

STARK Friendly Hash – Survey and Recommendation

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Abstract

A report on the selection process of the STARK friendly hash (SFH) function for standardization by the Ethereum Foundation. The outcome of this process, described here, is our recommendation to use the Rescue function over a prime field of size $\approx 2^{61}$ in sponge mode with 12 field elements per state.

With an Appendix by Jean-Charles Faugère and Ludovic Perret of [CryptoNext Security](#).

1 Executive Summary

On July 2, 2018, the Ethereum Foundation gave StarkWare a 2-year milestone-based grant to select a STARK friendly hash (SFH) function, to be used in combination with transparent and plausibly post-quantum secure proof systems within blockchains, and release an open source efficient STARK system for it. Under the grant agreement, StarkWare committed to *surveying and benchmarking a shortlist of STARK-friendly hash function candidates (giving thorough consideration to MiMC), discussing and getting feedback on the various candidates and deciding on a single STARK-friendly hash function around which StarkWare will build optimized tooling*. This report answers this requirement.

Timeline A quick survey of pre-existing hash functions led to conclude that standard functions like SHA2/3 will require exorbitant proving time and that new constructions should be solicited and collected. A list of relatively new SFH candidates was collected including MiMC [AGRRT16] (which existed prior to this grant), GMiMC [AGPRRRRS19], HadesMiMC [GKKRRS19] and MARVELLous [AABDS19] – the latter two projects received funding from this grant. A set of specific members for consideration was curated (see Section 3), and their STARK complexity and computational efficiency were measured (see Section 6). Simultaneously, these candidates were exposed to public and scientific examination via three different efforts:

- A public [SFH challenge](#) for attacking candidate constructions at various security levels between 45 and 256 bits of security.
- Algebraic cryptanalysis was applied to all candidates by the CryptoNext Security (CNS) team, arguably the world-wide leading experts in this area [FP19; Fau20].
- Symmetric cryptanalysis was applied to the constructions on two separate occasions by two teams of expert cryptographers [Can+20; Bey+20; BCLNPPW20].

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Recommendations Based on the results of the security analysis described in Section 4 StarkWare “pruned” the list of candidates, focusing on two of the original contenders: the MiMC construction and the MARVELlous family. STARK arithmetization considerations led us to prefer constructions over small prime fields rather than large fields and/or binary ones, as explained in Section 5. **Consequently, we recommend a particular version of the Rescue hash function – a member of the MARVELlous family – over a prime field of size $\approx 2^{61}$ as the SFH function for standardization by the Ethereum Foundation. This particular version of Rescue is denoted as Rescue_{122} in Section 3.**

We stress that like all new cryptographic constructions, the level of trust placed in the security of Rescue_{122} and the amount of value that should be staked on its security should depend on its tenure and ability to withhold further security scrutiny. Therefore, we recommend a moderated pace of adoption and usage over the next 12 months, alongside continued support for researching the security of this SFH function. Any party staking significant value on the security of Rescue_{122} must take into consideration the potential impact of security flaws in it that may be discovered with time.

Our choice of Rescue_{122} is to a large extent the result of the short and aggressive timeline decided upon for this project. Our choice does not imply that other SFH candidates and/or members of the MiMC, GMiMC, HadesMiMC and MARVELlous are inadequate. Despite security concerns regarding certain variants of the SFH candidates [ACGKLR19; Bon19; EGLRSW20; KR20; Bey+20], all of the families considered plausibly contain parameter settings that make them secure hash functions, and some of these variants are more efficient computationally and have lower (i.e., better) STARK complexity than Rescue_{122} . However, suggesting such variants and studying them properly would have exceeded the time given to the project.

Finally, we note that the aggressive timeline of the project was shorter than commonly recommend for soliciting and selecting new cryptographic primitives. Therefore, we hope that the Ethereum Foundation and other blockchain and ZKP communities will view Rescue_{122} as a preliminary time-limited SFH construction, one meant to serve until the community concludes a longer process of selecting an SFH function that will improve on our current selection.

Overview of report Section 2 surveys the project timeline in detail. Section 3 describes the shortlist of specific SFH candidates examined by the public and by expert cryptographers. Section 4 discusses the results of cryptanalytic efforts carried out on the SFH candidates and our pruning of the list of candidates based on it. Section 5 explains how we decided on the field type and size for the SFH recommendation. Section 6 surveys the STARK complexity and computational efficiency of new and pre-existing hash functions. Finally, Section 7 integrates the information from previous sections and explains our final recommendation.

2 SFH project timeline

We recount the evolution of the project over time, explaining how we reached our recommendation (see Fig. 1).

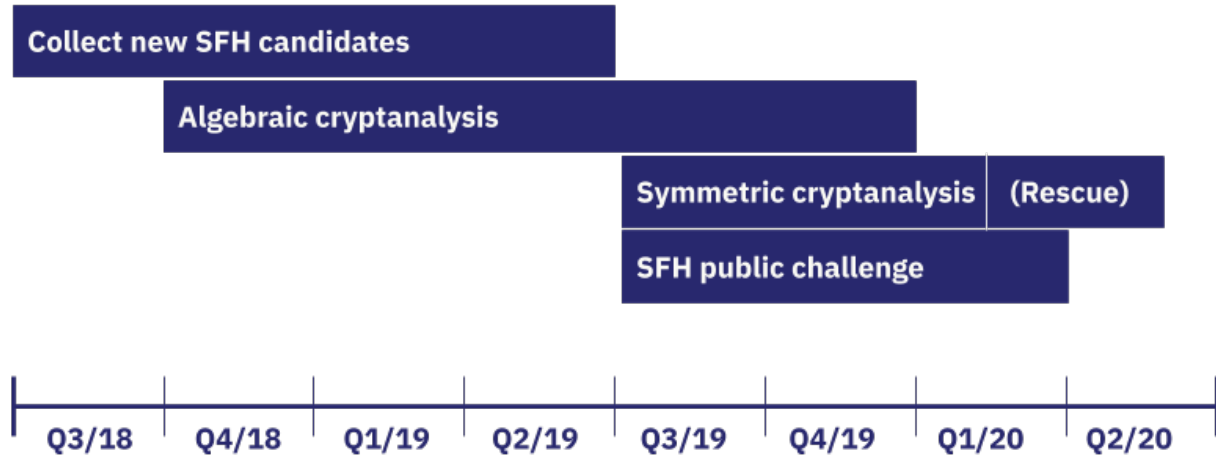


Figure 1: Timeline of SFH project.

- **July 2, 2018:** The 2-year grant agreement between StarkWare and the Ethereum Foundation is signed.
- **Q3/2018:** The Computer Security and Industrial Cryptography (COSIC) group at KU Leuven is contracted to curate a list of existing SFH candidates. It is quickly realized that, with the exclusion of MiMC [AGRRT16], the STARK complexity of existing hash functions (most notably, standard ones like SHA2/3) is prohibitively large. An alternative approach is agreed upon between StarkWare and the Ethereum Foundation: soliciting the construction of *new* SFH candidates.
- **Q3/2018–Q2/2019** New constructions of SFH candidates are collected. Several, like Pedersen¹ and MiMC, have existed before the project started. Others appeared independently during this time, like GMiMC [AGPRRRRS19]. Finally, two separate teams received support under the Ethereum Foundation grant to speedily advance the design of new families of primitives, leading to:
 1. **MARVELlous** – a family which includes Jarvis and Vision (binary fields) and Pepper and Rescue (prime fields), designed by a team of researchers from COSIC [AD18; AABDS19].
 2. **HadesMiMC** – a family which includes Starkad (binary fields) and Poseidon (prime fields), designed by a team of researchers from the IAIK (Institute for Applied Information Processing and Communication at the Graz University of Technology) [GLRRS20; GKKRRS19].

¹The Pedersen hash is quantum-susceptible, thus unsuitable for the EF goals of post-quantum secure constructions. However, it serves as a good benchmark for SFH functions.

- **Q4/2018–Q4/2019** To further understand the algebraic security of the SFH candidates, StarkWare contracted the CryptoNext Security (CNS) company led by Professors Jean-Charles Faugère and Ludovic Perret, who are members of the POLSYS project-team (Solvers for Algebraic Systems and Applications) of the Laboratoire d’Informatique de Paris 6 (LIP6), headed by Prof. Faugère. They examined the algebraic security of the above mentioned SFH candidates and submitted a series of reports to StarkWare and the Ethereum Foundation [FP19; Fau20], attached as Appendices **A** and **B** at the end of this report.
- **Q3/2019–Q1/2020** By the end of Q2/2019 it was clear that the list of SFH contenders should include MiMC, GMiMC, the MARVELlous family and the HadesMiMC family. Dr. Tomer Ashur from KU Leuven and Cryptomeria B.V. was contracted by StarkWare to help fix specific parameters to all constructions in a manner that would be comparable among different families and yet agreeable to the co-authors of those constructions. To foster examination of the constructions, a set of public challenges has been posted **online**, engaging the public in attempts to break the various candidates. A set of smart contracts have been deployed on Ethereum that reward the first finder of a collision in one of the constructions. The challenges cover several security levels – from 45 bits to 256 bits of security. At time of writing, a few of the 45-bit challenges have been broken (as anticipated) but none of the challenges at higher security levels (80-bits and above).
- **Nov 2019** A STARK-friendly hash selection committee, led by Prof. Anne Canteaut of INRIA Paris and including ten expert symmetric cryptographers convened in Paris for three days (November 11-13, 2019) to conduct cryptanalysis to the SFH candidates. The report submitted by this team [Can+20] (cf. [Bey+20]) recommended using the MARVELlous family, pending (i) further algebraic cryptanalysis of a specific question – the constrained input, constrained output (CICO) problem – and (ii) targeted cryptanalytic examination of Rescue.
- **Q2/2020** The CryptoNext Security company was contracted again to perform the needed algebraic analysis of the CICO problem for Rescue [Fau20] (cf. Appendix **B**). Additionally, a dedicated cryptanalytic effort targeted at Rescue was performed by a team of experts convened by the cryptosolutions company, led by Prof. Gregor Leander and Dr. Friedrich Wiemer [BCLNPPW20]. Both reports reached favorable results regarding the security of Rescue, leading to StarkWare recommending it for standardization as the SFH function for the Ethereum Foundation.

3 Specific SFH candidates considered

We assume familiarity with the candidate SFH families – MiMC, GMiMC, HadesMiMC and MARVELlous (see [Bey+20] for a concise description). Each of these can be instantiated in a variety of ways, based on the security level, choice of field, capacity and rate of sponge constructions, etc. For cryptanalytic purposes, all these parameters must be fixed. Dr. Tomer Ashur of Cryptomeria B.V. curated a list of specific parameters that was agreeable to the authors of the various SFH candidates. This list was used in the **public SFH challenge phase** and in the algebraic and symmetric cryptanalytic studies mentioned above. The specific candidates at the security level of ≈ 120 bits were the following constructions:



- **MiMC** Instantiated as a sponge construction using the MiMC– $2n/n$ Feistel construction described in [AGRR16, Section 2.1] over the prime field of size $2^{253} + 2^{199} + 1$ with 320 rounds. We denote this specific construction by MiMC_{126} since its conjectured security is at least 126 bits (subscripts reflect conjectured security also below).
- **Other** All other constructions were instantiated over fields of size $\approx 2^{64}$: the binary field used in Starkad and Vision was of size 2^{63} (using a standard basis over $\text{GF}(2)$, defined by the irreducible polynomial $x^{63} + x + 1$) and the prime field used by GMiMC, Poseidon and Rescue was of size $p = 2^{61} + 20 \times 2^{32} + 1$; elements of this field require 62 bits to be written, but later on, when considering security, we only assume the field size is at least 2^{61} . All constructions were in sponge mode with a state of 12 field elements, with a capacity of 4 field elements and rate of 8 field elements. Regarding the number of rounds:
 - **Starkad** was instantiated over the binary field with 8 full rounds and 43 partial rounds (total of 51 rounds). Denote this specific version by Starkad_{126} .
 - **Poseidon** was instantiated over the prime field with 8 full rounds and 40 partial rounds (total of 48 rounds). Denote this specific version by Poseidon_{122} .
 - **Vision** was instantiated over the binary field with 10 rounds, denoted Vision_{126} .
 - **Rescue** was instantiated over the prime field with 10 rounds, denoted Rescue_{122} .
 - **GMiMC** using the expanding round function (erf) version described in [AGPRRRS19, Section 2.1.2] was instantiated with the prime field and 101 rounds. We denote this version by GMiMC_{122} .

4 Security

The confidence we have in the security of a symmetric cryptographic primitive scales (roughly) as a function of the time that has lapsed since its invention, and the amount of usage it receives and value it supports (the “**Lindy Effect**”) and the quantity of public scrutiny it has received from cryptographers. According to these criteria, standard hash functions like SHA2, SHA3 and hash functions derived securely from block-ciphers like AES/Rijndael are deemed most secure. Closely following in terms of security are constructions that, though not selected as standards by national and international bodies, have been reviewed publicly for a large duration of time, and have been deployed in various settings. This family includes contenders for standardization like the SHA3 finalists, Blake, Grøstl, JH and Skein. (Jumping ahead, all of these constructions are extremely efficient but have prohibitively large STARK complexity.)

Next, we summarize our understanding of the security of the new SFH candidates. Our analysis is based entirely on the external sources written by expert cryptographers, namely, Professors Faugère and Perret for algebraic cryptanalysis [Appendices A and B], the reports of Prof. Canteaut’s team [Can+20] and the cryptosolutions team [BCLNPPW20], and the published papers [ACGKLR19; Bon19; Bey+20; KR20; EGLRSW20].

MiMC_{126} Among the SFH candidates, MiMC has received the largest amount of scrutiny due to its tenure and simplicity. To date, we are aware of two attacks on variants of it: the first, appearing in [Bon19], applies only to block-ciphers using the Feistel construction, but not the hash version, of which MiMC_{126} is a specific case. The second, recent, attack, applies to binary fields, and, to a

lesser extent, to extension fields [EGLRSW20] but not to the large prime field underlying MiMC₁₂₆. **Therefore, we consider MiMC₁₂₆ to be secure enough for consideration as the EF SFH.**

GMiMC₁₂₂, Starkad₁₂₆ and Poseidon₁₂₂ To date, several papers have revealed weaknesses in certain variants of the GMiMC family and the HadesMiMC family (to which Starkad₁₂₆ and Poseidon₁₂₂ belong) [Bon19; KR20; Bey+20]. The recent [Bey+20], which was written by the members of the SFH cryptanalytic committee (and is in line with their report [Can+20]) concludes by saying:

Our analysis of STARK-friendly primitives clearly shows that the concrete instances of GMiMC and HadesMiMC proposed in the StarkWare challenges present several major weaknesses, independently from the choice of the underlying finite field. At a first glance, the third contender involved in the challenges, namely Vision for the binary field and Rescue for the prime fields, seems more resistant to the cryptanalytic techniques we have used against the other two primitives. This seems rather expected since Vision and Rescue follow a more classical substitution permutation network (SPN) construction with full Sbox layers; for similar parameters, they include a larger number of Sboxes which may prevent them from the unsuitable behaviors we have exhibited on the other primitives. [Bey+20, Section 6]

Consequently, we removed GMiMC₁₂₂, Starkad₁₂₆ and Poseidon₁₂₂ from further consideration as the EF SFH. This does not mean the families GMiMC and HadesMiMC are insecure. Rather, our removal only implies that given the short timeline of our project and the desire for high security we would not feel comfortable recommending use of the very specific constructions GMiMC₁₂₂, Starkad₁₂₆ and Poseidon₁₂₂.

Vision₁₂₆ and Rescue₁₂₂ A precursor to Vision, named Jarvis, was shown to be insecure by both the algebraic cryptanalysis report [FP19] and an external publication [ACGKLR19], and this attack line could also be mounted on the unpublished precursor to Rescue, named Pepper (see [AD20] for a discussion of it). However, to date we are unaware of attacks on Vision₁₂₆ and Rescue₁₂₂ (as echoed in the report cited above). Furthermore, the dedicated algebraic cryptanalysis applied to the CICO problem for Rescue₁₂₂ (see Appendix B) and the dedicated cryptanalytic effort targeted at it led by the cryptosolutions team [BCLNPPW20] have not revealed any weaknesses in Rescue₁₂₂. To quote from the final report on Rescue by the latter team:

We considered several attack vectors and did not find attacks that pose a direct risk to the current design. Time was limited and we cannot exclude that other attack vectors or variants of the ones we considered lead to a full break of Rescue. [BCLNPPW20, Section 1]

Based on this, we find Vision₁₂₆ and Rescue₁₂₂ to be secure enough for consideration as the EF SFH, alongside MiMC₁₂₆.

5 Choice of finite field

Our recommendation takes into account the (i) type of field – prime vs. binary and (ii) ease of creating inputs to the hash function. Both aspects are discussed next.

Field type — binary vs. prime Binary fields have simpler arithmetic and are ideally suited for mapping bit-strings to field elements. Prime fields require arithmetic that is slightly more complicated and the mapping between bit strings and field elements is more problematic. However, prime fields offer an advantage in the context of STARK constructions: The evaluation domain of an algebraic execution trace can be picked to be a multiplicative subgroup of size precisely 2^k whereas no multiplicative sub-group of a binary field can be of such a size (but for the trivial group, that contains only the identity element). This means that over certain prime fields one may encode each column of the algebraic execution trace as a mapping from a group of size 2^k to the field, and then carry out quasi-linear time interpolation and multi-point evaluation via FFT and iFFT algorithms. Furthermore, the Fast RS IOPP (FRI) protocol [BBHR18] underlying STARK efficiency requires groups of size 2^k , and the same group used for arithmetization and encoding of the algebraic execution trace (AET), defined later in Definition 1, can be used in the FRI protocol. In the case of binary fields, the FRI protocol is carried out over an additive subgroup whereas the encoding of the AET is done over a subset that “embeds” a group of size $2^k - 1$ in a manner that is not as clean from an algebraic viewpoint (see [BBHR19] for details). **Due to this consideration, we find constructions over prime fields to be more efficient and focus our attention on these constructions².** Among the SFH contenders, the relevant ones are MiMC₁₂₆ and Rescue₁₂₂ (GMiMC₁₂₂ and Poseidon₁₂₂ have been excluded from the discussion for security reasons, as explained in Section 4).

