

# Optimal encodings to elliptic curves of $j$ -invariants 0, 1728

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**Abstract.** This article provides new constant-time encodings  $\mathbb{F}_q^* \rightarrow E(\mathbb{F}_q)$  to ordinary elliptic  $\mathbb{F}_q$ -curves  $E$  of  $j$ -invariants 0, 1728 having a small prime divisor of the Frobenius trace. Therefore all curves of  $j = 1728$  are covered. This circumstance is also true for the Barreto–Naehrig curves BN512, BN638 from the international cryptographic standards ISO/IEC 15946-5, TCG Algorithm Registry, and FIDO ECDA Algorithm. Many  $j = 1728$  curves as well as BN512, BN638 are not appropriate for the most efficient prior encodings. So, in fact, only universal SW (Shallue–van de Woestijne) one was previously applicable to them. However this encoding (in contrast to the new ones) cannot be computed at the cost of one exponentiation in the field  $\mathbb{F}_q$ .

**Key words:** congruent elliptic curves, encodings to (hyper)elliptic curves, isogenies,  $j$ -invariants 0, 1728, median value curves, optimal covers, Weil pairing

## 0 Introduction

Let  $\mathbb{F}_q$  be a finite field of characteristic  $p > 5$  and  $E = E_{a,b}: y^2 = x^3 - ax + b$  be an elliptic  $\mathbb{F}_q$ -curve. Many protocols of elliptic cryptography use a *hash function* [1, §3] of the form  $\mathcal{H}: \{0, 1\}^* \rightarrow E(\mathbb{F}_q)$ . It is often constructed with the help of an auxiliary map  $h: \mathbb{P}^1(\mathbb{F}_q) \rightarrow E(\mathbb{F}_q)$ , called *encoding*, such that  $\#\text{Im}(h) \geq (q+1)/n$  for some  $n \in \mathbb{N}$ . Clearly, the smaller the value  $n$ , the better, because  $h$  covers more  $\mathbb{F}_q$ -points. In this regard, it is worth recalling Hasse’s bound, which says that  $|t| \leq 2\sqrt{q}$  for the Frobenius trace  $t = q + 1 - N$ , where  $N := \#E(\mathbb{F}_q)$ . Good surveys on how to hash into elliptic curves are presented in [2, §8], [3].

In practice,  $h$  needs to be computed in constant time, otherwise it is vulnerable to timing attacks [2, §8.2.2, §12.1.1]. Besides, it is more convenient to restrict  $h$  to the multiplicative group  $\mathbb{F}_q^* = \mathbb{P}^1(\mathbb{F}_q) \setminus \{0, \infty\}$ , where  $0 := (0 : 1)$  and  $\infty := (1 : 0)$ . The point is that, as a rule,  $h(0)$ ,  $h(\infty)$  are of small orders in the group  $E(\mathbb{F}_q)$ . Even if this is not the case, we miss maximum two “interesting” points. There are already (e.g., in [3, §5]) standard hash functions  $\eta: \{0, 1\}^* \rightarrow \mathbb{F}_q^*$ , hence the composition  $\mathcal{H} = h \circ \eta$  gives the desired hash function. If we additionally require  $\mathcal{H}$  to be a *random oracle* [1, §3.7], then according to [4] it is enough to apply  $h$  twice, varying  $\eta$ , and to sum the resulting points. In this case,  $h$  must be *well-distributed*, but we do not know of a single natural example that would not meet this criterion.

There is the *SW encoding* [2, §8.3.4], [3, §6.6.1], which is applicable to any elliptic curve. Exactly two exponentiations in  $\mathbb{F}_q$  are required to evaluate it if one takes into account folklore batching techniques from [5, §4.2] to avoid computing the Legendre symbol and inverse

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element during square root extraction in  $\mathbb{F}_q$ . In that article the case  $q \equiv 3 \pmod{4}$  is processed, but in fact the reasoning holds true at least for  $q \equiv 5 \pmod{8}$  as shown in [6, §2]. In contrast, all other known encodings, including those constructed in this article, make use of only one exponentiation. This is what is meant whenever we talk about the encoding efficiency throughout the article. Since the exponentiation operation dominates the cost of encodings, the achieved speed up seems meaningful. Moreover, some modern cryptographic protocols (like the aggregated BLS signature [7]) call the hash function  $\mathcal{H}$  many times. So the cumulative gain may be conspicuous.

If  $j$ -invariant of  $E$  is different from 0, 1728, i.e.,  $ab \neq 0$ , then one can apply the *simplified SWU encoding*  $h$  (see, e.g., [5, §2.4, §4.1]), which appears to be optimal. Also, consider the curves  $E_b: y^2 = x^3 + b$  and  $E_a: y^2 = x^3 - ax$  of  $j$ -invariants 0, 1728 respectively. Having an  $\mathbb{F}_q$ -isogeny of small degree  $\varphi: E \rightarrow E_b$  (resp.  $\varphi: E \rightarrow E_a$ ) that is vertical (i.e.,  $j(E) \neq 0, 1728$ ), we obviously obtain the fast encoding  $\varphi \circ h$  to  $E_b$  (resp.  $E_a$ ). This was first seen in [5, §4]. In particular, it is a simple exercise that such an isogeny of degree 2 exists if and only if  $\sqrt[3]{b} \in \mathbb{F}_q$  (resp.  $\sqrt{a} \in \mathbb{F}_q$ ). Therefore, without loss of generality, we can focus only on curves  $E_b, E_a$  not satisfying the last conditions. We need encodings to such curves as well, because among them exist many *pairing-friendly* ones [2, §4], [8].

For  $q \equiv 2 \pmod{3}$  (resp.  $q \equiv 3 \pmod{4}$ ) there is in [2, §8.3.2] (resp. [9]) a bijective encoding to  $E_b$  (resp.  $E_a$ ). The former is said to be the *Boneh–Franklin encoding*. The given curves are so-called *median value curves* [10, §3.4], that is for them  $N = q + 1$  or, equivalently,  $t = 0$ . As a consequence of [1, Theorem 9.11.2], they are supersingular. Although there are supersingular curves  $E_b, E_a$  with other orders  $N$ , to be definite in this article we will deal only with ordinary curves. The fact is that in discrete logarithm cryptography supersingular ones are considered to be weak [2, §4.3, §9.1.3]. However, many of our results for  $E_b$  (resp.  $E_a$ ) seem to hold true if  $q \equiv 1 \pmod{3}$  (resp.  $q \equiv 1 \pmod{4}$ ).

Let again  $E$  be any elliptic  $\mathbb{F}_q$ -curve. It is natural to wonder about non-constant  $\mathbb{F}_q$ -covers  $\varphi: C \rightarrow E$  of small degree by (smooth projective absolutely irreducible) curves  $C$  of greater genus  $g$  for which there is an efficient encoding  $\mathbb{P}^1(\mathbb{F}_q) \rightarrow C(\mathbb{F}_q)$ . To our knowledge, there are two types of such curves, namely *cyclic trigonal curves*  $T$ , also known as *trielliptic*, (see, e.g., [11, §2]) for  $q \equiv 2 \pmod{3}$  and so-called *odd hyperelliptic curves*  $H$  [9, §2] for  $q \equiv 3 \pmod{4}$ . One of the covers of the first type is implicitly proposed by Icart in [12, §2] (see also [13]). In turn, covers of the second type (with  $g = \deg(\varphi) = 2$ ) are constructed by Fouque–Joux–Tibouchi in [14, §3] under the additional condition  $4 \mid N$ . The encodings to  $T, H$  are trivial generalizations of the encodings to the median value curves  $E_b, E_a$  respectively.

More precisely, the curves are given by equations  $H: y^2 = f(x)$  and  $T: x^3 = f_1(y)f_2(y)^2$ , where  $f, f_1, f_2$  are  $\mathbb{F}_q$ -polynomials without multiple roots ( $f$  is in addition odd). Whenever  $q \equiv 2 \pmod{3}$ , the projection to the  $y$ -coordinate  $pr_y: T \rightarrow \mathbb{P}^1$  is *exceptional*, i.e.,  $pr_y: T(\mathbb{F}_q) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{F}_q)$ . And its inverse map  $pr_y^{-1}$  is nothing but an encoding to  $T$ . In turn, the projection to the  $x$ -coordinate  $pr_x: H \rightarrow \mathbb{P}^1$  is evidently not exceptional. As far as we know, there is a still unsolved problem [10, Question 3.10] under what conditions (besides  $\#H(\mathbb{F}_q) = q + 1$ ) the curve  $H$  has an exceptional cover to  $\mathbb{P}^1$ . Nevertheless, a response to this question is neither sufficient nor necessary for existence of a fast bijective encoding to  $H$ . By the way, the curves  $T, H$  are not necessarily supersingular, because in contrast to the genus 1 case this property equally depends on their orders over extensions of  $\mathbb{F}_q$  (cf. [10, Example 3.15]).

