\mathbf{A} + \mathbf{e}$ for a short vector $\mathbf{e}$, $\mathsf{td}_{\mathbf{A}}$ allows to recover $\mathbf{x}$.

Then, along with its message, the sender S sends a position $j \in [k']$ to indicate which is the receiver's input being used in that execution of the protocol.

This variant improves in communication efficiency since, if R has several inputs, the parties don't need to run the protocol multiple times in parallel. Instead, they can just use the same message $\mathbf{a}'$ for several inputs of the receiver, as long as the LWE assumption holds for dimension $k - k'$.

Comparison with previous works. The idea of removing a row to a matrix $\mathbf{A}$ to *hide* something and then *recovering* it during *decryption* was already used in a previous work [DGHM18] to construct Hash with Encryption. However, in [DGHM18], the value to hash is chosen selectively by the adversary and thus security follows easily from the Extended LWE assumption [AP12].

Our case presents much more subtle technical challenges since the value $\mathbf{a}'$ sent by the receiver is chosen *after* seeing the matrix $\mathbf{A}$, and thus it has an *adaptive* flavor.

2.2 Extending to Malicious Adversaries

In the scheme above, it is information-theoretically impossible for a UC-simulator to extract the sender's input. In this section, we show how to modify the scheme to support UC-security against malicious senders.

In a nutshell, the idea to make the protocol secure against corrupted senders is to use a *cut-and-choose*-style approach using a two-round OT protocol, which exist under various assumptions [PVW08, DGH$^+$20]. Using the OT, the receiver is able to check if the matrices $\mathbf{C}_j = \mathbf{A}_{-i}\mathbf{R}_j$ sent by the sender are well-formed. More precisely, our protocol works as follows:

1. R computes $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$ as in the previous protocol and, additionally, it sends λ first-messages of the OT run in parallel (playing the role of the receiver) with input bits $(b_1, \ldots, b_\lambda)$ where half of them are equal to 0 and the remaining are equal to 1.

2. For $j \in [\lambda]$, S computes the matrix $\mathbf{C}_j = \mathbf{A}_{-i}\mathbf{R}_j$ for a short matrix $\mathbf{R}_j$. It chooses two random vectors $(\mathbf{u}_{0,j}, \mathbf{u}_{1,j})$ and inputs two messages $(M_{0,j}, M_{1,j})$ in the OT, where $M_{0,j} = \mathbf{R}_j$ and $M_{1,j} = (\mathbf{t}_{0,j}, \mathbf{t}_{1,j}, \mathbf{u}_{0,j} + \mathbf{z}_0, \mathbf{u}_{1,j} + \mathbf{z}_1)$ where $\mathbf{t}_{0,j} = \mathbf{a}'\mathbf{R}_j + \tilde{\mathbf{e}}_j + \hat{\mathbf{u}}_{0,j}$ and $\mathbf{t}_{1,j} = \mathbf{a}_i\mathbf{R}_j + \hat{\mathbf{u}}_{1,j}$, where $\hat{\mathbf{u}}_{0,j}$ and $\hat{\mathbf{u}}_{1,j}$ are the encodings of $\mathbf{u}_{0,j}$ and $\mathbf{u}_{0,j}$, respectively.

3. When $b_j = 0$, R can check that the matrix $\mathbf{C}_j$ is indeed well-formed. When $b_j = 1$, R can use $M_{1,j}$ to compute the *same* linear combination $\mathbf{w} = \mathbf{z}_0 + x_i\mathbf{z}_1$ for every position j (it aborts if there is at least one different from the others).

Security against an unbouded receiver in the OT-hybrid model essentially follows the same reasoning as in the previous protocol.

We now argue how we can build the simulator Sim against a corrupted sender. By simulating the OT functionality, the simulator Sim can extract the

sender's input using a single position j for which S inputs the *right* messages $M_{0,j}$ and $M_{1,j}$. This event always happens, except with negligible probability. Moreover, if R accepts the messages $M_{1,j}$, when $b_j = 1$, then this means that S input the same pair $(\mathbf{z}_0, \mathbf{z}_1)$ in these positions given that the LWE assumption holds.

The main drawback of this approach is that we lose reusability. It is trivial to see that this extraction strategy fails if S knows which are the bits that R sends to the OT functionality. Moreover, after several executions of the protocol, S is able to correctly guess the values of these bits. In fact, it is known that reusability is impossible in the OT-hybrid model [CDI+19].

Instantiating the OT functionality. When we instantiate our protocol with the OT scheme from [PVW08] we obtain several nice properties for the OLE scheme, namely: i) The protocol is still two-round. ii) The protocol preserves statistical security against a corrupted receiver, since the OT of [PVW08] is also statistically secure against a corrupted receiver. And iii) the OLE scheme is secure based solely on the LWE assumption.

3 Preliminaries

Throughout this work, λ denotes the security parameter and PPT stands for "probabilistic polynomial-time".

Let $\mathbf{A} \in \mathbb{Z}_q^{k \times n}$ and $\mathbf{x} \in \mathbb{Z}_q^n$. We denote by $\mathbf{A}_{-i}$ the matrix $\mathbf{A}$ with the i-th row removed. Similarly, $\mathbf{x}_{-i}$ denotes the vector $\mathbf{x}$ with the i-th coordinate removed. Moreover, $\|\mathbf{x}\|$ denotes the usual ℓ_2 norm of a vector $\mathbf{x}$. For a vector $\mathbf{b} \in \{0,1\}^k$, we denote its weight, that is the number of non-null coordinates, by $wt(\mathbf{b})$.

If S is a (finite) set, we denote by $x \leftarrow\!\!\$\, S$ the denotes an element $x \in S$ sampled according to a uniform distribution. Moreover, we denote by $U(S)$ the uniform distribution over S. If D is a distribution over S, $x \leftarrow\!\!\$\, D$ denotes an element $x \in S$ sampled according to D. If $\mathcal{A}$ is an algorithm, $y \leftarrow \mathcal{A}(x)$ denotes the output y after running $\mathcal{A}$ on input x.

A negligible function $\mathsf{negl}(n)$ in n is a function that vanishes faster than any polynomial in n.

Given two distributions D_1 and D_2, we say that they are ε-statistically indistinguishable, denoted by $D_1 \approx_\varepsilon D_2$, if the statistical distance is at most ε.

For two random variable X, Y (possibly correlated), we denote the min-entropy of X by $H_\infty(X)$ and the conditional-average min-entropy by $\tilde{H}_\infty(X|Y)$ (see [DORS08]).

Lemma 1 (Leftover Hash Lemma)**.** *Let $n, k, q \in \mathbb{N}$ such that $n \geq k$. Then, for a random variable $\mathbf{x} \in \mathbb{Z}_q^n$ and $\mathbf{A} \leftarrow\!\!\$\, \mathbb{Z}_q^{k \times n}$ we have that*

$$(\mathbf{A}, \mathbf{A}\mathbf{x}, Y) \approx_\varepsilon (\mathbf{A}, \mathbf{u}, Y)$$

where $\varepsilon = \sqrt{q^k \cdot 2^{-\tilde{H}_\infty(\mathbf{x}|Y)}}$, $\mathbf{u} \leftarrow\!\!\$\, \mathbb{Z}_q^k$ *and* Y *is a random variable (possibly) correlated with* $\mathbf{x}$.

Error-Correcting Codes. We define Error-Correcting Codes (ECC). An ECC over $\mathbb{Z}_q$ is composed by the following algorithms $\mathsf{ECC}_{\ell,k,\delta} = (\mathsf{Encode}, \mathsf{Decode})$ such that: i) $\mathbf{c} \leftarrow \mathsf{Encode}(\mathbf{m})$ takes as input a message $\mathbf{m} \in \mathbb{Z}_q^\ell$ and outputs a codeword $\mathbf{c} \in \mathbb{Z}_q^k$. ii) $\mathbf{m} \leftarrow \mathsf{Decode}(\tilde{\mathbf{c}})$ takes as input corrupted codeword $\tilde{\mathbf{c}} \in \mathbb{Z}_q^k$ and outputs a message $\mathbf{m} \in \mathbb{Z}_q^\ell$ if $\|\tilde{\mathbf{c}} - \mathbf{c}\| \leq \delta$ where $\mathbf{c} \leftarrow \mathsf{Encode}(\mathbf{m})$. In this case, we say that ECC corrects up to δ errors. We say that ECC is linear if any linear combination of codewords of ECC is also a codeword of ECC.

3.1 Universal Composability

UC-framework [Can01] allows to prove security of protocols even under arbitrary composition with other protocols. Let $\mathcal{F}$ be a functionality, π a protocol that implements $\mathcal{F}$ and $\mathcal{Z}$ be a environment, an entity that oversees the execution of the protocol in both the real and the ideal worlds. Let $\mathsf{IDEAL}_{\mathcal{F},\mathsf{Sim},\mathcal{Z}}$ be a random variable that represents the output of $\mathcal{Z}$ after the execution of $\mathcal{F}$ with adversary Sim. Similarly, let $\mathsf{REAL}^{\mathcal{G}}_{\pi,\mathcal{A},\mathcal{Z}}$ be a random variable that represents the output of $\mathcal{Z}$ after the execution of π with adversary $\mathcal{A}$ and with access to the functionality $\mathcal{G}$.