Ease of creating inputs for the hash function Future uses of the SFH function will create STARK proofs that involve claims of computational integrity in addition to assertions about the correct invocation of the SFH. This is already the case for all instances of deployed ZKPs, like the Sprout and Sapling circuits of Zcash and StarkWare’s StarkEx system. Natural encoding via R1CS and AIR systems use elements of an ambient field to describe the computation whose integrity is being proved. Moving from an encoding in one field to a different encoding is costly. In this respect, MiMC₁₂₆ with its state consisting of just two field elements suffers from a disadvantage with respect to other constructions: it is not clear how to efficiently add new inputs to the hash function, since the internal state of the hash is comprised of a small number of large field elements. In contrast, all other SFH candidates are instantiated with a state comprised of many elements in smaller fields and thus can incorporate new inputs without necessarily moving between field representations.

To summarize, sponge constructions using a state with several field elements from a small field, like GMiMC₁₂₂, Starkad₁₂₆, Poseidon₁₂₂, Vision₁₂₆ and Rescue₁₂₂, are better suited to deal with general computational integrity statements than constructions, like MiMC₁₂₆, whose state holds a small number of large field elements.

²Another reason to prefer prime over binary fields is the current lack of native support in the Ethereum Virtual Machine for binary field multiplication, which would entail high gas costs for operations over binary fields.

6 STARK complexity and hash function efficiency

In this section we compare the computational efficiency and the proving complexity of the different SFH candidates (and pre-existing hash functions).

6.1 A simple STARK complexity measure

When discussing STARK complexity we focus on prover complexity because it is the major bottleneck in this project, and also because verification time and proof size scale poly-logarithmically with proof size and are thus less affected by the efficiency of different constructions. In particular, verification time of under 10ms and a proof size of at most 200KB for STARK soundness error of at most 2^{-80} — as defined in the grant agreement — are attainable for all hash functions mentioned in this section.

We start by defining a simple metric to gauge the “STARK friendliness” of different cryptographic primitives. Before reaching this metric in Definition 2 we need a preliminary definition. Later on we shall use this metric to compare different SFH candidates (and other hash functions). We limit ourselves to the minimal requirements and refer the interested reader to prior research on post-quantum secure scalable and transparent proof systems in, e.g., [AHIV17; BBHR19; BCG-GRS19; COS20].

Definition 1 (Constraint systems and satisfiability). *Let \mathbb{F} be a finite field and w, t, d be integers. An (\mathbb{F}, w, t, d) -constraint is a multivariate polynomial C of total degree d over variables $X[i, j], i \in \{1, \dots, t\}, j \in \{1, \dots, w\}$.*

An (\mathbb{F}, w, t, d) -constraint system S is a set of (\mathbb{F}, w, t, d') -constraints for $d' \leq d$, where d , the degree of S , is the maximal degree of a constraint in S .

An (\mathbb{F}, w, t) -algebraic execution trace (AET) is an array A with t rows and w columns, each entry of which is an element of \mathbb{F} . The i -th row represents the state of a computation at time i and the j -th column tracks the state of an algebraic register over time. We denote the i, j entry of an AET A by $A[i, j]$. We say that A satisfies S if and only if for every constraint $C(X[i_1, j_1], \dots, X[i_k, j_k])$ in S we have $C(A[i_1, j_1], \dots, A[i_k, j_k]) = 0$.

With this definition in hand we proceed to define STARK complexity for purposes of measuring different SFH candidates.

Definition 2 (STARK complexity). *Let $f : \mathbb{F}^m \rightarrow \mathbb{F}^n$ be a function. We say that an (\mathbb{F}, w, t, d) -constraint system S implements f if all of the following conditions hold:*

- **i/o mapping** *There exist $n + m$ distinct indices $I_1, \dots, I_n, O_1, \dots, O_m \in [t] \times [w]$ called the input and output indices. Informally, we expect an (\mathbb{F}, t, w) -AET A to contain the input and output of f in these locations.*
- **completeness** *For every input $x = (x_1, \dots, x_n) \in \mathbb{F}^n$ and output $y = (y_1, \dots, y_m) = f(x) \in \mathbb{F}^m$, there exists an (\mathbb{F}, t, w) -AET A_x that satisfies S and furthermore $A_x[I_i] = x_i$ for all $i \in [n]$ and $A_x[O_j] = y_j$ for all $j \in [m]$.*
- **efficiency** *There exists an efficient algorithm that, given $x \in \mathbb{F}^n$, will produce an AET A_x that satisfies the completeness condition above.*

- **soundness** Given $A \in \mathbb{F}^{t \times w}$ denote $x(A) = (A[I_1], \dots, A[I_n])$ and $y(A) = (A[O_1], \dots, A[O_m])$. If $y(A) \neq f(x(A))$ then A does not satisfy S .

The STARK complexity of S is now defined as

$$c(S) := \lceil \log_2 |\mathbb{F}| / 64 \rceil^2 \cdot (d + w) \cdot t \log_2 t \quad (1)$$

and the STARK complexity of f , denoted $c(f)$, is the minimal STARK complexity of S taken over all S that implements f .

A few remarks are in place regarding the definition above.

- A trivial optimization is to convert a $t \times w$ table to a different one with $t' = 1$ and $w' = t \cdot w$. While this slightly reduces the complexity measure in Eq. (1), it also increases verifier complexity, not discussed here. The exact choices of t and w appearing in Table 1 reflect a setting that we believe optimizes the tradeoff between proving and verification time (details omitted). Readers interested in optimizing t and w in various settings are referred to [BBHR19], where the tradeoff is explained.
- Practically speaking, for all but trivial functions we cannot determine $c(f)$ exactly because proving lower bounds on $c(f)$ for nontrivial functions is beyond our reach. Thus, we can only rely on ever decreasing upper bounds on $c(f)$. Indeed, since we first examined the question of STARK complexity, a number of new techniques like holography [COS20; CHMMVW20] and lookup tables [GW20] may lead to better (lower) complexity measures for various functions. Relatedly, notice that determining the exact finite field over which to define the constraint system could drastically affect complexity. For practical reasons we limited our attention to a few “natural” candidates for the different hash functions.
- We intentionally left the notion of “efficiency” undefined in Definition 2. Asymptotically, it could be defined in the usual way, as polynomial running time. Practically, the exact polynomial time for this item did not enter our considerations and estimates
- The actual STARK proving time also depends on the rate chosen to encode the AET. Since its effect on all hash functions is similar, for the sake of comparing SFH candidates it can be omitted.

6.2 STARK complexity

Table 1 summarizes the STARK complexity of various hash functions according to Eq. (1) for the statement requested by the Ethereum Foundation in the grant agreement. The statement refers to a hash-chain of length 10^5 , i.e., involving 10^5 invocations of the hash function, and colors reflect STARK efficiency (green is good, red is bad).

Explanation of Table 1 The hash functions are sorted by families. We start with the standard functions SHA2 and SHA3 which did not have STARK complexity as their design goal, leading to large complexity measure c . The subscripts in all hash function names designate the number of bits of security that they claim. For SHA2, the estimate over the 64-bit binary field is taken from [BBHR19] and that paper also gives the estimates for the AES-based hash functions which are

Family	Name	\mathbb{F}	t / invocation	w	d	c / 10^5 invocations
Standard	SHA2 ₁₂₈	64-bit prime	1000	20	2	5.6×10^{10}
		$\text{GF}(2^{64})$	3762	56	11	7.2×10^{11}
	SHA3 ₁₂₈	$\text{GF}(2^{64})$	1536	25	2	1.1×10^{11}
AES-DM	AES ₆₄	$\text{GF}(2^{64})$	48	62	8	7.5×10^9
	Rijndael ₈₀	$\text{GF}(2^{64})$	58	68	8	9.9×10^9
Algebraic Sponge	Pedersen ₁₂₈	256-bit prime	128	16	2	8.7×10^{10}
	MiMC ₁₂₆	253-bit prime	320	2	3	6.4×10^{10}
	GMiMC ₁₂₂	61-bit prime	101	1	3	9.4×10^8
	Starkad ₁₂₆	$\text{GF}(2^{63})$	10	14	3	3.4×10^8
	Poseidon ₁₂₂	61-bit prime	8	17	3	3.1×10^8
	Rescue ₁₂₂	61-bit prime	10	12	3	3×10^8
	Vision ₁₂₆	$\text{GF}(2^{63})$	20	12	6	7.5×10^8
40			12	4	1.4×10^9	

Table 1: The STARK complexity (c) of various hash functions for 100,000 invocations, computed according to Eq. (1). See Section 6.2 for explanations.

built using the Davies–Meyer (DM) construction; we denote this family by AES-DM (see [BBHR19, Figure 4] for details). The family of algebraic functions, used in sponge mode, includes pre-existing functions like Pedersen (which is not post-quantum secure) and MiMC as well as the new SFH candidates. In the table, the third column (\mathbb{F}) is the field over which the function is defined (notice that for SHA2 we provide two options). The fourth column specifies the length of the AET for a single invocation and the next column specifies the number of columns (see the discussion following Definition 2 for potential tradeoffs between t and w). The final column gives the STARK complexity c according to Eq. (1) for 100,000 invocations of the hash function.

6.3 Efficiency as hash functions

One of the most important parameters in the selection of standard hash functions (like SHA2/3) is their efficiency and speed. In the context of this project efficiency considerations take second place compared with STARK complexity (c) and security but are nevertheless important to consider. Table 2 approximate the efficiency of the various SFH candidates on current CPUs, using the number of 64-bit multiplication operations as a rough estimate. We stress that we have not dedicated considerable efforts to optimizing any of these constructions. The number of rounds is denoted by R ; For the HadesMiMC members (Starkad₁₂₆ and Poseidon₁₂₂) we use R_f for full rounds and R_p for partial rounds. We explain the formulas used in the fifth column below:

- **Rescue₁₂₂**: Each round involves two steps. Each step involves a linear transformation, costing m^2 multiplications, one step requires a cubing operation (2 multiplications) per state variable and the other step requires a single cube-root operation per state variable (involving 68 multiplications).
- **MiMC₁₂₆**: We use the $2n/n$ Feistel construction described at the end of Section 2.1 of [AGRRT16]. This involves 320 rounds over a state of size 2. Each field multiplication over the large (≈ 256 -bit) field is approximated by 16 field operations over a 64-bit field.

SFH	$\log \mathbb{F} $	Rounds R	State m	# 64-bit multiplications	Total
Rescue ₁₂₂	61	10	12	$(2m^2 + 70m) \times R$	11280
MiMC ₁₂₆	256	320	2	$16 \times 2 \times R$	10240
Vision ₁₂₆	63	10	12	$(2m^2 + 8m + 2(\log \mathbb{F} + 3m)) \times R$	5820
Starkad ₁₂₆	63	$R_f = 8, R_p = 43$	12	$(m^2 + 2m) \times R_f + (2m + 2) \times R_p$	2462
Poseidon ₁₂₂	61	$R_f = 8, R_p = 40$	12	$(m^2 + 2m) \times R_f + (2m + 2) \times R_p$	2384
GMiMC ₁₂₂	61	101	12	$2 \times R$	202

Table 2: The estimated efficiency of new SFH candidates, sorted in decreasing order (lower is better). See Section 6.3 for explanations.

- **Vision₁₂₆**: Each round of Vision involves two steps. In each step we apply inversion for each state variable. However, using Montgomery inversion this amounts to a single inversion ($\log |F|$ multiplications) and 3 additional multiplications per state variable. Additionally, we need to apply a linear transformation (m^2 multiplications per step) and a linearized polynomial of degree 4 (4 multiplications per state variable), or its inverse, which we assume amounts to 4 multiplications as well.
- **Starkad₁₂₆** and **Poseidon₁₂₂**: Each full round requires one cubing per state variable (2 multiplications) and one linear transformation on all m state variables (m^2 multiplications). Each partial round involves only cubing a single variable and adding a linear combination of other state variables, totaling $2m + 2$ multiplications per (partial) round.
- **GMiMC₁₂₂**: Each round of GMiMC involves only a single cubing (2 multiplications), leading to extreme efficiency (10× better than all other constructions discussed above).

7 Rescue₁₂₂ as the recommended SFH for the EF

We end the report by summarizing the reasoning that led us to recommend Rescue₁₂₂ as the SFH candidate for standardization by the Ethereum Foundation.

Previous standard constructions like SHA2/3 have exorbitant STARK complexity (see Table 1). Among newer constructions, we focus on constructions over prime fields, as explained in Section 5. While the MiMC hash family is simplest and oldest, the reliance of MiMC₁₂₆ on a large prime field makes its STARK complexity worse than other candidates and its ability to incorporate new inputs in an efficient algebraic manner makes it unsuitable for standardization. The security analysis described in Section 4 suggests avoiding GMiMC₁₂₂ and Poseidon₁₂₂. This leaves us with Rescue₁₂₂ as the only remaining choice, by way of elimination.

Rescue is a new construction and thus has not received as much scrutiny as MiMC. However, its similarity to the familiar SPN construction (used by AES) and the preliminary algebraic and symmetric cryptanalysis results indicate it seems suitable for use. In terms of efficiency, Rescue₁₂₂ lags behind Poseidon₁₂₂ but not by a large factor, and when considering STARK complexity, Rescue₁₂₂ is slightly better (i.e., has smaller c) than Poseidon₁₂₂.

We close by pointing out that like all new cryptographic constructions, more scrutiny is essential for improving confidence in the security of Rescue₁₂₂. We therefore strongly recommend to the

Ethereum Foundation to consider further funding of public challenges and cryptanalytic research focused on Rescue₁₂₂, as well as on the large families of GMiMC, HadesMiMC and MARVELlous.

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A CryptoNext Security Report

Algebraic Attacks against STARK-Friendly Ciphers (Version 1.2)

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Abstract. This report presents a security analysis of the so-called STARK-friendly ciphers : **Jarvis**, **MiMC**, **Pepper**, **Vision**, **Rescue**, **Poseidon** and **Starkad**. These primitives are currently candidates for being the default cipher in STARK applications. The goal of this report is to assess the security of these primitives with respect to some algebraic attacks. This report has been prepared at the demand of Starkware Industries. The analysis has been performed during two periods : from November 2018 to September 2020 and in February 2020 (for an additional assessment on **Rescue**).

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1 Introduction

The rise of practical applications of advanced cryptographic protocols such as homomorphic encryption, Multi-Party Computation (MPC) or Zero-Knowledge (ZK) proof system emphasized the needs to design symmetric primitives that are optimized with respect to specific metrics related to the applications considered.

In particular, the ZK-STARK proof system [10] offers zero-knowledge proofs in which verification scales exponentially faster than data size. ZK-STARK proposed then a viable solution to the scalability problem in cryptocurrencies. In particular, the ZK-STARK technology is expected to be deployed on top of the Ethereum¹ blockchain in 2020.

The security and efficiency of ZK-STARK proof systems rely, in particular but crucially, on the security of a hash function. However, it turns that standard constructions – such as SHA2 [42] or the Davies–Meyer hash function based on Rijndael [20] – are too costly for the ZK-STARK technology. Regarding the potential applications of the such technology, it is of primary importance to design secure hash functions optimized for this ZK-proof system.

As formalized in [5], this leads to revisit the classical design rationale of block ciphers and hash functions based on such block ciphers. Following the terminology of [5], we shall call *Arithmetization-Oriented (AO) ciphers* this new design paradigm. In contrast to traditional block ciphers design, the main attacks to consider for AO ciphers are not of statistical nature but of algebraic nature.

The goal of this report is to investigate the security of STARK-friendly ([2,5,3,32]) primitives against algebraic attacks. More precisely, we have considered :

Jarvis [5]	The first generation of block cipher optimized for STARK-proofs
MiMC [2]	A lightweight block cipher well suited for the STARK technology
Pepper/Rescue/ Vision [3]	Second generation of AO ciphers that fix some weaknesses of Jarvis
Poseidon/Starkad [32]	Latest design of STARK-friendly hash functions

Table 1. List of STARK-friendly primitives considered in this report.

1.1 Overview of the Results

This report presents a security analysis of AO ciphers listed in Table 1 against algebraic attacks. An algebraic attack has typically two steps :

- *modelling* the cipher as a set of algebraic equations and,
- *solving* the non-linear equations to recover a secret of the cipher (secret-key or pre-image in our context). In this document, we use a Gröbner basis [14,13] (Section 2.1) for the solving step.

It is well known that algebraic attacks against modern block ciphers have had – so far – a limited impact on the security. This can be explained by at least two facts. First, it is challenging to perform practical experiments on sufficiently relevant reduced version of the

¹<https://medium.com/starkware/stark-friendly-hash-tire-kicking-8087e8d9a246>

ciphers [18]. Secondly, it is difficult to derive precise complexity bounds on the complexity of such attacks [6]; in particular on the solving step.

However, as emphasised in [5], the issue is central for STARK-friendly ciphers:

“Consequently, the question of security against Gröbner basis attacks is the crucial concern raised by arithmetization-oriented ciphers, and no such proposal is complete without explicitly addressing it”.

In this report, we present new results on the algebraic cryptanalysis of STARK-friendly ciphers (those listed in Table 1). We emphasize that a follow-up report considered [23] a broader set of attacks against AO ciphers; including the ciphers listed in Table 1.

1.1.1 Methodology. Starkware Industries provided a description and a SAGE ([43]) reference code of the primitive to analyse. We then implemented the primitive in MAGMA [12] an/or MAPLE [39] and always compared the output of the function with the reference SAGE implementation.

