Beauty of Cryptography: the Cryptographic Sequences and the Golden Ratio *

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Abstract: In this paper, the authors construct a new type of cryptographic sequence which is named an extra-super increasing sequence, and give the definitions of the minimal super increasing sequence $\{a_1, a_2, \cdots, a_n\}$ and minimal extra-super increasing sequence $\{z_1, z_2, \cdots, z_n\}$. Prove that the minimal extra-super increasing sequence is the odd-positioned subsequence of the Fibonacci sequence, namely $\{z_1, z_2, \ldots, z_n, \ldots\} = \{F_1, F_3, \ldots, F_{2n-1}, \ldots\}$, which indicates that approaching the golden ratio ϕ through the term difference ratio $(z_{n+1} - z_n)/z_n$ is stabler and faster than through the term ratio F_{n+1}/F_n. Further prove that the limit of the term ratio difference between the two cryptographic sequences equals the golden ratio conjugate Φ, namely $lim_{n\to\infty}(z_{n+1}/z_n - a_{n+1}/a_n) = \Phi$, which reveals the beauty of cryptography.

Keywords: Minimal extra-super increasing sequence, Fibonacci sequence, Golden ratio, Golden ratio conjugate, Term ratio difference

1 Introduction

In the world of numbers, there exist some interesting and unobservable phenomena.

A super increasing sequence and analogous sequences are ones about cryptography [1]. The Fibonacci sequence is a fabulous progression [2], and the golden ratio is a glorious irrational number [3]. In this paper, the authors discuss the triangular relations among these three things, and propose and prove the two new theorems, which designate a smooth and expeditious approach to the golden ratio, and display the beauty of cryptography on the golden ratio conjugate.

Throughout the paper, unless otherwise specified, the sign ∞ denotes the infinity, ϕ signifies the golden ratio, Φ signifies the golden ratio conjugate, $\sum$ symbolizes an accumulative sum, and $\surd x$ represents the square root of a positive real number x.

2 Definitions of the Minimal Super Increasing Sequence and Minimal Extra-super Increasing Sequence

In cryptography, we always consider finite sequences. An infinite sequence may be truncated to a fitly finite sequence.

2.1 Four Definitions Relevant to Cryptographic Sequences

The concept of a super increasing sequence was firstly proposed in 1978 by R. C. Merkle and M. E. Hellman [1], and used for the design of the MH knapsack cryptosystems [4].

Definition 1: For n positive integers $a_1, a_2, \ldots$, and a_n (n goes infinite), if a_i ($2 \le i \le n$) satisfies

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$$a_i > \sum_{j=1}^{i-1} a_j, \tag{1}$$

then this integer progression is called a super increasing sequence, denoted by $\{a_1, a_2, \ldots, a_n\}$, and shortly $\{a_i\}$, which is a known concept.

For example, $\{a_1, a_2, \ldots, a_8\} = \{2, 3, 7, 13, 29, 57, 113, 226\}$ is a super increasing sequence.

In what follows, we construct a new type of super increasing sequence.

Definition 2: For n positive integers $z_1, z_2, \ldots$, and z_n (n goes infinite), if z_i ($2 \le i \le n$) satisfies

$$z_i > \sum_{j=1}^{i-1} (i-j)z_j, \tag{2}$$

then this integer progression is called an extra-super increasing sequence, denoted by $\{z_1, z_2, \ldots, z_n\}$, and shortly $\{z_i\}$.

For example, $\{z_1, z_2, \ldots, z_8\} = \{1, 3, 8, 21, 54, 139, 367, 960\}$ is an extra-super increasing sequence.

Definition 3: For n positive integers $\underline{a}_1, \underline{a}_2, \ldots$, and $\underline{a}_n$ (n goes infinite), if the first term $\underline{a}_1 = 1$, and $\underline{a}_i$ ($2 \le i \le n$) satisfies

$$\underline{a}_i = 1 + \sum_{j=1}^{i-1} \underline{a}_j, \tag{3}$$

then this integer progression is called the minimal super increasing sequence, denoted by $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$, and shortly $\{\underline{a}_i\}$, which is a special case of super increasing sequences.