A protocol π *UC-realizes* $\mathcal{F}$ *in the* $\mathcal{G}$*-hybrid model* if for every PPT adversary $\mathcal{A}$ there is a PPT simulator Sim such that for all PPT environments $\mathcal{E}$, the distributions $\mathsf{IDEAL}_{\mathcal{F},\mathsf{Sim},\mathcal{Z}}$ and $\mathsf{REAL}^{\mathcal{G}}_{\pi,\mathcal{A},\mathcal{Z}}$ are computationally indistinguishable.

We now present the ideal functionalities that we will use in this work.

CRS functionality. This functionality generates a crs and distributes it between all the parties involved in the protocol. Here, we present the ideal functionality as in [PVW08].

$\mathcal{G}_{\mathsf{CRS}}$ functionality

Parameters: An algorithm D.

- Upon receiving $(\mathsf{sid}, \mathsf{P}_i, \mathsf{P}_j)$ from P_i, $\mathcal{G}_{\mathsf{CRS}}$ runs $\mathsf{crs} \leftarrow \mathsf{D}(1^\kappa)$ and returns $(\mathsf{sid}, \mathsf{crs})$ to P_i.
- Upon receiving $(\mathsf{sid}, \mathsf{P}_i, \mathsf{P}_j)$ from P_j, $\mathcal{G}_{\mathsf{CRS}}$ returns $(\mathsf{sid}, \mathsf{crs})$ to P_j.

OT functionality. Oblivious Transfer (OT) can be seen as a particular case of OLE. We show the ideal OT functionality below.

$\mathcal{F}_{\mathsf{OT}}$ functionality

Parameters: $\mathsf{sid} \in \mathbb{N}$ known to both parties.

- Upon receiving $(\mathsf{sid}, (M_0, M_1))$ from S, $\mathcal{F}_{\mathsf{OT}}$ stores (M_0, M_1) and ignores future messages from S with the same sid;
- Upon receiving $(\mathsf{sid}, b \in \{0,1\})$ from R, $\mathcal{F}_{\mathsf{OT}}$ checks if it has recorded $(\mathsf{sid}, (M_0, M_1))$. If so, it returns (sid, M_b) to R and $(\mathsf{sid}, \mathsf{receipt})$ to S, and halts. Else, it sends nothing but continues running.

OLE functionality. We now present the OLE functionality. This functionality involves two parties: the sender S and the receiver R.

$\mathcal{F}_{\mathsf{OLE}}$ functionality

Parameters: $\mathsf{sid}, q, k \in \mathbb{N}$ and a finite field $\mathbb{F}_q$ known to both parties.

- Upon receiving $\left(\mathsf{sid}, (\mathbf{a}, \mathbf{b}) \in \mathbb{F}_q^k \times \mathbb{F}_q^k\right)$ from S, $\mathcal{F}_{\mathsf{OLE}}$ stores (a, b) and ignores future messages from S with the same sid;
- Upon receiving $(\mathsf{sid}, x \in \mathbb{F}_q)$ from R, $\mathcal{F}_{\mathsf{OLE}}$ checks if it has recorded $(\mathsf{sid}, (\mathbf{a}, \mathbf{b}))$. If so, it returns $(\mathsf{sid}, \mathbf{z} = \mathbf{a} + x\mathbf{b})$ to R and $(\mathsf{sid}, \mathsf{receipt})$ to S, and halts. Else, it sends nothing but continues running.

Reusable OLE functionality. We now present the reusable OLE functionality as it was presented in [CDI+19].

$\mathcal{F}_{\mathsf{rOLE}}$ functionality

Parameters: $\mathsf{sid}, q, k \in \mathbb{N}$ and a finite field $\mathbb{F}_q$ known to both parties.

Setup phase: Upon receiving $(\mathsf{sid}, x \in \mathbb{F}_q)$ from R, $\mathcal{F}_{\mathsf{rOLE}}$ sends $(\mathsf{sid}, \mathsf{init})$ to the adversary and ignores future messages from R with the same sid.

Send phase:

- Upon receiving $\left(\mathsf{sid}, i, (\mathbf{a}, \mathbf{b}) \in \mathbb{F}_q^k \times \mathbb{F}_q^k\right)$ from S, $\mathcal{F}_{\mathsf{rOLE}}$ sends $(\mathsf{sid}, i, \mathsf{sent})$ to the adversary and ignores future messages from S with the same sid and $i \in \mathbb{N}$.
- Upon receiving $(\mathsf{sid}, i, \mathsf{deliver})$ from the adversary, $\mathcal{F}_{\mathsf{rOLE}}$ checks if it has stored (sid, x) from R and $(\mathsf{sid}, i, (\mathbf{a}, \mathbf{b}))$ from S. If so, it sends $(\mathsf{sid}, \mathbf{z} = \mathbf{a} + x\mathbf{b})$ to R.

Batch reusable OLE functionality. Here we define a batch version of the functionality defined above. In this functionality, the receiver inputs several

OLE inputs at the same time. The sender can then input an affine function together with an index corresponding to which input the receiver should receive the linear combination.

$\mathcal{F}_{\mathsf{brOLE}}^{k'}$ functionality

Parameters: $\mathsf{sid}, q, k, k' \in \mathbb{N}$ and a finite field $\mathbb{F}_q$ known to both parties.

Setup phase: Upon receiving $(\mathsf{sid}, \{x_j\}_{j\in[k]} \in \mathbb{F}_q^{k'})$ from R, $\mathcal{F}_{\mathsf{brOLE}}^{k'}$ sends $(\mathsf{sid}, \mathsf{init})$ to the adversary and ignores future messages from R with the same sid.

Send phase:

- Upon receiving $\big(\mathsf{sid}, i, (\mathbf{a}, \mathbf{b}) \in \mathbb{F}_q^k \times \mathbb{F}_q^k, j\big)$ from S, $\mathcal{F}_{\mathsf{brOLE}}^{k'}$ sends $(\mathsf{sid}, i, \mathsf{sent}, j)$ to the adversary and ignores future messages from S with the same sid and $i \in \mathbb{N}$.
- Upon receiving $(\mathsf{sid}, i, \mathsf{deliver}, j)$ from the adversary, $\mathcal{F}_{\mathsf{brOLE}}^{k'}$ checks if it has stored $(\mathsf{sid}, \{x_j\}_{j\in[k]})$ from R and $(\mathsf{sid}, i, (\mathbf{a}, \mathbf{b}), j)$ from S. If so, it sends $(\mathsf{sid}, \mathbf{z} = \mathbf{a} + x_j\mathbf{b})$ to R.

3.2 Lattices and Hardness Assumptions

Notation. Let $\mathbf{B} \in \mathbb{R}^{k\times n}$ be a matrix. We denote the lattice generated by $\mathbf{B}$ by $\Lambda = \Lambda(\mathbf{B}) = \{\mathbf{xB} : \mathbf{x} \in \mathbb{Z}^k\}$.[3] The dual lattice Λ^* of a lattice Λ is defined by $\Lambda^* = \{\mathbf{x} \in \mathbb{R}^n : \forall y \in \Lambda, \mathbf{x}\cdot\mathbf{y} \in \mathbb{Z}\}$. It holds that $(\Lambda^*)^* = \Lambda$.

A lattice Λ is said to be q-ary if $(q\mathbb{Z})^n \subseteq \Lambda \subseteq \mathbb{Z}^n$. For every q-ary lattice Λ, there is a matrix $\mathbf{A} \in \mathbb{Z}_q^{k\times n}$ such that

$$\Lambda_q(\mathbf{A}) = \{y \in \mathbb{Z}^n : \exists \mathbf{x} \in \mathbb{Z}_q^k, \mathbf{y} = \mathbf{xA} \mod q\}.$$

The orthogonal lattice $\Lambda_q^\perp$ is defined by $\{\mathbf{y} \in \mathbb{Z}_q^n : \mathbf{A}\mathbf{y}^T = 0 \mod q\}$. It holds that $\frac{1}{q}\Lambda_q^\perp = \Lambda_q^*$

Let $\rho_s(\mathbf{x})$ be probability distribution of the Gaussian distribution over $\mathbb{R}^n$ with parameter s and centered in 0. We define the discrete Gaussian distribution $D_{S,s}$ over S and with parameter s by the probability distribution $\rho_s(\mathbf{x})/\rho(S)$ for all $\mathbf{x} \in S$ (where $\rho_s(S) = \sum_{\mathbf{x}\in S}\rho_s(\mathbf{x})$).