	Jarvis	Jarvis-v2	Pepper-v1	Pepper-v2	Rescue	Vision	MiMC	Poseidon	Starkad
Sage (ref)	✗	✗	✗	✗	✗	✗	✗	✗	✗
Maple	✗	✗	✗	✗	✗	✗	✗		
Magma	✗	✗	✗	✗	✗	✗	✗	✗	

Table 2. Summary of the codes developed.

In addition, we developed a new dedicated C program to attack some of the primitives.

1.1.2 How to Fix the Number of Rounds. Given a security level s , the main question addressed in this work is to derive a minimal condition on the number N of rounds of STARK-friendly ciphers to reach a security of s bits. To do so, we developed a common methodology to attack STARK-friendly ciphers.

Let $D(N)$ be the the number of solutions of the algebraic system modelling a STARK-friendly cipher with N rounds. We present an algebraic attack whose complexity is polynomial in the number of solutions, i.e.

$$O(D(N)^\omega), \text{ where } \omega \text{ is } 1 \text{ or } 2.$$

This leads to the following methodology for deriving the minimal number of rounds.

- 1. Estimate the number $D(N)$ of solutions.** To do so, we performed practical experiments and used theoretical bounds such as the *Bézout bound* (Definition 1) and the so-called *multi-homogeneous Bézout bound* (Definition 2, [35,38,22]). The former bound is especially interesting since it takes advantage of the structure and leads to tighter complexity results. To our knowledge, this is the first time that such *multi-homogeneous Bézout bounds* are used cryptography.

2. **Choose** N such that :

$$D(N)^\omega > 2^s.$$

Thanks to fast linear algebra techniques (Section 2.2.2, [25,24]), we can always assume that $\omega = 2$. We will see that for some primitives ω can be even chosen equal 1.

In the following tables, we summarize the value of $D(N)$ for STARK-friendly ciphers, the number of rounds N initially suggested by the designers to reach a security level of 128 bits and the number of rounds N^* derived from our new attacks.

1.2 Algebraic Key-Recovery

Here, we provide the results obtained for an algebraic key-recovery against various AO block ciphers.

	MiMC	Jarvis	Jarvis-v2	Pepper-v1	Pepper-v2	Rescue
Field	\mathbb{F}_{2^n}	\mathbb{F}_{2^n}	\mathbb{F}_{2^n}	\mathbb{F}_p	\mathbb{F}_p	\mathbb{F}_p
D	3^N	$(N+1)4^N$	$(5N+3)4^{N-2}$	$4N3^{N-1}$	3^{N+1}	$3^{(N+1)m}$
N	82	10	10			32
N^*	≥ 81	≥ 62	≥ 62	≥ 77	≥ 80	$\geq 20(m=2)$

Table 3. Minimal number of rounds for MiMC, Jarvis, Pepper and Rescue.

	Vision		
Field	\mathbb{F}_{2^n}		
m	2	3	4
D	$\approx \frac{15}{256} 2^{6N}$	$\approx \frac{4615N}{589824} (2^6 3^2)^N$	$\approx 0.0022(2^{10} 3^2)^N$
N^*	12	≥ 8	≥ 6

Table 4. Minimal number of rounds for Vision.

1.3 Algebraic Pre-Image

Here, we describe the results obtained for an algebraic pre-image on the permutation induced by AO ciphers. In this context, we consider the so-called *Constrained-Input Constrained-Output* (CICO) problem introduced in [32]. The CICO problem is then capturing a pre-image attack against the permutation defining a sponge hash function [11].

Let $\text{perm} : \mathbb{K}^m \rightarrow \mathbb{K}^m$ be a permutation. The CICO problem considered here is defined as follows:

<p>CICO(k) Find $(x_{k+1}, \dots, x_t) \in \mathbb{K}^{m-k}$ such that $\mathbf{x} = (0, \dots, 0, x_{k+1}, \dots, x_t) \in \mathbb{K}^m$ // Constrained input $\mathbf{y} = \text{perm}_{\text{Hades}}(\mathbf{x})$ $y_1 = \dots = y_k = 0$ // Constrained output</p>

Fig. 1. The CICO problem.

CICO *Problem on Poseidon and Starkad*. The results are summarized in Table 5.

Name	Security level [32]	Field	n	R_F	R_P	t	Complexity of our attack	Cost
Poseidon-256	2^{128}	\mathbb{F}_p	256	8	84	6	$3^{R_F+R_P-1}$	$2^{144.2}$
Starkad-252	2^{126}	\mathbb{F}_{263}	63	12	39	24	$3^{2(4R_F+R_P-8)+2}$	$2^{253.6}$

Table 5. Complexity of our attack against Poseidon and Starkad. n is the bit size of the field, R_F (resp. R_P) refers to the numbers of full (resp. plain) rounds and t is the number of field elements (parameters of Poseidon and Starkad defined in Section 4.4).

CICO *Problem on Rescue*. In the particular case of Rescue, we considered algebraic key-recovery on the block cipher (Table 3) as well as algebraic pre-image against the Rescue permutation (Table 6).

k	1	2	> 2
Complexity for Rescue	$3^{(N-1)m+k}$	$3^{(N-1)m+k}$	$3^{2((N-1)m+k)}$

Table 6. Complexity of our attack against Rescue. k is the CICO parameter, N is the number of rounds and m is the number of field elements (parameters of Rescue are defined in Section 4.3.2).

This yields, in particular, for the 128-bit challenge parameters defined for the STARK-Friendly Hash (SFH) Challenge²:

Name	Field	$n = \log_2(p)$	N	m	Exhaustive search	Algebraic attack
Rescue128a	\mathbb{F}_p	125	16	4	2^{250}	$2^{98.3}$
Rescue128b	\mathbb{F}_p	253	22	12	2^{253}	$2^{401.3}$
Rescue128c	\mathbb{F}_p	125	10	12	2^{250}	$2^{174.3}$
Rescue128d	\mathbb{F}_p	61	10	12	2^{244}	2^{355}
Rescue128e	\mathbb{F}_p	253	10	11	2^{253}	$2^{158.5}$

Table 7. Challenge parameters for the Rescue-based hash function.

²<https://starkware.co/hash-challenge/>

1.4 Organisation of the Report

After this introduction, the paper is organized as follows. In Section 2, we introduce the algebraic tools used to analyse STARK-friendly ciphers such as Gröbner bases (Section 2.1) and multi-homogeneous systems (Section 2.3). In Section 3, we briefly review known results in the algebraic cryptanalysis of block ciphers. Section 4 describes the primitives analysed in this report (the ciphers listed in Table 1). We then describe key-recovery attacks against Jarvis (Section 5), MiMC (Section 6), Pepper (Section 7), Rescue (Section 8) and Vision (Section 9). Finally, we present pre-image attacks in Section 10 : against Starkad/Poseidon (Section 10.1) and Rescue (Section 10.2).

This report has been prepared at the demand of Starkware Industries. The analysis has been performed during two periods (see Section 11 an history of the document) : from November 2018 to September 2019 and then in March 2020 (the latest period was dedicated to investigate the CICO problem on Rescue(Section 10.2)). This final document is a compilation of several intermediate reports. The goal of this final report is to present the complexity of our attacks and to detail the main ideas. On the other hand, some proofs are only sketched and few technical details are omitted.

2 Polynomial System System Solving and Gröbner Bases

A central task in the algebraic cryptanalysis of STARK-friendly primitives is to solve a system of non-linear equations over a finite field \mathbb{F}_q (with $q = p^s$, p prime and $s > 0$). This classical NP-hard problem [31] is defined as follows:

Polynomial System Solving over a Finite Field (PoSSo_q)

Input. $p_1(x_1, \dots, x_n), \dots, p_m(x_1, \dots, x_n) \in \mathbb{F}_q[x_1, \dots, x_n]$.

Goal. Find – if any – a vector $(z_1, \dots, z_n) \in \mathbb{F}_q^n$ such that:

$$p_1(z_1, \dots, z_n) = 0, \dots, p_m(z_1, \dots, z_n) = 0.$$

In this work, we will use Gröbner bases [14,13] for solving the instances of PoSSo_q arising in our algebraic attacks. The rational is that we have efficient algorithms, such as the F₄ and F₅ algorithms [26,27]), as well as software (typically, Magma and Fgb [12,28]) to compute these bases and perform large scale experiments. In addition, a rich set of tools from computer algebra/algebraic geometry allow to understand the complexity of computing such bases. Finally, Gröbner bases algorithms turn to be quite flexible for taking advantage of *structured systems* that naturally appear in algebraic cryptanalysis. All these features will be illustrated in this report.

Before that, we briefly review some basic facts about Gröbner bases, recall basic complexity results and finally introduce the concept of multi-homogeneous systems.

2.1 Gröbner Bases

A *monomial* in $\mathbb{F}_q[x_1, \dots, x_n]$ is a power product of the variables, i.e. an element of the form $x_1^{\alpha_1} \dots x_n^{\alpha_n}$. The *leading monomial* of $p \in \mathbb{F}_q[x_1, \dots, x_n]$ – denoted by $\text{LM}(p, \prec)$ – is the largest monomial w.r.t. some admissible monomial ordering \prec among the monomials of p .

Remark 1. LEX and DRL are two widely used examples of monomial orderings. Given $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, they are defined as follows:

- $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \prec_{\text{LEX}} x_1^{\beta_1} \cdots x_n^{\beta_n}$ if the first left-most non-zero entry of $\beta - \alpha$ is positive.
- $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \prec_{\text{DRL}} x_1^{\beta_1} \cdots x_n^{\beta_n}$ if $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$, or $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and the right-most non-zero entry of $\beta - \alpha$ is negative.

We can now define Gröbner basis [14,13].

Definition 2.1 (Gröbner basis). *Let $\mathcal{I} = \langle p_1, \dots, p_m \rangle \subset \mathbb{F}_q[x_1, \dots, x_n]$ be a polynomial ideal. A subset $G \subset \mathcal{I}$ is a Gröbner basis – w.r.t. an admissible monomial ordering \prec – of \mathcal{I} if:*

$$\forall p \in \mathcal{I}, \text{ there exists } g \in G \text{ such that } \text{LM}(g, \prec) \text{ divides } \text{LM}(p, \prec).$$

It is clear from Definition 2.1, that the notion of Gröbner bases depends on a admissible monomial ordering. Gröbner bases have different computational and algorithmic properties with respect to the monomial ordering considered. Typically, LEX-Gröbner bases (i.e. Gröbner bases w.r.t. the lexicographical ordering) allow to eliminate variables whilst DRL is a more computational-friendly ordering as we will see.

In algebraic cryptanalysis, we are usually not interested in the Gröbner basis itself but rather in the set of solutions, or *variety*.

Definition 2.2 (Variety). *Let $\mathbb{F}_q \subset \mathbb{L}$ and $\mathcal{I} = \langle p_1, \dots, p_m \rangle \subset \mathbb{F}_q[x_1, \dots, x_n]$ be an ideal. We denote by*

$$V_{\mathbb{L}}(\mathcal{I}) = V_{\mathbb{L}}(p_1, \dots, p_m) = \{ \mathbf{z} = (z_1, \dots, z_n) \in \mathbb{L}^n \mid p_i(\mathbf{z}) = 0, \forall i, 1 \leq i \leq m \},$$

the \mathbb{L} -variety associated to \mathcal{I} , i.e. the common zeroes over \mathbb{L}^n of p_1, \dots, p_m . When $\mathbb{L} = \overline{\mathbb{F}_q}$, we simply denote $V(\mathcal{I}) = V_{\overline{\mathbb{F}_q}}(\mathcal{I})$. We shall say that the ideal $\mathcal{I} \subset \mathbb{F}_q[x_1, \dots, x_n]$ is zero-dimensional if $|V(\mathcal{I})| < \infty$.

Let $\mathcal{I} \subset \mathbb{F}_q[x_1, \dots, x_n]$ be an ideal. A Gröbner basis allows to explicitly compute in the quotient space $\mathbb{F}_q[x_1, \dots, x_n]/\mathcal{I}$. In the zero-dimensional case, this quotient is a finitely generated vector-space.

Proposition 1. *Let $\mathcal{I} = \langle p_1, \dots, p_m \rangle \subset \mathbb{F}_q[x_1, \dots, x_n]$ be a zero-dimensional ideal and $G \subset \mathcal{I}$ be a Gröbner basis of \mathcal{I} w.r.t. an admissible monomial ordering \prec . $\mathbb{F}_q[x_1, \dots, x_n]/\mathcal{I}$ is a finite-dimensional vector space generated by :*

$$\{ \mathbf{x}_\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \mathbf{x}^\alpha \notin \langle \text{LM}(g, \prec) \mid g \in G \rangle \}.$$

A LEX-Gröbner basis permits to eliminate variables.

Theorem 1 (Elimination theorem). *Let $\mathcal{I} = \langle p_1, \dots, p_m \rangle \subset \mathbb{F}_q[x_1, \dots, x_n]$ be a polynomial ideal and $G \subset \mathcal{I}$ be a LEX-Gröbner basis of \mathcal{I} . It holds that:*

$$\forall \ell, 0 \leq \ell < n, G_\ell = G \cap \mathbb{F}_q[x_{\ell+1}, \dots, x_n],$$

is a LEX-Gröbner basis of $\mathcal{I} \cap \mathbb{F}_q[x_{\ell+1}, \dots, x_n]$.

In the zero-dimensional case, the variety can be then easily computed from a LEX-Gröbner basis. Indeed, we can deduce from the elimination property (Theorem 1) that a LEX-Gröbner

basis of a zero-dimensional ideal has always the following triangular shape:

$$\left\{ \begin{array}{l} h_n(x_n) \\ h_{n-1}^{(1)}(x_n, x_{n-1}) \\ \dots \\ h_{n-1}^{(k_1)}(x_n, x_{n-1}) \\ h_{n-2}^{(1)}(x_n, x_{n-1}, x_{n-2}) \\ \vdots \end{array} \right.$$

To compute the variety from a LEX-Gröbner basis, we then simply have to successively eliminate variables by computing zeroes of univariate polynomials and back-substituting the results.

2.2 Zero-Dimensional Solving

From a practical point of view, computing (directly) a LEX-Gröbner basis is usually slower than computing a Gröbner basis w.r.t. another monomial ordering. On the other hand, it is known that computing Gröbner bases w.r.t. to a degree reverse lexicographical (DRL-Gröbner bases) is much faster in practice.

The FLGM algorithm [29] permits – in the zero-dimensional case – to efficiently solve this issue. This algorithm uses the knowledge of a Gröbner basis computed for a given order to construct a Gröbner for another order. The complexity of FGLM is polynomial in the number of solutions of the considered ideal. This suggests the following strategy for computing the solutions of a zero-dimensional system $p_1 = 0, \dots, p_m = 0$.

-
1. Compute a DRL-Gröbner basis G_{DRL} of $\mathcal{I} = \langle p_1, \dots, p_m \rangle \subset \mathbb{F}_q[x_1, \dots, x_n]$.
 2. Compute a LEX-Gröbner basis of \mathcal{I} using FGLM on G_{DRL} .
-

Fig. 2. Zero-dim solving steps.

This is for instance the default strategy used in the MAGMA computer algebra system. We detail now the complexity of each part.

2.2.1 Complexity of Computing a DRL-Gröbner Bases. The complexity of computing a DRL-Gröbner bases will be related to the quantity defined below.

Definition 2.3 (Degree of regularity). Let $p_1, \dots, p_m \in \mathbb{F}_q[x_1, \dots, x_n]$ be homogeneous polynomials. We shall call degree of regularity of p_1, \dots, p_m , denoted by $D_{\text{reg}}(p_1, \dots, p_m)$, the smallest integer $D_0 \geq 0$ such that the polynomials of degree D_0 in $\mathcal{I} = \langle p_1, \dots, p_m \rangle$ generate – as a \mathbb{F}_q vector space – the set $\text{Monomials}_q(n, D_0)$ of all monomials of degree D_0 in n variables over \mathbb{F}_q , i.e.

$$D_{\text{reg}}(p_1, \dots, p_m) = \min \{ D_0 \geq 0 \mid \dim_{\mathbb{F}_q} (\{ p \in \mathcal{I} \mid \deg(p) = D_0 \}) = \#\text{Monomials}_q(n, D_0) \}.$$

Remark 2. The degree of regularity can be generalized to non-homogeneous polynomials $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{F}_q[x_1, \dots, x_n]^m$ by considering the degree of regularity of the homogeneous components of highest degrees $\mathbf{p}^H = (p_1^H, \dots, p_m^H) \in \mathbb{F}_q[x_1, \dots, x_n]^m$. Namely :

$$D_{\text{reg}}(p_1, \dots, p_m) = D_{\text{reg}}(p_1^H, \dots, p_m^H).$$

Once this notion fixed, we can rather easily establish an upper bound on the cost of computing a DRL-Gröbner basis [37,33,6,9,8].

Theorem 2. *Let $\omega, 2 \leq \omega \leq 3$ be the linear algebra constant and $p_1, \dots, p_m \in \mathbb{F}_q[x_1, \dots, x_n]$ such that $\mathbf{p}^H = (p_1^H, \dots, p_m^H)$ is a zero-dimensional system. Let then $D_{\text{reg}} = D_{\text{reg}}(p_1, \dots, p_m)$ and \prec be a total degree monomial ordering. We can compute a Gröbner basis of $\langle p_1, \dots, p_m \rangle$ with respect to \prec in*

$$O\left(m \cdot \binom{n + D_{\text{reg}}}{D_{\text{reg}}}\right)^\omega \text{ arithmetic operations over } \mathbb{F}_q. \quad (1)$$

with $2 \leq \omega < 2.3728639$ the linear algebra constant [44,30].

The complexity of computing a DRL-Gröbner basis is then exponential in the degree of regularity. Unfortunately, this degree of regularity is difficult to compute in general; as difficult as computing the Gröbner basis. Fortunately, there is a particular class of systems for which this degree can be computed efficiently : (*regular* and) *semi-regular sequences*.

