Limits of Polynomial Packings for $\mathbb{Z}_{p^k}$ and $\mathbb{F}_{p^k}$

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Abstract. We formally define polynomial packing methods and initiate a unified study of related concepts in various contexts of cryptography. This includes homomorphic encryption (HE) packing and reverse multiplication-friendly embedding (RMFE) in information-theoretically secure multi-party computation (MPC). We prove several upper bounds and impossibility results on packing methods for $\mathbb{Z}_{p^k}$ or $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$ in terms of (i) packing density, (ii) level-consistency, and (iii) surjectivity. These results have implications on recent development of HE-based MPC over $\mathbb{Z}_{2^k}$ secure against actively corrupted majority and provide new proofs for upper bounds on RMFE.

Keywords: Packing method · Homomorphic encryption · Secure multi-party computation · Reverse multiplication-friendly embedding · $\mathbb{Z}_{p^k}$.

1 Introduction

HE Packing. Homomorphic encryption (HE), which allows computations on ciphertexts without decryption, is such a versatile tool that it is often referred as the holy grail of cryptography. After Gentry's breakthrough [Gen09], HE has undergone extensive study and development. HE is now considered to be exploitable in real-life applications (e.g. privacy-preserving machine learning [KSK+18]) and regarded as a core building block in various cryptographic primitives (e.g. secure multi-party computation [DPSZ12]).

One drawback of contemporary lattice-based HE schemes [BGV12, FV12] is that their plaintext space is of the form $\mathbb{Z}_q[x]/\Phi_M(x)$, as their security is based on Ring Learning with Errors (RLWE) [LPR10]. That is, these schemes are homomorphic with regards to the addition and multiplication of polynomial ring $\mathbb{Z}_q[x]/\Phi_M(x)$. This raises a question of how to *homomorphically* encode messages into the plaintexts, as our data are usually binary bits, integers, fixed/floating point numbers, or at least $\mathbb{Z}_p$ and $\mathbb{F}_{p^k}$.

Among a number of works on how to encode data into HE plaintexts [CJLL17, CLPX18, CIV18, CKKS17], Smart-Vercauteren [SV10, SV14] first introduced the idea of *packing* several $\mathbb{Z}_p$ (or $\mathbb{F}_{p^k}$) elements into the HE plaintext space $\mathbb{Z}_p[x]/\Phi_M(x)$ via CRT[3] ring isomorphism with *well-chosen* prime p. Their simple yet powerful technique enables SIMD[4]-like optimizations and enhances amor-

[3] Chinese Remainder Theorem

[4] Single Instruction, Multiple Data

tized performance. That is, with a polynomial packing method, we can securely compute on *multiple* $\mathbb{Z}_p$-messages simultaneously by homomorphically computing on a *single* packed HE plaintext in $\mathbb{Z}_p[x]/\Phi_M(x)$. In particular, through the packing, the complex multiplicative structure of $\mathbb{Z}_p[x]/\Phi_M(x)$ embeds the more handy coordinate-wise multiplication (a.k.a. Hadamard product) of $\mathbb{Z}_p^n$, where n denotes the number of packed messages. Packing has now become a standard technique in HE research, and it is not too much to say that the performance of HE applications are determined by how well packings are utilized.

However, this conventional packing method has a limitation: it cannot (efficiently) pack $\mathbb{Z}_{2^k}$-messages.[5] This limitation has recently attracted attention due to development of secure multi-party computation (MPC) over $\mathbb{Z}_{2^k}$ secure against actively corrupted majority by SPD$\mathbb{Z}_{2^k}$ [CDE$^+$18]. SPD$\mathbb{Z}_{2^k}$ follows the framework of HE-based MPC protocol SPDZ [DPSZ12], while targeting $\mathbb{Z}_{2^k}$-messages rather than prime field $\mathbb{Z}_p$-messages, with a motivation from the fact that $\mathbb{Z}_{2^k}$ arithmetic matches closely what happens on standard CPUs. In this context, Overdrive2k [OSV20] and MHz2k [CKL21], whose goal are efficient constructions of HE-based MPC over $\mathbb{Z}_{2^k}$, came up with new and more involved polynomial packing methods for $\mathbb{Z}_{2^k}$-messages (Section 4).

RMFE in Perfectly Secure MPC. Another context where polynomial packings appear is *information-theoretically secure MPC* (or perfectly secure MPC). A main tool in this area is Shamir's linear secret sharing scheme(LSSS). A cumbersome fact when using LSSS is that the number of shares is restricted by the field where computation takes place.[6] Thus, it is standard to *lift* the computation to a larger field which supports enough number of shares, but this causes substantial overheads. In their seminal work [CCXY18], Cascudo-Cramer-Xing-Yuan first defined and studied *reverse multiplication-friendly embedding (RMFE)* which is, roughly speaking, an embedding of several elements of small finite field into a larger finite field while providing *somewhat* homomorphism of degree-2. Note that an RMFE can be indeed viewed as a polynomial packing $\mathbb{F}_{p^k}^n \to \mathbb{F}_{p^d} \cong \mathbb{F}_p[x]/f(x)$, where p is a prime and $f(x) \in \mathbb{F}_p[x]$ is an irreducible polynomial of degree d. Surprisingly, [CCXY18] constructed *constant-rate* RMFEs, leveraging algebraic geometry, and applied them to remove logarithmic overhead in amortized communication complexity which appears to enable Shamir's secret sharing. Since [CCXY18], RMFE has become a standard tool in information-theoretically secure MPC, to achieve *linear* amortized communication cost while preserving optimal corruption tolerance: [BMN18, DLN19, CG20, CXY20, DLSV20, PS21].

In [CRX21], the notion of RMFE was extended to *over Galois rings* for construction of efficient perfectly secure MPC over $\mathbb{Z}_{p^k}$. Again, RMFE over Galois rings for $\mathbb{Z}_{p^k}$-messages can be viewed as a polynomial packing $\mathbb{Z}_{p^k}^n \to$

[5] The original method of [SV10] does not consider packings for $\mathbb{Z}_{p^k}$. Gentry-Halevi-Smart [GHS12] later generalized the method to support such packing. However, this method achieves only considerably low efficiency. See Section 4.1.

[6] Indeed, the number of evaluation points is bounded by the size of the field.

$GR(p^k, d) \cong \mathbb{Z}_{p^k}[x]/f(x)$, where p is a prime and $f(x) \in \mathbb{Z}_{p^k}[x]$ is a degree-d irreducible polynomial in $\mathbb{F}_p[x]$.

Other Contexts. Other than HE and perfectly secure MPC, there are still more areas where polynomial packings are used for amortization: correlation extraction for secure computation [BMN17], zk-SNARK [CG21], etc. Moreover, we believe that polynomial packing will be even more prominent and universal tool for efficiency and practicality in the future: (i) RLWE-based cryptosystems are emerging, where plaintexts are $\mathbb{Z}_q[x]/\Phi_M(x)$; (ii) Secure computation is emerging, where some parts of protocols need to be large or of certain form due to security or mathematical properties required, whereas where we actually want to compute in is (extremely) small and typical such as $\mathbb{F}_2$ or $\mathbb{Z}_{2^{32}}$.

1.1 Our Contribution

Unified Definition and Survey. In this work, we formally define polynomial packing methods, which can be understood as (somewhat) homomorphic encoding for copies of a small ring, e.g. $\mathbb{Z}_p$ or $\mathbb{F}_{p^k}$, into a larger ring, e.g. $\mathbb{Z}_q[x]/f(x)$, (Section 3.1). The notion of polynomial packing unifies forementioned concepts in various contexts of cryptography, including HE packing and RMFE in perfectly secure MPC. Then, we gather existing packing methods in one place. This includes RMFE (Section 2.3 and 3.1), classic HE packing methods (Section 3.1), and recent development occurred in HE-based MPC over $\mathbb{Z}_{2^k}$ (Section 4). We also provide *decomposition* lemmas which suggest that it is enough to study packing methods for $\mathbb{Z}_{p^k}^n$ (or $\mathbb{F}_{p^k}^n$) into $\mathbb{Z}_{p^t}[x]/f(x)$ where $t \geq k$ and p is prime, instead of general case of $\mathbb{Z}_P^n$ (or $\mathbb{F}_P^n$) into $\mathbb{Z}_Q[x]/f(x)$ where $P, Q \in \mathbb{Z}^+$ (Section 3.2). The results also rule out the possibility of using composite modulus for better packing.

Upper Bounds and Impossibility. We prove several upper bounds and impossibility results on packing methods for $\mathbb{Z}_{p^k}$ or $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$.

- Upper Bounds on Packing Density (Section 5): We evaluate the efficiency of packing methods by packing density which measures how densely the messages are packed in (plaintext) polynomials (Def. 3.3).[7] We prove that, when a packing method provides somewhat homomorphism upto degree-D polynomials, the packing density is roughly upper bounded by $1/D$ (Thm. 5.1 and 5.3). These results have several implications:

[7] We note that packing density differs from *ciphertext rate* which is the main interest of recent developments in compressible or optimal-rate FHE [BDGM19, GH19]. Ciphertext rate measures the ratio of *plaintext* size to *ciphertext* size, whereas packing density measures the ratio of *message* size to *plaintext* size. On the distinction of message and plaintext, please refer to Section 2.1.

  - The packing method of MHz2k [CKL21] achieves nearly optimal density in some sense when using their parameters (Example 5.1). Our results justify the *lifting* of MHz2k packing (See Section 4.4).
  - We provide the first upper bound on RMFE over Galois ring for $\mathbb{Z}_{p^k}$-messages (Example 5.2).
  - We provide a new proof for upper bound on RMFE, which can be extended to higher-degree settings unlike the previous proof (Example 5.7).
- Impossibility of Level-consistency (Section 6): The notion of level-consistency captures the property whether packings are decodable in an identical way at different multiplicative levels (Def. 6.1). The level-consistency is a desirable feature as it allows homomorphic computation between different packing levels. We prove sufficient and necessary conditions on parameters to allow a level-consistent packing method. These results have the following implications:
  - HELib packing [HS15] (a.k.a. GHS packing [GHS12], See Section 4.1) is essentially the optimal method to use in *fully* homomorphic encryption(FHE) (Example 6.1).
  - It is impossible to construct efficient level-consistent packing methods in most cases. This justifies the use of *level-dependent* packings in SPDZ-like MPC protocols over $\mathbb{Z}_{2^k}$ [OSV20, CKL21] and highlights the usefulness of the trick proposed by MHz2k [CKL21], which closed the gap between the level-consistent and level-dependent packing methods in so-called *reshare* protocol. (See Section 6.1.)
- Impossibility of Surjectivity (Section 7): For a packing method into $\mathcal{R}$, the notion of surjectivity captures the condition whether every element of $\mathcal{R}$ is decodable (Def. 7.1). This distiction is essential when designing a cryptographic protocol with the packing method in a malicious setting, where an adversary might freely deviate from the protocol. If there is an element in $\mathcal{R}$ which fails to decode, a malicious adversary might make use of the element to illegitimately learn information of other parties, if such invalid packings are not properly handled. We prove sufficient and necessary conditions on parameters to allow a surjective packing method. Our results suggest that it is impossible to construct a meaningful surjective packing method in most cases. This justifies the use of *non*-surjective packings and the need of ZKPoMK[8], which ensures an HE ciphertext encrypts a validly packed plaintext, in SPDZ-like MPC protocols over $\mathbb{Z}_{2^k}$ [OSV20, CKL21].

1.2 Organization

In Section 2, we introduce our notations (Section 2.1) and some preliminaries on polynomial factorization (Section 2.2) and RMFE (Section 2.3). We highly

[8] Zero-knowledge proof of message knowledge

recommend readers to read at least the first three items of Section 2.1, as we will be slightly abusing some terminologies for simplicity.

In Section 3, we formally define the concept of packing methods and introduce basic examples. In Section 3.2, we prove *decomposition* lemmas, which enable us to focus on packing methods into $\mathbb{Z}_{p^t}[x]/f(x)$. Continuing Section 3.1, in Section 4, we introduce more involved examples of packing methods. These examples are not necessary to follow our main theorems. However, most of definitions and statements in this paper are motivated from these examples, they are helpful for understanding implications of our theorems.

In Section 5, 6, and 7, we present our main results on packing density, level-consistency, and surjectivity, respectively. These sections can be read independently, except Thm. 7.2 and 7.4 in Section 7 which refer the notion of level-consistency of Section 6. In these sections, we defer lengthy proofs of our main results to the last subsections of each sections for readability. However, we did the efforts to keep key ideas and lemmas of the proofs in the main body. Each section is structured in four parts: (i) definitions and basic facts (ii) main theorem for $\mathbb{Z}_{p^k}$ case and its implications (iii) main theorem for $\mathbb{F}_{p^k}$ case and its implications and (iv) deferred proofs. Subsections (ii) and (iii) can be read independently, so whom only interested in $\mathbb{Z}_{p^k}$ case can skip the subsection on $\mathbb{F}_{p^k}$ case, and vice versa. Whom only interested in the results and implications can skip the subsections on proofs. Finally, in Section 8, we list remaining open problems.

2 Preliminaries

2.1 Notations and Terminologies

- In this paper, we only consider finite commutative rings with unity. Thus, we omit the long description and simply refer them as rings. Readers must understand the term *ring* as finite commutative rings with unity, even if not explicitly stated.
- Likewise, since we only consider monic polynomials in this paper, we omit description on *monic* property throughout the paper. Readers must understand any polynomials as a monic polynomial, even if not explicitly stated.
- This paper carefully distinguishes between the use of the terms *message* and *plaintext*. Messages are those we really want to compute with. On the other hand, plaintexts are defined by encryption scheme (particularly, HE schemes) we are using. In this paper, messages are in $\mathbb{Z}_{p^k}$ or $\mathbb{F}_{p^k}$ and plaintexts are in $\mathbb{Z}_q[x]/f(x)$.
- For prime fields, we use both notations $\mathbb{F}_p$ and $\mathbb{Z}_p$, depending on whether we want to emphasize that it is a field or that it is the ring of integer modulo p.
- The multiplicative order of b modulo a is denoted as $\mathrm{ord}_a(b)$.
- We use $\mathrm{Inv}_a(b)$ to denote the smallest positive integer which is a multiplicative inverse of b modulo a.
- We use $\odot$ to denote the coordinate-wise multiplication (a.k.a. Hadamard product) in products of rings.
- In a product of rings R^n, the element $\boldsymbol{e}_i$ denotes a standard unit vector whose i-th coordinate is 1 and the other coordinates are 0.
- We denote the M-th cyclotomic polynomial as $\Phi_M(x)$ and the Euler's totient function as $\phi(\cdot)$.
- We use $GR(p^k, d)$ to denote the Galois ring, a degree-d extension of $\mathbb{Z}_{p^k}$.
- We use notations $[n] := \{1, 2, \cdots, n\}$ and $[0, n] := \{0, 1, \cdots, n\}$.

2.2 Polynomial Factorizations

Here, we briefly review some basic facts on polynomial factorizations in $\mathbb{Z}_{p^k}[x]$. First, recall Hensel lifting (or Hensel's lemma). For a proof and detailed discussions, refer to [Wan03] or any other textbook.

Lemma 2.1 (Hensel Lifting). *Let $f(x) \in \mathbb{Z}_{p^k}[x]$ be a monic polynomial which factors into $\prod_{i=1}^{r} g_i(x)^{\ell_i}$ in $\mathbb{F}_p[x]$, where $g_i(x)$ are distinct irreducible polynomials. Then there exist pairwise coprime monic polynomials $f_1(x), \cdots, f_r(x) \in \mathbb{Z}_{p^k}[x]$ such that $f(x) = \prod_{i=1}^{r} f_i(x)$ in $\mathbb{Z}_{p^k}[x]$ and $f_i(x) = g_i(x)^{\ell_i} \pmod p$, for all $i \in [r]$.*

When $\gcd(M, p) = 1$, $\Phi_M(x)$ factors into $\prod_{i=1}^{r} g_i(x)$ in $\mathbb{F}_p[x]$, where $g_i(x)$ are distinct irreducible polynomials of degree $d := \mathrm{ord}_M(p)$. Thus, $\phi(M) = r \cdot d$ holds. To see this, consider a primitive M-th root of unity in a sufficiently large extension field of $\mathbb{F}_p$. Then, it is easy to see that the number of its conjugates is d which coincides with the degree of its minimal polynomial. Applying Hensel's lemma, we have a factorization $\Phi_M(x) = \prod_{i=1}^{r} f_i(x)$ in $\mathbb{Z}_{p^k}[x]$, where $\deg(f_i) = d$ and $f_i(x) = g_i(x) \pmod{p}$. Accordingly, we have a CRT ring isomorphism $\mathbb{Z}_{p^k}[x]/\Phi_M(x) \cong \prod_{i=1}^{r} \mathbb{Z}_{p^k}[x]/f_i(x)$. Each $\mathbb{Z}_{p^k}[x]/f_i(x)$ is often referred to as a CRT *slot* of $\mathbb{Z}_{p^k}[x]/\Phi_M(x)$.

2.3 RMFE

Reverse multiplication-friendly embeddings (RMFE) were first defined and studied in-depth by [CCXY18].[9] At a high level, RMFEs are embeddings of several elements of small finite field into a larger finite field, while providing *somewhat* homomorphism of degree-2.

Definition 2.1 (RMFE). *A pair of maps (φ, ψ) is called a $(n, d)_{p^k}$-reverse multiplication-friendly (RMFE) if it satisfies the following.*

- *The map $\varphi : \mathbb{F}_{p^k}^n \to \mathbb{F}_{p^{kd}}$ is $\mathbb{F}_{p^k}$-linear.*
- *The map $\psi : \mathbb{F}_{p^{kd}} \to \mathbb{F}_{p^k}^n$ is $\mathbb{F}_{p^k}$-linear.*
- *For all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{F}_{p^k}^n$, it holds $\psi(\varphi(\boldsymbol{a}) \cdot \varphi(\boldsymbol{b})) = \boldsymbol{a} \odot \boldsymbol{b}$*

Surprisingly, [CCXY18] constructed families of $(n, d)_{p^k}$-RMFE where the density n/d converges to some *constant*, for arbitrary prime power p^k, leveraging algebraic geometry. That is, [CCXY18] constructed *constant-rate* RMFEs. For instance, we have a family of $(n, d)_2$-RMFE with $n/d \to 0.203$ from [CCXY18].[10]. Since this seminal work, RMFE has become a standard tool in information-theoretically secure MPC, to achieve *linear* amortized communication cost while preserving optimal corruption tolerance: [CCXY18, BMN18, DLN19, CG20, CXY20, DLSV20, PS21]. RMFE was also leveraged in zk-SNARK context recently [CG21].

Recently in [CRX21], RMFE *over Galois rings* was first defined and studied. It is a natural generalization of RMFE over fields to Galois rings.

Definition 2.2 (RMFE over Galois Ring). *A pair of maps (φ, ψ) is called an $(n, d)_{p^r}$-RMFE over modulus p^k if it satisfies the following.*

- *The map $\varphi : GR(p^k, r)^n \to GR(p^k, d)$ is $GR(p^k, r)$-linear.*
- *The map $\psi : GR(p^k, d) \to GR(p^k, r)^n$ is $GR(p^k, r)$-linear.*
- *For all $\boldsymbol{a}, \boldsymbol{b} \in GR(p^k, r)^n$, it holds $\psi(\varphi(\boldsymbol{a}) \cdot \varphi(\boldsymbol{b})) = \boldsymbol{a} \odot \boldsymbol{b}$*

The authors also showed that any $(n, d)_{p^r}$-RMFE over fields can be naturally lifted upto an $(n, d)_{p^r}$-RMFE over modulus p^k. That is, there are *asymptotically good* RMFE also in the Galois ring setting.

[9] Nonetheless, this object was also previously studied in [BMN17], to amortize oblivious linear evaluations (OLE) into a larger extension field for correlation extraction problem in MPC. However, their construction achieved only sublinear density (See Section 4.3).

[10] We have found out that we can slightly improve this rate by the hybrid approach with *3-free sets* (Section 4.3), but we omit here for simplicity.