For $\varepsilon > 0$, the smoothing parameter $\eta_\varepsilon(\Lambda)$ of a lattice Λ is the least real $\sigma > 0$ such that $\rho_{1/\sigma}(\Lambda^* \setminus \{0\}) \leq \varepsilon$ [MR07].

Useful Lemmata. The following lemmas are well-known results on discrete Gaussians over lattices.

[3] The matrix $\mathbf{B}$ is called a basis of $\Lambda(\mathbf{B})$.

Lemma 2 ([Ban93]). *Let $\sigma > 0$ and $\mathbf{x} \hookleftarrow_{\$} D_{\mathbb{Z}^n,\sigma}$. Then we have that*

$$\Pr\left[\|\mathbf{x}\| \geq \sigma\sqrt{n}\right] \leq \mathsf{negl}(n)\,.$$

The following lemma states how we can statistically hide an error vector with a sample from a much wider distribution.

Lemma 3. *Let q, B, σ such that $q = \lambda^{\omega(1)}$ and $\sigma/B = \lambda^{\omega(1)}$. Then the distributions D_σ and $D_\sigma + e$ are statistically close, where e is sampled from a B-bounded distribution.*

The next lemma gives us a lower-bound on the min-entropy of a discrete Gaussian distribution.

Lemma 4 ([PR06, GPV08]). *For a lattice Λ of dimension n, $\varepsilon > 0$ and $\sigma > 2\eta_\varepsilon(\Lambda)$, we have that*

$$D_{\Lambda,\sigma}(\mathbf{x}) \leq 2^{n-1}\frac{1+\varepsilon}{1-\varepsilon}$$

for all $\mathbf{x} \in \Lambda$. In particular, $H_\infty(D_{\Lambda,\sigma}) \geq n - 1$, if $\varepsilon < 1/3$.

For $\Lambda = \mathbb{Z}^n$, we just need to consider $\sigma > \omega(\log n)$ [GPV08] in order to have $H_\infty(D_{\Lambda,\sigma}) \geq n - 1$.

Finally, the next lemma is a consequence of the smoothing lemma [MR07] and it tells us that $\mathbf{Ae}$ is uniform, when $\mathbf{e}$ comes from a discrete Gaussian and for a proper choice of parameters.

Lemma 5 ([GPV08]). *Let $q \in \mathbb{N}$ and $\mathbf{A} \in \mathbb{Z}_q^{k\times n}$ be a matrix whose columns generate $\mathbb{Z}_q^k$. Moreover, let $\varepsilon \in (0, 1/2)$ and $\sigma \geq \eta_\varepsilon(\Lambda_q^\perp(\mathbf{A}))$. Then, for $\mathbf{e} \hookleftarrow_{\$} D_{\mathbb{Z}^m,\sigma}$,*

$$\mathbf{Ae} \mod q \approx_{2\varepsilon} \mathbf{u}$$

where $\mathbf{u} \hookleftarrow_{\$} \mathbb{Z}_q^k$.

LWE Assumption. The Learning with Errors assumption was first presented in [Reg05]. The assumption roughly states that it should be hard to solve a set linear equations by just adding a little noise to it.

Definition 1 (Learning with Errors). *Let $q, k \in \mathbb{N}$ where $k = k(\lambda)$, $\mathbf{A} \in \mathbb{Z}_q^{k\times n}$ and $\beta \in \mathbb{R}$. For any $n = \mathsf{poly}(k \log q)$, the $\mathsf{LWE}_{k,\beta,q}$ assumption holds if for every PPT algorithm $\mathcal{A}$ we have*

$$|\Pr[1 \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{sA} + \mathbf{e})] - \Pr[1 \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{y})]| \leq \mathsf{negl}(\lambda)$$

for $\mathbf{s} \hookleftarrow_{\$} \{0,1\}^k$, $\mathbf{e} \hookleftarrow_{\$} D_{\mathbb{Z}^n,\beta}$ and $\mathbf{y} \hookleftarrow_{\$} \{0,1\}^n$.

Regev proved in [Reg05] that there is a (quantum) worst-case to average-case reduction from some problems on lattices which are believed to be hard even in the presence of a quantum computer.

Trapdoors for Lattices. Recent works [GPV08, MP12] have presented trapdoors functions based on the hardness of LWE.

Lemma 6 ([GPV08, MP12])**.** *Let $\tau(k) \in \omega\left(\sqrt{\log k}\right)$ be a function and τ a sufficiently large constant. There is a pair of algorithms $(\mathsf{TdGen}, \mathsf{Invert})$ such that if $(\mathbf{A}, \mathsf{td}) \leftarrow \mathsf{TdGen}(q, k)$ then:*

- *$\mathbf{A} \in \mathbb{Z}_q^{k \times n}$ where $n \in \mathcal{O}(k \log q)$ is a matrix whose distribution is 2^{-k} close to the uniform distribution over $\mathbb{Z}_q^{k \times n}$.*
- *For any $\mathbf{s} \in Z_q^k$ and $\mathbf{e} \in \mathbb{Z}_q^n$ such that $\|\mathbf{e}\| < q/(\sqrt{n}\tau(k))$, we have that*

$$\mathbf{s} \leftarrow \mathsf{Invert}(\mathbf{sA} + \mathbf{e}, \mathsf{td}).$$

Observe that, if $(\mathbf{A}, \mathsf{td}_{\mathbf{A}}) \leftarrow \mathsf{TdGen}(1^\lambda, n, k, q)$, then $\Lambda(\mathbf{A})$ has no *short* vectors. That is, for all $\mathbf{y} \in \Lambda(\mathbf{A})$, then $\|\mathbf{y}\| > B = q/(\sqrt{n}\tau(k))$ [MP12]. If this does not happen, then the algorithm Invert would not output the right $\mathbf{s}$ for a non-negligible number of cases.

4 Reusable Oblivious Linear Evaluation Secure Against Corrupted Receiver

In this section, we present a semi-honest protocol for OLE based on the hardness of the LWE assumption. The protocol is reusable, meaning that it implements functionality $\mathcal{F}_{\mathsf{rOLE}}$ defined in Section 3.

4.1 Protocol

We begin by presenting the protocol.

Construction 1. *The protocol is composed by the algorithms $(\mathsf{GenCRS}, \mathsf{R}_1, \mathsf{S}, \mathsf{R}_2)$. Let $k, n, \ell, \ell', q \in \mathbb{Z}$ such that $q = 2^{\omega(\log \lambda)}$ is a prime and $n = \mathsf{poly}(k \log q)$, and let $\beta, \delta_0, \delta_1, \xi \in \mathbb{R}$ such that $\frac{q}{\sqrt{n}\tau(k)} > \beta \geq \delta_0 \geq 1$ (where $\tau(k) = \omega(\sqrt{\log k})$, as in Lemma 6) and $\frac{\beta\delta_0}{\delta_1} = \mathsf{negl}(\lambda)$. We present the protocol in full detail.*

$\mathsf{GenCRS}(1^\lambda)$**:**

- *Sample $\mathbf{A} \leftarrow\!\!\$\, \mathbb{Z}_q^{k \times n}$.*
- *Choose a linear ECC $\mathsf{ECC}_{\ell',\ell,\xi} = (\mathsf{ECC.Encode}, \mathsf{ECC.Decode})$ over $\mathbb{Z}_q$.*
- *Output $\mathsf{crs} = (\mathbf{A}, \mathsf{ECC}_{\ell',\ell,\xi})$.*

$\mathsf{R}_1\,(\mathsf{crs}, x \in \mathbb{Z}_q)$**:**

- *Parse* crs *as* $(\mathbf{A}, \mathsf{ECC}_{\ell',\ell,\xi})$.
- *Sample* $\mathbf{x} = (x_1, \ldots, x_k) \leftarrow_\$ \mathbb{Z}_q^k$ *such that* $x_i = x$ *and a small error vector* $\mathbf{e} \leftarrow_\$ D_{\mathbb{Z}^n,\beta}$*, for a uniformly chosen index* $i \leftarrow_\$ [k]$.
- *Compute* $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$.
- *Output* $\mathsf{ole}_1 = (\mathbf{a}', i)$ *and* $\mathsf{st} = \mathbf{x}$.