The notion of regular sequences is classical [37,36] but holds only for $m \leq n$. Semi-regular sequences, that we recall below, have been introduced in [6,9] for over-defined systems. In essence, the algebraic systems encountered in the cryptographic context are naturally over-defined due to the field equations motivating then such a notion.

Definition 2.4. *We assume that $m > n$ and $q > 2$. Let $p_1, \dots, p_m \in \mathbb{F}_q[x_1, \dots, x_n]$ be homogeneous polynomials of degrees d_1, \dots, d_m respectively. This sequence is semi-regular if:*

1. $\langle p_1, \dots, p_m \rangle \neq \mathbb{F}_q[x_1, \dots, x_n]$,
2. for all $i, 1 \leq i \leq m$ and $g \in \mathbb{F}_q[x_1, \dots, x_n]$:

$$\deg(g \cdot p_i) < D_{\text{reg}}(p_1, \dots, p_m) \text{ and } g \cdot p_i \in \langle p_1, \dots, p_{i-1} \rangle \Rightarrow g \in \langle p_1, \dots, p_{i-1} \rangle.$$

Now, let $p_1, \dots, p_m \in \mathbb{F}_q[x_1, \dots, x_n]$ be polynomials of degrees d_1, \dots, d_m respectively. We shall say that the sequence p_1, \dots, p_m is semi-regular if the sequence p_1^H, \dots, p_m^H is semi-regular.

Regular sequences are defined almost as in Definition 2.4. The only difference is that $m \leq n$ and the second condition is simply: $g \cdot p_i \in \langle p_1, \dots, p_{i-1} \rangle \Rightarrow g \in \langle p_1, \dots, p_{i-1} \rangle$. Definition 2.4 of semi-regular sequences requires to be adapted for the Boolean polynomial ring $\mathbb{F}_2[x_1, \dots, x_n] / \langle x_i^2 - x_i \rangle_{1 \leq i \leq n}$ [6,9] to take into account the field equations.

Remark 3. Note that semi-regular sequences can be also defined from a more algorithmic point of view. Semi-regular sequences can be defined as the sequences for which all the matrices generated during a Gröbner basis computation with F_5 are of maximal possible rank [27].

The degree of regularity can be computed explicitly for semi-regular sequences [6,7]. It is derived from a particular coefficient in a power series.

Definition 2.5. Let $S(z) = \sum_{k \geq 0} c_k z^k \in \mathbb{N}[[z]]$ be a formal power series. We define the index $\text{Ind}(S)$ as the first k such that $c_k \leq 0$ (if no such k exist, then $\text{Ind}(S) = \infty$). We denote by:

$$[S(z)]_+ = \sum_{k=0}^{\text{Ind}(S)-1} c_k z^k, \text{ the series truncated at } \text{Ind}(S).$$

We have then:

Property 1. A sequence $p_1, \dots, p_m \in \mathbb{F}_q[x_1, \dots, x_n]$ of respective degrees d_1, \dots, d_m is semi-regular if and only if its Hilbert series is given by:

$$\text{HS}_q(t) = \left[\frac{\prod_{i=1}^m (1 - z^{d_i})}{(1 - z)^n} \right]_+, \text{ if } m > n \text{ and } q > 2, \text{ and } \text{HS}_2(t) = \left[\frac{(1 + z)^n}{\prod_{i=1}^m (1 + z^{d_i})} \right]_+, \text{ if } q = 2.$$

The degree of regularity of a semi-regular sequence $p_1, \dots, p_m \in \mathbb{F}_q[x_1, \dots, x_n]$ is given by $1 + \deg(\text{HS}_q(t))$.

Remark 4 (Macaulay bound). For regular sequences ($m \leq n$), the degree of regularity is given by the so-called *Macaulay bound* [37,33]:

$$\sum_{i=1}^m (d_i - 1) + 1. \quad (2)$$

For quadratic polynomials, the bound is $n + 1$.

2.2.2 Change of Ordering. We now consider the last step in the zero-dim solving process. To date, FLGM [29] is the fastest technique for the change of ordering. We recall below the complexity of the initial FLGM.

Theorem 3. Let $\mathcal{I} \subset \mathbb{F}_q[x_1, \dots, x_n]$ be a zero-dimensional ideal and $G_{\prec_{\text{old}}}$ be a $G_{\prec_{\text{old}}}$ -Gröbner basis of \mathcal{I} (w.r.t. to an admissible monomial ordering \prec_{old}). FGLM [29] permits to compute a \prec_{old} -Gröbner basis $G_{\prec_{\text{new}}}$ of \mathcal{I} knowing $G_{\prec_{\text{old}}}$ in

$$\mathcal{O}(n \cdot D^3),$$

with D being the dimension of the \mathbb{F}_q -vector space $\mathbb{F}_q[x_1, \dots, x_n]/\mathcal{I}$ (which is equivalent to the number of zeroes of \mathcal{I} counted with their multiplicities).

The complexity of the version described in [29] can be improved using fast linear algebra techniques [25,24]. These lead to a version of FGLM whose complexity is :

$$\mathcal{O}(n \cdot D^\omega), \text{ with } 2 \leq \omega < 2.3728639.$$

In this report, we can always assume that ω is at most two. For some STARK-friendly primitives, the cost of the change of ordering can be further reduced to:

$$\mathcal{O}(n \cdot D).$$

This is linear in the number of solutions.

In any case, it is clear that the value of D is crucial for the complexity of FGLM and solving PoSSo_q . We recall below a classical bound, known as *Bézout bound*, on the number of solutions.

Definition 1 (Bézout bound). Let $\mathcal{I} = \langle p_1, \dots, p_n \rangle \subset \mathbb{F}_q[x_1, \dots, x_n]$ be a zero-dimensional ideal and d_1, \dots, d_n be the degrees of p_1, \dots, p_n respectively. We have :

$$\#V(\mathcal{I}) \leq \prod_{i=1}^n d_i.$$

In the case of STARK-friendly primitives, the change of ordering is the dominant part of the solving process. It is then of primary importance to find tighter bounds on the number of solutions.

2.3 Multi-Homogeneous Systems

The systems arising in algebraic cryptanalysis behave usually differently than semi-regular systems (Definition 2.4). In this report, we shall see that the systems derived from many STARK-friendly ciphers have a multi-homogeneous structure [35,38,22].

Definition 2.6. Let $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{F}_q[x_1, \dots, x_n]^m$ be homogeneous polynomials. We consider a partition $\{\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(\ell)}\}$ of the variables $\mathbf{X} = \{x_1, \dots, x_n\}$.

- We shall say that \mathbf{p} is multi-homogeneous if the polynomials p_i are homogeneous w.r.t. the $\mathbf{X}^{(j)}$'s.
- The finite sequence, denoted by $\text{mdeg}(f)$, of degrees with respect to each block is called the multi-degree of f .
- Let $f \in \mathbb{K}[\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(\ell)}]$ be a multi-homogeneous polynomial of multi-degree (d_1, \dots, d_ℓ) . It holds that:

$$\forall (\alpha_1, \dots, \alpha_\ell) \in (\mathbb{F}_q^*)^\ell = f \left(\alpha_1 \mathbf{X}^{(1)}, \dots, \alpha_\ell \mathbf{X}^{(\ell)} \right) = \left(\prod_{i=1}^{\ell} \alpha_i^{d_i} \right) f \left(\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(\ell)} \right).$$

In the case of multi-homogeneous systems, we have a dedicated version of the Bézout bound (Definition 1).

Definition 2 (Multi-homogeneous Bézout bound). Let $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{F}_q[\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(\ell)}]^n$ be a set of multi-homogeneous polynomials of multi-degrees $\text{mdeg}(f_i) = (d_{i,1}, \dots, d_{i,\ell})$. Then, the number of zeroes of \mathbf{p} is bounded from above by the coefficient of $\alpha_1^{|\mathbf{X}^{(1)}|} \cdot \alpha_2^{|\mathbf{X}^{(2)}|} \dots \alpha_\ell^{|\mathbf{X}^{(\ell)}|}$ in the polynomial :

$$\prod_{i=1}^n (d_{i,1} + d_{i,2} + \dots + d_{i,\ell}).$$

The multi-homogeneous Bézout bound is a central tool that will be used to derive tighter complexity bounds of our algebraic attacks.

Remark 5. Note that the problem of finding an optimal multi-homogeneous structure minimizing the multi-homogeneous Bézout bound is NP-Hard.

3 Algebraic Cryptanalysis of Block Ciphers

Algebraic cryptanalysis of block ciphers has been popularized by [19]. The authors demonstrated that key-recovery on AES [20] can be reduced to the problem of solving a system of quadratic equations over \mathbb{F}_2 .

Given a pair of message/ciphertext, the basic idea of such modelling is to introduce intermediate variables for each round as well as variables corresponding to the secret-key. In AES, the Sbox $S_{\text{AES}} : \mathbb{F}_{2^8} \rightarrow \mathbb{F}_{2^8}$ is the inverse function in \mathbb{F}_{2^8} . That is:

$$S_{\text{AES}}(X) = \begin{cases} 1/X, & \text{if } X \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

If $X \neq 0$, then $X \times S_{\text{AES}}(X) = 1$. This yields 8 quadratic equations over \mathbb{F}_2 .

Initially, it was claimed in [19] that the complexity of solving the whole key-recovery system modelling AES

“does not grow exponentially with the number of rounds”.

This claim was proved to be false in a series of papers, e.g. [17,21,4].

Despite this false claims on the complexity, the results of [19] motivated to further investigate the behaviour of algebraic attacks against block ciphers. We recall below a selection of these results.

Alternative modelling. In [40], the authors introduced the so-called *Big Encryption Standard* (BES). This is an *embedding* of AES over \mathbb{F}_{2^8} leading to an alternative algebraic modelling of AES with multivariate polynomials over \mathbb{F}_{2^8} . these equations are of higher degree than in the previous modelling [19]. In the case of BES, the SBox is represented as

$$S_{\text{BES}}(X) = X^{2^8-2} = X^{254}.$$

Practical experiments. In order to understand the practical behaviour of algebraic attacks, [18] proposed a general framework to perform such attacks on so-called *small scale variants* of AES. Even for such small scale variants, it turns to be difficult to perform relevant practical experiments.

A Gröbner basis for AES. In [16], the authors proved a rather surprising result. The algebraic equations G_{AES} describing a key-recovery in AES already forms a zero-dimensional Gröbner basis for a suitable monomial ordering (the degree-lexicographical ordering). To do so, [16] considered the BES-style modelling of AES over \mathbb{F}_{2^8} [40].

As explained in Section 2.2.1, the cost of computing the variety is the cost of the change of ordering. Thus, the complexity of this approach is essentially polynomial in number of solutions of the algebraic system over \mathbb{F}_{2^8} . A bound on the number of solutions can be easily deduced from the shape of the leading terms of G_{AES} . In the case of AES128, the number of solutions D (counted with multiplicities) is:

$$D = 254^{200} \approx 2^{1598}. \tag{3}$$

Remark that 254 is the degree of the polynomial representation of the inverse function in $\mathbb{F}_{2^8} = \mathbb{F}_{256}$. Note also that the cost of the change of ordering is polynomial in (3). Thus, the attack has no direct impact on the security of AES. Still, this approach provides a first rigorous (upper) bound on the complexity of an algebraic attack against AES. It also emphasizes the importance of the number of solutions D in the cost of algebraic attacks against block ciphers.

Block Ciphers Sensitive to Gröbner bases. The idea of [16] has been generalized in [15] to a large class of block ciphers. In particular, the authors introduces two parametrized families of block ciphers – **Curry** (SPN) and **Flurry** (Feistel) – that have a sound design strategy against linear and differential attacks but which can be modelled by very simple polynomial equations. Then, they generalized the approach described in the previous paragraph for **AES** on **Curry** and **Flurry**. This demonstrates the existence of block ciphers resistant against differential and linear cryptanalysis but vulnerable against Gröbner basis attacks.

4 STARK-Friendly Ciphers

We briefly describes here the STARK-friendly ciphers that have been considered in this report. Namely:

Jarvis [5]	The first generation of block cipher optimized for STARK-proofs
MiMC [2]	A lightweight block cipher well suited for the STARK technology
Pepper/Rescue/ Vision [3]	A second generation of AO ciphers fixing some weaknesses of Jarvis
Starkad/Poseidon [32]	Latest design of STARK-friendly hash functions

We review here the main components of these primitives and refer to the initial papers for details regarding the design rational. As mentioned in the introduction, we describe the ciphers with the specification provided by **Starkware** Industries.

4.1 Jarvis

The state of **AES** is a matrix composed by elements in \mathbb{F}_{2^8} transformed through a certain number of rounds with operations in \mathbb{F}_{2^8} or in \mathbb{F}_2 . **Jarvis** is a generalization of **AES**, or more precisely of **BES**, whose state is reduced to a single field element of \mathbb{F}_{2^n} and involves few simple operations in \mathbb{F}_{2^n} . It borrows to **AES**, the use of the inverse as a non-linear component as well as *quasi-linear* polynomials for the linear layer. We recall that $L(X) \in \mathbb{F}_{2^n}[X]$ is a quasi-linear polynomial if it is defined as follows :

$$L(X) = c_{-1} + \sum_{i=0}^{n-1} c_i X^{2^i}.$$

In particular, **Jarvis** uses a special polynomial $A(X) \in \mathbb{F}_{2^n}[X]$ chosen such that:

$$A(X) = C(B^{-1}(X)) \tag{4}$$

where $C(X) \in \mathbb{F}_{2^n}[X]$ and $B(X) \in \mathbb{F}_{2^n}[X]$ quasi-linear polynomials of degree 4 which define a permutation.

Algorithm 1 Jarvis

1: **Input:** plaintext $M \in \mathbb{F}_{2^n}$ and a secret-key $K_0 \in \mathbb{F}_{2^n}$
2: $K_1, \dots, K_N \leftarrow \text{KeyExpand}(K_0)$
3: $S_0 \leftarrow M + K_0$
4: **for** r **from** 1 **to** N **do**
5: $S_r \leftarrow A(1/S_{r-1}) + K_r$ (with $A(X) \in \mathbb{F}_{2^n}[X]$ being defined as in (4))
6: **end for**

Algorithm 2 KeyExpand

1: **Input:** $K_0 \in \mathbb{F}_{2^n}$
2: **Internal constant:** $C_0 \in \mathbb{F}_{2^n}$
3: $C_1, \dots, C_N \leftarrow \text{ConsGen}(C_0)$
4: **for** r **from** 1 **to** N **do**
5: $K_r \leftarrow 1/K_{r-1} + C_r$
6: **end for**
7: **return** $K_1, \dots, K_N \in \mathbb{F}_{2^n}$

Algorithm 3 ConsGen

```

1: Input:  $C_0 \in \mathbb{F}_{2^n}$ 
2: Internal constants:  $a, b \in \mathbb{F}_{2^n}$ 
3: for  $r$  from 1 to  $N$  do
4:    $C_r \leftarrow a C_{r-1} + b$ 
5: end for
6: return  $C_1, \dots, C_N \in \mathbb{F}_{2^n}$ 

```

Security analysis. We give below the parameters initially recommended by the designers of Jarvis [5] for various security levels.

Name	n	N
Jarvis128	128	10
Jarvis192	192	12
Jarvis256	256	14

Recently, and in parallel of this work, the authors of [1] presented an improved algebraic attack against Jarvis. In particular, [1] has been able to exploit the particular structure of the linear layer, i.e. the polynomial A as defined in (4), to improve the algebraic modelling of Jarvis. They then rely on generic bounds on the degree of regularity and number of solutions (Section 2) to derive new complexity bounds on the cost of an algebraic attack.

Name	n	N	New security bound [1]
Jarvis128	128	10	72
Jarvis192	192	12	85
Jarvis256	256	14	92

Table 8. Complexity of the attack of [1] (assuming $\omega = 2$).

The authors [1] also mentioned that at least $N = 18$ rounds are necessary to reach a security level of 128 bit according to this new attack. With our new algebraic attack, we show that number of rounds should be, at least, $N = 62$ to reach this security level (Table 3).

Finally, the authors [1] observed that their algebraic attack behaves better in practice than expected. In particular, they observed that the number of solutions $D(N) \approx 2 \times 5^N$. In Theorem 5.3, we prove that $D(N) = (N + 1)4^N$.

4.2 MiMC

The MiMC family operates on finite field \mathbb{K} that can be a binary extension \mathbb{F}_{2^n} or a prime field \mathbb{F}_p . In this report, we have considered the (SPN) version of MiMC where the non-linear component is the cubic Sbox (i.e. $d = 3$).

Algorithm 4 MiMC

```

1: Input: plaintext  $M \in \mathbb{K}$  and a secret-key  $K \in \mathbb{K}$ 
2: Internal constants:  $C_1, \dots, C_N \in \mathbb{K}$ 
3:  $S_1 \leftarrow (M + (K + C_1))^d$ 
4: for  $r$  from 1 to  $N$  do
5:    $S_r \leftarrow (S_{r-1} + (K + C_r))^d$ 
6: end for
7: return  $S_N$ 

```

The internal constants $C_1, \dots, C_N \in \mathbb{K}$ are chosen randomly.

Remark 6. We provide in Appendix A some reference codes for implementing Algorithm 4 in SAGE, MAGMA and MAPLE.

Number of Rounds. According to the designers [2], the number of rounds N required to reach a security level s is equal to :

$$N = \left\lceil \frac{s}{\log_2 3} \right\rceil.$$

More recently, the authors of [1] quickly analysed MiMC against algebraic attacks. In particular, they observed that a natural algebraic modelling of MiMC has a number of solutions $D(N) \approx 3^N$. We also prove this result in Theorem 5.3.

4.3 Pepper/Rescue/Vision

In [5], the authors set the basis on the design of AO ciphers. This leads to the proposal of two new block ciphers **Vision** and **Rescue** optimized for the STARK-technology. These are SPN block ciphers operating on fields of odd and even characteristic, respectively. In contrast to Jarvis (Section 4.1), the state space in **Vision** and **Rescue** are composed of more than one element.