## 0.1 Background for the new encodings

Recall a series of notions and results, which can be found in [15, §1], [16, §1-2] and which will soon be used. Elliptic  $\mathbb{F}_q$ -curves  $E, E'$  are called *n-congruent* (where  $p \nmid n \in \mathbb{N}$ ) if there is an isomorphism  $\tau: E[n] \simeq E'[n]$  of the Frobenius modules. Then  $\tau$  is said to be an *anti-isometry* (and  $E, E'$  are *reversely n-congruent*) with respect to the Weil pairing  $e_n$  whenever  $e_n(\tau(P_0), \tau(P_1)) = e_n^{-1}(P_0, P_1)$  for all points  $P_0, P_1 \in E[n]$ . The last identity exactly means that the graph  $\Gamma_\tau$  of  $\tau$  is a *maximal isotropic subgroup* with respect to the Weil pairing on  $A := E \times E'$ . Therefore the quotient map  $\widehat{\Phi}: A \rightarrow A/\Gamma_\tau$  is an  $\mathbb{F}_q$ -isogeny to a *principally polarized* abelian surface  $A/\Gamma_\tau$ . The mentioned construction is also referred to as *gluing* (or *tying*)  $E, E'$  along their *n-torsion subgroups* via  $\tau$ .

If  $A/\Gamma_\tau$  is isomorphic as PPAS to the Jacobian  $J$  of some curve  $H$ , then  $\tau$  is called *irreducible*. There is in [15, §2] the powerful Kani criterion of irreducibility. In this case, the dual isogeny  $\Phi: J \rightarrow A$  is the natural extension of some  $\mathbb{F}_q$ -covers  $\varphi: H \rightarrow E, \varphi': H \rightarrow E'$  of degree  $n$ . Moreover, they are *optimal*, i.e., there is no decomposition into non-trivial  $\mathbb{F}_q$ -covers  $H \rightarrow F, F \rightarrow E$  (resp.  $F \rightarrow E'$ ) for some elliptic curve  $F$ . In the literature one may also encounter the terms *maximal*, or vice versa, *minimal*. In addition to the optimality,  $\varphi, \varphi'$  are *complementary covers* to each other in the sense of [16, §2]. Conversely, any pair of such covers induces an  $\mathbb{F}_q$ -isogeny  $\Phi: J \rightarrow A$  and hence its dual  $\widehat{\Phi}: A \rightarrow J$ . Besides, the kernel of  $\widehat{\Phi}$  is the graph  $\Gamma_\tau$  of some (irreducible)  $\mathbb{F}_q$ -anti-isometry  $\tau: E[n] \simeq E'[n]$ .

From now on let  $E': cy^2 = x^3 - ax + b \simeq_{\mathbb{F}_q} E_{ac^2, bc^3}$  denote a twist of  $E$  of degree at most 2, where  $c \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$ . Surprisingly, for  $E = E_a$  and  $q \equiv 3 \pmod{4}$  this twist is trivial, i.e.,  $E \simeq_{\mathbb{F}_q} E'$ , because, without loss of generality, we can take  $c = -1$ . Due to [17, Lemma 1] this case is exceptional, that is why authors often neglect it. An equation of the non-trivial quadratic twist of  $E_a$  and other useful information about twists (not necessarily quadratic) of elliptic curves are provided in [18, §X.5-X.6]. As is known (e.g., from [1, Exercise 9.5.4]),  $-t$  is the Frobenius trace of  $E'$ . Consequently, the curves  $E, E'$  are not  $\mathbb{F}_q$ -isogenous whenever  $t \neq 0$  as assumed above. Therefore this pair of curves is never *trivial*, that is, a congruence  $E[n] \simeq E'[n]$  (if any) is not the restriction of an  $\mathbb{F}_q$ -isogeny  $E \rightarrow E'$ .

In the new terms, the Fouque–Joux–Tibouchi approach consists of tying  $E, E'$  (with the restrictions on  $q, N$ ) along the subgroup  $E[2] = E'[2]$  via an irreducible  $\mathbb{F}_q$ -(anti)-isometry  $\tau$ . Curiously, for the curves  $E_b, E_a$  such  $\tau$  exists if and only if, as before,  $\sqrt[3]{b} \in \mathbb{F}_q, \sqrt{a} \in \mathbb{F}_q$  respectively (see §3.1, §2.1). In fact, by virtue of [19, Proposition 3] the required  $\tau$  is easily constructed depending on  $\#E(\mathbb{F}_q)[2]$  for all elliptic  $\mathbb{F}_q$ -curves  $E$  of  $j \neq 0, 1728$ . The fact is that they do not have non-trivial automorphisms, hence any non-identical  $\tau$  is automatically irreducible. Furthermore, the resulting genus 2 curve  $H$  is a median value one according to [20, Equality 3.5].

The given article tries to extend the considered approach to greater degrees  $n$  in order to cover remaining curves  $E_b, E_a$ . First of all, we will analyse in what situation this is possible. Fortunately, for any curve  $E_a$  it is sufficient to take  $n \leq 4$  due to §2 (the general case  $n = 4$  is treated in §2.3). At the same time, for curves  $E_b$  the situation is more complicated. Among other things, we will generalize in §4 the class of odd hyperelliptic curves to a much wider one of median value curves. Moreover, for every representative  $H$  of this class we still have an efficient encoding  $h: \mathbb{P}^1(\mathbb{F}_q) \simeq H(\mathbb{F}_q)$ . We are interested in the smallest possible  $n$ , because obviously  $\#\text{Im}(\varphi \circ h) \geq (q+1)/n$ , not to mention that for smaller  $n$  formulas of the cover  $\varphi$

are more compact and faster to compute.

Although the second cover  $\varphi'$  is not used for encoding, it plays an important role. Indeed, if  $E$  is tied with an elliptic curve  $F$  not  $\mathbb{F}_q$ -isomorphic to  $E'$  (even if  $F$  is  $\mathbb{F}_q$ -isogenous to  $E'$ ), then the covering  $H$  of  $E, F$  is not quite “symmetric” (cf. Remark 2). At the moment, nobody knows how to encode into such  $H$  through one exponentiation in the field  $\mathbb{F}_q$ .

It is explained in §3.3 that, dealing with curves  $E_b$ , it is enough to restrict ourselves to prime degrees  $\ell = n \geq 5$ . In accordance with our Theorem 1 degree  $\ell$  (optimal) covers  $\varphi: H \rightarrow E_b, \varphi': H \rightarrow E'_b$  exist if and only if  $\ell \mid t$ . Unfortunately, there are curves  $E_b$  (even pairing-friendly) without small divisors of  $t$ . Nevertheless, in §3.2 we will study in detail the case  $\ell = 5$ , which is valid for some standardized *Barreto–Naehrig*  $\mathbb{F}_p$ -curves [2, Example 4.2]. We are talking about BN512 ( $b = 3$ ) and BN638 ( $b = 257$ ) from the standards [21], [22, §5.2.8], [23, §4.1]. By means of [1, Theorem 25.4.6] one can determine that the smallest (prime) degree of a vertical  $\mathbb{F}_p$ -isogeny for BN512 (resp. BN638) equals 1291 (resp. 1523). Thus the new encodings are the best known ones, as far as is known.

We will essentially improve results from the author’s article [24] (resp. preprint [6]), which implicitly provides non-optimal  $\mathbb{F}_q$ -covers of degree 8 (resp. 20) to the curves  $E_a, E'_a$  (resp.  $E_b, E'_b$  for the case  $5 \mid t$ ). This non-optimality was not noticed earlier. So in the light of the current article the previous ones lose relevance. By the way, there the author uses the language of rational  $\mathbb{F}_q$ -curves (and their parametrizations) on the *Kummer surface*  $A/[-1]$ . However, as is known (e.g., from [25]), it is equivalent to the language of  $\mathbb{F}_q$ -covers by hyperelliptic curves (not necessarily of genus 2).