$\mathsf{S}\left(\mathsf{crs}, (\mathbf{z}_0, \mathbf{z}_1) \in (\mathbb{Z}_q^{\ell'})^2, m_{\mathsf{R}}\right)$**:**

- *Parse* crs *as* $(\mathbf{A}, \mathsf{ECC}_{\ell',\ell,\xi})$ *and* ole_1 *as* $(\mathbf{a}', i)$.
- *Sample* $\mathbf{R} \in \mathbb{Z}_q^{n\times\ell}$ *where each column* $\mathbf{r}^{(j)} \leftarrow_\$ D_{\mathbb{Z}^n,\delta_0}$ *for* $j \in [\ell]$.
- *Compute* $\mathbf{s}_0, \mathbf{s}_1 \in \mathbb{Z}_q^\ell$ *such that* $\mathbf{s}_0 = \mathbf{a}'\mathbf{R}$*,* $\mathbf{s}_1 = -\mathbf{a}_i\mathbf{R}$*, and* $\mathbf{C} = \mathbf{A}_{-i}\mathbf{R} \in \mathbb{Z}_q^{(k-1)\times\ell}$.
- *Compute* $\mathbf{t}_0 = \mathbf{s}_0 + \tilde{\mathbf{e}} + \mathsf{ECC.Encode}(\mathbf{z}_0)$ *and* $\mathbf{t}_1 = \mathbf{s}_1 + \mathsf{ECC.Encode}(\mathbf{z}_1)$*, where* $\tilde{\mathbf{e}} \leftarrow_\$ D_{\mathbb{Z}^\ell,\delta_1}$.
- *Output* $\mathsf{ole}_2 = (\mathbf{C}, \mathbf{t}_0, \mathbf{t}_1)$.

$\mathsf{R}_2(\mathsf{crs}, \mathsf{st}, m_{\mathsf{S}})$**:**

- *Parse* crs *as* $(\mathbf{A}, \mathsf{ECC}_{\ell',\ell,\xi})$*,* ole_2 *as* $(\mathbf{C}, \mathbf{t}_0, \mathbf{t}_1)$ *and* st *as* $\mathbf{x} \in \mathbb{Z}_q^k$.
- *Compute* $\tilde{\mathbf{y}} = \mathbf{x}_{-i}\mathbf{C} - (\mathbf{t}_0 + x_i\mathbf{t}_1)$ *and then run* $\mathbf{y} \leftarrow \mathsf{ECC.Decode}(\tilde{\mathbf{y}})$*. If* $\mathbf{y} = \perp$*, abort the protocol.*
- *Output* $\mathbf{y} \in \mathbb{Z}_q^{\ell'}$.

4.2 Analysis of the Protocol

Theorem 1 (Correctness)**.** *Let* $\mathsf{ECC}_{\ell',\ell,\xi}$ *be a linear ECC where* $\xi \geq \sqrt{\ell}(\beta\delta_0 n + \delta_1)$*. Then the protocol presented in Construction 1 is correct.*

Proof. To prove correctness, we have to prove that R_2 outputs $\mathbf{z}_0 + x_i\mathbf{z}_1$, where $(\mathbf{z}_0, \mathbf{z}_1)$ is the input of S.

First, note that

$$\begin{aligned}\mathbf{x}_{-i}\mathbf{C} &= \mathbf{x}_{-i}\mathbf{A}_{-i}\mathbf{R}\\ &= (\mathbf{x}\mathbf{A} - x_i\mathbf{a}_i)\mathbf{R}\\ &= (\mathbf{x}\mathbf{A} + \mathbf{e})\mathbf{R} - (x_i\mathbf{a}_i)\mathbf{R} - \mathbf{e}\mathbf{R} = \mathbf{s}_0 + x_i\mathbf{s}_1 + \mathbf{e}'\end{aligned}$$

where $\mathbf{e}' = -\mathbf{e}\mathbf{R}$ is a small error vector. The receiver computes

$$\tilde{\mathbf{y}} = \mathbf{x}_{-i}\mathbf{C} - (\mathbf{t}_0 + x_i\mathbf{t}_1) = \mathsf{ECC.Encode}(\mathbf{z}_0 + x_i\mathbf{z}_1) + \mathbf{e}''$$

where $\mathbf{e}'' = -\mathbf{e}\mathbf{R} - \tilde{\mathbf{e}}$ and the last equality holds because ECC is linear.

Finally, by Lemma 2, we have that $\|\mathbf{e}\| \leq \beta\sqrt{n}$. Moreover, if $\mathbf{r}^{(i)}$ is a column of $\mathbf{R}$, then $\left\|\mathbf{r}^{(i)}\right\| \leq \delta_0\sqrt{n}$. Therefore, each coordinate of $\mathbf{e}'$ has norm at most $\|\mathbf{e}\| \cdot \left\|\mathbf{r}^{(i)}\right\| \leq \beta\delta n$. On the other hand, $\|\tilde{\mathbf{e}}\| \leq \delta_1\sqrt{\ell}$ We conclude that $\|\mathbf{e}''\| \leq \sqrt{\ell}(\beta\delta_0 n + \delta_1)$. Since ECC corrects up to $\xi \geq \sqrt{\ell}(\beta\delta_0 n + \delta_1)$ errors, the output of $\mathsf{ECC.Decode}(\tilde{\mathbf{y}})$ is exactly $\mathbf{z}_0 + x_i\mathbf{z}_1$. □

Theorem 2 (Security)**.** *Assume that the* $\mathsf{LWE}_{k,\beta,q}$ *assumption holds,* $\frac{q}{\sqrt{n}\tau(k)} > \beta \geq \delta_0 \geq 1$ *(where* $\tau(k) = \omega(\sqrt{\log k})$ *as in Lemma 6), and* $\delta_0\beta/\delta_1 = \mathsf{negl}(\lambda)$*. The protocol presented in Construction 1 securely realizes the functionality* $\mathcal{F}_{\mathsf{rOLE}}$ *in the* $\mathcal{G}_{\mathsf{CRS}}$*-hybrid model against:*

- *a semi-honest sender given that the* $\mathsf{LWE}_{k,\beta,q}$ *assumption holds;*
- *a static malicious receiver where security holds statistically.*

Proof. We begin by proving security against a computationally unbounded corrupted receiver.

Simulator for corrupted receiver: We describe the simulator Sim. Let $(\mathsf{TdGen}, \mathsf{Invert})$ be the algorithms described in Lemma 6.

- **CRS generation:** Sim generates $(\mathbf{A}, \mathsf{td}_{\mathbf{A}}) \leftarrow \mathsf{TdGen}(1^\lambda, q, k, n)$. It chooses an ECC $\mathsf{ECC}_{\ell',\ell,\xi}$. It publishes $\mathsf{crs} = (\mathbf{A}, \mathsf{ECC})$ and keeps $\mathsf{td}_{\mathbf{A}}$ to itself.
- Upon receiving a message $\mathbf{a}'$ from R, Sim runs $\tilde{\mathbf{x}} \leftarrow \mathsf{Invert}(\mathsf{td}_{\mathbf{A}}, \mathbf{a}')$. There are two cases to consider:
  - If $\tilde{\mathbf{x}} = \perp$, then Sim samples $\mathbf{t}_0, \mathbf{t}_1 \leftarrow_\$ \mathbb{Z}_q^\ell$ and $\mathbf{C} \leftarrow_\$ \mathbb{Z}_q^{k-1\times\ell}$. It sends $\mathsf{ole}_2 = (\mathbf{C}, \mathbf{t}_0, \mathbf{t}_1)$.
  - Else if, $\tilde{\mathbf{x}} \neq \perp$, then Sim sets $x = x_i$ where x_i is the i-th coordinate of $\tilde{\mathbf{x}}$. It sends x to $\mathcal{F}_{\mathsf{rOLE}}$. Whenever it receives a $\mathbf{y}$ from $\mathcal{F}_{\mathsf{rOLE}}$, Sim chooses uniformly at random two vectors $\mathbf{z}_0, \mathbf{z}_1 \in \mathbb{Z}_q^{\ell'}$ such that $\mathbf{z}_0 + x\mathbf{z}_1 = y$. Then, it samples a uniform matrix $\mathbf{U} \leftarrow_\$ \mathbb{Z}_q^{k\times\ell}$ and an error vector $\tilde{\mathbf{e}} \leftarrow_\$ D_{\mathbb{Z}^\ell,\sigma_1}$ and sets

$$\begin{aligned}\mathbf{C} &= \mathbf{U}_{-i}\\ \mathbf{t}_0 &= \tilde{\mathbf{x}}\mathbf{U} + \tilde{\mathbf{e}} + \mathsf{ECC.Encode}(\mathbf{z}_0)\\ \mathbf{t}_1 &= -\mathbf{u}_i + \mathsf{ECC.Encode}(\mathbf{z}_1)\end{aligned}$$

    where $\mathbf{u}_i$ is the i-th row of $\mathbf{U}$. It sends $\mathsf{ole}_2 = (\mathbf{C}, \mathbf{t}_0, \mathbf{t}_1)$.

We now proceed to show that the real-world and the ideal-world executions are indistinguishable. The following lemma shows that the CRS generated in the simulation is indistinguishable from one generated in the real-world execution. Then, the next two lemmas deal with the two possible cases in the simulation.

Lemma 7. *The CRS generated above is statistically indistinguishable from a CRS generated according to* GenCRS.