The goal of [5] was also to fix initial weaknesses quickly identified on Jarvis. In this process, we also analysed intermediate versions such as Jarvis-v2 (one dimensional version of **Vision**) and **Pepper** (evolution of Jarvis that removed the linear layer and uses two Sboxes).

4.3.1 Pepper. The first version of **Pepper-v1** operates on a prime field $\mathbb{K} = \mathbb{F}_p$. We set $d = 3, c = (2p - 1)/3$ and $N' = 2N$.

Algorithm 5 Pepper-v1

```

1: Input: plaintext  $M \in \mathbb{K}$  and a
   secret-key  $K_0 \in \mathbb{K}$ 
2:  $K_1, \dots, K_{N'} \leftarrow$ 
   KeyExpand( $K_0$ )
3:  $S_0 \leftarrow M$ 
4: for  $r$  from 1 to  $N$  do
5:    $S_r \leftarrow (S_{r-1} + K_{2r-1})^c +$ 
      $K_{2r}$ 
6: end for
7: return  $S_N$ 

```

Algorithm 6 KeyExpand

```

1: Input:  $K_0 \in \mathbb{K}$ 
2: Internal constant:  $C_0 \in \mathbb{K}$ 
3:  $C_1, \dots, C_{N'} \leftarrow$  ConsGen( $C_0$ )
4: for  $r$  from 1 to  $N'$  do
5:    $K_r \leftarrow 1/K_{r-1} + C_r$ 
6: end for
7: return  $K_1, \dots, K_{N'} \in \mathbb{K}$ 

```

Algorithm 7 ConsGen

```

1: Input:  $C_0 \in \mathbb{K}$ 
2: Internal constants:  $a, b \in$ 
    $\mathbb{F}_{2^n}$ 
3: for  $r$  from 1 to  $N'$  do
4:    $C_r \leftarrow aC_{r-1} + b$ 
5: end for
6: return  $C_1, \dots, C_{N'} \in \mathbb{K}$ 

```

The second version, denoted **Pepper-v2**, simplifies the key expansion and add a key addition on the message.

Algorithm 8 Pepper-v2

```

1: Input: plaintext  $M \in \mathbb{K}$  and a
   secret-key  $K_0 \in \mathbb{K}$ 
2:  $K_1, \dots, K_{N'} \leftarrow$ 
   KeyExpand( $K_0$ )
3:  $S_0 \leftarrow M + K_0$ 
4: for  $r$  from 1 to  $N$  do
5:    $S_r \leftarrow (S_{r-1}^d + K_{2r-1})^c +$ 
      $K_{2r}$ 
6: end for
7: return  $S_N$ 

```

Algorithm 9 KeyExpand

```

1: Input:  $K_0 \in \mathbb{K}$ 
2: Internal constants:  $a, b \in$ 
    $\mathbb{F}_{2^n}$ 
3: for  $r$  from 1 to  $N'$  do
4:    $K_r \leftarrow aK_{r-1} + b$ 
5: end for
6: return  $K_1, \dots, K_{N'} \in \mathbb{K}$ 

```

4.3.2 Rescue. In this case, the finite field is \mathbb{F}_p . The Sboxes are the power maps X^d and X^c , where d is the smallest prime such that $\gcd((p-1), d) = 1$. In most cases, we can set $d = 3$ and $c = (2p-1)/3$. This is the situation analysed in this report. Finally, we set $N' = 2N$.

Algorithm 10 Rescue

```

1: Input: plaintext  $M \in \mathbb{F}_p^m$  and a secret-key  $K_0 \in$ 
    $\mathbb{F}_p^m$ 
2:  $K_1, \dots, K_{N'} \leftarrow$  KeyExpand( $K_0$ )
3:  $S_0 \leftarrow M + K_0$ 
4: for  $r$  from 1 to  $N$  do
5:    $S \leftarrow {}^T[S_{r-1}[1]^d, \dots, S_{r-1}[m]^d]$ 
6:    $S' \leftarrow \mathcal{M}S + K_{2r-1}$ 
7:    $S'' \leftarrow {}^T[S'[1]^c, \dots, S'[m]^c]$ 
8:    $S_r \leftarrow \mathcal{M}S'' + K_{2r}$ 
9: end for
10: return  $S_N$ 

```

Algorithm 11 KeyExpand

```

1: Input:  $K_0 \in \mathbb{F}_p^m$ 
2: Internal constants:  $a, b \in \mathbb{F}_p$ 
3:  $B \leftarrow {}^T[b, \dots, b]$ 
4: for  $r$  from 1 to  $N'$  do
5:    $K_r \leftarrow aK_{r-1} + b$ 
6: end for
7: return  $K_1, \dots, K_{N'} \in \mathbb{F}_p^m$ 

```

Number of Rounds. According to [5], the number of rounds N required to reach a security level $m \cdot \log_2(p)$ is equal to :

$$\max \left(10, 2 \left\lceil \frac{\log_2(p)}{4} \right\rceil \right).$$

4.3.3 Hash Functions. Once the secret-key is fixed in Rescue, then Algorithm 10 defines a permutation, denoted as:

$$\text{perm}_{\text{Rescue}} : \mathbb{F}_q^m \longrightarrow \mathbb{F}_q^m. \quad (5)$$

As explained in [5], $\text{perm}_{\text{Rescue}}$ can be processed into a hash function thanks to the sponge construction [11]. In this case, the value of the secret-key is fixed to a random value. The state of a sponge function is defined as $m = r + c$ bits, where r and c are called the *rate* and the *capacity* of the sponge, respectively. In particular, we focused our attention the following 128-bit challenge parameters³.

4.3.4 Vision. This block cipher [5] operates on the finite field $\mathbb{F}_{2^{n/m}}$. The non-linear component is the inverse function in $\mathbb{F}_{2^{n/m}}$ and it uses a quasi-linear permutation $B(X) \in \mathbb{F}_{2^{n/m}}[X]$ of degree d (d is usually 4). Finally, the state is also mixed thanks to an invertible matrix $\mathcal{M} \in \mathbb{F}_{2^{n/m}}^{m \times m}$. Finally, we set $N' = 2N$.

³<https://starkware.co/hash-challenge/>

Name	Field	$n = \log_2(p)$	N	m	c
Rescue128a	\mathbb{F}_p	125	16	4	2
Rescue128b	\mathbb{F}_p	253	22	12	1
Rescue128c	\mathbb{F}_p	125	10	12	2
Rescue128d	\mathbb{F}_p	61	10	12	4
Rescue128e	\mathbb{F}_p	253	10	11	1

Table 9. Challenge parameters for the Rescue-based hash function.

Algorithm 12 $\text{Vision}(N, d, m, n)$

1: **Input:** plaintext $M \in \mathbb{F}_{2^{n/m}}^m$ and a secret-key $K_0 \in \mathbb{F}_{2^{n/m}}^m$
2: $K_1, \dots, K_{N'} \leftarrow \text{KeyExpand}(K_0)$
3: $S_0 \leftarrow M + C_0$
4: **for** r **from** 1 **to** $N - 1$ **do**
5: $S \leftarrow {}^T[B(1/S_{r-1}[1]), \dots, B(1/S_{r-1}[m])]$
6: $S \leftarrow \mathcal{M}S + K_{2r-1}$
7: $S \leftarrow {}^T[B^{-1}(1/S[1]), \dots, B^{-1}(1/S[m])]$
8: $S_r \leftarrow \mathcal{M}S + K_{2r}$
9: **end for**
10: $S \leftarrow {}^T[1/S_{N-1}[1], \dots, 1/S_{N-1}[m]]$
11: $S_N \leftarrow \mathcal{M}S + K_{2N-1}$
12: **return** S_N

Algorithm 13 KeyExpand

1: **Input:** $K_0 \in \mathbb{F}_{2^{n/m}}^m$
2: **Internal constants:** $a, b \in \mathbb{F}_{2^{n/m}}^m$
3: $B \leftarrow {}^T[b, \dots, b]$
4: **for** r **from** 1 **to** N' **do**
5: $K_r \leftarrow aK_{r-1} + B$
6: **end for**
7: **return** $K_1, \dots, K_{N'} \in \mathbb{F}_{2^{n/m}}^m$

Number of Rounds. According to [5], the number of rounds N required to reach a security level n is equal to :

$$\max\left(10, 2 \left\lceil \frac{n}{5.5m} \right\rceil\right).$$

Variant of Jarvis. We will have $\text{Jarvis-v2}(N, n) = \text{Vision}(N, d, 1, n)$.

4.4 Poseidon and Starkad

To date, Poseidon and Starkad this are the last primitives especially optimized for STARK applications. Starkad and Poseidon [32] are hash functions that operate on a binary field and prime field respectively. More precisely, Starkad and Poseidon are constructed from a permutation and transformed into a hash function thanks to the sponge construction [32].

To describe the permutations, we use the following notations :

- \mathbb{K} a finite field
- MDS is $t \times t$ matrix,
- $C[\dots]$ are round constants
- P is an extra parameter and is usually equal to 1.
- d is the degree of the S-BOX and is usually equal to 3.
- The field is chosen so that $x \rightarrow x^d$ is a *permutation*.

Algorithm 14 Starkad/Poseidon(R_F, R_P, t, d, P)

```

1: Input:  $\mathbf{s} \in \mathbb{K}^t$ 
2:  $R_f \leftarrow R_F/2$ 
3: for  $r$  from 1 to  $R_f$  do
4:    $\mathbf{s}[i] \leftarrow (\mathbf{s}[i] + C[t(r-1) + i])^d$  for  $i = 1, \dots, t$ 
5:    $\mathbf{s} \leftarrow \mathbf{s} \times \text{MDS}$ 
6: end for
7: for  $r$  from 1 to  $R_P$  do
8:    $\mathbf{s}[i] \leftarrow (\mathbf{s}[i] + C[t(R_f + r - 1) + i])^d$  for  $i = 1, \dots, P$ 
9:    $\mathbf{s}[i] \leftarrow \mathbf{s}[i] + C[t(R_f + r - 1) + i]$  for  $i = (P + 1), \dots, t$ 
10:   $\mathbf{s} \leftarrow \mathbf{s} \times \text{MDS}$ 
11: end for
12: for  $r$  from 1 to  $R_f - 1$  do do
13:   $\mathbf{s}[i] \leftarrow (\mathbf{s}[i] + C[t(R_f + R_P + r - 1) + i])^d$  for  $i = 1, \dots, t$ 
14:   $\mathbf{s} \leftarrow \mathbf{s} \times \text{MDS}$ 
15:   $\mathbf{s}[i] \leftarrow (\mathbf{s}[i] + C[t(R_f + R_P + r - 1) + i])^d$  for  $i = 1, \dots, t$ 
16: end for
17: Output :  $\mathbf{s}$ 

```

Remark 7. A reference code for Algorithm 14 is provided in Appendix B.

We report some typical values (S-Box is $x \rightarrow x^3$). We also give another value so that $n k \approx 256$. To simplify the description we introduce the following full round and partial round

Name	Security	Field	n	R_F	R_P	t
Poseidon-256	2^{128}	\mathbb{F}_p	256	8	84	6
STARK-252	2^{126}	$\mathbb{F}_{2^{63}}$	63	12	39	24
Sage Reference Code	$2^{128?}$	$\mathbb{F}_{2^{63}}$	63	6	42	24

Table 10. Parameters for Poseidon and Starkad.

functions:

Algorithm 15 $R_r(s)$

```

1: Input:  $\mathbf{s} \in \mathbb{K}^t$ 
2:  $\mathbf{s}[i] \leftarrow (\mathbf{s}[i] + C[t(r-1) + i])^d$  for  $i = 1, \dots, t$ 
3: Output :  $\mathbf{s} \times \text{MDS}$ 

```

Algorithm 16 $R'_r(s)$

```

1: Input:  $\mathbf{s} \in \mathbb{K}^t$ 
2:  $\mathbf{s}[i] \leftarrow (\mathbf{s}[i] + C[t(r-1) + i])^d$  for  $i = 1, \dots, P$ 
3:  $\mathbf{s}[i] \leftarrow \mathbf{s}[i] + C[t(r-1) + i]$  for  $i = (P + 1), \dots, t$ 
4: Output :  $\mathbf{s} \times \text{MDS}$ 

```

Algorithm 17 $L(s)$

```

1: Input:  $s \in \mathbb{K}^t$ 
2:  $\mathbf{s}[i] \leftarrow (s[i] + C[t(r-1) + i])^d$  for  $i = 1, \dots, P$ 
3:  $\mathbf{s}[i] \leftarrow s[i] + C[t(r-1) + i]$  for  $i = (P + 1), \dots, t$ 
4: Output :  $\mathbf{s} \times \text{MDS}$ 

```

In the following, we always define $R_f = R_F/2$. The permutation function `perm` in `Starkad` and `Poseidon` is defined by:

$$\text{perm}_{\text{Hades}}(\mathbf{s}) = L(\mathbf{s}''') \text{ where } \begin{cases} \mathbf{s}' = R_{R_f} \circ \cdots \circ R_1(\mathbf{s}) \\ \mathbf{s}'' = R'_{R_f+R_P} \circ \cdots \circ R'_{R_f+1}(\mathbf{s}') \\ \mathbf{s}''' = R_{R_f+R_P-1} \circ \cdots \circ R_{R_f+R_P+1}(\mathbf{s}'') \end{cases} \quad (6)$$

We then have four maps:

$$\begin{aligned} \text{perm}_{\text{Hades}} &: \mathbb{K}^t \longrightarrow \mathbb{K}^t \\ R_r &: \mathbb{K}^t \longrightarrow \mathbb{K}^t \\ R'_r &: \mathbb{K}^t \longrightarrow \mathbb{K}^t \\ L &: \mathbb{K}^t \longrightarrow \mathbb{K}^t \end{aligned}$$

5 Algebraic Cryptanalysis of Jarvis

We present in this part an improved algebraic attack against `Jarvis` (Section 4.1). Our modelling is given in Section 5.1. Then, we show that the equations modelling `Jarvis` have a multi-homogeneous structure (Section 2.3). This allows to derive the bound on the number of solutions given in Table 3 (Theorem 5.3). We then present experimental results in Section 5.3 and derive an experimental upper bound on the maximal degree reached in the computation of DRL-Gröbner basis; leading the complexity results obtained in Section 5.4.

5.1 Algebraic Modelling

We describe in this part the algebraic modelling of `Jarvis`. To this end, we introduce several extra variables. Namely:

- s_0, s_1, \dots, s_{N-1} corresponding to intermediate state variables,
- a variable k_0 corresponding to the master-key and variables k_1, \dots, k_N corresponding to the sub-keys.

According to Algorithm 1 and the very definition of the linear layer $A(X) \in \mathbb{F}_{2^n}[X]$ (as in (4)), we have for each round r :

$$s_r = A(1/s_{r-1}) + k_r = C(B^{-1}(1/s_{r-1})) + k_r. \quad (7)$$

For each $r, 1 \leq r \leq N$, we introduce a new variable

$$u_{r-1} = B^{-1}(1/s_{r-1}). \quad (8)$$

Combining (8) with (7), we get:

$$s_r = C(u_{r-1}) + k_r.$$

Finally, (8) is equivalent to:

$$B(u_{r-1}) = 1/s_{r-1}$$

By taking the numerator, this leads to:

$$B(u_{r-1}) s_{r-1} - 1 = 0.$$

Let $K_0 \in \mathbb{F}_{2^n}$ and $(M, C = \text{Jarvis}(M, K_0)) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$ be a pair message/ciphertext encrypted under the secret-key K_0 . Our key-recovery system for `Jarvis` is as follows:

$$\mathcal{S}_{\text{Jarvis}}(M, C) = \begin{cases} s_0 + k_0 + M, \\ B(u_0) s_0 + 1, \\ C(u_0) + s_1 + k_1, \\ \dots \\ B(u_{r-1}) s_{r-1} + 1, \\ C(u_{r-1}) + s_r + k_r, \\ \dots \\ B(u_{N-1}) s_{N-1} + 1, \\ C(u_{N-1}) + C + k_N \end{cases} \quad (9) \quad \mathcal{K}_{\text{Jarvis}} = \begin{cases} k_1 k_0 + k_0 C_0 + 1, \\ \vdots \\ k_r k_{r-1} + k_{r-1} C_r + 1, \\ \vdots \\ k_N k_{N-1} + k_{N-1} C_N + 1, \end{cases}$$

According to Section 4.1, the polynomials $B, C \in \mathbb{F}_{2^n}[X]$ are of degree 4, and constants $C_1, \dots, C_N \in \mathbb{F}_{2^n}$ are known and explicitly given. From now on, we denote by $\text{Sys}_{\text{Jarvis}}(M, C)$ the key-recovery system for Jarvis composed by the equations of $\mathcal{S}_{\text{Jarvis}}(M, C)$ and from $\mathcal{K}_{\text{Jarvis}}$.

Lemma 1. *The key-recovery system $\mathcal{S}_{\text{Jarvis}}(M, C)$ has $3N + 1$ equations over \mathbb{F}_{2^n} (N of degree 2, N of degree 4, N of degree 5 and one linear equation) and $3N + 1$ variables.*

Example 1. Let's consider $\text{Sys}_{\text{Jarvis}}(M, C)$ for $\mathbb{F}_{2^n} = \mathbb{F}_{2^{31}}$, $N = 3$ and with ($M = 539787998$, $C = 1389316418$).

$$s_0 + k_0 + 539787998,$$

$$s_0 u_0^4 + 283125965 s_0 u_0^2 + 1450602990 s_0 u_0 + 424342659 s_0 + 1,$$

$$u_0^4 + 1781139933 u_0^2 + k_1 + s_1 + 498270819 u_0 + 761232961,$$

$$s_1 u_1^4 + 283125965 s_1 u_1^2 + 1450602990 s_1 u_1 + 424342659 s_1 + 1, u_1^4 + 1781139933 u_1^2 + k_2 + s_2 + 498270819 u_1 + 761232961,$$

$$s_2 u_2^4 + 283125965 s_2 u_2^2 + 1450602990 s_2 u_2 + 424342659 s_2 + 1,$$

$$u_2^4 + 1781139933 u_2^2 + k_3 + 498270819 u_2 + 2140157699,$$

$$k_0 k_1 + 1824939501 k_0 + 1,$$

$$k_1 k_2 + 1240992103 k_1 + 1,$$

$$k_2 k_3 + 1828068674 k_2 + 1.$$

5.2 Bounding the Number of Solutions

As emphasized in Sections 2 and 3, the number of solutions $D(N)$ of $\text{Sys}_{\text{Jarvis}}(M, C)$ is a key quantity to understand the complexity of an algebraic attack. The Bézout bound (Definition 1, Section 2.2.2) on $\text{Sys}_{\text{Jarvis}}(M, C)$ yields :

$$D(N) \leq 5^N \times 4^N \times 2^N = 40^N.$$

However, this bound is not tight since it does not take into account the structure of the equations.

