

# MAKE: a Matrix Action Key Exchange

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ABSTRACT. We offer a public key exchange protocol based on a semidirect product of two cyclic (semi)groups of matrices over  $\mathbb{Z}_p$ . One of the (semi)groups is additive, the other one multiplicative. This allows us to take advantage of both operations on matrices to diffuse information. We note that in our protocol, no power of any matrix or of any element of  $\mathbb{Z}_p$  is ever exposed, so all standard attacks on Diffie-Hellman-like protocols (including Shor's quantum algorithm attack) are not applicable.

## 1. INTRODUCTION

We start by recalling the classical Diffie-Hellman protocol [1]. The simplest, and original, implementation of this protocol uses the multiplicative group of integers modulo  $p$ , where  $p$  is prime and  $g$  is primitive modulo  $p$ . A more general description of the protocol uses an arbitrary finite cyclic group:

- (1) Alice and Bob agree on a finite cyclic group  $G$  of order  $q$  and a generating element  $g$  in  $G$ . We will write the group  $G$  multiplicatively.
- (2) Alice picks a random natural number  $m < q$  and sends  $g^m$  to Bob.
- (3) Bob picks a random natural number  $n < q$  and sends  $g^n$  to Alice.
- (4) Alice computes  $K_A = (g^n)^m = g^{nm}$ .
- (5) Bob computes  $K_B = (g^m)^n = g^{mn}$ .

Since  $mn = nm$ , both Alice and Bob are now in possession of the same group element  $K = K_A = K_B$  which can serve as the shared secret key.

The protocol is considered secure against eavesdroppers if  $G$  and  $g$  are chosen properly. The eavesdropper must solve the *Diffie-Hellman problem* (recover  $g^{mn}$  from  $g$ ,  $g^m$ , and  $g^n$ ) to obtain the shared secret key. This is currently considered difficult for a “good” choice of parameters (see e.g. [5] for details).

In [3], a new key exchange protocol was offered, based on a semidirect product of multiplicative matrix semigroups. That protocol is similar to the Diffie-Hellman protocol, but it differs in one essential detail: at the last two steps, Alice and Bob use multiplication instead of exponentiation. In other words, the equality  $K_A = K_B$  in that case is based on the identity  $m + n = n + m$  in the ring of integers, instead of the identity  $mn = nm$ . Also, even though the parties do compute a large power of a public element (as in the classical Diffie-Hellman protocol), they do not transmit the whole result, but rather just part of it, so no power of a public element is ever exposed.

The generic protocol in [3] can be based on any (semi)group, in particular on any non-commutative (semi)group. For a particular instantiations of the generic protocol, the authors of [3] used conjugation as an action of the multiplicative semigroup of matrices on itself, and matrices were considered over the group ring  $\mathbb{Z}_7[A_5]$ . A ring of such a small size was selected to enhance efficiency; however, because of this small size, the protocol turned out to be vulnerable to a “linear algebra attack”, similar to the attack on Stickel's protocol [11] offered in [10], albeit more sophisticated, see [6], [8].

In this paper, we use matrices over  $\mathbb{Z}_p$  for a large  $p$ . The size of the exponents used in the protocol should guarantee security against brute force attacks. We also note that in our protocol, no power of any matrix is ever exposed, so all standard attacks on Diffie-Hellman-like protocols (including Shor's quantum algorithm attack [9]) are not applicable.

## 2. SEMIDIRECT PRODUCTS AND EXTENSIONS BY AUTOMORPHISMS

We include this section to make the exposition more comprehensive. The reader who is uncomfortable with group-theoretic constructions can skip to subsection 2.1.