Their goal was to construct efficient $(n, d)_p$-RMFEs over modulus p^k for $\mathbb{Z}_{p^k}$-messages as a building block for more efficient information-theoretically secure MPC over $\mathbb{Z}_{p^k}$. More generally, it seems there are very limited applications where messages in Galois ring (except $\mathbb{Z}_{p^k}$ or $\mathbb{F}_{p^k}$) play important roles. Thus, in our work, we focus on $(n, d)_p$-RMFE over modulus p^k for $\mathbb{Z}_{p^k}$-messages. Note that this case can be interpreted as packing $\mathbb{Z}_{p^k}$-messages into $GR(p^k, d) \cong \mathbb{Z}_{p^k}[x]/f(x)$ for some degree-d $f(x) \in \mathbb{Z}_{p^k}[x]$ which is irreducible modulo p.

3 Packing: Definitions and Basic Facts

In this section, we formally define *packings* and related concepts which are our main interests in this work. Some basic examples of packing methods are introduced for illustrative purpose. We also present some propositions which allow us to modularize our study of packing methods. We begin with a formal definition of packing.

3.1 Definitions and Basic Examples

Definition 3.1 (Packing). *Let R and $\mathcal{R}$ be rings. We call a pair of algorithms* $(\mathsf{Pack}, \mathsf{Unpack})$ *a packing method for n R-messages into $\mathcal{R}$, if it satisfies the following.*

- Pack *is an algorithm (possibly probabilistic) which, given $\boldsymbol{a} \in R^n$ as an input, outputs an element of $\mathcal{R}$.*
- Unpack *is a deterministic algorithm which, given $a(x) \in \mathcal{R}$ as an input, outputs an element of R^n or $\perp$ denoting a failure.*
- $\mathsf{Unpack}(\mathsf{Pack}(\boldsymbol{a})) = \boldsymbol{a}$ *holds for all $\boldsymbol{a} \in R^n$ with probability* 1.

For simplicity, the definition is presented a bit generally. In this paper, we are mostly interested in the cases where R is $\mathbb{Z}_p$ with $p \in \mathbb{Z}^+$ (or a finite field $\mathbb{F}_{p^k}$) and $\mathcal{R}$ is a polynomial ring $\mathbb{Z}_q[x]/f(x)$ with $q \in \mathbb{Z}^+$ and monic $f(x)$.

Notice that in Def. 3.1 the ring structure is not considered. Packing methods are interesting only when algebraic structures of the rings come in, since otherwise a packing is nothing more than a vanilla data encoding. The following definition of *degree* captures quality of (somewhat) homomorphic correspondence between packed messages and a packing. In this work, we are interested in packings of at least degree-2.

Definition 3.2 (Degree-D Packing). *Let $\mathcal{P} = (\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ be a collection of packing methods for R^n into $\mathcal{R}$. We call $\mathcal{P}$ a degree-D packing method, if it satisfies the following for all $1 \leq i \leq D$:*

- *If $a(x), b(x)$ satisfy $\mathsf{Unpack}_i(a(x)) = \boldsymbol{a}$, $\mathsf{Unpack}_i(b(x)) = \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in R^n$, then $\mathsf{Unpack}_i(a(x) \pm b(x)) = \boldsymbol{a} \pm \boldsymbol{b}$ holds;*
- *If $a(x), b(x)$ satisfy $\mathsf{Unpack}_s(a(x)) = \boldsymbol{a}$, $\mathsf{Unpack}_t(b(x)) = \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in R^n$ and $s, t \in \mathbb{Z}^+$ such that $s + t = i$, then $\mathsf{Unpack}_i(a(x) \cdot b(x)) = \boldsymbol{a} \odot \boldsymbol{b}$ holds.*

Notice that the definition is heavy on the use of Unpack rather than Pack. Some readers might find it unnatural to define a property of *packing* methods with their *unpacking* structures. However, this is how things are. For instance, given that a collection of unpacking algorithms $(\mathsf{Unpack}_i)_{i=1}^{D}$ allows a degree-D packing method, it is trivial to find an appropriate collection of packing algorithms $(\mathsf{Pack}_i)_{i=1}^{D}$: we can just define Pack_i as an algorithm which randomly outputs an preimage of the input regarding Unpack_i. On the other hand, if a collection of packing algorithms $(\mathsf{Pack}_i)_{i=1}^{D}$ is given, it requires non-trivial computations to find an appropriate collection of packing algorithms $(\mathsf{Unpack}_i)_{i=1}^{D}$ in this case. In this regard, definitions and proofs coming up are also aligned to Unpack rather than Pack.

Here are some direct but noteworthy consequences of the definition.

Remark 3.1. Note that the definition implies that $\mathsf{Unpack}_i(c \cdot a(x)) = c \cdot \boldsymbol{a}$ holds for all $c \in \mathbb{Z}$ with probability 1. In particular, $\mathsf{Unpack}_i(0) = \mathbf{0}$.

Remark 3.2. A packing method $\mathcal{P} = (\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ is of degree-D, only if $\mathcal{P}' = (\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D'}$ is a degree-D' packing method for all $D' < D$.

The following are some basic examples of packing methods. More sophisticated examples are introduced in Section 4.

Example 3.1 (Coefficient Packing). Let $f(x)$ be a degree-d monic polynomial in $\mathbb{Z}_p[x]$. Define Pack as a bijection which maps $(a_0, \cdots, a_{d-1}) \in \mathbb{Z}_p^d$ to $\sum_{i=0}^{d-1} a_i \cdot x^i \in \mathbb{Z}_p[x]/f(x)$. Define Unpack as the inverse of Pack. Then, $(\mathsf{Pack}, \mathsf{Unpack})$ is a degree-1 packing method for $\mathbb{Z}_p^d$ into $\mathbb{Z}_p[x]/f(x)$. We often refer this method as *coefficient packing.* As coefficient packing is already too good, we do not further examine degree-1 packing methods in this paper. Note that this method also applies to $\mathbb{F}_{p^k}$-messages if degree-1 is sufficient, since $\mathbb{F}_{p^k}^n$ is isomorphic to $\mathbb{Z}_p^{kn}$ as $\mathbb{Z}_p$-modules.

Example 3.2 (Conventional HE Packing). When making use of lattice-based HE schemes, where the plaintext space is of the form $\mathbb{Z}_p[x]/\Phi_M(x)$, it is standard to choose prime p such that $p = 1 \pmod{M}$ (and M as a power-of-two to enable efficient implementations). Then, $\Phi_M(x)$ fully splits in $\mathbb{Z}_p[x]$, and $\mathbb{Z}_p[x]/\Phi_M(x) \cong \mathbb{Z}_p^{\phi(M)}$ holds. The isomorphism induces a natural packing method, which is of degree-∞, i.e. degree-D for any $D \in \mathbb{Z}^+$. This packing is more than good in several aspects, but has quite heavy restrictions on parameters. In particular, the method does not allow packing $\mathbb{Z}_{2^k}$-messages.

Example 3.3 (HE Packing for $\mathbb{F}_{p^d}$). If one want to pack $\mathbb{F}_{p^d}$-messages when making use of lattice-based HE schemes, we often choose M so that $\Phi_M(x)$ factorizes into r distinct degree-d irreducible polynomials in $\mathbb{Z}_p[x]$. Then, we have $\mathbb{Z}_p[x]/\Phi_M(x) \cong \mathbb{F}_{p^d}^r$. As Example 3.2, this isomorphism induces a natural packing method which is of degree-∞, but has even heavier restriction on parameters.

Example 3.4 (RMFE). Essentially, an RMFE is nothing more than a degree-2 packing method for copies of a finite field $\mathbb{F}_{p^k}$ into a larger finite field $\mathbb{F}_{p^d} \cong \mathbb{Z}_p[x]/f(x)$, where p is a prime and $f(x)$ is a monic degree-d irreducible polynomial in $\mathbb{Z}_p[x]$. The only additional requirement is that the packing algorithm at level-1 and unpacking algorithm at level-2 must be $\mathbb{Z}_p$-linear functions. However, any degree-2 packing method can be easily transformed to satisfy the requirement.

Example 3.5 (RMFE over Galois Ring). Essentially, an RMFE over Galois ring for $\mathbb{Z}_{p^k}$-messages is nothing more than a degree-2 packing method for copies of $\mathbb{Z}_{p^k}$ into a larger Galois ring $GR(p^k, d) \cong \mathbb{Z}_{p^k}[x]/f(x)$, where p is a prime and $f(x)$ is a degree-d irreducible polynomial in $\mathbb{Z}_p[x]$. The only additional requirement is that the packing algorithm at level-1 and unpacking algorithm at level-2 must be $\mathbb{Z}_{p^k}$-linear functions. However, any degree-2 packing method can be easily transformed to satisfy the requirement.

Lastly, we define *packing density* which measures efficiency of packing methods. It measures how dense messages are packed in a single packing.

Definition 3.3 (Packing Density). *For a packing method for R^n into $\mathcal{R}$, we define its* packing density *as* $\log(|R|^n)/\log(|\mathcal{R}|)$.

Example 3.1, 3.2, and 3.3 have perfect packing density of 1. However, we will see that these are very special cases. In most cases such perfect packing density is not achievable, and even moderate packing density is hard to achieve.

3.2 Decomposition Lemmas

In this subsection, we state and prove several necessary conditions on existence of certain packing methods. The following propositions allow us to modularize our study and focus on the case of packings into $\mathbb{Z}_{p^t}[x]/f(x)$.

Proposition 3.1. *Let R be a ring with characteristic p and $\mathcal{R}$ be a ring with characteristic q. There exists a degree-0 packing method* $(\mathsf{Pack}, \mathsf{Unpack})$ *for R^n into $\mathcal{R}$ only if p divides q.*

Proof. Let $a(x)$ be an output of $\mathsf{Pack}(\mathbf{1})$. Then, $\mathsf{Unpack}(q \cdot a(x)) = q \cdot \mathbf{1}$. Meanwhile, $q \cdot a(x) = 0$ in $\mathcal{R}$. Thus, $q \cdot \mathbf{1} = \mathbf{0}$ in R^n. □

Proposition 3.2. *Let R be a ring with characteristic p. Let $q = q_1 \cdot q_2$, where $p|q_1$ and $\gcd(q_1, q_2) = 1$. There exists a degree-D packing method $\mathcal{P}$ for R^n into $\mathbb{Z}_q[x]/f(x)$, if and only if there exists a degree-D packing method $\mathcal{P}'$ for R^n into $\mathbb{Z}_{q_1}[x]/f(x)$.*

Proof. Suppose $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ is a degree-D packing method $\mathcal{P}$ for R^n into $\mathbb{Z}_q[x]/f(x)$. Let $a(x)$ satisfy $\mathsf{Unpack}_i(a(x)) = \boldsymbol{a}$ for some $\boldsymbol{a} \in R^n$ and $1 \leq i \leq D$. We can identify $a(x)$ with $(a_1(x), a_2(x)) \in \mathbb{Z}_{q_1}[x]/f(x) \times \mathbb{Z}_{q_2}[x]/f(x)$ via CRT isomorphism. Now, consider multiplying a constant $\mathrm{Inv}_{q_1}(q_2) \cdot q_2$. Observe the following.

- $(\mathrm{Inv}_{q_1}(q_2) \cdot q_2) \cdot \boldsymbol{a} = (\mathrm{Inv}_p(q_2) \cdot q_2) \cdot \boldsymbol{a} = \boldsymbol{a} \in R^n$
- $(\mathrm{Inv}_{q_1}(q_2) \cdot q_2) \cdot a_1(x) = 1 \cdot a_1(x) = a_1(x) \in \mathbb{Z}_{q_1}[x]/f(x)$
- $(\mathrm{Inv}_{q_1}(q_2) \cdot q_2) \cdot a_2(x) = \mathrm{Inv}_{q_1}(q_2) \cdot 0 = 0 \in \mathbb{Z}_{q_2}[x]/f(x)$

Thus, if $\mathsf{Unpack}_i(a(x)) = \mathsf{Unpack}_i(a_1(x), a_2(x)) = \boldsymbol{a}$ then $\mathsf{Unpack}_i(a_1(x), 0) = \boldsymbol{a}$.

Let π_{q_1} and ι_{q_1} denote the projection and injection between $\mathbb{Z}_q[x]/f(x)$ and $\mathbb{Z}_{q_1}[x]/f(x)$ respectively. Then, for all $a(x) \in \mathbb{Z}_q[x]/f(x)$, $\mathsf{Unpack}_i(a(x))$ is fully determined by $\pi_{q_1}(a(x))$, given it does not output a failure $\perp$.

Define $\mathsf{Pack}'_i := \pi_{q_1} \circ \mathsf{Pack}_i$ and $\mathsf{Unpack}'_i := \mathsf{Unpack}_i \circ \iota_{q_1}$ (Fig. 1). Then, it is straightforward that $(\mathsf{Pack}'_i, \mathsf{Unpack}'_i)_{i=1}^{D}$ is a degree-D packing method for R^n into $\mathbb{Z}_{q_1}[x]/f(x)$.

$$\begin{tikzcd}
R^n \arrow[r, "\mathsf{Pack}_i"] \arrow[dr, "\mathsf{Pack}'_i"'] & \mathbb{Z}_q[x]/f(x) \arrow[d, "\pi_{q_1}"] \\
 & \mathbb{Z}_{q_1}[x]/f(x)
\end{tikzcd}$$

(a) Pack'_i

$$\begin{tikzcd}
R^n & \mathbb{Z}_q[x]/f(x) \arrow[l, "\mathsf{Unpack}_i"'] \\
 & \mathbb{Z}_{q_1}[x]/f(x) \arrow[u, "\iota_{q_1}"'] \arrow[ul, "\mathsf{Unpack}'_i"]
\end{tikzcd}$$

(b) Unpack'_i

Fig. 1: Definitions of Pack'_i and Unpack'_i in Prop. 3.2

On the other hand, suppose that $(\mathsf{Pack}'_i, \mathsf{Unpack}'_i)_{i=1}^{D}$ is a degree-D packing method for R^n into $\mathbb{Z}_{q_1}[x]/f(x)$. Define $\mathsf{Pack}_i := \iota_{q_1} \circ \mathsf{Pack}'_i$ and $\mathsf{Unpack}_i := \mathsf{Unpack}'_i \circ \pi_{q_1}$ (Fig. 2). Then, it is straightforward that $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ is a degree-D packing method for R^n into $\mathbb{Z}_q[x]/f(x)$. □

$$\begin{tikzcd}
R^n \arrow[r, "\mathsf{Pack}_i"] \arrow[dr, "\mathsf{Pack}'_i"'] & \mathbb{Z}_q[x]/f(x) \\
 & \mathbb{Z}_{q_1}[x]/f(x) \arrow[u, "\iota_{q_1}"']
\end{tikzcd}$$

(a) Pack_i

$$\begin{tikzcd}
R^n & \mathbb{Z}_q[x]/f(x) \arrow[l, "\mathsf{Unpack}_i"'] \arrow[d, "\pi_{q_1}"] \\
 & \mathbb{Z}_{q_1}[x]/f(x) \arrow[ul, "\mathsf{Unpack}'_i"]
\end{tikzcd}$$

(b) Unpack_i

Fig. 2: Definitions of Pack_i and Unpack_i in Prop. 3.2

Proposition 3.3. *Let $p = p_1 \cdot p_2$ and $q = q_1 \cdot q_2$, where $p_1 | q_1$, $p_2 | q_2$ and $\gcd(q_1, q_2) = 1$. There exists a degree-D packing method $\mathcal{P}$ for $\mathbb{Z}_p^n$ into $\mathcal{R} := \mathbb{Z}_q[x]/f(x)$, if and only if there exist degree-D packing methods $\mathcal{P}^{(j)}$ for $\mathbb{Z}_{p_j}^n$ into $\mathcal{R}_j := \mathbb{Z}_{q_j}[x]/f(x)$ for $j = 1, 2$.*

Proof. Suppose $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ is a degree-D packing method $\mathcal{P}$ for $\mathbb{Z}_p^n$ into $\mathcal{R}$. Let $a(x) \in \mathcal{R}$ satisfy $\mathsf{Unpack}_i(a(x)) = \boldsymbol{a}$ for some $\boldsymbol{a} \in \mathbb{Z}_p^n$ and $1 \le i \le D$. We can identify $a(x)$ with $(a_1(x), a_2(x)) \in \mathcal{R}_1 \times \mathcal{R}_2$ and $\boldsymbol{a}$ with $(\boldsymbol{a}_1, \boldsymbol{a}_2) \in \mathbb{Z}_{p_1}^n \times$

$\mathbb{Z}_{p_2}^n$ via CRT isomorphisms. Now, consider multiplying a constant $\mathrm{Inv}_{q_1}(q_2) \cdot q_2$. Observe the following.

- $(\mathrm{Inv}_{q_1}(q_2) \cdot q_2) \cdot \boldsymbol{a}_1 = (\mathrm{Inv}_{p_1}(q_2) \cdot q_2) \cdot \boldsymbol{a}_1 = \boldsymbol{a}_1 \in \mathbb{Z}_{p_1}^n$
- $(\mathrm{Inv}_{q_1}(q_2) \cdot q_2) \cdot \boldsymbol{a}_2 = \mathrm{Inv}_{q_1}(q_2) \cdot \mathbf{0} = \mathbf{0} \in \mathbb{Z}_{p_2}^n$
- $(\mathrm{Inv}_{q_1}(q_2) \cdot q_2) \cdot a_1(x) = 1 \cdot a_1(x) = a_1(x) \in \mathcal{R}_1$
- $(\mathrm{Inv}_{q_1}(q_2) \cdot q_2) \cdot a_2(x) = \mathrm{Inv}_{q_1}(q_2) \cdot 0 = 0 \in \mathcal{R}_2$

That is, if $\mathsf{Unpack}_i(a_1(x), a_2(x)) = (\boldsymbol{a}_1, \boldsymbol{a}_2)$ then $\mathsf{Unpack}_i(a_1(x), 0) = (\boldsymbol{a}_1, \mathbf{0})$. The similar holds for $j = 2$.

Let π_{p_j} and ι_{p_j} denote the projection and injection between $\mathbb{Z}_p^n$ and $\mathbb{Z}_{p_j}^n$ respectively. Also let π_{q_j} and ι_{q_j} denote the projection and injection between $\mathcal{R}$ and $\mathcal{R}_j$ respectively. Then, for all $a(x) \in \mathcal{R}$, $\pi_{p_j} \circ \mathsf{Unpack}_i(a(x))$ is fully determined by $\pi_{q_j}(a(x))$, given it does not output a failure $\perp$.

Define $\mathsf{Pack}_i^{(j)} := \pi_{q_j} \circ \mathsf{Pack}_i \circ \iota_{p_j}$ and $\mathsf{Unpack}_i^{(j)} := \pi_{p_j} \circ \mathsf{Unpack}_i \circ \iota_{q_j}$ (Fig. 3). Then, it is straightforward that $(\mathsf{Pack}_i^{(j)}, \mathsf{Unpack}_i^{(j)})_{i=1}^D$ is a degree-D packing method for $\mathbb{Z}_{p_j}^n$ into $\mathcal{R}_j$.

$$\begin{tikzcd}
\mathbb{Z}_p^n \arrow[r, "\mathsf{Pack}_i"] & \mathcal{R} \arrow[d, "\pi_{q_j}"] \\
\mathbb{Z}_{p_j}^n \arrow[u, "\iota_{p_j}"] \arrow[r, "\mathsf{Pack}_i^{(j)}"] & \mathcal{R}_j
\end{tikzcd}$$

(a) $\mathsf{Pack}_i^{(j)}$

$$\begin{tikzcd}
\mathbb{Z}_p^n \arrow[d, "\pi_{p_j}"'] & \mathcal{R} \arrow[l, "\mathsf{Unpack}_i"'] \\
\mathbb{Z}_{p_j}^n & \mathcal{R}_j \arrow[u, "\iota_{q_j}"'] \arrow[l, "\mathsf{Unpack}_i^{(j)}"']
\end{tikzcd}$$

(b) $\mathsf{Unpack}_i^{(j)}$

Fig. 3: Definitions of $\mathsf{Pack}_i^{(j)}$ and $\mathsf{Unpack}_i^{(j)}$ in Prop. 3.3

On the other hand, suppose $(\mathsf{Pack}_i^{(j)}, \mathsf{Unpack}_i^{(j)})_{i=1}^D$ are degree-D packing methods for $\mathbb{Z}{p_j}^n$ into $\mathcal{R}_j$, for $j = 1, 2$. Let ψ_p denote the CRT ring isomorphism from $\mathbb{Z}_p^n$ to $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n$. Also, let ψ_q denote the CRT ring isomorphism from $\mathcal{R}$ to $\mathcal{R}_1 \times \mathcal{R}_2$. Define $\mathsf{Pack}_i := \psi_q^{-1} \circ (\mathsf{Pack}_i^{(1)} \times \mathsf{Pack}_i^{(2)}) \circ \psi_p$ and $\mathsf{Unpack}_i := \psi_p^{-1} \circ (\mathsf{Unpack}_i^{(1)} \times \mathsf{Unpack}_i^{(2)}) \circ \psi_q$ (Fig. 4). Then, it is straightforward that $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ is a degree-D packing method for $\mathbb{Z}_p^n$ into $\mathcal{R}$. □

According to Prop. 3.1 and 3.2, to study degree-D packing methods for copies of a finite field $\mathbb{F}_{p^k}$ into $\mathbb{Z}_q[x]/f(x)$, it is enough to study degree-D packing methods into $\mathbb{Z}_{p^t}[x]/f(x)$ for some $t \geq 1$. The similar holds for packing methods for copies of $\mathbb{Z}_p$ according to Prop. 3.1, 3.2, and 3.3. That is, to study degree-D packing methods for copies of $\mathbb{Z}_p$ into $\mathbb{Z}_q[x]/f(x)$ where p is an arbitrary integer, it is enough to study degree-D packing methods for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ for some $t \geq k$ where p is a prime.