The only thing that differs in both CRS's is that the matrix $\mathbf{A}$ is generated via TdGen in the simulation (instead of being chosen uniformly). By Lemma 6, it follows that the CRS is statistically indistinguishable from one generated using GenCRS. □

Lemma 8. *Assume that* $\tilde{\mathbf{x}} = \perp$. *Then, the simulated execution is indistinguishable from the real-world execution.*

We prove that no (computationally unbounded) adversary can distinguish both executions, except with negligible probability. First, note that, if $\perp = \tilde{\mathbf{x}} \leftarrow \mathsf{Invert}(\mathsf{td}_{\mathbf{A}}, \mathbf{a}')$, then $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$ where $\|\mathbf{e}\| > \beta\sqrt{n}$ since only in this case algorithm Invert fails to *invert* $\mathbf{a}'$.

Consider the matrix $\mathbf{A}' = \begin{pmatrix} \mathbf{A} \\ \mathbf{a}' \end{pmatrix}$. If $\mathbf{a}'$ is of the form described above, then the matrix $\mathbf{A}'$ has no *short* vectors in its row-span. In other words, there is no vector $\mathbf{v} \neq 0$ in the row-span of $\mathbf{A}'$ such that $\|\mathbf{v}\| \leq \beta\sqrt{n}$. This comes from the fact that $\mathbf{A}$ has no short vectors whose norm is smaller than $\beta\sqrt{n}$

Hence $\rho_\beta(\Lambda_q(\mathbf{A}') \setminus \{0\}) \leq \mathsf{negl}(\lambda)$. Moreover, we have that

$$\begin{aligned}
\rho_\beta(\Lambda_q(\mathbf{A}') \setminus \{0\}) &\geq \rho_{\delta_0}(\Lambda_q(\mathbf{A}') \setminus \{0\}) \\
&\geq \rho_{1/\delta_0}(\Lambda_q(\mathbf{A}') \setminus \{0\}) \\
&\geq \rho_{1/(q\delta_0)}(\Lambda_q(\mathbf{A}') \setminus \{0\}) \\
&= \rho_{1/\delta_0}(q\Lambda_q(\mathbf{A}') \setminus \{0\}) \\
&= \rho_{1/\delta_0}((\Lambda_q^\perp(\mathbf{A}'))^* \setminus \{0\})
\end{aligned}$$

where the first and the second inequalities hold because $\beta > \delta_0 > 1$ by hypothesis and the last equality holds because $\frac{1}{q}\Lambda_q^\perp(\mathbf{A}') = \Lambda_q(\mathbf{A}')^*$. Since

$$\rho_{1/\delta_0}((\Lambda_q^\perp(\mathbf{A}'))^* \setminus \{0\}) \leq \mathsf{negl}(\lambda)$$

then $\delta_0 \geq \eta_\varepsilon(\Lambda^\perp(\mathbf{A}'))$, for $\varepsilon = \mathsf{negl}(\lambda)$, and the conditions of Lemma 5 are met.

Therefore, we can switch to a hybrid experiment where $\mathbf{A}'\mathbf{R} \mod q$ is replaced by $\mathbf{U} \leftarrow_\$ \mathbb{Z}^{(k+1)\times\ell}$ incurring only negligible statistical distance. That is,

$$\begin{pmatrix} \mathbf{C} \\ \mathbf{t}_1 \\ \mathbf{t}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{-i} \\ \mathbf{a}_i \\ \mathbf{a}' \end{pmatrix} \mathbf{R} + \begin{pmatrix} 0 \\ \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_0 + \tilde{\mathbf{e}} \end{pmatrix} \approx_{\mathsf{negl}(\lambda)} \mathbf{U} + \begin{pmatrix} 0 \\ \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_0 + \tilde{\mathbf{e}} \end{pmatrix} \approx_{\mathsf{negl}(\lambda)} \mathbf{U}$$

where $\hat{\mathbf{z}}_j$ is the encoding of $\mathsf{ECC.Encode}(\mathbf{z}_j)$ for $j \in \{0, 1\}$.

We conclude that, in this case, the real-world and the ideal-world execution (where Sim just sends a uniformly chosen triple $(\mathbf{C}, \mathbf{t}_0, \mathbf{t}_1)$) are statistically indistinguishable. □

Lemma 9. *Assume that* $\tilde{\mathbf{x}} \neq \perp$. *Then, the simulated execution is indistinguishable from the real-world execution.*

The proof follows the following sequence of hybrids:

Hybrid $\mathcal{H}_0$. This is the real-world protocol. In particular, in this hybrid, the simulator behaves as the honest sender and computes

$$\begin{aligned}
\mathbf{t}_0 &= \mathbf{a}'\mathbf{R} + \tilde{\mathbf{e}} + \mathsf{ECC.Encode}(\mathbf{z}_0) = \mathbf{xAR} + \mathbf{eR} + \tilde{\mathbf{e}} + \mathsf{ECC.Encode}(\mathbf{z}_0) \mod q \\
\mathbf{t}_1 &= \mathbf{a}_i\mathbf{R} + \mathsf{ECC.Encode}(\mathbf{z}_1) \mod q \\
\mathbf{C} &= \mathbf{A}_{-i}\mathbf{R} \mod q.
\end{aligned}$$

Hybrid $\mathcal{H}_1$. This hybrid is similar to the previous one, except that Sim computes $\mathbf{t}_0$ as $\mathbf{xAR} + \tilde{\mathbf{e}} + \mathsf{ECC.Encode}(\mathbf{z}_0) \mod q$, where $\tilde{\mathbf{e}} \leftarrow\!\!\$\, D_{\mathbb{Z}^\ell,\delta_1}$.

Claim 1. $|\Pr[1 \leftarrow \mathcal{A} : \mathcal{A} \textit{ plays } \mathcal{H}_0] - \Pr[1 \leftarrow \mathcal{A} : \mathcal{A} \textit{ plays } \mathcal{H}_1]| \leq \mathsf{negl}(\lambda)$.

This claim holds statistically and it is a direct consequence of Lemma 3 since $\delta_0\beta/\delta_1 = \mathsf{negl}(\lambda)$ by hypothesis.

Hybrid $\mathcal{H}_2$ This hybrid is similar to the previous one, except that Sim computes $\mathbf{t}_0$ as $\mathbf{U} + \tilde{\mathbf{e}} + \mathsf{ECC.Encode}(\mathbf{z}_0)$, $\mathbf{C}$ by $\mathbf{U}_{-i}$ and $\mathbf{t}_1$ by $\mathbf{u}_i + \mathsf{ECC} : \mathsf{Encode}(\mathbf{z}_1)$, where $\mathbf{u}_i$ is the i-th row of $\mathbf{U} \leftarrow\!\!\$\, \mathbb{Z}_q^{k\times\ell}$.

Note that this hybrid is similar to the simulator for the corrupted receiver.

Claim 2. $|\Pr[1 \leftarrow \mathcal{A} : \mathcal{A} \textit{ plays } \mathcal{H}_1] - \Pr[1 \leftarrow \mathcal{A} : \mathcal{A} \textit{ plays } \mathcal{H}_2]| \leq \mathsf{negl}(\lambda)$.

By Lemma 4, each column of $\mathbf{R}$ has min-entropy of at least $(n-1)$. Thus, by the Leftover Hash Lemma (Lemma 1) and given that $n = \mathsf{poly}(k \log q)$, the claim holds statistically.

Claim 3. *Let* R *be any receiver. The values* $(\mathbf{z}_0, \mathbf{z}_1)$ *are perfectly hidden from* R *given that* $\mathbf{z}_0 + x\mathbf{x}_1 = \mathbf{y}$.

To see this, consider again the values computed by Sim in this hybrid

$$\begin{aligned}
\mathbf{C} &= \mathbf{U}_{-i} \\
\mathbf{t}_0 &= \tilde{\mathbf{x}}\mathbf{U} + \tilde{\mathbf{e}} + \mathsf{ECC.Encode}(\mathbf{z}_0) \\
\mathbf{t}_1 &= -\mathbf{u}_i + \mathsf{ECC.Encode}(\mathbf{z}_1)
\end{aligned}$$

There are two cases to consider: either $x_i = 0$ or $x_i \neq 0$. In the first case, neither $\mathbf{C}$ nor $\mathbf{t}_0$ carry information about $\mathbf{u}_i$, which is a uniformly chosen vector. Hence, $\mathbf{z}_1$ is perfectly hidden from R.

When $x_i \neq 0$, consider another pair $(\mathbf{z}'_0, \mathbf{z}'_1)$ such that $\mathbf{z}_0 + x_i\mathbf{z}_1 = \mathbf{z}'_0 + x_i\mathbf{z}'_1$. Then, the probability that Sim runs the protocol on input $(\mathbf{z}_0, \mathbf{z}_1)$ or $(\mathbf{z}'_0, \mathbf{z}'_1)$ is exactly the same from the point-of-view of R, given that $\mathbf{U} \leftarrow\!\!\$\, \mathbb{Z}_q^{k\times\ell}$. To see this, note that given $\mathbf{C}, \mathbf{t}_0, \mathbf{t}_1$, we can set $\mathbf{u}_i = \mathbf{s}_1 - \mathsf{ECC.Encode}(\mathbf{z}'_1) =$

$x_i^{-1}(\mathbf{s}\_\mathsf{ECC.Encode}(\mathbf{z}'_0))$, which is uniform in $\mathbb{Z}_q$ since q is prime. A simple calculation shows that correctness still holds in this case.

