Graph-Based Construction for Non-Malleable Codes

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Abstract

Non-malleable codes protect the communication against adversarial tampering of data, as a relaxation of error-correcting codes and error-detecting codes. In this paper we provide several explicit constructions of non-malleable codes based on bipartite expander graphs such as Ramanujan graphs, Wenger graphs and generalized quadrangles. The resulted codes can either work for a more flexible split-state model or reduce the encoding space cost in comparison with the existing results.

1 Introduction

Non-malleable codes, introduced by Dziembowski, Pietrzak and Wichs [21, 22], are resilient to adversarial tampering on *arbitrary* number of symbols which is beyond the scope of error-correcting and error-detecting codes. Consider the following "tampering experiment". A message $m \in \mathcal{M}$ is encoded via a (randomized) encoding function $\mathrm{enc} : \mathcal{M} \to \mathcal{X}$, yielding a codeword $c = \mathrm{enc}(m)$. However the codeword c is modified by an adversary using some tampering function $f \in \mathcal{F}$ with $f : \mathcal{X} \to \mathcal{X}$ to an erroneous word $\tilde{c} = f(c)$, and $\tilde{c}$ is decoded using a deterministic function dec, resulting $\tilde{m} = \mathrm{dec}(\tilde{c})$. In terms of the practical application, the reliability $\tilde{m} = m$ is desired. An error-correcting code with minimum distance d can guarantee the reliable communication with respect to the family $\mathcal{F}$ which satisfies that for $f \in \mathcal{F}$ the Hamming distance between $\tilde{c} = f(c)$ and c is at most $\lfloor (d-1)/2 \rfloor$. However it is impossible to achieve the reliability using error-correcting codes if the tampering family $\mathcal{F}$ is large. In order to deal with this, Dziembowski *et al.* [21] proposed the non-malleable codes (with respect to $\mathcal{F}$), which ensure that either the tampered codeword is correctly decoded, i.e., $\tilde{m} = m$, or the decoded message $\tilde{m}$ is completely unrelated to the original message m. As remarked in [21, 22], the concept of non-malleable codes is in a spirit of non-malleablity

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proposed by Dolev, Dwork and Naor [17] in cryptographic primitives. Informally speaking, the non-malleability in the context of encryption requires that given the ciphertext it is impossible to generate a different ciphertext so that the respective plaintexts are related [17].

It is known that no non-malleable code exists if the tampering family $\mathcal{F}$ is the entire space of functions. Thus the study on non-malleable codes has focused on the specific families $\mathcal{F}$. One typical tampering family is with the *split-state model*, which has also been investigated in the context of leakage cryptography [13, 20]. Roughly speaking, this model assumes that the encoded memory/state of the system is partitioned into two parts and adversaries can arbitrarily tamper the data stored in each part independently. More precisely, each message is encoded into a word $c = (L, R) \in \mathcal{L} \times \mathcal{R}$ and adversaries try to tamper it using some functions $g : \mathcal{L} \to \mathcal{L}$ and $h : \mathcal{R} \to \mathcal{R}$ which change c to $\tilde{c} = (g(L), h(R)) \in \mathcal{L} \times \mathcal{R}$. Moreover, if $|\mathcal{L}| = |\mathcal{R}|$, we call it an *equally-sized* split-state model.

To explicitly construct non-malleable codes is a fundamental and challenging problem. In the literature, explicit non-malleable codes for the split-state model have been derived based on two-source extractors and additive combinatorics, see [1, 2, 3, 4, 5, 6, 7, 10, 11, 18, 29, 30] for example. Notably, Dziembowski, Kazana and Obremski [18] pointed out: "This brings a natural question if we could show some relationship between the extractors and the non-malleable codes in the split-state model. Unfortunately, there is no obvious way of formalizing the conjecture that non-malleable codes need to be based on extractors". Recently, Rasmussen and Sahai [36] discovered that (non-bipartite) expander graphs could provide non-malleable codes for the split-state model, which in some sense answers Dziembowski-Kazana-Obremski's question in [18]. Inspired by [36], we are interested with exploring more graph-theoretic constructions for split-state non-malleable codes. More precisely, we shall study the following problem.