Therefore, from now on, we focus on packing methods for $\mathbb{Z}_{p^k}^n$ or $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ where p is a prime. (Afterwards, p is a fixed prime, even if it is

$$\begin{tikzcd}
\mathbb{Z}_p^n \arrow[r, "\mathsf{Pack}_i"] \arrow[d, "\psi_p"'] & \mathcal{R} \\
\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \arrow[r, "\mathsf{Pack}_i^{(1)} \times \mathsf{Pack}_i^{(2)}"] & \mathcal{R}_1 \times \mathcal{R}_2 \arrow[u, "\psi_q^{-1}"']
\end{tikzcd}$$

(a) Pack_i

$$\begin{tikzcd}
\mathbb{Z}_p^n & \mathcal{R} \arrow[l, "\mathsf{Unpack}_i"'] \arrow[d, "\psi_q"] \\
\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \arrow[u, "\psi_p^{-1}"] & \mathcal{R}_1 \times \mathcal{R}_2 \arrow[l, "\mathsf{Unpack}_i^{(1)} \times \mathsf{Unpack}_i^{(2)}"']
\end{tikzcd}$$

(b) Unpack_i

Fig. 4: Definitions of Pack_i and Unpack_i in Prop. 3.3

not explicitly stated.) This is not only because they are the most interesting case containing $\mathbb{Z}_{2^k}$ and $\mathbb{F}_{2^k}$, but also because they play roles as building blocks when constructing general packing methods (Prop. 3.2, 3.3). We note that the properties of packing methods, which we examine in the following sections (level-consistency in Sect. 6 and surjectivity in Sect. 7), are preserved by the constructions in Prop. 3.2 and 3.3.

4 More Examples

In continuation of Section 3.1, we give more examples on packing methods. The following examples are degree-2 packing methods for $\mathbb{Z}_{2^k}$-messages, which are (or can be) used to construct HE-based MPC protocol over $\mathbb{Z}_{2^k}$ following the approach of SPDZ [DPSZ12]. Most of definitions and statements in this paper are motivated from these examples.

4.1 HELib Packing

In Example 3.2, we introduced the conventional HE packing method for $\mathbb{Z}_q$-messages into $\mathbb{Z}_q[x]/\Phi_M(x)$, where M is a power-of-two and $q = 1 \pmod{M}$. However, it is not always applicable, e.g. if we consider $\mathbb{Z}_{2^k}$-messages. The problem here is that $\Phi_M(X)$ never fully splits in $\mathbb{Z}_{2^k}$. One way to detour this problem is the following. It was first proposed by Gentry-Halevi-Smart [GHS12] and generalized by Halevi-Shoup [HS15] to optimize *bootstrapping* procedure for fully homomorphic encryption (particularly, for HELib [HS14]). In this paper, we will refer this method as HELib packing.

To construct a packing method for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^k}[x]/\Phi_M(x)$, choose M to satisfy $\gcd(M,p) = 1$. Let $\Phi_M(x)$ factor into r distinct degree-d irreducible polynomials in $\mathbb{Z}_p[x]$, where $d := \mathrm{ord}_M(p)$. Then, we have the factorization $\Phi_M(x) = \prod_{i=1}^{r} f_i(x)$ in $\mathbb{Z}_{p^k}[x]$ via Hensel lifting and the CRT ring isomorphism $\mathbb{Z}_{p^k}[x]/\Phi_M(x) \cong \prod_{i=1}^{r} \mathbb{Z}_{p^k}[x]/f_i(x)$. The packing algorithm Pack put i-th $\mathbb{Z}_{p^k}$-message at the constant term of $\mathbb{Z}_{2^k}[x]/f_i(x)$ and put zeroes at the other coefficients. Define Unpack as the inverse of Pack. It is easy to see that $(\mathsf{Pack}, \mathsf{Unpack})$ defines a degree-∞ packing method. However, the HELib packing achieves very low packing density $1/d$.

4.2 Overdrive2k Packing

To design an efficient HE-based MPC protocol over $\mathbb{Z}_{2^k}$, Overdrive2k [OSV20] constructed a degree-2 packing method for $\mathbb{Z}_{2^k}^n$ into $\mathbb{Z}_{2^k}[x]/\Phi_M(x)$, where M is odd (so yielding a CRT ring isomorphism $\mathbb{Z}_{2^k}[x]/\Phi_M(x) \cong \prod_{i=1}^{r} \mathbb{Z}_{2^k}[x]/f_i(x)$ with $\deg(f_i) = d$). For construction, they considered the following problem. Consider a subset A of $[0, d-1]$ with $A = \{a_1, \cdots, a_m\}$ so that $2a_i \neq a_j + a_k$ for all $(i,i) \neq (j,k)$ and $a_i + a_j < d$ for all i, j. The problem is to find the maximum value of $m = |A|$ with A for given d. Given a solution m and A for given d, the packing algorithm of Overdrive2k at level-1 put i-th m messages in $\mathbb{Z}_{2^k}$ at the coefficients of x^{a_i} of an element in $\mathbb{Z}_{2^k}[x]/f_i(x)$ for $a_i \in A$ and put zeroes at the other coefficients. Then, via the ring homomorphism, we can pack $r \cdot m$ messages into a plaintext achieving the packing density of m/d. The authors Overdrive2k noted that the packing density of their method seems to follow the trend of approximately $d^{0.6}/d$.

Since the set A is carefully designed, if we multiply two packed plaintexts, the $(2 \cdot a_i)$-th coefficient of the result equals the multiplied value of a_i-th coefficients of the original plaintexts. That is, Overdrive2k packing is of degree-2. Note that

Overdrive2k packing naturally extends to arbitrary degree-2 packing methods for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^k}[x]/f(x)$.

4.3 Notes on Overdrive2k Packing

There are other cryptography literature those considering similar problems of Overdrive2k packing [Lip12, BMN17, DLSV20]. They are also interested in embedding several elements into a larger polynomial ring for amortizing computations while providing one multiplication. Even though the authors of Overdrive2k did not present detailed discussions, behind the scene of [Lip12], [BMN17], [DLSV20], and Overdrive2k [OSV20], there is one of the central problems in additive number theory.

3-free Set Problem. A set of numbers no three of which form an arithmetic progression is called 3-free set (a.k.a. progression-free set or Salem-Spencer Set). Especially, we are most interested in 3-free subset of $[n]$. We denote the size of a largest 3-free subset of $[n]$ by $r_3(n)$.

After Erdős and Turán first considered 3-free set and stated the famous Erdős-Turán conjecture on arithmetic progression [ET36], 3-free set, its variants, and its generalizations have been researched extensively. The strongest lower bound on $r_3(n)$ until now is given by Behrend [Beh46]: $r_3(n) = n/e^{O(\sqrt{\log n})}$. On the other hand, an upper bound by Bloom [Blo16] is known: $r_3(n) = O(n(\log\log n)^4/\log n)$. Meanwhile, a recent manuscript by Bloom and Sisask [BS20] claimed a proof of a stronger upper bound: $r_3(n) = O(\frac{n}{\log^{1+c} n})$ for some $c > 0$.

Recall that Overdrive2k considered the following problem to embed ring elements as much as they can into a polynomial ring. Consider a subset A of $\{0, 1, \cdots, d-1\}$ with $A = \{a_1, \cdots, a_m\}$ so that $2a_i \neq a_j + a_k$ for all $(i, i) \neq (j, k)$ and $a_i + a_j < d$ for all i, j. The problem is to find the maximum value of $m = |A|$ with A for given d. We denote the solution to this problem for d by $\rho_3(d)$. Clearly, this problem is closely related to 3-free sets. It is easy to see that $\rho_3(d) = r_3(\lfloor \frac{d+1}{2} \rfloor)$.

Constructing 3-free Sets. There is an elementary method constructing 3-free sets via ternary representations of nonnegative integers. If we construct a set composed of ternary numbers that use only the digits 0 and 1, not 2, such a set must be a 3-free set. If two of its elements a_1 and a_2 are the first and the second of an arithmetic progression of length three, the third a_3 must have the digit two at the position of the least significant digit where a_1 and a_2 differ. Using this method, we can obtain a 3-free subset of $[n]$ with size approximately $n^{\log_3(2)} \approx n^{0.631}$, which is considerably smaller than the lower bound by Behrend. Observing the paper, the authors of Overdrive2k seem to have only considered this ternary construction.

Note that ternary construction can be naturally extended to $(D+1)$-ary construction, yielding a degree-D packing method of density roughly

$$\frac{(d/D)^{\log_{D+1}(2)}}{d}.$$

Meanwhile, Behrend's contruction does not well extend to be used for degree-D packing methods. We also note that constructing an optimal 3-free subset requires an intense amount of computation at the current stage of research. The optimal solutions are known only for small input n's: Gasarch, Glenn, and Kruskal [GGK08] found the exact size of the largest 3-free subset of $[n]$ for $n \leq 187$.

Generalized 3-free Set Problem. To achieve a better packing density, Block, Maji, and Nguyen [BMN17] proposed a generalized version of the 3-free set problem. The idea is to consider two subsets A and B of $\{0, 1, \cdots, d-1\}$ with $A = \{a_1, \cdots, a_m\}$ and $B = \{b_1, \cdots, b_m\}$ so that $a_i + b_i \neq a_j + b_k$ for all $(i, i) \neq (j, k)$ and $a_i + b_j < d$ for all i, j. The generalized problem is to find the maximum value of $m = |A| = |B|$ with A and B for given d. We denote the solution to this generalized problem for d by $\hat{\rho_3}(d)$. Obviously, $\hat{\rho_3}(d)$ is greater than $\rho_3(d)$. Applying the solution of generalized 3-free set problem on polynomial multiplication, we can use two different embedding methods for right operands and left operands and directly improve the capacity of Overdrive2k. However, the asymmetric nature of the generalized problem significantly reduces freedom in homomorphic computations between packed plaintexts. Therefore, we exclude this approach from the scope of our study.

4.4 MHz2k Packing

To further improve the packing density of Overdrive2k, MHz2k [CKL21] construct a degree-2 packing method for $\mathbb{Z}_{2^k}$-messages into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$, where t is slightly larger than k. Their core idea is to pack messages at *evaluation points* via interpolation unlike Overdrive2k which rather pack at coefficients. The caveat here is, however, that the polynomial interpolation on $\mathbb{Z}_{2^k}$ is not always possible, e.g. there is no $f(x) \in \mathbb{Z}_{2^k}$ satisfying $f(0) = 1$ and $f(2) = 0$ simultaneously. In this context, they propose the *tweaked interpolation*, where they we lift the target points of $\mathbb{Z}_{2^k}$ upto a larger ring $\mathbb{Z}_{2^{k+\delta}}$, multiplying an appropriate power-of-two to eliminate the effect of non-invertible elements.

Let $t = k + 2\delta$ and $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ factors into $\prod_{i=1}^{r} \mathbb{Z}_{2^t}[x]/f_i(x)$ via CRT, where $f_i(x)$ are all of degree-d. The packing algorithm at level-1 perform tweaked interpolation on i-th $\lfloor \frac{d+1}{2} \rfloor$ $\mathbb{Z}_{2^k}$-messages $\{\mu_{ij}\}$, so that we have $L_i(x) \in \mathbb{Z}_{2^t}[x]$ which satisfies (i) $\deg(L_i) \le \lfloor \frac{d-1}{2} \rfloor$ and (ii) $L_i(j) = \mu_{ij} \cdot 2^{\delta}$. Then, put $L_i(x)$ in the i-th CRT slot of $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$, i.e. $\mathbb{Z}_{2^t}[x]/f_i(x)$. This gives us a packing density of roughly $k/(2k + 2d)$.[11] Since the degree condition on $L_i(x)$ and extra δ in the modulus are designed to avoid degree overflow and modulus overflow, when the product of two packings is given, we can decode the homomorphically multiplied messages without any loss of information. That is, we can unpack at level-2 by evaluating points on each CRT slot and observing the upper k bits of outputs. For detailed and precise description of MHz2k packing, refer to [CKL21].

Note that MHz2k packing can be naturally extended to a degree-D packing method for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/\Phi_M(x)$ with $\gcd(M, p) = 1$ of density roughly

$$\frac{k}{D \cdot (k + \frac{d}{p-1})}.$$

4.5 Comparison

In this subsection, we compare some properties of the examples previously given in this section. These features are motivations of the definitions and results in later sections. This subsection is summarized in Table 1.

Notice that, in HELib packing which is of degree-∞, packing algorithms and unpacking algorithms are identical for all level. We will later refer these kind of packings as *level-consistent* packings (Section 6). However, in Overdrive2k and MHz2k packing, the packing algorithm differs for each level. For example, in Overdrive2k packing, messages are coefficients of x^{a_i}'s at level-1, and coefficients of $x^{2 \cdot a_i}$'s at level-2. We will later refer these kind of packings as *level-dependent* packings (Section 6).

One big difference of MHz2k packing from the previous packings is that it uses larger modulus for polynomial ring than that of messages. The other

[11] We have noticed that one can slightly increase the density by employing point-at-infinity technique of RMFE constructions [CCXY18]. However, for simplicity, we omit the details here.

Table 1: Comparisons on degree-2 packing methods for $\mathbb{Z}_{2^k}$-messages

Method	HELib	Overdrive2k	MHz2k
Level-consistency	consistent	dependent	dependent
$t \overset{?}{=} k$	$t = k$	$t = k$	$t > k$
Density	$1/d$	$\approx d^{0.6}/d$	$\approx k/(2k+2d)$

packing methods are sort of coefficient packing, making it no use of increasing the modulus for polynomial ring. This difference will serve as one of the topics in Section 5 (e.g. Example 5.1).

Note that degree-2 MHz2k packing reaches density of nearly $1/2$ when k is sufficiently larger than d. This is true for typical parameters used in HE-based MPC over $\mathbb{Z}_{2^k}$: $k = 64, 128, 196$ and $d \leq 20$. In Section 5, we will show that MHz2k packing achieves a certain form of near-optimality (Example 5.1).

We now examine common features of these methods. Note that there are *invalid* packings regarding to these packing methods. For example, in HELib packing, $a(x) \in \mathbb{Z}_{2^k}[x]/\Phi_M(x)$ is not a valid packing, i.e. $\mathsf{Unpack}(a(x)) = \perp$, if $a(x)$ modulo $f_i(x)$ is not a constant. We will later refer these kind of packings as non-*surjective* packings (Section 7).

Also notice that all these packings leverage CRT ring isomorphism, which is a natural and convenient way to achieve parallelism. They pack messages into each CRT slot in an identical and independent manner. We refer packing methods following this approach as *CRT packings*. However, we shed light on the possibility that this CRT approach might be hindering us to achieve a better packing density (Example 5.5).

5 Bounds on Packing Density

In this section, we examine upper bounds on packing density of degree-D packing methods for $\mathbb{Z}_{p^k}$ and $\mathbb{F}_{p^k}$.

5.1 Algebraic Background

We first remark some algebraic facts, which enable proofs in the following subsections.

Proposition 5.1. *When R is a principal ideal ring (PIR), every submodule of a free R-module of rank n can be finitely generated with n generators.*

Proof. See Section 5.4 □

Remark 5.1. Note that $\mathbb{Z}_{p^t}$ is a local PIR. Consider $\mathcal{R} := \mathbb{Z}_{p^t}[x]/f(x)$ as a free $\mathbb{Z}_{p^t}$-module with the rank $\deg(f)$. Then by Nakayama's lemma, the cardinality of minimal generating sets is a well-defined invariant for submodules of $\mathcal{R}$.

Let $\mathcal{A}$ be a linearly independent subset of $\mathcal{R}$. Then, since the span $\langle \mathcal{A} \rangle$ is a submodule of $\mathcal{R}$ with a minimal generating set $\mathcal{A}$, inequality $\deg(f) \geq |\mathcal{A}|$ holds by Prop. 5.1.

5.2 Packing Density of $\mathbb{Z}_{p^k}$-Message Packings

In this subsection, we examine upper bounds on packing density of degree-D $\mathbb{Z}_{p^k}$-message packings. We begin with an upper bound for degree-1 packing methods: we cannot pack copies of $\mathbb{Z}_{p^k}$ more than the degree of the quotient polynomial. Unlike the simple and plausible statement, the proof is quite involved. In particular, it depends on Remark 5.1. The following proposition says that we cannot reduce the degree of quotient polynomial significantly and tower the packings along a large modulus. Notice that there are no restriction on t and $f(x)$.

Proposition 5.2. *There exists a degree-1 packing method for $\mathbb{Z}_{p^k}^n$ into $\mathcal{R} := \mathbb{Z}_{p^t}[x]/f(x)$ with $k \leq t$, only if $n \leq \deg(f)$.*

Proof. Let $(\mathsf{Pack}_1, \mathsf{Unpack}_1)$ be a degree-1 packing method for $\mathbb{Z}_{p^k}^n$ into $\mathcal{R}$. For each $i \in [n]$, choose $a_i(x) \in \mathcal{R}$ such that $\mathsf{Unpack}_1(a_i(x)) = \boldsymbol{e}_i$. View $\mathcal{R}$ as a free $\mathbb{Z}_{p^t}$-module of rank $\deg(f)$, and consider the submodule $\langle a_1(x), \cdots, a_n(x) \rangle$. By linear homomorphic property (Remark 3.1), when $\sum_{i=1}^n c_i \cdot a_i(x) = 0$ for some $c_i \in \mathbb{Z}_{p^t}$, then $c_i = 0 \pmod{p^k}$ must hold. Thus, $\{a_1(x), \cdots, a_n(x)\}$ is a minimal generating set of $\langle a_1(x), \cdots, a_n(x) \rangle$, and therefore $n \leq \deg(f)$ holds (Remark 5.1). □

In the rest of this subsection, we narrow our scope to packing methods for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^k}[x]/f(x)$ with the same modulus. Indeed, this setting is less general. Nonetheless, our results still have interesting consequences (See Example 5.1 - 5.6). The following is a small remark on packings of non-zero elements modulo p in this setting.

Remark 5.2. Let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ be a degree-D packing method for $\mathbb{Z}_{p^k}^n$ into $\mathcal{R} := \mathbb{Z}_{p^k}[x]/f(x)$. For any $i \in [D]$, if $\mathsf{Unpack}_i(a(x)) = \boldsymbol{a}$ for some $\boldsymbol{a} \in \mathbb{Z}_{p^k}^n$ which is non-zero modulo p, then $a(x)$ is also non-zero modulo p. Otherwise, $\mathsf{Unpack}_i(p^{k-1} \cdot a(x)) = \mathsf{Unpack}_i(0) = \mathbf{0} \neq p^{k-1} \cdot \boldsymbol{a}$, contradicting the linear homomorphic property (Remark 3.1). In particular, when $f(x)$ is an irreducible polynomial in $\mathbb{Z}_p[x]$, such $a(x)$ is a unit in $\mathcal{R}$.