Simulator for corrupted sender. We describe how the simulator Sim proceeds: It takes S's inputs $(\mathbf{z}_0, \mathbf{z}_1)$ and sends them to the ideal functionality $\mathcal{F}_{\mathsf{rOLE}}$, which returns nothing. It simulates the dummy R by sampling $\mathbf{a}' \leftarrow_\$ \mathbb{Z}_q^n$ and sending it to the corrupted sender.

It is trivial to see that both the ideal and the real-world executions are indistinguishable given that the $\mathsf{LWE}_{k,q,\beta}$ assumption holds. □

4.3 Batch Reusable OLE

We now show how we can extend the protocol described above in order to implement a batch reusable OLE protocol, that is, in order to implement the functionality $\mathcal{F}_{\mathsf{brOLE}}$ described in Section 3.

This variant improves the efficiency of the protocol since the receiver R can *commit* to a batch of inputs $\{x_i\}_{i\in[k']}$, and not just one input, using the same first message of the two-round OLE. Hence, the size of the first message can be amortized over the number of R's inputs, to achieve a better communication complexity.

Construction 2. *The protocol is composed by the algorithms* $(\mathsf{GenCRS}, \mathsf{R}_1, \mathsf{S}, \mathsf{R}_2)$. *Let* $k, n, \ell, \ell', q, k' \in \mathbb{Z}$ *such that* $q = 2^{\omega(\log\lambda)}$ *is a prime and* $n = \mathsf{poly}(k \log q)$, *and let* $\beta, \delta_0, \delta_1, \xi \in \mathbb{R}$ *such that* $\frac{q}{\sqrt{n}\tau(k)} > \beta \geq \delta_0 \geq 1$ *(where* $\tau(k) = \omega(\sqrt{\log k})$, *as in Lemma 6) and* $\frac{\beta\delta_0}{\delta_1} = \mathsf{negl}(\lambda)$.

$\mathsf{GenCRS}(1^\lambda)$**:** *This algorithm is similar to the one described in Construction 1.*

$\mathsf{R}_1\left(\mathsf{crs}, \{x_j\}_{j\in[k']} \in \mathbb{Z}_q\right)$**:** *The algorithm is similar to the one described in Construction 1, except that it outputs* $\mathsf{ole}_1 = (\mathbf{a}', k')$ *and* $\mathsf{st} = \mathbf{x}$, *where* $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$ *and* $(x_1, \ldots, x_{k'})$ *corresponds to the first* k' *of* $\mathbf{x}$.

$\mathsf{S}\left(\mathsf{crs}, (\mathbf{z}_0, \mathbf{z}_1) \in (\mathbb{Z}_q^{\ell'})^2, m_{\mathsf{R}}, j \in [k']\right)$**:** *This algorithm is similar to the one described in Construction 1, except that it additionally uses* j *as the index* i *and that it outputs* j *(which corresponds to which* x_j *the receiver* R *is supposed to use in that particular execution of the protocol).*

$\mathsf{R}_2(\mathsf{crs}, \mathsf{st}, m_{\mathsf{S}})$**:** *This algorithm is similar to the one described in Construction 1, except that it outputs* $\mathbf{z}_0 + x_j\mathbf{z}_1 = \mathbf{y} \leftarrow \mathsf{ECC.Decode}(\mathbf{x}_{-j}\mathbf{C} - (\mathbf{t}_0 + x_j\mathbf{t}_1))$.

It is clear that correctness still holds.

Theorem 3 (Security)**.** *Assume that the* $\mathsf{LWE}_{k-k',\beta,q}$ *assumption holds,* $\frac{q}{\sqrt{n}\tau(k)} > \beta \geq \delta_0 \geq 1$ *(where* $\tau(k) = \omega(\sqrt{\log k})$ *as in Lemma 6), and* $\delta_0\beta/\delta_1 = \mathsf{negl}(\lambda)$. *The protocol presented in Construction 1 securely realizes the functionality* $\mathcal{F}_{\mathsf{brOLE}}^{k'}$ *in the* $\mathcal{G}_{\mathsf{CRS}}$ *hybrid model against:*

- *a semi-honest sender given that the* $\mathsf{LWE}_{k-k',\beta,q}$ *assumption holds;*
- *a static malicious receiver where security holds statistically.*

The proof of the theorem stated above essentially follows the same blueprint as the proof of Theorem 3, except that the simulator for the corrupted receiver extracts the first k' coordinates $\{x_j\}_{j\in[k']}$ of $\mathbf{x}$ and sends these values to $\mathcal{F}_{brOLE}$. From now on, it behaves exactly as the simulator in the proof of Theorem 3. Indistinguishability of executions follows exactly the same reasoning.

4.4 Parameters to Implement the OLE.

Our protocol assumes the hardness of LWE with a super-polynomial modulus-to-noise ratio. This implies that the modulus q needs to be chosen as $2^{\omega(\log\lambda)}$. Thus, our protocol implements OLE for fields of size super-polynomial in the security parameter.

5 OLE from LWE secure against Malicious Adversaries

In this section, we extend the construction of the previous section to support malicious sender. The idea is to use a *cut-and-choose* approach via the use of an OT scheme in two rounds and extract the sender's input via the OT simulator.

5.1 Protocol

Construction 3. *The protocol is composed by the algorithms* $(\mathsf{GenCRS}, \mathsf{R}_1, \mathsf{S}, \mathsf{R}_2)$. *Let* $k, n, \ell, \ell', q \in \mathbb{Z}$ *such that* $q = 2^{\omega(\log\lambda)}$ *is a prime and* $n = \mathsf{poly}(k \log q)$*, and let* $\beta, \delta_0, \delta_1, \xi \in \mathbb{R}$ *such that* $\frac{q}{\sqrt{n}\tau(k)} > \beta \geq \delta_0 \geq 1$ *(where* $\tau(k) = \omega(\sqrt{\log k})$*, as in Lemma 6) and* $\frac{\beta\delta_0}{\delta_1} = \mathsf{negl}(\lambda)$. $\mathcal{F}_{\mathsf{OT}}$ *is the OT functionality as in Section 3. We now present the protocol in full detail.*

$\mathsf{GenCRS}(1^\lambda)$**:**

- *Sample* $\mathbf{A} \leftarrow\!\!\$\ \mathbb{Z}_q^{k\times n}$.
- *Choose a linear ECC* $\mathsf{ECC}_{\ell',\ell,\xi} = (\mathsf{ECC.Encode}, \mathsf{ECC.Decode}, \xi)$ *over* $\mathbb{Z}_q$.
- *Output* $\mathsf{crs} = (\mathbf{A}, \mathsf{ECC}_{\ell',\ell,\xi})$.

$\mathsf{R}_1(\mathsf{crs}, x \in \mathbb{Z}_q)$**:**

- *Parse* crs *as* $(\mathbf{A}, \mathsf{ECC}_{\ell',\ell,\xi})$.
- *Sample* $\mathbf{x} = (x_1, \ldots, x_k) \leftarrow\!\!\$\ \mathbb{Z}_q^k$ *such that* $x_i = x$ *and a small error vector* $\mathbf{e} \leftarrow\!\!\$\ \chi_\beta^n$*, for some index* $i \leftarrow\!\!\$\ [k]$.

- *Compute* $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$.
- *Additionally choose uniformly at random* $\mathbf{b} = (b_1, \ldots, b_\lambda) \leftarrow\!\!\$\, \{0,1\}^\lambda$ *such that the weight* $wt(\mathbf{b}) = \lambda/2$ *(i.e., half of the coordinates are* 0*). Send* $\{b_j\}_{j\in[\lambda]}$ *to the OT functionality* $\mathcal{F}_{\mathsf{OT}}$.
- *Output* $\mathsf{ole}_1 = (i, \mathbf{a}')$ *and* $\mathsf{st} = \left(\mathbf{x}, i, \{b_j\}_{j\in[\lambda]}\right)$.

