The Bitcoin Cash Backbone Protocol

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Abstract

Bitcoin Cash, created in 2017, is a "hard fork" from Bitcoin responding to the need for allowing a higher transaction volume. This is achieved by a larger block size, as well as a new difficulty adjustment (target recalculation) function(s) that acts more frequently (as opposed to Bitcoin's difficulty adjustment happening about every two weeks), resulting in a potentially different target *for each block*. While seemingly achieving its goal in practice, to our knowledge there is no formal analysis to back this proposal up.

In this paper we provide the first formal cryptographic analysis of Bitcoin Cash's target recalculation functions against all possible adversaries. We follow the analytical approach developed in the *Bitcoin backbone protocol* [Eurocrypt 2015 and follow-ups], of first establishing basic properties of the blockchain data structure, from which the properties of a robust transaction ledger (namely, *Consistency* and *Liveness*) can be derived. However, the more active target recalculation mechanism as well as the more pronounced fluctuation of the mining population (due in part to miners' behavior of switching chains towards achieving higher expected rewards) require new analytical tools.

We perform our analysis in the bounded-delay network model with dynamic participation of miners, of both ASERT and SMA (Bitcoin Cash's current and former recalculation functions, respectively) and conclude that in order to satisfy security (namely, properties satisfied except with negligible probability in the security parameter) considerably larger parameter values should be used with respect to the ones used in practice.

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1 Introduction

While opening up a new era in the area of cryptocurrencies, Nakamoto's Bitcoin protocol [Nak09b, Nak09a] has been critized for its heavy use of the computational resources needed by its underlying proof of work (PoW) mechanism as well as its relatively long settlement time of transactions. As a consequence, a number of alternative cryptocurrencies have been proposed with the purpose of ameliorating the above issues. One such proposal is Bitcoin Cash (BCH)[1], created in August 2017 as a "hard fork" of Bitcoin, with the original motivation of increasing the size of blocks, and thus allowing more transactions to be processed.

Another consequence of the lower computational "investment" required by these alternate cryptocurrencies (for example, the hashing power invested on Bitcoin Cash is approximately 5% of that on Bitcoin) is that miners are able to evaluate their expected reward and rapidly switch among different blockchains in order to achieve a higher profit, giving rise to an environment where the number of participating miners may fluctuate wildly, which in turn has a direct effect on suitable difficulty (of PoWs) recalculation mechanisms[2].

The above two aspects – desired higher transaction throughput and higher participation variation – are the motivation for this work. We focus on Bitcoin Cash as a representative of a newly proposed target recalculation function, and perform a formal analysis of the protocol's security under such dynamic environment. The importance of an adequate target recalculation mechanism has already been pointed out in [GKL17], where it is observed that if it is removed, the blockchain protocol becomes insecure in the dynamic setting *even* if all parties behave honestly, resulting in a blockchain that will diverge substantially (i.e., spawning "forks") as the number of miners keeps increasing, thus becoming vulnerable to many known cryptographic attacks. Furthermore, an inadequate target recalculation function may break the balance between miners' invested hashing power and reward, thus reducing their confidence in the system and leading them to quit the mining pool, arguably a situation that should be avoided.

Bitcoin Cash's target recalculation algorithm has gone through three stages. When created, the recalculation mechanism was a combination of Bitcoin's target recalculation function and an *emergency difficulty adjustment* (EDA) mechanism, which would suddenly enlarge targets if block generating intervals became too long. In November 2017, the initial function was replaced by a new function called *SMA* (for *Simple Moving Average*, or "cw-144"). At a high level, SMA is analogous to Bitcoin's recalculation function in the sense that it determines the next target based on an "epoch" of blocks, except that in the new algorithm the target value is recalculated more frequently – in fact, the target for *each block* varies. Moreover, the epoch of blocks changes with every block in a "sliding window" fashion.

Finally, the recent (November 2020) update introduces a control-theory-inspired recalculation function called *ASERT* (for *Absolutely Scheduled Exponentially Rising Targets*, [WISK20]), which is completely different from the previous ones. Specifically, ASERT is not epoch-based, and adjusts the difficulty level simply based on the current timestamp and the block height (details in Section 3.1). The rate at which the target changes is controlled by a "smoothing factor" m, which we will show is a crucial parameter in our analysis.

Overview of our results. Our Bitcoin Cash analysis is based on the approach introduced in [GKL15] which extracts and analyzes Bitcoin's basic data structure properties, namely, *common prefix* and *chain quality*, based on which the properties of a transaction ledger can be established. The analysis in [GKL15] is in the *static* setting (i.e., the protocol is executed by a *fixed* number

[1] https://www.bitcoincash.org/.

[2] As a reference, Bitcoin adjusts the PoWs' difficulty level every 2016 blocks – approximately every two weeks.

of parties, whose exact number, however, is unknown), where the target value T for each block is predetermined and hardcoded into the protocol at the very beginning. Following the above initial formulation, the above analysis was extended in [GKL17] to a *dynamic* environment, where parties (miners) may come and go, while still considering a synchronous network execution. Finally, in [GKL20] the authors extend the dynamic-network analysis to the bounded-delay network regime [DLS88, PSS17]. This last result concludes that with suitably selected parameters (albeit differing from real-world values), the Bitcoin backbone protocol abstraction achieves the required properties of a robust transaction ledger – namely, *consistency* and *liveness*.

Our main goal is to investigate to what extent our abstraction of the Bitcoin Cash protocol can implement a robust transaction ledger. Similarly to the above results, an important intermediate step is to show that, using Bitcoin Cash's target recalculation mechanism, the protocol satisfies the basic blockchain properties. As in prior work on dynamic environments, bounds on the ways that miners come in and drop out of the computation are necessary for things to work. We suggest a new methodology to capture how the number of parties can fluctuate. In a nutshell, our definition is comprised of two parts concerning *both* short-term and long-term participation. Specifically, consider a relatively long sequence of rounds S such that $|S| < \Sigma$. We then we require that $\max_{r \in S} n_r < \Gamma \cdot \min_{r \in S} n_r$, where n_r denotes the total number of active parties in round r and $\Gamma \in \mathbb{R}^+$. (This long period of time bound is similar to the one in [GKL17, GKL20].) In addition, we consider the variation of number of parties within a *short* period of time, imposing an upper bound on it. Specifically, for any consecutive subsequence $S' \preccurlyeq S$ such that $|S'| < \sigma$, $\max_{r \in S'} n_r < \gamma \cdot \min_{r \in S'} n_r$, for $\gamma \in \mathbb{R}^+$. We call an environment satisfying the above bounds a $(\langle \gamma, \sigma \rangle, \langle \Gamma, \Sigma \rangle)$-*respecting environment* (cf. Definition 1).

We now present a high-level overview of our analytical approach, starting, following [GKL17, GKL20], by defining the notion of a *typical execution*, wherein the probabilistic processes deriving from PoW attempts behave according to expectation. Crucial in this determination is the use of martingale bounds to establish that the variance is not too high, concluding that almost all polynomially bounded (in the security parameter κ) executions are typical. Next, we show that any chain held by an honest party enjoys (1) approximately accurate timestamps (i.e., no adversarial blocks are present with a timestamp that deviates too much from its real creation time) and (2) reasonable "goodness" (cf. [GKL20]), which means that for a round r, the probability of block generation given current target T_r and number of miners n_r, is very close to the initial block generation rate f (which is fixed with the genesis block and assumed to be "good").

Next, we demonstrate that a protocol's typical execution satisfies both the common prefix and chain quality properties, which as mentioned above serve as an intermediate step to build a robust transaction ledger. As a conclusion, (our abstraction of) the Bitcoin Cash protocol with chains of variable difficulty running in a bounded-delay network and suitably parameterized, satisfies, with overwhelming probability, the properties of consistency and liveness.

We emphasize that after Bitcoin Cash replaced the existing Bitcoin-like target recalculation function with ASERT, the analyses in [GKL17, GKL20] cannot be directly applied. The main distinction is that ASERT is no longer epoch based, and as such previous analyses regarding the duration of an epoch cease to hold, and so does the argument for goodness. We overcome this technical obstacle by introducing a new set of analytical tools which focus on the "calibration" of blocks' timestamps, and establish the conditions under which goodness holds in a sliding window (Lemmas 8), and then extend it to the entire execution (Lemma 6, 9 and Theorem 10). Moreover, Bitcoin adjusts the target values every 2016 blocks, while in Bitcoin Cash, since its recalculation mechanism is invoked for every block, the target values in the same epoch keep changing. We deal with this modification by classifying the distances from the calibrated timestamp into states, and then presenting a probabilistic convergence argument to the state including (and closely surround-

ing) the calibrated timestamp.

We also present our analysis of the former SMA recalculation function in Appendix C. For this part, we highlight that the approach of bounding the target variation by the "dampening" filter (τ – see Section C.1) in prior work fails. Furthermore, in contrast to prior work, the relationship between the target and the accumulated difficulty, which plays a significant role in the proof of "goodness," cannot be directly derived either. Our new treatment is based on bounding the variation of targets with the fluctuation of number of parties. Afterwards, by selecting blocks with appropriate targets in an epoch and employing a new condition (C4, Section C.2) to reverse the advantage of targets against accumulated difficulties (see proofs of Lemmas 24 and 25), we overcome the technical obstacles described above.

Finally, we compare our results with data from the Bitcoin Cash network, from which we extract the actual party fluctuation rate and network delay. Our main conclusion is that in order to satisfy security (namely, properties satisfied except with negligible probability) increased parameter values should be used with respect to the ones used in practice – specifically, a larger value of m (the smoothing factor in ASERT and epoch length in SMA) – concretely, $m = 432$, compared to the value $m = 288$ that is being used in ASERT (which in turn corresponds to 2 days) – should be adopted. In addition, regarding the SMA function, a larger dampening filter $\tau = 8$ in the SMA function should be used, instead of $\tau = 2$, which is the value that was last used (cf. Section D). Lastly, our comparison with the existing Bitcoin Cash network shows that the ASERT function performs better than the SMA function under a pronounced party fluctuation.

2 Preliminaries

2.1 Model and Definitions

We will present protocol our protocol analysis in the bounded-delay network, as in [GKL20]. The model extends the static setting model with a fixed number of parties used in [GKL14, GKL15] for the analysis of the Bitcoin backbone protocol, to the dynamic setting with a varying number of parties. These models are in turn based on Canetti's formulation of "real world" notion of protocol execution [Can00a, Can00b, Can01] for multi-party protocols. We describe the parts that are common to these two models and highlight the differences.

Round structure and protocol execution. As in previous formulations, which we follow almost *verbatim*, we let the protocol execution proceed in rounds (note these are not message passing rounds). An environment program denoted $\mathcal{Z}$ provides inputs to parties that execute the protocol Π. Our adversarial model is both *adaptive* and *rushing*. *Adaptive* means that the adversary $\mathcal{A}$ is allowed to take control of parties on the fly; while *rushing* means that $\mathcal{A}$ gets to see all honest parties' messages before deciding his strategy in any given round. The parties' communication mechanism is captured by a diffusion ("gossiping") functionality. It allows the order of messages to be controlled by $\mathcal{A}$, i.e., there is no atomicity guarantees in message broadcast [HT94], and, furthermore, the adversary is allowed to "spoof" the source information on every message (i.e., communication is not authenticated). Still, the adversary cannot change the contents of the message nor prevent them from being delivered (note that in the bounded delay model, honest parties' message delay has a upper bound of Δ rounds).

A system of interactive Turing machines (ITM's) in the sense of [Can00b] is formed by a pair of $(\mathcal{Z}, C)$ where C denotes a control program encoding the parties that *may* become active in a protocol execution(parties come from a universe $\mathcal{U}$ of parties), and $\mathcal{Z}$ denotes an environment program that interacts with other instances of programs that it spawns at the discretion of the control program

C. An execution is w.r.t. a program Π, an adversary $\mathcal{A}$ (which is another ITM) and the universe of parties $\mathcal{U}$. In addition, C maintains a flag for each instance of an ITM (abbreviated ITI in the terminology of [Can00b]), that is called the `ready` flag and is initially set to false for all parties. Observe that parties are unaware of the set of activated parties.

The joint hash function/network functionality. In order to ensure "fairness" to all miners, we present the dual functionality that is available to all parties running the protocol and the adversary and abstracts the hash function and the network.

- *The hash function functionality.* We adopt a joint random oracle [BR93] functionality to capture the parties' access to the hash function. It accepts queries of the form (compute, x) and (verify, x, y). For the first type of query, assuming x was never queried before, a value y is sampled from $\{0,1\}^\kappa$ and it is entered to a table T_H. If x was queried before the pair (x, y) is recovered from T_H. The value y is provided as an answer. For the second type of query, a membership test is performed on the table. In each round, each honest party P_i is allowed to ask q queries, and is given unlimited queries for "verification" for the function $H(\cdot)$. On the other hand, $\mathcal{A}$ is given a number of queries that cannot exceed $t_r \cdot q$ the upper bound on the number of corrupted parties that may be activated in round r; no verification queries are provided to $\mathcal{A}$ (the adversary can easily simulate those locally). The bound for the adversary is determined as follows. Whenever a corrupted party is activated the bound is increased by q; whenever a query is asked the bound is decreased by 1 (it is not necessary that the specific corrupted party makes the query). The value q is a polynomial function of κ in synchronous model but 1 in bounded delay network.
- *The diffusion functionality.* Message passing and round bookkeeping is maintained by this functionality. A round variable *round* is initialized to 0. For each party a string denoted by Receive() is maintained and the party is allowed to fetch the contents of its corresponding Receive() at any time. The functionality records all messages of the form $(\mathsf{Diffuse}, m)$ it receives from the parties. Completion of a round for a party is indicated by sending a special message (RoundComplete). The adversary $\mathcal{A}$ is allowed to receive all the currently recorded Diffuse messages at any time and messages to the Receive() strings as desired. The round is completed when the adversary submits its (RoundComplete) message. In such case, the functionality inspects the contents of all Receive() strings and includes any messages m that were diffused by the parties but not contributed by the adversary to the Receive() tapes (in the bounded-delay network, the adversary includes such messages Δ rounds ago thus guaranteeing message delivery up to Δ rounds). It also flushes any diffuse records that are placed in the Receive() string of all parties. The variable round is then incremented and a new round begins.

The dynamic setting. Given the functionalities as described above, the adversary can choose the termination of the round thus deciding on the spot how many honest parties were activated adaptively. In each round, the number of parties that are active in the protocol is denoted by n_r and is equal to the total number of parties that have submitted the (RoundComplete) indicator to the diffusion functionality and have their internal flag ready set to true. Determining n_r can only be done by examining the view of all honest parties and is not a quantity that is accessible to any of the honest parties individually. The number of corrupt parties controlled by $\mathcal{A}$ in a round r is similarly denoted by t_r. Note that the corrupt parties come from two sources—they are activated as corrupted or adaptively corrupted (at any time)—and once an honest party is corrupted, it is considered as controlled by the adversary in all subsequent rounds (if activated).

Parties, when activated, are able to read their input tape Input() and communication tape Receive() from the diffusion functionality. If a party finds that its ready flag is false, it enters a "bootstrapping" mode where it will diffuse a discovery message and synchronize (in the case of Nakamoto consensus, the party will send a request for the latest blockchains, will collect all of them

until a time-out parameter is reached and then will pick the most difficult one to start mining). When the synchronization phase terminates, the party will set its `ready` flag to true and after this point it will be counted among the honest parties. An honest party goes "offline" when it misses a round, i.e., the adversary issues a (RoundComplete) but that party misses the opportunity to complete its computation. To record this action, whenever this happens we assume that the party's ready flag is set to false (in particular this means that a party is aware that it went offline; note, however, that the party does not need to report it to anyone). Also observe that parties are unaware of the set of activated parties. As in previous works, we assume, without loss of generality, that each honest party has the same computational power.

The protocol class that we will analyze will not be able to preserve its properties for arbitrary sequences of parties. Thus, we restrict the way the number of parties fluctuate *both* over a long period of time and over a short period of time.

Definition 1. For $\gamma, \Gamma \in \mathbb{R}^+$, we call a sequence $(n_r)_{r\in\mathbb{N}}$ $(\langle\gamma,\sigma\rangle, \langle\Gamma,\Sigma\rangle)$-*respecting* if it holds that in a sequence of rounds S with $|S| \leq \Sigma$ rounds, $\max_{r\in S} n_r \leq \Gamma \cdot \min_{r\in S} n_r$ and for any consecutive sub-sequence rounds $S' \preccurlyeq S$ with $|S'| \leq \sigma$ rounds, $\max_{r\in S'} n_r \leq \gamma \cdot \min_{r\in S'} n_r$.

Remark 1. Definition 1 is consistent with the notion introduced in [GKL17], except that there the party fluctuation is expressed with respect to the number of honest parties. Note that the two fluctuations are related by a constant (concretely, $2-\delta$; see Table 1). Yet, it turns out that expressing statements in terms of honest-party fluctuation will considerably simplify them. Thus, in a few cases we will slightly overload notation and keep using γ, Γ, while we are actually referring to honest-party fluctuation.

Furthermore, note the addition to the definition of the parameters for short-term fluctuation.

An environment that is $(\langle\gamma,\sigma\rangle, \langle\Gamma,\Sigma\rangle)$-respecting should satisfy $\gamma^{\lfloor\Sigma/\sigma\rfloor} \geq \Gamma$, i.e., the accumulation of short-term fluctuation values could surpass the long-term fluctuation value. Further, note that although the sequence can capture exponential growth, the total run time is bounded by a polynomial (in κ), and thus the Σ/σ ratio is also polynomially bounded.

The term $\{\text{VIEW}^P_{\Pi,\mathcal{A},\mathcal{Z}}(z)\}_{z\in\{0,1\}^*}$ denotes the random variable ensemble describing the view of party P after the completion for an execution running protocol Π with environment $\mathcal{Z}$ and adversary $\mathcal{A}$, on input $z \in \{0,1\}^*$. We will consider a "standalone" execution without any auxiliary information, and restrict ourselves to executions with $z = 1^\kappa$; thus, we will simply refer to the ensemble by $\text{VIEW}^P_{\Pi,\mathcal{A},\mathcal{Z}}$.

Properties of protocols. In our theorems we will be concerned with *properties* of protocols Π running in the above setting. Such properties will be defined as predicates over the random variable $\text{VIEW}_{\Pi,\mathcal{A},\mathcal{Z}}$ by quantifying over all possible adversaries $\mathcal{A}$ and environments $\mathcal{Z}$. Note that all our protocols will only satisfy properties with a small probability of error in κ as well as in a parameter k that is selected from $\{1,\ldots,\kappa\}$ (note that in practice one may choose k to be much smaller than κ, e.g., $k = 6$).