We now recall the definition of a semidirect product:

**Definition 1.** *Let  $G, S$  be two groups, let  $\text{Aut}(G)$  be the group of automorphisms of  $G$ , and let  $\rho : S \rightarrow \text{Aut}(G)$  be a homomorphism. Then the semidirect product of  $G$  and  $S$  is the set*

$$\Gamma = G \rtimes_{\rho} S = \{(g, h) : g \in G, h \in S\}$$

with the group operation given by

$$(g, h)(g', h') = (g^{\rho(h')} \cdot g', h \cdot h').$$

Here  $g^{\rho(h')}$  denotes the image of  $g$  under the automorphism  $\rho(h')$ , and when we write a product  $h \cdot h'$  of two morphisms, this means that  $h$  is applied first.

In this paper, we focus on a special case of this construction, where the group  $S$  is just a sub(semi)group of the semigroup of self-homomorphisms of  $G$ , typically a cyclic sub(semi)group. We give more details about this special case in Section 2.1 below.

**2.1. Extensions by automorphisms.** A particularly simple special case of the semidirect product construction is where the group  $S$  is just a subgroup of the group  $\text{Aut}(G)$  of automorphisms of  $G$ . Then the semidirect product is the set of all pairs  $(g, \phi)$ , where  $g \in G$ ,  $\phi \in S$ , with the group operation given by

$$(1) \quad (g, \phi) \cdot (g', \phi') = (\phi'(g) \cdot g', \phi \cdot \phi').$$

One can also use this construction if  $G$  is not necessarily a group, but just a semigroup, and/or consider endomorphisms (i.e., self-homomorphisms) of  $G$ , not necessarily automorphisms. Then the resulting semidirect product will be a semigroup, not a group, but this is quite sufficient for being the platform of a Diffie-Hellman-like key exchange protocol.

## 3. ACTION

Our platform semigroup will be a semidirect product of two semigroups of matrices over  $\mathbb{Z}_p$ . In the notation of Section 2.1, the semigroup  $G$  will be the *additive* semigroup of matrices, and the semigroup  $S$  will be a cyclic subsemigroup of the *multiplicative* semigroup of matrices.

The action of  $S$  on  $G$  will be as follows:  $M^H = HMH$  for  $M \in G$ ,  $H \in S$ . Note that this is an action only if we restrict to a *commutative* (in particular, cyclic) sub(semi)group  $S$  generated by a matrix  $H$ .

Thus, in the semidirect product of the additive semigroup of matrices with the multiplicative cyclic semigroup generated by  $H$ , the multiplication looks like this (cf. formula (1)):

$$\begin{aligned} (M, H)^2 &= (M, H) \cdot (M, H) = (HMH + M, H^2); \\ (M, H)^3 &= (HMH + M, H^2) \cdot (M, H) = (H^2MH^2 + HMH + M, H^3), \text{ etc.} \end{aligned}$$

## 4. PROTOCOL DESCRIPTION

Below is the protocol description. Parameters are discussed separately, in Section 5.

1. (*key selection*) **(i)** Alice and Bob agree on two invertible public matrices,  $M$  and  $H$ , over  $\mathbb{Z}_p$ , such that  $MH \neq HM$ .  
**(ii)** Alice selects a private integer  $m$  and Bob selects a private integer  $n$ .
2. Alice computes  $(M, H)^m$  and sends **only the first component** (call it  $A$ ) of the result to Bob.
3. Bob computes  $(M, H)^n$  and sends **only the first component** (call it  $B$ ) of the result to Alice.
4. Alice computes  $(B, x) \cdot (A, H^m) = (H^m B H^m + A, ?)$ . Her key is now  $K_A = H^m B H^m + A$ .
5. Bob computes  $(A, y) \cdot (B, H^n) = (H^n A H^n + B, ?)$ . His key is now  $K_B = H^n A H^n + B$ .
6. Since  $(M, H)^{m+n} = (B, x) \cdot (A, H^m) = (A, y) \cdot (B, H^n) = (K, H^{m+n})$ , we should have  $K_A = K_B = K$ , the shared secret key.