Roughly speaking, our main result is that we cannot pack more than d/D $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^k}[x]/f(x)$ while satisfying degree-D homomorphic property, where $d = \deg(f)$. Intuitively, the statement can be understood as that we must pack the inputs into lower d/D coefficients since reduction by the quotient polynomial act as randomization and will ruin the structure of packing. However, the proof is much more involved since we have to handle all possible packing methods. Notice that the following theorem subsumes Prop. 5.2 as the $D = 1$ case in the $t = k$ setting. The essence of the proof is a generic construction of a large set which is required to be linearly independent regardless of specific structures of packing methods.

Theorem 5.1. *There exists a degree-D packing method for $\mathbb{Z}_{p^k}^n$ into $\mathcal{R} := \mathbb{Z}_{p^k}[x]/f(x)$ where $f(x) \in \mathbb{Z}_{p^k}[x]$ is a degree-d irreducible polynomial modulo p, only if $d \geq D \cdot (n-1) + 1$.*

Proof. Let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ be a degree-D packing method for $\mathbb{Z}_{p^k}^n$ into $\mathcal{R}$. For each $i \in [n]$, choose $a_i(x) \in \mathcal{R}$ such that $\mathsf{Unpack}_1(a_i(x)) = \boldsymbol{e}_i$. Let us denote $\mathcal{A}^{(r,s)} := \{a_1(x)^r \cdot a_j(x)^s\}_{1<j\leq n}$. For example, $\mathcal{A}^{(0,D)} = \{a_2(x)^D, \cdots, a_n(x)^D\}$, $\mathcal{A}^{(D,0)} = \{a_1(x)^D\}$, and $\mathcal{A}^{(1,D-1)} = \{a_1(x)a_2(x)^{D-1}, \cdots, a_1(x)a_n(x)^{D-1}\}$.

Step 1: Consider the following set of level-t packings.

$$\mathcal{A}_t := \bigcup_{\substack{r+s=t \\ 0<s}} \mathcal{A}^{(r,s)}$$

We will show that $\mathcal{A}_t$ is linearly independent in $\mathcal{R}$ for all $t \leq D$ by induction on t. The case where $t = 1$ is true by the linear homomorphic property at level-1 (Remark 3.1): $\mathcal{A}_1 = \{a_2(x), \cdots, a_n(x)\}$ (See also Prop. 5.2).

Suppose $\mathcal{A}_t$ is linearly independent for some $t < D$. View $\mathcal{A}_{t+1}$ as $\mathcal{A}^{(0,t+1)} \cup a_1(x) \cdot \mathcal{A}_t$. Suppose $\sum_{a_\alpha(x) \in \mathcal{A}_{t+1}} (c_\alpha \cdot a_\alpha(x)) = 0$, for some $c_\alpha \in \mathbb{Z}_{p^k}$. Then, by linear homomorphic property at level-$(t+1)$, $c_\alpha = 0$ must hold for all $a_\alpha(x) \in \mathcal{A}^{(0,t+1)}$, since elements of $a_1(x) \cdot \mathcal{A}_t$ unpack to $\mathbf{0}$ and $\mathcal{A}^{(0,t+1)}$ unpacks to a linearly independent set by construction. Subsequently, we have again the following equality:

$$\sum_{a_\alpha(x) \in a_1(x) \cdot \mathcal{A}_t} (c_\alpha \cdot a_\alpha(x)) = 0.$$

Meanwhile, since $a_1(x)$ is a unit in $\mathcal{R}$ (Remark 5.2) and $\mathcal{A}_t$ is linearly independent by induction hypothesis, $c_\alpha = 0$ must also hold for all $a_\alpha(x) \in a_1(x) \cdot \mathcal{A}_t$. Thus, $\mathcal{A}_t$ is linearly independent in $\mathcal{R}$ for all $t \leq D$.

Step 2: Now consider the set $\mathcal{A} := \mathcal{A}_D \cup \{a_1(x)^D\}$, which coincides with $\{a_1(x)^D, \cdots, a_n(x)^D\} \cup a_1(x) \cdot \mathcal{A}_{D-1}$. Suppose $\sum_{a_\alpha(x)\in\mathcal{A}}(c_\alpha \cdot a_\alpha(x)) = 0$, for some $c_\alpha \in \mathbb{Z}_{p^k}$. Then, by linear homomorphic property at level-D, $c_\alpha = 0$ must hold for all $a_\alpha(x) \in \{a_1(x)^D, \cdots, a_n(x)^D\}$, since elements of $a_1(x) \cdot \mathcal{A}_{D-1}$ unpack to $\mathbf{0}$ and $\{a_1(x)^D, \cdots, a_n(x)^D\}$ unpacks to a linearly independent set by construction. Subsequently, we have again the following equality:

$$\sum_{a_\alpha(x)\in a_1(x)\cdot\mathcal{A}_{D-1}} (c_\alpha \cdot a_\alpha(x)) = 0.$$

Meanwhile, since $a_1(x)$ is a unit in $\mathcal{R}$ and $\mathcal{A}_{D-1}$ is linearly independent by Step 1, $c_\alpha = 0$ must also hold for all $a_\alpha(x) \in a_1(x) \cdot \mathcal{A}_{D-1}$. Thus, $\mathcal{A}$ is linearly independent, and therefore $d \geq |\mathcal{A}| = D(n-1)+1$ must hold (Remark 5.1). □

The following are direct consequences of our theorem.

Example 5.1. Degree-D packing methods for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^k}[x]/f(x)$, where $f(x)$ is a degree-d irreducible polynomial modulo p, have packing density of no larger than $\frac{1}{D} + \frac{1}{d}\cdot(1-\frac{1}{D})$. Consequently, degree-$D$ *CRT* packing methods for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^k}[x]/f(x)$, where $f(x)$ factors into r distinct irreducible factors modulo p, have packing density of no larger than $\frac{1}{D} + \frac{r}{\deg(f)} \cdot (1 - \frac{1}{D})$ (Section 4.5). In particular, degree-D CRT packing methods for $\mathbb{Z}_{2^k}$-messages into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$, where M is odd and $\Phi_M(x)$ factors into distinct degree-d irreducible factors modulo p, have packing density of no larger than $\frac{1}{D}+\frac{1}{d}\cdot(1-\frac{1}{D})$.

That is, when parameters are carefully chosen, the MHz2k packing already nearly reach the optimal packing density for packing methods for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^k}[x]/f(x)$ (Section 4.4). Thus, if one wants to construct a degree-D packing method for $\mathbb{Z}_{2^k}$-messages into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ with substantially better density than the MHz2k packing, the only possibility is choosing $t > k$ or not employing the CRT approach. (See also Example 5.6)

Example 5.2 (RMFE over Galois Ring). Consider RMFE *over Galois rings* for copies of $\mathbb{Z}_{p^k}$ into a larger Galois ring isomorphic to $\mathbb{Z}_{p^k}[x]/f(x)$, which is exactly the setting of Thm. 5.1. The theorem states that such RMFE cannot have packing density larger than $\frac{1}{2} + \frac{1}{2\deg(f)}$. To the best of our knowledge, this is the first upper bound result on packing density of RMFE over Galois rings. Our theorem also yields upper bounds on packing density of degree-D generalization of RMFE over Galois rings.

Example 5.3. For $D > 1$, consider degree-D packing methods for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$, where $f(x)$ is irreducible modulo p. By Prop. 5.2, when $t > k$, we cannot achieve a perfect packing density 1. When $t = k$, we cannot achieve a perfect packing density 1 unless $\deg(f) = 1$, by Thm. 5.1. That is, there is no perfect degree-D packing method for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$, when $f(x)$ is irreducible modulo p and $\deg(f) > 1$.

Example 5.4. For $D > 1$, consider degree-D packing methods for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$, where $f(x)$ is square-free modulo p. By Example 5.3, there is no perfect degree-D CRT packing method for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$, unless $f(x)$ splits into distinct linear factors. In particular, there is no perfect degree-D CRT packing method for $\mathbb{Z}_{2^k}$-messages into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ when M is odd.

The following theorem is a bit more general version of Thm. 5.1 which has no restriction on the quotient polynomial. However, it assumes the existence of a unit of $\mathcal{R}$ which unpacks to an element of $\mathbb{Z}_{p^k}^n$.

Theorem 5.2. *Let* $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ *be a degree-D packing method for* $\mathbb{Z}_{p^k}^n$ *into* $\mathcal{R} := \mathbb{Z}_{p^k}[x]/f(x)$*. Suppose a linear combination of* $\{a_i(x)\}_{i\in I}$ *is a unit in* $\mathcal{R}$*, where each* $a_i(x) \in \mathcal{R}$ *satisfies* $\mathsf{Unpack}_1(a_i(x)) = \boldsymbol{e}_i$*. Then,* $d \geq D \cdot (n - |I|) + |I|$ *holds.*

Proof (Sketch). Assume $u(x) := \sum_{i\in I} c_i \cdot a_i(x)$ is a unit, for some $c_i \in \mathbb{Z}_{p^k}$. The proof is exactly same as that of Thm. 5.1, but with only difference in the definition of $\mathcal{A}^{(r,s)}$ and $\mathcal{A}$. Here, we define $\mathcal{A}^{(r,s)} := \{u(x)^r \cdot a_j(x)^s\}_{j\notin I}$ and $\mathcal{A} := \mathcal{A}_D \cup \{a_i(x)^D\}_{i\in I}$. □

The following are some consequences of Thm. 5.2.

Example 5.5. We can revisit upper bound on packing density of *CRT* packings (Section 4.5) using Thm. 5.2. Consider degree-D CRT packing methods $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ for $\mathbb{Z}_{p^k}$-messages into $\mathcal{R} := \mathbb{Z}_{p^k}[x]/f(x)$, where $f(x)$ factors into r distinct irreducible factors modulo p. Let $f(x) = \prod_{i=1}^r f_i(x)$ via Hensel lifting.

By Remark 5.2, for each $i \in [r]$, we have $a^{(i)}(x)$ such that (i) $a^{(i)}(x)$ is a unit modulo $f_\imath(x)$ if and only if $\imath = i$ and (ii) $\mathsf{Unpack}_1(a^{(i)}(x)) = \boldsymbol{e}_j$ for some distinct $j \in [n]$. Then, $\sum_{i=1}^r a^{(i)}(x)$ is a unit in $\mathcal{R}$. That is, we have $|I| = r$ for Thm. 5.2, yielding the upper bound $\frac{1}{D} + \frac{r}{\deg(f)} \cdot (1 - \frac{1}{D})$ previously shown in Example 5.1.

Example 5.6. Suppose one wants to design a degree-D packing method for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^k}[x]/f(x)$ which has a packing density substantially larger than $1/D$. The only possibility is designing a packing method where every unit element of $\mathbb{Z}_{p^k}[x]/f(x)$ unpacks to elements of $\mathbb{Z}_{p^k}^n$ with very few zero coordinates or fails to unpack at level-1.

5.3 Packing Density of $\mathbb{F}_{p^k}$-Message Packings

In this subsection, we examine upper bounds on packing density of degree-D $\mathbb{F}_{p^k}$-message packings. We begin with an upper bound for degree-1 packing methods, which is an analogue of Prop. 5.2. Unlike the simple and plausible statement, the proof is quite involved. In particular, it depends on Remark 5.1. The following proposition says that we cannot reduce the degree of quotient polynomial significantly and tower the packings along a large modulus. Notice that there are no restriction on t and $f(x)$.

Proposition 5.3. *There exists a degree-*1 *packing method for* $\mathbb{F}_{p^k}^n$ *into* $\mathcal{R} := \mathbb{Z}_{p^t}[x]/f(x)$*, only if* $n \cdot k \leq \deg(f)$.

Proof. Let $(\mathsf{Pack}_1, \mathsf{Unpack}_1)$ be a degree-1 packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R}$. Fix a basis of $\mathbb{F}_{p^k}$ as $\{\beta_1, \cdots, \beta_k\}$. For each $i \in [n]$ and $j \in [k]$, choose $a_{ij}(x) \in \mathcal{R}$ such that $\mathsf{Unpack}_1(a_{ij}(x)) = \beta_j \cdot \boldsymbol{e}_i$. View $\mathcal{R}$ as a free $\mathbb{Z}_{p^t}$-module of rank $\deg(f)$, and consider the submodule $\langle a_{ij}(x) \rangle_{i \in [n], j \in [k]}$. By linear homomorphic property (Remark 3.1), when $\sum_{i=1}^n c_{ij} \cdot a_{ij}(x) = 0$ for $c_i \in \mathbb{Z}_{p^t}$, then $c_i = 0 \pmod p$ must hold. Thus, $\{a_{ij}(x)\}_{i \in [n], j \in [k]}$ is a minimal generating set of $\langle a_{ij}(x) \rangle_{i \in [n], j \in [k]}$, and therefore $n \cdot k \leq \deg(f)$ holds (Remark 5.1). □

In the rest of this subsection, we narrow our scope to packing methods for $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_p[x]/f(x)$ with the prime modulus. Indeed, this setting is less general. Nonetheless, our results still have interesting consequences (See Example 5.7 - 5.10).

Our main result in this subsection is the following theorem, which is a finite field analogue of Thm. 5.1. However, it is much more involved since we must also handle the multiplicative structure inside $\mathbb{F}_{p^k}$. Notice that our theorem subsumes Prop. 5.3 as the $D = 1$ case in the $t = 1$ setting. The essence of the proof is again a generic construction of a large set which is required to be linearly independent regardless of specific structures of packing methods.

Theorem 5.3. *Let* $\mathcal{B} := \{\beta_1, \cdots, \beta_k\}$ *be a basis of* $\mathbb{F}_{p^k}$ *as a* $\mathbb{F}_p$*-vector space. There exists a degree-*D *packing method for* $\mathbb{F}_{p^k}^n$ *into* $\mathcal{R} := \mathbb{Z}_p[x]/f(x)$ *where* $f(x) \in \mathbb{Z}_p[x]$ *is a degree-*d *irreducible polynomial modulo* p*, only if the following inequality holds.*

$$d \geq \dim\langle \beta_1^D, \cdots, \beta_k^D \rangle + (n-1) \sum_{t=1}^{D} \dim\langle \beta_1^t, \cdots, \beta_k^t \rangle$$

Proof. Let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ be a degree-D packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R}$. For each $i \in [n]$ and $j \in [k]$, choose $a_{ij}(x) \in \mathcal{R}$ such that $\mathsf{Unpack}_1(a_{ij}(x)) = \beta_j \cdot \boldsymbol{e}_i$. For each $s \in \mathbb{Z}^+$, fix a basis $\mathcal{B}_s := \{\beta^{(s)_j}\}_j$ of $\langle \beta_1^s, \cdots, \beta_k^s \rangle$. Then, there exist $a_{ij}^{(s)}(x) \in \mathcal{R}$ such that (i) $\mathsf{Unpack}_s(a_{ij}^{(s)}(x)) = \beta_j^{(s)} \cdot \boldsymbol{e}_i$ and (ii) $a_{ij}^{(s)}(x)$ is a linear combination of $\{a_{ij}(x)^s\}_{j \in [k]}$. Let us denote $\mathcal{A}^{(r,s)} := \{a_{11}(x)^r \cdot a_{ij}^{(s)}(x)\}_{1 < i \leq n \text{ \& } j \in [|\mathcal{B}_s|]}$.

Step 1: Consider the following set of level-t packings.

$$\mathcal{A}_t := \bigcup_{\substack{r+s=t \\ 0<s}} \mathcal{A}^{(r,s)}$$

We will show that $\mathcal{A}_t$ is linearly independent in $\mathcal{R}$ for all $t \leq D$ by induction on t. The case where $t = 1$ is true by the linear homomorphic property at level-1 (Remark 3.1): $\mathcal{A}_1 = \{a_{ij}(x)\}_{1 < i \leq n \text{ \& } j \in [k]}$ (See also Prop. 5.2).

Suppose $\mathcal{A}_t$ is linearly independent for some $t < D$. View $\mathcal{A}_{t+1}$ as $\mathcal{A}^{(0,t+1)} \cup a_{11}(x) \cdot \mathcal{A}_t$. Suppose $\sum_{a_\alpha(x) \in \mathcal{A}_{t+1}} (c_\alpha \cdot a_\alpha(x)) = 0$, for some $c_\alpha \in \mathbb{Z}_p$. Then, by linear homomorphic property at level-$(t+1)$, $c_\alpha = 0$ must hold for all $a_\alpha(x) \in \mathcal{A}^{(0,t+1)}$, since elements of $a_{11}(x) \cdot \mathcal{A}_t$ unpack to $\mathbf{0}$ and $\mathcal{A}^{(0,t+1)}$ unpacks to a linearly independent set by construction. Subsequently, we have again the following equality:

$$\sum_{a_\alpha(x) \in a_{11}(x) \cdot \mathcal{A}_t} (c_\alpha \cdot a_\alpha(x)) = 0.$$

Meanwhile, since $a_{11}(x)$ is non-zero (and hence a unit in $\mathcal{R}$) (Remark 3.1) and $\mathcal{A}_t$ is linearly independent by induction hypothesis, $c_\alpha = 0$ must also hold for all $a_\alpha(x) \in a_{11}(x) \cdot \mathcal{A}_t$. Thus, $\mathcal{A}_t$ is linearly independent in $\mathcal{R}$ for all $t \leq D$.

Step 2: Now consider the set $\mathcal{A} := \mathcal{A}_D \cup \{a_{1j}^{(D)}(x)\}_{j \in [|\mathcal{B}_D|]}$, which coincides with $\{a_{ij}^{(D)}(x)\}_{i \in [n] \text{ \& } j \in [|\mathcal{B}_D|]} \cup a_{11}(x) \cdot \mathcal{A}_{D-1}$. Suppose $\sum_{a_\alpha(x) \in \mathcal{A}} (c_\alpha \cdot a_\alpha(x)) = 0$, for some $c_\alpha \in \mathbb{Z}_p$. Then, by linear homomorphic property at level-D, $c_\alpha = 0$ must hold for all $a_\alpha(x) \in \{a_{ij}^{(D)}(x)\}_{i \in [n] \text{ \& } j \in [|\mathcal{B}_D|]}$, since elements of $a_{11}(x) \cdot \mathcal{A}_{D-1}$ unpack to $\mathbf{0}$ and $\{a_{ij}^{(D)}(x)\}_{i \in [n] \text{ \& } j \in [|\mathcal{B}_D|]}$ unpacks to a linearly independent set by construction. Subsequently, we have again the following equality:

$$\sum_{a_\alpha(x) \in a_{11}(x) \cdot \mathcal{A}_{D-1}} (c_\alpha \cdot a_\alpha(x)) = 0.$$

Meanwhile, since $a_{11}(x)$ is a unit in $\mathcal{R}$ and $\mathcal{A}_{D-1}$ is linearly independent by Step 1, $c_\alpha = 0$ must also hold for all $a_\alpha(x) \in a_{11}(x) \cdot \mathcal{A}_{D-1}$. Thus, $\mathcal{A}$ is linearly independent, and therefore $d \geq |\mathcal{A}|$ must hold. □

To have a more concrete bound, we prove the following proposition. Let $\sigma_{p^k}^{(t)}$ denote the multiplicative order of p modulo $\frac{p^k-1}{\gcd(p^k-1,t)}$.

Proposition 5.4. *Let β be a primitive element of $\mathbb{F}_{p^k}$. Let $\sigma_{p^k}^{(t)}$ be defined as the multiplicative order of p modulo $\frac{p^k-1}{\gcd(p^k-1,t)}$. Regarding the primitive element basis $\{1, \beta, \beta^2, \cdots, \beta^{k-1}\}$, the following equality holds.*

$$\dim\langle 1^t, \beta^t, \beta^{2t}, \cdots, \beta^{(k-1)t} \rangle = \sigma_{p^k}^{(t)}$$

Proof. Observe that $\dim\langle 1^t, \beta^t, \beta^{2t}, \cdots, \beta^{(k-1)t} \rangle$ is equal to the degree of the minimal polynomial of β^t in $\mathbb{F}_p[x]$. The degree of the minimal polynomial of β^t is again equal to the length of the orbit of β^t regarding Frobenius map $x \mapsto x^p$. Since β is a primitive element, we are finding the smallest $s \in \mathbb{Z}^+$ satisfying $t = t \cdot p^s \pmod{p^k - 1}$, which is $\sigma_{p^k}^{(t)}$ by definition. □

Corollary 5.1. *There exists a degree-D packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R} := \mathbb{Z}_p[x]/f(x)$ where $f(x) \in \mathbb{Z}_p[x]$ is a degree-d irreducible polynomial modulo p, only if the following inequality holds.*

$$d \geq \sigma_{p^k}^{(D)} + (n-1) \sum_{t=1}^{D} \sigma_{p^k}^{(t)}$$

Proof. Choose a primitive element β of $\mathbb{F}_{p^k}$ and apply Thm. 5.3 on the basis $\{1, \beta, \beta^2, \cdots, \beta^{k-1}\}$ with the help of Prop. 5.4. □

Remark 5.3. Since $\gcd(p^k - 1, t) \leq t$, we have $\sigma_{p^k}^{(t)} \geq \log_p\left(\frac{p^k-1}{t}\right)$. Subsequently, we have a very rough bound of $\sigma_{p^k}^{(t)} \gtrsim k - \log_p(t)$. Applying this bound to Cor. 5.1, we have the following bound.

$$d \geq k \cdot (D \cdot (n-1) + 1) - \log_p(D \cdot (D!)^{n-1})$$

The following are some consequences of our main result.