To estimate $D(N)$ more accurately, we first consider a variant of $\mathcal{S}_{\text{Jarvis}}(M, C)$ where we keep the variable s_{N-1} . The system is:

$$\mathcal{S}_{\text{Jarvis}}^h(M) = \begin{cases} s_0 + k_0 + M, \\ B(u_0) s_0 + 1, \\ C(u_0) + s_1 + k_1, \\ \dots \\ B(u_{r-1}) s_{r-1} + 1, \\ C(u_{r-1}) + s_r + k_r, \\ \dots \\ B(u_{N-1}) s_{N-1} + 1, \\ C(u_{N-1}) + s_N + k_N \end{cases} \quad (10)$$

We denote by $\text{Sys}_{\text{Jarvis}}^h(M)$ the corresponding *homogenized system* (that is, $\mathcal{S}_{\text{Jarvis}}^h(M) + \mathcal{K}_{\text{Jarvis}}$). This system has a multi-homogeneous structure (Definition 2.6, Section 2.3).

Theorem 4. *The system $\text{Sys}_{\text{Jarvis}}^h(M) \subset \mathbb{F}_{2^n}[s_0, s_1, \dots, s_N, k_0, k_1, \dots, k_N, u_0, \dots, u_{N-1}]$ is multi-homogeneous and its multi-homogeneous Bézout bound is:*

$$(N + 1)4^N.$$

Proof. $\text{Sys}_{\text{Jarvis}}^h(M)$ is multi-homogeneous w.r.t. to following partition of the variables:

$$V = [[s_0], [u_0], [s_1], [u_1], \dots, [s_{N-1}], [u_{N-1}], [k_0], \dots, [k_N]].$$

Let $s = \#\text{Sys}_{\text{Jarvis}}^h(M)$ be the number of polynomials and $n_h = 3N + 1$ be the number of variables. We build the matrix:

$$M = [\text{deg}_{V_j}(f_i)]_{1 \leq i \leq s, 1 \leq j \leq n_h} \quad (11)$$

where V_j is the j -th variable in the list V and f_i is the i -th equation in the system $\text{Sys}_{\text{Jarvis}}^h(M)$.

Example 2. In the case of Example 1, the matrix (11) is as follows:

$$M = \left[\begin{array}{cccc|ccc} 1 & & & & 1 & & \\ 1 & 4 & & & & & \\ & 4 & 1 & & & 1 & \\ & & 1 & 4 & & & \\ & & & 4 & 1 & & \\ & & & & 1 & 4 & \\ & & & & & 4 & 1 \\ \hline & & & & & 1 & 1 \\ & & & & & & 1 & 1 \\ & & & & & & & 1 & 1 \end{array} \right]$$

To ease the notations, we denote by x_1, \dots, x_{n_h} be variables of $\mathcal{S}_{\text{Jarvis}}(M, C)$. Let then :

$$W = M \times [x_1, \dots, x_{n_h}]^T.$$

The Bézout multi-homogeneous bound (Definition 2, Section 2) is the coefficient of $V_1 V_2 \cdots V_{n_h}$ in the polynomial

$$P_N = \prod_{i=1}^s W_i.$$

The proof is by induction that this coefficient is $(N + 1)4^N$.

Base case. When $N = 1$, the polynomial is:

$$P_1 = (k_0 + s_0)(s_0 + 4u_0)(4u_0 + k_1)(k_0 + k_1).$$

By expanding this expression, we get: $P_1 = k_0^2 k_1 s_0 + 4 k_0^2 k_1 u_0 + 4 k_0^2 s_0 u_0 + 16 k_0^2 u_0^2 + k_0 k_1^2 s_0 + 4 k_0 k_1^2 u_0 + k_0 k_1 s_0^2 + \boxed{8 k_0 k_1 s_0 u_0} + 16 k_0 k_1 u_0^2 + 4 k_0 s_0^2 u_0 + 16 k_0 s_0 u_0^2 + k_1^2 s_0^2 + 4 k_1^2 s_0 u_0 + 4 k_1 s_0^2 u_0 + 16 k_1 s_0 u_0^2$ and the bound is 8.

Induction. We assume that the bound is $N4^{N-1}$ for $N - 1$ rounds. Then it is easy to check that:

$$P_N = P_{N-1}(s_{N-1} + 4u_{N-1})(4u_{N-1} + k_N)(k_{N-1} + k_N)$$

We expand the polynomial starting with $(k_{N-1} + k_N)$.

1. We begin with k_{N-1} . Since the next products are $(k_0 + k_1)(k_1 + k_2) \cdots (k_{N-2} + k_{N-1})$ so we obtain successively k_{N-2}, \dots, k_1, k_0 . Next we have $(k_0 + s_0)(s_0 + 4u_0)(4u_0 + k_1 + s_1)(s_1 + 4u_1) \cdots$. Thus, we deduce (in this order): $s_0, 4u_0, s_1, 4u_1, \dots$. In total, the coefficient is 4^N .
2. We now begin with k_N . From the product $(s_{N-1} + 4u_{N-1})(4u_{N-1} + k_N)$ we obtain $4u_{N-1}$ and s_{N-1} so we have $4u_{N-1}s_{N-1}k_N P_{N-1}$. Hence, by induction the coefficient is $4N 4^{N-1} = N4^N$.

By mixing the two cases, we have $4^N + N4^N = (N + 1)4^N$. □

in 4.1, the polynomials $B, C \in \mathbb{F}_{2^n}[X]$ are

We can generalize the proof for any polynomials $B, C \in \mathbb{F}_{2^n}[X]$ as in (4) (Section 4.1) such that $\deg(B) = d_B$ and $\deg(C) = d_C$.

Theorem 5. *Let $\text{Sys}_{\text{Jarvis}}^h(M)$ be the key-recovery system generated from polynomials $B(X)/C(X)$ of degree d_B/d_C . The Bézout multi-homogeneous bound is equal to*

$$B = \begin{cases} \frac{d_B^{N+1} - d_C^{N+1}}{d_B - d_C} & \text{when } d_B \neq d_C \\ (N + 1)d^N & \text{when } d = d_B = d_C \end{cases}$$

5.3 Maximal Degree in DRL Gröbner Basis

In this part, we present experiment results showing that the number of solutions derived in Theorem indeed holds in practice.

Experimental Fact 1. *The maximal degree occurring in the computation of a DRL-Gröbner basis of $\mathcal{S}_{\text{Jarvis}}(M, C)$ is*

$$\max(5, N + 2). \tag{12}$$

Note that for any value of N , $\mathcal{S}_{\text{Jarvis}}(M, C)$ contains equations of degree 5 (Lemma 1). So, the maximal degree is always at least 5.

Remark 8. The experimental bound (12) is much better than the Macaulay bound (Remark 4) for $\mathcal{S}_{\text{Jarvis}}(M, C)$:

$$8N + 1.$$

We also provide some the timings on the two steps in zero-dim solving process (Section 2.2). In what follows, we denote by:

- T_{DRL} , the time for computing a DRL-Gröbner basis,
- D_{max} , the maximum degree reached in the computation of a DRL-Gröbner basis
- T_{FGLM} , the time for the change of ordering,

In Table 11, we present the experimental results obtained on **Jarvis** using the Fgb software. Most of the experiments can also be done with the MAGMA computer algebra system. The only

N	n	#Sol	T_{DRL}	D_{max}	T_{FGLM}	Density FGLM
2	31	24	0.0s	5	0.0s	52.9%
3	31	160	0.03s	5	0.01s	31.3%
4	31	896	1.85s	6	0.89s	26.0%
5	31	4608	169.7s	7	153.7s	35.1%
6	31	22528	23027s	8	17639s	37.6%

Table 11. Algebraic cryptanalysis of **Jarvis** with Fgb.

main difference is the implementation of the FGLM algorithm which can be the bottleneck in Magma.

N	n	#Sol	T_{DRL}	T_{FGLM}
2	31	24	0.02s	0.00s
3	31	160	0.14s	0.28s
4	31	896	9.35s	56.6s
5	31	4608	962.6s	5464.2s

Table 12. Algebraic cryptanalysis of **Jarvis** with MAGMA.

5.4 Complexity Analysis

According to the experiments of Section 5.3, we can assume that the maximum degree reached in the computation of a DRL-Gröbner basis of $\mathcal{S}_{\text{Jarvis}}(M, C)$ is $\max(5, N + 2)$ (Exp. Fact 1). According to Theorem 2, the cost of computing a DRL-Gröbner basis is then:

$$T_1 = O\left(\binom{4N + 3}{N + 2}^\omega\right).$$

This out of the scope of this report, but we demonstrated that the change of ordering (Section 2.2.2) can be done in :

$$T_2 = O((N + 1)4^N).$$

This is linear in the number of solutions $\mathcal{S}_{\text{Jarvis}}(M, C)$ (Theorem 5.3). The number of rounds in Table 3 is obtained by assuming that the complexity of solving is essentially equivalent to T_2 . From now, we will always use this approach.

6 Algebraic Cryptanalysis of MiMC

6.1 A First Simple Algebraic Attack

Let $K \in \mathbb{K}$ and $(M, C = \text{MiMC}(M, K)) \in \mathbb{K} \times \mathbb{K}$ be a pair message/ciphertext encrypted under the secret-key K . The algebraic modelling of MiMC (Section 4.2) is rather straightforward.

We need to introduce:

- variables s_0, s_1, \dots, s_{N-1} corresponding to the intermediate states, and
- a variable k corresponding to the master-key.

The key-recovery system is then as follows:

$$\mathcal{S}_{\text{MiMC}}(M, C) = \begin{cases} s_0 = (M + k + C_1)^d, \\ \vdots \\ s_r = (S_{r-1} + k + C_r)^d, \\ \vdots \\ C = k + S_{N-1} \end{cases}$$

Recall that there is no key schedule in MiMC. Thus :

Lemma 2. *The key-recovery system $\mathcal{S}_{\text{MiMC}}(M, C)$ has $N + 1$ equations over \mathbb{K} (N of degree d in one linear equation) and $N + 1$ variables.*

Example 3. We provide below the equations of the system $\mathcal{S}_{\text{MiMC}}(M, C)$ for $N = 3, d = 3$ and $\mathbb{K} = \mathbb{F}_{2^{31}}$.

$$\begin{cases} k^3 + 18k^2 + 260k + s_0 + 4680, \\ k^3 + s_0k^2 + s_0^2k + s_0^3 + 573341058k^2 + 573341058s_0^2 \\ \quad + 1561922980k + 1561922980s_0 + s_1 + 1910775587, \\ k^3 + s_1k^2 + s_1^2k + s_1^3 + 54203621k^2 + 54203621s_1^2 \\ \quad + 21629643k + 21629643s_1 + s_2 + 869220636, \\ s_2 + k + 2100078585 \end{cases}$$

We have 4 variables k, s_0, s_1, s_2 and 4 equations (3 cubic equations and 1 linear equation).

The following result on the number of solutions of $\mathcal{S}_{\text{MiMC}}(M, C)$ is obvious :

Theorem 6. *Let d be the degree of SBox used in MiMC (Algorithm 4, Section). The Bézout bound for $\mathcal{S}_{\text{MiMC}}(M, C)$ is:*

$$d^N.$$

Similarly than in the previous section, we can assume that the change of ordering can be done in linear time. This explains the number of rounds derived in Table 3.

6.2 Improved Algebraic Attacks

For MiMC, we also considered variants of the attack considered in Section 6.1.

6.2.1 Several Pairs. We consider the case when we have $\ell > 1$ pairs of message/ciphertext (instead of just one in the Section 6.1). For each pair, we need to introduce N new variables corresponding to the intermediate states (for each pair message/ciphertext). On the other hand, we can keep the same variable k for the secret-key. All in all, we have then $\ell(N+1)$ equations and $\ell N + 1$ variables. The algebraic system is then slightly overdetermined.

Remark 9. For Gröbner basis the more equations we have the best it is. More precisely, when m is the number of equations and n the number of variables, the complexity is exponential when $m \approx n$ but sub-exponential when $m \gg n$ (under some algebraic assumptions).

Conclusion. We perform some experiments. In comparison to the attack of Section 6.1, we can only reduce by one the maximal degree occurring in the Gröbner basis computation. Hence, since we increase a lot the number of variables there is no gain in practice.

6.2.2 Splitting the Equations. We assume here that the field $\mathbb{K} = \mathbb{F}_{2^n}$. The idea is to split the system $\mathcal{S}_{\text{MIMC}}(M, C)$ over the base field \mathbb{F}_2 . In addition, we can add the field equations. We have then $(N+1)n + n$ equations of degree 2 in \mathbb{F}_2 and $(N+1)n$ variables. This has to be compared with the direct attack (Section 6.1) that has much less variables, but whose equations are of degree 3.

We present below some practical results. In particular, we report the maximal degree D_{\max} reached during the computation of a DRL-Gröbner basis. In this context, the system has very few solutions making the cost of the change of ordering negligible (in contrast to the direct algebraic attack of Section 6.1 that has an exponential number of variables). The cost of this attack is then exponential in D_{\max} . The last column essentially corresponds to the time for computing a DRL-Gröbner basis.

N	n	D_{\max}	CPU Time
3	7	3	0.00s
3	17	3	2.75s
3	27	3	23.67s
3	50	3	177.45s
5	7	3	0.11s
5	10	3	1.41s
5	15	3	34.22s
5	17	3	80.49s
10	10	3	26.44s
15	10	3	1179.57s
6	10	3	4.2s
6	11	> 3	
6	12	> 3	
6	15	> 3	
8	15	> 3	
10	15	> 3	

- Experiments are difficult to conduct (due to the large number of variables). Hence, only partial experimental results are available.
- Surprisingly when n (say ≤ 10) is small the complexity does not seem to depend on the number of rounds.

- When $n > 10$ then there is a change in the complexity (maximal degree in the computation). Consequently the computation is probably exponential.
- The attack is not very efficient in practice, however it is difficult to evaluate precisely the *theoretical complexity*.

Thus, since it is difficult to evaluate precisely the complexity, it is recommended to take $\mathbb{K} = \mathbb{F}_p$ where p is a prime number to avoid this attack.

7 Algebraic Cryptanalysis of Pepper

Let $K_0 \in \mathbb{K}^m$ and $(M, C = \text{Pepper-vi}(M, K_0)) \in \mathbb{K}^m \times \mathbb{F}_q^m$ be a pair message/ciphertext encrypted under the secret-key K_0 . We denote by $\text{Sys}_{\text{Pepper-vi}}(M, C)$ the corresponding algebraic system.

In this case, we introduce:

- s_0, s_1, \dots, s_{N-1} corresponding to intermediate state variables,
- a variable k_0 corresponding to the master-key and variables $k_1, \dots, k_{N'}$ corresponding to the sub-keys.

For Pepper-v1 , the system $\text{Sys}_{\text{Pepper-v1}}(M, C)$ is as follows:

$$\text{Sys}_{\text{Pepper-v1}}(M, C) \begin{cases} S_1^3 - M^3 - k_1, \\ \vdots \\ S_r^3 - k_{2r+1} - (S_{r-1} + k_{2r})^3, \\ \vdots \\ C = k_{2N} + S_N \end{cases} \quad \mathcal{K}_{\text{Pepper}} \begin{cases} k_1 k_0 + C_0 k_0 - 1, \\ \dots \\ k_{2r-1} k_{2r} + C_{2r-1} k_{2r-1} - 1, \\ k_{2r} k_{2r+1} + C_{2r+1} k_{2r} - 1, \\ \dots \\ k_{2N} k_{2N-1} + C_{2N} k_0 - 1 \end{cases}$$

We note that the C_r are known constants. The system corresponding $\text{Sys}_{\text{Pepper-v2}}(M, C)$ is essentially similar. The main difference is that the equations corresponding to the key scheduling are linear.

Theorem 7. *For $\text{Pepper-v1}(N)$, the Bézout multi-homogeneous bound is*

$$(d+1)N d^{N-1}.$$

For $\text{Pepper-v2}(N)$, the Bézout bound is d^{N+1} .

8 Algebraic Cryptanalysis of Rescue

8.1 Analysis of the Block Cipher

Let $K_0 \in \mathbb{F}_q^m$ and $(M, C = \text{Rescue}(M, K_0)) \in \mathbb{F}_q^m \times \mathbb{F}_q^m$ be a pair message/ciphertext encrypted under the secret-key K_0 . We denote by $\text{Sys}_{\text{Rescue}}(M, C)$ the corresponding algebraic system.