$\mathsf{S}\left(\mathsf{crs}, (\mathbf{z}_0, \mathbf{z}_1) \in \mathbb{Z}_q^{\ell'}, \mathsf{ole}_1\right)$**:**

- *Parse* crs *as* $(\mathbf{A}, \mathsf{ECC}_{\ell',\ell,\xi})$ *and* ole_1 *as* $(i, \mathbf{a}')$.
- *For all* $j \in [\lambda]$*, do the following:*
  - *Sample* $\mathbf{R}_j \in \mathbb{Z}_q^{n\times\ell}$ *where each column is sampled according to* $D_{\mathbb{Z}^n,\delta_0}$.
  - *Compute* $\mathbf{s}_{0,j}, \mathbf{s}_{1,j} \in \mathbb{Z}_q^\ell$ *such that* $\mathbf{s}_{0,j} = \mathbf{a}'\mathbf{R}_j$, $\mathbf{s}_{1,j} = \mathbf{a}_i\mathbf{R}_j$, *and* $\mathbf{C}_j = \mathbf{A}_{-i}\mathbf{R}_j \in \mathbb{Z}_q^{(k-1)\times\ell}$.
  - *Compute* $\mathbf{t}_{0,j} = \mathbf{s}_{0,j} + \tilde{\mathbf{e}} + \mathsf{ECC.Encode}(\mathbf{u}_{0,j})$ *and* $\mathbf{t}_{1,j} = \mathbf{s}_{1,j} + \mathsf{ECC.Encode}(\mathbf{u}_{1,j})$*, for uniformly chosen* $\mathbf{u}_{0,j}, \mathbf{u}_{1,j} \leftarrow\!\!\$\, \mathbb{Z}_q^\ell$ *and* $\tilde{\mathbf{e}} \leftarrow\!\!\$\, D_{\mathbb{Z}^\ell,\delta_1}$.
  - *Send* $M_{0,j}, M_{1,j}$ *to* $\mathcal{F}_{\mathsf{OT}}$ *where* $M_{0,j} = \mathbf{R}_j$ *and* $M_{1,j} = (\mathbf{t}_{0,j}, \mathbf{t}_{1,j}, \mathbf{u}_{0,j} + \mathbf{z}_0, \mathbf{u}_{1,j} + \mathbf{z}_1)$.
- *Output* $\mathsf{ole}_2 = \{\mathbf{C}_j\}_{j\in[\lambda]}$.

$\mathsf{R}_2(\mathsf{crs}, \mathsf{st}, \mathsf{ole}_2)$**:**

- *Parse* crs *as* $(\mathbf{A}, \mathsf{ECC}_{\ell',\ell,\xi})$, ole_2 *as* $\{\mathbf{C}_j\}_{j\in[\lambda]}$ *and* st *as* $\left(\mathbf{x}, i, \{b_j\}_{j\in[\lambda]}\right)$.
- *If any of the matrices* $\mathbf{C}_j$ *is not full-rank, abort the protocol.*
- *For all* $j \in [\lambda]$*, do the following:*
  - *Recover* $M_{b_j,j}$ *from* $\mathcal{F}_{\mathsf{OT}}$.
  - *If* $b_j = 0$*, parse* $M_{0,j} = \tilde{\mathbf{R}}_j$. *If* $\mathbf{C}_j \neq \mathbf{A}_{-i}\tilde{\mathbf{R}}_j$ *and* $\tilde{\mathbf{R}}_j$ *is not small, abort the protocol.*
  - *If* $b_j = 1$*, parse* $M_{1,j}$ *as* $(\tilde{\mathbf{t}}_{0,j}, \tilde{\mathbf{t}}_{1,j}, \tilde{\mathbf{v}}_{0,j}, \tilde{\mathbf{v}}_{1,j})$. *Compute* $\tilde{\mathbf{y}}_j = \mathbf{x}_{-i}\mathbf{C}_j - (\tilde{\mathbf{t}}_{0,j} + x_i\tilde{\mathbf{t}}_{1,j})$ *and then run* $\mathbf{y}_j \leftarrow \mathsf{ECC.Decode}(\tilde{\mathbf{y}}_j)$. *If* $\mathbf{y}_j = \perp$*, abort the protocol. Else, compute* $\mathbf{w}_j = \tilde{\mathbf{v}}_{0,j} + x_i\tilde{\mathbf{v}}_{1,j} - \mathbf{y}_j$.
- *Check if* $\mathbf{w}_j = \mathbf{w}_{j'}$ *for all* (j, j') *such that* $b_j = b_{j'} = 1$. *If the test fails, abort the protocol.*
- *For any* j *such that* $b_j = 1$*, set* $\mathbf{w} = \mathbf{w}_j$. *Output* $\mathbf{w} \in \mathbb{Z}_q^{\ell'}$.

5.2 Analysis

We now proceed to the analysis of the protocol described above.

Theorem 4 (Correctness)**.** *Let* $\mathsf{ECC}_{\ell',\ell,\xi}$ *be a linear ECC where* $\xi \geq \sqrt{\ell}(\beta\delta_0 n + \delta_1)$*. Then the protocol presented in Construction 3 is correct.*

Proof. From the proof of Theorem 1, we have that $\mathbf{y}_j = \mathbf{u}_{0,j} + x_i\mathbf{u}_{1,j}$, except with negligible probability, when $b_j = 1$. Hence $\mathbf{w}_j = \tilde{\mathbf{v}}_{0,j} + x_i\tilde{\mathbf{v}}_{1,j} - \mathbf{y}_j = \mathbf{z}_0 + x_i\mathbf{z}_1$, which is equal to every other $\mathbf{w}_{j'}$ when $b_j = b_{j'} = 1$. □

Theorem 5 (Security)**.** *Assume that the* $\mathsf{LWE}_{n,k,\beta,q}$ *assumption holds* $\frac{q}{\sqrt{n}\tau(k)} > \beta \geq \delta_0 \geq 1$ *(where* $\tau(k) = \omega(\sqrt{\log k})$ *as in Lemma 6), and* $\delta_0\beta/\delta_1 = \mathsf{negl}(\lambda)$*. The protocol presented in Construction 3 securely realizes the functionality* $\mathcal{F}_{\mathsf{OLE}}$ *in the* $(\mathcal{G}_{\mathsf{CRS}}, \mathcal{F}_{\mathsf{OT}})$*-hybrid model against static malicious adversaries where:*

- *Security against corrupted sender holds given that the* $\mathsf{LWE}_{n,k,\beta,q}$ *assumption holds.*
- *Security against a corrupted receiver holds statistically in the* $(\mathcal{G}_{\mathsf{CRS}}, \mathcal{F}_{\mathsf{OT}})$*-hybrid model.*

Proof. In order to prove security against a static malicious receiver we have to show that there is a simulator that can simulate the view of the adversary in the ideal world.

Simulator for corrupted receiver: The proof is essentially the same as the proof of Theorem 3. Still, we present the simulator.

Let $(\mathsf{TdGen}, \mathsf{Invert})$ be the algorithms described in Lemma 6.

- **CRS generation:** Sim behaves as the simulator of the proof of Theorem 3 and additionally simulates $\mathcal{F}_{\mathsf{OT}}$.
- Upon receiving a message $\mathbf{a}'$ from R and $\{b_j\}_{j\in[\lambda]}$ through $\mathcal{F}_{\mathsf{OT}}$, Sim runs $\tilde{\mathbf{x}} \leftarrow \mathsf{Invert}(\mathsf{td}_{\mathbf{A}}, \mathbf{a}')$. There are two cases to consider:
  - If $\tilde{\mathbf{x}} = \perp$, then for each $j \in [\lambda]$ Sim does the following:
    - If $b_j = 0$, then Sim computes $\mathbf{C}_j = \mathbf{A}_{-i}\mathbf{R}_j$. It sets $M_{0,j} = \mathbf{R}_j$ and $M_{1,j} \leftarrow_{\$} \mathbb{Z}_q^{\ell} \times \mathbb{Z}_q^{\ell} \times \mathbb{Z}_q^{\ell'} \times \mathbb{Z}_q^{\ell'}$.
    - If $b_j = 1$, then Sim sets $\mathbf{C}_j \leftarrow_{\$} \mathbb{Z}_q^{k-1\times\ell}$, $M_{0,j} \leftarrow_{\$} \mathbb{Z}_q^{n\times\ell}$ and $M_{1,j} \leftarrow_{\$} \mathbb{Z}_q^{\ell} \times \mathbb{Z}_q^{\ell} \times \mathbb{Z}_q^{\ell'} \times \mathbb{Z}_q^{\ell'}$.
  - Else if $\tilde{\mathbf{x}} \neq \perp$, then Sim sets $x = x_i$ where x_i is the i-th coordinate of $\tilde{\mathbf{x}}$. It sends x to $\mathcal{F}_{\mathsf{OLE}}$. Upon receiving a $\mathbf{y}$ from $\mathcal{F}_{\mathsf{OLE}}$, Sim chooses uniformly at random two vectors $\mathbf{z}_0, \mathbf{z}_1 \in \mathbb{Z}_q^{\ell'}$ such that $\mathbf{z}_0 + x\mathbf{z}_1 = y$. Then, for every $j \in [\lambda]$, it does the following:
    - If $b_j = 0$, then Sim computes $\mathbf{C}_j = \mathbf{A}_{-i}\mathbf{R}_j$. It sets $M_{0,j} = \mathbf{R}_j$ and $M_{1,j} \leftarrow_{\$} \mathbb{Z}_q^{\ell} \times \mathbb{Z}_q^{\ell} \times \mathbb{Z}_q^{\ell'} \times \mathbb{Z}_q^{\ell'}$.