2.2 Blockchain Notation

We introduce the blockchain notations that we employ for Bitcoin Cash. Due to the hard fork, Bitcoin Cash's blockchain does not differ much from Bitcoin's chain. We use similar notation to that in [GKL17]. Let $G(\cdot)$ and $H(\cdot)$ denote cryptographic hash functions with output in $\{0,1\}^\kappa$. A block with target $T \in \mathbb{N}$ is a quadruple of the form $B = \langle r, st, x, ctr\rangle$ where $st \in \{0,1\}^\kappa, x \in \{0,1\}^*$, and $r, ctr \in \mathbb{N}$. The quadruple satisfies the predicate $\mathsf{validblock}^T_q(B)$ defined as

$$(H(ctr, G(r, st, x)) < T) \wedge (ctr \leq q).$$

In real-world situation, as showed above, the parameter $q \in \mathbb{N}$ is a bound on the size of register ctr; in our treatment we remove this bound and allow ctr to be arbitrary. Instead, we use q to denote the maximum allowed number of hash queries in a round. We do this for convenience and our analysis applies in a straightforward manner to the case that ctr is restricted to the range $0 \leq ctr < 2^{32}$ and q is independent of ctr.

A *blockchain*, or simply a *chain* is a sequence of *blocks*. We call the rightmost block the *head* of the chain, denoted by $\text{head}(\mathcal{C})$. Note that the empty string ε is also a chain, and by convention we set $\text{head}(\varepsilon) = \varepsilon$. Parties that would like to extend a chain $\mathcal{C}$ with $\text{head}(\mathcal{C}) = \langle r, st, x, ctr \rangle$ should compute a valid block $B = \langle r', st', x', ctr' \rangle$ that satisfies $st' = H(ctr, G(r, st, x))$ and $r' > r$, where r' is called the *timestamp* of block B. In case $\mathcal{C} = \varepsilon$, by convention any valid block of the form $\langle r', st', x', ctr' \rangle$ may extend it. In either case we get an extended chain $\mathcal{C}_{\mathsf{new}} = \mathcal{C}B$ that satisfies $\text{head}(\mathcal{C}_{\mathsf{new}}) = B$.

The *length* of a chain $\text{len}(\mathcal{C})$ is its number of blocks. Consider a chain $\mathcal{C}$ of length ℓ and any non-negative integer k. We denote by $\mathcal{C}^{\lceil k}$ the chain resulting from "pruning" the k rightmost blocks. Note that for $k > \text{len}(\mathcal{C})$, $\mathcal{C}^{\lceil k} = \varepsilon$. If $\mathcal{C}_1$ is a prefix of $\mathcal{C}_2$ we write $\mathcal{C}_1 \preceq \mathcal{C}_2$.

Given a chain $\mathcal{C}$ of $\text{len}(\mathcal{C}) = \ell$, we let $\mathbf{x}_{\mathcal{C}}$ denote the vector of ℓ values that is stored in $\mathcal{C}$ and starts with the value of the first block. Similarly, $\mathbf{r}_{\mathcal{C}}$ is the vector that contains the timestamps of the blockchain $\mathcal{C}$.

For a chain of variable difficulty, the target T is recalculated for each block based on the round timestamps of the previous blocks. Specifically, there is a function $D : \mathbb{Z}^* \to \mathbb{R}$ which receives an arbitrary vector of round timestamps and produces the next target. The value $D(\varepsilon)$ is the initial target of the system. The difficulty of each block is measured in terms of how many times the block is harder to obtain than a block of target T_0. To be precise, the difficulty of a block with target T will be equal to T_0/T (We denote the difficulty as $1/T$ for simplicity in the full analysis section). We will use $\text{diff}(\mathcal{C})$ to denote the difficulty of a chain. This is equal to the sum of the difficulties of all the blocks that comprise the chain.

2.3 Blockchain Properties

We define two properties of blockchain that it will establish. They are related to the protocol application properties of Consistency and Liveness (cf. Section 3).

The *common prefix* property, parameterized by a value $k \in \mathbb{N}$, considers an arbitrary environment and adversary, and it holds as long as any two parties' chains are different only in their most recent k blocks. It is actually helpful to define the property between an honest party's chain and another chain that may be adversarial. The definition is as follows.

Definition 2 (Common Prefix Property)**.** The *common prefix* property Q_{cp} with parameter $k \in \mathbb{N}$ states that, at any round of the execution, if a chain $\mathcal{C}$ belongs to an honest party, then for any valid chain $\mathcal{C}'$ in the same round such that either $\text{diff}(\mathcal{C}') > \text{diff}(\mathcal{C})$, or $\text{diff}(\mathcal{C}') = \text{diff}(\mathcal{C})$ and $\text{head}(\mathcal{C}')$ was computed no later than $\text{head}(\mathcal{C})$, it holds that $\mathcal{C}^{\lceil k} \preceq \mathcal{C}'$ and $\mathcal{C}'^{\lceil k} \preceq \mathcal{C}$.

The second property, called *chain quality*, expresses the number of honest-party contributions that are contained in a sufficiently long and continuous part of a party's chain. Because we consider chains of variable difficulty it is more convenient to think of parties' contributions in terms of the total difficulty they add to the chain as opposed to the number of blocks they add (as done in [GKL14]). The property states that adversarial parties are bounded in the amount of difficulty they can contribute to any sufficiently long segment of the chain.

Definition 3 (Chain Quality Property)**.** The chain quality property Q_{cq} with parameters $\mu \in \mathbb{R}$ and $\ell \in \mathbb{N}$ states that for any party P with chain $\mathcal{C}$ in $\text{VIEW}_{\Pi,\mathcal{A},\mathcal{Z}}$, and any segment of that chain of difficulty d such that the timestamp of the first block of the segment is at least ℓ smaller than the timestamp of the last block, the blocks the adversary has contributed in the segment have a total difficulty that is at most $\mu \cdot d$.

2.4 Ledger Properties

The (main) application of the protocol is a *robust transaction ledger* (cf. [GKL15]), aimed at maintaining a ledger $\mathcal{L}$ of a serialized transaction sequence organized in the form of a blockchain. We now introduce the distinction between $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, where $\mathcal{L}$ denotes the settled ledger in the view of the party, and $\widetilde{\mathcal{L}}$ denotes the settled ledger with a sequence of transactions appended that are still not settled in the view of the party. Note that it always holds that $\mathcal{L} \preceq \widetilde{\mathcal{L}}$. The properties that the protocol application must satisfy are as follows:

- **Consistency**: For any two honest parties P_1, P_2, reporting $\mathcal{L}_1, \mathcal{L}_2$ at rounds $r_1 \leq r_2$, resp., it holds that $\mathcal{L}_1$ is a prefix of $\widetilde{\mathcal{L}}_2$.
- **Liveness** (Parameterized by $u \in \mathbb{N}$, the "wait time" parameter): If a transaction tx is provided to all honest parties for u consecutive rounds, then it holds that for any player P, tx will be in $\mathcal{L}$.

3 Bitcoin Cash

In this section we present the latest Bitcoin Cash's target (re)calculation function (ASERT), elaborate on the main differences with respect to Bitcoin (in addition to the size of blocks), and then provide a high-level description of (an abstraction of) the Bitcoin Cash protocol, which we call the *Bitcoin Cash backbone protocol* (cf. [GKL15, GKL17]).

3.1 Bitcoin Cash's Target Calculation Function

The November 2020 Bitcoin Cash update introduced a new difficulty adjustment algorithm, called *ASERT* (for Absolutely Scheduled Exponentially Rising Targets), aimed at achieving a stable block generation interval and transaction confirmation time as well as reducing the advantage of non-steady mining strategies. The new algorithm is derived from the control theory literature, and, specifically, from the notion of *Exponentially Moving Average* (EMA), a type of moving average that places greater weight on the most recent data points (in contrast, to *Simple Moving Average*, which applies an equal weight to all the observations in the period). For more details, refer to [WISK20], where a mathematical derivation of this function based on exponential smoothing, a common technique for removing noise from time series data, is provided.

ASERT adjusts the target based on the *anchor block,* i.e., a block whose target is denoted by T_0, and is used as a reference for all subsequent blocks. We use $f_0 \in (0, 1)$ to denote the ideal block generation rate (and thus $1/f_0$ represents the ideal block generation interval). At a high level, for a given block, ASERT compares its timestamp with the scheduled timestamp, which is a product of the ideal block generating interval and the block's *height* difference (e.g., for the i-th block, it is $(i-1)/f_0$). If the block's timestamp is ahead of the scheduled time, which means that the number of miners is larger than that corresponding to the anchor block, ASERT decreases the target (i.e., raises the difficulty); if it falls behind the scheduled time, then the target is increased (i.e., the difficulty is reduced). The amount by which the target is changed is based on m the

decay/smoothing factor. Specifically, the target is adjusted exponentially based on the rate of time difference and the smoothing time (i.e., m/f_0). For example, if the smoothing time is 2 days, then a block with a timestamp 2 days ahead of the scheduled timestamp would have a target whose value is half of the anchor target's.

Formally, ASERT is defined as follows. (Note that we assume the anchor block to be the first block and timestamp starts at 0.)

Definition 4. For fixed constants m, T_0, the target calculation function $D : \mathbb{Z}^* \to R$ is defined as

$$D(\varepsilon) = T_0 \text{ and } D(r_1, \ldots, r_v) = T_0 \cdot 2^{(r_v - (v-1)/f_0)/(m/f_0)}, \tag{1}$$

where $(r_1, \ldots, r_v)$ corresponds to a chain of v blocks with r_i,$i = 1, ..., v$ the timestamp of the i-th block.

Note that, as opposed to the previous Bitcoin Cash's recalculation function (SMA) and Bitcoin's target recalculation algorithm, ASERT is no longer epoch based. Moreover, ASERT is memoryless—i.e., the target for one block is only decided by the current timestamp and the block's height. No matter what timestamps the previous blocks have, they would not influence the current block's target value.

Remark 2. This new recalculation function removes the previous function's "dampening" filter, which raises the question of whether it would suffer from Bahack's raising difficulty attack [Bah13]. It turns out that it does not, since Equation (1) intrinsically prevents the difficulty from a sudden sharp increase. More specifically, assuming monotonically increasing timestamps, even if the adversary produces m blocks with the same timestamp, he can only double the difficulty value.

3.2 The Bitcoin Cash Backbone Protocol

The main changes introduced by Bitcoin Cash's hard fork from Bitcoin were an increase of the block size, replacement of the difficulty adjustment algorithm, and modification of the transaction rules; the protocol structure remained unchanged. For the analysis, we adopt the protocol abstraction presented in [GKL17], consisting of the main algorithm (Algorithm 4), which at the beginning of a round receives input (new transactions as well as chains sent by other miners); validates them and compares them (according to their accumulated difficulty) against the miner's current chain, adopting the one with highest difficulty (it could be the party's own); and attempts to extend the adopted chain by generating a PoW with the current round's difficulty value.

As in [GKL15], in our description of the protocol we intentionally avoid specifying the type of values/content that miners try to insert in the chain, the type of chain validation they perform (beyond checking for its structural properties with respect to the hash functions $G(\cdot), H(\cdot)$, and the way they interpret the chain. These checks and operations are handled by the external functions $V(\cdot), I(\cdot)$ and $R(\cdot)$ (the content validation function, the input contribution function and the chain reading function, resp.) which are specified by the application that runs "on top" of the backbone protocol. The Bitcoin Cash protocol in the dynamic setting comprises three algorithms *chain validation*, *chain comparison* and *proof of work*.

Chain validation. The validate algorithm performs a validation of the structural properties of a given chain $\mathcal{C}$. It is given as input the value q, as well as hash functions $H(\cdot), G(\cdot)$. It is parameterized by the content validation predicate predicate $V(\cdot)$ as well as by $D(\cdot)$, the *target calculation function* (see Section 3.1 and Appendix C.1). For each block of the chain, the algorithm checks that the proof of work is properly solved (with a target that is suitable as determined by the target calculation function), and that the counter *ctr* does not exceed q. Furthermore it collects the

inputs from all blocks, $x_{\mathcal{C}}$, and tests them via the predicate $V(\mathbf{x}_{\mathcal{C}})$. Chains that fail these validation procedure are rejected. (Algorithm 1.)

Algorithm 1 The *chain validation predicate*, parameterized by q, T, the hash functions $G(\cdot), H(\cdot)$, and the *content validation predicate* $V(\cdot)$. The input is $\mathcal{C}$.

function $\mathsf{validate}(r_{\mathsf{now}}, \mathcal{C})$
  $valid \leftarrow V(\mathbf{x}_{\mathcal{C}}) \wedge (\mathcal{C} \neq \varepsilon)$
  if $valid =$ **true then** ▷ The chain is non-empty and meaningful w.r.t. $V(\cdot)$
    $r' \leftarrow r_{\mathsf{now}}$
    $\langle r, st, x, ctr\rangle \leftarrow \mathrm{head}(\mathcal{C})$
    $st' \leftarrow H(ctr, G(r, st, x))$
    repeat
      $\langle s, x, ctr\rangle \leftarrow \mathrm{head}(\mathcal{C})$
      $T \leftarrow D(\mathbf{r}_{\mathcal{C}^{\lceil 1}})$ ▷ Calculate target based on $\mathcal{C}^{\lceil 1}$
      if $\mathsf{validblock}_q^T(\langle st, x, ctr\rangle) \wedge (H(ctr, G(r, st, x)) = st') \wedge (r < r')$ **then**
        $r' \leftarrow r$ ▷ Retain round timestamp
        $st' \leftarrow st$ ▷ Retain hash value
        $\mathcal{C} = \mathcal{C}^{\lceil 1}$ ▷ Remove the head from $\mathcal{C}$
      else
        $valid \leftarrow$ False
      end if
    until $(\mathcal{C} = \varepsilon) \vee (b =$ False$)$
  end if
  return *valid*
end function

Chain Comparison. The objective of the second algorithm, called $\mathsf{maxvalid}$, is to find the "best possible" chain when given a set of chains. The algorithm is straightforward and is parameterized by a $\mathsf{max}(\cdot)$ function that applies some ordering in the space of chains. The most important aspect is the chains' difficulty in which case $\mathsf{max}(\mathcal{C}_1, \mathcal{C}_2)$ will return the most difficult of the two. In case $\mathrm{diff}(\mathcal{C}_1) = \mathrm{diff}(\mathcal{C}_2)$, some other characteristic can be used to break the tie. In our case, $\mathsf{max}(\cdot, \cdot)$ will always return the first operand to reflect the fact that parties adopt the first chain they obtain from the network. (Algorithm 2.)

Algorithm 2 The function that finds the "best" chain, parameterized by function $\max(\cdot)$. The input is $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$.

function $\mathsf{maxvalid}(r, \{\mathcal{C}_1, \ldots, \mathcal{C}_k\})$
  $temp \leftarrow \varepsilon$
  for $i = 1$ to k **do**
    if $\mathsf{maxvalid}(r, \mathcal{C}_i)$ **then**
      $temp \leftarrow \mathsf{max}(\mathcal{C}, temp)$
    end if
  end for
  return *temp*
end function

Proof of work. The third algorithm, called pow, is the proof of work-finding procedure. It

takes as input a chain and attempts to extend it via solving a proof of work. This algorithm is parameterized by two hash functions $H(\cdot), G(\cdot)$ as well as the parameter q. Moreover, the algorithm calls the target calculation function $D(\cdot)$ in order to determine the value T that will be used for the proof of work. The procedure, given a chain $\mathcal{C}$ and a value x to be inserted in the chain, hashes these values to obtain h and initializes a counter ctr. Subsequently, it increments ctr and checks to see whether $H(ctr, h) < T$; in case a suitable ctr is found then the algorithm succeeds in solving the POW and extends chain $\mathcal{C}$ by one block. (Algorithm 3.)

Algorithm 3 The proof of work function, parameterized by q and hash functions $H(\cdot), G(\cdot)$. The input is $(x, \mathcal{C})$.

function $\mathsf{pow}(r, x, \mathcal{C})$
  if $\mathcal{C} = \varepsilon$ **then** ▷ Determine proof of work instance.
    $st \leftarrow 0$
  else
    $\langle r', st', x', ctr' \rangle \leftarrow \text{head}(\mathcal{C})$
    $st \leftarrow H(ctr' G(r', st', x'))$
  end if
  $ctr \leftarrow 1$
  $B \leftarrow \varepsilon$
  $T \leftarrow D(\mathbf{r}_{\mathcal{C}})$ ▷ Calculate target for next block based on timestamps.
  $h \leftarrow G(r, st, x)$
  while $(ctr \leq q)$ **do**
    if $H(ctr, h) < T$ **then** ▷ Proof of work succeeds and a new block is created.
      $B \leftarrow \langle r, st, x, ctr \rangle$
      break
    end if
    $ctr \leftarrow ctr + 1$
  end while
  $\mathcal{C} \leftarrow \mathcal{C}B$ ▷ Chain is extended.
  return $\mathcal{C}$
end function

Bitcoin Cash backbone protocol. The core of the Bitcoin Cash backbone protocol with variable difficulty is similar to that in [GKL15], with several important distinctions. First is the procedure to follow when the parties become active. Parties check the `ready` flag they possess, which is false if and only if they have been inactive in the previous round. In case the `ready` flag is false, they diffuse a special message '**Join**' to request the most recent version of the blockchain(s). Similarly, parties that receive the special request message in their Receive() tape broadcast their chains. As before parties, run "indefinitely" (our security analysis will apply when the total running time is polynomial in κ). The input contribution function $I(\cdot)$ and the chain reading function $R(\cdot)$ are applied to the values stored in the chain. Parties check their communication tape Receive() to see whether any necessary update of their local chain is due; then they attempt to extend it via the POW algorithm pow. The function $I(\cdot)$ determines the input to be added in the chain given the party's state st, the current chain $\mathcal{C}$, the contents of the party's input tape Input() and communication tape Receive(). The input tape contains two types of symbols, Read and (Insert, *value*); other inputs are ignored. In case the local chain $\mathcal{C}$ is extended the new chain is diffused to the other parties. Finally, in case a Read symbol is present in the communication tape,

the protocol applies function $R(\cdot)$ to its current chain and writes the result onto the output tape Output(). The pseudocode of the backbone protocol is presented in Algorithm 4.

Algorithm 4 The Bitcoin Cash backbone protocol in the dynamic setting at round "`round`" on local state $(st, \mathcal{C})$ parameterized by the *input contribution function* $I(\cdot)$ and the *chain reading function* $R(\cdot)$. The `ready` flag is **false** if and only if the party was inactive in the previous round.