Example 5.7 (RMFE). Note that $\sigma_{p^k}^{(2)}$ is always k. Then, by Cor. 5.1, degree-2 packing methods for $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_p[x]/f(x)$, where $f(x)$ is a degree-d irreducible polynomial, have packing density of no larger than $\frac{1}{2} + \frac{k}{2d}$. That is, packing density of RMFE is upper bounded by $\frac{1}{2} + \frac{k}{2d}$. This is a known result (See [CXY20]). However, previous proofs do not extend to higher-degree cases (See Example 5.9) or to the Galois ring case (See Example 5.2).

Example 5.8 (Degree-2 Packing). By Example 5.7, degree-2 *CRT* packing methods for $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_p[x]/f(x)$, where $f(x)$ factors into r distinct irreducible factors, have packing density of no larger than $\frac{1}{2} + \frac{r \cdot k}{2\deg(f)}$ (Section 4.5). In particular, degree-2 CRT packing methods for $\mathbb{F}_{2^k}$-messages into $\mathbb{Z}_2[x]/\Phi_M(x)$, where M is odd and $\Phi_M(x)$ factors into distinct degree-d irreducible factors modulo 2, have packing density of no larger than $\frac{1}{2} + \frac{k}{2d}$.

Suppose one wants to design a degree-2 packing method for $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$ which has a packing density substantially larger than 1/2. Note that choosing $t \geq 2$ already yields packing density no larger than 1/2 by Prop. 5.3. Thus, only possibility is not employing the CRT approach (See also Remark 5.4).

Example 5.9 (Degree-3 Packing). Note that $\sigma_{p^k}^{(3)}$ is always k, except the case of $p^k = 4$. Then, by Cor. 5.1, degree-3 packing methods for $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_p[x]/f(x)$, where $f(x)$ is a degree-d irreducible polynomial, have packing density of no larger than $\frac{1}{3} + \frac{2k}{3d}$, unless $p^k = 4$. Consequently, degree-3 *CRT* packing methods for $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_p[x]/f(x)$, where $f(x)$ factors into r distinct irreducible factors, have packing density of no larger than $\frac{1}{3} + \frac{2r \cdot k}{3\deg(f)}$. In particular, degree-3 CRT packing methods for $\mathbb{F}_{2^k}$-messages into $\mathbb{Z}_2[x]/\Phi_M(x)$, where M is odd and $\Phi_M(x)$ factors into distinct degree-d irreducible factors modulo 2, have packing density of no larger than $\frac{1}{3} + \frac{2k}{3d}$, given $k \neq 2$.

Suppose one wants to design a degree-3 packing method for $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$ which has a packing density substantially larger than 1/3. Note that choosing $t \geq 3$ already yields packing density no larger than 1/3 by Prop. 5.3. Thus, only possibility is choosing $t = 2$ or not employing the CRT approach (See also Remark 5.4).

Example 5.10. By the same arguments as in Example 5.3 and 5.4, we have the following: For $D > 1$, there is no perfect degree-D packing method for $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$, when $f(x)$ is irreducible modulo p and $\deg(f) > 1$.

Thus, there is no perfect degree-D CRT packing method for $\mathbb{F}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$, unless $f(x)$ splits into distinct linear factors. In particular, there is no perfect degree-D CRT packing method for $\mathbb{F}_{2^k}$-messages into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ when M is odd.

The following theorem is a bit more general version of Thm. 5.3 which has no restriction on the quotient polynomial. However, it assumes the existence of a unit of $\mathcal{R}$ which unpacks to an element of $\mathbb{F}_{p^k}^n$.

Theorem 5.4. *Let $\mathcal{B} := \{\beta_1, \cdots, \beta_k\}$ be a basis of $\mathbb{F}_{p^k}$ as a $\mathbb{F}_p$-vector space. Let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ be a degree-D packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R} := \mathbb{Z}_p[x]/f(x)$. Suppose a linear combination of $\{a_{ij}(x)\}_{i\in I \ \& \ j\in[k]}$ is a unit in $\mathcal{R}$, where each $a_{ij}(x) \in \mathcal{R}$ satisfies $\mathsf{Unpack}_1(a_{ij}(x)) = \beta_j \cdot \boldsymbol{e}_i$. Then, the following inequality holds.*

$$d \geq |I| \cdot \dim\langle \beta_1^D, \cdots, \beta_k^D\rangle + (n - |I|) \sum_{t=1}^{D} \dim\langle \beta_1^t, \cdots, \beta_k^t\rangle$$

Proof (Sketch). Assume $u(x) := \sum_{i\in I \ \& \ j\in[k]} c_{ij} \cdot a_{ij}(x)$ is a unit, for some $c_i \in \mathbb{Z}_p$. The proof is exactly same as that of Thm. 5.3, but with only difference in the definition of $\mathcal{A}^{(r,s)}$ and $\mathcal{A}$. Here, we define $\mathcal{A}^{(r,s)} := \{u(x)^r \cdot a_{ij}^{(s)}(x)\}_{i\notin I \ \& \ j\in[|\mathcal{B}_s|]}$ and $\mathcal{A} := \mathcal{A}_D \cup \{a_{ij}^{(D)}(x)\}_{i\in I \ \& \ j\in[|\mathcal{B}_D|]}$. □

Remark 5.4. As Cor. 5.1, and Rem. 5.3, we can apply Prop. 5.4 to have a more concrete version of Thm. 5.4. The theorem has analogous consequences of Example 5.5 and 5.6.

5.4 Proof of Prop. 5.1

We believe the following proposition is a classic fact in algebra. Nonetheless, since we could not find a proper reference containing the statement, we give a proof. Our proof is a more or less verbatim of the proof given in [Con] for the analogous fact on principal ideal *domains* (PID).

Proposition 5.1. *When R is a principal ideal ring (PIR), every submodule of a free R-module of rank n can be finitely generated with n generators.*

Proof. A free R-module of rank n is isomorphic to R^n, so we can assume the free R-module is R^n without loss of generality. We proceed by induction on n. The case where $n = 0$ is trivial. The case where $n = 1$ is true since R is a PIR: every R-submodule of R is a principal ideal, i.e. can be finitely generated with 1 generator.

Suppose the statement is proved for all free R-modules of rank not larger than n. Let M be a submodule of R^{n+1}. Let $\pi : R^{n+1} \to R^n$ be the projection which maps an element of R^{n+1} to its first n coordintes. First consider the image of $\pi|_M$, the restriction of π to M. Indeed, the image is $\pi_M(R^{n+1}) = \pi(M)$, which is a submodule of R^n and therefore has at most n generators by the inductive hypothesis. Thus, we can put $\pi(M) = \sum_{i=1}^{k} R \cdot \boldsymbol{b}_i$ for some $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_k \in R^n$ where $k \leq n$. Let $\boldsymbol{b}_i = \pi(\boldsymbol{a}_i)$ for some $\boldsymbol{a}_i \in M$. Then, $\pi_M(R^{n+1}) = \pi(M) = \sum_{i=1}^{k} R \cdot \pi(\boldsymbol{a}_i)$. And $\ker(\pi|_M) = M \cap \ker(\pi)$. Notice $\ker(\pi) \cong R$ as R-modules. Since R is a PIR, $\ker(\pi|_M) = R \cdot \boldsymbol{a}_0$ for some $\boldsymbol{a}_0 \in M$.

We will show $M = \sum_{i=0}^{k} R \cdot \boldsymbol{a}_i$, and therefore M can be generated by $k+1$ generators $\boldsymbol{a}_0, \cdots, \boldsymbol{a}_k$ with $k+1 \leq n+1$. It is clear that $\sum_{i=0}^{k} R \cdot \boldsymbol{a}_i \subset M$. For the other direction, choose an arbitrary $\boldsymbol{a} \in M$. Then, from the above discussions, $\pi|_M(\boldsymbol{a}) = r_1\pi(\boldsymbol{a}_1) + \cdots + r_k\pi(\boldsymbol{a}_k) = \pi(r_1\boldsymbol{a}_1 + \cdots + r_k\boldsymbol{a}_k)$ for some $r_1, \cdots, r_k \in R$. Therefore $\boldsymbol{a} - \sum_{i=1}^{k} r_i\boldsymbol{a}_i \in \ker(\pi|_M)$, and $\boldsymbol{a} - \sum_{i=1}^{k} r_i\boldsymbol{a}_i = r_0\boldsymbol{a}_0$ for some $r_0 \in R$. Thus $\boldsymbol{a} = r_0\boldsymbol{a}_0 + r_1\boldsymbol{a}_1 + \cdots + r_k\boldsymbol{a}_k \in \sum_{i=0}^{k} R \cdot \boldsymbol{a}_i$, and $M \subset \sum_{i=0}^{k} R \cdot \boldsymbol{a}_i$. □

6 Level-consistency

In this section, we define and examine the concept of *level-consistency*, which is a favorable property for a packing method to have. Our main results are necessary and sufficient conditions for a polynomial ring to allow a level-consistent packing method for $\mathbb{Z}_{p^k}$ and $\mathbb{F}_{p^k}$. They limit the achievable efficiency of level-consistent packing methods, yielding the impossiblity of designing an efficient packing methods while satisfying level-consistency. We begin with the definition.

6.1 Definition and Basic Facts

Definition 6.1. *For $D > 1$, a degree-D packing method* $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ *is called* level-consistent *if* Unpack_i *is all identical for* $1 \le i \le D$*. Otherwise, we say a packing method is* level-dependent*.*

The notion of level-consistency captures the property whether packings are decodable in an identical way at different levels (Prop. 6.1). The level-consistency is a desirable feature, as it allows homomorphic computation between different packing levels. On the other hand, when working with level-dependent packing methods, we must be careful about whether the operands are packed in the same packing level as we perform homomorphic computation on packed messages.

For instance, Overdrive2k [OSV20] and MHz2k [CKL21] design and utilize $\mathbb{Z}_{2^k}$-message packing methods, which are *level-dependent*, to construct HE-based MPC protocols over $\mathbb{Z}_{2^k}$ following the approach of SPDZ [DPSZ12]. Their level-dependency complicates the so-called *reshare* protocol which re-encrypts a *level-zero* HE ciphertext to a *fresh* ciphertext allowing two-level HE to be sufficient for their purpose. The problem here is that a masking HE ciphertext is used twice in the reshare protocol: once to mask the input ciphertext of level-zero and once to reconstruct the fresh ciphertext of level-one by subtracting it. While the difference of HE levels can be managed easily with modulus-switching, that of the packing levels seems to be problematic.

In order to remedy this issue caused by level-dependency, Overdrive2k and MHz2k had to come up with their own solutions. Overdrive2k provides two masking ciphertexts having the *same messages* but in *different packing*: one with level-zero packing and the other with level-one packing. However, this solution substantially degrades the efficiency of the protocol. MHz2k resolves this issue by a technical trick which does not cause any extra cost, closing the gap between the level-consistent and level-dependent packing methods in this case.

This issue does not arise in SPDZ-family [DPSZ12, DKL$^+$13, KPR18, BCS19] over a finite field $\mathbb{Z}_p$, where the conventional packing method is already level-consistent (See Example 3.2). For detailed discussion, refer to [CKL21]. In a later subsection, we prove the impossibility of designing an efficient $\mathbb{Z}_{2^k}$-message packings while satisfying level-consistency. This justifies the use of *level-dependent* packings in SPDZ-like MPC protocols over $\mathbb{Z}_{2^k}$ and highlights the usefulness of the trick proposed by MHz2k [CKL21].

The following proposition says that a level-consistent packing method can be trivially extended to an arbitrary degree.

Proposition 6.1. *A level-consistent degree-D packing method $\mathcal{P}$ can be extended to a level-consistent degree-D' packing method $\mathcal{P}'$ for arbitrary $D' > D$.*

Proof. When $\mathcal{P}$ is $(\mathsf{Pack}_i, \mathsf{Unpack})_{i=1}^{D}$, just define $\mathcal{P}'$ as $(\mathsf{Pack}_1, \mathsf{Unpack})_{i=1}^{D'}$. □

A crucial tool when dealing with a level-consistent packing method is idempotents. We extensively leverage the concept of idempotents and their properties when proving our main results on level-consistency. Here, we list and prove the properties of idempotents related to level-consistent packing methods, which are used afterwards.

The following proposition on idempotents is a classic result in finite ring theory. Nevertheless, for completeness, we give a proof.

Proposition 6.2. *Let R be a finite ring. For all $a \in R$, there exists a positive integer s such that a^s is idempotent, i.e. $a^{2s} = a^s$.*

Proof. Consider the sequence $(a^i)_{i \in \mathbb{Z}^+}$ of R-elements. Since R is finite, there is an element of R which appears infinitely many times in the sequence. Thus, we can choose $i, j \in \mathbb{Z}^+$ satisfying $a^i = a^j$ and $2i \leq j$. Letting $s = j - i$ proves the proposition: $a^s = a^{j-2i}a^i = a^{j-2i}a^j = a^{2s}$. □

The following proposition says that any idempotent $\boldsymbol{a}$ must have an idempotent packing $a(x)$, regarding to a level-consistent method.

Proposition 6.3. *Let R and $\mathcal{R}$ be rings. Let $\mathcal{P}$ be a level-consistent packing method for R^n into $\mathcal{R}$ with identical unpacking algorithms* Unpack*. For any idempotent $\boldsymbol{a} \in R^n$, there exists an idempotent $a(x) \in \mathcal{R}$ such that* $\mathsf{Unpack}(a(x)) = \boldsymbol{a}$.

Proof. First, extend $\mathcal{P}$ to a degree-D packing method for a sufficiently large D (Prop. 6.1). Let $\boldsymbol{a} \in R^n$ be idempotent. Choose an element $\tilde{a}(x) \in \mathcal{R}$ such that $\mathsf{Unpack}(\tilde{a}(x)) = \boldsymbol{a}$. By Prop. 6.2, there exists $s \in \mathbb{Z}^+$ such that $a(x) := \tilde{a}(x)^s$ is idempotent in $\mathcal{R}$. Then, $\mathsf{Unpack}(a(x)) = \mathsf{Unpack}(\tilde{a}(x)^s) = \boldsymbol{a}^s = \boldsymbol{a}$ holds. □

The following proposition is a slight generalization of the property of Galois rings having only 0 and 1 as idempotents.

Proposition 6.4. *For a prime p, let $\mathcal{R} := \mathbb{Z}_{p^t}[x]/f(x)$ and $f(x) = g(x)^\ell \pmod{p}$, where $g(x)$ is an irreducible polynomial in $\mathbb{F}_p[x]$. Then, an idempotent element of $\mathcal{R}$ is either* 0 *or* 1.

Proof. Suppose $a(x) \in \mathcal{R}$ is idempotent. Then, $f(x)$ divides $a(x)^2 - a(x)$ in $\mathbb{Z}_{p^t}[x]$, and therefore $g(x)^\ell$ divides $a(x)(a(x)-1)$ in $\mathbb{F}_p[x]$. Since $g(x)$ is irreducible and $a(x)$ and $a(x) - 1$ are coprime in $\mathbb{F}_p[x]$, $a(x)$ equals 0 or 1 in $\mathcal{R}/p\mathcal{R}$.

Suppose $a(x) = 1$ in $\mathcal{R}/p\mathcal{R}$. We can represent $a(x)$ as $1 + p^s \cdot \tilde{a}(x)$ for some $t \geq s > 0$, where $\tilde{a}(x)$ is not divisible by p. Then, $0 = a(x)^2 - a(x) = p^{2s} \cdot \tilde{a}(x)^2 + p^s \cdot \tilde{a}(x)$ in $\mathcal{R}$. Since $s > 0$ and $p \nmid \tilde{a}(x)$, s must be t and therefore $a(x) = 1$ in $\mathcal{R}$. We can similarly show that if $a(x) = 0$ in $\mathcal{R}/p\mathcal{R}$ then $a(x) = 0$ in $\mathcal{R}$. □

Another tool which is useful when dealing with level-consistent packing methods is nilpotents. The following proposition says any nilpotent must unpack to a nilpotent, given it is a valid packing regarding to a level-consistent method.

Proposition 6.5. *Let R and $\mathcal{R}$ be rings, and let $\mathcal{P}$ be a level-consistent packing method for R^n into $\mathcal{R}$ with identical unpacking algorithms* Unpack. *For any nilpotent $a(x) \in \mathcal{R}$,* $\mathsf{Unpack}(a(x))$ *outputs a nilpotent $\boldsymbol{a} \in R^n$ or a failure $\perp$.*

Proof. Suppose $\mathsf{Unpack}(a(x))$ outputs $\boldsymbol{a} \in R^n$. Let s be a positive integer such that $a(x)^s = 0$ in $\mathcal{R}$. Extend $\mathcal{P}$ to a degree-s packing method (Prop. 6.1). Then, $\boldsymbol{a}^s = \mathsf{Unpack}(a(x)^s) = \mathsf{Unpack}(0) = \boldsymbol{0}$ holds. □

Lastly, we introduce the notion of *one-to-one* packing which plays an important role in the proof of our main result.

Definition 6.2 (One-to-one Packing). *Let R and $\mathcal{R}$ be rings. We say a packing method* $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ *for R^n into $\mathcal{R}$ is* one-to-one*, if there is unique $a(x) \in \mathcal{R}$ such that* $\mathsf{Unpack}_i(a(x)) = \boldsymbol{a}$ *for all $\boldsymbol{a} \in R^n$ and $i \in [D]$.*

6.2 Level-consistency in $\mathbb{Z}_{p^k}$-Message Packings

Our main result on level-consistency in $\mathbb{Z}_{p^k}$-message packings is the following theorem. Our theorem illustrates a necessary condition for a surjective packing method for $\mathbb{Z}_{p^k}$-messages to exist. As mentioned, the proof regards the notion of idempotents (Prop. 6.3, 6.4).

Theorem 6.1. *For a prime p, let $f(x) \in \mathbb{Z}_{p^t}[x]$ have exactly r distinct irreducible factors in $\mathbb{Z}_p[x]$. There exists a level-consistent packing method for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ only if $n \leq r$.*

Proof. Let $f(x)$ be factorized into $\prod_{i=1}^{r} \bar{f}_i(x)$ in $\mathbb{Z}_p[x]$, where each $\bar{f}_i(x)$ is a power of a distinct irreducible polynomial in $\mathbb{Z}_p[x]$. The factorization can be lifted upto $\mathbb{Z}_{p^t}[x]$ via Hensel lifting. Let $f(x) = \prod_{i=1}^{r} f_i(x)$, where $f_i(x) \in \mathbb{Z}_{p^t}[x]$ is the Hensel lift of $\bar{f}_i(x)$ satisfying $\bar{f}_i(x) = f_i(x) \pmod p$. By Prop. 6.4, there are 2^r idempotents in $\mathbb{Z}_{p^t}[x]/f(x) \approx \prod_{i=1}^{r} \mathbb{Z}_{p^t}[x]/f_i(x)$, namely $\{0,1\}^r$. Also note that there are 2^n idempotents in $\mathbb{Z}_{p^k}^n$, namely $\{0,1\}^n$.