**Remark 1.** *Note that, in contrast with the “standard” Diffie-Hellman key exchange, correctness here is based on the equality  $x^m \cdot x^n = x^n \cdot x^m = x^{m+n}$  rather than on the equality  $(x^m)^n = (x^n)^m = x^{mn}$ . In the “standard” Diffie-Hellman set up, our trick would not work because, if the shared key  $K$  was just the product of two openly transmitted elements, then anybody, including the eavesdropper, could compute  $K$ .*

## 5. PARAMETERS AND KEYS SAMPLING

The basic parameter  $p$  should have the same properties and the same magnitude as recommended for the classical Diffie-Hellman protocol. In particular,  $p$  should be a *safe prime*, i.e., a prime of the form  $p = 2q + 1$  see Proposition 1 in Section 6.

The exponents  $m$  and  $n$  should be of the same magnitude as  $q = \frac{p-1}{2}$ .

For the public matrices  $M$  and  $H$ , we require that they are invertible and that  $MH \neq HM$ . Sampling of invertible non-commuting matrices can be done as follows. Build a matrix  $M$  with entries selected uniformly at random from  $\mathbb{Z}_p$ . Check if it is invertible; with high probability it will be. If not, then start over. Then do the same for the matrix  $H$ . Then check if  $MH \neq HM$ ; with high probability this will be the case. If not, then start the whole sampling over.

Then, since the matrix  $H$  is raised to a large power, we should worry that the order of  $H$  in the (multiplicative) group  $GL_k(\mathbb{Z}_p)$  is not too small. We suggest that the size of all matrices in our scheme be  $3 \times 3$ . In that case, the order of the group  $GL_3(\mathbb{Z}_p)$  is  $(p^3 - 1)(p^3 - p)(p^3 - p^2) = p^3(p - 1)^3(p + 1)(p^2 + p + 1) = 8p^3q^3(p + 1)(p^2 + p + 1)$ .

Recall that by Lagrange’s theorem, the order of any element of a group should divide the order of the group. Since we do not have control over prime factors of  $(p + 1)$   $(p^2 + p + 1)$  and those may be rather small, we just have to check that  $H^{(p+1)(p^2+p+1)} \neq I$ . Then it will follow that the order of  $H$  is at least  $q$ , i.e., is large.

## 6. SECURITY

In this section, we address security of the protocol described in Section 4.

Our security assumption here, analogous to the *computational Diffie-Hellman assumption*, is that it is computationally infeasible to retrieve the shared secret key  $K$  from the quadruple  $(M, H, A, B)$ . The matrices  $A$  and  $B$  are expressed in terms of the matrices  $M$  and  $H$  as follows:

$$(2) \quad A = H^{m-1}MH^{m-1} + H^{m-2}MH^{m-2} + \dots + HMH + M.$$

$$(3) \quad B = H^{n-1}MH^{n-1} + H^{n-2}MH^{n-2} + \dots + HMH + M.$$

The shared secret key  $K$  is

$$(4) \quad K = H^{n+m-1}MH^{n+m-1} + H^{n+m-2}MH^{n+m-2} + \dots + HMH + M.$$

What makes our scheme compare favorably to, say, the scheme of [3] is that in computing  $A$  and  $B$ , both operations on matrices (addition and multiplication) are employed, which is good for security because neither multiplicative (e.g. the determinant) nor additive (e.g. the trace) functions of a matrix can be used to reduce the problem to a problem in  $\mathbb{Z}_p$ .

Also, the expressions for  $A$  and  $B$  cannot be factored into products of simpler expressions. To compare, factoring would be possible if the action of  $H$  on  $M$  was a one-sided multiplication. For example, if the action of  $H$  on  $M$  was given by  $M \cdot H$ , then the matrix  $A$  would be equal to  $MH^m + MH^{m-1} + \dots + MH + M = M(H^m + H^{m-1} + \dots + H + I)$ . The adversary could multiply this on the right by  $(H - I)$  and get  $M(H^{m+1} - I)$ . Since the matrices  $M$  and  $H$  are public, the adversary could then recover  $MH^{m+1}$  or even  $H^{m+1}$  if  $M$  is invertible. Then, if  $H$  is invertible, too, the adversary can recover  $H^m$ , and this breaks the scheme.