For example, $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_8\} = \{1, 2, 4, 8, 16, 32, 64, 128\}$ is the 8th minimal super increasing sequence.

Definition 4: For n positive integers $\underline{z}_1, \underline{z}_2, \ldots$, and $\underline{z}_n$ (n goes infinite), if the first term $\underline{z}_1 = 1$, and $\underline{z}_i$ ($2 \le i \le n$) satisfies

$$\underline{z}_i = 1 + \sum_{j=1}^{i-1} (i-j)\underline{z}_j, \tag{4}$$

then this integer progression is called the minimal extra-super increasing sequence, denoted by $\{\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_n\}$, and shortly $\{\underline{z}_i\}$, which is a special case of extra-super increasing sequences.

For example, $\{\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_8\} = \{1, 2, 5, 13, 34, 89, 233, 610\}$ is the 8th minimal extra-super increasing sequence.

Owing to

$$\begin{aligned}
\underline{z}_i &= 1 + \sum_{j=1}^{i-1} (i-j)\underline{z}_j \\
&= 1 + (i-1)\underline{z}_1 + (i-2)\underline{z}_2 + \ldots + 2\cdot\underline{z}_{i-2} + 1\cdot\underline{z}_{i-1} \\
&= (1 + ((i-1)-1)\underline{z}_1 + ((i-1)-2)\underline{z}_2 + \ldots + 1\cdot\underline{z}_{i-2}) + (\underline{z}_1 + \underline{z}_2 + \ldots + \underline{z}_{i-1}),
\end{aligned}$$

Formula (4) may be expressed as

$$\underline{z}_i = \underline{z}_{i-1} + \sum_{j=1}^{i-1} \underline{z}_j, \tag{4'}$$

which will be employed in the following text.

2.2 Cryptographic Meanings of Extra-super Increasing Sequences

An extra-super increasing sequence bears a useful property.

Property 1: Assume that $\{z_1, z_2, \ldots, z_n\}$ is an extra-super increasing sequence. Then, for the term z_i ($1 < i \le n$) and any positive integer k, there exists

$$(k+1)z_i > \sum_{j=1}^{i-1}(k+i-j)z_j. \tag{5}$$

Proof:

According to Definition 2, we have $z_i > \sum_{j=1}^{i-1} (i-j)z_j$, and its development is

$$z_i > (i-1)z_1 + (i-2)z_2 + \ldots + 1\cdot z_{i-1}. \tag{2'}$$

Naturally,

$$z_i > z_1 + z_2 + \ldots + z_{i-1}.$$

Further, there exists

$$kz_i > kz_1 + kz_2 + \ldots + kz_{i-1}, \tag{5'}$$

where k (> 0) is an arbitrary integer.

Adding (5′) to (2′) obtains

$$kz_i + z_i > (kz_1 + (i-1)z_1) + (kz_2 + (i-2)z_2) + \ldots + (kz_{i-1} + 1 \cdot z_{i-1}),$$

namely

$$(k+1)z_i > \textstyle\sum_{j=1}^{i-1}(k+i-j)z_j.$$

Hence, Property 1 holds. □

Property 1 makes extra-super increasing sequences (including the minimal extra-super increasing sequence) possess cryptographic meanings.

Let $b_1b_2\ldots b_8 = 11011001$ be a plaintext block, and $L_i = \sum_{j=i}^{8} b_j$ $(1 \le i \le 8)$.

Again let $\{z_1, z_2, \ldots, z_8\} = \{1, 2, 5, 13, 34, 89, 233, 610\}$ be an extra-super increasing sequence (also the 8th minimal extra-super increasing sequence).

Then the anomalous subset sum

$$\begin{aligned} S &= \textstyle\sum_{i=1}^{8} b_i L_i z_i \\ &= 1\cdot5\cdot1 + 1\cdot4\cdot2 + 0\cdot3\cdot5 + 1\cdot3\cdot13 + 1\cdot2\cdot34 + 0\cdot1\cdot89 + 0\cdot1\cdot233 + 1\cdot1\cdot610 \\ &= 730 \end{aligned}$$

is a ciphertext.