By Prop. 6.3, for each idempotent $\boldsymbol{a}$ of $\mathbb{Z}_{p^k}^n$, there is a distinct idempotent $a(x)$ of $\mathbb{Z}_{p^t}[x]/f(x)$ such that $\mathsf{Unpack}(a(x)) = \boldsymbol{a}$. Thus, the number of idempotents in $\mathbb{Z}_{p^k}^n$ cannot be larger than that of $\mathbb{Z}_{p^t}[x]/f(x)$, and $n \leq r$ holds. □

The following are some consequences of Thm. 6.1. We begin with an optimality result for HELib packing (Section 4.1).

Example 6.1. Essentially, Thm. 6.1 asserts that HELib packing offers the optimal packing density if level-consistency is required. As level-consistency is more than a favorable feature for *fully* homomorphic encryption(FHE), our result reassures that HELib packing is an excellent packing method to use for FHE, and it strongly justifies long line of researches based on such packing method [GHS12, HS15, CH18].

The following examples illustrate the hardness of designing an efficient HE packing method for $\mathbb{Z}_{2^k}$-messages while satisfying level-consistency. We have similar results for $\mathbb{Z}_{p^k}$-messages with $p \neq 2$.

Example 6.2. When $M = 2^m$, since $\Phi_M(x) = (x+1)^{2^{m-1}}$ in $\mathbb{F}_2[x]$, we can pack at most one copy of $\mathbb{Z}_{2^k}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ while satisfying level-consistency.

Example 6.3. When M is an odd, $\Phi_M(x)$ factors into a product of distinct irreducible polynomials of degree $d = \mathrm{ord}_M(2)$ in $\mathbb{F}_2[x]$. Let $\phi(M) = r \cdot d$. Then, we can pack at most r copies of $\mathbb{Z}_{2^k}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ while satisfying level-consistency. Note that, since $d > \log M$ by definition, $r < \phi(M)/\log M$.

Example 6.4. When $M = 2^s \cdot M'$, where M' is an odd, $\Phi_M(x) = \Phi_{M'}(-x^{2^{s-1}}) = \Phi_{M'}(x)^{2^{s-1}}$ in $\mathbb{F}_2[x]$. Thus, we cannot pack more copies of $\mathbb{Z}_{2^k}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ than $\mathbb{Z}_{2^t}[x]/\Phi_{M'}(x)$ while satisfying level-consistency.

Thm. 6.1 also yields the impossibility of level-consistent RMFEs over Galois ring for $\mathbb{Z}_{p^k}$-messages.

Example 6.5. In $GR(p^t, d) \cong \mathbb{Z}_{p^t}[x]/f(x)$ with a degree-d $f(x)$ which is irreducible modulo p, we can pack at most one copy of $\mathbb{Z}_{p^k}$ while satisfying level-consistency. That is, there is no meaningful level-consistent RMFE over Galois ring for $\mathbb{Z}_{p^k}$-messages.

On the other side, we have the following theorem with a constructive proof, which asserts that the necessary condition in Thm. 6.1 is also a sufficient one.

Theorem 6.2. *If there are r distinct irreducible factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{F}_p[x]$, then there is a level-consistent packing method for $\mathbb{Z}_{p^k}^r$ into $\mathbb{Z}_{p^t}[x]/f(x)$.*

Proof. Let $f(x)$ be factorized into $\prod_{i=1}^r g_i(x)^{\ell_i}$ in $\mathbb{F}_p[x]$, where each $g_i(x)$ is distinct irreducible polynomial in $\mathbb{F}_p[x]$. The factorization can be lifted upto $\mathbb{Z}_{p^k}[x]$ via Hensel lifting. Let $f(x) = \prod_{i=1}^r f_i(x)$, where $f_i(x) \in \mathbb{Z}_{p^k}[x]$ is the Hensel lift of $g_i(x)^{\ell_i}$ satisfying $f_i(x) = g_i(x)^{\ell_i} \pmod p$. Then, we can identify $\mathbb{Z}_{p^k}[x]/f(x)$ with $\prod_{i=1}^r \mathbb{Z}_{p^k}[x]/f_i(x)$ via the CRT ring isomorphism.

There is a trivial ring monomorphism $\psi : \mathbb{Z}_{p^k}^r \to \mathbb{Z}_{p^k}[x]/f(x)$ defined as the following.

$$\psi(a_1, \cdots, a_r) = (a_1, \cdots, a_r) \in \prod_{i=1}^r \mathbb{Z}_{p^k}[x]/f_i(x)$$

Define the function $\psi^{-1} : \mathbb{Z}_{p^k}[x]/f(x) \to \mathbb{Z}_{p^k}^r \cup \{\perp\}$ as the following.

$$\psi^{-1}(a(x)) = \begin{cases} \boldsymbol{a}, & \text{if there is } \boldsymbol{a} \in \mathbb{Z}_{p^k}^r \text{ such that } \psi(\boldsymbol{a}) = a(x) \\ \perp, & \text{otherwise} \end{cases}$$

Let π_k and ι_k denote the projection and injection between $\mathbb{Z}_{p^t}[x]/f(x)$ and $\mathbb{Z}_{p^k}[x]/f(x)$ respectively. Define $\mathsf{Pack} := \iota_k \circ \psi$ and $\mathsf{Unpack} := \psi^{-1} \circ \pi_k$ (Fig. 5). Then, it is straightforward that $(\mathsf{Pack}, \mathsf{Unpack})$ is a level-consistent packing method. □

$$\begin{tikzcd}
\mathbb{Z}_{p^k}^r \arrow[r, "\mathsf{Pack}"] \arrow[dr, "\psi"'] & \mathbb{Z}_{p^t}[x]/f(x) \\
& \mathbb{Z}_{p^k}[x]/f(x) \arrow[u, "\iota_k"']
\end{tikzcd}$$

(a) Pack

$$\begin{tikzcd}
\mathbb{Z}_{p^k}^r & \mathbb{Z}_{p^t}[x]/f(x) \arrow[l, "\mathsf{Unpack}"'] \arrow[d, "\pi_k"] \\
& \mathbb{Z}_{p^k}[x]/f(x) \arrow[ul, "\psi^{-1}"]
\end{tikzcd}$$

(b) Unpack

Fig. 5: Definitions of Pack and Unpack in Thm. 6.2

Corollary 6.1. *For a prime p, let $f(x) \in \mathbb{Z}_{p^t}[x]$ have exactly r distinct irreducible factors in $\mathbb{Z}_p[x]$. There exists a level-consistent packing method for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ if and only if $n \leq r$.*

Proof. Straightforward from Thm. 6.1 and 6.2. □

6.3 Level-consistency in $\mathbb{F}_{p^k}$-Message Packings

Our main result on level-consistency in $\mathbb{F}_{p^k}$-message packings is the following theorem. It is a finite field analogue of Thm. 6.1 which is on $\mathbb{Z}_{p^k}$-message packings. Our theorem illustrates a necessary condition for a level-consistent packing method for $\mathbb{F}_{p^k}$-messages to exist.

Theorem 6.3. *Let r be the number of distinct irreducible factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{F}_p[x]$ whose degrees are multiples of k. There exists a level-consistent packing method $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ only if $n \leq r$.*

Proof. See Section 6.4. □

The following are some consequences of Thm. 6.3. They illustrate the hardness of designing an efficient HE packing method for $\mathbb{F}_{2^k}$-messages while satisfying level-consistency. We have similar results for $\mathbb{F}_{p^k}$-messages with $p \neq 2$.

Example 6.6. When $M = 2^m$, since $\Phi_M(x) = (x+1)^{2^{m-1}}$ in $\mathbb{F}_2[x]$, we can only pack copies of $\mathbb{F}_2$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ while satisfying level-consistency. Even in that case, we can pack at most one copy of $\mathbb{F}_2$.

Example 6.7. When M is an odd, $\Phi_M(x)$ factors into a product of distinct irreducible polynomials of degree $d = \mathrm{ord}_M(2)$ in $\mathbb{F}_2[x]$. Let $\phi(M) = r \cdot d$. Then, we can only pack copies of $\mathbb{F}_{2^k}$ such that $k|d$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ while satisfying level-consistency. In that case, we can pack at most r copies of $\mathbb{F}_{2^k}$. Note that, since $d > \log M$ by definition, $r < \phi(M)/\log M$. For instance, if one wants to pack $\mathbb{F}_{2^8}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ with an odd M while satisfying level-consistency, then one must choose M such that $\mathrm{ord}_M(2)$ is a multiple of 8.

Example 6.8. When $M = 2^s \cdot M'$, where M' is an odd, $\Phi_M(x) = \Phi_{M'}(-x^{2^{s-1}}) = \Phi_{M'}(x)^{2^{s-1}}$ in $\mathbb{F}_2[x]$. Thus, we cannot pack more copies of $\mathbb{F}_{2^k}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ than $\mathbb{Z}_{2^t}[x]/\Phi_{M'}(x)$ while satisfying level-consistency.

Thm. 6.3 also yields the impossibility of level-consistent RMFEs.

Example 6.9. In $\mathbb{F}_{p^d} \cong \mathbb{Z}_p[x]/f(x)$ with a degree-d irreducible $f(x)$, we can pack at most one copy of $\mathbb{F}_{p^k}$ while satisfying level-consistency. Furthermore, if $k \nmid d$, we cannot pack even a single copy of $\mathbb{F}_{p^k}$ into $\mathbb{F}_{p^d}$ while satisfying level-consistency. That is, there is no meaningful level-consistent RMFE.

On the other side, we have the following theorem with a constructive proof, which asserts that the necessary condition in Thm. 6.3 is also a sufficient one.

Theorem 6.4. *Suppose there are r distinct irreducible factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{F}_p[x]$ whose degrees are multiples of k. Then, there exists a level-consistent packing method $\mathbb{F}_{p^k}^r$ into $\mathbb{Z}_{p^t}[x]/f(x)$.*

Proof. Let $g(x) \in \mathbb{F}_p[x]$ be the product of r distinct irreducible factors of $f(x)$ in $\mathbb{F}_p[x]$ whose degrees are multiples of k. Then, there is a ring monomorphism $\psi : \mathbb{F}_{p^k}^r \to \mathbb{F}_p[x]/g(x)$. Define the function $\psi^{-1} : \mathbb{F}_p[x]/g(x) \to \mathbb{F}_{p^k}^r \cup \{\perp\}$ as the following.

$$\psi^{-1}(a(x)) = \begin{cases} \boldsymbol{a}, & \text{if there is } \boldsymbol{a} \in \mathbb{F}_{p^k}^r \text{ such that } \psi(\boldsymbol{a}) = a(x) \\ \perp, & \text{otherwise} \end{cases}$$

Let π_p and ι_p denote the projection and injection between $\mathbb{Z}_{p^k}[x]/f(x)$ and $\mathbb{F}_p[x]/f(x)$, and let π_g and ι_g denote those of $\mathbb{F}_p[x]/f(x)$ and $\mathbb{F}_p[x]/g(x)$ respectively.

Define Pack $:= \iota_p \circ \iota_g \circ \psi$ and Unpack $:= \psi^{-1} \circ \pi_h \circ \pi_p$ (Fig. 6). Then, it is straightforward that (Pack, Unpack) is a level-consistent packing method. □

$$\begin{tikzcd}
\mathbb{F}_{p^k}^r \arrow[r, "\mathsf{Pack}"] \arrow[rdd, "\psi"'] & \mathbb{Z}_{p^t}[x]/f(x) \\
& \mathbb{F}_p[x]/f(x) \arrow[u, "\iota_p"'] \\
& \mathbb{F}_p[x]/g(x) \arrow[u, "\iota_g"']
\end{tikzcd}$$

(a) Pack

$$\begin{tikzcd}
\mathbb{F}_{p^k}^r & \mathbb{Z}_{p^t}[x]/f(x) \arrow[l, "\mathsf{Unpack}"'] \arrow[d, "\pi_p"] \\
& \mathbb{F}_p[x]/f(x) \arrow[d, "\pi_g"] \\
& \mathbb{F}_p[x]/g(x) \arrow[luu, "\psi^{-1}"]
\end{tikzcd}$$

(b) Unpack

Fig. 6: Definitions of Pack and Unpack in Thm. 6.4

Corollary 6.2. *Let r be the number of distinct irreducible factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{F}_p[x]$ whose degrees are multiples of k. There exists a level-consistent packing method $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ if and only if $n \leq r$*

Proof. Straightforward from Thm. 6.3 and 6.4. □

6.4 Proof of Thm. 6.3

In this subsection, we prove Thm. 6.3. The proof is elementary, but consists of a number of steps. As mentioned, idempotents (Prop. 6.3) and nilpotents (Prop. 6.5) are at the core of the proof. Even if they are not directly referred, many parts of the proof are motivated from the concepts. Also notice the crucial role of one-to-one property in the proof of Lem. 6.4.

Theorem 6.3. *Let r be the number of distinct irreducible factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{F}_p[x]$ whose degrees are multiples of k. There exists a level-consistent packing method $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ only if $n \leq r$.*

Proof. Straightforward from Lem. 6.1, 6.2, 6.3, and 6.4. □

Lemma 6.1. *Suppose there exists a level-consistent packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$. Then, there exists a level-consistent packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/f(x)$.*

Proof. Let $(\mathsf{Pack}, \mathsf{Unpack})$ be a level-consistent packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$. Suppose $a(x), b(x) \in \mathbb{Z}_{p^t}[x]/f(x)$ satisfy $\mathsf{Unpack}(a(x)) = \boldsymbol{a}$ and $\mathsf{Unpack}(b(x)) = \boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{F}_{p^k}^n$. If $a(x) = b(x)$ modulo p, then $\boldsymbol{a} = \boldsymbol{b}$ since (i) $\mathsf{Unpack}(a(x) - b(x)) = \boldsymbol{a} - \boldsymbol{b}$, (ii) $a(x) - b(x)$ is nilpotent in $\mathbb{Z}_{p^t}[x]/f(x)$, and (iii) $\boldsymbol{0}$ is the only nilpotent element in $\mathbb{F}_{p^k}^n$ (Prop. 6.5). Thus, for all $a(x) \in \mathbb{Z}_{p^t}[x]/f(x)$, $\mathsf{Unpack}(a(x))$ is fully determined by $a(x) \bmod p$, given it does not output a failure $\perp$.

Let $\mathsf{Pack}' = \pi_p \circ \mathsf{Pack}$ where π_p denotes the projection from $\mathbb{Z}_{p^t}[x]/f(x)$ to $\mathbb{F}_p[x]/f(x)$. Let $\mathsf{Unpack}' : \mathbb{F}_p[x]/f(x) \to \mathbb{F}_{p^k}^n \cup \{\perp\}$ be defined as the following.

$$\mathsf{Unpack}(a(x)) = \begin{cases} \boldsymbol{a}, & \text{if there is } \tilde{a}(x) \in \mathbb{Z}_{p^t}[x]/f(x) \text{ such that } \pi_p(\tilde{a}(x)) = a(x) \\ & \text{and } \mathsf{Unpack}(\tilde{a}(x)) = \boldsymbol{a} \text{ for some } \boldsymbol{a} \in \mathbb{F}_{p^k}^n \\ \perp, & \text{otherwise} \end{cases}$$

Then, it is straightforward that $(\mathsf{Pack}', \mathsf{Unpack}')$ is a level-consistent packing method. □

Lemma 6.2. *Suppose there exists a level-consistent packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/f(x)$. Then, there exists a level-consistent packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/\hat{g}(x)$, where $\hat{g}(x)$ is the largest square-free factor of $f(x)$.*

Proof. Let $(\mathsf{Pack}, \mathsf{Unpack})$ be a level-consistent packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/f(x)$. Note that $a(x) \in \mathbb{F}_p[x]/f(x)$ is nilpotent if and only if it is divisible by $\hat{g}(x)$. We can use the same argument used in the proof of Lem. 6.1 with the help of Prop. 6.5. Then, for all $a(x) \in \mathbb{F}_p[x]/f(x)$, $\mathsf{Unpack}(a(x))$ is fully determined by $a(x) \bmod \hat{g}(x)$, given it does not output a failure $\perp$. Consequently, we can design a level-consistent packing method $(\mathsf{Pack}', \mathsf{Unpack}')$ for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/\hat{g}(x)$ as in the proof of Lem. 6.1. □

Lemma 6.3. *Suppose there exists a level-consistent packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/\hat{g}(x)$ where $\hat{g}(x)$ is square-free. Then, there exists a factor $g(x)$ of $\hat{g}(x)$ which allows a level-consistent one-to-one packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/g(x)$.*

Proof. Let $(\mathsf{Pack}, \mathsf{Unpack})$ be a level-consistent packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R} := \mathbb{F}_p[x]/\hat{g}(x)$. Let $\hat{g}(x)$ factorizes into r distinct irreducible polynomials $\{\hat{g}_i(x)\}_{i=1}^r$. We identify $\mathcal{R}$ with $\prod_{i=1}^r \mathbb{F}_p[x]/\hat{g}_i(x)$.

Step 1: For a subset $A \subset [r]$, let $e_A(x) \in \mathcal{R}$ denote the element which is 1 modulo $\hat{g}_i(x)$ for $i \in A$ and 0 modulo $\hat{g}_i(x)$ for $i \notin A$. Note that $e_{A\cup B}(x) = e_A(x) + e_B(x) - e_A(x) \cdot e_B(x)$. Thus, if $\mathsf{Unpack}(e_A(x)) = \mathbf{0}$ and $\mathsf{Unpack}(e_B(x)) = \mathbf{0}$, then $\mathsf{Unpack}(e_{A\cup B}(x))$ is also $\mathbf{0}$ by the level-consistency. We can therefore choose the maximal set $I \subset [r]$ such that $\mathsf{Unpack}(e_I(x)) = \mathbf{0}$.

Step 2: Let $g(x) := \prod_{i \notin I} \hat{g}_i(x)$. Consider the ideal $Z \subset \mathcal{R}$ generated by $e_I(x)$, which coincides with $g(x) \cdot \mathcal{R}$. Then, for any $a(x) \in Z$, $\mathsf{Unpack}(a(x))$ outputs $\mathbf{0}$ or a failure $\perp$, since $a(x) \cdot e_I(x) = a(x)$. Thus, for all $a(x) \in \mathcal{R}$, $\mathsf{Unpack}(a(x))$ is fully determined by $a(x) \bmod g(x)$, given it does not output a failure $\perp$.

Step 3: Let $a(x) \in \mathcal{R}$ satisfies $\mathsf{Unpack}(a(x)) = \mathbf{0}$. Suppose $a(x)$ is non-zero modulo $\hat{g}_i(x)$ if $i \in A$ and 0 modulo $\hat{g}_i(x)$ if $i \notin A$, for some $A \subset [r]$. Since $\mathbb{F}_p[x]/\hat{g}_i(x)$ are fields, there exists $s \in \mathbb{Z}^+$ such that $a(x)^s = e_A(x)$. By definition, $A \subset I$ holds. Thus, for any $a(x) \in \mathcal{R}$ satisfying $\mathsf{Unpack}(a(x)) = \mathbf{0}$, it holds that $a(x) \in Z$, i.e. $a(x) = 0 \pmod{g(x)}$.

Step 4: Following the proof of Lem. 6.1 together with Step 2, we can design a level-consistent packing method $(\mathsf{Pack}', \mathsf{Unpack}')$ for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/g(x)$. Moreover, such $(\mathsf{Pack}', \mathsf{Unpack}')$ is a one-to-one packing method by Step 3. □

Lemma 6.4. *Let $g(x) \in \mathbb{F}_p[x]$ be square-free and r be the number of distinct irreducible factors of $g(x)$ whose degrees are multiples of k. There exists a level-consistent one-to-one packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/g(x)$, only if $r \le n$.*

Proof. Let $\mathcal{P} = (\mathsf{Pack}, \mathsf{Unpack})$ be a level-consistent one-to-one packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R} := \mathbb{F}_p[x]/g(x)$. Let $g(x)$ factorizes into $\hat{r}$ distinct irreducible polynomials $\{g_i(x)\}_{i=1}^{\hat{r}}$ and let $d_i := \deg(g_i)$. We identify $\mathcal{R}$ with $\prod_{i=1}^r \mathbb{F}_p[x]/g_i(x)$. For a subset $A \subset [\hat{r}]$, let $e_A(x) \in \mathcal{R}$ denote the element which is 1 modulo $g_i(x)$ for $i \in A$ and 0 modulo $g_i(x)$ for $i \notin A$. Let $\boldsymbol{e}_i \in \mathbb{F}_{p^k}^n$ denote the element with 1 in its i-th coordinate and 0 in the others.