However, with the action  $M \rightarrow HMH$ , no factorization of  $A$  (or  $B$ ) is possible, and there is no visible way to recover  $K$  from  $(M, H, A, B)$ .

We also note again that no power of any matrix is ever exposed, so all standard attacks on Diffie-Hellman-like protocols are not applicable in our situation. We also note that the analog of the discrete logarithm problem for our protocol is at least as hard as it is for the classical Diffie-Hellman protocol.

**Proposition 1.** *Let a prime  $p$  be of the form  $p = 4n + 3$ . (In particular, safe primes have this property.) Suppose it is computationally feasible for the adversary to recover, from the  $3 \times 3$  public matrices  $A$ ,  $M$ , and  $H$  over  $\mathbb{Z}_p$ , the private exponent  $m$  in the protocol in Section 4. Then it is computationally feasible for the adversary to recover the private exponent  $k$  from  $g$  and  $g^k$  in the classical Diffie-Hellman protocol.*

*Proof.* In the notation we used for the Diffie-Hellman protocol in the Introduction, suppose we are given  $g$  and  $g^m$ , for some  $g \in \mathbb{Z}_p$  and  $m \in \mathbb{Z}$ . Make a  $3 \times 3$  matrix  $M$  with  $a_{11} = g$ ;  $a_{22}$ ,  $a_{23}$ , and  $a_{33}$  random. Let the matrix  $H$  be a diagonal matrix, with  $h_{22} = 1$ ;  $h_{11}$  and  $h_{33}$  random.

Then, for any  $i \geq 0$ , the matrix  $H^i$  will have  $h_{11}^i$ , 1, and  $h_{33}^i$  on the diagonal. The matrix  $H^iMH^i$  will have  $g^{2i}$  in the upper left corner, and zeros as other entries in the first row and in the first column. Therefore, the matrix  $A$  (see the formula (2)), which is the sum of  $H^iMH^i$ , will have  $g^{2\sum i}$  in the upper left corner, and zeros as other entries in the first row and in the first column. The summation in the exponent on  $g$  is from  $i = 0$  to  $i = m - 1$  and the whole exponent on  $g$  is therefore equal to  $(m - 1)(m - 2)$ . Clearly, one knows  $m$  if and only if one knows  $(m - 1)(m - 2)$ . Other entries of the matrix  $A$  are as follows. The entry (2,2) is  $m \cdot a_{22}$ , i.e., it is random since  $a_{22}$  is random. Similarly, the entry (2,3) is  $m \cdot a_{23}$ , i.e., is random. The entry (3,2) is 0. Finally, the entry (3,3) is  $a_{33} \cdot \sum h_{33}^j$ , where  $j$  in the exponent on  $h_{33}$  runs from  $j = 0$  to  $j = m - 1$ . Again, this is random since  $a_{33}$  is random.

Thus, the matrix  $A$  has  $g^{(m-1)(m-2)}$  in the upper left corner, and other entries are either zeros or random. Hence our strategy for recovering  $k$  from  $g$  and  $g^k$  will be as follows. Put  $g^k$  in the upper

left corner of  $A$ , and make other entries either zeros or random, according to what we wrote in the previous paragraph. Then recover  $m$ , if possible, using an algorithm that is assumed to exist by the Proposition hypothesis. We say “if possible” because  $k$  may not have the form  $(m-1)(m-2)$ , in which case the algorithm may fail.