If $\{z_1, z_2, \cdots, z_8\} = \{1, 2, 5, 13, 34, 89, 233, 610\}$ is known, and $L = 0$ set, the plaintext $b_1b_2\ldots b_8$ can be recovered as follows:

Due to $S = 730 \ge (L + 1)$ $(z_8$ $(= 610))$, there are $b_8 = 1$, $L = 0 + 1 = 1$, and $S = 730 - 610 = 120$.

Due to $S = 120 < (L + 1)$ $(z_7$ $(= 233))$, there are $b_7 = 0$, $L = 1 + 0 = 1$, and $S = 120 - 0 = 120$.

Due to $S = 120 < (L + 1)$ $(z_6$ $(= 89))$, there are $b_6 = 0$, $L = 1 + 0 = 1$, and $S = 120 - 0 = 120$.

Due to $S = 120 \ge (L + 1)$ $(z_5$ $(= 34))$, there are $b_5 = 1$, $L = 1 + 1 = 2$, and $S = 120 - 2\cdot34 = 52$.

Due to $S = 52 \ge (L + 1)$ $(z_4$ $(= 13))$, there are $b_4 = 1$, $L = 2 + 1 = 3$, and $S = 52 - 3\cdot13 = 13$.

Due to $S = 13 < (L + 1)$ $(z_3$ $(= 5))$, there are $b_3 = 0$, $L = 3 + 0 = 3$, and $S = 13 - 0 = 13$.

Due to $S = 13 \ge (L + 1)$ $(z_2$ $(= 2))$, there are $b_2 = 1$, $L = 3 + 1 = 4$, and $S = 13 - 4\cdot2 = 5$.

Due to $S = 5 \ge (L + 1)$ $(z_1$ $(= 1))$, there are $b_1 = 1$, $L = 4 + 1 = 5$, and $S = 5 - 5\cdot1 = 0$.

In this way, the ciphertext $b_1b_2\ldots b_8 = 11011001$ is recovered.

During the above process, $\{z_1, z_2, \ldots, z_8\}$ is equivalent to a private key. In practicable asymmetrical cryptosystem, a private key should be converted into a related public key.

Understandably, super increasing sequences and extra-super increasing sequences are referred as the cryptographic sequences.

3 The Fibonacci Sequence and the Golden Ratio

The Fibonacci sequence is a famous progression, and the golden ratio is a famous irrational number. There exists a relation between them.

3.1 The Fibonacci Sequence and Its Three Properties

The Fibonacci sequence may be defined recursively.

Definition 5: Let $\{F_0, F_1, \ldots, F_n, \ldots\}$ be a sequence, where

① $F_n = F_{n-1} + F_{n-2}$ (recursion formula)

② $F_0 = 0$, $F_1 = 1$ (initial conditions),

and then $\{F_0, F_1, \ldots, F_n, \ldots\}$ is called the Fibonacci sequence [5][6].

Under the circumstances of having no divergence, $F_0 = 0$ is ignored.

Generally, we have the Fibonacci sequence

$$\{F_1, F_2, \ldots, F_{17}, \ldots\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots\}.$$

The Fibonacci sequence has many properties, of which three are concerned.

Property 2: There are

$$F_{n+1} / F_n \approx \phi, \text{ and } lim_{n\to\infty}(F_{n+1} / F_n) = \phi,$$

where n is comparatively large, and ϕ is the golden ratio [5].

Property 3: Summing consecutive odd-positioned Fibonacci numbers, beginning with F_1, will yield a number which is the next Fibonacci number following the last term in the sum [5]. Namely there is

$$F_1 + F_3 + \dots + F_{2n-1} = F_{2n},$$

where n is equal to or bigger than 1.

For example, there is $F_1 + F_3 + F_5 + F_7 = 1 + 2 + 5 + 13 = 21 = F_8$.

Property 4: For the Fibonacci sequence $\{F_1, F_2, \dots, F_n, \dots\}$, there is

$$F_n = (\sqrt{5})^{-1}(\phi^n - (-\phi^{-1})^n),$$

where F_n is n-th Fibonacci number, and ϕ is likewise the golden ratio [5].