1: **if** `ready` = **true then**
2: Diffuse('`ready`')
3: $\widetilde{\mathcal{C}} \leftarrow \mathsf{maxvalid}(\mathcal{C}$ all chains $\mathcal{C}'$ found in Receive())
4: $\langle st, x\rangle \leftarrow I(st, \widetilde{\mathcal{C}}, \texttt{round}, \text{Input}(), \text{Receive}())$
5: $\mathcal{C}_{\mathsf{new}} \leftarrow \mathsf{pow}(\texttt{round}, x, \widetilde{\mathcal{C}})$
6: **if** $(\mathcal{C} \neq \mathcal{C}_{\mathsf{new}}) \vee$ ('**Join**' $\in$ Receive()) **then**
7: $\mathcal{C} \leftarrow \mathcal{C}_{\mathsf{new}}$
8: Diffuse($\mathcal{C}$) ▷ chain is diffused when it is updated or when someone wants to join.
9: **end if**
10: **if** Input() contains Read **then**
11: **write** $R(\mathbf{x}_{\mathcal{C}})$ to Output()
12: Diffuse(RoundComplete)
13: **end if**
14: **else**
15: `ready` $\leftarrow$ **true**
16: Diffuse(Join, RoundComplete)
17: **end if**

4 Protocol Analysis

In this section we present the analysis of the Bitcoin Cash backbone protocol using the recalculation function ASERT in bounded-delay networks and dynamic environments, with the ultimate goal of establishing under what conditions the transaction ledger's properties are satisfied. While the basic assumptions and general approach are common to those in [GKL20] (albeit with different parameters), we introduce new tools in order to carry out our analysis.

First, we highlight the difference of notations in our analysis with respect to [GKL17, GKL20]. We use n_r to denote the total number of parties at round r ([GKL17, GKL20] uses n_r to denote the number of *honest* parties), and t_r to denote the number of corrupted parties. Thus, the number of honest parties at round r is $n_r - t_r$. For simplicity, we use $h_r = n_r - t_r$. This follows the tradition in the secure multiparty computation literature and the notation in [GKL15]. In addition, instead of considering the fluctuation ratio of the honest parties, as in [GKL17, GKL20], in our model we consider the fluctuation of the total number of parties (cf. Definition 1 and Fact 1).

For convenience, Table 1 summarizes the set of parameters introduced so far. Note that our security parameter is κ, and $\varphi = \Theta(m) = \mathsf{polylog}(\kappa)$.

Our probability space is over all executions of length at most some polynomial in κ. We will denote by $\mathbf{Pr}$ the probability measure of this space. We also define the random variable $\mathcal{E}$ taking values on this space and with a distribution induced by the random coins of all entities (adversary, environment, parties) and the random oracle.

Recall that in our modeling of a bounded-delay network, each party's query bound to the random oracle (RO) per round is $q = 1$. Now suppose that at round r exactly n parties query the

Table 1: *Summary of parameters (ASERT).*

- δ: Advantage of honest parties, $\forall r(t_r/h_r < 1 - \delta)$.
- $\gamma, \sigma, \Gamma, \Sigma$: Determine how the number of parties fluctuates across rounds in a period (cf. Definition 1 and Fact 1).
- f: Probability that at least one honest party succeeds generating a PoW in a round assuming h_0 parties and target T_0 (the protocol's initialization parameters).
- m: Smoothing factor (cf. Definition 4).
- τ: Parameter that regulates the target that the adversary could query the PoW with.
- ϵ: Quality of concentration of random variables (cf. Definition 7).
- κ: The length of the hash function output.
- φ: Related to the properties of the protocol.
- L: The total number of rounds in the execution of the protocol.

RO with a target T. Then the probability that at least one of them will succeed is

$$f(T, h) = 1 - (1 - pT)^h \leq phT, \text{ where } p = 1/2^\kappa.$$

As in prior work, we denote $f_0 = f(T_0, h_0)$, where T_0 and h_0 are the initial target and estimate of number of honest parties, respectively. The objective of the target recalculation mechanism is to maintain a target T for each party such that $f(T, h_r) \approx f_0$ for all rounds r. For notational simplicity, we will drop the subscript from f_0, and will always specify the two arguments of $f(\cdot, \cdot)$ to avoid confusion.

During a round r of an execution E, the honest parties might be split and work on different chains, and thus might query the RO on different targets. Denote by $T_r^{\min}$ and $T_r^{\max}$ the minimum and maximum of these targets, respectively. We say r is a target-recalculation point of a valid chain $\mathcal{C}$, if there is a block with timestamp r (recall that the target is adjusted for every block).

Next, and following [GKL17], we define the properties *goodness* and *accuracy*, which we will then show most executions satisfy, and which will help achieve the desired application's properties. First, we say that round r is a *target-recalculation point* of a chain $\mathcal{C}$, if $\mathcal{C}$ has a block with timestamp r.

Definition 5 (Goodness)**.** Round r is *good* if $f/2\gamma(2-\delta)\Gamma^3 \leq ph_r T_r^{\min}$ and $ph_r T_r^{\max} \leq 2\gamma\Gamma^3 f$. A target-recalculation point r is *good* if the target T for the next block satisfies $f/2(2-\delta)\Gamma^3 \leq ph_r T \leq 2\Gamma^3 f$. A chain is *good* if all its target-recalculation points are good.

Definition 6 (Accuracy)**.** A block created at round u is *accurate* if it has a timestamp v such that $|u - v| \leq \ell + 2\Delta$. A chain is *accurate* if all its blocks are accurate. A chain is *stale*, if for some $u \geq \ell + 2\Delta$, it does not contain an honest block with timestamp $v \geq u - \ell - 2\Delta$.

For a given round r, we let $\mathcal{S}_r$ denote the set of chains that belong, or could potentially belong to an honest party. Being explicit about this set of chains will help in the formulation of a number of predicates (see below). Specifically, $\mathcal{S}_r$ includes[3]:

- Chain $\mathcal{C}$ that belongs to an honest party;
- chain $\mathcal{C}$ with $\text{diff}(\mathcal{C}) > \text{diff}(\mathcal{C}')$ for some chain $\mathcal{C}'$ of an honest party; and

[3]Note that these chains should exist and be valid at round r.

– chain $\mathcal{C}$ with $\mathrm{diff}(\mathcal{C}) = \mathrm{diff}(\mathcal{C}')$ for some chain $\mathcal{C}'$ of an honest party and $\mathrm{head}(\mathcal{C})$ was computed no later than $\mathrm{head}(\mathcal{C}')$.

Random variables. We are interested in estimating the difficulty acquired by honest parties during a sequence of rounds. For a given round r, the following real-valued random variables are defined in [GKL20]:

– D_r: Sum of the difficulties of all blocks computed by honest parties.
– Y_r: Maximum difficulty among all blocks computed by honest parties.
– Q_r: Equal to Y_r when $D_u = 0$ for all $r < u < r + \Delta$ and 0 otherwise.

A round r such that $D_r > 0$ is called *successful* and one where $Q_r > 0$ *isolated-successful*.

Regarding the adversary, in order to overcome the fact that he can query the oracle for arbitrarily low targets and thus obtain blocks of arbitrarily high difficulty, we would like to upper-bound the difficulty he can acquire during a set J of queries. This is achieved by associating a set of consecutive adversarial queries J with the target of its first query. We denote this target $T(J)$, and say that $T(J)$ is *associated* with J. We then define $A(J)$ and $B(J)$ to be equal to the sum of the difficulties of all blocks computed by the adversary during queries in J for target at least $T(J)/\tau$ and $T(J)$, respectively – i.e., queries in J for targets less than $T(J)/\tau$ (resp. $T(J)$) do not contribute to $A(J)$ (resp. $B(J)$).

For simplicity, we write $h(S) = \sum_{r \in S} h_r$ for a set of rounds S and queries J (similarly, $t(S)$, $D(S)$, $Y(S)$, $Q(S)$, $A(J)$ and $B(J)$).

Regarding protocol executions, let $\mathcal{E}_{r-1}$ fix the execution just before round r. In particular, a value E_{r-1} of $\mathcal{E}_{r-1}$ determines the adversarial strategy and so determines the targets against which every party will query the oracle at round r and the number of parties h_r and t_r, but it does not determine D_r or Q_r. For an adversarial query j we will use, slightly overloading notation, $E_{j-1}^{(J)}$ to denote the execution just before this query.

The following fact is a consequence of Definition 1 (respecting environments):

Fact 1. Let S be a set of at most Σ consecutive rounds in a $(\langle \gamma, \sigma \rangle, \langle \Gamma, \Sigma \rangle)$-respecting environment and $U \subseteq S$.
(a) If $U \preccurlyeq S$ and $|U| < \sigma$,

$$\frac{h_S}{\Gamma} \leq \frac{h(S)}{|S|} \leq \Gamma \cdot h_S \quad \text{and} \quad \frac{h_U}{\gamma} \leq \frac{h(U)}{|U|} \leq \gamma \cdot h_U,$$

where $h_S \in \{h_r : r \in S\}$ and $h_U \in \{h_r : r \in U\}$.
(b)

$$h(S) \leq (1 + \frac{\Gamma |S \backslash U|}{|U|}) h(U) \quad \text{and} \quad |S| \sum_{r \in S} (p h_r)^2 \leq \Gamma (\sum_{r \in S} p h_r)^2.$$

Proof. (a) As proved in [GKL17], the average of several numbers (i.e., $h(S)/|S|$) is bounded by their minimum and maximum, we get the desired inequality.

(b) We learn by (a) that $h(S \backslash U) \leq |S \backslash U| \cdot \Gamma \cdot h(U)/|U|$. Note that $h(S) = h(S \backslash U) + h(U)$, by adding $h(U)$ we get the first inequality. The second inequality is a simple implementation of Theorem 18 with $a_k = p h_r$, $b_k = 1$, $w_k = 1$ and $M = \Gamma' \cdot m$. □

In order to obtain meaningful concentration of random variables, we have to consider a sufficiently long sequence with a number of rounds at least

$$\ell = \frac{4(2-\delta)(1+3\epsilon)}{\epsilon^2 f [1 - 2\gamma \Gamma^3 f]^{\Delta+1}} \cdot \max\{\Delta, \tau\} \cdot \gamma \Gamma^4 \cdot \varphi. \tag{2}$$

We will assume that ℓ is appropriately small compared to the length m of a sliding interval/window. Specifically,

$$2\ell + 6\Delta \leq \frac{\epsilon m}{2\gamma\Gamma^3 f}. \tag{C1}$$

In addition, we would like the advantage δ of the honest parties over adversarial parties to be large enough to absorb error factors. Thus, we require the following inequalities:

$$[1 - 2\gamma\Gamma^3 f]^{\Delta} \geq 1 - \epsilon \text{ and } \epsilon \leq \delta/8 \leq 1/8. \tag{C2}$$

Next, we show a chain-growth lemma referring to accumulated difficulty (cf. [GKL20]), as opposed to number of blocks in the original formulations [GKL15, KP15].

Lemma 1 (Chain Growth)**.** *Suppose that at round u of an execution E an honest party diffuses a chain of difficulty d. Then, by round v, every honest party has received a chain of difficulty at least $d + Q(S)$, where $S = \{r : u + \Delta \leq r \leq v - \Delta\}$.*

Proof. If two blocks are obtained at rounds which are at distance at least Δ, then we are certain that the later block increased the accumulated difficulty. To be precise, assume $S^* \subseteq S$ is such that, for all $i, j \in S^*$, $|i - j| \geq \Delta$ and $Y_i > 0$. We argue that, by round v, every honest party has a chain of difficulty at least

$$d + Y(S^*) \geq d + Q(S).$$

Observe first that every honest party will receive the chain of difficulty d by round $u + \Delta$ and so the first block obtained in S^* extends a chain of weight at least d. Next, note that if a block obtained in S^* is the head of a chain of weight at least d', then the next block in S^* extends a chain of weight at least d'. □

4.1 Typical Executions

The notion of *typical executions* was introduced in the analysis framework we are following [GKL15] in order to capture situations where an execution E's progress does not deviate too much from its expected (desired) progress. Since executions consist of rounds, and within rounds parties perform Bernoulli trials, we can calculate the expected progress when given the corresponding probabilities. On this basis, if the difference between the real execution and its expectation is reasonable, E is declared "typical." Note that besides expectation, the variance should also be taken into consideration. We will later show (applying Theorem 16 – martingale inequality) that either the variance is too high with respect to a set of rounds, or the parties have made progress during these rounds as expected.

In addition to the behavior of the random variables described above, bad events may occur related to the underlying hash function $H(\cdot)$, which is modeled as a random oracle and used to obtain PoWs. The bad events are *insertion* (of a block in between two consecutive blocks), *copy* (same block exists in two different position of the blockchain), and *prediction* (a block extends one with an earlier creation time). Refer to [GKL20] for a precise definition. A typical execution will rule out these bad events as well.

We are now ready to specify what is a typical execution in our setting (compare with [GKL20]'s).

Definition 7 (Typical execution)**.** An execution E is *typical* if the following hold:

(a) For any set S of at least ℓ consecutive good rounds,

$$(1 - \epsilon)[1 - 2\gamma\Gamma^3 f]^{\Delta} ph(S) < Q(S) \leq D(S) < (1 + \epsilon) ph(S).$$

(b) For any set J indexing a set of consecutive adversarial queries and $\alpha(J) = 2(\frac{1}{\epsilon} + \frac{1}{3})\varphi/T(J)$,

$$A(J) < p|J| + \max\{\epsilon p|J|, \tau\alpha(J)\} \text{ and } B(J) < p|J| + \max\{\epsilon p|J|, \alpha(J)\}.$$

(c) No insertions, no copies, and no predictions occurred in E.

The next proposition is a simple application of Definition 7 and the honest-majority assumption.

Proposition 2. *Let E be a typical execution in a $(\langle\gamma, \sigma\rangle, \langle\Gamma, \Sigma\rangle)$-respecting environment. Let $S = \{r \mid (u \leq r \leq v) \wedge (v - u \geq \ell)\}$ be a set of consecutive good rounds and J the set of adversarial queries in $U = \{r \mid u - \Delta \leq r \leq v + \Delta\}$. Then the following inequalities hold:*

(a) $(1 + \epsilon)p|J| \leq Q(S) \leq D(U) < (1 + 5\epsilon)Q(S)$.
(b) If w is a good round such that $|w - r| \leq \Sigma$ for any $r \in S$, then $Q(S) > (1 + \epsilon)[1 - 2\gamma\Gamma^3 f]^{\Delta}|S|ph_w/\Gamma$. If, in addition, $T(J) \geq T_w^{\min}$, then $A(J) < (1 - \delta + 3\epsilon)Q(S)$.

Proof. (a) The middle inequality directly follows the definition of the random variables Q and D. For the other two inequalities, since $|J| \leq (1 - \delta)h(U)$ we only have to compare the value of $h(S)$ and $h(U)$. Note that Fact 1(b) implies that $h(U\backslash S) \leq \Gamma \cdot |U\backslash S| \cdot \frac{h(S)}{|S|} \leq \Gamma \cdot \frac{h(S)}{\ell} \cdot 2\Delta$. Thus,

$$h(U) \leq (1 + \frac{2\Gamma\Delta}{\ell})h(S) < (1 + \frac{\epsilon^2}{2})h(S).$$

(b) The first inequality is a direct application of Fact 1(a). For the second one, round w is good implies $ph_w T(J) \geq ph_w T_w^{\min} \geq f/2\gamma(2 - \delta)\Gamma^3$. Condition (C2) implies that

$$\epsilon(1 - 2\epsilon)ph(S) \geq \epsilon(1 - 2\epsilon)|S| \cdot \frac{ph_w T_w^{\min}}{\Gamma T(J)} \geq \frac{\epsilon(1 - 2\epsilon)f\ell}{2\gamma(2 - \delta)\Gamma^4 T(J)} > \tau\alpha(J).$$

As showed in (a), $p|J| \leq (1 - \delta + \epsilon^2/2)ph(S)$. For either $\epsilon p|J|$ or $\tau\alpha(J)$ in Definition 7(b), we obtain $A(J) < (1 - \delta + 3\epsilon)Q(S)$. □

We are now able to show that almost all Bitcoin Cash backbone protocol executions polynomially bounded (in κ) are typical (the proof is presented in Appendix B). Formally:

Theorem 3. *Assuming the Bitcoin Cash backbone protocol runs for L rounds, the event "E is not typical" is bounded by* $\mathsf{poly}(L) \cdot e^{-\Omega(\mathsf{polylog}(\kappa))}$.

4.2 Accuracy and Goodness

Next, we consider accuracy and goodness over the space of typical executions in a $(\langle\gamma, \ell + 2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$-respecting environment, as well as implications between the two. We assume that all the requirements for the initialization parameters h_0 and T_0 are satisfied.

Lemma 4. *Let E be a typical execution in a $(\langle\gamma, \ell+2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$-respecting environment. If E_{r-1} is good, then there are no stale chains in $\mathcal{S}_r$.*

Proof. For the sake of a contradiction, consider a chain $\mathcal{C} \in \mathcal{S}_r$ which has not been extended by the honest party for at least $\ell + 2\Delta$ rounds. Without loss of generality, let r denote the least round with this property. Let B be the last block of $\mathcal{C}$ computed by the honest party (possibly the genesis block) and let w be its timestamp. Set $S = \{u : w + \Delta \leq u \leq r - \Delta\}$ and $U = \{u : w \leq u \leq r\}$.

Therefore, to reach a contradiction it suffices to show that the adversary's accumulated difficulty $d < Q(S)$.

Let J denote the queries in U starting from the first adversarial query attempting to extend B. We learn from Condition (C1) that targets the adversary can query during U is bounded by $T(J)/\tau$. Thus, $d < A(J)$. If $A(J) < (1+\epsilon)p|J|$, then $A(J) < Q(S)$ is obtained by Proposition 2(a). Otherwise, $A(J) < (\frac{1}{\epsilon}+1)\tau\alpha(J) = 2(\frac{1}{\epsilon}+1)(\frac{1}{\epsilon}+\frac{\epsilon}{3})\tau\varphi/T(J)$. However, by considering only the first ℓ rounds in S, $h(S) \geq h_u\ell/\gamma$. We have

$$Q(S) > (1-\epsilon)[1-2\gamma\Gamma^3 f]^{\Delta} \cdot \frac{ph_u\ell T(J)}{\gamma T(J)} > \frac{(1-\epsilon)[1-2\gamma\Gamma^3 f]^{\Delta} f\ell}{2\gamma(2-\delta)\Gamma^4 T(J)} \geq \frac{2(1-\epsilon)(1+3\epsilon)\tau\varphi}{\epsilon^2 T(J)} \geq A(J).$$

In either situation, we obtain the desired inequality $d < Q(S)$. □

Corollary 5. *Let E be a typical execution in a $(\langle\gamma, \ell+2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$-respecting environment. If E_{r-1} is good, then all chains in $\mathcal{S}_r$ are accurate.*

Proof. Suppose—towards a contradiction—that, for some $w \leq r$, $\mathcal{C} \in \mathcal{S}_r$ contains a block which is not accurate and let $u \leq w$ be the timestamp of this block and v its creation time. If $u - v > \ell + 2\Delta$, then every honest party would consider $\mathcal{C}$ to be invalid during rounds $v, v+1, \ldots, u$. If $v - u > \ell + 2\Delta$, then in order for $\mathcal{C}$ to be valid it should not contain any honest block with timestamp in $u, u+1, \ldots, v$. (Note that we are using Definition 7(c) here as a block could be inserted later.) In either case, $\mathcal{C} \in \mathcal{S}_r$ but is stale, contradicting Lemma 4. □

Next, we move to goodness. We remark that, in contrast to SMA and Bitcoin's recalculation function, ASERT is no longer an epoch-based algorithm. Thus, previous analysis regarding the duration of an epoch (cf. [GKL20]) do not hold, and neither does the argument for goodness, so a new analysis is required.