Problem 1. *Based on graph theory, provide explicit constructions of non-malleable codes for the split-state model.*

Indeed, Rasmussen and Sahai [36] provided an elegant answer to Problem 1. However we noticed that the construction in [36] cannot directly be transferred to the general split-state model. Inspired by this, we initially extend the construction in [36] to bipartite graphs. Specifically, in this paper, we first establish a coding scheme based on *bipartite graphs*. Then we prove that when the underlying bipartite graph is an (r, s)-*biregular graph* with the second largest eigenvalue μ, our coding scheme provides $O\big(\frac{\mu^{3/2}}{\sqrt{rs}}\big)$-non-malleable codes for the split-state model which is not necessarily to be equally-sized (see Theorem 9). This can be seen as an extension of the coding scheme in [36] in the sense that we could deduce the codes for equally-sized split-state model in [36] as special cases (see Remark 8). Based on this, we provide several more solutions to Problem 1 by means of Ramanujan graphs, Wenger graphs and generalized quadrangles (see Table 1). In particular, the resulted non-malleable codes

can either work for a more flexible non-equally sized split-state model (see Theorems 16, 17, 19) or reduce the encoding space cost (see Theorem 14, Corollary 15) in comparison with the non-malleable codes in [36, Section C].

Ref.	$\lvert\mathcal{L}\rvert$	$\lvert\mathcal{R}\rvert$	encoding space cost	equally-sized?	comments
[36, Section C]	q^3	q^3	$24\log(1/\varepsilon)+O(1)$	Yes	$q=p^2$, p is a prime
Corollary 15	$\Theta(p^{5/2}\log(p))$	$\Theta(p^{5/2}\log(p))$	$20\log(1/\varepsilon)+O(\log\log(1/\varepsilon))$	Yes	p is an odd prime
Theorem 16	q^3	q^3	$24\log(1/\varepsilon)+O(1)$	Yes	$q\neq 2$ is a prime power
Theorem 17	$(q+2)q^2$	q^3	$24\log(1/\varepsilon)+O(1)$	No	q is a prime power
Theorem 19	$(q^2+1)(q^5+1)$	$(q^3+1)(q^5+1)$	$60\log(1/\varepsilon)+O(1)$	No	q is a prime power

Table 1: Explicit graph-based ε-non-malleable codes in this paper and [36]

In addition, our results show that some related error-correcting codes have potential applications to constructing non-malleable codes for the split-state model. For example, it is well-known in coding theory and combinatorics that a low-density parity-check (LDPC) code has an associated bipartite graph called *Tanner graph*. Precisely, the Tanner graph of an LDPC code with parity-check matrix $H=(h_{ij})$ is a bipartite graph such that the vertex set is the index set of rows and columns of H, and two vertices i and j are adjacent if and only if $h_{ij}\neq 0$, see [39]. It is shown that the algebraic or combinatorial constructions of LDPC codes often provide Tanner graphs with small second largest eigenvalue, see [16, 25, 31, 38, 28] for example. According to the bipartite graph based coding scheme proposed in this paper, a connection between LDPC codes and non-malleable codes can be accordingly established. Particularly, the constructions in Theorems 16, 17 and 19 are based on several typical bipartite graphs realized as Tanner graphs of LDPC codes.

The remainder of this paper is organized as follows. Section 2 briefly reviews non-malleable codes and basics in graph theory. Section 3 provides the coding scheme based on bipartite graphs and discusses its non-malleability. Section 4 presents explicit constructions of non-malleable codes. Section 5 concludes this paper.

2 Preliminaries

In this section we recall the notion of non-malleable codes and some useful basics in graph theory.

Throughout this paper, let $x\leftarrow\mathcal{X}$ denote that the random variable x sampled uniformly from a set $\mathcal{X}$. Let $\perp$ denote a special symbol.

2.1 Non-malleable codes

A *coding scheme* is a pair of functions $(\mathrm{enc}, \mathrm{dec})$, where $\mathrm{enc} : \mathcal{M} \to \mathcal{X}$ is a randomized encoding function, and $dec : \mathcal{X} \to \mathcal{M} \cup \{\perp\}$ is a deterministic decoding function. Assume that for all $m \in \mathcal{M}$,

$$\Pr[\mathrm{dec}(\mathrm{enc}(m)) = m] = 1,$$

where the probability is taken over the randomness of enc.