Since $\mathcal{P}$ is one-to-one, there is only one element which unpacks to $\boldsymbol{e}_i$, which we can set as $e_{A_i}(x)$ for some $A_i \subset [\hat{r}]$ by Prop. 6.3 and 6.4. Moreover, $A_i \cap A_j = \emptyset$ for distinct i, j since $e_{A_i}(x) \cdot e_{A_j}(x)$ must be 0 to be unpacked to $\mathbf{0}$ by one-to-one property.

Let $u \in \mathbb{F}_{p^k}$ be a multiplicative generator of $\mathbb{F}_{p^k}$, and let $u_i(x) \in \mathcal{R}$ be the element which unpacks to $u \cdot \boldsymbol{e}_i$. Observe that $u_i(x) \bmod g_j(x)$ is non-zero if and only if $j \in A_i$, since $(u \cdot \boldsymbol{e}_i)^{p^k-1} = \boldsymbol{e}_i$ and consequently $u_i(x)^{p^k-1} =$

$e_{A_i}(x)$. Moreover, for $j \in A_i$, the multiplicative order s of $u_i(x) \bmod g_j(x)$ in $\mathbb{F}_p[x]/g_j(x) \cong \mathbb{F}_{p^{d_j}}$ must divide $p^k - 1$.

Meanwhile, if the multiplicative order s is less than $p^k - 1$, then $u_i(x)^s = e_{A_i}(x) = 1 \pmod{g_j(x)}$. This contradicts the one-to-one property, since there must be another element of $\mathcal{R}$, a power of $u_i(x)^s - e_{A_i}(x)$, which unpacks to $\boldsymbol{e}_i$ and is 0 modulo $g_j(x)$. Consequently, $s = p^k - 1$ must hold. To allow such conditions on the orders, d_j's must be multiples of k for $j \in A_i$. Thus, for each $\boldsymbol{e}_i$, we can choose distinct $g_j(x)$ whose degree is a multiple of k, and $r \leq n$ holds. □

7 Surjectivity

In this section, we define and examine the concept of *surjectivity*, which is a favorable property for a packing method to have. Our main results are necessary and sufficient conditions for a polynomial ring to allow a surjective packing method for $\mathbb{Z}_{p^k}$ and $\mathbb{F}_{p^k}$. They limit the achievable efficiency of surjective packing methods, yielding the impossiblity of designing an efficient packing methods while satisfying surjectivity. We begin with the definition.

7.1 Definition and Basic Facts

Definition 7.1 (Surjective Packing). *Let $\mathcal{R}$ be a ring. We say a degree-D packing method* $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ *into $\mathcal{R}$ is* surjective[12] *if there is no $a(x) \in \mathcal{R}$ such that* $\mathsf{Unpack}_1(a(x)) =\perp$.

For a packing method for R^n into $\mathcal{R}$, the notion of surjectivity captures the condition whether every element of $\mathcal{R}$ is decodable. This distiction is essential when designing a cryptographic protocol with the packing method in a malicious setting, where an adversary might freely deviate from the protocol. If there is $a(x) \in \mathcal{R}$ such that $\mathsf{Unpack}_1(a(x)) =\perp$, a malicious adversary might make use of $a(x)$, when one is supposed to use a valid packing according to the protocol. The deviation may not only harm the correctness of the protocol, but also may leak information of honest parties, if such invalid packings are not properly handled.

For instance, Overdrive2k [OSV20] and MHz2k [CKL21] design and utilize $\mathbb{Z}_{2^k}$-message packings which is *not* surjective to construct HE-based MPC protocols over $\mathbb{Z}_{2^k}$ following the approach of SPDZ [DPSZ12]. In order to mitigate the *invalid* packings, they perform ZKPoMK (Zero-Knowledge Proof of Message Knowledge) to ensure an HE ciphertext encrypts a validly packed plaintext.[13] ZKPoMK does not appear in SPDZ-family [DPSZ12, DKL$^+$13, KPR18, BCS19] over a finite field $\mathbb{Z}_p$, where the conventional packing method is already surjective with perfect packing density (See Example 3.2). In a later subsection, we prove the impossibility of designing an efficient $\mathbb{Z}_{2^k}$-message packings while satisfying surjectivity. This justifies the use of *non*-surjective packings and the need of ZKPoMK in SPDZ-like MPC protocols over $\mathbb{Z}_{2^k}$.

The following proposition says that the definition of surjectivity trivially extends to all levels. The fact is used throughout this section.

Proposition 7.1. *Suppose* $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ *is a degree-D surjective packing method for R^n into $\mathcal{R}$. Then, there is no $a(x) \in \mathcal{R}$ such that* $\mathsf{Unpack}_i(a(x)) =\perp$, *for all $i \in [D]$.*

Proof. By surjectivity and multiplicative homomorphic property, it holds that $\mathsf{Unpack}_2(a(x)) = \mathsf{Unpack}_1(1) \odot \mathsf{Unpack}_1(a(x)) \in R^n$, for all $a(x) \in \mathcal{R}$. Likewise, we can proceed inductively upto $\mathsf{Unpack}_D(\cdot)$. □

[12] In a sense that any element of $\mathcal{R}$ *could* be an image of $\mathsf{Pack}_1(\cdot)$.

[13] ZKPoMK was first conceptualized in MHZ2k [CKL21], but it is also performed in Overdrive2k [OSV20] implicitly. For detailed discussion, refer to [CKL21].

A crucial fact when dealing with a surjective packing method is the following proposition on zero-sets. We extensively use the proposition when proving our main results on surjectivity.

Proposition 7.2 (Zero-set Ideal). *Let R and $\mathcal{R}$ be rings. For $D > 1$, let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ be a degree-D surjective packing method for R^n into $\mathcal{R}$. Let Z_i be the set consisting of elements $a(x) \in \mathcal{R}$ such that $\mathsf{Unpack}_i(a(x)) = \mathbf{0}$. Then, $Z = Z_1 = \cdots = Z_D$ for some ideal Z of $\mathcal{R}$. Moreover, $|Z| = |\mathcal{R}|/|R|^n$.*

Proof. By Prop. 7.1 and multiplicative homomorphic property, $\mathcal{R} \cdot Z_i \subset Z_{i+1}$ holds for $i < D$. Since $1 \in \mathcal{R}$, $Z_i \subset \mathcal{R} \cdot Z_i$ holds, and therefore $Z_i \subset \mathcal{R} \cdot Z_i \subset Z_{i+1}$. By Prop. 7.1 and additive homomorphic property, Z_i's have the same size, namely $|Z_i| = |\mathcal{R}|/|R|^n$. Thus, $Z_i = \mathcal{R} \cdot Z_i = Z_{i+1}$ holds. We can now put $Z := Z_1 = \cdots = Z_D$. Moreover, since $\mathcal{R} \cdot Z = Z$ holds, Z is an ideal of $\mathcal{R}$. □

7.2 Surjectivity in $\mathbb{Z}_{p^k}$-Message Packings

Our main result on surjectivity in $\mathbb{Z}_{p^k}$-message packings is the following theorem. Our theorem illustrates a necessary condition for a surjective packing method for $\mathbb{Z}_{p^k}$-messages to exist.

Theorem 7.1. *Let $\check{r}$ be the number of linear factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{Z}_{p^k}[x]$ which are mutually distinct modulo p. For $D > 1$, there exists a degree-D surjective packing method $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ only if $n \leq \check{r}$.*

Proof. See Section 7.4. □

Before we proceed, we state a simple fact on irreducibility of $\Phi_{2^m}(x)$ over a power-of-two modulus.

Proposition 7.3 (Irreducibility of $\Phi_{2^m}(x)$). *For $M = 2^m$, cyclotomic polynomial $\Phi_M(x)$ is irreducible modulo 4, i.e. there are no $f(x), g(x) \in \mathbb{Z}_4[x]$ such that $f(x) \cdot g(x) = \Phi_M(x) \pmod 4$ and $\deg(f), \deg(g) \geq 1$.*

Proof. Suppose such $f(x)$ and $g(x)$ exist. Let $f(x) := \sum_{i=0}^{d_f} f_i \cdot x^i$ and similarly for $g(x)$, with $d_f + d_g = 2^{m-1}$. Since $\Phi_M(x)$ factors into $(x+1)^{2^{m-1}}$ in $\mathbb{F}_2[x]$, $f(x)$ and $g(x)$ must be $x^{d_f} + 1 = (x+1)^{d_f}$ and $x^{d_g} + 1 = (x+1)^{d_g}$ in $\mathbb{F}_2[x]$, respectively. Thus, $f_i = 0 \pmod 2$ for $0 < i < d_f$, and $g_i = 0 \pmod 2$ for $0 < i < d_g$. Meanwhile, we can assume $f_{d_f} = g_{d_g} = 1 \pmod 4$ without loss of generality. Also note that, since $f_0 \cdot g_0 = 1 \pmod 4$, either $f_0 = g_0 = 1$ or $f_0 = g_0 = 3$ must hold modulo 4.

Suppose $d_f \neq d_g$, and without loss of generality assume $d_f > d_g$. Consider the d_g-th coefficient of $\Phi_M(x)$. It is 0 mudulo 2 as $\Phi_M(x) = x^{2^{m-1}} + 1$. However, computing it as $\sum_{i=0}^{d_g} f_i \cdot g_{d_g - i} = f_0 \cdot g_{d_g} \pmod 2$, it is 1 modulo 2 and leads to a contradiction. Thus, $d_f = d_g$ must hold.

Consider the d_g-th coefficient of $\Phi_M(x)$, again. It is 0 mudulo 4 as $\Phi_M(x) = x^{2^{m-1}} + 1$. However, computing it as $\sum_{i=0}^{d_g} f_i \cdot g_{d_g - i} = f_0 \cdot g_{d_g} + f_{d_f} \cdot g_0 \pmod 4$, it is 2 modulo 4 and leads to a contradiction. Thus, such $f(x)$ and $g(x)$ do not exist. □

The following are some consequences of Thm. 7.1. They illustrate the impossibility of designing a surjective HE packing method for $\mathbb{Z}_{2^k}$-messages with cyclotomic polynomials. We have similar results for $\mathbb{Z}_{p^k}$-messages with $p \neq 2$.

Example 7.1. When $M = 2^m$, by Prop. 7.3, we cannot pack any copies of $\mathbb{Z}_{2^k}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ while satisfying surjectivity and degree-2 homomorphism.

Example 7.2. When M is an odd, $\Phi_M(x)$ factors into a product of distinct irreducible polynomials of degree $d = \mathrm{ord}_M(2)$ in $\mathbb{F}_2[x]$. Thus, we cannot pack any copies of $\mathbb{Z}_{2^k}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ while satisfying surjectivity and degree-2 homomorphism.

Example 7.3. When $M = 2^s \cdot M'$, where M' is an odd, $\Phi_M(x) = \Phi_{M'}(-x^{2^{s-1}})$ in $\mathbb{Z}[x]$. Thus, by Example 7.2, we cannot pack any copies of $\mathbb{Z}_{2^k}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ while satisfying surjectivity and degree-2 homomorphism.

Thm. 7.1 also yields the impossibility of surjective RMFEs over Galois ring for $\mathbb{Z}_{p^k}$-messages.

Example 7.4. In $GR(p^t, d) \cong \mathbb{Z}_{p^t}[x]/f(x)$ with a degree-d $f(x)$ which is irreducible modulo p, we cannot pack any copy of $\mathbb{Z}_{p^k}$ while satisfying surjectivity, unless $d = 1$. That is, there is no meaningful surjective RMFE over Galois ring for $\mathbb{Z}_{p^k}$-messages.

On the other side, we have the following theorem with a constructive proof, which asserts that the necessary condition in Thm. 7.1 is also a sufficient one.

Theorem 7.2. *Suppose there are r linear factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{Z}_{p^k}[x]$ which are mutually distinct modulo p. Then, there exists a level-consistent surjective packing method $\mathbb{Z}_{p^k}^r$ into $\mathbb{Z}_{p^t}[x]/f(x)$.*

Proof. Let $g(x) \in \mathbb{Z}_{p^k}[x]$ be the product of such r linear factors of $f(x)$ in $\mathbb{Z}_{p^k}[x]$. Then, there is a CRT ring isomophism $\psi : \mathbb{Z}_{p^k}^r \xrightarrow{\cong} \mathbb{Z}_{p^k}[x]/g(x)$. Let π_k and ι_k denote the projection and injection between $\mathbb{Z}_{p^t}[x]/f(x)$ and $\mathbb{Z}_{p^k}[x]/f(x)$, and let π_g and ι_g denote those of $\mathbb{Z}_{p^k}[x]/f(x)$ and $\mathbb{Z}_{p^k}[x]/g(x)$ respectively.

Define $\mathsf{Pack} := \iota_k \circ \iota_g \circ \psi$ and $\mathsf{Unpack} := \psi^{-1} \circ \pi_h \circ \pi_k$ (Fig. 7). Then, it is straightforward that $(\mathsf{Pack}, \mathsf{Unpack})$ is a level-consistent surjective packing method. □

Corollary 7.1. *Let r be the number of linear factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{Z}_{p^k}[x]$ which are mutually distinct modulo p. For $D > 1$, there exists a degree-D surjective packing method $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ if and only if $n \leq r$.*

Proof. Straightforward from Thm. 7.1 and 7.2. □

The following corollary suggests that surjectivity is a somewhat stronger notion than level-consistency for $\mathbb{Z}_{p^k}$-message packings.

Corollary 7.2. *For $D > 1$, if there exists a degree-D surjective packing method for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$, then there exists a level-consistent surjective packing method for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$.*

Proof. Straightforward from Thm. 7.1 and 7.2. □

$$\begin{tikzcd}
\mathbb{Z}_{p^k}^r \arrow[r, "\mathsf{Pack}"] \arrow[ddr, "\psi"'] & \mathbb{Z}_{p^t}[x]/f(x) \\
 & \mathbb{Z}_{p^k}[x]/f(x) \arrow[u, "\iota_k"'] \\
 & \mathbb{Z}_{p^k}[x]/g(x) \arrow[u, "\iota_g"']
\end{tikzcd}$$

(a) Pack

$$\begin{tikzcd}
\mathbb{Z}_{p^k}^r & \mathbb{Z}_{p^t}[x]/f(x) \arrow[l, "\mathsf{Unpack}"'] \arrow[d, "\pi_k"] \\
 & \mathbb{Z}_{p^k}[x]/f(x) \arrow[d, "\pi_g"] \\
 & \mathbb{Z}_{p^k}[x]/g(x) \arrow[uul, "\psi^{-1}"]
\end{tikzcd}$$

(b) Unpack

Fig. 7: Definitions of Pack and Unpack in Thm. 7.2

7.3 Surjectivity in $\mathbb{F}_{p^k}$-Message Packings

Our main result on surjectivity in $\mathbb{F}_{p^k}$-message packings is the following theorem. It is a finite field analogue of Thm. 7.1 which is on $\mathbb{Z}_{p^k}$-message packings. Our theorem illustrates a necessary condition for a surjective packing method for $\mathbb{F}_{p^k}$-messages to exist.

Theorem 7.3. *Let r be the number of distinct degree-k irreducible factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{F}_p[x]$. For $D > 1$, there exists a degree-D surjective packing method $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ only if $n \leq r$.*

Proof. See Section 7.5. □

The following are some consequences of Thm. 7.3. They illustrate the hardness of designing an efficient HE packing method for $\mathbb{F}_{2^k}$-messages while satisfying surjectivity. We have similar results for $\mathbb{F}_{p^k}$-messages with $p \neq 2$.

Example 7.5. When $M = 2^m$, since $\Phi_M(x) = (x+1)^{2^{m-1}}$ in $\mathbb{F}_2[x]$, we can only pack copies of $\mathbb{F}_2$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ while satisfying surjectivity and degree-2 homomorphism. Even in that case, we can pack at most one copy of $\mathbb{F}_2$.

Example 7.6. When M is an odd, $\Phi_M(x)$ factors into a product of distinct irreducible polynomials of degree $d = \mathrm{ord}_M(2)$ in $\mathbb{F}_2[x]$. Let $\phi(M) = r \cdot d$. Then, we can only pack copies of $\mathbb{F}_{2^d}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ while satisfying surjectivity and degree-2 homomorphism. In that case, we can pack at most r copies of $\mathbb{F}_{2^d}$. Note that, since $d > \log M$ by definition, $r < \phi(M)/\log M$.

For instance, if one wants to pack $\mathbb{F}_{2^8}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ with an odd M while satisfying the conditions, then one must choose M such that $\mathrm{ord}_M(2) = 8$. However, such M cannot be larger than $(2^8 - 1)$ and might be too small for a secure parameter of HE.

Example 7.7. When $M = 2^s \cdot M'$, where M' is an odd, $\Phi_M(x) = \Phi_{M'}(-x^{2^{s-1}}) = \Phi_{M'}(x)^{2^{s-1}}$ in $\mathbb{F}_2[x]$. Thus, we cannot pack more copies of $\mathbb{F}_{2^k}$ into $\mathbb{Z}_{2^t}[x]/\Phi_M(x)$ than $\mathbb{Z}_{2^t}[x]/\Phi_{M'}(x)$ while satisfying surjectivity and degree-2 homomorphism.

Meanwhile, using such M can be useful when packing copies of a small field: it enables to meet certain level of HE security by enlarging the degree of the ring. See Example 7.6.

Thm. 7.3 also yields the impossibility of surjective RMFEs.

Example 7.8. In $\mathbb{F}_{p^d} \cong \mathbb{Z}_p[x]/f(x)$ with a degree-d irreducible $f(x)$, we cannot pack even a single copy of $\mathbb{F}_{p^k}$ while satisfying surjectivity and degree-2 homomorphism, if $k \neq d$. That is, there is no meaningful surjective RMFE.

On the other side, we have the following theorem with a constructive proof, which asserts that the necessary condition in Thm. 7.3 is also a sufficient one.

Theorem 7.4. *If there are r distinct degree-k irreducible factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{F}_p[x]$, then there exists a level-consistent surjective packing method $\mathbb{F}_{p^k}^r$ into $\mathbb{Z}_{p^t}[x]/f(x)$.*

Proof. Let $g(x) \in \mathbb{F}_p[x]$ be the product of r distinct degree-k irreducible factors of $f(x)$ in $\mathbb{F}_p[x]$. Then, there is a ring isomophism $\psi : \mathbb{F}_{p^k}^r \xrightarrow{\cong} \mathbb{F}_p[x]/g(x)$. Let π_p and ι_p denote the projection and injection between $\mathbb{Z}_{p^k}[x]/f(x)$ and $\mathbb{F}_p[x]/f(x)$, and let π_g and ι_g denote those of $\mathbb{F}_p[x]/f(x)$ and $\mathbb{F}_p[x]/g(x)$ respectively.

Define $\mathsf{Pack} := \iota_p \circ \iota_g \circ \psi$ and $\mathsf{Unpack} := \psi^{-1} \circ \pi_h \circ \pi_p$ (Fig. 8). Then, it is straightforward that $(\mathsf{Pack}, \mathsf{Unpack})$ is a level-consistent surjective packing method. □

$$\begin{tikzcd}
\mathbb{F}_{p^k}^r \arrow[r, "\mathsf{Pack}"] \arrow[rdd, "\psi"'] & \mathbb{Z}_{p^t}[x]/f(x) \\
 & \mathbb{F}_p[x]/f(x) \arrow[u, "\iota_p"'] \\
 & \mathbb{F}_p[x]/g(x) \arrow[u, "\iota_g"']
\end{tikzcd}$$

(a) Pack

$$\begin{tikzcd}
\mathbb{F}_{p^k}^r & \mathbb{Z}_{p^t}[x]/f(x) \arrow[l, "\mathsf{Unpack}"'] \arrow[d, "\pi_p"] \\
 & \mathbb{F}_p[x]/f(x) \arrow[d, "\pi_g"] \\
 & \mathbb{F}_p[x]/g(x) \arrow[luu, "\psi^{-1}"]
\end{tikzcd}$$

(b) Unpack

Fig. 8: Definitions of Pack and Unpack in Thm. 7.4

Corollary 7.3. *Let r be the number of distinct degree-k irreducible factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{F}_p[x]$. For $D > 1$, there exists a degree-D surjective packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ if and only if $n \leq r$.*

Proof. Straightforward from Thm. 7.3 and 7.4. □

The following corollary suggests that surjectivity is a somewhat stronger notion than level-consistency, also in the $\mathbb{F}_{p^k}$ case.