We start by presenting preliminary observations regarding the ASERT function. Note that the target for the next block in ASERT is now merely related to the block's timestamp and height. For the i-th block with timestamp r and corresponding number of honest parties h_r, it is not hard to see that if $r = (i-1)/f + (m/f)\log(h_0/h_r)$, the i-th block would have block generating rate exactly f. We call this timestamp r the *calibrated timestamp* for block $\mathcal{B}_i$. Additionally, r is a good target recalculation point if it satisfies

$$\frac{i-1}{f} + \frac{m}{f}\log(2(2-\delta)\Gamma^3 \cdot \frac{h_0}{h_r}) \leq r \leq \frac{i-1}{f} + \frac{m}{f}\log(2\Gamma^3 \cdot \frac{h_0}{h_r}). \tag{3}$$

We now define a new random variable to describe the deviation of timestamps: we use X_i to express by how much the i-th block deviates from its *calibrated timestamp*. For i-th block with timestamp r_i and number of honest parties h_i,

$$X_1 = 0 \text{ and } X_{i+1} = X_i + (r_{i+1} - r_i) - \frac{1}{f} - \frac{m}{f}\log\frac{h_{i+1}}{h_i} \text{ for } i \geq 0.$$

The three parts in the definition of X_{i+1} are as follows: (1) $(r_{i+1} - r_i)$ represents the difference of their timestamps; (2) $1/f$ is the ideal block interval; (3) $(m/f)\log(h_{i+1}/h_i)$ is the difference of the respective number of honest parties. For good blocks, the variable should satisfy that $-(m/f)\log 2(2-\delta)\Gamma^3 \leq X_i \leq (m/f)\log 2\Gamma^3$.

As defined, X_i is sensitive to the fluctuation of number of parties. Since we can only bound the fluctuation rate during a fixed number of rounds, X_i is not suitable for the analysis. To overcome this, we consider a new random variable W_i within a $(\langle\gamma, \ell+2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$-respecting

environment, and then show that if this new random variable satisfies certain conditions, then X_i presented above satisfies the ideal bound (3).

In more detail, we consider the calibrated timestamp $r = (i-1)/f + (m/f)\log(h_0/h_r)$, and a sliding window of $4(m/f)\log\Gamma$ rounds starting with block B_u and number of honest parties h_u. For each subsequent block in this window, we replace h_r with h_u and call $r' = (i-1)/f + (m/f)\log(h_0/h_u)$ the *relatively calibrated timestamp with respect to* B_u for i-th block $B_i, i \geq u$. We can now define a new random variable W_i expressing by how much the i-th block deviates from its relatively calibrated timestamp (wrt B_u). That is, for i-th block with timestamp r_i,

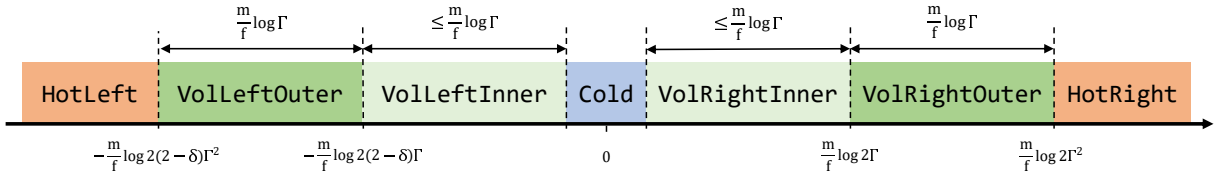
$$W_u = X_u \text{ and } W_{i+1} = W_i + (r_{i+1} - r_i) - \frac{1}{f} \text{ for } i \geq u.$$

Now the definition of the random variable only consists of two parts: the difference of their timestamps, and the ideal block interval. For good target recalculation points, W_i should satisfy

$$-\frac{m}{f}\log 2(2-\delta)\Gamma^2 \leq W_i \leq \frac{m}{f}\log 2\Gamma^2.$$

Next, we define seven states based on values of the random variable W_i. Studying the possible transitions between them allows us to establish that typical executions are good (as well as accurate). Refer to Figure 1. Let $h_{\mathsf{max}}, h_{\mathsf{min}}$ denote the maximum and minimum of the number of parties during the sliding window, respectively.

Figure 1: *The states based on the values of random variable W_i.*



$$\begin{aligned}
\mathsf{HotLeft}_i &\triangleq W_i < -\frac{m}{f}\log 2(2-\delta)\Gamma^2 \\
\mathsf{VolatileLeftOuter}_i &\triangleq -\frac{m}{f}\log 2(2-\delta)\Gamma^2 \leq W_i < -\frac{m}{f}\log 2(2-\delta)\Gamma \\
\mathsf{VolatileLeftInner}_i &\triangleq -\frac{m}{f}\log 2(2-\delta)\Gamma \leq W_i < -\frac{m}{f}\log\frac{2(2-\delta)h_{\mathsf{max}}}{h_u} \\
\mathsf{Cold}_i &\triangleq -\frac{m}{f}\log\frac{2(2-\delta)h_{\mathsf{max}}}{h_u} \leq W_i \leq \frac{m}{f}\log\frac{2h_u}{h_{\mathsf{min}}} \\
\mathsf{VolatileRightInner}_i &\triangleq \frac{m}{f}\log\frac{2h_u}{h_{\mathsf{min}}} < W_i \leq \frac{m}{f}\log 2\Gamma \\
\mathsf{VolatileRightOuter} &\triangleq \frac{m}{f}\log 2\Gamma < W_i \leq \frac{m}{f}\log 2\Gamma^2 \\
\mathsf{HotRight}_i &\triangleq W_i > \frac{m}{f}\log 2\Gamma^2
\end{aligned}$$

States VolatileLeftOuter and VolatileRightOuter are of fixed length $(m/f)\log\Gamma$, while states VolatileLeftInner and VolatileRightInner are of length at most $(m/f)\log\Gamma$. These lengths will play a significant role in the following analysis of goodness.

We aim to show that for blocks $B_u, \ldots, B_v$ generated in an interval of length $4(m/f)\log\Gamma$ rounds, the following holds:

- For a block $B_i,\ i > u$ with W_i (w.r.t. B_u) in state Cold, we can construct a new $4(m/f)\log\Gamma$-round sliding window with W_i (w.r.t. B_i) in state $\mathsf{VolatileLeftInner}$, $\mathsf{VolatileRightInner}$ or Cold.
- If W_u is in state $\mathsf{VolatileLeftInner}$, $\mathsf{VolatileRightInner}$ or Cold, the probability of $W_i,\ i > u$ reaching $\mathsf{HotLeft}$ or $\mathsf{HotRight}$ is negligible.
- If W_u is in state $\mathsf{VolatileLeftInner}, \mathsf{VolatileRightInner}$ or Cold, $W_i,\ i > u$ will return to Cold with overwhelming probability.

Lemma 6 below follows from the definition of each state and party fluctuation; Lemma 7 establishes a basic property the volatile states satisfy.

Lemma 6. *For a block B_v, if W_v w.r.t. B_u is in state Cold, then W_v w.r.t. B_v is in state $\mathsf{VolatileLeftInner}, \mathsf{VolatileRightInner}$ or Cold.*

Proof. While w.r.t. B_u, $X_v = W_v + (m/f)\log(h_v/h_u)$. Combine it with Cold_v as well as $h_{\mathsf{max}} \leq \Gamma \cdot h_v, h_v \leq \Gamma \cdot h_{\mathsf{min}}$, we get

$$\begin{aligned} -\frac{m}{f}\log 2(2-\delta)\Gamma &\leq -\frac{m}{f}\log\frac{2(2-\delta)h_{\mathsf{max}}}{h_u} + \frac{m}{f}\log\frac{h_v}{h_u} \\ &\leq X_v = W_v + \frac{m}{f}\log\frac{h_v}{h_u} \leq \frac{m}{f}\log\frac{2h_u}{h_{\mathsf{min}}} + \frac{m}{f}\log\frac{h_v}{h_u} \leq \frac{m}{f}\log 2\Gamma. \end{aligned}$$

By the definition of W_v w.r.t. B_v, $W_v = X_v$, therefore in state $\mathsf{VolatileLeftInner}$, $\mathsf{VolatileRightInner}$ or Cold. □

Lemma 7. *Until the next block is produced, if W_i is in the state $\mathsf{VolatileLeftOuter}$ or $\mathsf{VolatileLeftInner}$, the block generation rate is always below $f/2$; if W_i is in the state $\mathsf{VolatileRightInner}$ or $\mathsf{VolatileRightOuter}$, the block generation rate is always above $2f$.*

Proof. For the first part, our goal is to show that even if the adversary and the honest parties join force, they cannot achieve the block generation rate over $f/2$. Suppose i-th block has timestamp r and number of honest parties h_r. If $W_i < -(m/f)\log[2(2-\delta)h_{\mathsf{max}}/h_u]$, the target of B_i satisfies

$$T_r < T_0 \cdot 2^{(\frac{i-1}{f} + \frac{m}{f}\log\frac{h_0}{h_u} - \frac{m}{f}\log\frac{2(2-\delta)h_{\mathsf{max}}}{h_u} - \frac{i-1}{f})/(m/f)} = \frac{T_0}{2}\cdot\frac{h_0}{(2-\delta)h_{\mathsf{max}}} \leq \frac{T_0}{2}\cdot\frac{h_0}{(2-\delta)h_r}.$$

Therefore, the block generating rate at round r is $pT_rh_r < f/[2(2-\delta)]$. Note that $pT_rh_{\mathsf{max}} < f/[2(2-\delta)]$ as well, which implies that while the number of honest parties may raise during the rounds till the next block will be produced, the block generating rate will never exceed $f/[2(2-\delta)]$. Moreover, the adversary may join force to accelerate the block production. Recall that $\forall r, t_r \leq (1-\delta)h_r$, after the adversary joins, the block production rate is still below $f/2$.

For the second part, we prove that the block generating rate will not fall below $2f$ when the honest parties keep solely working. Similarly, assuming i-th block has timestamp r and number of honest parties h_r as well as $W_i \geq (m/f)\log(2h_u/h_{\mathsf{min}})$. Thus, the corresponding target

$$T_r > \cdot 2^{(\frac{i-1}{f} + \frac{m}{f}\log\frac{h_0}{h_u} + \frac{m}{f}\log\frac{2h_u}{h_{\mathsf{min}}} - \frac{i-1}{f})/(m/f)} = 2T_0\cdot\frac{h_0}{h_{\mathsf{min}}} \geq 2T_0\cdot\frac{h_0}{h_r}.$$

By $pT_rh_{\mathsf{min}} > 2f$, we learn that the block generating rate is always above $2f$. □

We are now ready to establish the probability of "escaping" from a volatile state to a hot state. (Recall that our security parameter is κ, and m is selected as a polylogarithmic function of κ.)

Lemma 8. *Consider blocks $B_u, \ldots, B_v$ with timestamps $r_v - r_u \leq 4(m/f)\log\Gamma$ in a $(\langle\gamma, \ell + 2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$-respecting environment. If W_u is in state* VolatileLeftInner, VolatileRightInner *or* Cold*, the probability of W_i, $i > u$ reaching* HotLeft *or* HotRight *is negligible.*

Proof. Regarding the probability of escaping from VolatileLeftOuter, by Lemma 7, at every round it will succeed producing a block with probability at most $f/2$. We view the number of blocks produced as a binomial distribution with success probability $f/2$. And, for worst case, W_i begins at the leftmost point in VolatileRightInner, it has to go leftwards for $(m/f)\log\Gamma$ in order to reach HotLeft.

Since $W_{i+1} = W_i + (r_{i+1} - r_i) - 1/f$, we get $W_v = W_u + r_v - r_u - (u-v)/f$. Assume now $W_u = -(m/f)\log 2(2-\delta)\Gamma$, if W_v is in HotLeft, $r_v - r_u - (u-v)/f < -(m/f)\log\Gamma$. Obviously, this will never happen in the first $(m/f)\log\Gamma$ rounds. For the rounds with index in $\{(m/f)\log\Gamma + 1, \ldots, 4(m/f)\log\Gamma\}$, split them into segments with length $1/f$. For r in i-th segment with index $\{(m/f)\log\Gamma + (i-1)/f + 1, \ldots, (m/f)\log\Gamma + i/f\}$, suppose we produce a block B_v, in expectation, parties will succeed for $\lfloor(m\log\Gamma + i - 1)/2\rfloor$ times in these rounds. If they succeed for more than $(m\log\Gamma + i)$ times, then $r_v - r_u - (u-v)/f < -(m/f)\log\Gamma$, thus reach HotLeft.

Note that $m\log\Gamma + i \geq 2\cdot\lfloor(m\log\Gamma + i - 1)/2\rfloor$ always holds. By Theorem 17, let $Z_i, \ldots, Z_T$ ($T = r$) be independent variables with $\mathrm{E}[Z_i] = f/2$ and $Z_i \in \{0,1\}$. Let $Z = \sum_{i=1}^{T} Z_i$, $\mu = \sum_{i=1}^{T} f/2 = \mathrm{E}[Z] = \lfloor(m\log\Gamma + i - 1)/2\rfloor \geq m\log\Gamma$. Then, for $\Lambda = 1$, we get

$$\Pr[Z \geq (1+\Lambda)\mu] \leq \exp\Big[-\frac{\Lambda^2}{2+\Lambda}\cdot m\log\Gamma\Big] \leq 2^{-\Omega(m)}.$$

Eventually, this may happen for $3(m/f)\log\Gamma$ times, thus we get the negligible escape probability

$$1 - (1 - 2^{-\Omega(m)})^{3\frac{m}{f}\log\Gamma} \geq 3\frac{m}{f}\log\Gamma\cdot 2^{-\Omega(m)} = 2^{-\Omega(m)}.$$

Consider the the probability of escaping from VolatileRightOuter, by Lemma 7, at every round it will succeed producing a block with probability at least $2f$. We view the number of blocks produced as a binomial distribution with success probability $2f$. And, for worst case, W_i begins at the rightmost point in VolatileRightInner, it has to go rightwards for $(m/f)\log\Gamma$ in order to reach HotRight.

Since $W_{i+1} = W_i + (r_{i+1} - r_i) - 1/f$, we get $W_v = W_u + r_v - r_u - (u-v)/f$. Assume now $W_u = (m/f)\log 2\Gamma$, if W_v is in HotRight, $r_v - r_u - (u-v)/f > (m/f)\log\Gamma$. Obviously, this will never happen in the first $(m/f)\log\Gamma$ rounds. For the rounds with index in $\{(m/f)\log\Gamma + 1, \ldots, 4(m/f)\log\Gamma\}$, split them into segments with length $1/f$. For r in i-th segment with index $\{(m/f)\log\Gamma + (i-1)/f + 1, \ldots, (m/f)\log\Gamma + i/f\}$ suppose we produce a block B_v, in expectation, parties will succeed for $2(m\log\Gamma + i - 1)$ times in these rounds. If they succeed for less than i times, then $r_v - r_u - (u-v)/f > (m/f)\log\Gamma$, thus reach HotRight.

Note that $i \leq (1/2)\cdot 2(m\log\Gamma + i - 1)$ always holds. By Theorem 17, let $Z_i, \ldots, Z_T$ ($T = r$) be independent variables with $\mathrm{E}[Z_i] = 2f$ and $Z_i \in \{0,1\}$. Let $Z = \sum_{i=1}^{T} Z_i$, $\mu = \sum_{i=1}^{T} 2f = \mathrm{E}[Z] = 2(m\log\Gamma + i - 1) \geq 2\log\Gamma$. Then, for $\Lambda = 1/2$, we get

$$\Pr[Z \leq (1-\Lambda)\mu] \leq \exp\Big[-\frac{\Lambda^2}{2+\Lambda}\cdot 2m\log\Gamma\Big] \leq 2^{-\Omega(m)}.$$

Eventually, this may happen for $3(m/f)\log\Gamma$ times, and the final escape probability

$$1 - (1 - 2^{-\Omega(m)})^{3\frac{m}{f}\log\Gamma} \geq 3\frac{m}{f}\log\Gamma\cdot 2^{-\Omega(m)} = 2^{-\Omega(m)}$$

is still negligible. □

Next, we focus on the "return" probability. Since W_i will travel far away from the *relatively calibrated timestamp* (i.e., HotLeft or HotRight) only with negligible probability, and once it reaches Cold we are done, we consider the following two bad events:

- During $4(m/f)\log\Gamma$ rounds, beginning at $-(m/f)\log 2(2-\delta)\Gamma$ (the leftmost point of VolatileLeftInner), W_i stays in state VolatileLeftOuter and VolatileLeftInner.
- During $4(m/f)\log\Gamma$ rounds, beginning at $(m/f)\log 2\Gamma$ (the rightmost point of VolatileRightInner), W_i stays in the state VolatileRightInner and VolatileRightOuter.

We show that by the concentration of the binomial distribution, these two bad events happen only with negligible probability. Note that our results are achieved considering the largest distance W_i needs to travel and with worst success probability. For those events that start closer to the relatively calibrated timestamp, the events' probability will be much lower.

Lemma 9. *Consider blocks $B_u, \ldots, B_v$ with $r_v - r_u \leq 4(m/f)\log\Gamma$ in a $(\langle\gamma, \ell+2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$-respecting environment. If W_u is in state* VolatileLeftInner, VolatileRightInner *or* Cold*, W_i, $i > u$ will return to* Cold *with overwhelming probability.*

Proof. We consider the two bad events that makes W_i fail to return Cold.

For the first event of staying in the left-side states, Lemma 7 shows that at every round it will succeed producing a block with probability at most $f/2$. We view the number of blocks produced as a binomial distribution with success probability $f/2$. And, since we assume it begins at the leftmost point in VolatileLeftInner, it has to go rightwards for at most $(m/f)\log\Gamma$ in order to reach Cold.

Consider the first $(4m/f)\log\Gamma$ rounds with blocks $\{B_u, \ldots, B_v\}$. Since $W_{i+1} = W_i + (r_{i+1} - r_i) - 1/f$, we get $W_v = W_u + r_v - r_u - (u-v)/f$. Assume now $W_u = -(m/f)\log 2(2-\delta)\Gamma$, if W_v is in Cold, $r_v - r_u - (u-v)/f > (m/f)\log\Gamma$. In expectation, parties will succeed for $(2m\log\Gamma)$ times in $(4m/f)\log\Gamma$ rounds. If they succeed for more than $(3m\log\Gamma)$ times, it cannot satisfy $r_v - r_u - (u-v)/f > (m/f)\log\Gamma$, thus fail to reach Cold, i.e., W_i still falls in VolatileLeftInner.