Let A, B be two random variables over the same set $\mathcal{X}$. Then the *statistical distance* between A and B is defined as

$$\Delta(A, B) := \frac{1}{2} \sum_{x \in \mathcal{X}} \Big| \Pr[A = x] - \Pr[B = x] \Big|.$$

Definition 2 (Split-state non-malleable codes)**.** In the split-state model, assume $\mathcal{X} = \mathcal{L} \times \mathcal{R}$ is the product set of sets $\mathcal{L}$ and $\mathcal{R}$. Let $\mathcal{F}$ be a set of functions from $\mathcal{L} \times \mathcal{R}$ to itself, where each $f \in \mathcal{F}$ can be represented as $f(L, R) = (g(L), h(R))$ for all $(L, R) \in \mathcal{L} \times \mathcal{R}$ with some $g : \mathcal{L} \to \mathcal{L}$ and $h : \mathcal{R} \to \mathcal{R}$. Then a coding scheme $(\mathrm{enc}, \mathrm{dec})$ such that $\mathrm{enc} : \mathcal{M} \to \mathcal{L} \times \mathcal{R}$ and $\mathrm{dec} : \mathcal{L} \times \mathcal{R} \to \mathcal{M} \cup \{\perp\}$ is called an *ε-non-malleable code with respect to $\mathcal{F}$* if for every $f \in \mathcal{F}$, there exists a distribution D_f on $\mathcal{M} \cup \{\mathrm{same}^*, \perp\}$ such that for every $m \in \mathcal{M}$, we have $\Delta(A_f^m, B_f^m) \leq \varepsilon$, where

$$A_f^m := \left\{ \begin{array}{c} (L, R) \leftarrow \mathrm{enc}(m); \\ \text{Output } \mathrm{dec}(g(L), h(R)) \end{array} \right\},$$

$$B_f^m := \left\{ \begin{array}{c} \tilde{m} \leftarrow D_f; \\ \text{If } \tilde{m} = \mathrm{same}^* \text{ output } m \text{ else output } \tilde{m} \end{array} \right\}.$$

Hereafter, as in [18] and [36], the symbol "$\perp$" from Definition 2 will be dropped since it usually denotes the situation when the decoding function detects tampering and outputs an error message, which is not dealt in this paper. As mentioned in [18], this would be not so problematic for practical applications.

This paper focuses on *single-bit* non-malleable codes, i.e., $\mathcal{M} = \{0, 1\}$. It is shown in [18] that single-bit non-malleable codes can also be formulated as in the following Theorem 3.

Theorem 3 ([18, 19])**.** *Let* $(\mathrm{enc}, \mathrm{dec})$ *be a coding scheme with* $\mathrm{enc} : \{0, 1\} \to \mathcal{X}$ *and* $\mathrm{dec} : \mathcal{X} \to \{0, 1\}$*. Let $\mathcal{F}$ be a set of functions from $\mathcal{X}$ to itself. Then* $(\mathrm{enc}, \mathrm{dec})$ *is an ε-non-malleable code with respect to $\mathcal{F}$ if and only if it holds for every $f \in \mathcal{F}$ that*

$$\frac{1}{2} \sum_{b \in \{0,1\}} \Pr\Big[\mathrm{dec}\big(f(\mathrm{enc}(b))\big) = 1 - b\Big] \leq \frac{1}{2} + \varepsilon$$

where the probability is over the uniform choice of b and the randomness of enc*.*

2.2 Expander graphs

Throughout this paper, we assume that all graphs are undirected and simple, i.e., without multiple edges and loops. Let $G = (V, E)$ denote a graph G with vertex set V and edge set E. Let $G = (V_1, V_2, E)$ be a bipartite graph with a partition (V_1, V_2) of vertex set and edge set $E \subset \{\{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2\}$. For convenience, we identify $G = (V_1, V_2, E)$ with an orientation $\overrightarrow{G} = (V_1, V_2, \overrightarrow{E})$ where

$$\overrightarrow{E} = \{(v_1, v_2) : \{v_1, v_2\} \in E\} \subset V_1 \times V_2.$$

We call $\overrightarrow{G}$ the *associated orientation of* G.