Re-write  $(m-1)(m-2)$  as  $(m - \frac{3}{2})^2 - \frac{9}{4}$ . Here  $\frac{3}{2}$  means  $3 \cdot 2^{-1}$ . The inverse of 2 exists since  $p \neq 2$ . Similarly,  $\frac{9}{4}$  is a particular fixed element of  $\mathbb{Z}_p$ ; denote it by  $r$ . Thus,  $k - r = (m - \frac{3}{2})^2$ , i.e.,  $(k - r)$  should be a quadratic residue modulo  $p$  for our hypothetical algorithm to work. If  $(k - r)$  is not a quadratic residue modulo  $p$ , then, since  $p$  is of the form  $p = 4n + 3$ , it is known that  $(r - k)$  should be a quadratic residue. Then we replace  $g^k$  in the matrix  $A$  by  $g^r(g^k)^{-1} = g^{r-k}$  and run our algorithm on this updated matrix as well.

Thus, we will have two similar algorithms running in parallel on two different matrices  $A$ : one with  $g^k$  in the upper left corner, the other one with  $g^{r-k}$ . One of these algorithms will recover  $m$ , and therefore  $k$  (or  $(r - k)$ ), and this completes the proof.  $\square$

We have also run some tests to see if the matrix  $K$  (the shared secret key) is indistinguishable from random. Figure 1 shows a histogram of values of the (1,1) entry of  $K$ , for a 200-bit  $p$ . Here values from 0 to  $p$  are split into 10 groups (“bins”) of size  $\lfloor \frac{p}{10} \rfloor$  and the number of values in each bin, out of 100,000 trials, is recorded. The histogram shows essentially uniform distribution of values between the bins. Histograms for values of other entries of the matrix  $K$  look the same, so  $K$  passes at least this simple randomness test.

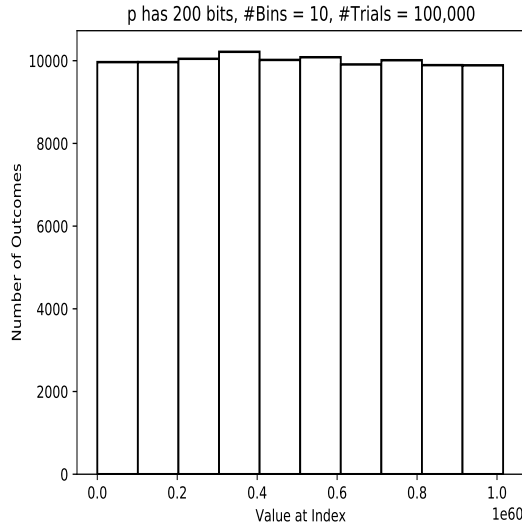
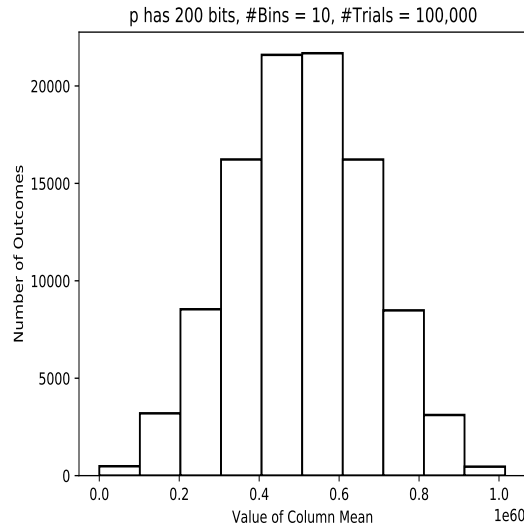
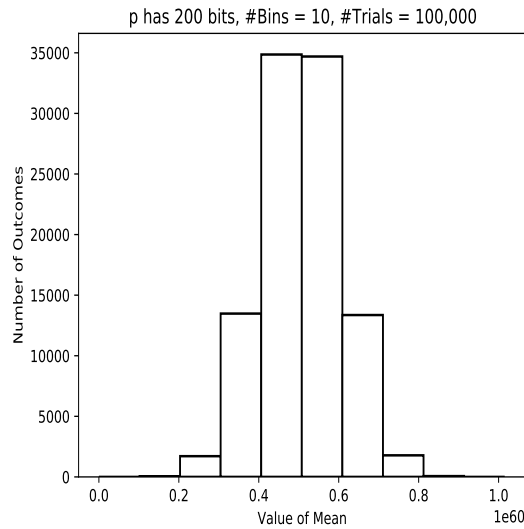
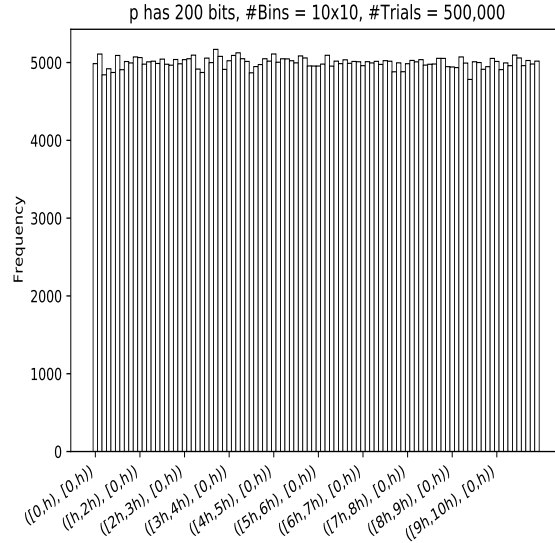