The proofs of Property 2, 3, and 4 can be found in Reference [5].

3.2 The Golden Ratio and Golden Ratio Conjugate

The golden ratio exists in art, nature, and science.

Definition 6: The golden ratio is such a ratio of two quantities that it satisfies equivalently that the ratio of the sum of the two quantities to the larger is equal to the ratio of the larger to the smaller [7][8].

Let $\phi = a / b$ be a ratio with $a > b > 0$. Then, a and b are said to be in the golden ratio if

$$(a + b) / a = a / b = \phi,$$

namely

$$1 + 1 / \phi = \phi, \tag{6}$$

and further

$$\phi^2 = \phi + 1. \tag{6$'$}$$

Formula (6) indicates that ϕ is an irrational constant number, and equals approximately to 1.618033988749 [3]. Other names frequently used for the golden ratio are the golden section, golden mean, divine proportion, etc [9].

Again let $\Phi = b / a$ be a ratio with $a > b > 0$. Then, we can derive from Formula (6) the equation

$$1 + \Phi = 1 / \Phi. \tag{7}$$

Commonly, Φ is called the golden ratio conjugate [10]. Furthermore, we have

$$\Phi = \phi - 1, \tag{7$'$}$$

which can easily be inferred from Formula (6) and (7).

Besides, ϕ may be represented as a continued fraction [11]

$$\phi = [1; 1, 1, 1, \cdots], \tag{8}$$

and also represented as an infinite series [12]

$$\phi = 13 / 8 + \textstyle\sum_{n=0}^{\infty}((-1)^{n+1}(2n + 1)!) / (4^{2n+3}n!(n + 2)!). \tag{9}$$

Evidently, an approximation of ϕ can be fetched from Formula (8) or (9).

4 Relations among Cryptographic Sequences, Fibonacci Sequence, and Golden Ratio

The Fibonacci sequence is a bridge between the cryptographic sequences and the golden ratio.

4.1 Term Ratio of the Minimal Extra-super Increasing Sequence

A term ratio of one positive integer sequence is a current term to its preceding term.

The i-th term ratio of the minimal super increasing sequence is defined as $\underline{a}_{i+1} / \underline{a}_i$ (for $i = 1, 2, \ldots, n, \ldots$).

Property 5: Let $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n, \ldots\}$ be the minimal super increasing sequence, and then there exists

$$\underline{a}_{i+1} / \underline{a}_i = 2 \text{ (for } i = 1, 2, \ldots, n, \ldots),$$

that is to say, the term ratio $\underline{a}_{i+1} / \underline{a}_i$ (for $i = 1, 2, \ldots, n, \ldots$) is a constant number evaluated as 2.

Proof:

According to Definition 3, we have

$$\underline{a}_i = 1 + \textstyle\sum_{j=1}^{i-1} \underline{a}_j \text{ (for } i = 2, 3, \ldots, n, \ldots),$$

and its development is

$$\underline{a}_i = 1 + (\underline{a}_1 + \underline{a}_2 + \ldots + \underline{a}_{i-1}).$$

Adding $\underline{a}_i$ to either of the above expression obtains

$$\underline{a}_i + \underline{a}_i = 1 + (\underline{a}_1 + \underline{a}_2 + \ldots + \underline{a}_{i-1} + \underline{a}_i),$$

namely

$$2\underline{a}_i = \underline{a}_{i+1}, \Rightarrow \underline{a}_{i+1} / \underline{a}_i = 2.$$

Hence, the term ratio $\underline{a}_{i+1} / \underline{a}_i = 2$ (for $i = 1, 2, \ldots, n, \ldots$) is a constant number. □

Similarly, the i-th term ratio of the minimal extra-super increasing sequence is defined as $\underline{z}_{i+1} / \underline{z}_i$ (for $i = 1, 2, \ldots, n, \ldots$).

In addition, the i-th term difference ratio of the minimal extra-super increasing sequence is defined as $(\underline{z}_{i+1} - \underline{z}_i) / \underline{z}_i$ (for $i = 1, 2, \ldots, n, \ldots$).