A graph G is called a *d-regular graph* if every vertex of G connects exactly d edges. A bipartite graph $G = (V_1, V_2, E)$ is called an *(r, s)-biregular graph* if every vertex of V_1 and V_2 connects exactly r and s edges, respectively. Clearly, for an (r, s)-biregular graph $G = (V_1, V_2, E)$ and its associate orientation $\overrightarrow{G} = (V_1, V_2, \overrightarrow{E})$, the following equation holds.

$$|E| = |\overrightarrow{E}| = r|V_1| = s|V_2|. \tag{2.1}$$

Let $G = (V, E)$ be a graph with n vertices. Then the *adjacency matrix* of G, denoted by $A(G)$, is a $|V| \times |V|$ binary matrix such that the (u, w)-entry is 1 if and only if $\{u, w\} \in E$. Clearly, $A(G)$ is a symmetric matrix and thus has exactly n real eigenvalues with multiplicity, denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Lemma 4 (e.g. [9])**.** *Let G be a graph with n vertices.*

1. *If G is d-regular, then $\lambda_1 = d$ and $\lambda_n \geq -d$, where $\lambda_n = -d$ if and only if G is bipartite.*
2. *If G is (r, s)-biregular, then $\lambda_1 = \sqrt{rs}$ and $\lambda_n = -\sqrt{rs}$.*

By Lemma 4, the largest eigenvalue of a (bi-)regular graph is always determined. However, the second largest eigenvalue typically has rich properties. For a d-regular graph G, denote $\lambda(G) := \max\limits_{2 \leq i \leq n} |\lambda_i|$. For an (r, s)-biregular graph G, denote

$$\mu(G) := \max_{2 \leq i \leq n-1} |\lambda_i|.$$

An (r, s)-biregular graph G is a *μ-spectral expander* if $\mu(G) \leq \mu$. It has the following nice expansion property.

Proposition 5 ([40])**.** *Let $G = (V_1, V_2, E)$ be an (r, s)-biregular graph which is a μ-spectral expander. For a subset $S \subset V_1$, define the neighbour of S as*

$$N(S) := \{u \in V_2 : u \text{ is adjacent to some vertex in } S\},$$

and let $\rho(S) := \frac{|S|}{|V_1|}$. Then for every subset $S \subset V_1$,

$$\frac{|N(S)|}{|S|} \geq \frac{r^2}{\rho(S)(rs - \mu^2) + \mu^2}.$$

By Proposition 5, it is readily seen that if G is a μ-spectral expander with small μ, then G has a good expansion property and thus we are interested in how $\mu(G)$ can be small.

Lemma 6 ([27]). *Suppose that G is a sufficiently large graph. Then the followings hold.*

(1) *If G is d-regular, then $\lambda(G) = \Omega(\sqrt{d})$.*

(2) *If G is (r,s)-biregular, then $\mu(G) = \Omega(\sqrt{r+s})$.*

3 Codes from bipartite graphs

In this section we provide a bipartite graph based coding scheme and show that it produces non-malleable codes.

3.1 A coding scheme

First we propose a coding scheme based on bipartite graphs.

Definition 7. Let $G = (V_1, V_2, E)$ be a bipartite graph and $\overrightarrow{G} = (V_1, V_2, \overrightarrow{E})$ the associated orientation of G. Then the associated graph code $(\mathrm{enc}_G, \mathrm{dec}_G)$ consists of the functions

$$\mathrm{enc}_G : \{0,1\} \to V_1 \times V_2, \quad \mathrm{dec}_G : V_1 \times V_2 \to \{0,1\}$$

such that

$$\mathrm{enc}_G(b) := \begin{cases} (u,w) \leftarrow (V_1 \times V_2) \setminus \overrightarrow{E} & \text{if } b = 0; \\ (u,w) \leftarrow \overrightarrow{E} & \text{if } b = 1, \end{cases}$$

$$\mathrm{dec}_G(v_1, v_2) := \begin{cases} 0 & \text{if } (v_1, v_2) \notin \overrightarrow{E}; \\ 1 & \text{if } (v_1, v_2) \in \overrightarrow{E}. \end{cases}$$

Remark 8. Rasmussen and Sahai [36] designed a coding scheme based on a graph $G = (V, E)$ so that the space of codewords is $V \times V$, but it works only for equally-sized split-state model with $|\mathcal{L}| = |\mathcal{R}| = |V|$. On the other hand, our code can be applied to a more flexible split-state model, i.e. $|\mathcal{L}| = |V_1|$ may not be necessary equal to $|\mathcal{R}| = |V_2|$.