By Theorem 17, let $Z_i, \ldots, Z_T (T = (4m/f)\log\Gamma)$ be independent variables with $\mathrm{E}[Z_i] = f/2$ and $Z_i \in \{0,1\}$. Let $Z = \sum_{i=1}^{T} Z_i$ and $\mu = \sum_{i=1}^{T} f/2 = \mathrm{E}[Z] = 2m\log\Gamma$. Then, for $\Delta = 1/2$, we get

$$\Pr[Z \geq (1+\Lambda)\mu] \leq \exp\Big(-\frac{\Lambda^2}{2+\Lambda}\cdot 2m\log\Gamma\Big) = 2^{-\Omega(m)}.$$

For the second event of staying in the right-side states, Lemma 7 shows that at every round it will succeed producing a block with probability at least $2f$. We view the number of blocks produced as a binomial distribution with success probability $2f$. And, since we assume it begins at the rightmost of VolatileRightInner, it has to go leftwards for at most $(m/f)\log\Gamma$ in order to reach Cold.

Consider the first $(2m/f)\log\Gamma$ rounds with blocks $\{B_u, \ldots, B_v\}$. Since $W_{i+1} = W_i + (r_{i+1} - r_i) - 1/f$, we get $W_v = W_u + r_v - r_u - (u-v)/f$. Assume now $W_u = (m/f)\log 2\Gamma$, if W_v is in Cold, $r_v - r_u - (u-v)/f < -(m/f)\log\Gamma$. In expectation, parties will succeed for $(4m\log\Gamma)$ times in $(2m/f)\log\Gamma$ rounds. If they succeed for less than $(3m\log\Gamma)$ times, it cannot satisfy $r_v - r_u - (u-v)/f > -(m/f)\log\Gamma$, thus fail to reach Cold, i.e., W_i still falls in VolatileRightInner.

By Theorem 17, let $Z_i, \ldots, Z_T (T = (2m/f)\log\Gamma)$ be independent variables with $\mathrm{E}[Z_i] = 2f$ and $Z_i \in \{0,1\}$. Let $Z = \sum_{i=1}^{T} Z_i$ and $\mu = \sum_{i=1}^{T} 2f = \mathrm{E}[Z] = 4m\log\Gamma$. Then, for $\Delta = 1/4$, we get

$$\Pr[Z \leq (1-\Lambda)\mu] \leq \exp\Big(-\frac{\Lambda^2}{2+\Lambda}\cdot 4m\log\Gamma\Big) = 2^{-\Omega(m)}.$$

Therefore, the return probability is $1 - 2^{-\Omega(m)}$. □

Remark 3. We note that if we consider a respecting environment that allows more time for the same party fluctuation, then the goodness parameter—i.e., the upper bound and lower bound of the target recalculation point—can be closer to f. For example, the good target recalculation parameter can be changed to $(1+\Gamma^3)f$ and $f/[(2-\delta)(1+\Gamma^3)]$, with sliding window length $2\Gamma^3(m/f)\log\Gamma$. In the analysis, this is a trade-off. The presentation in this section chooses the more intuitive values ($2f$ and $f/2$) for simplicity.

Theorem 10. *A typical execution in a $(\langle\gamma, \ell+2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$-respecting environment is accurate and good.*

Proof. Accuracy directly follows Corollary 5.

Regarding goodness, we show that it is maintained for all target recalculation points in a sliding window of length $4(m/f)\log\Gamma$, and that it can be iterated from the first round to the whole execution. Note that in our assumption, the first block satisfies $pT_1 n_{r_1} = f$ (i.e., the state is Cold), thus, we can establish the first sliding window starting from B_1. The iteration works as follows: We choose the last block with state Cold in the sliding window, and then establish a new sliding window starting from it. According to Lemma 9, such block always exists. For blocks in each sliding window, Lemma 8 ensures that all the target recalculation points are good (i.e., never go into the state HotLeft or HotRight).

For those rounds which are not target recalculation points, it follows from Lemma 4 that the maximum interval between two blocks is $\ell + 2\Delta$, thus, by Fact 1(a) $h_u/\gamma \leq h_r \leq \gamma h_u$ holds. Combining it with $f/2(2-\delta)\Gamma^3 \leq ph_uT \leq 2\Gamma^3 f$, we obtain the desired inequality. □

4.3 Blockchain and Ledger Properties

We now show that the Bitcoin Cash backbone protocol satisfies the two properties common prefix and chain quality (Section 2), for a suitable respecting environment. First, a preliminary lemma:

Lemma 11. *For any round r of a typical execution in a $(\langle\gamma, \ell+2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$-respecting environment and any two chains $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{S}_r$, the timestamp of head($\mathcal{C}\cap\mathcal{C}'$) is at least $r-2\ell-4\Delta$.*

Proof. Let v be the timestamp of head($\mathcal{C}\cap\mathcal{C}'$) and $u \leq v$ the greatest timestamp among those blocks on $\mathcal{C}\cap\mathcal{C}'$ that was computed by an honest party. Let $U = \{i : u < i \leq r\}$, $S = \{i : u+\Delta \leq i \leq r-\Delta\}$, and let J denote the adversarial queries that correspond to the rounds in U. We claim that if $r - v \geq \ell + 2\Delta$, then

$$2Q(S) \leq D(U) + A(J).$$

This contradicts Proposition 2 for $\delta \geq 8\epsilon$, which implies $D(U) < (1+5\epsilon)Q(S)$ and $A(J) < (1-\delta+3\epsilon)Q(S)$.

Towards proving the claim, associate with each $r \in S$ such that $Q_r > 0$ an arbitrary honest block that is computed at round r for difficulty Q_r. Let $\mathcal{B}$ be the set of these blocks and note that their difficulties sum to $Q(S)$. We argue the existence of a set of blocks $\mathcal{B}'$ computed in U such

that $\mathcal{B} \cap \mathcal{B}' = \emptyset$ and $\{d \in B : B \in \mathcal{B}\} \subseteq \{d \in B : B \in \mathcal{B}'\}$. This suffices, because each block in $\mathcal{B}'$ contributes either to $D(U) - Q(S)$ or to $A(J)$ and so $Q(S) \leq D(U) - Q(S) + A(J)$.

Consider, then, a block $B \in \mathcal{B}$ extending a chain $\mathcal{C}^*$ and let $d = \text{diff}(\mathcal{C}^* B)$. If $d \leq \text{diff}(\mathcal{C} \cap \mathcal{C}')$ (note that $u < v$ in this case and $\text{head}(\mathcal{C} \cap \mathcal{C}')$ is adversarial), let B' be the block of $\mathcal{C} \cap \mathcal{C}'$ containing d. Such a block clearly exists and has a timestamp greater than u. Furthermore, $B' \notin \mathcal{B}$, since B' was computed by the adversary. If $d > \text{diff}(\mathcal{C} \cap \mathcal{C}')$, note that there is a unique $B \in \mathcal{B}$ such that $d \in B$ (recall the argument in Chain Growth Lemma). Since B cannot simultaneously be on $\mathcal{C}$ and $\mathcal{C}'$, there is a $B' \notin \mathcal{B}$ either on $\mathcal{C}$ or on $\mathcal{C}'$ that contains d. □

Theorem 12 (Common-Prefix)**.** *For a typical execution in a* $(\langle \gamma, \ell + 2\Delta \rangle, \langle \Gamma, 4(m/f) \log \Gamma \rangle)$*-respecting environment, the common-prefix property holds for parameter* ϵm*.*

Proof. Suppose—towards a contradiction—common prefix fails for two chains $\mathcal{C}_1$ and $\mathcal{C}_2$ at rounds $r_1 \leq r_2$. It is easy to see that there exist a round $r \leq r_2$ and two chains $\mathcal{C}$ and $\mathcal{C}'$ such that each had at least k blocks after $\text{head}(\mathcal{C} \cap \mathcal{C}')$. In view of Lemma 11, it suffices to show that these blocks were computed within at least $\ell + 2\Delta$ rounds. Suppose the honest parties query during a set of rounds S of size $\ell + 2\Delta$, the accumulated difficulty is $D(S)$, and the number of blocks produced in $|S|$ is at most $T^{\max} \cdot D(S)$ where $T^{\max}$ denote the maximum target among blocks in $|S|$. For the target recalculation round r^* associated with target $T^{\max}$, we have

$$T^{\max} \cdot D(S) < (1+\epsilon) p T^{\max} h(S) \leq (1+\epsilon) p \gamma T^{\max} h_{r^*} |S| < (1+\epsilon)\epsilon m/2.$$

For the last inequality, note that by Condition (C1) $|S| < \epsilon m/(4\gamma\Gamma^3 f)$ and by Theorem 10 $pT^{\max} h_{r^*} \leq 2\Gamma^3 f$. By proposition 2 the adversary contributed to less than $(1+\epsilon)\epsilon m/2(1+\delta)$ blocks, for a total of less than ϵm. □

Theorem 13 (Chain-Quality)**.** *For a typical execution in a* $(\langle \gamma, \ell + 2\Delta \rangle, \langle \Gamma, 4(m/f) \log \Gamma \rangle)$*-respecting environment, the chain-quality property holds with parameters* $\ell + 2\Delta$ *and* $\mu = \delta - 3\epsilon$*.*

Proof. Let us denote by B_i the i-th block of $\mathcal{C}$ so that $\mathcal{C} = B_1 \dots B_{len(\mathcal{C})}$ and consider K consecutive blocks $B_u, \dots, B_v$. Define K' as the least number of consecutive blocks $B_{u'} \dots B_{v'}$ that include the K given ones (i.e., $u' \leq u$ and $v \leq v'$) and have the properties (1) that the block $B_{u'}$ was computed by an honest party or is B_1 in case such block does not exist, and (2) that there exists a round r' that $B_1 \dots B_{v'} \in \mathcal{S}_{r'}$. Denote by d' the total difficulty of these K' blocks. Define $U = \{r, \dots, r'\}$, $S = \{r + \Delta, \dots, r' - \Delta\}$, and J the adversarial queries in U starting with the first to obtain one of the K' blocks. Let x denote the total difficulty of all the blocks from honest parties that are included in the K blocks and—towards a contradiction—assume $x < \mu d'$. In a typical execution, all the K' blocks $B_j : u' \leq j \leq v'$ have been computed in U. But then we have the following contradiction to Proposition 2(b).

$$(1 - \delta + 3\epsilon) Q(S) > A(J) \geq d' - x \geq (1-\mu) d' \geq (1-\mu) Q(S) = (1 - \delta + 3\epsilon) Q(S).$$

The inequalities follow from the definitions of x and d' and Chain-Growth Lemma. Note that if $U > 4(m/f) \log \Gamma$, we may use Lemma 4 to partition U appropriately and apply Proposition 2(b) to each part. Its validity come from that a block computed by an honest party provides both properties (1) and (2) required for K'. □

We conclude by showing that a typical execution of the Bitcoin Cash backbone protocol in a $(\langle \gamma, \ell + 2\Delta \rangle, \langle \Gamma, 4(m/f) \log \Gamma \rangle)$-respecting environment, satisfies the two properties of a robust transaction ledger presented in Section 2. They are the direct consequence of the blockchain properties shown above, following the approach in [GKL17, GKL20].

Theorem 14 (Consistency)**.** *For a typical execution in a* $(\langle\gamma, \ell + 2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$*-respecting environment, Consistency is satisfied by setting the settled transactions to be those reported more than* ϵm *blocks deep.*

Theorem 15 (Liveness)**.** *For a typical execution in a* $(\langle\gamma, \ell + 2\Delta\rangle, \langle\Gamma, 4(m/f)\log\Gamma\rangle)$*-respecting environment, Liveness is satisfied for depth* ϵm *with wait-time* $(4\Gamma^4 + 1)\epsilon m/f$*.*

Proof. We claim that the chain $\mathcal{C}$ of any honest party has at least ϵm blocks that where computed in the last $2(2-\delta)\epsilon\Gamma^4 m/(1-2\epsilon)f + 2\Delta$ rounds. Denote by $T^{\min}$ the lowest target among these blocks and r^* a round associated with this target. If S is its subset without the first and last Δ rounds, by Chain-Growth Lemma, the length of the segment is at least

$$T^{\min} \cdot Q(S) > (1-\epsilon)[1-2\gamma\Gamma^3 f]^{\Delta} p T^{\min} h(S) \geq \frac{(1-2\epsilon)f|S|}{2(2-\delta)\Gamma^4} \geq \epsilon m.$$

Furthermore, if a transaction tx is included in any block computed by an honest party for the first $\ell + 2\Delta$ rounds, by Lemma 4, the chain $\mathcal{C}$ of any honest party contains tx in a block B. Thus, the honest parties will have

$$\ell + 6\Delta + \frac{2(2-\delta)\epsilon\Gamma^4 m}{(1-2\epsilon)f} \leq \frac{\epsilon m}{2\gamma\Gamma^3 f} + \frac{2(2-\delta)\epsilon\Gamma^4 m}{(1-2\epsilon)f} \leq (4\Gamma^4+1)\cdot\frac{\epsilon m}{f}$$

as the total wait-time. □

5 Comparison with the Bitcoin Cash Network

Our analysis above shows that for a sufficiently long execution in a somewhat idealized network environment and without severe external disturbances, Bitcoin Cash would achieve the desired security properties (with overwhelming probability). Comparison of our analysis with the existing Bitcoin Cash network data reveals that the ASERT function performs better than the SMA function in an environment where party fluctuation is large. Further, both SMA and ASERT suffer from undersized parameters. In this section we expand on the above comparison(s). (Here we only present conclusions regarding ASERT; refer to Appendix D for those regarding SMA.)

Party fluctuation. To perform our comparison, we need to determine the fluctuation (ratio) of the number of parties in the real network. We can extract it from the public statistical data – the *hashrate*, which is the number of hash queries that all miners in the network perform per second. In our model, the queries that every party can make in one round is bounded. Thus, for the purpose of our comparison we assume the hashrate to be identical to the number of parties, and its fluctuation ratio also reflects the the number of parties'. To be more precise, the available hashrate does not reveal the entire Bitcoin Cash network, but instead the hashing power invested in some well-known mining pools. Nonetheless, since most of the computational power is concentrated in these mining pools, this hashrate can very closely approximate the entire network's.

In addition, instead of the the exact fluctuation ratio with respect to a short period of time (e.g., 10 minutes), we adopt the ratio based on the *daily average hashrate*, which shows the average queries per second in one day; this measure would correspond to parameter Γ in our analysis. Since our analysis should be applied to a relatively long execution, this measure replacement seems reasonable.[4]

[4]We note that if we consider the fluctuation rate over 10-minute intervals, the overall party fluctuation will become extremely pronounced (parameter Γ will exceed 4), making our analysis inapplicable.

Figure 2: *The daily average hashrate of Bitcoin Cash from July 18 to August 16, 2020.*

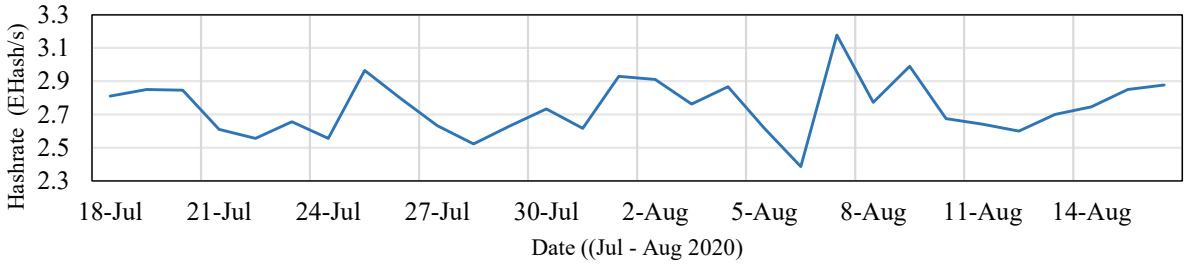


Figure 2 shows the *daily average hashrate* in one month period (Jul 18 2020 – Aug 17 2020)[5], which we adopt as the representative case of the Bitcoin Cash network execution under the SMA function. The maximum value in this period is 3.1783EHash/s (on Aug 07 2020) and the minimum is 2.3865EHash/s (on Aug 06 2020), so it suffices to set $\Gamma = 1.398$ as the party fluctuation ratio. Regarding the ratio during a small period of time, note that as we are discussing the average hashrate, $\gamma = 1.057$, which satisfies $\gamma^{\lfloor \Sigma/\sigma \rfloor} > \Gamma$, would be a suitable value.

We note that the assumed behavior may not hold when some type of external disturbance occurs, such as the halving of the block reward or sharp gains and declines in BCH's monetary value, which might cause a large number of miners abruptly joining or dropping out of a mining pool in a short period of time. As it turns out, these events do not happen very frequently (e.g., block reward halves about every 4 years), so we are comfortable extrapolating the execution of the Bitcoin Cash network from what Figure 2 depicts.

Figure 3: *The daily average hashrate of Bitcoin Cash from Dec 1 to Dec 30, 2020.*

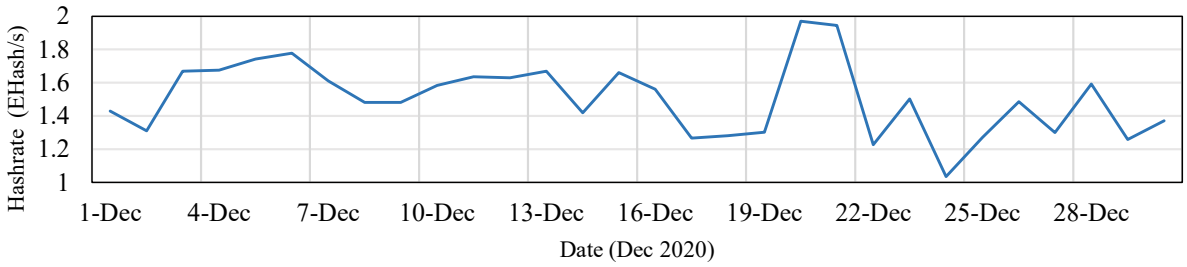


Figure 3 shows the daily average hashrate in one month period (Dec 1 2020 – Dec 30 2020), after the Nov. 2020 network update when the ASERT function was adopted. The maximum value in this period is 1.9451EHash/s (on Dec 21 2020) and the minimum is 1.0349EHash/s (on Dec 24 2020). Therefore, Γ should be to 1.88. This fluctuation rate is relatively high, and we speculate was caused by two reasons: (1) the network could not return to a quiet state soon after the update happens, and (2) the price of Bitcoin Cash experienced substantial ups and downs during this period. We thus set the value of $\gamma = 1.099$.

Network delay. Another important aspect to validate our analysis in a bounded-delay network

[5]Source: https://bitinfocharts.com/comparison/hashrate-bch.html

environment is the actual message delay in the network. It is shown in [DW13] that the delay in the Bitcoin network mainly stems from its multi-hop broadcast ("diffuse" in our model terminology) and block propagation mechanism (Bitcoin Cash employs a similar mechanism to Bitcoin), which we now expand on. As a mining node would typically maintain a limited connections with other nodes, when the node wants to broadcast one block after receiving and verifying it, the node will first send an *inv* message containing the block hash to all its neighbors. If the neighbors do not have the block locally, they will issue a *getdata* message to the sender of *inv*. Then the actual block transfer begins. Thus, at each hop during the broadcast the message incurs a propagation delay consisting of the transmission time and the local verification of the block. Decker and Wattenhofer [DW13] evaluate the distribution of the block propagation time since the first block announcement, which shows a median time of 6.5 seconds as well as a mean time of 12.6 seconds. For more recent data, Bitcoin Monitoring [6] carried out by the German Federal Ministry of Education and Research shows that 90% of the block propagation time is below 6 seconds. Thus, assuming a round duration of 6 seconds, it is reasonable to let the delay bound (Δ) in our analysis approximately equal to 1 round.