## 1 Preliminaries

We continue to work with an ordinary elliptic curve  $E: y^2 = x^3 - ax + b$  over a finite field  $\mathbb{F}_q$  of characteristic  $p > 5$ . As is customary, let’s use the same symbol for  $E \subset \mathbb{A}_{(x,y)}^2$  and  $E \cup \mathcal{O} \subset \mathbb{P}^2$ , where  $\mathcal{O} := (0 : 1 : 0)$ . As said before,  $E': cy^2 = x^3 - ax + b$ , where  $c \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$ , stands for the (non-trivial) quadratic twist of  $E$ . In formulas instead of  $E'$  we will use the fairly standard notation  $E^c$  in order to stress the choice  $c$ . The corresponding  $\mathbb{F}_{q^2}$ -isomorphism has the form

$$\sigma: E \xrightarrow{\sim} E^c \quad (x, y) \mapsto (x, y/\sqrt{c}).$$

Let  $t$  (resp.  $-t$ ) be the Frobenius trace of  $E$  (resp.  $E'$ ). Recall that  $p \nmid t$  for ordinary curves according to one of their equivalent definitions. Since the traces of  $n$ -congruent elliptic curves coincide modulo  $n$ , we obtain the elementary

**Lemma 1.** *If the curves  $E, E'$  are  $n$ -congruent for  $n \in \mathbb{N}$  such that  $p \nmid n$ , then  $n \mid 2t$ .*

Also, we have

**Theorem 1.** *For every prime  $\ell \neq 2, p$  the following statements are equivalent:*

1.  $\ell \mid t$ ;
2. the curves  $E, E'$  are reversely  $\ell$ -congruent;

3. there is an irreducible  $\mathbb{F}_q$ -anti-isometry  $E[\ell] \simeq E'[\ell]$ ;
4.  $E$  has a vertical (in the sense of [1, Definition 25.4.2]) degree  $\ell$  isogeny defined over  $\mathbb{F}_{q^2}$ , but not over  $\mathbb{F}_q$ .

*Proof.* Denote by  $\text{Fr}$ ,  $\text{Fr}'$  the Frobenius endomorphisms on  $E$ ,  $E'$  respectively. By definition, the curves are  $\ell$ -congruent if and only if there is a group isomorphism  $\tau: E[\ell] \xrightarrow{\sim} E'[\ell]$  such that  $\text{Fr}' \circ \tau = \tau \circ \text{Fr}$ . Since  $E[\ell] \simeq E'[\ell] \simeq (\mathbb{Z}/\ell)^2$  as abstract groups, we can represent the maps  $\text{Fr}$ ,  $\text{Fr}'$ ,  $\tau$  by means of matrices  $M_{\text{Fr}}, M_{\text{Fr}'}, M_{\tau} \in \text{GL}_2(\mathbb{Z}/\ell)$ . Given a basis  $\{P_0, P_1\}$  of  $E[\ell]$ , it is natural to take  $\{\sigma(P_0), \sigma(P_1)\}$  as a basis of  $E'[\ell]$ . Without loss of generality, assume that  $M_{\text{Fr}}$  is the rational canonical form (also known as the Frobenius normal form). Since the characteristic polynomial of  $\text{Fr}$  equals  $\chi_{\text{Fr}}(x) = x^2 - tx + q$  and  $\text{Fr}' \circ \sigma = -\sigma \circ \text{Fr}$ , we obtain

$$M_{\text{Fr}} = \lambda I_2 \quad \text{or} \quad M_{\text{Fr}} = \begin{pmatrix} 0 & -q \\ 1 & t \end{pmatrix}, \quad M_{\text{Fr}'} = -M_{\text{Fr}}, \quad M_{\tau} = \begin{pmatrix} m_0 & m_1 \\ m_2 & m_3 \end{pmatrix}$$

for the unit matrix  $I_2$  and some  $\lambda, m_k \in \mathbb{Z}/\ell$ ,  $\lambda \neq 0$ . As is known,  $M_{\text{Fr}}$  depends on whether  $\chi_{\text{Fr}}$  coincides with the minimal polynomial of  $\text{Fr}$ .

To be clear, all the next equations are modulo  $\ell$ . The condition  $\text{Fr}' \circ \tau = \tau \circ \text{Fr}$  means that  $-M_{\text{Fr}'} \cdot M_{\tau} = M_{\tau} \cdot M_{\text{Fr}}$ , i.e.,  $M_{\text{Fr}} \neq \lambda I_2$  and

$$\begin{cases} -qm_2 = -m_1, \\ -qm_3 = qm_0 - tm_1, \\ m_0 + tm_2 = -m_3, \\ m_1 + tm_3 = qm_2 - tm_3. \end{cases} \Leftrightarrow \begin{cases} m_1 = qm_2, \\ -qm_3 = qm_0 - tqm_2, \\ m_0 + tm_2 = -m_3, \\ 2tm_3 = 0. \end{cases} \Leftrightarrow \begin{cases} m_1 = qm_2, \\ -m_3 = m_0 - tm_2, \\ -m_3 = m_0 + tm_2, \\ tm_3 = 0. \end{cases} \Leftrightarrow \begin{cases} m_1 = qm_2, \\ m_0 = -m_3, \\ t = 0. \end{cases}$$

The fact  $\text{tr}(\lambda I_2) = 2\lambda \neq 0$  implies that  $1 \Leftrightarrow 2$  whenever  $M_{\text{Fr}} = \lambda I_2$ . In the opposite case, it remains to prove the implication  $1 \Rightarrow 2$ . Notice that the congruence  $\tau$  is an anti-isometry if and only if  $\det(M_{\tau}) = -1$ . For  $m_0, m_1$  from the last linear system we get  $\det(M_{\tau}) = -(m_3^2 + qm_2^2)$ , hence it is sufficient to assign  $m_2 = 0$ ,  $m_3 = 1$ .

Putting  $F := E'$  in [26, Lemma 4.5], we conclude that  $\tau$  is never reducible, because the isomorphism  $\sigma: E[\ell] \xrightarrow{\sim} E'[\ell]$  is Frobenius equivariant only for  $\ell = 2$ . Therefore we established the criterion  $2 \Leftrightarrow 3$ .

Further, we show the equivalence  $1 \Leftrightarrow 4$ . Let's borrow results from [1, §25.4.1]. Denote by  $f_0$  the conductor of the endomorphism ring  $\text{End}(E)$  and by  $D < 0$  the discriminant of the imaginary quadratic field  $\text{End}(E) \otimes \mathbb{Q}$ . The discriminant of  $\chi_{\text{Fr}}$  equals  $D_1 = t^2 - 4q = Df_1^2$ , where  $f_1 \in \mathbb{N}$  (s.t.  $f_0 \mid f_1$ ) is the conductor of the order  $\mathbb{Z}[\text{Fr}]$ . Since over  $\mathbb{F}_{q^2}$  the Frobenius endomorphism  $\text{Fr}^2$  has the trace  $t_2 = t^2 - 2q$  [1, Exercise 9.10.9], the discriminant of its characteristic polynomial equals  $t_2^2 - 4q^2 = D_1 t^2 = Df_2^2$ , where  $f_2 = f_1 t$  is the conductor of  $\mathbb{Z}[\text{Fr}^2]$ . In other words,  $t = [\mathbb{Z}[\text{Fr}] : \mathbb{Z}[\text{Fr}^2]]$ .

Our next reasoning is based on [1, Theorem 25.4.6]. Assume that  $E$  has a degree  $\ell$  vertical  $\mathbb{F}_{q^2}$ -isogeny not defined over  $\mathbb{F}_q$ . It is descending, because the (unique) ascending isogeny of  $E$  (if it exists) is always defined over  $\mathbb{F}_q$ . As a result,  $\ell \mid \frac{f_2}{f_0}$  and  $\ell \nmid \frac{f_1}{f_0}$ , hence  $\ell \mid t$ . Conversely, from  $\ell \mid t$  it follows that  $\ell \mid \frac{f_2}{f_0}$ . By our assumption,  $\ell \neq 2, p$ , hence  $\ell$  does not divide simultaneously  $f_1$  and  $t$  (look at the formula for  $D_1$ ). Thus we have the desired isogeny.  $\square$

## 2 Covers $\varphi: H \rightarrow E_a, \varphi': H \rightarrow E'_a$

Throughout all this section we deal with curves  $E_a: y^2 = x^3 - ax$  over a finite field  $\mathbb{F}_q$  such that  $q \equiv 1 \pmod{4}$  or, equivalently,  $i := \sqrt{-1} \in \mathbb{F}_q$ . The formulas of covers presented below are verified in [27] with the help of Magma [28].