3.2 Non-malleability

The following theorem shows that the coding scheme in Definition 7 based on biregular spectral expanders can produce non-malleable codes for the split-state model.

Theorem 9. *Let $G = (V_1, V_2, E)$ be a sufficiently large (r,s)-biregular graph which is a μ-spectral expander. Suppose that $|E| = \Omega\Big(\frac{(rs)^2 \log(rs)}{\mu}\Big)$. Let $\mathcal{F}$ be the set of all functions $f = (g,h)$ with $g : V_1 \to V_1$ and $h : V_2 \to V_2$, where $f(v_1, v_2) := (g(v_1), h(v_2))$ for any $(v_1, v_2) \in V_1 \times V_2$. Then $(\mathrm{enc}_G, \mathrm{dec}_G)$ is an $O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big)$-non-malleable code with respect to $\mathcal{F}$.*

The proof is based on the discussion in [36] and the expander-mixing lemma for biregular graphs, please refer to Appendix for more details.

Remark 10. Suppose that G is an (r,s)-biregular graph with $s \geq r = \omega(\sqrt{s})$ and $\mu(G) = O(\sqrt{s})$. Then Theorem 9 guarantees that $(\mathrm{enc}_G, \mathrm{dec}_G)$ is an $O(s^{1/4}/r^{1/2})$-non-malleable code, where $s^{1/4}/r^{1/2} = o(1)$ by the assumption on r and s. On the other hand, according to Lemma 4, the quantity $O(s^{1/4}/r^{1/2})$ in Theorem 9 is best possible up to a constant.

The following corollary follows from Theorem 9 and (2.1).

Corollary 11. *Let $G = (V_1, V_2, E)$ be a bipartite d-regular graph with $|V_1| = |V_2| = n$ which is a μ-spectral expander. Suppose that $n = \Omega\Big(\frac{\log(d)\cdot d^3}{\mu}\Big)$ and $\mathcal{F}$ is as in Theorem 9. Then $(\mathrm{enc}_G, \mathrm{dec}_G)$ is an $O\Big(\frac{\mu^{3/2}}{d}\Big)$-non-malleable code with respect to $\mathcal{F}$.*

Remark 12. Corollary 11 actually includes the explicit construction of non-malleable codes by Rasmussen and Sahai [36, Section C]. In fact, for a finite abelian group X and a subset S of X, the *Cayley graph* $Cay(X,S)$ is an $|S|$-regular graph with vertex set X in which two vertices x and y are adjacent if and only if $xy^{-1} \in S$. Note that from $Cay(X,S)$, a bipartite $|S|$-regular graph can be easily obtained as follows. Take two disjoint copies X_1 and X_2 of X and construct bipartite graph so that $x_1 \in X_1$ and $x_2 \in X_2$ are adjacent if and only if $x_1 x_2^{-1} \in S$; such a bipartite regular graph is called a *bi-Cayley graph* [35]. For a prime p let $\mathbb{F}_p$ denote the p-element field and $q = p^2$. Rasmussen and Sahai [36] constructed $O(q^{-1/4})$-non-malleable codes from a non-bipartite graph $Cay(\mathbb{F}_p^6, S)$ with some $S \subset \mathbb{F}_p^6$ such that $|S| = q$. According to Corollary 11, the corresponding bi-Cayley graph provides the same non-malleable code as in [36, Section C].

4 Explicit constructions

In this section, we present explicit non-malleable codes based on specific biregular spectral expanders.

4.1 Via Ramanujan graphs

In this subsection, we construct non-malleable codes based on suitably chosen graphs from known families of Ramanujan graphs. The resulted codes can reduce the encoding space cost in comparison with the codes in [36] (see also Remark 12).

To show our construction based on Corollary 11, we need the following claim.

Claim 13. For a given large prime p, there exist explicit (bipartite) $(p+1)$-regular graphs G with $\Theta(p^{5/2}\log(p))$ vertices and $\mu(G) \leq 2\sqrt{p}$.