Example 4. We consider $N = 3, m = 2, p = 65519, M = \begin{bmatrix} 13612 \\ 47071 \end{bmatrix}, C = \begin{bmatrix} 55389 \\ 56921 \end{bmatrix}, a = 20801,$

$b = 275,$ and $k = \begin{bmatrix} 64238 \\ 8057 \end{bmatrix}$ In this case, $\text{Sys}_{\text{Rescue}}(M, C)$ is as follows:

$$\begin{aligned} & [s_{0,1}^3 - k_0 - 13612, s_{0,2}^3 - k_1 + 18448, \\ & s_{1,1}^3 + 11(-11s_{0,1} - 132s_{0,2} + 20801k_0 + 275)^3 + 132(12s_{0,1} + 133s_{0,2} + 20801k_1 + 275)^3 + 5875k_0 - 20397, s_{1,2}^3 - \\ & 12(-11s_{0,1} - 132s_{0,2} + 20801k_0 + 275)^3 - 133(12s_{0,1} + 133s_{0,2} + 20801k_1 + 275)^3 + 5875k_1 - 20397, s_{2,1}^3 + \\ & 11(-11s_{1,1} - 132s_{1,2} - 12940k_0 - 22772)^3 + 132(12s_{1,1} + 133s_{1,2} - 12940k_1 - 22772)^3 + 12888k_0 - 22273, s_{2,2}^3 - \\ & 12(-11s_{1,1} - 132s_{1,2} - 12940k_0 - 22772)^3 - 133(12s_{1,1} + 133s_{1,2} - 12940k_1 - 22772)^3 + 12888k_1 - 22273, -11 \\ & (-11s_{2,1} - 132s_{2,2} + 20460k_0 + 16099)^3 - 132(12s_{2,1} + 133s_{2,2} + 20460k_1 + 16099)^3 - 22964k_0 + 18095, \\ & 12(-11s_{2,1} - 132s_{2,2} + 20460k_0 + 16099)^3 + 133(12s_{2,1} + 133s_{2,2} + 20460k_1 + 16099)^3 - 22964k_1 + 16563]. \end{aligned}$$

The following result on the number of solutions of $\mathcal{S}_{\text{Rescue}}(M, C)$ is obvious :

Theorem 8. *Let d be the degree of the first SBox used in Rescue (Algorithm 10, Section 4.3.2). The Bézout bound for $\mathcal{S}_{\text{Rescue}}(M, C)$ is:*

$$d^{m \cdot (N+1)}. \quad (13)$$

Proof. We have :

- $N + 1$ rounds,
- m equations of degree d at each round.

So, $\mathcal{S}_{\text{Rescue}}(M, C)$ has in total $m(N + 1)$ equations of degree d . The result follows from Definition 1. \square

The result in Table 3 is obtained by assuming that the change of ordering is quadratic in the Bézout bound (13).

8.2 Analysis of the Hash Function

As explained in [5], **Rescue** (Section 4.3) can be transformed into a hash function thanks to the sponge construction. In this part, we consider the security of **Rescue**-based hash function against an algebraic attacks. To do so, we consider the so-called *Constrained-Input Constrained-Output* (CICO) problem. This problem, introduced in [32], is related to the the problem of finding a pre-image of the hash function.

Assuming that the secret-key is fixed in **Rescue**, then Algorithm 10 (Section 4.3) defines a permutation. Let $\text{perm}_{\text{Rescue}} : \mathbb{F}_q^m \rightarrow \mathbb{F}_q^m$ be the corresponding permutation.

In this context, the CICO problem is as follows:

<p>CICO (k) Find $x_{k+1}, \dots, x_t \in \mathbb{K}^{t-k}$ such that $\mathbf{x} = (0, \dots, 0, x_{k+1}, \dots, x_t)$ // Constrained input $\mathbf{y} = \text{perm}_{\text{Rescue}}(\mathbf{x})$ $y_1 = \dots = y_k = 0$ // Constrained output</p>
--

9 Algebraic Cryptanalysis of Vision

9.1 Algebraic Modelling

Let $K_0 \in \mathbb{F}_{2^{n/m}}$ and $(M, C = \text{Vision}(M, K_0)) \in \mathbb{F}_{2^{n/m}} \times \mathbb{F}_{2^{n/m}}$ be a pair message/ciphertext encrypted under the secret-key K_0 . We denote by $\text{Sys}_{\text{Vision}}(M, C)$ the algebraic system modelling **Rescue** (Algorithm 12) where the variables are:

- $s_{0,1}, \dots, s_{0,m}, s_{1,1}, \dots, s_{1,m}, \dots, s_{N-1,1}, \dots, s_{N-1,m}$ corresponding to intermediate state variables,
- additional variables $u_{0,1}, \dots, u_{0,m}, u_{1,1}, \dots, u_{1,m}, \dots, u_{N-1,1}, \dots, u_{N-1,m}$ corresponding to intermediate state variables,
- variables $k_{0,1}, \dots, k_{0,m}$ corresponding to the master-key.

We give below a pseudo-code for generating the equations of $\text{Sys}_{\text{Vision}}(M, C)$.

```

Input:  $(M, C = \text{Vision}(M, K_0)) \in \mathbb{F}_{2^{n/m}} \times \mathbb{F}_{2^{n/m}}$ 
 $K := {}^T[k_{0,1}, \dots, k_{0,m}]$ 
 $S_0 := M + K_0$ 
 $\text{Eqs} := []$ 
 $K_1, \dots, K_{2N} := \text{KeyExpand}(K_0)$  //  $K_i$  depends on  $\{k_{0,i}\}$ 
for  $r$  from 1 to  $N - 1$  do
   $\text{Eqs} := \text{Eqs} \cup [\text{Numerator}(B(1/S_{r-1}[i]))u_{r,i} \mid i = 1, \dots, m]$ 
   $S := \mathcal{M}^T[u_{r,1}, \dots, u_{r,m}] + K_{2r-1}$ 
   $\text{Eqs} := \text{Eqs} \cup [B(s_{r,i})S[i] - 1 \mid i = 1, \dots, m]$ 
   $S_r := \mathcal{M}^T[s_{r,1}, \dots, s_{r,m}] + K_{2r}$ 
EndFor
 $\text{Eqs} := \text{Eqs} \cup [u_{N,i}S_{N-1}[i] - 1 \mid i = 1, \dots, m]$ 
 $\text{Eqs} := \text{Eqs} \cup [u_{N,i} + K_{2N-1}[i] - C[i] \mid i = 1, \dots, m]$ 

```

Fig. 3. MAGMA pseudo-code for $\text{Sys}_{\text{Vision}}(M, C)$.

9.2 Bounding the Number of Solutions

Similarly to the algebraic attack against **Jarvis** (Section 5), the algebraic system $\text{Sys}_{\text{Vision}}(M, C)$ (Figure 3) has a multi-homogeneous structure (Definition 2.6, Section 2.3). In particular, we can derive a precise bound on the number of solutions of $\text{Sys}_{\text{Vision}}(M, C)$ for $m = 2, 3, 4$. Remark that the technique can be generalized to any m .

Theorem 9. *Let d be the degree of B (Algorithm 12). $\text{Sys}_{\text{Vision}}(M, C)$ has a multi-homogeneous structure. Let then $B_{N,m}(d)$ be the multi-homogeneous Bézout bound of $\text{Vision}(N, d, m, n)$ (with the modelling of Figure 3). We have :*

$$B_{N,2}(d) = \frac{2 \left(- (3d^2 + 12d + 3)N + (11d + 1)(d - 1) \right) (d^2)^N + (14d^2 + 11d + 2) (4d^2)^N}{18d^4},$$

$$B_{N,3}(d) = -\frac{(-249d^3 + 33d^2 + 9d + 47) (d^3)^N}{64d^6} + \frac{(51d^3 + 77d^2 + 29d + 3) (N + 1) (9d^3)^N}{144d^6},$$

$$-\frac{(201d^3 + 655d^2 + 295d + 33) (9d^3)^N}{576d^6} + \frac{(7d^3 + 9d^2 + 9d + 7) (N + 1) (d^3)^N}{16d^6}.$$

$$B_{N,4}(d) \approx \frac{(377d^2 + 231d + 22)(66d^2 + 63d + 11)(36d^4)^N}{61250d^8}.$$

In the table below, we report experimental results on the exact number of solutions of $\text{Sys}_{\text{Vision}}(M, C)$ for $m = 2$. We also compute the bound derived in Theorem 9. We can notice that the bound is perfectly sharp.

(N, m, d)	Exact Value	$B_{N,2}(d)$
(1, 2, 4)	4	4
(2, 2, 4)	233	233
(3, 2, 4)	15072	15072
(2, 2, 2)	65	65
(3, 2, 2)	1096	1096
(4, 2, 2)	17968	17968
(2, 2, 1)	20	20
(3, 2, 1)	90	90
(4, 2, 1)	376	376
(5, 2, 1)	1526	1526

It is easy to obtain an asymptotic estimates:

$$B_{N,2}(d) \approx \frac{(14d^2 + 11d + 2)(4d^2)^n}{18d^4}.$$

In particular when $d = \deg(B) = 4$, we deduce the value given in Table 4:

$$B_{N,2}(4) \approx \frac{15}{256} 64^N = \frac{15}{256} 2^{6N}.$$

In the next table, we provide experimental results for $m = 3$. In this case, The bound of Theorem 9 is tight but no longer exact.

(N, m, d)	Exact Value	$B_{N,3}(d)$
(1, 3, 4)	8	8
(2, 3, 2)	667	679
(2, 3, 4)	4645	4693
(2, 3, 1)	109	112
(3, 3, 1)	1626	1752

For $m = 3$, we get:

$$B_{3,N}(d) \approx \frac{(3d + 1)(4(d + 1)(17d + 3)n + d^2 - 116d - 21)(9d^3)^n}{576d^6}.$$

In particular when $d = 4$, we get:

$$B_{3,N}(d) \approx \frac{(18460n - 6097)576^n}{2359296}.$$

10 Algebraic Pre-Image Attacks

In this part, we consider the CICO problem (Figure 1, [32]) against Starkad, Poseidon (Section 10.1) and Rescue (Section 10.2). This is a preimage attack against the permutations underlying these ciphers.

10.1 The CICO Problem for Starkad and Poseidon

Let $\text{perm}_{\text{Hades}} : \mathbb{K}^t \rightarrow \mathbb{K}^t$ be the permutation defined as in (6) (Section 4.4). In this context, the CICO problem is defined as follows:

$$\begin{array}{l}
\text{CICO}_{\text{Hades}}(k) \\
\text{Find } (x_{k+1}, \dots, x_t) \in \mathbb{K}^{t-k} \text{ such that} \\
\mathbf{x} = (0, \dots, 0, x_{k+1}, \dots, x_t) \in \mathbb{K}^t \text{ // Constrained input} \\
\mathbf{y} = \text{perm}_{\text{Hades}}(\mathbf{x}) \\
y_1 = \dots = y_k = 0 \quad \text{// Constrained output}
\end{array}$$

Remark 10. We assume that \mathbb{K} is of size n . We then fix nk bits of the input and the output of the permutation function. In practice, we will have $nk \approx 256$ bits.

In the CICO problem, we want to find $\mathbf{x} = (0, \dots, 0, x_{k+1}, \dots, x_t)$ such that:

$$\text{perm}_{\text{Hades}}(\mathbf{x}) = (0, \dots, 0, \star, \dots, \star)$$

It is equivalent to find $\mathbf{z} = (z_1, \dots, z_k)$ such that $L(z_1, \dots, z_k) = (0, \dots, 0, \star, \dots, \star)$ and $\mathbf{z} = \text{perm}'_{\text{Hades}}(\mathbf{x})$ where the $\text{perm}'_{\text{Hades}}$ function is:

$$\text{perm}'_{\text{Hades}}(\mathbf{s}) = \mathbf{s}''' \text{ where } \begin{cases} \mathbf{s}' = R_{R_f} \circ \dots \circ R_1(\mathbf{s}) \\ \mathbf{s}'' = R'_{R_f+R_P} \circ \dots \circ R'_{R_f+1}(\mathbf{s}') \\ \mathbf{s}''' = R_{R_f+R_P-1} \circ \dots \circ R_{R_f+R_P+1}(\mathbf{s}'') \end{cases}$$

Since there is no matrix multiplication, we can recover explicitly the values of z_1, \dots, z_k . We then define a variant of the CICO problem by:

$$\begin{array}{l}
\text{CICO}'_{\text{Hades}}(k, (a_1, \dots, a_k) \in \mathbb{K}^k) \\
\text{Find: } (x_{k+1}, \dots, x_t) \in \mathbb{K}^{t-k} \text{ such that} \\
\mathbf{x} = (0, \dots, 0, x_{k+1}, \dots, x_t) \in \mathbb{K}^t \text{ // Constrained input} \\
\mathbf{y} = \text{perm}'_{\text{Hades}}(\mathbf{x}) \\
y_1 = a_1, \dots, y_k = a_k \quad \text{// Constrained output}
\end{array}$$

10.1.1 Algebraic Modelling of the CICO Problem – Starkad and Poseidon. Let $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{K}^k$. The algebraic system $\mathcal{S}_{\text{CICO}'_{\text{Hades}}}(\mathbf{a}) \subset \mathbb{K}[x_{k+1}, \dots, x_t]$ corresponding to the $\text{CICO}'_{\text{Hades}}$ problem is given below :

$$\mathcal{S}_{\text{CICO}'_{\text{Hades}}}(\mathbf{a}) = \begin{cases} y_1 = a_1 \\ \vdots \\ y_k = a_k \end{cases} \text{ where } \mathbf{y} = \text{perm}'_{\text{Hades}}(0, \dots, 0, x_{k+1}, \dots, x_t).$$

Hence, we have k equations and $t - k$ variables. For practical parameters (Table 10), we see that $t - k$ is much larger than k . Hence the system $\mathcal{S}_{\text{CICO}'_{\text{Hades}}}(\mathbf{a})$ is under-defined. As a consequence, we can fix arbitrarily some values of the input \mathbf{x} .

By randomly fixing $r_{2k+1}, \dots, r_t \in \mathbb{K}$, we define a square algebraic system of k equations and k variables:

$$\mathcal{S}_{\text{CICO}'_{\text{Hades}}}(\mathbf{a}) = \begin{cases} y_1 = a_1 \\ \vdots \\ y_k = a_k \end{cases} \quad \text{where } y = \text{perm}'_{\text{Hades}}((0, \dots, 0, x_{k+1}, \dots, x_{2k}, r_{2k+1}, \dots, r_t)).$$

Remark 11. The complexity of solving $\mathcal{S}_{\text{CICO}'_{\text{Hades}}}(\mathbf{a})$ does not depend on t . The attack is more efficient when the field (n) is big.

10.1.2 Practical Experiments We present here experimental results on the complexity of solving $\mathcal{S}_{\text{CICO}'_{\text{Hades}}}(\mathbf{a})$. For the experiments, we took $d = 2$ and a prime field $\mathbb{K} = \mathbb{F}_p$. We have done the experiments for $k = 1, 2, 3$ (we also checked that the results are the same when $\mathbb{K} = \mathbb{F}_{2^n}$ and $d = 3$). In each case, we compute a Gröbner basis of $\mathcal{S}_{\text{CICO}'_{\text{Hades}}}(\mathbf{a})$ w.r.t. a lexicographical monomial ordering.

We report below the degree of the non linear leading terms in the Gröbner basis.

R_F	R_P	t	degree
2	0	5	x_3^2, x_4^2
2	1	5	x_3^2, x_4^4
2	2	5	x_3^2, x_4^8
4	0	5	x_3^2, x_4^{32}
4	1	5	x_3^2, x_4^{64}
4	2	5	x_3^2, x_4^{128}
6	0	5	x_3^2, x_4^{512}
6	1	5	x_3^2, x_4^{1024}

$k = 2.$

R_F	R_P	t	degree
2	0	6	x_4^2, x_5^2, x_6^2
2	1	6	x_4^2, x_5^2, x_6^4
2	2	6	x_4^2, x_5^2, x_6^8
4	0	6	x_4^2, x_5^2, x_6^{128}
4	1	6	x_4^2, x_5^2, x_6^{256}
4	2	6	x_4^2, x_5^2, x_6^{512}
6	0	6	x_4^2, x_5^2, x_6^{8192}

$k = 3.$

R_F	R_P	t	degree
2	0	4	x_2^2
2	1	4	x_2^4
2	2	4	x_2^8
4	0	4	x_2^8
4	1	4	x_2^{16}
4	2	4	x_2^{32}
6	0	4	x_2^{32}
6	1	4	x_2^{64}
10	0	4	x_2^{512}
11	1	4	x_2^{1024}
14	0	4	x_2^{8192}

$k = 1.$

10.1.3 Bounding the Number of Solutions Starkad and Poseidon. We can prove the following results:

Theorem 10. *The degree of the univariate polynomial of the lexicographical Gröbner basis of $\mathcal{S}_{\text{CICO}'_{\text{Hades}}}(\mathbf{a})$ is*

$$d^{k R_F + P R_P - 2k + 1}$$

For the permutation of Starkad/Poseidon ($d = 3$ and $P = 1$), it holds in particular that the number of solutions D is equal to:

$$D = 3^{k R_F + R_P - 2k + 1}.$$

Remark 12. As expected, when $k \neq 1$ we see that the number of full rounds is the most important factor in the complexity.

The complexity of the attack is polynomial in the degree D of the univariate polynomial: $O(D^\omega)$. When $k = 1$ or 2 , we take $\omega = 1$; when $k > 2$ then $\omega = 2$.

The complexity of the attack is then:

k	1	2	> 2
Complexity for Starkad/Poseidon	$3^{R_F+R_P-1}$	$3^{2R_F+R_P-3}$	$3^{2(kR_F+R_P-2k)+2}$

This allows to derive the security estimates of Table 6 (Section 1.3).