### 2.1 Degree $n = 2$

This case is well studied in the literature, but we shortly discuss it for the sake of completeness. There is on  $E_a$  the order 4 automorphism  $[i]: (x, y) \mapsto (-x, iy)$ , which is known to generate  $\text{Aut}(E_a)$ . Regardless of a quadratic non-residue  $c$ , obviously,

$$E_a[2] = E_a^c[2] = \{P_0, P_\pm, \mathcal{O}\}, \quad \text{where} \quad P_0 := (0, 0), \quad P_\pm := (\pm\sqrt{a}, 0).$$

Also, note that  $[i](P_0) = P_0$  and  $[i](P_\pm) = P_\mp$ .

If  $\sqrt{a} \in \mathbb{F}_q$ , that is  $E_a[2] \subset E_a(\mathbb{F}_q)$ , then we have the  $\mathbb{F}_q$ -(anti)-isometry

$$\tau: E_a[2] \simeq E_a^c[2] \quad P_0 \mapsto P_+, \quad P_+ \mapsto P_0, \quad P_- \mapsto P_-.$$

This isometry is irreducible according to [19, Proposition 3], because it is not the restriction of an element from  $\text{Aut}(E_a)$ . Using the given proposition also in the opposite case, we obtain

**Lemma 2.** *There is an irreducible  $\mathbb{F}_q$ -(anti)-isometry  $E_a[2] \simeq E'_a[2]$  if and only if  $\sqrt{a} \in \mathbb{F}_q$ .*

Moreover, after simplifying the formulas of [19, Proposition 4] applied to  $\tau$ , we get the quadratic  $\mathbb{F}_q$ -covers

$$\begin{aligned} \varphi: H \rightarrow E_a & \quad (x, y) \mapsto \left( \frac{\sqrt{a}(cx^2 - 2)}{-3cx^2}, \frac{2\sqrt{a}}{3^2c^2x^3} \cdot y \right), \\ \varphi': H \rightarrow E_a^c & \quad (x, y) \mapsto \left( \frac{\sqrt{a}(2cx^2 - 1)}{3}, \frac{2\sqrt{a}}{3^2c} \cdot y \right) \end{aligned}$$

by the genus 2 curve

$$H: y^2 = 3c\sqrt{a}(2c^3x^6 - 3c^2x^4 - 3cx^2 + 2).$$

### 2.2 Degree $n = 3$

Due to §2.1 hereafter we suppose that  $c = a \notin (\mathbb{F}_q^*)^2$ . In addition to the Legendre symbol  $\left(\frac{x}{q}\right) = x^{(q-1)/2}$  for  $x \in \mathbb{F}_q^*$ , we will need the 4-th power residue one  $\left(\frac{x}{q}\right)_4 := x^{(q-1)/4}$ .

**Lemma 3.** *Under the condition  $\sqrt{a} \notin \mathbb{F}_q$  there is an irreducible  $\mathbb{F}_q$ -anti-isometry  $E_a[3] \simeq E'_a[3]$  if and only if  $\sqrt{3}, \sqrt{2\sqrt{3}} \in \mathbb{F}_q$ .*

*Proof.* As is known (e.g., from [18, Proposition X.5.4]), among all curves of  $j = 1728$  the quadratic twist  $E_{a'}$  of  $E_a$  (for  $a' \in \mathbb{F}_q^*$ ) is uniquely characterized by the equality  $\left(\frac{a'/a}{q}\right)_4 = -1$ . Consequently, by virtue of [29, Theorem 1.1] the curves  $E_a, E'_a$  are reversely 3-congruent

if and only if exists a point  $(\lambda : \mu) \in \mathbb{P}^1(\mathbb{F}_q)$  such that  $B^+(\lambda, \mu) = 0$  and  $\left(\frac{A^-(\lambda, \mu)/c_4}{q}\right)_4 = -1$ , where  $c_4 := a/27$ .

It is readily checked that for  $c_6 = 0$  we have

$$A^-(x, y) = -\frac{4}{c_4^3}(x^4 - 6c_4x^2y^2 - 3c_4^2y^4), \quad B^+(x, y) = 6c_4^2xy(x^4 + 3c_4^2y^4).$$

First,  $A^-(0, 1) = 12/c_4$  and  $A^-(1, 0) = -4/c_4^3$ . Therefore

$$\left(\frac{A^-(0, 1)/c_4}{q}\right)_4 = \left(\frac{12c_4^2}{q}\right)_4 = \left(\frac{4a^2/3}{q}\right)_4, \quad \left(\frac{A^-(1, 0)/c_4}{q}\right)_4 = \left(\frac{-4}{q}\right)_4 = \left(\frac{2i}{q}\right).$$

Since  $(i+1)^2 = 2i$ , the last symbol equals 1. In turn,  $\left(\frac{4a^2/3}{q}\right)_4 = -1$  if and only if  $\sqrt{3} \in \mathbb{F}_q$  and  $\left(\frac{2a/\sqrt{3}}{q}\right) = \left(\frac{2\sqrt{3}a}{q}\right) = -1$ , that is  $\sqrt{2\sqrt{3}} \in \mathbb{F}_q$ .

Second, let  $\lambda^4 = -3c_4^2$ , that is  $\lambda^2 = \pm i\sqrt{3}c_4$ . Then

$$A^-(\lambda, 1) = -2^4 3\omega^k / c_4, \quad \left(\frac{A^-(\lambda, 1)/c_4}{q}\right)_4 = \left(\frac{-3c_4^2}{q}\right)_4,$$

where  $1 \leq k \leq 2$ . The symbol  $\left(\frac{-3c_4^2}{q}\right)_4 = -1$  if and only if  $\sqrt{3} \in \mathbb{F}_q$  and  $\left(\frac{i\sqrt{3}c_4}{q}\right) = -1$ . However in this case  $\lambda \notin \mathbb{F}_q$ . Finally, the lemma is proved according to the equivalence  $2 \Leftrightarrow 3$  of Theorem 1.  $\square$

Based on this lemma the author found the cubic  $\mathbb{F}_q$ -covers (where  $s := \sqrt{2\sqrt{3}}$ )

$$\begin{aligned} \varphi: H \rightarrow E_a & \quad (x, y) \mapsto \left( \frac{3(2x^3 - \sqrt{3}ax)}{sa}, \frac{s(2\sqrt{3}x^2 - a)}{2^2a^2} \cdot y \right), \\ \varphi': H \rightarrow E_a^a & \quad (x, y) \mapsto \left( \frac{sx^3}{3(\sqrt{3}x^2 - 2a)}, \frac{x^3 - 2\sqrt{3}ax}{3sa(\sqrt{3}x^2 - 2a)^2} \cdot y \right) \end{aligned}$$

by the genus 2 curve

$$H: y^2 = 2sa(2\sqrt{3}x^5 - 7ax^3 + 2\sqrt{3}a^2x).$$

Similar formulas are contained in [20, Algorithm 5.4, Appendix A] even for any pair of elliptic curves glued along their 3-torsion subgroups via an irreducible anti-isometry.

The implication  $3 \Rightarrow 4$  of Theorem 1 allows to derive these formulas in the same way as in §2.3. In order to save space let's not repeat the intermediate computations. The only difference is that, in contrast to §2.3, the endomorphism  $e = [2]$  (up to  $\text{Aut}(E_a)$ ), because  $\deg(\tilde{\varphi}) = \deg(\tilde{\varphi}') = 12$  and curves  $E_a$  do not possess cyclic endomorphisms of degree 4.

### 2.3 Degree $n = 4$

**Theorem 2.** *Under the condition  $\sqrt{a} \notin \mathbb{F}_q$  there is always an irreducible  $\mathbb{F}_q$ -anti-isometry  $E_a[4] \simeq E'_a[4]$ . Moreover, we have the optimal  $\mathbb{F}_q$ -covers*

$$\begin{aligned} \varphi: H \rightarrow E_a & \quad (x, y) \mapsto \left( \frac{2^4ia^2x}{3(3x^2 - a)^2}, \frac{2(i-1)a(3^2x^2 + a)}{3^2(3x^2 - a)^3} \cdot y \right), \\ \varphi': H \rightarrow E_a^a & \quad (x, y) \mapsto \left( \frac{2^4iax^3}{3(x^2 - 3a)^2}, \frac{2(i-1)(x^3 + 3^2ax)}{3^2(x^2 - 3a)^3} \cdot y \right) \end{aligned}$$

by the genus 2 curve

$$H: y^2 = 2 \cdot 3a(3^2x^5 - 2 \cdot 7ax^3 + 3^2a^2x).$$

*Proof.* The existence of an  $\mathbb{F}_q$ -anti-isometry  $\tau: E_a[4] \simeq E'_a[4]$  stems from [16, Corollary 7.4]. Indeed, the discriminant  $D(x^3 - ax) = 2^6a^3 \notin (\mathbb{F}_q^*)^2$ . Now let's start to derive (using Magma [27]) the described formulas, thereby showing the irreducibility of some  $\tau$ . For this purpose one can apply the Fisher approach [30, §3], but we will consider a more elegant one.