Our construction of graphs is based on the following *Ramanujan graphs* due to Lubotzky, Phillips and Sarnak [32], and Margulis [33] (see also [14, Theorem 4.2.2]). Let p, r be two distinct odd primes such that $r > 2\sqrt{p}$ and p is a *quadratic non-residue* modulo r. Then one can explicitly construct a bipartite $(p+1)$-regular graph $X^{p,r}$ with $r(r^2-1)$ vertices and $\mu(X^{p,r}) \leq 2\sqrt{p}$ for every $r > 2\sqrt{p}$. Indeed the graph $X^{p,r}$ is constructed as a Cayley graph $Cay(\mathrm{PGL}_2(\mathbb{F}_r), S)$ with some explicit generating set $S \subset \mathrm{PGL}_2(\mathbb{F}_r)$ of size $p+1$, where $\mathrm{PGL}_2(\mathbb{F}_r)$ denotes the projective general linear group of rank 2 over the r-element field $\mathbb{F}_r$. The details of the construction can be found in [14]. Note that (e.g. [34]) for each prime p, one can check whether two given vertices are adjacent in $X^{p,r}$ in $\mathrm{poly}(\log(r))$-time, and hence the graph can be constructed in $\mathrm{poly}(r)$-time.

Proof of Claim 13. To prove Claim 13, it suffices to take the graph $X^{p,r}$ with $r = \Theta(p^{5/6}\log^{1/3}(p))$. Indeed for each sufficiently large prime p, by Bertrand's postulate, there exists a prime $r = \Theta(p^{5/6}\log^{1/3}(p)) > 2\sqrt{p}$, which can be found in $\mathrm{poly}(p)$-time. If p is a quadratic non-residue modulo r, then $X^{p,r}$ is a bipartite $(p+1)$-regular graph with $\Theta(p^{5/2}\log(p))$ vertices and $\mu(X^{p,r}) \leq 2\sqrt{p}$. □

Thus we now obtain the following theorem.

Theorem 14. *For any sufficiently large prime p, suppose that $r = \Theta(p^{5/2}\log(p))$ is a prime such that p is a quadratic non-residue modulo r. Then $(\mathrm{enc}_G, \mathrm{dec}_G)$ with $G = X^{p,r}$ is an $O(p^{-1/4})$-non-malleable code for the split-state model with $|\mathcal{L}| = |\mathcal{R}| = \Theta(p^{5/2}\log(p))$.*

Note that Theorem 14 cannot deal with the case when $r = \Theta(p^{5/2}\log(p))$ is a prime such that p is a *quadratic residue* modulo r. However, in this case, one can instead explicitly construct a *non-bipartite* $(p+1)$-regular graph $Y^{p,r}$ with $r(r^2-1)/2$ vertices and $\lambda(Y^{p,r}) \leq 2\sqrt{p}$ (see [14, 32, 33]). By [36, Theorem 7], the graph $Y^{p,r}$ with $r = \Theta(p^{5/2}\log(p))$ provides an $O(p^{-1/4})$-non-malleable code for the split-state model with $|\mathcal{L}| = |\mathcal{R}| = \Theta(p^{5/2}\log(p))$ (see also Theorem 27 in Appendix). By this fact and Theorem 14, we immediately obtain the following corollary.

Corollary 15. *For any sufficiently large prime p, there exists an explicit $(p+1)$-regular graph G with $\Theta(p^{5/2}\log(p))$ vertices which provides an $O(p^{-1/4})$-non-malleable code for the split-state model with $|\mathcal{L}| = |\mathcal{R}| = \Theta(p^{5/2}\log(p))$. In particular, for every $0 < \varepsilon < 1$, there exists an explicit ε-non-malleable code with the encoding space cost $20\log(1/\varepsilon) + O(\log\log(1/\varepsilon))$.*

The last statement of Corollary 15 directly follows from the discussion in [36, Section 1.3]. Note that the explicit codes derived in [36] (see also Remark 12) need encoding space $24\log(1/\varepsilon) + O(1)$. In other words, the resulted codes here can reduce the encoding space cost in comparison with the codes in [36].

4.2 Via Wenger graphs

In this subsection, we construct non-malleable codes based on *Wenger graphs*. Wenger graphs play a significant role in extremal graph theory [26, 42] and also have been appeared as Tanner graphs of LDPC codes (e.g. [12, 38, 28]). As shown below, Wenger graphs provide ε-non-malleable codes with encoding space cost $24\log(1/\varepsilon) + O(1)$, which is the same as that of Theorem 14, however, Wenger graphs and the corresponding codes could be constructed more simply.