10.2 The CICO Problem for Rescue

Let $\text{perm}_{\text{Rescue}} : \mathbb{F}_q^m \rightarrow \mathbb{F}_q^m$ be the **Rescue** permutation (Section 4.3.3). As in the previous part, we consider:

CICO_{Rescue}(k)
Find $(x_{k+1}, \dots, x_t) \in \mathbb{K}^{m-k}$ such that
 $\mathbf{x} = (0, \dots, 0, x_{k+1}, \dots, x_t) \in \mathbb{K}^m$ // Constrained input
 $\mathbf{y} = \text{perm}_{\text{Rescue}}(\mathbf{x})$
 $y_1 = \dots = y_k = 0$ // Constrained output

This is not detailed in the report but, as in Section 10.1, we can define an algebraic set of equations $\mathcal{S}_{\text{CICO}_{\text{Rescue}}}(k)$ modelling **CICO**_{Rescue}(k). Also, it holds that:

Theorem 11. *The number of solutions D of $\mathcal{S}_{\text{CICO}_{\text{Rescue}}}(k)$ is equal to:*

$$D = 3^{(N-1)m+k}.$$

The complexity of the attack is polynomial in the degree D of the univariate polynomial: $O(D^\omega)$. When $k = 1$ or 2 , we take $\omega = 1$; when $k > 2$ then $\omega = 2$.

k	1	2	> 2
Complexity for Rescue	$3^{(N-1)m+k}$	$3^{(N-1)m+k}$	$3^{2((N-1)m+k)}$

This allows to derive the security estimates of Table 7 (Section 1.3).

11 History of the Report

Version	Comment	Date
	Algebraic analysis of STARK-friendly ciphers	11/2018-09/2019
1.0	First version of the document	09/2019
1.1	Minor typos corrected	03/2020
	Algebraic analysis of the CICO problem against Rescue	03/2020
1.2	New part (Section 10.2) dedicated to the CICO problem against Rescue . Report updated and re-organised accordingly.	04/2020

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A Reference codes for MiMC

A.1 SAGE code for MiMC

```

n = 129
K = GF(2**n, "a")
K.inject_variables()
constants = [.....]
def mimc_encryption(p, k, num_rounds):
    state = (p + (k + K.fetch_int(constants[0])))^3
    for i in range(1, num_rounds):
        state = (state + (k + K.fetch_int(constants[i])))^3
    state = state + k
    return state
num_rounds = ceil(n / log(3, 2))
k = K.fetch_int(0x42424242424242424242424242424242)
p = K.random_element()
c = mimc_encryption(p, k, num_rounds)

print "Plaintext:", p.integer_representation()
print "Ciphertext:", c.integer_representation()

```

A.2 MAPLE code MiMC

```

Nb:=80; # number of rounds
d:=3; # degree (3 by default)
Vs:=proc(i) cat('s',i): end:
encryption_symb:=proc(msg, cyp, Nb)
    local state,eqs,i;
    state:=(msg + (k + C[1]))^d;
    eqs:=[Vs(0)-state];
    state:=Vs(0);
    for i from 1 to (Nb-1) do
        state:=(state + (k + C[i+1]))^d;
        eqs:=[op(eqs),Vs(i)-state];
        state:=Vs(i);
    od:
    return [op(eqs),state+k-cyp];
end:

# Algebraic Equations:
Eqs:=encryption_symb(M,C,Nb):

```

A.3 MAGMA code for MiMC

```

n:=129;
d:=3;
Nb:=Ceiling(n / Log(2, 3));

```

```

K<a>:=GF(2,n);

constants:=[...];
encryption:=function(p, sk, Nb)
  state:=(p + (sk + constants[1]))^d;
  for i in [1..(Nb-1)] do
    state:=(state + (sk + constants[i+1]))^d;
  end for;
  state:=state + sk;
  return state;
end function;
sk:=Random(K);
msg:=Random(K);
cyp:=encryption(msg, sk, Nb);

```

B MAGMA code for Poseidon/Starkad

The following MAGMA code permits to:

- easily change the ground field to be \mathbb{F}_p , \mathbb{F}_{2^n} or even \mathbb{Q} .
- change the degree of the equations between 2 and 3.
- try to modify P the number of S-boxes which are applied in the middle rounds (partial rounds).

```

perm:=function(input_words,dd,part)
  R_f := Z!(R_F / 2);
  state_words := Vector(F,input_words);
  for r in [1..R_f] do
    for i in [1..t] do
      state_words[i] := (state_words[i] + round_constants_field[t*(r-1)+i])^dd;
    end for;
    state_words := state_words * MDS_matrix_field;
  end for;
  for r in [1..R_P] do
    for i in [1..t] do
      state_words[i] := state_words[i] + round_constants_field[t*(R_f+r-1)+i];
    end for;
    for i in [1..part] do
      state_words[i] := (state_words[i])^dd;
    end for;
    state_words := state_words * MDS_matrix_field;
  end for;
  for r in [1..(R_f-1)] do
    for i in [1..t] do
      state_words[i] := (state_words[i] +
        round_constants_field[t*(R_f+R_P+r-1)+i])^dd;
    end for;
  end for;

```

```
    state_words := state_words * MDS_matrix_field;
end for;
for i in [1..t] do
    state_words[i] := (state_words[i] + round_constants_field[t*(R_F+R_P-1)+i])^dd;
end for;
return state_words;
end function;
```

B CICO Report

Rescue
CICO Problem

Prepared at the request of StarkWare Industries

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March 16,2020

Short description of Rescue

Rescue_d(N, m, p)

Arithmetic: finite field $\mathbb{K} = \mathbb{F}_p$, $c = \frac{1}{d} \bmod (p-1)$, $N = 2N$
 \mathcal{M} is an invertible $m \times m$ matrix.

Rescue:

Input: $M \in \mathbb{K}^m$

$S_0 = M + C_0$

for r **from** 1 **to** N **do**

$S = {}^T [S_{r-1}[1]^d, \dots, S_{r-1}[m]^d]$

$S = \mathcal{M}S + C_{2r-1}$

$S = {}^T [S[1]^c, \dots, S[m]^c]$

$S_r = \mathcal{M}S + C_{2r}$

Output: $S_N \in \mathbb{K}^m$

CstesGeneration:

Input: $C_0 \in \mathbb{K}^m, a \in \mathbb{K}, b \in \mathbb{K}$

$B = {}^T [b, \dots, b]$

for r **from** 1 **to** N **do**

$C_r = aC_{r-1} + B$

Output: C_0, \dots, C_N

In practice: $d = 3$ and $c = (2p-1)/3$

Rescue overview

Since we want to study the CICO problem, C_0 is no longer a secret key but a randomly-chosen vector:

- 1 Choose randomly $a \in \mathbb{K}, b \in \mathbb{K}$ and $C_0 \in \mathbb{K}^m$
- 2 Run CstesGeneration(C_0, a, b)
- 3 Given a message $M \in \mathbb{K}^m$ run $C := \text{Rescue}(M)$
- 4 Send $C \in \mathbb{K}^m$

A sponge construction generates a hash function from an underlying permutation by iteratively applying it to a large state. The state of a sponge function is defined to consist of $m = r + c$ bits, where r and c are called the rate and the capacity of the sponge, respectively.

Algebraic Equations

Variables/equations

The CICO Problem

The **CICO** Problem (Constrained-Input Constrained-Output) is defined by:

CICO Problem(k):

Find: $x_{k+1}, \dots, x_m \in \mathbb{K}^{m-k}$ such that

$x = [0, \dots, 0, x_{k+1}, \dots, x_m]$ // Constrained input

$y = \text{Rescue}(x)$

$y_1 = \dots = y_k = 0$ // Constrained output

Hence if we assume that \mathbb{K} is of size n we fix nk bits of the input **and** the output of the permutation function. In practice we will have $nk \approx 256$ bits.

Remark

Thanks to the last Matrix multiplication \mathcal{M} at the end, we cannot remove the last round. However, we will see how to **simplify** the algebraic system.

CICO first example and optimizations

We take, $N = 3$, $m = 2$, $k = 1$ and $\mathcal{M} = \begin{bmatrix} -11 & -132 \\ 12 & 133 \end{bmatrix}$

We take for the input/output vectors:

$$X = [0, x_1] \text{ and } Y = [0, y_1]$$

We have the following equations:

$$[s_{0,2}^3 - x_1 + 24825,$$

$$s_{1,1}^3 + 11 (7615 - 132s_{0,2})^3 + 132 (31617 + 133s_{0,2})^3 - 29116,$$

$$s_{1,2}^3 - 12 (7615 - 132s_{0,2})^3 - 133 (31617 + 133s_{0,2})^3 - 21978,$$

$$s_{2,1}^3 + 11 (-11s_{1,1} - 132s_{1,2} - 15445)^3 + 132 (12s_{1,1} + 133s_{1,2} - 26929)^3 + 31513,$$

$$s_{2,2}^3 - 12 (-11s_{1,1} - 132s_{1,2} - 15445)^3 - 133 (12s_{1,1} + 133s_{1,2} - 26929)^3 + 27923,$$

$$-11 (-11s_{2,1} - 132s_{2,2} + 15957)^3 - 132 (12s_{2,1} + 133s_{2,2} - 113)^3 + 2578,$$

$$12 (-11s_{2,1} - 132s_{2,2} + 15957)^3 + 133 (12s_{2,1} + 133s_{2,2} - 113)^3 + 8446 - y_1]$$

Remark

We can remove the first and the last equations: these equations are linear in x_1 and y_1 . In general we can **remove** the first k equations and the last k equations.

CICO second example and optimizations

We take, $N = 2$, $m = 3$, $k = 1$ and $\mathcal{M} = \begin{bmatrix} -11 & -132 \\ 12 & 133 \end{bmatrix}$

We take for the input/output vectors:

$$X = [0, x_1, x_2] \text{ and } Y = [0, y_1, y_2]$$

We have the following equations (removing the linear equations):

$$[s_{1,1}^3 - 1331 (29260 - 19534s_{0,2} - 24464s_{0,3})^3 + 19534 (-31963 + 3309s_{0,2} + 25225s_{0,3})^3 +$$

$$24464 (-3157 + 16226s_{0,2} - 760s_{0,3})^3 - 29116,$$

$$s_{1,2}^3 + 1463 (29260 - 19534s_{0,2} - 24464s_{0,3})^3 - 3309 (-31963 + 3309s_{0,2} + 25225s_{0,3})^3 -$$

$$25225 (-3157 + 16226s_{0,2} - 760s_{0,3})^3 - 21978,$$

$$s_{1,3}^3 - 133 (29260 - 19534s_{0,2} - 24464s_{0,3})^3 - 16226 (-31963 + 3309s_{0,2} + 25225s_{0,3})^3 +$$

$$760 (-3157 + 16226s_{0,2} - 760s_{0,3})^3 - 11587,$$

$$1331 (1331s_{1,1} - 19534s_{1,2} - 24464s_{1,3} - 15445)^3 - 19534 (-1463s_{1,1} + 3309s_{1,2} + 25225s_{1,3} - 26929)^3 -$$

$$24464 (133s_{1,1} + 16226s_{1,2} - 760s_{1,3} - 22939)^3 - 31513]$$

Remark

Clearly we have 4 equations and 5 variables: we can fix arbitrarily $s_{0,3}$ (or equivalently x_2). In general we can **fix** the last $m - 2k$ variables of the input vector $X = [0, \dots, 0, x_1, \dots, x_k, \dots,]$.

Structure of the new system of equations

CICO problem (k) for $\text{Rescue}_d(N, m, p)$

Assuming that $X = [0, \dots, 0, x_1, \dots, x_k, \dots,]$

We have linear equations (that we can ignore):

$$\left\{ \begin{array}{l} s_{0,i}^d - x_{i-k} - \alpha_i \text{ for } i=(k+1), \dots, m \end{array} \right.$$

where $\alpha_i \in \mathbb{K}$ and we have non linear equations:

$$(\mathcal{S}) \left\{ \begin{array}{l} \frac{s_{1,i}^d - P_{1,i}(s_{0,k+1}, \dots, s_{0,2k})}{s_{2,i}^d - P_{2,i}(s_{1,1}, \dots, s_{1,m})} \text{ for } i=1, \dots, m \\ \dots \\ \frac{s_{N-1,i}^d - P_{N-1,i}(s_{N-2,1}, \dots, s_{N-2,m})}{P_{N,i}(s_{N-1,1}, \dots, s_{N-1,m})} \text{ for } i=1, \dots, m \end{array} \right.$$

where $P_{i,j}$ are polynomials of degree d .

Parameters

Name	Field	$n = \log_2 p$	N	m	k
Rescue128a	\mathbb{F}_p	125	16	4	2
Rescue128b	\mathbb{F}_p	253	22	12	1
Rescue128c	\mathbb{F}_p	125	10	12	2
Rescue128d	\mathbb{F}_p	61	10	12	4
Rescue128e	\mathbb{F}_p	253	10	11	1

We remark that $m \geq 2k$, so that we can apply the previous strategy and fix $m - 2k$ variables.

We report now some Gröbner bases experiments: as usual this is $TDRL + TFGLM$ where TDRL (resp. TFGLM) is the CPU time of the DRL Gröbner basis (resp. the time to change the monomial ordering).

When $k = 1$ or 2 is possible to speedup the computations by using the sparse structure of the equations.

Experiments: standard tools $d=3$

#	Nb	m	k	#Sol	TDRL	TFGLM	Dmax
#	2	2	1	3^3	0.0s	0.0s	6
#	2	3	1	3^4	0.0s	0.0s	7
#	2	4	1	3^5	0.0s	0.0s	9
#	2	4	2	3^6	0.09s	0.04s	12
#	3	2	1	3^5	0.0s	0.0s	8
#	3	3	1	3^7	0.42s	0.73s	13
#	3	4	1	3^9	124.9s	501.6s	16
#	4	2	1	3^7	0.22s	0.77s	12
#	5	2	1	3^9	44.4s	543.1s	16

Experiments: standard tools $d=2$

#	Nb	m	k	#Sol	TDRL	TFGLM	Dmax
#	2	4	2	2^6	0s	0s	7
#	2	5	2	2^7	0.01s	0s	8
#	3	2	1	2^5	0s	0s	5
#	3	4	2	2^{10}	0.47s	0.14s	11
#	3	5	2	2^{12}	19.8s	6.5 s	12
#	4	2	1	2^7	0s	0s	7
#	4	3	1	2^8	0.29s	0.13s	9
#	5	2	1	2^9	0.04s	0.02s	9

CICO Problem: complexity of the attack

From the experimental results, we see that all the algebraic systems behave like **regular systems**. So that we can derive sharp complexity bounds.

We observe similar results for $d = 2$ or $d = 3$ (but computations are much faster for $d = 2$ of course).

We count the number of solutions of the system (\mathcal{S}) by counting the number of equations: In (\mathcal{S}) we have,

$$E = m + (N - 2)m + k = (N - 1)m + k \text{ equations of degree } d.$$

Consequently,

Theorem (Number of Solutions)

The number of solutions of the CICO problem (k) for $\text{Rescue}_d(N, m, p)$:

$$D = 3^{(N-1)m+k}$$

CICO Problem: comparison with exhaustive search

The complexity of the attack will be polynomial in this number $D = 3^{(N-1)m+k}$. It is important to compare D with the cost of the exhaustive search which is $p^k \approx 2^{256}$.

Name	$n = \log_2 p$	N	m	k	p^k	D
Rescue128a	125	16	4	2	2^{250}	$2^{98.3}$
Rescue128b	253	22	12	1	2^{253}	$2^{401.0}$
Rescue128c	125	10	12	2	2^{250}	$2^{174.3}$
Rescue128d	61	10	12	4	2^{244}	$2^{177.5}$
Rescue128e	253	10	11	1	2^{253}	$2^{158.5}$

We see that the number D vary significantly. We have to investigate precisely to complexity of the attack.

CICO Problem: complexity of the attack

The complexity of the attack is polynomial in the degree D of the univariate polynomial: $O(D^\omega)$

When $k = 1$ or 2 , we take $\omega = 1$; when $k > 2$ then $\omega = 2$.

Proposition

The complexity of the attack is:

k	1 or 2	> 2
Complexity	$3^{(N-1)m+k}$	$3^{2((N-1)m+k)}$

Name	$n = \log_2 p$	N	m	k	p^k	CICO
Rescue128a	125	16	4	2	2^{250}	$2^{98.3}$
Rescue128b	253	22	12	1	2^{253}	$2^{401.0}$
Rescue128c	125	10	12	2	2^{250}	$2^{174.3}$
Rescue128d	61	10	12	4	2^{244}	2^{355}
Rescue128e	253	10	11	1	2^{253}	$2^{158.5}$

Hybrid method (1)

For Rescue128d the size of the ground field $\mathbb{K} = GF(2^{61} + 202^{32} + 1)$ is “small”. An **hybrid method** is a combination between exhaustive search and Gröbner bases computation.

For the system (\mathcal{S}) the maximal degree occurring in the (DRL) Gröbner basis is given by

$$d_{max} = 1 + (d - 1)E = 2((N - 1)m + k) + 1$$

By exhaustive search, we can find the value of one variable of (\mathcal{S}): we keep the same the number of equations but we decrease by one the number of variables. Under semi-regularity assumption of the system, the new maximum degree is given by:

$$d_{max} = \left\lceil \frac{d_{max}}{2} \right\rceil = E + 1 = ((N - 1)m + k) + 1$$

Experiments: standard tools $d=3$

We check experimentally the semi-regularity of the system when we fix one variable:

#	Nb	m	k	TDRL	Dmax
#	2	4	2	0.0s	7
#	2	5	2	0.08s	8
#	2	6	2	0.95s	9
#	3	4	2	259.4s	11

Conclusion: the system behaves exactly as expected.

Hybrid method (2)

$$d_{max} = \left\lceil \frac{d_{max}}{2} \right\rceil = E + 1 = ((N - 1)m + k) + 1$$

However, the total complexity estimate of this computation is very pessimistic: it is equal to $\binom{E + E - 1 + 1}{E - 1}^\omega = \binom{2E}{E - 1}^\omega$

In total, the complexity of the exhaustive search and the Gröbner basis is $C = 2^n \binom{2E}{E - 1}^\omega$.

For Rescue128d, then $N=10, m=12, k=4$ we have $E = 114$, so that $C = 2^{61+223.7\omega}$. Since $k - 1 = 3 \gg 1$ we cannot expect to have $\omega = 2$.

👉 The Hybrid method is **not efficient** in this case.