The beginning is as in [24, §3], but here we work at the level of abelian surfaces rather than Kummer ones. First of all, with the help of *Vélu's formulas* [1, 25.1.1] one can explicitly write out the  $\mathbb{F}_q$ -conjugate isogenies  $\widehat{\varphi}_\pm: E_a \rightarrow E_\pm := E_a/P_\pm$  to the elliptic curves

$$E_\pm: y^2 = x^3 - 11ax \mp 2 \cdot 7a\sqrt{a}$$

of  $j$ -invariant  $(2 \cdot 3 \cdot 11)^3$ . Note that

$$E_+[2] = \{Q_0^{(0)}, Q_\pm^{(0)}, \mathcal{O}\}, \quad E_-[2] = \{Q_0^{(1)}, Q_\pm^{(1)}, \mathcal{O}\},$$

where

$$Q_0^{(k)} := ((-1)^{(k+1)}2\sqrt{a}, 0), \quad Q_\pm^{(k)} := ((-1)^k(1 \pm 2\sqrt{2})\sqrt{a}, 0).$$

Again applying Vélu's formulas to  $Q_0^{(k)}$ , we determine the dual isogenies

$$\varphi_\pm: E_\pm \rightarrow E_a \quad (x, y) \mapsto \left( \frac{(x \pm \sqrt{a})^2}{2^2(x \pm 2\sqrt{a})}, \frac{x^2 \pm 2^2\sqrt{a}x + 3a}{2^3(x \pm 2\sqrt{a})^2} \cdot y \right).$$

Further, making use of [19, Proposition 4] with respect to the irreducible (anti)-isometry

$$\tau: E_+[2] \simeq E_-[2] \quad Q_0^{(0)} \mapsto Q_0^{(1)}, \quad Q_\pm^{(0)} \mapsto Q_\mp^{(1)},$$

we obtain quadratic covers  $\chi'_\pm: H' \rightarrow E_\pm$ . This isometry is  $\pi$ -invariant in the sense of [31, §1] regardless of whether  $\sqrt{2} \in \mathbb{F}_q$  or not. Consequently, the genus 2 curve  $H'$  is also  $\pi$ -invariant. Thus it is isomorphic to some  $\mathbb{F}_q$ -curve  $H$  by means of the isomorphism  $\psi: H \simeq H'$  from [31, §1] (substitute  $\sqrt{a}$  instead of  $i$ ). After simplifying the formulas of  $\chi_\pm := \chi'_\pm \circ \psi$ , we get the desired equation of  $H$  and the  $\mathbb{F}_q$ -conjugate covers

$$\chi_\pm: H \rightarrow E_\pm \quad (x, y) \mapsto \left( \frac{\mp 2\sqrt{a}(3x^2 \pm 5\sqrt{a}x + 3a)}{3(x \pm \sqrt{a})^2}, \frac{\mp \sqrt{a}}{3^2(x \pm \sqrt{a})^3} \cdot y \right).$$

Based on the auxiliary  $\mathbb{F}_q$ -conjugate covers  $\theta_\pm := \varphi_\pm \circ \chi_\pm$  of degree 4, we obtain the  $\mathbb{F}_q$ -morphisms

$$\tilde{\varphi}: H \rightarrow E_a \quad P \mapsto \theta_+(P) + \theta_-(P), \quad \tilde{\varphi}': H \rightarrow E_a^a \quad P \mapsto \sigma(\theta_+(P) - \theta_-(P)).$$

Using the classical addition-subtraction formulas on elliptic curves (e.g., from [1, §9.1]), we



actually get the  $\mathbb{F}_q$ -covers

$$\begin{aligned} \tilde{\varphi}: H \rightarrow E_a & \begin{cases} X_0 := \frac{(3^2x^2 + a)^2(3^2x^4 - 2 \cdot 7ax^2 + 3^2a^2)}{2^5 3ax(3x^2 - a)^2}, \\ Y_0 := \frac{(3^6x^8 - 2^2 3^5 ax^6 + 2 \cdot 3^5 a^2 x^4 - 2^2 7 \cdot 13a^3 x^2 + 3^2 a^4)(3^2x^2 + a)}{2^8 3^2 a^2 x^2 (3x^2 - a)^3} \cdot y, \end{cases} \\ \tilde{\varphi}': H \rightarrow E_a^a & \begin{cases} X_1 := \frac{(x^2 + 3^2a)^2(3^2x^4 - 2 \cdot 7ax^2 + 3^2a^2)}{2^5 \cdot 3x^3(x^2 - 3a)^2}, \\ Y_1 := \frac{(3^2x^8 - 2^2 7 \cdot 13ax^6 + 2 \cdot 3^5 a^2 x^4 - 2^2 3^5 a^3 x^2 + 3^6 a^4)(x^2 + 3^2a)}{2^8 3^2 ax^5 (x^2 - 3a)^3} \cdot y. \end{cases} \end{aligned}$$

Moreover,  $\deg(\tilde{\varphi}) = \deg(\tilde{\varphi}') = \deg(X_k) = 8$ . Functions similar to  $X_0, X_1$  are given in [24, §3.1]. There the author stops at this stage, however it turns out that  $\tilde{\varphi}, \tilde{\varphi}'$  are not optimal covers. More precisely, below we will prove that over  $\mathbb{F}_q$  exist elliptic curves  $E, E'$ , isomorphisms  $\eta: E \xrightarrow{\sim} E_a, \eta': E' \xrightarrow{\sim} E_a^a$ , and degree 4 covers  $\bar{\varphi}: H \rightarrow E, \bar{\varphi}': H \rightarrow E'$  such that

$$\varphi = \eta \circ \bar{\varphi}, \quad \tilde{\varphi} = [i] \circ e \circ \varphi, \quad \varphi' = \eta' \circ \bar{\varphi}', \quad \tilde{\varphi}' = [i] \circ e \circ \varphi'.$$

Here

$$e: E_a \rightarrow E_a = E_a/P_0 \quad (x, y) \mapsto \left( \frac{i(x^2 - a)}{2x}, \frac{(1-i)(x^2 + a)}{(2x)^2} \cdot y \right)$$

is the unique (up to  $\text{Aut}(E_a)$ ) endomorphism on  $E_a$  of degree 2. In order not to complicate the notation,  $e$  equally denotes the same endomorphism on  $E_a^a$ .

First of all, there are the decompositions

$$x_0 := \frac{x}{(3x^2 - a)^2}, \quad X_0 = \frac{2^8 a^3 x_0^2 + 3^2}{2^5 3ax_0}, \quad x_1 := \frac{x^3}{(x^2 - 3a)^2}, \quad X_1 = \frac{2^8 ax_1^2 + 3^2}{2^5 3x_1}.$$

In order to determine them we make use of the standard Magma function ‘‘Decomposition’’. Unfortunately, it does not work over the function field in  $a$ , hence first we substitute in  $a$  a large prime and after we check the correctness for general  $a$ .

In addition to 0, the remaining 4 roots of the polynomial  $f$  (where  $H: y^2 = f(x)$ ) are equal to

$$r_{\pm} := \frac{(\pm i + 2\sqrt{2})\sqrt{a}}{3}, \quad r'_{\pm} := \frac{(\pm i - 2\sqrt{2})\sqrt{a}}{3}.$$

It is readily checked that

$$x_0(0) = x_1(0) = 0, \quad x_0(r_{\pm}) = x_0(r'_{\pm}) = \frac{\mp 3i\sqrt{a}}{2^4 a^2}, \quad x_1(r_{\pm}) = x_1(r'_{\pm}) = \frac{\pm 3i\sqrt{a}}{2^4 a}.$$

Consider the polynomials

$$g_k(x) := x(x - x_k(r_+))(x - x_k(r_-)) = x^3 + \frac{3^2}{2^8 a^{3-2k}} x.$$

It turns out that in the function field  $\mathbb{F}_q(H)$  there are the square roots of  $f_k(x) := 6g_k(x_k(x))$ , namely

$$\sqrt{f_0(x)} = \frac{3^2 x^2 + a}{2^4 a^2 (3x^2 - a)^3} \cdot y, \quad \sqrt{f_1(x)} = \frac{x^3 + 3^2 ax}{2^4 a (x^2 - 3a)^3} \cdot y.$$

As a consequence,  $E: y^2 = 6g_0(x)$ ,  $E': y^2 = 6g_1(x)$  and the corresponding covers are nothing but

$$\bar{\varphi}: H \rightarrow E \quad (x, y) \mapsto (x_0(x), \sqrt{f_0(x)}), \quad \bar{\varphi}': H \rightarrow E' \quad (x, y) \mapsto (x_1(x), \sqrt{f_1(x)}).$$

Composing these covers with the  $\mathbb{F}_q$ -isomorphisms

$$\begin{aligned} \eta: E &\simeq E_a & (x, y) &\mapsto \left( \frac{2^4 i a^2}{3} \cdot x, \frac{2^5 (i-1) a^3}{3^2} \cdot y \right), \\ \eta': E' &\simeq E_a^a & (x, y) &\mapsto \left( \frac{2^4 i a}{3} \cdot x, \frac{2^5 (i-1) a}{3^2} \cdot y \right), \end{aligned}$$

we obtain the desired  $\mathbb{F}_q$ -covers  $\varphi: H \rightarrow E_a$ ,  $\varphi': H \rightarrow E_a^a$ . This is a computational exercise to show that  $\tilde{\varphi} = [i] \circ e \circ \varphi$  and  $\tilde{\varphi}' = [i] \circ e \circ \varphi'$  as stated above.