Conforming to Definition 4, we are able to easily calculate

$\{\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_{17}\}$

$= \{1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, 75025, 196418, 514229, 1346269, 3524578\}$.

Now, compute the term ratio $\underline{z}_{i+1} / \underline{z}_i$ ($i = 1, 2, \ldots, 16$).

As $i = 1$, $\underline{z}_2 / \underline{z}_1 = 2 / 1 = 2$.

As $i = 2$, $\underline{z}_3 / \underline{z}_2 = 5 / 2 = 2.5$.

As $i = 3$, $\underline{z}_4 / \underline{z}_3 = 13 / 5 = 2.6$.

As $i = 4$, $\underline{z}_5 / \underline{z}_4 = 34 / 13 \approx 2.61538$.

As $i = 5$, $\underline{z}_6 / \underline{z}_5 = 89 / 34 \approx 2.6176471$.

As $i = 6$, $\underline{z}_7 / \underline{z}_6 = 233 / 89 \approx 2.617977528$.

As $i = 7$, $\underline{z}_8 / \underline{z}_7 = 610 / 233 \approx 2.61802575107$.

As $i = 8$, $\underline{z}_9 / \underline{z}_8 = 1597 / 610 \approx 2.6180327868852$.

As $i = 9$, $\underline{z}_{10} / \underline{z}_9 = 4181 / 1597 \approx 2.618033813400125$.

As $i = 10$, $\underline{z}_{11} / \underline{z}_{10} = 10946 / 4181 \approx 2.61803396316670652$.

As $i = 11$, $\underline{z}_{12} / \underline{z}_{11} = 28657 / 10946 \approx 2.6180339850173579389$.

As $i = 12$, $\underline{z}_{13} / \underline{z}_{12} = 75025 / 28657 \approx 2.618033988205325051471$.

As $i = 13$, $\underline{z}_{14} / \underline{z}_{13} = 196418 / 75025 \approx 2.61803398867044318560479$.

As $i = 14$, $\underline{z}_{15} / \underline{z}_{14} = 514229 / 196418 \approx 2.6180339887383030068527324$.

As $i = 15$, $\underline{z}_{16} / \underline{z}_{15} = 1346269 / 514229 \approx 2.618033988748203621343798191$.

As $i = 16$, $\underline{z}_{17} / \underline{z}_{16} = 3524578 / 1346269 \approx 2.6180339887496481015309718934 3$.

Afterward, calculate the term difference ratio $(\underline{z}_{i+1} - \underline{z}_i) / \underline{z}_i$ ($i = 1, 2, \ldots, 16$).

As $i = 1$, $(\underline{z}_2 - \underline{z}_1) / \underline{z}_1 = (2 - 1) / 1 = 1$.

As $i = 2$, $(\underline{z}_3 - \underline{z}_2) / \underline{z}_2 = (5 - 2) / 2 = 1.5$.

As $i = 3$, $(\underline{z}_4 - \underline{z}_3) / \underline{z}_3 = (13 - 5) / 5 = 1.6$.

As $i = 4$, $(\underline{z}_5 - \underline{z}_4) / \underline{z}_4 = (34 - 13) / 13 \approx 1.61538$.

As $i = 5$, $(\underline{z}_6 - \underline{z}_5) / \underline{z}_5 = (89 - 34) / 34 \approx 1.6176471$.

As $i = 6$, $(\underline{z}_7 - \underline{z}_6) / \underline{z}_6 = (233 - 80) / 89 \approx 1.617977528$.

As $i = 7$, $(\underline{z}_8 - \underline{z}_7) / \underline{z}_7 = (610 - 233) / 233 \approx 1.61802575107$.

As $i = 8$, $(z_9 - z_8) / z_8 = (1597 - 610) / 610 \approx 1.6180327868852$.

As $i = 9$, $(z_{10} - z_9) / z_9 = (4181 - 1597) / 1597 \approx 1.618033813400125$.

As $i = 10$, $(z_{11} - z_{10}) / z_{10} = (10946 - 4181) / 4181 \approx 1.61803396316670652$.