Conclusions. Based on the considerations above, for the purpose of our comparison, we parameterize the real-world Bitcoin Cash network with network bounded-delay $\Delta = 1$ (round), honest advantage $\delta = 0.99$, quality of concentration $\epsilon = 0.123$, long term party fluctuation ratio $\Gamma = 1.88$, short term party fluctuation ratio $\gamma = 1.099$ and ideal block generating rate $f = 0.01$.

On one hand, the resulting probabilities with respect to the goodness parameters are very tight, when considered in the real network parameters (i.e., $m = 288$ and $\Gamma = 1.88$). To be precise, in a sliding window, the probability of the block generation rate exceeding the goodness bound is below 10^{-12}, and the probability of not returning to Cold is less than 10^{-9}. Therefore, it is safe to conclude that the ASERT function achieves a relatively stable block generation rate.

On the other hand, we cannot directly plug in in these parameter values to satisfy Condition (C2), since the party fluctuation is too pronounced, thus making it hard for the concentration parameter ϵ to absorb the errors. Condition (C2), however, can be satisfied in the following two scenarios: (1) party fluctuation ratio in a relatively quiet environment (e.g., $\Gamma = 1.398$ as shown in Fig 2). Then all the requirements are satisfied and so does our analysis. (2) Balancing the block generation rate bound and the sliding window length we consider (cf. Remark 3), then all the requirements are satisfied if the upper bound for goodness is $(1 + \Gamma^3)f$.

Finally, we observe that the choice of the smoothing factor (m) is too low. While the current value $m = 288$ can satisfy the basic requirements in Condition (C1), it is better to let $m = 432$ to increase the value of ℓ, thus making it closer to the real execution. However, we note that such value of m still fails to give a tight typical execution probability. This is because in order to satisfy the (desired) blockchain properties, the length of a sliding window should be much larger (of the order of years!) so that the martingale probability in Theorem 3 can be negligible.

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[6] Source: https://dsn.tm.kit.edu/bitcoin/

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A Martingale Sequences and Other Mathematical Facts

Definition 8. [MU05, Chapter 12] A sequence of random variables $X_0, X_1, \ldots$ is a martingale with respect to sequence $Y_0, Y_1, \ldots$, if, for all $n \geq 0$, (1)X_n is a function of $Y_0, \ldots, Y_n$, (2)$\mathbb{E}[|X_n|] < \infty$, and (3) $\mathbb{E}[X_{n+1}|Y_0, \ldots, Y_n] = X_n$.

Theorem 16. [McD98, Theorem 3.15] *Let $X_0, X_1, \ldots$ be a martingale with respect to the sequence $Y_0, Y_1, \ldots$. For $n \geq 0$, let $V = \sum_{i=1}^{n} \mathrm{var}(X_i - X_{i-1}|Y_0, \ldots, Y_{i-1})$ and $b = \max_{1 \leq i \leq n} \sup(X_i - X_{i-1}|Y_0, \ldots, Y_{i-1})$, where* sup *is taken over all possible assignments to $Y_0, \ldots, Y_{i-1}$. Then, for any $t, v \geq 0$,*

$$\Pr[(X_n \geq X_0 + t) \wedge (V \leq v)] \leq \exp\Big\{ - \frac{t^2}{2v + 2bt/3} \Big\}.$$

Theorem 17 (Chernoff bound)**.** *Let $X_i, \ldots, X_T$ be independent random variables with $\mathrm{E}[X_i] = p_i$ and $X_i \in [0, 1]$. Let $X = \sum_{i=1}^{T}$ and $\mu = \sum_{i=1}^{T} p_i = \mathrm{E}[X]$. Then, for all $\Lambda > 0$,*

$$\Pr[X \geq (1 + \Lambda)\mu] \leq \exp(-\frac{\Lambda^2}{2 + \Lambda}\mu) \text{ and } \Pr[X \leq (1 - \Lambda)\mu] \leq \exp(-\frac{\Lambda^2}{2 + \Lambda}\mu).$$

Fact 2. Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers. Then,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \frac{n}{1/x_1 + 1/x_2 + \cdots + 1/x_n}.$$

Theorem 18 (Cassel's inequality)**.** [WAT55] *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two positive n-tuples with $0 < m \leq \frac{a_k}{b_k} \leq M < \infty$ for each $k \in \{1, \ldots, n\}$ and constants m, M. Then,*

$$(\sum_{k=1}^{n} w_k a_k^2)(\sum_{k=1}^{n} w_k b_k^2) \leq \frac{(M + m)^2}{4mM} \cdot (\sum_{k=1}^{n} w_k a_k b_k)^2.$$

Theorem 19. *Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers. If $\max_{i \in [n]} x_i < \gamma \cdot \min_{i \in [n]} x_i$, then*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \frac{(\gamma + 1)^2}{4\gamma} \cdot \frac{n}{1/x_1 + 1/x_2 + \cdots + 1/x_n}.$$

Proof. Let $a_i = \sqrt{n_i}$ and $b_i = 1/\sqrt{n_i}$. $\{a_i/b_i\}$ forms a tuple which is identical to $\{n_i\}$. Since $\max_{i \in [n]} T_i < \gamma \cdot \min_{i \in [n]} T_i$, there exist appropriate constants m and $M = \gamma \cdot m$ that satisfies the condition in Theorem 18. Let $w_k = 1$ for all k, then

$$(\sum_{k=1}^{n} n_k)(\sum_{k=1}^{n} \frac{1}{n_k}) = (\sum_{k=1}^{n} w_k a_k^2)(\sum_{k=1}^{n} w_k b_k^2) \leq \frac{(M + m)^2}{4mM} \cdot (\sum_{k=1}^{n} w_k a_k b_k)^2 = \frac{(\gamma + 1)^2}{4\gamma} \cdot n^2.$$

By rearranging we obtain the desired inequality. □

B Proof of All Executions are Typical

Proof of Theorem 3: Since the length of the execution, L, is fixed we will prove the stated bound for a fixed set of consecutive rounds S—or, with respect to the adversary, a fixed set of consecutive queries J—and then apply a union bound over all such sets in the length of the execution. Furthermore, we may assume $|S| \leq \Sigma = \Theta(m)$. Note that $\ell < \Sigma/2$ and we may partition S into parts with size between ℓ and Σ. In the end, we sum over all parts to obtain the desired bound.

Before we consider the random variables, we first propose some useful inequalities that hold for any round r: $[1 - f(T_r^{\max}, h_r)] \leq \mathbf{E}[Y_r|\mathcal{E}_{r-1} = E_{r-1}] \leq \mathbf{E}[D_r|\mathcal{E}_{r-1} = E_{r-1}] = ph_r$, $\mathbf{E}[Y_r^2|\mathcal{E}_{r-1} = E_{r-1}] \leq ph_r/T_r^{\min}$, and $\mathbf{var}[D_r|\mathcal{E}_{r-1} = E_{r-1}] \leq ph_r/T_r^{\min}$. Let us drop the subscript r for convenience. Suppose that the h honest parties at round r query for targets $T_1, \ldots, T_h$. Observe that all these variables are determined by E_{r-1}. We have

$$\begin{aligned}\mathbf{E}[Y_r|\mathcal{E}_{r-1} = E_{r-1}] &= \sum_{i\in[h]} \frac{1}{T_i} \cdot \frac{T_i}{2^\kappa} \prod_{i<j} [1 - f(T_j, 1)] \geq \sum_{i\in[h]} p \prod_{j\in[h]} [1 - f(T_j, 1)] \\ &\geq \sum_{i\in[h]} p \prod_{j\in[h]} [1 - f(T^{\max}, 1)] = \sum_{i\in[h]} p[1 - f(T^{\max}, h)] = ph[1 - f(T^{\max}, h)],\end{aligned}$$

where the third inequality holds because $f(T, h)$ is increasing in T. For the upper bound on variance,

$$\mathbf{var}[D_r|\mathcal{E}_{r-1} = E_{r-1}] \leq \sum_{i\in[h]} \frac{1}{T_i^2} \cdot \frac{T_i}{2^\kappa} = \sum_{i\in[h]} \frac{p}{T_i} \leq \frac{ph}{T^{\min}}$$

and $\mathbf{E}[Y_r^2|\mathcal{E}_{r-1} = E_{r-1}]$ is bounded like.

For starter, we fix an execution E_0 just before the beginning of S (or J). We will prove that the statements fail with exponentially small probability for an arbitrary E_0. Note that E_0 determines the number of parties h_0 and t_0 at the beginning of S (or J) and the target $T(J)$ associated with the first query in J.

For each round $i \in S$, we define a Boolean random variable F_i equal to 1 exactly when all h_i hash values that were returned to the queries of the honest parties were above $\min\{T : f(T, h_i) \geq 2\gamma\Gamma^3 f\}$; define $Z_i = Y_i \cdot F_{i+1} \cdots F_{i+\Delta-1}$. Let G denote the event that the rounds in S are good. Given G, for any $i \in S$, $(F_i = 1) \Longrightarrow (D_i = 0)$ and so $Q_i \geq Z_i$. Thus, for any d,

$$\Pr\big[G \wedge Q(S) \leq d\big] \leq \Pr\big[G \wedge Z(S) \leq d\big],$$

and we may focus on the right-hand side. Identify S with $\{1, \ldots, |S|\}$ and partition it with sets of the form $S_j = \{j, j + \Delta, j + 2\Delta, \ldots\}$ for $j \in \{0, 1, \ldots, \Delta - 1\}$. We will show that, for each part S_j,

$$\Pr\big[G \wedge Z(S_j) \leq (1 - \epsilon)[1 - 2\gamma\Gamma^3 f]^\Delta ph(S_j)\big] \leq e^{-\varphi}.$$

Let us fix a set $S_j = \{s_1, s_2, \ldots, s_\nu\}$ with $\nu \geq \lfloor |S|/\Delta \rfloor$, and define the event G_t as the conjunction of the events G and $t = (1 - 2\gamma\Gamma^3 f)^\Delta ph(S_j)$. Note that $|S_j| \leq L$ and so t ranges over a discrete set of size at most L and we can afford a union bound on it. Thus, it is sufficient to show that for any such t,

$$\mathbf{Pr}\big[G_t \wedge Z(S_j) \leq [1 - 2\gamma\Gamma^3 f]^\Delta ph(S_j) - t\big] \leq e^{-\varphi}.$$

To that end, consider the sequence of random variables

$$X_0 = 0;\ X_k = \sum_{i\in[k]} Z_{s_i} - \sum_{i\in[k]} \mathbf{E}[Z_{s_i}|\mathcal{E}_{s_i-1}], k \in [\nu].$$

This is a martingale with respect to sequence $\mathcal{E}_{s_1-1}(\mathcal{E}_0 = E_0), \ldots, \mathcal{E}_{s_v-1}, \mathcal{E}$, because $\mathbf{E}[X_k|\mathcal{E}_{s_k-1}] = \mathbf{E}\big[Z_{s_k} - \mathbf{E}[Z_{s_k}|\mathcal{E}_{s_k-1}]|\mathcal{E}_{s_k-1}\big] + \mathbf{E}[X_{k-1}|\mathcal{E}_{s_k-1}] = X_{k-1}$ (recall basic properties of conditional expectation [McD98]).

Specifically, the above follows from linearity of conditional expectation and the fact that X_{k-1} is a deterministic function of $\mathcal{E}_{s_{k-1}+\Delta-1} = \mathcal{E}_{s_k-1}$. Furthermore, given an execution E satisfying G_t,

$$\epsilon \sum_{i \in S_j} \mathbf{E}[Z_i|\mathcal{E}_{s_k-1} = E_{s_k-1}] \geq \epsilon \sum_{i \in S_j} [1 - 2\gamma\Gamma^3 f]^{\Delta} p h_i = t.$$

Thus, our goal is to show $\mathbf{Pr}[-X_\nu \geq t \wedge G_t] \leq e^{-\varphi}$.

We now provide the details relevant to Theorem 16. Consider an execution E satisfying G_t and let B denote the event $\mathcal{E}_{s_k-1} = E_{s_k-1}$. Note that $Z_{s_k}^2 = Y_{s_k}^2 \cdot F_{s_k+1} \ldots F_{s_k+\Delta-1}$ and all these random variables are independent given B. Since $X_k - X_{k-1} = Z_{s_k} - \mathbf{E}[Z_{s_k}|\mathcal{E}_{s_k-1}]$ and

$$Z_{s_k} - \mathbf{E}[Z_{s_k}|B] \leq \frac{1}{T_{s_k}^{\min}} \cdot \frac{p h_{s_k}}{p h_{s_k}} \leq \frac{\Gamma p h(S_j)}{p h_{s_k} T_{s_k}^{\min} |S_j|} \leq \frac{\Gamma p h(S_j)}{\nu f/(2(2-\delta)\gamma\Gamma^3)} \leq \frac{2(2-\delta)\gamma\Gamma^4 t}{\epsilon(1-2\gamma\Gamma^3 f)^{\Delta} f\nu} \stackrel{\text{def}}{=} b,$$

we see that the event G implies $X_k - X_{k-1} \leq b$. With respect to $V = \sum_k \mathbf{var}[X_k - X_{k-1}|\mathcal{E}_{s_k-1}] \leq \sum_k \mathbf{E}[Z_{s_k}^2|\mathcal{E}_{s_k-1}]$, using the independence of the random variables and inequalities showed at the beginning of the proof,

$$\sum_{k \in [\nu]} [Z_{s_k}^2|B] \leq (1-2\gamma\Gamma^3 f)^{\Delta-1} \sum_{k \in [\nu]} \frac{(p h_{s_k})^2}{p h_{s_k} T_{s_k}^{\min}} \leq \frac{(1-2\gamma\Gamma^3 f)^{\Delta-1}}{f/[2(2-\delta)\gamma\Gamma^3]} \cdot \sum_{k \in [\nu]} (p h_{s_k})^2.$$

Applying Fact 1(b) on this bound, we see that event G_t implies

$$V \leq \frac{2(2-\delta)\gamma\Gamma^4[1-2\gamma\Gamma^3 f]^{\Delta-1}}{f|S_j|} \cdot \Big(\sum_{k \in [\nu]} p h_{s_k}\Big)^2 \leq \frac{2(2-\delta)\gamma\Gamma^4 t^2}{\epsilon^2 f(1-2\gamma\Gamma^3 f)^{\Delta+1}\nu} \stackrel{\text{def}}{=} b.$$

In view of these bounds (note that $bt < \epsilon v$), by Theorem 16,

$$\mathbf{Pr}[-X_\nu \geq t \wedge G_t] \leq \exp\Big\{-\frac{t^2}{2v(1+\frac{\epsilon}{3})}\Big\} \leq \exp\Big\{-\frac{\epsilon^2 f[1-2\gamma\Gamma^3 f]^{\Delta+1}\nu}{2(2-\delta)\gamma\Gamma^4(1+\frac{\epsilon}{3})}\Big\} \leq e^{-\varphi}, \tag{4}$$

where for the last inequality we used the condition on ℓ (recall that $\nu \geq \ell/\Delta$).

For the bound on $D(S)$, ti will be convenient to work per query. Let J denote the queries in S, $\nu = |J|$, and Z_i the difficulty of any block obtained from query $i \in J$. Define the martingale sequence

$$X_0 = 0;\ X_k = \sum_{i \in [k]} Z_i + \sum_{i \in [k]} \mathbf{E}[Z_i|\mathcal{E}_{i-1}], k \in [\nu].$$

With similar calculations above we obtain that G_t (with $t = \epsilon p \nu$) implies

$$X_k - X_{k-1} \leq \frac{2\gamma\Gamma^2 t}{\epsilon f|S|} \stackrel{\text{def}}{=} b \text{ and } V \leq \frac{2\gamma\Gamma^2 t^2}{\epsilon^2 f|S|} \stackrel{\text{def}}{=} v.$$

Applying Theorem 16 we obtain

$$\mathbf{Pr}[X_\nu \geq t \wedge G_t] \leq \exp\Big\{-\frac{\epsilon t}{2b(1+\frac{\epsilon}{3})}\Big\} \leq e^{-\varphi}. \tag{5}$$

We next focus on part (b). For each $j \in J$, let A_j be equal to the difficulty of the block obtained with the j-th query as long as the target was at least $T(J)/\tau$; thus $A(J) = \sum_{j \in J} A_j$. If $|J| = \nu$, identify J with $[\nu]$ and define the martingale

$$X_0 = 0;\ X_k = \sum_{j \in [k]} A_j - \sum_{j \in [k]} \mathbf{E}[A_j | \mathcal{E}_{j-1}], k \in [\nu].$$

For all $k \in [\nu]$ we have $X_k - X_{k-1} \leq \tau/T(J)$, $\mathbf{var}[X_k - X_{k-1}|\mathcal{E}_{k-1}] \leq p\tau/T(J)$, and $\mathbf{E}[A_j|\mathcal{E}_{j-1}] \leq p$. We may apply Theorem 16 with $b = \tau/T(J)$, $v = bp\nu \leq bt/\epsilon$, and $t = \max\{\epsilon p\nu, 2(\frac{1}{\epsilon} + \frac{1}{3})b\varphi\}$. We obtain

$$\mathbf{Pr}\Big[\sum_{jinJ} A_j \geq p\nu + t\Big] \leq \exp\Big\{-\frac{t}{2b(\frac{1}{3} + \frac{1}{\epsilon})}\Big\} \leq e^{-\varphi}. \tag{6}$$

Based on the three martingale inequalities (4)(5)(6), the probability of one partition S is not typical is bounded by $1 - (1 - e^{-\varphi})^3 \leq 3e^{-\varphi}$. For a system that runs for L steps, the total number of partitions is $\binom{L}{2}$. Thus, the probability for the entire execution is not typical is bounded by $3\binom{L}{2}e^{-\varphi}$.

For part (c) and $i \in \{0, 1, 2, 3\}$, let $B_i = \langle r_i, st_i, x_i, ctr_i \rangle$ and $g_i = G(r_i, st_i, x_i)$. If a block extends two distinct blocks, then a collision has occurred. To see this, suppose block B_3 extents two distinct blocks B_1 and B_2, then $st_3 = H(ctr_1, g_1) = H(ctr_2, g_2)$; implying a collision either in H or in G, since B_1 and B_2 are distinct. The existence of an insertion or a copy implies a collision as well. If the total running time of the system of ITM's is L then it holds that there are at most L queries posed to G, H. It follows that the probability of a collision occurring is $\binom{L}{2}2^{-\kappa+1} \leq 2^{-\kappa+1+2\log L}$. Note that, for polynomially many rounds in κ, the probability that a guessed block occurs is exponentially small in κ.