It remains to prove the optimality of  $\varphi$ ,  $\varphi'$ . The only possible non-trivial decomposition of  $\varphi$  (up to an  $\mathbb{F}_q$ -isomorphism) has the form  $\varphi = e \circ \varphi_2$  for some quadratic  $\mathbb{F}_q$ -cover  $\varphi_2: H \rightarrow E_a$ . For  $\varphi_2$  there is the quadratic complementary  $\mathbb{F}_q$ -cover  $\varphi_2': H \rightarrow F$  whose the construction is explained, e.g., in [16, §2]. Here  $F$  is a certain elliptic  $\mathbb{F}_q$ -curve. In other words, there are  $\mathbb{F}_q$ -isogenies  $J \stackrel{\Phi_2}{\simeq} E_a \times F \stackrel{\Psi}{\simeq} E_a \times E_a'$  with the kernels isomorphic to  $(\mathbb{Z}/2)^2$ , where  $J$  is the Jacobian of  $H$ . According to uniqueness (up to an  $\mathbb{F}_q$ -isogeny) of the Jacobian decomposition into simple components,  $F \sim_{\mathbb{F}_q} E_a'$ . As a result, the restriction  $\Psi|_F: F \rightarrow E_a'$  is an  $\mathbb{F}_q$ -isogeny such that  $\ker(\Psi|_F) \hookrightarrow (\mathbb{Z}/2)^2$ , that is  $\Psi|_F \in \{\text{id}, e, [2]\}$  up to an  $\mathbb{F}_q$ -isomorphism. In all the cases,  $F \simeq_{\mathbb{F}_q} E_a'$  and hence  $\Phi_2$  induces an irreducible  $\mathbb{F}_q$ -(anti)-isometry  $E_a[2] \simeq E_a'[2]$ . Taking into account §2.1 and the condition  $\sqrt{a} \notin \mathbb{F}_q$ , we come to a contradiction. The same reasoning is equally correct for  $\varphi'$ .  $\square$

### 3 Covers $\varphi: H \rightarrow E_b$ , $\varphi': H \rightarrow E_b'$

Throughout all this section we deal with curves  $E_b: y^2 = x^3 + b$  over a finite field  $\mathbb{F}_q$  such that  $q \equiv 1 \pmod{3}$ , i.e.,  $\omega := \sqrt[3]{1} \in \mathbb{F}_q$ ,  $\omega \neq 1$  or, equivalently,  $\sqrt{-3} \in \mathbb{F}_q$ . The formulas of covers presented below are verified in [27] with the help of Magma [28].

#### 3.1 Degree $n = 2$

This case is well studied in the literature, but we shortly discuss it for the sake of completeness. There is on  $E_b$  the order 6 automorphism  $[-\omega]: (x, y) \mapsto (\omega x, -y)$ , which is known to generate  $\text{Aut}(E_b)$ . Regardless of a quadratic non-residue  $c$ , obviously,

$$E_b[2] = E_b^c[2] = \{P_k\}_{k=0}^2 \cup \{\mathcal{O}\}, \quad \text{where} \quad P_k := (-\omega^k \sqrt[3]{b}, 0).$$

Also, note that  $[-\omega](P_k) = P_{k+1}$ .

If  $\sqrt[3]{b} \in \mathbb{F}_q$ , that is  $E_b[2] \subset E_b(\mathbb{F}_q)$ , then we have the  $\mathbb{F}_q$ -(anti)-isometry

$$\tau: E_b[2] \simeq E_b^c[2] \quad P_0 \mapsto P_1, \quad P_1 \mapsto P_0, \quad P_2 \mapsto P_2.$$

This isometry is irreducible according to [19, Proposition 3], because it is not the restriction of an element from  $\text{Aut}(E_b)$ . Using the given proposition also in the opposite case, we obtain

**Lemma 4.** *There is an irreducible  $\mathbb{F}_q$ -(anti)-isometry  $E_b[2] \simeq E'_b[2]$  if and only if  $\sqrt[3]{b} \in \mathbb{F}_q$ .*

Moreover, after simplifying the formulas of [19, Proposition 4] applied to  $\tau$ , we get the quadratic  $\mathbb{F}_q$ -covers

$$\begin{aligned}\varphi: H &\rightarrow E_b & (x, y) &\mapsto \left( \frac{\sqrt[3]{b}}{cx^2}, \frac{y}{c^2x^3} \right), \\ \varphi': H &\rightarrow E_b^c & (x, y) &\mapsto \left( c\sqrt[3]{b}x^2, \frac{y}{c} \right)\end{aligned}$$

by the genus 2 curve

$$H: y^2 = bc(c^3x^6 + 1).$$

### 3.2 Degree $n = 5$

The degrees 3, 4, and  $> 5$  are discussed in §3.3. Due to §3.1 hereafter one can suppose that  $\sqrt[3]{b} \notin \mathbb{F}_q$ , although we do not use this. In addition to the Legendre symbol  $\left(\frac{x}{q}\right) = x^{(q-1)/2}$  for  $x \in \mathbb{F}_q^*$ , we will need the  $k$ -th power residue one  $\left(\frac{x}{q}\right)_k := x^{(q-1)/k}$ , where  $k \in \{3, 6\}$ .

**Lemma 5.** *There is an irreducible  $\mathbb{F}_q$ -anti-isometry  $E_b[5] \simeq E'_b[5]$  if and only if  $\sqrt{5} \notin \mathbb{F}_q$  and  $\sqrt[3]{b/10} \in \mathbb{F}_q$ .*

*Proof.* As is known (e.g., from [18, Proposition X.5.4]), among all curves of  $j = 0$  the quadratic twist  $E_{b'}$  of  $E_b$  (for  $b' \in \mathbb{F}_q^*$ ) is uniquely characterized by the equality  $\left(\frac{b'/b}{q}\right)_6 = -1$ . Consequently, by virtue of [32, §13] the curves  $E_b, E'_b$  are reversely 5-congruent if and only if exists a point  $(\lambda : \mu) \in \mathbb{P}^1(\mathbb{F}_q)$  such that  $\mathbf{c}_4(\lambda, \mu) = 0$  and  $\left(\frac{\mathbf{c}_6(\lambda, \mu)/c_6}{q}\right)_6 = -1$ , where  $c_6 := -b/54$ . Here  $\mathbf{c}_4, \mathbf{c}_6$  are the *dual Hesse polynomials* for  $n = 5$  from [32, §9].