Corollary 7.4. *For $D > 1$, if there exists a degree-D surjective packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$, then there exists a level-consistent surjective packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$.*

Proof. Straightforward from Thm. 7.3 and 7.4. □

7.4 Proof of Thm. 7.1

In this subsection, we prove Thm. 7.1. The proof is elementary, but consists of a number of steps. As mentioned, Prop. 7.2 plays an important role in the proof.

Theorem 7.1. *Let $\check{r}$ be the number of linear factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{Z}_{p^k}[x]$ which are mutually distinct modulo p. For $D > 1$, there exists a degree-D surjective packing method $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ only if $n \leq \check{r}$.*

Proof. Straightforward from Lem. 7.1, 7.2, and 7.3. □

Lemma 7.1. *For $D > 1$, if there exists a degree-D surjective packing method for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$, then there exists a degree-D surjective packing method for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^k}[x]/f(x)$.*

Proof. Let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ be a degree-D surjective packing method for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$. For all $b(x) \in \mathbb{Z}_{p^t}[x]/f(x)$, since $\mathsf{Unpack}_i(b(x)) = \boldsymbol{b}$ for some $\boldsymbol{b} \in \mathbb{Z}_{p^k}^n$ by surjectivity(Prop. 7.1), $\mathsf{Unpack}_i(p^k \cdot b(x)) = \mathbf{0}$ holds. Thus, at any level-i and for all $a(x) \in \mathbb{Z}_{p^t}[x]/f(x)$, $\mathsf{Unpack}_i(a(x))$ is fully determined by $a(x) \bmod p^k$.

Let $\mathsf{Pack}'_i = \pi_k \circ \mathsf{Pack}_i$ and $\mathsf{Unpack}'_i = \mathsf{Unpack}_i \circ \iota_k$, where π_k and ι_k denote the projection and injection between $\mathbb{Z}_{p^t}[x]/f(x)$ and $\mathbb{Z}_{p^k}[x]/f(x)$ respectively. Then, it is straightforward that $(\mathsf{Pack}'_i, \mathsf{Unpack}'_i)_{i=1}^D$ is a degree-D surjective packing method. □

For the remaining parts of this subsection, let $f(x)$ be a monic polynomial in $\mathbb{Z}_{p^k}[x]$, and let $\mathcal{R} := \mathbb{Z}_{p^k}[x]/f(x)$. Let $f(x)$ be factorized into $\prod_{i=1}^r g_i(x)^{\ell_i}$ in $\mathbb{F}_p[x]$, where each $g_i(x)$ is distinct irreducible polynomial in $\mathbb{F}_p[x]$. The factorization can be lifted upto $\mathbb{Z}_{p^k}[x]$ via Hensel lifting. Let $f(x) = \prod_{i=1}^r f_i(x)$, where $f_i(x) \in \mathbb{Z}_{p^k}[x]$ is the Hensel lift of $g_i(x)^{\ell_i}$ satisfying $f_i(x) = g_i(x)^{\ell_i} \pmod p$. Let $d_j := \deg(f_i)$. Then, we can identify $\mathcal{R}$ with $\prod_{i=1}^r \mathbb{Z}_{p^k}[x]/f_i(x)$ via the CRT ring isomorphism. We denote as $\mathcal{R}_i$ for the subring of $\mathcal{R}$ which is isomorphic to $\mathbb{Z}_{p^k}[x]/f_i(x)$ according to the CRT isomorphism. Let Z be the zero-set ideal defined in Prop. 7.2

Lemma 7.2. *Let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ be a degree-D surjective packing method for $\mathbb{Z}_{p^k}^n$ into $\mathcal{R}$, for some $D > 1$. For each standard unit vector $\boldsymbol{e}_i \in \mathbb{Z}_{p^k}^n$, there exists $a_i(x)$ such that $\mathsf{Unpack}_1(a_i(x)) = \boldsymbol{e}_i$ and $a_i(x) \in \mathcal{R}_j$ for some $j \in [r]$. Moreover, such $a_i(x)$ is a unit in $\mathcal{R}_j$, and such j is distinct for all i's.*

Proof. Let $a_i(x)$ satisfy $\mathsf{Unpack}_1(a_i(x)) = \boldsymbol{e}_i$, and let $A_i \subset [r]$ be the set of j's such that $a_i(x)$ is non-zero modulo $f_j(x)$. Without loss of generality, assume that $a_i(x)$ has the smallest such subset of $[r]$, among the elements satisfying $\mathsf{Unpack}_1(\cdot) = \boldsymbol{e}_i$.

Step 1: Suppose, for $j \in A_i$, $a_i(x) \pmod{f_j(x)}$ is a non-unit in $\mathbb{Z}_{p^k}[x]/f_j(x)$. Let $\mathsf{Unpack}_1(e_j(x))$ outputs an element in $\mathbb{Z}_{p^k}^n$ with c_i in its i-th coordinate.

Then, at level-1, $(e_j(x) - c_i \cdot a_i(x))$ unpacks to an element with 0 in its i-th coordinate. Thus, the following holds.

$$\mathsf{Unpack}_2\Big(a_i(x) \cdot \big(e_j(x) - c_i \cdot a_i(x)\big)\Big) = \mathbf{0}$$

That is, $a_i(x) \cdot (e_j(x) - c_i \cdot a_i(x)) \in Z$. Meanwhile, notice that $(e_j(x) - c_i \cdot a_i(x))$ (mod $f_j(x)$) is a unit, since $\mathbb{Z}_{p^k}[x]/f_j(x)$ is a local ring. Therefore, $a_i(x) \cdot e_j(x) \in Z$ and $a_i(x) - a_i(x) \cdot e_j(x)$ unpacks to $\boldsymbol{e}_i$ at level-1, contradicting the assumption on the size of A_i. Thus, for all $j \in A_i$, $a_i(x) \pmod{f_j(x)}$ must be a multiplicative unit in $\mathbb{Z}_{p^k}[x]/f_j(x)$.

Step 2: Consider $e_j(x) \cdot a_i(x) \in \mathcal{R}_j$, and let $\mathsf{Unpack}_1(e_j(x) \cdot a_i(x))$ outputs an element in $\mathbb{Z}_{p^k}^n$ with $\tilde{c}_i$ in its i-th coordinate. Then, at level-1, $(e_j(x) \cdot a_i(x) - \tilde{c}_i \cdot a_i(x))$ unpacks to an element with 0 in its i-th coordinate. Thus, the following holds.

$$\mathsf{Unpack}_2\Big(a_i(x) \cdot \big(e_j(x) \cdot a_i(x) - \tilde{c}_i \cdot a_i(x)\big)\Big) = \mathbf{0}$$

That is, $a_i(x)^2 \cdot (e_j(x) - \tilde{c}_i) \in Z$, and therefore $a_i(x) \cdot (e_j(x) - \tilde{c}_i) \in Z$ since $a_i(x)$ (mod $f_\jmath(x)$) is a multiplicative unit in $\mathbb{Z}_{p^k}[x]/f_\jmath(x)$ for all $\jmath \in A_i$ by Step 1. Consequently, it holds that $\mathsf{Unpack}_1(e_j(x) \cdot a_i(x)) = \mathsf{Unpack}_1(\tilde{c}_i \cdot a_i(x)) = \tilde{c}_i \cdot \boldsymbol{e}_i$.

Suppose $\tilde{c}_i \in \mathbb{Z}_{p^k}$ is a non-unit. Then, $(1 - \tilde{c}_i)$ is a unit, and $(1 - \tilde{c}_i)^{-1} \cdot (a_i(x) - e_j(x) \cdot a_i(x))$ unpacks to $\boldsymbol{e}_i$ at level-1 contradicting the assumption on the size of A_i. Thus, $\tilde{c}_i$ is a unit. Then, $\tilde{c}_i^{-1} \cdot e_j(x) \cdot a_i(x)$ unpacks to $\boldsymbol{e}_i$ at level-1 satisfying the desired conditions.

Step 3: Suppose $a_{i'}(x)$ is also in $\mathcal{R}_j$ and unpacks to a standard unit vector $\boldsymbol{e}_{i'}$ at level-1. Then, $a_i(x) \cdot a_{i'}(x)$ unpacks to $\mathbf{0}$ at level-2, and therefore $a_i(x) \cdot a_{i'}(x) \in Z$. However, since $a_i(x)$ and $a_{i'}(x)$ are both units in $\mathcal{R}_j$, all elements of $\mathcal{R}_j$ must be included in Z, leading to a contradiction. □

Lemma 7.3. *Let* $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ *be a degree-D surjective packing method for* $\mathbb{Z}_{p^k}^n$ *into* $\mathcal{R}$*, for some* $D > 1$*. If there exists* $a_i(x) \in \mathcal{R}_j$ *which satisfies* $\mathsf{Unpack}_1(a_i(x)) = \boldsymbol{e}_i$ *for a standard unit vector* $\boldsymbol{e}_i \in \mathbb{Z}_{p^k}^n$*, then* $f_j(x)$ *has a linear factor in* $\mathbb{Z}_{p^k}[x]$*.*

Proof. Consider $x \cdot a_i(x) \in \mathcal{R}_j$. Suppose $\mathsf{Unpack}_1(x \cdot a_i(x))$ outputs an element in $\mathbb{Z}_{p^k}^n$ with c_i in its i-th coordinate. Then, at level-1, $(x \cdot a_i(x) - c_i \cdot a_i(x))$ unpacks to an element with 0 in its i-th coordinate. Thus, the following holds.

$$\mathsf{Unpack}_2\Big(\big(x \cdot a_i(x) - c_i \cdot a_i(x)\big) \cdot a_i(x)\Big) = \mathsf{Unpack}_2\Big((x - c_i) \cdot a_i(x)^2\Big) = \mathbf{0}$$

That is, $(x - c_i) \cdot a_i(x)^2 \in Z$, and therefore $(x - c_i) \cdot e_j(x) \in Z$ as $a_i(x)$ is a unit in $\mathcal{R}_j$ (Lem. 7.2).

Now consider $(x - c_i) \in \mathbb{Z}_{p^k}[x]/f_j(x)$ and the ideal $\langle x - c_i \rangle \subset \mathbb{Z}_{p^k}[x]/f_j(x)$ generated by it. The ideal $\langle x - c_i \rangle$ contains at least $p^{k \cdot (d_j - 1)}$ elements, namely $(x - c_i) \cdot h(x)$'s for $h(x) \in \mathbb{Z}_{p^k}[x]$ with $\deg(h) < d_j - 1$, which are multiples of $(x - c_i)$ in $\mathbb{Z}[x]$. On the other hand, $\langle x - c_i \rangle$ cannot contain more than $p^{k \cdot d_j}/p^k$

elements: this is because $\mathcal{R}_j$ must contain p^k distinct elements modulo Z, namely $c \cdot a_i(x)$'s for $c \in \mathbb{Z}_{p^k}$. Thus, $|\langle x - c_i \rangle| = p^{k \cdot (d_j - 1)}$ holds. In particular, it must hold that $(x - c_i)^{d_j} - f_j(x)$ is a multiple of $(x - c_i)$ in $\mathbb{Z}[x]$. Consequently, $f_j(x)$ has a linear factor $(x - c_i)$ in $\mathbb{Z}_{p^k}[x]$. □

7.5 Proof of Thm. 7.3

In this subsection, we prove Thm. 7.3. The proof is elementary, but consists of a number of steps. As mentioned, Prop. 7.2 plays an important role in the proof.

Theorem 7.3. *Let r be the number of distinct degree-k irreducible factors of $f(x) \in \mathbb{Z}_{p^t}[x]$ in $\mathbb{F}_p[x]$. For $D > 1$, there exists a degree-D surjective packing method $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$ only if $n \le r$.*

Proof. Straightforward from Lem. 7.4, 7.5, and 7.6. □

Lemma 7.4. *For $D > 1$, if there exists a degree-D surjective packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$, then there exists a degree-D surjective packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/f(x)$.*

Proof. Let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ be a degree-D surjective packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$. For all $b(x) \in \mathbb{Z}_{p^t}[x]/f(x)$, since $\mathsf{Unpack}_i(b(x)) = \boldsymbol{b}$ for some $\boldsymbol{b} \in \mathbb{F}_{p^k}^n$ by surjectivity, $\mathsf{Unpack}_i(p \cdot b(x)) = \mathbf{0}$ holds. Thus, at any level-i and for all $a(x) \in \mathbb{Z}_{p^t}[x]/f(x)$, $\mathsf{Unpack}_i(a(x))$ is fully determined by $a(x) \bmod p$.

Let $\mathsf{Pack}'_i = \pi_p \circ \mathsf{Pack}_i$ and $\mathsf{Unpack}'_i = \mathsf{Unpack}_i \circ \iota_p$, where π_p and ι_p denote the projection and injection between $\mathbb{Z}_{p^t}[x]/f(x)$ and $\mathbb{F}_p[x]/f(x)$ respectively. Then, it is straightforward that $(\mathsf{Pack}'_i, \mathsf{Unpack}'_i)_{i=1}^D$ is a degree-D surjective packing method. □

Lemma 7.5. *For $D > 1$, if there exists a degree-D surjective packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R} := \mathbb{F}_p[x]/f(x)$, then there exists $g(x) \in \mathbb{F}_p[x]$ which divides $f(x)$, is of degree $k \cdot n$, and allows a degree-D packing method for $\mathbb{F}_{p^k}^n$ into $\mathbb{F}_p[x]/g(x)$.*

Proof. Let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^D$ be a degree-D surjective packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R}$. By Prop. 7.2, the sets Z_i consisting of elements $a(x) \in \mathcal{R}$ such that $\mathsf{Unpack}_i(a(x)) = \mathbf{0}$ coincide with an ideal $Z = \check{g}(x) \cdot \mathcal{R}$ for some $\check{g}(x) \in \mathbb{F}_p[x]$ which divides $f(x)$, as cR is a principal ideal ring. Let $g(x) := f(x)/\check{g}(x)$. Then, $\mathcal{R}/Z \cong \mathbb{F}_p[x]/g(x)$, and therefore $\deg(g) = k \cdot n$ since $|\mathcal{R}/Z| = p^{kn}$. Moreover, at any level-i and for all $a(x) \in \mathbb{F}_p[x]/f(x)$, $\mathsf{Unpack}_i(a(x))$ is fully determined by $a(x) \bmod g(x)$.

Let $\mathsf{Pack}'_i = \pi_g \circ \mathsf{Pack}_i$ and $\mathsf{Unpack}'_i = \mathsf{Unpack}_i \circ \iota_g$, where π_g and ι_g denote the projection and injection between $\mathbb{F}_p[x]/f(x)$ and $\mathbb{F}_p[x]/g(x)$ respectively. Then, it is straightforward that $(\mathsf{Pack}'_i, \mathsf{Unpack}'_i)_{i=1}^D$ is a degree-D packing method. □

Lemma 7.6. *For $D > 1$, there exists a degree-D packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R} := \mathbb{F}_p[x]/g(x)$, where $g(x) \in \mathbb{F}_p[x]$ is a polynomial of degree $k \cdot n$, only if $g(x)$ factors into n distinct degree-k irreducible polynomials.*

Proof. **Step 1:** Let $(\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ be a degree-D packing method for $\mathbb{F}_{p^k}^n$ into $\mathcal{R}$. Since $|\mathbb{F}_{p^k}^n| = |\mathcal{R}|$, all Pack_i and Unpack_i are bijective, and $0 \in \mathcal{R}$ is the only element which packs $\mathbf{0}$ at each level-i. Thus, $a(x) \cdot b(x) = 0$ if and only if $\boldsymbol{a} \odot \boldsymbol{b} = \mathbf{0}$, where $\boldsymbol{a} := \mathsf{Unpack}_1(a(x))$ and similar for $\boldsymbol{b}$, since $\mathsf{Unpack}_1(a(x)) \odot \mathsf{Unpack}_1(b(x)) = \mathsf{Unpack}_2(a(x) \cdot b(x))$.

Step 2: Suppose $g(x)$ is not square-free. Then, there exists a non-zero $a(x) \in \mathcal{R}$ such that $a(x)^2 = 0$. By Step 1, $\boldsymbol{a}^2 = \mathbf{0}$, where $\boldsymbol{a} := \mathsf{Unpack}_1(a(x))$. However, there is no non-zero $\boldsymbol{a} \in \mathbb{F}_{p^k}^n$ satisfying $\boldsymbol{a}^2 = \mathbf{0}$. Thus, $g(x)$ is square-free. Let $g(x)$ factorizes into r distinct irreducible polynomials $\{g_i(x)\}_{i=1}^r$ and let $d_i := \deg(g_i)$. We identify $\mathbb{F}_p[x]/g(x)$ with $\prod_{i=1}^r \mathbb{F}_p[x]/g_i(x)$.

Step 3: Note that for any $\boldsymbol{a} \in \mathbb{F}_{p^k}^n$ with s zero-coordinates, there are p^{ks} elements in $\mathbb{F}_{p^k}^n$ whose Hadamard product with $\boldsymbol{a}$ is $\mathbf{0}$. Then, consider $\check{e}_i(x) \in \mathbb{F}_p[x]/g(x)$ which corresponds to the vector of polynomials in $\prod_{i=1}^r \mathbb{F}_p[x]/g_i(x)$ with 0 in its i-th coordinate and 1 in the others. Observe that there are p^{d_i} elements in $\mathbb{F}_p[x]/g(x)$ whose product with $\check{e}_i(x)$ is 0. By Step 1 and the above facts, $p^{d_i} = p^{ks}$ for some s. Thus, the degree d_i is a positive multiple of k and we can let $d_i := kc_i$ where $\sum_{i=1}^r c_i = n$.

Step 4: By Step 1, the number of zero-divisors in $\mathbb{F}_{p^k}^n$ and $\mathcal{R}$ must be same. The number of elements which are not zero-divisors in $\mathbb{F}_{p^k}^n$ is $(p^k - 1)^n$. Meanwhile, the number of elements which are not zero-divisors in $\mathcal{R}$ is $\prod_{i=1}^r (p^{d_i} - 1)$. Thus, the following must hold.

$$\prod_{i=1}^{r} (p^{d_i} - 1) = (p^k - 1)^n = \prod_{i=1}^{r} (p^k - 1)^{c_i}$$

Observe that $p^{d_i} - 1 \geq (p^k - 1)^{c_i}$ holds, where the equality holds if and only if $c_i = 1$. Thus, $d_i = k$ for all $1 \leq i \leq r$. □

8 Open Problems

(i) The authors believe that the statement of Thm. 5.1 and 5.3 are also true in the general case of packing methods for $\mathbb{Z}_{p^k}$-messages into $\mathbb{Z}_{p^t}[x]/f(x)$ with $t \neq k$. However, we could not eventually prove it or come up with a counterexample (See also Example 5.1, 5.8, and 5.9). We would be happy to see a proof for these generalized conjectures or a counterexample (e.g. a degree-2 packing method for $\mathbb{Z}_{p^k}^n$ into $\mathbb{Z}_{p^t}[x]/f(x)$, where $f(x)$ is irreducible in $\mathbb{F}_p[x]$, achieving density of larger than $1/2 + 1/(2\deg(f))$).

(ii) The authors believe that CRT approach is optimal for packing density (Section 4.5). However, we could not eventually prove it or come up with a counterexample (See also Example 5.6 and Remark 5.4). We would be happy to see a stronger version of Thm. 5.2 and 5.4, or a packing method which is not based on CRT approach but outperforms known packing methods.

(iii) Although MHz2k packing offers nearly optimal packing density of 1/2 in some circumstances, its density is highly dependent on the degree of the quotient polynomials. On the other hand, recent RMFE constructions [CCXY18] offer asymptotically optimal density, namely $\Theta(1)$ regarding the degree of quotient polynomial. However, in a concrete manner they are substantially below the optimal density of 1/2, namely below 1/4 (when packing prime field elements[14]). There is a gap between asymptotically optimal constructions and concretely (near)-optimal constructions. Also, there is a substantial gap between upper bounds and the achieved packing densities in general parameters. We wound be happy to see better upper bounds on packing density or new constructions of packing methods with better density, reducing these gaps.

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