Recall that φ is a polylogarithmic function of κ (e.g., $\varphi = O(\log^2 \kappa)$). Combining probabilities on both martingale and collision, the overall error probability is $\mathsf{poly}(L) \cdot e^{-\Omega(\mathsf{polylog}(\kappa))}$.

C Bitcoin Cash (Cont'd)

C.1 The SMA Target Calculation Function

The target calculation function $D(\cdot)$ aims at maintaining the block production rate constant. The probability $f(T, n)$ with which n parties produce a new block with target T is approximated by

$$f(T, n) \approx \frac{qTn}{2^\kappa}.$$

To achieve the above goal Bitcoin Cash tries to keep $qTn/2^\kappa$ close to f. To that end, Bitcoin Cash watches on an epoch of m previous blocks and based on their difficulty as well as how fast these blocks were generated, it computes the next target. More specifically, say the last m blocks of a chain $\mathcal{C}$ with targets $\{T_i\}_{i \in [m]}$ were produced in Λ rounds. For every block in the epoch and n participants, it holds that $f(T_i, n) \approx qT_in/2^\kappa$. For m consecutive blocks, the average block generating rate $f^* = qn\sum_{i \in [m]} T_i/(m \cdot 2^\kappa)$, and the entire generating time equals $\Lambda = m/f^* = m^2 \cdot 2^\kappa/(qn\sum_{i \in [m]} T_i)$.

Consider the case where a number of players

$$n(T_1, \ldots, T_m, \Lambda) = \frac{m^2 \cdot 2^\kappa}{q\Lambda \sum_{i \in [m]} T_i}$$

attempt to produce m blocks of target $\{T_i\}_{i\in[m]}$ in Λ rounds (in expectation). Such number of players can be estimated as $n(T_1,\ldots,T_m,\Lambda) = m^2\cdot 2^\kappa/(q\Lambda\sum_{i\in[m]} T_i)$; then the next target T' is set so that $n(T_1,\ldots,T_m,\Lambda)$ players would need $1/f$ rounds in expectation to produce the next block of target T'. Therefore, it makes sense to set

$$T' = \frac{\Lambda}{m^2/f}\cdot\sum_{i\in[m]} T_i$$

because if the number of players is indeed $n(T_1,\ldots,T_m,\Lambda)$ and remains unchanged, it will take them $1/f$ rounds in expectation to produce next block. If the estimate of the initial number of parties is n_0, we will assume T_0 is appropriately set so that $f \approx qT_0n_0/2^\kappa$ and then

$$T' = \frac{n_0}{n(T_1,\ldots,T_m,\Lambda)}\cdot T_0.$$

Based on the above, we can now give a formal definition of the target (re)calculation function. In the definition, parameter τ serves as the "dampening filter" to deal with the case $n_0T_0/n(T^{\text{avg}},\Lambda)\notin[T/\tau,\tau T]$. This mechanism is necessary given an efficient attack presented by Bahack [Bah13] for Bitcoin, which also applies to Bitcoin Cash.

Definition 9. For fixed constants κ,τ,m,n_0,T_0, the target calculation function $D:\mathbb{Z}^*\to\mathbb{R}$ is defined as

$$D(\varepsilon)=T_0 \text{ and } D(r_1,\ldots,r_v)=\begin{cases}\frac{1}{\tau}\cdot T^{\text{avg}}, & \text{if } \frac{n_0\cdot T_0}{n(T^{\text{avg}},\Lambda)} < \frac{1}{\tau}\cdot T^{\text{avg}}\\ \tau\cdot T^{\text{avg}}, & \text{if } \frac{n_0\cdot T_0}{n(T^{\text{avg}},\Lambda)} > \tau\cdot T^{\text{avg}}\\ \frac{n_0}{(T^{\text{avg}},\Lambda)}\cdot T_0, & \text{otherwise}\end{cases}$$

where $n(T^{\text{avg}},\Lambda)=\frac{m\cdot 2^\kappa}{q\Lambda T^{\text{avg}}}$, with $\Lambda = r_{v-1}-r_{v-m}$ and $T^{\text{avg}}=\frac{1}{m}\cdot\sum_{i=1}^{m} D(r_1,...,r_{v-i})$.

Remark 4. At the start of the protocol execution – the first m blocks – parties cannot acquire enough previous blocks for target recalculation. While several alternate approaches exist (e.g., maintain a static target or switch to other target recalculation function), we will assume a "safe start", i.e., there exist blocks with an ideal target value prior to the first round.

C.2 Protocol Analysis (SMA)

In this section we present the analysis of the Bitcoin Cash protocol with SMA function. In order to make this section self-contained, we state all the preliminaries. However, we omit some of the proofs which generally follow those in Section 4.

For convenience, Table 2 summarizes the set of parameters introduced so far. Our security parameter is κ, and note that $\varphi=\Theta(m)=\mathsf{polylog}(\kappa)$.

Our probability space is over all executions of length at most some polynomial in κ. We will denote by $\mathbf{Pr}$ the probability measure of this space. We also define the random variable $\mathcal{E}$ taking values on this space and with a distribution induced by the random coins of all entities (adversary, environment, parties) and the random oracle.

Recall that in our modeling of a bounded-delay network, each party's query bound to the random oracle (RO) per round is $q=1$. Now suppose at round r exactly n parties query the RO with a target T. Then the probability that at least one of them will succeed is

$$f(T,h)=1-(1-pT)^h\le phT, \text{ where } p=1/2^\kappa.$$

Table 2: *Summary of parameters (SMA).*

- δ: Advantage of honest parties, $\forall r(t_r/h_r < 1 - \delta)$.
- $\gamma, \sigma, \Gamma, \Sigma$: Determines how the number of parties fluctuates across rounds in a period, cf. Definition 1 and Fact 1.
- f: Probability at least one honest party succeeds in a round assuming h_0 parties and target T_0 (the protocol's initialization parameters).
- m: the length of an epoch in number of blocks.
- τ: The dampening filter, cf. Definition 9.
- λ: The ratio of target range, with relation to target recalculation algorithm.
- ϵ: Quality of concentration of random variables, cf. Definition 13.
- κ: The length of the hash function output.
- φ: Related to the properties of the protocol.
- L: The total number of rounds in the execution.

As in prior work, we denote $f_0 = f(T_0, h_0)$ where T_0 and h_0 are the initial target and estimate of number of honest parties. The objective of the target recalculation mechanism would be to maintain a target T for each party such that $f(T, h_r) \approx f_0$ for all rounds r. For simplicity, we will drop the subscript from f_0, and will always specify the two arguments of $f(\cdot, \cdot)$ to avoid confusion.

During a round r of an execution E, the honest parties might be split and work on different chains, and thus might query the RO on different targets. Denote by $T_r^{\min}$ and $T_r^{\max}$ the minimum and maximum of these targets, respectively. We say r is a target-recalculation point of a valid chain $\mathcal{C}$, if there is a block with timestamp r (recall that the target is adjusted for every block).

Next, and following [GKL17] we define properties *goodness* and *accuracy*, which we will then show most executions satisfy, and which will help achieve the desired application. In addition,

Definition 10 (Goodness)**.** Round r is *good* if $f/2\gamma\Gamma \leq ph_r T_r^{\min}$ and $ph_r T_r^{\max} \leq (1+\delta)\gamma\Gamma f$. Round r is a *target-recalculation point* of a chain $\mathcal{C}$, if $\mathcal{C}$ has a block with timestamp r. A target-recalculation point r is *good* if the target T for the next block satisfies $f/2\Gamma \leq ph_r T \leq (1+\delta)\Gamma f$. Finally, a chain is *good* if all its target-recalculation points are good.

Definition 11 (Accuracy)**.** A block created at round u is *accurate* if it has a timestamp v such that $|u - v| \leq \ell + 2\Delta$. A chain is *accurate* if all its blocks are accurate. A chain is *stale*, if for some $u \geq \ell + 2\Delta$ it does not contain an honest block with timestamp $v \geq u - \ell - 2\Delta$.

We already mentioned that an epoch a sequence of m blocks. Let u and v be the corresponding initial and last target-recalculation points. Thus, the *duration* of an epoch is $v - u$.

For a given round r, we let $\mathcal{S}_r$ denote a set of chains that belong, or could potentially belong to an honest party. Being explicit about this set of chains will help expressing a number of predicates (see below). Specifically, $\mathcal{S}_r$ includes:

- Chain $\mathcal{C}$ that belongs to an honest party;
- Chain $\mathcal{C}$ with $\text{diff}(\mathcal{C}) > \text{diff}(\mathcal{C}')$ for some chain $\mathcal{C}'$ of an honest party;
- chain $\mathcal{C}$ with $\text{diff}(\mathcal{C}) = \text{diff}(\mathcal{C}')$ for some chain $\mathcal{C}'$ of an honest party and $\text{head}(\mathcal{C})$ was computed no later than $\text{head}(\mathcal{C}')$.

Note that these chains should exist and be valid at round r. As in [GKL17], we now define the following series of predicates.

Definition 12. For a round r, let:
- $\textsc{GoodChains}(r) \triangleq$ "For all $u \leq r$, every chain in $\mathcal{S}_u$ is good."
- $\textsc{GoodRounds}(r) \triangleq$ "All rounds $u \leq r$ are good."
- $\textsc{NoStaleChains}(r) \triangleq$ "For all $u \leq r$, there are no stale chains in $\mathcal{S}_u$."
- $\textsc{Accurate}(r) \triangleq$ "For all $u \leq r$, all chains in $\mathcal{S}_u$ are accurate."
- $\textsc{Duration}(r) \triangleq$ "For all $u \leq r$ and duration Λ of any epoch in $\mathcal{C} \in \mathcal{S}_u$, $\frac{1}{2(1+\delta)\Gamma^2} \cdot \frac{m}{f} \leq \Lambda \leq 2(1+\delta)\Gamma^2 \cdot \frac{m}{f}$."

Random variables. We are interested in estimating the difficulty acquired by honest parties during a sequence of rounds. For a given round r, the following real-valued random variables are defined in [GKL20]:

- D_r: "Sum of the difficulties of all blocks computed by honest parties."
- Y_r: "Maximum difficulty among all blocks computed by honest parties."
- Q_r: "Equal to Y_r when $D_u = 0$ for all $r < u < r + \Delta$ and 0 otherwise."

A round r such that $D_r > 0$ is called *successful* and one where $Q_r > 0$ *isolated successful*.

Regarding the adversary, in order to overcome the fact that he can query the oracle for arbitrarily low targets and thus obtain blocks of arbitrarily high difficulty, we would like to upper-bound the difficulty he can acquire during a set J of queries. This is achieved by associating a set of consecutive adversarial queries J with the target of its first query. We denote this target $T(J)$, and say that $T(J)$ is *associated* with J. We then define $A(J)$ and $B(J)$ to be equal to the sum of the difficulties of all blocks computed by the adversary during queries in J for target at least $T(J)/\lambda\Gamma$ and $T(J)$, respectively – i.e., queries in J for targets less than $T(J)/\lambda\Gamma$ (resp. $T(J)$) do not contribute to $A(J)$ (resp. $B(J)$).

For simplicity, we write $h(S) = \sum_{r \in S} h_r$ for a set of rounds S and queries J (similarly, $t(S)$, $D(S)$, $Y(S)$, $Q(S)$, $A(J)$ and $B(J)$).

Regarding protocol executions, let $\mathcal{E}_{r-1}$ fix the execution just before round r. In particular, a value E_{r-1} of $\mathcal{E}_{r-1}$ determines the adversarial strategy and so determines the targets against which every party will query the oracle at round r and the number of parties h_r and t_r, but it does not determine D_r or Q_r. For an adversarial query j we will write E_{j-1} for the execution just before this query.

Furthermore, we are interested in the range of targets during an execution in a $(\langle\gamma, \sigma\rangle, \langle\Gamma, \Sigma\rangle)$-respecting environment. Intuitively, the range of target T is related to the number of parties $\mathbf{n}$. It is not hard to see that the ratio T_{max}/T_{min} should be larger than that on number of parties, or the protocol would not be able to achieve the goal of maintaining a stable block generating rate ($f \approx pTn$). In addition, recall that according to the target recalculation function $T' = \frac{\Delta}{m^2 f} \cdot \sum_{i \in [m]} T_i$, the targets of two neighboring blocks will not differ by too much, thus forming a relatively "smooth" oscillation target curve. Finally, we define a new variable $\lambda \geq 1$ that helps bound the fluctuation on the targets. Specifically, we assume that the targets during consecutive rounds $|S| \leq \Sigma$ satisfy $T_{max} \leq \lambda\Gamma \cdot T_{min}$).

In order to obtain meaningful concentration of random variables, we have to consider a sufficiently long sequence with a number of rounds at least

$$\ell = \frac{4(1+3\epsilon)}{\epsilon^2 f[1-(1+\delta)\gamma\Gamma f]^{\Delta+1}} \cdot \max\{\Delta, \lambda\Gamma\} \cdot \gamma\Gamma^2 \cdot \varphi. \tag{7}$$

We will assume that ℓ is appropriately small compared to the duration of an epoch. Specifically,

$$2\ell + 6\Delta \leq \frac{\epsilon m}{2(1+\delta)\Gamma^2 f}. \tag{C3}$$

In addition, we would like the advantage δ of the honest parties over adversarial parties to belarge enough to absorb error factors. Thus, we require the following inequalities:

$$\frac{4\lambda\Gamma}{(\lambda\Gamma+1)^2}\cdot[1-(1+\delta)\gamma\Gamma f]^{\Delta}>1-\epsilon \text{ and } \epsilon\leq\delta/8\leq 1/8. \tag{C4}$$

Note that the chain-growth lemma still works here (cf. Lemma 1).

Typical executions. We have introduced the notion of *typical executions* in Section 4. Regarding SMA, we note that Bitcoin Cash (as well as other cryptocurrencies) adopts strategies to reduce the impact of bad events to the minimal extent, such as rearranging the blocks temporarily before recalculating the target (as opposed to the Bitcoin implementation, which directly adopts the last block's timestamp). Since our definition automatically gets rid of these bad events, we will not consider such mechanisms in our analysis.

We are now ready to specify the typical execution associated with SMA algorithm.

Definition 13 (Typical execution)**.** An execution E is *typical* if the following hold:

(a) For any set S of at least ℓ consecutive good rounds,

$$(1-\epsilon)[1-(1+\delta)\gamma\Gamma f]^{\Delta}ph(S)<Q(S)\leq D(S)<(1+\epsilon)ph(S).$$

(b) For any set J indexing a set of consecutive queries of the adversary and $\alpha(J)=2(\frac{1}{\epsilon}+\frac{1}{3})\varphi/T(J)$,

$$A(J)<p|J|+\max\{\epsilon p|J|,\lambda\Gamma\alpha(J)\} \text{ and } B(J)<p|J|+\max\{\epsilon p|J|,\alpha(J)\}.$$

(c) No insertions, no copies, and no predictions occurred in E.

The next proposition is a simple application of Definition 13 as well as the honest-majority-environment assumption.

Proposition 20. *Let E be a typical execution in a $(\langle\gamma/(2-\delta),\sigma\rangle,\langle\Gamma/(2-\delta),\Sigma\rangle)$-respecting environment. Let $S=\{r\mid(u\leq r\leq v)\wedge(v-u\geq\ell)\}$ be a set of consecutive good rounds and J the set of adversarial queries in $U=\{r\mid u-\Delta\leq r\leq v+\Delta\}$. The following inequalities hold:*

(a) $(1+\epsilon)p|J|\leq Q(S)\leq D(U)<(1+5\epsilon)Q(S)$.
(b) $T(J)A(J)<\epsilon m/4(1+\delta)$ *or* $A(J)<(1+\epsilon)p|J|$ *and* $\lambda\Gamma\cdot T(J)B(J)<\epsilon m/4(1+\delta)$ *or* $B(J)<(1+\epsilon)p|J|$.
(c) If w is a good round such that $|w-r|\leq\Sigma$ for any $r\in S$, then $Q(S)>(1+\epsilon)[1-(1+\delta)\gamma\Gamma f]^{\Delta}|S|ph_w/\Gamma$. If in addition $T(J)\geq T_w^{\min}$, then $A(J)<(1-\delta+3\epsilon)Q(S)$.

Proof. We only consider part (b): if $\epsilon p|J|\geq\lambda\Gamma\alpha(J)$, Definition 13(b) applies directly. Otherwise, $p|J|<\lambda\Gamma\alpha(J)/\epsilon$ we get

$$T(J)\cdot A(J)<\frac{2}{\epsilon^2}\cdot(1+\epsilon)(1+\frac{\epsilon}{3})\lambda\Gamma\varphi<\frac{f\ell}{2\gamma\Gamma^2}<\frac{\epsilon m}{4(1+\delta)\gamma\Gamma^2}.$$

This follows Equation 7 and Condition (C3). □

We are now able to show that almost all Bitcoin Cash protocol executions polynomially bounded (in κ) are typical. Formally:

Theorem 21. *Assuming the Bitcoin Cash backbone protocol runs for L rounds, the event "E is not typical" is bounded by* $\mathsf{poly}(L) \cdot e^{-\Omega(\mathsf{polylog}(\kappa))}$.

Next, we consider the validity of the predicates in Definition 12 over the space of typical execution in a $(\langle \gamma/(2-\delta), \sigma \rangle, \langle \Gamma/(2-\delta), \Sigma \rangle)$-respecting environment, as well as implications among them. Furthermore, we assume that all the requirements hold for the initialization parameters h_0 and T_0. (Some of the statements' proofs require the execution to have a safe start [cf. Remark 4].)

Lemma 22. $\textsc{GoodRounds}(r-1) \implies \textsc{NoStaleChains}(r)$.

Proof. For the sake of a contradiction, consider a chain $\mathcal{C} \in \mathcal{S}_r$ which has not been extended by the honest party for at least $\ell + 2\Delta$ rounds. Without loss of generality, let r denote the least round with this property. Let B be the last block of $\mathcal{C}$ computed by the honest party (possibly the genesis block) and let w be its timestamp. Set $S = \{u : w + \Delta \leq u \leq r - \Delta\}$ and $U = \{u : w \leq u \leq r\}$. Therefore, to reach a contradiction it suffices to show that the adversary's accumulated difficulty $d < Q(S)$.