As $i = 11$, $(z_{12} - z_{11}) / z_{11} = (28657 - 10946) / 10946 \approx 1.6180339850173579389$.

As $i = 12$, $(z_{13} - z_{12}) / z_{12} = (75025 - 28657) / 28657 \approx 1.618033988205325051471$.

As $i = 13$, $(z_{14} - z_{13}) / z_{13} = (196418 - 75025) / 75025 \approx 1.6180339886704431856047 9$.

As $i = 14$, $(z_{15} - z_{14}) / z_{14} = (514229 - 196418) / 196418 \approx 1.6180339887383030068527324$.

As $i = 15$, $(z_{16} - z_{15}) / z_{15} = (1346269 - 514229) / 514229 \approx 1.6180339887482036213437981 91$.

As $i = 16$, $(z_{17} - z_{16}) / z_{16} = (3524578 - 1346292) / 1346269 \approx 1.6180339887496481015309718 9343$.

In what follows, we evaluate the term ratio difference $z_{i+1} / z_i - a_{i+1} / a_i$ ($i = 1, 2, \ldots, 16$).

When $i = 1$, $z_2 / z_1 - a_2 / a_1 = 0$.

When $i = 2$, $z_3 / z_2 - a_3 / a_2 = 0.5$.

When $i = 3$, $z_4 / z_3 - a_4 / a_3 = 0.6$.

When $i = 4$, $z_5 / z_4 - a_5 / a_4 = 0.61538$.

When $i = 5$, $z_6 / z_5 - a_6 / a_5 = 0.6176471$.

When $i = 6$, $z_7 / z_6 - a_7 / a_6 = 0.617977528$.

When $i = 7$, $z_8 / z_7 - a_8 / a_7 = 0.61802575107$.

When $i = 8$, $z_9 / z_8 - a_9 / a_8 = 0.6180327868852$.

When $i = 9$, $z_{10} / z_9 - a_{10} / a_9 = 0.618033813400125$.

When $i = 10$, $z_{11} / z_{10} - a_{11} / a_{10} = 0.61803396316670652$.

When $i = 11$, $z_{12} / z_{11} - a_{12} / a_{11} = 0.6180339850173579389$.

When $i = 12$, $z_{13} / z_{12} - a_{13} / a_{12} = 0.618033988205325051471$.

When $i = 13$, $z_{14} / z_{13} - a_{14} / a_{13} = 0.6180339886704431856047 9$.

When $i = 14$, $z_{15} / z_{14} - a_{15} / a_{14} = 0.6180339887383030068527324$.

When $i = 15$, $z_{16} / z_{15} - a_{16} / a_{15} = 0.6180339887482036213437981 91$.

When $i = 16$, $z_{17} / z_{16} - a_{17} / a_{16} = 0.6180339887496481015309718 9343$.

From the above data, we observe that $lim_{n\to\infty}(z_{n+1}/z_n - a_{n+1}/a_n)$ is approaching a certain number, and $lim_{n\to\infty}((z_{n+1} - z_n)/z_n)$ is also approaching a certain number.

4.2 The Minimal Extra-super Increasing Sequence Being a Regular Subsequence of the Fibonacci Sequence

Let $\{z_1, z_2, \ldots, z_9, \ldots\} = \{1, 2, 5, 13, 34, 89, 233, 610, 1597, \ldots\}$ is the minimal extra-super increasing sequence.

$\{F_1, F_2, \ldots, F_{17}, \ldots\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots\}$ is the Fibonacci sequence.

It is not difficult to watch $z_1 = F_1$, $z_2 = F_3$, $z_3 = F_5$, …, and $z_9 = F_{17}$.

Theorem 1: The minimal extra-super increasing sequence $\{z_1, z_2, \ldots, z_n, \ldots\}$ is the odd-positioned subsequence of the Fibonacci sequence $\{F_1, F_2, \ldots, F_n, \ldots\}$, namely there exists $\{z_1, z_2, \ldots, z_n, \ldots\} = \{F_1, F_3, \ldots, F_{2n-1}, \ldots\}$.