* If $b_j = 1$, then Sim samples a uniform matrix $\mathbf{U}_j \leftarrow_\$ \mathbb{Z}_q^{k\times\ell}$ and an error vector $\tilde{\mathbf{e}}_j \leftarrow_\$ D_{\mathbb{Z}^\ell,\sigma_1}$ and sets

$$\begin{aligned}\mathbf{C}_j &= \mathbf{U}_{-i,j}\\ \mathbf{t}_0 &= \tilde{\mathbf{x}}\mathbf{U}_j + \tilde{\mathbf{e}}_j + \mathsf{ECC.Encode}(\mathbf{u}_{0,j})\\ \mathbf{t}_1 &= -\mathbf{u}_{i,j} + \mathsf{ECC.Encode}(\mathbf{u}_{1,j})\end{aligned}$$

  where $\mathbf{u}_{i,j}$ is the i-th row of $\mathbf{U}_j$. It sets $M_{0,j} \leftarrow_\$ \mathbb{Z}_q^{n\times\ell}$ and $M_{1,j} = (\mathbf{t}_0, \mathbf{t}_1, \mathbf{u}_{0,j} + \mathbf{z}_0, \mathbf{u}_{1,j} + \mathbf{z}_1)$.
* It sends $\mathsf{ole}_2 = \{\mathbf{C}_j\}$.

The proof of indistinguishability of executions is essentially the same as the proof in Theorem 3. We stress that security still holds statistically in the $\mathcal{F}_{\mathsf{OT}}$ model.

Simulator for corrupted sender: We describe the simulator Sim against a corrupted sender.

- **CRS generation:** Sim generates crs as in GenCRS.
- Sim computes $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e} \in \mathbb{Z}_q^n$ for a uniformly chosen $\mathbf{x} \leftarrow_\$ \mathbb{Z}_q^k$ and $\mathbf{e} \leftarrow_\$ \mathbb{D}_{\mathbb{Z}^n,\beta}$, $i \leftarrow_\$ [k]$ and sends $(i, \mathbf{a}')$ to the sender.
- Upon receiving $(M_{0,j}, M_{1,j})$ from the sender (a message intended to $\mathcal{F}_{\mathsf{OT}}$), Sim does the following:
  - It chooses a partition of size $\lambda/2$ of $[\lambda]$. Let us denote this partition by $I_0 \cup I_1 = [\lambda]$.
  - It parses $M_{0,j}$ as $\mathbf{R}_j$ and $M_{1,j}$ as $(\mathbf{t}_{0,j}, \mathbf{t}_{1,j}, \mathbf{v}_{0,j}, \mathbf{v}_{1,j})$.
  - It checks if $\mathbf{C}_j = \mathbf{A}_{-i}\mathbf{R}_j$ for $j \in I_0$. If this does not happen for at least one index $j \in I_0$, it aborts.
  - Similarly, it checks if $\mathbf{w}_j = \mathbf{w}_{j'}$ for any pair of indices $(j, j') \in I_1 \times I_1$, where $\mathbf{w}_j = \mathbf{v}_{0,j} + x_i\mathbf{v}_{1,j} - \mathsf{ECC.Decode}(\mathbf{x}_{-i}\mathbf{C}_j - (\mathbf{t}_{0,j} + x_i\mathbf{t}_{1,j}))$. If this does not happen for at least one pair of indices, it aborts.
  - Check if there are positions j for which both $M_{0,j}$ and $M_{1,j}$ values are *correct*, meaning that the honest receiver does not abort for any of these values. Let j' be one of these positions. Then, Sim extracts $\mathbf{z}_0, \mathbf{z}_1$ in the following way: It computes $\mathbf{s}_{0,j}$, which allows to computes $\mathbf{u}_{0,j}$. Finally, it recovers $\mathbf{z}_0$ from $\mathbf{u}_{0,j} + \mathbf{z}_0$. The value $\mathbf{z}_1$ can be recovered similarly. It sends $\mathbf{z}_0, \mathbf{z}_1$ to $\mathcal{F}_{\mathsf{OLE}}$.

We first show that the Sim described above is correct and efficient.

Lemma 10. Sim *succeeds in extracting a unique pair* $(\mathbf{z}_0, \mathbf{z}_1)$*, except with negligible probability.*

The probability of Sim not extracting any pair $(\mathbf{z}_0, \mathbf{z}_1)$ is equal to the probability of not existing any position j' such that both values in the OT are correct given that Sim has not aborted before.

Note that this case only happens if the corrupted sender inputs malformed values for $M_{0,j}$ in $\lambda/2$ positions and malformed values for $M_{1,j}$ in the remaining $\lambda/2$ positions. Let E be the event where Sim aborts given this scenario. This happens with probability

$$\Pr[E] = 1 - \Pr[X = 0]$$

where $X \leftarrow\!\!\$\ \mathit{HyperGeom}(\lambda/2, \lambda/2, \lambda)$ where $\mathit{HyperGeom}$ is a hypergeometric distribution with parameters $(\lambda/2, \lambda/2, \lambda)$. Thus,

$$\Pr[E] = 1 - \frac{\binom{\lambda/2}{0}\binom{\lambda/2}{\lambda/2}}{\binom{\lambda}{\lambda/2}} = 1 - \mathsf{negl}(\lambda)\,.$$

We conclude that Sim aborts, except with negligible probability. Thus, it extracts a pair $(\mathbf{z}_0, \mathbf{z}_1)$, except with negligible probability. □

Lemma 11. *The extracted pair $(\mathbf{z}_0, \mathbf{z}_1)$ is unique, given that the $\mathsf{LWE}_{k,n,\beta,q}$ assumption holds.*

Assume that there is an adversary $\mathcal{A}$ corrupting the sender that is able to embed two pairs $(\mathbf{z}_0, \mathbf{z}_1) \neq (\mathbf{z}'_0, \mathbf{z}'_1)$ such that $\mathbf{z}_0 + x_i\mathbf{z}_1 = \mathbf{z}'_0 + x_i\mathbf{z}'_1$ in the execution of the protocol, with non-negligible probability ε. We show that, if this happens, then we can build an algorithm $\mathcal{B}$ that attacks the $\mathsf{LWE}_{k,\beta,q}$. The algorithm $\mathcal{B}$ takes as input an LWE challenge $(\mathbf{A}, \mathbf{a}') \in \mathbb{Z}_q^{k \times (n\lambda/\varepsilon)}$, parses $\mathbf{A} = (\mathbf{A}_1 | \ldots | \mathbf{A}_{\lambda/\varepsilon})$ and $\mathbf{a}' = (\mathbf{a}'_1 | \ldots | \mathbf{a}'_{\lambda/\varepsilon})$. Now, $\mathcal{B}$ runs λ/ε executions of $\mathcal{A}$ in parallel where, at each execution $j \in [\lambda/\varepsilon]$, it embeds $\mathbf{A}_j$ in the crs and simulate the receiver in the protocol by sending $(i, \mathbf{a}'_j)$, for $i \leftarrow\!\!\$\ [k]$.

Then, $\mathcal{B}$ extracts two different pairs $(\mathbf{z}_0, \mathbf{z}_1) \neq (\mathbf{z}'_0, \mathbf{z}'_1)$ while simulating $\mathcal{F}_{\mathsf{OT}}$ and computes for which x_i we have $\mathbf{z}_0 + x_i\mathbf{z}_1 = \mathbf{z}'_0 + x_i\mathbf{z}'_1$. If $\mathbf{a}' = \mathbf{x}\mathbf{A} + \mathbf{e}$, then after repeating this process λ/ε, we get that x_i is expected to be the same in λ executions. If this happens, then output 1. Else if $\mathbf{a}' \leftarrow\!\!\$\ \mathbb{Z}_q^n$ there is no information about $\mathbf{x}$ (and thus x_i). Hence the probability of extracting the same x_i in λ/ε executions is $\lambda/(\varepsilon|\mathbb{Z}_q|) = \mathsf{negl}(\lambda)$.

Hence, the advantage of $\mathcal{B}$ is non-negligible in attacking the $\mathsf{LWE}_{k,\beta,q}$. □

Lemma 12. *The ideal-world and real-world executions are indistinguishable, given that the $\mathsf{LWE}_{k,\beta,q}$ assumption holds.*

The simulator behaves and aborts with exactly the same probability as the real-world receiver. □

5.3 Instantiating the Functionality $\mathcal{F}_{\mathsf{OT}}$

Several two-round OT protocols exist in the literature that are proven to be UC-secure [PVW08, DGH+20]. When we instantiate our protocol with the OT scheme from [PVW08] we obtain several nice properties for the OLE scheme, namely:

1. The protocol is still two-round.
2. The protocol preserves statistical security against a corrupted receiver, since the OT of [PVW08] is also statistically secure against a corrupted receiver.
3. The OLE scheme is secure based only on the LWE assumption.

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