Let J denote the queries in U starting from the first adversarial query attempting to extend B. We learn from Condition (C3) that targets the adversary can query during U is bounded by $T(J)/\lambda\Gamma$. Thus, $d < A(J)$. If $A(J) < (1+\epsilon)p|J|$, then $A(J) < Q(S)$ is obtained by Proposition 20(a). Otherwise, $A(J) < (\frac{1}{\epsilon}+1)\lambda\Gamma\alpha(J) = 2(\frac{1}{\epsilon}+1)(\frac{1}{\epsilon}+\frac{\epsilon}{3})\lambda\Gamma\varphi/T(J)$. However, by considering only the first ℓ rounds in S, $h(S) \geq h_u\ell/\gamma$. We have

$$\begin{aligned} Q(S) &> (1-\epsilon)[1-(1+\delta)\gamma\Gamma f]^{\Delta} \cdot \frac{ph_u\ell T(J)}{\gamma T(J)} \\ &> \frac{(1-\epsilon)[1-(1+\delta)\gamma\Gamma f]^{\Delta} f\ell}{2\gamma\Gamma^2 T(J)} \geq \frac{2(1-\epsilon)(1+3\epsilon)\lambda\Gamma\varphi}{\epsilon^2 T(J)} \geq A(J). \end{aligned}$$

In either situation, we obtain the desired inequality $d < Q(S)$. □

Corollary 23. $\textsc{GoodRounds}(r-1) \implies \textsc{Accurate}(r)$.

Lemma 24. $\textsc{GoodRounds}(r-1) \wedge \textsc{GoodChains}(r-1) \implies \textsc{Duration}(r)$.

Proof. Assume—towards a contradiction—that $\textsc{Duration}(r)$ is false. Without loss of generality, assume $w \leq r$ associated with a chain $\mathcal{C} \in \mathcal{S}_w$ and duration Λ is the smallest round that does not satisfy

$$\frac{1}{2(1+\delta)\Gamma^2} \cdot \frac{m}{f} \leq \Lambda \leq 2(1+\delta)\Gamma^2 \cdot \frac{m}{f}.$$

For the upper bound, we will show that the honest parties can by themselves obtain m blocks. Note that Lemma 22 implies that two honest blocks in this epoch exist with timestamps u and v that $v - u \geq \Lambda - 2\ell + \Delta$. We define $S = \{r : u + \Delta \leq r \leq v + \Delta\}$ the time that the honest parties can contribute to the difficulty. Assuming $\Lambda > 2(1+\delta)\Gamma^2 m/f$, Condition (C3) implies that $|S| \geq \Lambda - 2\ell - 6\Delta \geq 2(1+\delta)(1-\epsilon)\Gamma^2 m/f$. We denote $r^* \in S$ the timestamp of the block with lowest target (i.e., highest difficulty) and T_{r^*} its associated target. Thus we have

$$Q(S) > (1-\epsilon)[1-(1+\delta)\gamma\Gamma f]^{\Delta} \cdot \frac{f|S|}{2\Gamma^2 T_{r^*}} \geq (1+\delta)(1-\epsilon)^3 \cdot \frac{m}{T_{r^*}} > \sum_{r\in[m]} \frac{1}{T_i}.$$

For the first inequality, we used Definition 13(a) and that r^* is a good target-recalculation point; the lower bound for $|S|$ above for the next and Condition (C4) for the last one. Result here contradicts

Chain Growth, since the honest parties at round v have already accumulated more than $\sum_{i\in[m]} 1/T_i$ difficulty on top of u.

For the sake of proving the lower bound, we are going to argue that even if the honest parties and the adversary join forces they still cannot obtain m blocks. We denote u, v timestamps of the the first and last block in the epoch, respectively. Then, the honest parties can contribute difficulty during $S = \{u, \ldots, v\}$, and queries are available for the adversary during $S' = \{u - \ell - 2\Delta, \ldots, v + \ell + 2\Delta\}$. Note that S' contains all rounds that the adversary can contribute, since we have Corollary 23 and then $\textsc{Accurate}(r)$ holds. Similarly, we denote $r^* \in S'$ the timestamp of the block with highest target (i.e., lowest difficulty) and T_{r^*} its associated target. We claim that the adversary start with the first query for target T_{r^*} (so that $T(J) = T_{r^*}$). Since r^* is a good target-recalculation point, it follows Fact 1(a) that $ph(S) \le p\Gamma h_{r^*}|S| \le (1+\delta)\Gamma^2 f|S|/T_{r^*}$. Thus,

$$D(S) < (1+\epsilon)ph(S) \le (1+\epsilon)(1+\delta)\cdot\frac{\Gamma^2 f|S|}{T_{r^*}} \le (1+\epsilon)\cdot\frac{m}{2T_{r^*}}.$$

With respect too the adversary, if $\tau T(J)B(J) < \epsilon m/4$, then the total number of blocks is less than m and we are done. Otherwise, by Proposition 20(b) we have

$$\begin{aligned} B(J) &< (1+\epsilon)p|J| \le (1+\epsilon)(1-\delta)ph(S') \le (1+\epsilon)(1-\delta)p\Gamma h_{r^*}|S'| \\ &\le (1-\delta^2)(1+\epsilon)(1+\frac{2\ell+4\Delta}{|S|})|S|\cdot\frac{\Gamma^2 f}{T_{r^*}} \le (1-\delta)(1+\epsilon)^2\cdot\frac{m}{2T_{r^*}}. \end{aligned}$$

For the last inequality, note that since Condition (C4) we have $\epsilon \le \delta/4$ and T_{r^*} is the maxmium target in the epoch,

$$B(J) + D(S) < \frac{(1+\epsilon)+(1-\delta)(1+\epsilon)^2}{2}\cdot\frac{m}{T_{r^*}} < \frac{m}{T_{r^*}} \le \sum_{r\in[m]}\frac{1}{T_i},$$

which means the total accumulated difficulty cannot produce m blocks. □

Lemma 25. $\textsc{GoodRounds}(r-1) \Longrightarrow \textsc{GoodChains}(r)$.

Proof. For starter, we recall our assumption that all the target recalculation points in the first epoch are good. Therefore, it suffices to show that the next target-recalculation point will be good, when given an epoch with m good recalculation points. We denote u, v timestamps of the epoch beginning and ending, respectively (a.k.a. $v = u + \Lambda < r$). Let T' denote the target of the next block, our goal is to show that $f/2\Gamma \le ph_v T' \le (1+\delta)\Gamma f$.

We begin with the lower bound. If $\Lambda \ge \Gamma m/f$, i.e., $T' \ge \Gamma \cdot T^{\text{avg}}$, we pick the block in the epoch with lowest target and denote r^* its timestamp and $T^{\min}$ its associated target. Thus, $ph_v T' \ge ph_{r^*}T'/\Gamma \ge ph_{r^*}T^{\text{avg}} \ge ph_{r^*}T^{\min} \ge f/2\Gamma$, because r^* is assumed to be a good target-recalculation point.

Now we assume $\Lambda < \Gamma m/f$, which implies $\Lambda \le (T'/T^{\text{avg}})(m/f)$. It is clear that all the blocks are computed in either $S = \{u, \ldots, v\}$ by the honest parties or $S' = \{u - \ell - 2\Delta, \ldots, v + \ell + 2\Delta\}$ by the adversary. Let J denote the set of queries available to the adversary in S'. Note that, by Condition (C3) and Lemma 24, $|S'| = \Lambda + 2\ell + 4\Delta \le (1+\epsilon)\Lambda$. We have

$$B(J) < (1-\delta)(1+\epsilon)ph(S') \le (1-\delta)(1+\epsilon)p\Gamma h_v|S'| \le (1-\delta)(1+\epsilon)^2 p\Gamma h_v \Lambda.$$

Similarly, $D(S) < (1+\epsilon)ph(S) \le (1+\epsilon)p\Gamma h_v\Lambda$. Assuming $ph_vT' < f/2\Gamma$ and recalling Fact 2, we obtain the contradiction

$$2\Gamma ph_v\Lambda \le 2\Gamma ph_v\cdot\frac{T'}{T^{\text{avg}}}\cdot\frac{m}{f} < \frac{m}{T^{\text{avg}}} \le \sum_{i\in[m]}\frac{1}{T_i} \le D(S) + B(J) < (2+4\epsilon-\delta)\Gamma ph_v\Lambda \le 2\Gamma ph_v\Lambda.$$

Then we move to the upper bound. Note that the adversary can be silent, so here we only consider the honest parties contributing difficulty during $S = \{u + \ell + 3\Delta, \ldots, v - \ell - 3\Delta\}$. If $\Lambda \leq m/\Gamma f$, then $T' \leq T^{\text{avg}}/\Gamma$. We pick the block in the epoch with highest target and denote r^* its timestamp and $T^{\max}$ its target. Thus, $ph_v T' \leq p\Gamma h_{r^*} T' \leq ph_{r^*} T^{\text{avg}} \leq ph_{r^*} T^{\max} \leq (1+\delta)\Gamma f$ where r^* is a good target-recalculation point.

Next, we may assume $\Lambda > m/\Gamma f$, which implies $\Lambda \geq (T'/T^{\text{avg}})(m/f)$ and $|S| = |\Lambda - 2\ell - 6\Delta| \geq (1-\epsilon)\Lambda$. Assuming $ph_v T' > (1+\delta)\Gamma f$, we obtain the following contradiction

$$\begin{aligned}\frac{ph_v\Lambda}{(1+\delta)\Gamma} &\geq \frac{ph_v}{(1+\delta)\Gamma} \cdot \frac{T'}{T^{\text{avg}}} \cdot \frac{m}{f} > \frac{m}{T^{\text{avg}}} \geq \frac{4\lambda\Gamma}{(\lambda\Gamma+1)^2} \sum_{i\in[m]} \frac{1}{T_i} \\ &\geq \frac{4\lambda\Gamma}{(\lambda\Gamma+1)^2} \cdot Q(S) > \frac{4\lambda(1-\epsilon)[1-(1+\delta)\gamma\Gamma f]^{\Delta}}{(\lambda\Gamma+1)^2} \cdot ph_v|S| \geq \frac{ph_v\Lambda}{(1+\delta)\Gamma}.\end{aligned}$$

The third inequality follows Theorem 19. For the next one, recall that $\mathcal{C} \in \mathcal{S}_r$ and by Lemma 22 there is a block computed by an honest party among the first and last $\ell + 2\Delta$ rounds of the epoch, respectively. Thus this inequality follows by Chain Growth. The last inequality is a consequence of Condition (C4). □

Corollary 26. GoodRounds$(r-1) \Longrightarrow$ GoodRounds(r).

Proof. Consider any $\mathcal{C} \in \mathcal{S}_r$ and let u denote its last recalculation point before r and T the associated target. If r is a recalculation point, it follows directly by Lemma 25. Otherwise, we learn from Lemma 22 that the maximum interval between two blocks is $\ell + 2\Delta$, thus by Fact 1(a) $h_u/\gamma \leq h_r \leq \gamma h_u$ holds. Combining it with $f/2\Gamma \leq ph_u T \leq (1+\delta)\Gamma f$ we obtain the desired inequality. □

Theorem 27. *Consider a typical execution in a $(\langle\gamma/(2-\delta), \ell+2\Delta\rangle, \langle\Gamma/(2-\delta), 2(1+\delta)\Gamma^2 m/f\rangle)$-respecting environment, and assume conditions C3 and C4 are satisfied. Then all predicates in Definition 12 hold.*

Theorem 27 follows directly follows from our safe-start assumption (Remark 4).

Blockchain properties. We now show that Bitcoin Cash satisfies the two properties common prefix and chain quality (cf. Section 2), for a suitable respecting environment. First, a preliminary lemma:

Lemma 28. *For any round r of a typical execution in a $(\langle\gamma/(2-\delta), \ell+2\Delta\rangle, \langle\Gamma/(2-\delta), 2(1+\delta)\Gamma^2 m/f\rangle)$-respecting environment and any two chains $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{S}_r$, the timestamp of head($\mathcal{C} \cap \mathcal{C}'$) is at least $r - 2\ell - 4\Delta$.*

Theorem 29 (Common-Prefix)**.** *For a typical execution in a $(\langle\gamma/(2-\delta), \ell+2\Delta\rangle, \langle\Gamma/(2-\delta), 2(1+\delta)\Gamma^2 m/f\rangle)$-respecting environment, the common-prefix property holds for parameter $\epsilon\gamma^2 m/2\Gamma$.*

Proof. Suppose—towards a contradiction—common prefix fails for two chains $\mathcal{C}_1$ and $\mathcal{C}_2$ at rounds $r_1 \leq r_2$. It is easy to see that there exist a round $r \leq r_2$ and two chains $\mathcal{C}$ and $\mathcal{C}'$ such that each had at least k blocks after head($\mathcal{C} \cap \mathcal{C}'$). In view of Lemma 28, it suffices to show that these blocks were computed within at least $\ell + 2\Delta$ rounds. Suppose the honest parties query during a set of rounds S of size $\ell + 2\Delta$, the accumulated difficulty is $D(S)$, and the number of blocks produced in $|S|$ is at most $T^{\max} \cdot D(S)$ where $T^{\max}$ denote the maximum target among blocks in $|S|$. For any round r^* associated with target $T^{\max}$, we have

$$T^{\max} \cdot D(S) < (1+\epsilon)pT^{\max}h(S) \leq (1+\epsilon)p\gamma T^{\max}h_{r^*}|S| < (1+\epsilon)\epsilon\gamma^2 m/4\Gamma.$$

For the last inequality, note that by Condition (C3) $|S| < \epsilon m/4(1+\delta)\Gamma^2 f$ and by Theorem 27 $pT^{\max} h_{r^*} \leq (1+\delta)\gamma\Gamma f$. By proposition 20 the adversary contributed to less than $(1+\epsilon)\epsilon\gamma^2 m/2(1+\delta)\Gamma$ blocks, for a total of less than $\epsilon\gamma^2 m/2\Gamma$. □

Theorem 30 (Chain-Quality)**.** *For a typical execution in a* $(\langle\gamma/(2-\delta), \ell+2\Delta\rangle, \langle\Gamma/(2-\delta), 2(1+\delta)\Gamma^2 m/f\rangle)$*-respecting environment, the chain-quality property holds with parameters* $\ell+2\Delta$ *and* $\mu = \delta - 3\epsilon$*.*

Transaction ledger. We conclude this section by showing that a typical execution of the (abstraction of the) Bitcoin Cash protocol we are considering, in a $(\langle\gamma/(2-\delta), \ell+2\Delta\rangle, \langle\Gamma/(2-\delta), 2(1+\delta)\Gamma^2 m/f\rangle)$-respecting environment, satisfies the two properties of a robust transaction ledger presented in Section 2. They are the direct consequence of the blockchain properties shown above, following the approach in [GKL17, GKL20].

Theorem 31 (Consistency)**.** *For a typical execution in a* $(\langle\gamma/(2-\delta), \ell+2\Delta\rangle, \langle\Gamma/(2-\delta), 2(1+\delta)\Gamma^2 m/f\rangle)$*-respecting environment, Consistency is satisfied by setting the settled transactions to be those reported more than* $\epsilon\gamma^2 m/2\Gamma$ *blocks deep.*

Theorem 32 (Liveness)**.** *For a typical execution in a* $(\langle\gamma/(2-\delta), \ell+2\Delta\rangle, \langle\Gamma/(2-\delta), 2(1+\delta)\Gamma^2 m/f\rangle)$*-respecting environment, Liveness is satisfied for depth* $\epsilon\gamma^2 m/2\Gamma$ *with wait-time* $(\gamma^2+1)\epsilon m/f$*.*

Proof. We claim that the chain $\mathcal{C}$ of any honest party has at least $\epsilon\gamma^2 m/2\Gamma$ blocks that where computed in the last $\epsilon\gamma^2 m/(1-2\epsilon)f + 2\Delta$ rounds. Denote by $T^{\min}$ the lowest target among these blocks and r^* a round associated with this target. If S is its subset without the first and last Δ rounds, by Chain-Growth Lemma, the length of the segment is at least

$$T^{\min} \cdot Q(S) > (1-\epsilon)[1-(1+\delta)\gamma\Gamma f]^{\Delta} pT^{\min} h(S) \geq \frac{(1-2\epsilon)f|S|}{2\Gamma} \geq \frac{\epsilon\gamma^2 m}{2\Gamma}.$$

Furthermore, if a transaction tx is included in any block computed by an honest party for the first $\ell+2\Delta$ rounds, by Lemma 22, the chain $\mathcal{C}$ of any honest party contains tx in a block B. Thus, the honest parties will have

$$\ell + 4\Delta + \frac{\epsilon\gamma^2 m}{(1-2\epsilon)f} \leq \frac{\epsilon m}{2(1+\delta)\Gamma^2 f} + \frac{\epsilon\gamma^2 m}{(1-2\epsilon)f} \leq (\gamma^2+1)\cdot\frac{\epsilon m}{f}$$

as the total wait-time. □

D Comparison with the Bitcoin Cash Network (Cont'd)

In this section we present the comparison of our protocol abstraction using the SMA function with the (corresponding) Bitcoin Cash network data. The comparison reveals an undersized epoch length and dampening filter. Most of the parameters have been considered in Section 5, here we only need to evaluate one more parameter λ.

Target variation. Our formulation bounds the target variation by a fixed parameter λ and the parties' fluctuation ratio. This is a reasonable idea, since the number of parties, accumulated difficulty and number of blocks (chain growth) are correlated. In the real network, however, the hash power invested in minor blockchains varies wildly, and as such it is unlikely that the number of parties would stay high for a long period of time, and thus target values would seldom vary

more pronouncedly than the number of parties. Compared with other parameters, the criteria for selecting λ does not seem as strict. As we can cover more executions if we set a larger value for λ, we choose $\lambda = 1.201$ when applying our analysis. As a result, when compared with previous work [GKL17, GKL20], our analysis can tolerate situations where targets vary more wildly (i.e., they exceed the $\lambda\Gamma$ bound).

Conclusions.

Based on the considerations above, by parameterizing the real-world Bitcoin Cash network with network bounded-delay $\Delta = 1$ (round), honest advantage $\delta = 0.99$, quality of concentration $\epsilon = 0.123$, long term party fluctuation ratio $\Gamma = 1.398$, short term party fluctuation ratio $\gamma = 1.057$, ideal block generating rate $f = 0.01$, and target fluctuation parameter $\lambda = 1.201$, we obtain that Condition (C4) is satisfied under these parameters.

Regarding the protocol parameters m and τ in use, $m = 144$ meets the lowest criteria to support our analysis, $\tau = 2$ may fail in some situations, which can be avoided by these two parameters. More specifically, with regard to the epoch length m, from Condition (C3) we get an upper bound for ℓ of about 640 seconds, which would not be desirable as it exceeds the expected block production interval (600s). We would like that the actual value of ℓ amounts at least to a few blocks' total expected generation time. Thus,setting $m = 432$ (almost 3 days) would yield an ideal epoch length. Regarding the dampening filter τ, it follows from our analysis that it should hold that $\tau \geq 2(1+\delta)\Gamma^2 \approx 7.8$. Thus, $\tau = 8$ would be a suitable value.

Unfortunately, let $\lambda = 1.201$ is not a universal choice for all the executions. And we can see that parameters in Condition (C4) are almost impossible to be enlarged, i.e., our analysis is hard to take effect under the environment that party fluctuation ratio $\Gamma > 1.4$ (if we consider a larger value for λ, then the party fluctuation it can tolerate is even smaller). Compared with the ASERT function (notably, ASERT can work well when $\Gamma = 1.88$), obviously SMA performs worse when party fluctuation rate is relatively high.