FIGURE 1. Values distribution for the (1,1) entry of  $K$

Another test we have run was computing the mean of the entries of  $K$  in each single row and each single column, as well as the mean of *all* entries of the matrix  $K$ . Figure 2 below shows a histogram of the means of the entries of the first column, out of 100,000 trials. Figure 3 shows a histogram of the means of all entries of the matrix  $K$ . Again, values from 0 to  $p$  are split into 10 groups (“bins”) of size  $\lfloor \frac{p}{10} \rfloor$ . By the central limit theorem, if several random variables are independent and identically distributed, then their mean is approximately normally distributed. Thus, our Figure 2 and Figure 3 are, though indirect, still an evidence of different entries of  $K$  being independent.

FIGURE 2. Mean distribution of the first column entries of  $K$ FIGURE 3. Mean distribution of all entries of  $K$ 

Another evidence of the matrix  $K$  being indistinguishable from random is independent distribution of values of different entries of  $K$ . This is illustrated by Figure 4. It shows that joint distribution of values of a pair of different entries of  $K$  is very close to uniform, i.e., for any two possible values  $(x, y)$  of the entries in such a pair, the probability to occur is  $\frac{1}{p^2} = \frac{1}{p} \cdot \frac{1}{p}$ , evidencing independence of  $x$  and  $y$ . The ticks on the x-axis of the histogram in Figure 4 split  $p$  possible values of the first entry in a pair into 10 bins corresponding to sets of values with an increment of  $h = \lfloor \frac{p}{10} \rfloor$ . Each of these 10 bins is split again into 10 bins of equal size to accommodate possible values of the second entry in a pair.

FIGURE 4. Distribution of values of a pair of entries of  $K$ 

## 7. IMPLEMENTATION AND PERFORMANCE

The scheme of this paper was implemented using Python. The code is available online, along with a challenge, see [7].

We note that the formulas (2) and (3) may create an impression that to compute the matrices  $A$  and  $B$ , Alice and Bob have to compute all powers of the matrix  $H$ , from 1 to  $(m - 1)$  or  $(n - 1)$ . However, this is not the case; a large power of  $(M, H)$  can be computed with the usual square-and-multiply method, and this will produce  $A$  or  $B$  as the corresponding first component.

The number of multiplications in  $\mathbb{Z}_p$  needed to compute  $H^n$  is, of course, larger than that to compute  $g^n$  for  $g \in \mathbb{Z}_p$ . To compute the square of a  $3 \times 3$  matrix, one needs 24 multiplications of elements of  $\mathbb{Z}_p$ , so one can expect our protocol to run at least 24 times slower than the classical Diffie-Hellman protocol. This is still fast enough to be practical; with a 2000-bit  $p$  the run time of the protocol is about 2 sec on a very basic computer. Besides, given that at this time there are no visible approaches whatsoever (other than brute force) to attack the protocol in this paper, the basic parameter  $p$  can probably be taken smaller than what is recommended for standard Diffie-Hellman protocols, and this will reduce the run time.