It is readily checked that for  $c_4 = 0$  we have the decomposition

$$\mathbf{c}_4(x, y) = -2^2 5 c_6^{13} xy \cdot Q_0(x^3, y^3) Q_1(x^3, y^3) Q_2(x^3, y^3),$$

where

$$Q_0 := x^2 - 2^6 5 c_6 xy - 2^6 5 c_6^2 y^2, \quad Q_1 := x^2 - 5 c_6 xy + 2^3 5 c_6^2 y^2, \quad Q_2 := x^2 + 2^3 5 c_6 xy + 2^9 5 c_6^2 y^2.$$

First,  $\mathbf{c}_6(0, 1) = 2^{30} 5^5 c_6^{29}$  and  $\mathbf{c}_6(1, 0) = c_6^{19}$ . Therefore

$$\left(\frac{\mathbf{c}_6(0, 1)/c_6}{q}\right)_6 = \left(\frac{c_6^4/5}{q}\right)_6, \quad \left(\frac{\mathbf{c}_6(1, 0)/c_6}{q}\right)_6 = 1$$

and hence

$$\left(\frac{\mathbf{c}_6(0, 1)/c_6}{q}\right) = \left(\frac{5}{q}\right), \quad \left(\frac{\mathbf{c}_6(0, 1)/c_6}{q}\right)_3 = \left(\frac{c_6/5}{q}\right)_3 = \left(\frac{b/10}{q}\right)_3.$$

Second, the discriminants of the quadratic forms are equal to

$$D(Q_0) = 2^8 3^4 5 c_6^2, \quad D(Q_1) = -3^3 5 c_6^2, \quad D(Q_2) = -2^6 3^3 5 c_6^2.$$

As a result, for  $y = 1$  their roots are

$$x_{0,\pm} = 2^3(2^25 \pm 3^2\sqrt{5})c_6, \quad x_{1,\pm} = \frac{(5 \pm 3\sqrt{-3}\sqrt{5})c_6}{2}, \quad x_{2,\pm} = 2^2(-5 \pm 3\sqrt{-3}\sqrt{5})c_6.$$

It is easily shown that

$$\mathbf{c}_6(\sqrt[3]{x_{0,\pm}}, 1) = -2^{30}3^{15}5^5\alpha_0^2c_6^{29}, \quad \mathbf{c}_6(\sqrt[3]{x_{1,\pm}}, 1) = -3^{15}5^5\alpha_1^2c_6^{29}, \quad \mathbf{c}_6(\sqrt[3]{x_{2,\pm}}, 1) = -2^{30}3^{15}5^5c_6^{29}$$

for some  $\alpha_0, \alpha_1 \in \mathbb{F}_q(\sqrt{5})$ . No matter the index  $k$ , the element  $\mathbf{c}_6(\sqrt[3]{x_{k,\pm}}, 1)/c_6$  is a quadratic residue in  $\mathbb{F}_q$  whenever 5 is so. However only in this case  $x_{k,\pm} \in \mathbb{F}_q$ . Finally, the lemma is proved according to the equivalence 2  $\Leftrightarrow$  3 of Theorem 1.  $\square$

Based on this lemma the author found the optimal  $\mathbb{F}_q$ -covers

$$\begin{aligned} \varphi: H \rightarrow E_{10} & \quad (x, y) \mapsto \left( \frac{5^2(x^3 - 2^3)}{x^2(2x^3 - 5^2)}, \frac{2^2x^6 - 5 \cdot 11x^3 + 2^45^2}{x^3(2x^3 - 5^2)^2} \cdot y \right), \\ \varphi': H \rightarrow E_{10}^5 & \quad (x, y) \mapsto \left( \frac{x^2(2^3x^3 - 5^3)}{5^2(x^3 - 2 \cdot 5)}, \frac{2^4x^6 - 5^211x^3 + 2^25^4}{5^4(x^3 - 2 \cdot 5)^2} \cdot y \right) \end{aligned}$$

by the genus 2 curve

$$H: y^2 = 5(2x^6 - 3^25x^3 + 2 \cdot 5^3).$$

The implication 3  $\Rightarrow$  4 of Theorem 1 allows to derive these formulas in the same way as in §2.3. In order to save space let's not repeat the intermediate computations. Nevertheless, it is necessary to emphasize that, in contrast to §2.3, in the current situation  $j(E_{\pm}) \notin \mathbb{F}_q$  and the definition field  $\mathbb{F}_q(E_{\pm}[2]) = \mathbb{F}_{q^6}$  (see details in [6, §1]). So as an irreducible anti-isometry  $\tau: E_+[2] \simeq E_-[2]$  we should take  $\chi$  from [31, §1]. Besides, the endomorphism  $e = [2]$  (up to  $\text{Aut}(E_{10})$ ), because  $\deg(\tilde{\varphi}) = \deg(\tilde{\varphi}') = 20$  and curves of  $j = 0$  do not possess cyclic endomorphisms of degree 4.

For  $B := b/10$  we have the  $\mathbb{F}_q$ -isomorphisms

$$E_{10} \simeq E_b, \quad E_{10}^5 \simeq E_b^5 \quad (x, y) \mapsto (\sqrt[3]{B} \cdot x, \sqrt{B} \cdot y)$$

if  $\sqrt{B} \in \mathbb{F}_q$  and

$$E_{10} \simeq E_b^5 \quad (x, y) \mapsto (\sqrt[3]{B} \cdot x, \sqrt{B/5} \cdot y), \quad E_{10}^5 \simeq E_b \quad (x, y) \mapsto (\sqrt[3]{B} \cdot x, \sqrt{5B} \cdot y)$$

otherwise. Correctly composing these isomorphisms with  $\varphi, \varphi'$ , we obtain  $\mathbb{F}_q$ -covers  $H \rightarrow E_b, H \rightarrow E_b^5$  of degree 5 for any  $b$ .

### 3.3 Other degrees $n$

According to Lemma 1 the condition  $n \mid 2t$  is necessary for the existence of an  $n$ -congruence between the curves  $E_b, E_b'$ . Since  $q$  and  $\#E_b(\mathbb{F}_q)$  are odd by our assumptions, the trace  $t = q + 1 - \#E_b(\mathbb{F}_q)$  is so. Conversely, by virtue of Theorem 1 there is an irreducible  $\mathbb{F}_q$ -anti-isometry  $E_b[\ell] \simeq E_b'[\ell]$  for any prime divisor  $\ell \mid t$ . Therefore it is enough to

consider primes  $n = \ell$ , because we are interested in  $n$  as small as possible. Recall that the discriminant of the Frobenius characteristic polynomial on  $E_b$  (and  $E'_b$ ) equals  $t^2 - 4q = -3f^2$  for some  $f \in \mathbb{N}$  (see details in [2, §4.2.1]). Since  $3 \nmid q$  in this article, the case  $\ell = 3$  does not arise.

It remains to treat  $\ell \geq 7$ . Unfortunately, for such numbers the *modular curves*  $X_{E_b}^-(\ell)$  (from [32, §13], [33, §1.1]) are no longer rational. So we cannot provide (similarly to §2.2, §3.2) necessary and sufficient conditions under which  $E_b, E'_b$  are reversely  $\ell$ -congruent. Besides, the author did not find in today's real-world cryptography  $\mathbb{F}_q$ -curves  $E_b$  with a greater trace divisor and without an efficient encoding. Thus we stop at  $\ell = 5$ .

Formally, all our Magma computations are over fields of characteristic 0 so that the derived formulas of the covers  $\varphi, \varphi'$  are valid independently of  $\mathbb{F}_q$  (except for maybe a finite number of degenerate cases). However the *strong Frey–Mazur conjecture* [30, §1] predicts that at least over the field  $\mathbb{Q}$  there is no  $\ell$ -congruent pair  $E_b, E'_b$  no matter  $b, c \in \mathbb{Q}, \ell > 13$ . Hence one may try to construct  $\varphi, \varphi'$  to some curves  $E_b, E'_b$  only for  $\ell \in \{7, 11, 13\}$ .

## 4 Encodings $h: \mathbb{P}^1(\mathbb{F}_q) \xrightarrow{\sim} H(\mathbb{F}_q)$ and $\varphi \circ h: \mathbb{P}^1(\mathbb{F}_q) \rightarrow E(\mathbb{F}_q)$

Note that all genus 2 curves previously encountered in this article are given in the affine  $\mathbb{F}_q$ -form

$$H: y^2 = f(x) := f_6x^6 + f_5x^5 + f_4x^4 + f_3x^3 + df_4x^2 + d^2f_5x + d^3f_6$$

for some  $d \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$ . Its precise values are contained in Table 1. As usual,  $H$  has the smooth completion in the weighted projective plane  $\mathbb{P}(1, 3, 1)$  with respect to variables  $X, Y, Z$  such that  $x = X/Z, y = Y/Z^3$ . At infinity  $H$  contains the points  $\mathcal{O}_\pm := (1 : \pm\sqrt{f_6} : 0)$ . In compliance with [1, Definition 10.1.11] the equation of  $H$  is a ramified model if  $f_6 = 0$ , a split model if  $\sqrt{f_6} \in \mathbb{F}_q^*$ , and an inert one otherwise.