Proof (by induction):

① When $n = 1$, it holds that $\{z_1\} = \{1\} = \{F_{2\cdot 1-1}\} = \{F_1\}$.

② When $n = 2$, it holds that $\{z_1, z_2\} = \{1, 2\} = \{F_{2\cdot 1-1}, F_{2\cdot 2-1}\} = \{F_1, F_3\}$.

③ When $n = 3$, it holds that $\{z_1, z_2, z_3\} = \{1, 2, 5\} = \{F_{2\cdot 1-1}, F_{2\cdot 2-1}, F_{2\cdot 3-1}\} = \{F_1, F_3, F_5\}$.

④ Assume that when $n = k$, it holds that $\{z_1, z_2, \ldots, z_k\} = \{F_1, F_3, \ldots, F_{2\cdot k-1}\}$.

⑤ In succession, we need to proof that when $n = k + 1$, it holds that

$$\{z_1, z_2, \ldots, z_k, z_{k+1}\} = \{F_1, F_3, \ldots, F_{2\cdot k-1}, F_{2\cdot k+1}\}.$$

In light of Formula (4′), there exists

$$\begin{aligned} z_{k+1} &= z_k + \textstyle\sum_{j=1}^{k} z_j \\ &= z_1 + z_2 + \ldots + z_k + z_k. \end{aligned}$$

Again in light of the assumption at ④, there exists

$$\begin{aligned} z_{k+1} &= z_1 + z_2 + \ldots + z_k + z_k \\ &= F_1 + F_3 + \ldots + F_{2\cdot k-1} + F_{2\cdot k-1}. \end{aligned}$$

By Property 3, we have

$$\begin{aligned} z_{k+1} &= F_1 + F_3 + \ldots + F_{2\cdot k-1} + F_{2\cdot k-1} \\ &= F_{2\cdot k} + F_{2\cdot k-1}. \end{aligned}$$

Again by Definition 5, we have

$$z_{k+1} = F_{2\cdot k} + F_{2\cdot k-1} = F_{2\cdot k+1}.$$

Namely

$$\{z_1, z_2, \ldots, z_k, z_{k+1}\} = \{F_1, F_3, \ldots, F_{2\cdot k-1}, F_{2\cdot k+1}\}.$$

Hence, $\{z_1, z_2, \ldots, z_n, \ldots\}$ is the odd-positioned subsequence of $\{F_1, F_2, \ldots, F_n, \ldots\}$. □

Section 4.1 and Theorem 1 show together that the term difference ratio $(z_{n+1}-z_n)/z_n$ is a stabler faster approach to ϕ than the term ratio F_{n+1}/F_n, which saves the half of storage space and addressing time.

4.3 Limit of Term Ratio Difference ($z_{n+1}/z_n - a_{n+1}/a_n$) Being Equal to Φ

By the medium of the Fibonacci sequence $\{F_1, F_2, \ldots, F_n, \ldots\}$, the term ratio difference $(z_{n+1}/z_n - a_{n+1}/a_n)$ between the two cryptographic sequences corresponds to the golden ratio conjugate Φ.

Theorem 2: Let $\{z_1, z_2, \ldots, z_n, \ldots\}$ be the minimal extra-super increasing sequence, and $\{a_1, a_2, \ldots, a_n, \ldots\}$ be the minimal super increasing sequence; then there is $lim_{n\to\infty}(z_{n+1}/z_n - a_{n+1}/a_n) = \Phi$.

Proof.

By Theorem 1, we have $\{z_1, z_2, \ldots, z_n, \ldots\} = \{F_1, F_3, \ldots, F_{2n-1}, \ldots\}$, and by Property 5, $a_{n+1}/a_n = 2$. Thus, there exists

$$\begin{aligned} lim_{n\to\infty}(z_{n+1}/z_n - a_{n+1}/a_n) &= lim_{n\to\infty}(F_{2n+1}/F_{2n-1} - a_{n+1}/a_n) \\ &= lim_{n\to\infty}(F_{2n+1}/F_{2n-1} - 2) \\ &= lim_{n\to\infty}(F_{2n+1}/F_{2n-1}) - 2. \end{aligned}$$