A particular 2000-bit safe prime  $p$  that we used in our computer simulations was 100458505468885003633418577656224333902553170484436983273607309963845847739507115860865964753239939027972338834707903941940188314348678981808910413754306718965087266944429878241410578991733762502442817585765598816431431108282071433256273345939973526837788093199292557721204590554061504359121574222368307048919809010489980961017706729222034791017130925070426893349814057145812995340991548906078333104951440614482037356443864699967124299012034397810342312642333550598174454039699165710636052240583294703998189114479917657125270697086234200442489544474659560583354052797579309573507121265302226528942789519.

## 8. CONCLUSIONS

- We have offered a key exchange protocol, resembling the classical Diffie-Hellman protocol, based on a semidirect product of two cyclic semigroups of matrices over  $\mathbb{Z}_p$ . One of the semigroups is additive, the other one multiplicative.

- In our protocol, no power of any matrix or any element of  $\mathbb{Z}_p$  is ever exposed, so all standard attacks on Diffie-Hellman-like protocols are not applicable.
- Security assumption, analogous to the *computational Diffie-Hellman assumption*, is computational infeasibility of recovering the value of a polynomial (of unknown degree) of a special form on two given matrices, from the values of two other polynomials (also of unknown degrees) on the same two matrices. A weaker security assumption is analogous to the discrete log assumption in  $\mathbb{Z}_p$  and is provably at least as hard.

## REFERENCES

- [1] W. Diffie and M. E. Hellman, *New Directions in Cryptography*, IEEE Transactions on Information Theory **IT-22** (1976), 644–654.
- [2] D. Grigoriev and V. Shpilrain, *Tropical cryptography II: extensions by homomorphisms*, Comm. Algebra. **47** (2019), 4224–4229.
- [3] M. Habeeb, D. Kahrobaei, C. Koupparis, V. Shpilrain, *Public key exchange using semidirect product of (semi)groups*, in: ACNS 2013, Lecture Notes Comp. Sc. **7954** (2013), 475–486.
- [4] D. Kahrobaei, V. Shpilrain, *Using semidirect product of (semi)groups in public key cryptography*, in: CiE 2016, Lecture Notes Comp. Sc. **9709** (2016), 132–141.
- [5] A. Menezes, P. van Oorschot, and S. Vanstone, *Handbook of Applied Cryptography*, CRC-Press 1996.
- [6] A. G. Myasnikov and V. Roman’kov, *A linear decomposition attack*, Groups, Complexity, Cryptology **7** (2015), 81–94.
- [7] Python code: MAKE, MAKE Challenge, <http://shpilrain.ccny.cuny.edu/MAKE.py.txt>  
[http://shpilrain.ccny.cuny.edu/MAKE\\_Challenge.py.txt](http://shpilrain.ccny.cuny.edu/MAKE_Challenge.py.txt)
- [8] V. Roman’kov, *Linear decomposition attack on public key exchange protocols using semidirect products of (semi)groups*, preprint. <http://arxiv.org/abs/1501.01152>
- [9] P. Shor, *Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer*, SIAM J. on Computing **26** (1997), 1484–1509.
- [10] V. Shpilrain, *Cryptanalysis of Stickel’s key exchange scheme*, in: Computer Science in Russia 2008, Lecture Notes Comp. Sc. **5010** (2008), 283–288.
- [11] E. Stickel, *A New Method for Exchanging Secret Keys*, in: Proc. of the Third Intern. Conf. on Information Technology and Applications (ICITA 05). Contemp. Math. **2** (2005), IEEE Computer Society, 426–430.

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