§	2.1	2.2	2.3	3.1	3.2
$d$	$1/c$	$a$	$a$	$1/c$	$5$

Table 1: The values of  $d \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$

It is readily checked that there are on  $H$  the involutions

$$\pm\alpha: H \xrightarrow{\sim} H \quad (X : Y : Z) \mapsto (dZ : \pm d\sqrt{d} \cdot Y : X)$$

or in the affine coordinates:

$$\pm\alpha: H \xrightarrow{\sim} H \quad (x, y) \mapsto \left( \frac{d}{x}, \pm \frac{d\sqrt{d}}{x^3} \cdot y \right).$$

In particular,  $P_\pm := (0, \pm d\sqrt{df_6}) \xleftarrow{\alpha} \mathcal{O}_\pm$ . In the case  $f_6 = 0$  the points  $P_+ = P_-$ ,  $\mathcal{O}_+ = \mathcal{O}_-$  are moreover Weierstrass points on  $H$ . Also, it is worth mentioning that the quotients  $H/(\pm\alpha)$  are  $\mathbb{F}_q$ -conjugate elliptic curves. By the way, over algebraically closed fields genus 2 curves with non-hyperelliptic involutions were actively studied, for example, in [34].

Consider any partition  $\mathbb{F}_q^* = Y \sqcup -Y$  (e.g., as in [24, §4]) and the corresponding modulus

$$\mathbb{F}_q \rightarrow Y \sqcup \{0\} \quad |y| := \begin{cases} y & \text{if } y \in Y \sqcup \{0\}, \\ -y & \text{otherwise, i.e., } y \in -Y. \end{cases}$$

As can be seen, the involution  $\alpha$  underlies the encoding

$$h: \mathbb{F}_q^* \rightarrow H(\mathbb{F}_q) \quad h(x) := \begin{cases} (x, |\sqrt{f(x)}|) & \text{if } \sqrt{f(x)} \in \mathbb{F}_q, \\ \left( \frac{d}{x}, -\left| \frac{d\sqrt{df(x)}}{x^3} \right| \right) & \text{otherwise, i.e., } \sqrt{df(x)} \in \mathbb{F}_q. \end{cases}$$

It can be naturally extended to  $\mathbb{P}^1(\mathbb{F}_q)$  as follows:

$$(h(0), h(\infty)) := \begin{cases} (P_+, \mathcal{O}_+) & \text{if } f_6 = 0, \\ (\mathcal{O}_+, \mathcal{O}_-) & \text{if } \sqrt{f_6} \in \mathbb{F}_q^*, \\ (P_+, P_-) & \text{otherwise, i.e., } \sqrt{df_6} \in \mathbb{F}_q^*. \end{cases}$$

**Lemma 6.** *The encoding  $h: \mathbb{P}^1(\mathbb{F}_q) \rightarrow H(\mathbb{F}_q)$  is bijective and hence  $\#H(\mathbb{F}_q) = q + 1$ .*

*Proof.* Obviously,  $h(0) \neq h(\infty)$  and  $h(\{0, \infty\})$  coincides with the set of all  $\mathbb{F}_q$ -points among  $P_\pm, \mathcal{O}_\pm$  regardless of the model of  $H$ . Since  $h(\mathbb{F}_q^*) \cap \{P_\pm, \mathcal{O}_\pm\} = \emptyset$ , it remains to prove the lemma for  $h$  restricted to  $\mathbb{F}_q^*$ . Further, the first condition in the definition of  $h$  also processes non-zero  $\mathbb{F}_q$ -roots of the polynomial  $f$  (if any). Consequently,  $h$  gives the bijection between them and Weierstrass  $\mathbb{F}_q$ -points on  $H$  different from  $P_+, \mathcal{O}_+$ .

Assume that  $h(x_0) = h(x_1)$  for some  $x_0, x_1 \in \mathbb{F}_q^*$  outside the set of roots of  $f$ . If in addition  $f(x_0)f(x_1) \in (\mathbb{F}_q^*)^2$ , then clearly  $x_0 = x_1$ . In the opposite case  $x_1 = d/x_0$ , from the modulus definition the contradiction  $f(x_0) = f(x_1) = 0$  follows. Thus the injectivity is proved. To show the surjectivity we need the property  $f(d/x)f(x) \notin (\mathbb{F}_q^*)^2$ , which stems from the equality  $f(d/x) = d^3 f(x)/x^6$  for  $x \in \mathbb{F}_q^*$ . Then given a point  $P = (x, y)$  from  $H(\mathbb{F}_q) \setminus \{P_\pm, \mathcal{O}_\pm\}$ , it is easily checked that  $h^{-1}(P) = x$  if  $|y| = y$  and  $h^{-1}(P) = d/x$  otherwise.  $\square$

Unfortunately, in addition to finding the square root the previous definition of  $h$  requires to initially determine the Legendre symbol  $\left(\frac{f(x)}{q}\right)$ . Fortunately, this is only an illusion. To be definite, let's suppose that  $q \equiv 3 \pmod{4}$ , but the other practical case  $q \equiv 5 \pmod{8}$  can be handled just as in [6, §2]. We put  $g(x) := f(x)^{(q+1)/4}$ . Abusing the notation, we will just write  $f, g$ . It is worth emphasizing that  $g$  should be computed, using the trick from [5, §4.2]. Note that  $g^2 = f^{(q+1)/2} = \left(\frac{f}{q}\right) \cdot f$  and  $\sqrt{-d} \in \mathbb{F}_q$ . Thus the encoding can be rewritten in the following way:

$$h: \mathbb{F}_q^* \rightarrow H(\mathbb{F}_q) \quad h(x) := \begin{cases} (x, |g|) & \text{if } g^2 = f, \\ \left( \frac{d}{x}, -\left| \frac{d\sqrt{-d} \cdot g}{x^3} \right| \right) & \text{otherwise, i.e., } g^2 = -f. \end{cases} \quad (1)$$

As before, denote by  $\varphi: H \rightarrow E$  any  $\mathbb{F}_q$ -cover of small degree to an elliptic curve  $E$ . Embedding  $H, E$  into (weighted) projective planes and representing  $\varphi$  in the new coordinates, we entirely avoid the inversion operation in  $\mathbb{F}_q$ . This is not cheating, because such coordinates are preferred in elliptic cryptography [2, §2.3.2 and §3.3]. Eventually, we get Algorithm 1 and Remark 1.

**Algorithm 1:** Evaluation of the encoding  $\varphi \circ h$

**Data:**  $x \in \mathbb{F}_q^*$  and an  $\mathbb{F}_q$ -cover  $\varphi: H \rightarrow E$

**Result:** the  $\mathbb{F}_q$ -point  $(\varphi \circ h)(x)$  on  $E$

**begin**

compute  $f$  and  $g$  as described above;

**if**  $g^2 = f$  **then**

$P :=$  the point of the first line of (1)

**else**

$P :=$  that of the second line in (weighted) projective coordinates

**end**

//  $P = h(x)$

**return**  $\varphi(P)$

**end**

**Remark 1.** Whenever  $q \not\equiv 1 \pmod{8}$ , the encodings  $h, \varphi \circ h$  are implemented in constant time through one exponentiation in  $\mathbb{F}_q$ .

**Remark 2.** It seems that not every median value curve (not to mention a general genus 2 curve) possesses an efficient encoding.

To argue the last words, consider, for example, the curve  $E'_a$  (such that  $\sqrt{a}, \sqrt{2} \in \mathbb{F}_q^*$ ) whose the quadratic twist  $E_a$  is  $\mathbb{F}_q$ -isogenous to the curve  $E_+$  (see details in §2.3). As we know, the irreducible  $\mathbb{F}_q$ -(anti)-isometry

$$E'_a[2] \simeq E_+[2] \quad P_0 \mapsto Q_0^{(0)}, \quad P_{\pm} \mapsto Q_{\pm}^{(0)}$$

gives rise to a covering  $H$  of  $E'_a, E_+$ . The author checked that (at least for some  $q$ ) all automorphisms of  $H$  are defined over  $\mathbb{F}_q$ . Since this fact is useless for us, the readers are left with their computer algebra system to see for themselves.

Finally, by virtue of Lemma 6 and [4, Theorem 7] we obtain

**Corollary 1.** The encoding  $h$  is 2-well-distributed (the same is true for  $\varphi \circ h$  if the cover  $\varphi$  is optimal). More formally, let  $k \in \{1, 2\}$ ,  $\psi_1 := \varphi, \psi_2 := \text{id}$ , and  $J$  be the Jacobian of  $H$ . Then

$$\left| \sum_{x \in \mathbb{P}^1(\mathbb{F}_q)} \chi_k((\psi_k \circ h)(x)) \right| \leq 2\sqrt{q}$$

for any non-trivial characters  $\chi_1: E(\mathbb{F}_q) \rightarrow \mathbb{C}^*$  and  $\chi_2: J(\mathbb{F}_q) \rightarrow \mathbb{C}^*$ .

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