Again by Property 4, we have

$$\begin{aligned} &lim_{n\to\infty}(F_{2n+1}/F_{2n-1}) \\ &= lim_{n\to\infty}((\sqrt{5})^{-1}(\phi^{2n+1} - (-\phi^{-1})^{2n+1})) / ((\sqrt{5})^{-1}(\phi^{2n-1} - (-\phi^{-1})^{2n-1})) \\ &= lim_{n\to\infty}(\phi^{2n+1} - (-\phi^{-1})^{2n+1}) / (\phi^{2n-1} - (-\phi^{-1})^{2n-1}) \\ &= lim_{n\to\infty}(\phi^2(\phi^{2n-1} - (-\phi^{-1})^{2n+3})) / (\phi^{2n-1} - (-\phi^{-1})^{2n-1}) \\ &= \phi^2\, lim_{n\to\infty}(\phi^{2n-1} - (-\phi^{-1})^{2n+3}) / (\phi^{2n-1} - (-\phi^{-1})^{2n-1}) \\ &= \phi^2\, lim_{n\to\infty}(\phi^{2n-1}(1 + (\phi^{-1})^{4n+2})) / (\phi^{2n-1}(1 + (\phi^{-1})^{4n-2})) \\ &= \phi^2\, lim_{n\to\infty}(1 + (\phi^{-1})^{4n+2}) / (1 + (\phi^{-1})^{4n-2}) \\ &= \phi^2\, lim_{n\to\infty}(1 + 0) / (1 + 0) \\ &= \phi^2. \end{aligned}$$

In terms of Formula (6′), there exists $\phi^2 = \phi + 1$.

Further, there exists

$$lim_{n\to\infty}(F_{2n+1}/F_{2n-1}) = \phi + 1.$$

Again In terms of Formula (7′), there exists

$$lim_{n\to\infty}(F_{2n+1}/F_{2n-1}) - 2 = \phi + 1 - 2 = \phi - 1 = \Phi.$$

Hence, the limit $lim_{n\to\infty}(z_{n+1}/z_n - a_{n+1}/a_n) = \Phi$ holds. □

Corollary 1: Let $\{z_1, z_2, \ldots, z_n, \ldots\}$ be the minimal extra-super increasing sequence; then there exists $lim_{n\to\infty}((z_{n+1} - z_n) / z_n) = \phi$.

Corollary 2: Let $\{F_1, F_2, \ldots, F_{2n-1}, F_{2n}, F_{2n+1}, \ldots\}$ be the Fibonacci sequence; then there exists $lim_{n\to\infty}((F_{2n+1} - F_{2n-1}) / F_{2n-1}) = \phi$.

Theorem 2 manifests very exquisite relations between the cryptographic sequences and the golden ratio. Corollary 1 or 2 suggests one new approach to the golden ratio through the freshly constructed increasing sequence or the odd-positioned numbers of the classical Fibonacci sequence.

5 Conclusion

The paper constructs a new type of cryptographic sequence which is named an extra-super increasing sequence, and gives the definitions of the minimal super increasing sequence $\{a_1, a_2, \ldots, a_n\}$ and minimal extra-super increasing sequence $\{z_1, z_2, \ldots, z_n\}$.

The paper discusses the triangular relations among the cryptographic sequences, the Fibonacci sequence, and the golden ratio. Theorem 1 tells us $\{z_1, z_2, \ldots, z_n, \ldots\} = \{F_1, F_3, \ldots, F_{2n-1}, \ldots\}$, which indicates that approaching ϕ through the term difference ratio $(z_{n+1} - z_n) / z_n$ is more smooth and expeditious than through the term ratio F_{n+1} / F_n. Theorem 2 tells us $lim_{n\to\infty}(z_{n+1} / z_n - a_{n+1} / a_n) = \Phi$, which reveals the beauty of cryptography.

Our intended tasks have not been completed by now. At the next step, we will research extra-super increasing sequences thoroughly, and devise new asymmetrical cryptosystems based on extra-super increasing sequences.

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