Let q be a prime power and $\mathbb{F}_q$ denote the q-element field. Take disjoint copies V_1 and V_2 of $\mathbb{F}_q^{m+1}$. Then the *Wenger graph* $W_m(q) = (V_1, V_2, E)$ is the bipartite graph such that an edge between $(p_1, p_2, \ldots, p_{m+1}) \in V_1$ and $(l_1, l_2, \ldots, l_{m+1}) \in V_2$ is in E if and only if the following m equations hold.

$$\begin{aligned} l_2 + p_2 &= l_1 p_1 \\ l_3 + p_3 &= l_2 p_1 \\ &\vdots \\ l_{m+1} + p_{m+1} &= l_m p_1 \end{aligned}$$

The graph $W_m(q)$ is a bipartite q-regular graph with $2q^{m+1}$ vertices.

Theorem 16. *For any prime power $q > 2$, $(\mathrm{enc}_G, \mathrm{dec}_G)$ with $G = W_2(q)$ is an $O(q^{-1/4})$-non-malleable code for the split-state model with $|\mathcal{L}| = |\mathcal{R}| = q^3$. In particular, for every $0 < \varepsilon < 1$, there exists an explicit ε-non-malleable code with the encoding space cost $24\log(1/\varepsilon) + O(1)$.*

Proof. All eigenvalues of the adjacency matrix $A(W_m(q))$ have been completely determined in [12] and consequently we have $\mu(W_m(q)) = \sqrt{mq}$ if $1 \le m \le q-1$. Now consider the case that $m = 2$ and $q \neq 2$. Then for the graph $W_2(q)$, we have $n = 2q^3 = \Theta(q^3)$ and $\frac{d^3 \log(d)}{\mu} = \Theta(q^{5/2} \log q)$. Combining with Corollary 11, the theorem follows. □

4.3 Via generalized quadrangles

In this subsection, we provide split-state non-malleable codes based on generalized quadrangles, which have also been used to derive LDPC codes [31] in terms of Tanner graphs.

A *generalized quadrangle* of order (α, β) is an $(\alpha+1, \beta+1)$-biregular graph $GQ(\alpha, \beta) = (V_1, V_2, E)$ such that

- for all $x, y \in V_1 \cup V_2$, there exists a path of length ≤ 4 connecting x and y;
- for all $x, y \in V_1 \cup V_2$, if the length of the shortest path connecting x and y is $h < 4$, then there exists only one path of length h connecting x and y;

- for every $x \in V_1 \cup V_2$, there exists $y \in V_1 \cup V_2$ such that there exists a path of length 4 connecting x and y.

More details of generalized quadrangles can be found in [37, 41]. Now based on generalized quadrangles, we can derive two families of non-malleable codes for the non-equally-sized scenario.

Theorem 17. *For any prime power q, $(\mathrm{enc}_G, \mathrm{dec}_G)$ with $G = GQ(q-1, q+1)$ is an $O(q^{-1/4})$-non-malleable code for the split state model with $|\mathcal{L}| = (q+2)q^2$ and $|\mathcal{R}| = q^3$. In particular, for every $0 < \varepsilon < 1$, there exists an explicit ε-non-malleable code with the encoding space cost $24\log(1/\varepsilon) + O(1)$.*

We need the following lemma to prove Theorem 17.

Lemma 18 ([37], [40], [41])**.** *For the graph $GQ(\alpha, \beta)$, we have*

- $|V_1| = (\alpha+1)(\alpha\beta+1)$,
- $|V_2| = (\beta+1)(\alpha\beta+1)$,
- $\mu(GQ(\alpha, \beta)) = \sqrt{\alpha+\beta}$.

Proof of Theorem 17. To obtain the theorem, we apply an explicit construction of $GQ(q-1, q+1)$ for every prime power q ([8, Sections 4 and 5]). According to (2.1) and Lemma 18, we have $|E| = r|V_1| = \Theta(q^4)$ and $\frac{(rs)^2\log(rs)}{\mu} = \Theta(q^{7/2}\log q)$. Thus by Theorem 9, $(\mathrm{enc}_G, \mathrm{dec}_G)$ with $G = GQ(q-1, q+1)$ gives the desired code. □

The following theorem can deal with more unbalanced non-equally-sized scenario at the expense of encoding space cost.

Theorem 19. *For any prime power q, $(\mathrm{enc}_G, \mathrm{dec}_G)$ with $G = GQ(q^2, q^3)$ is an $O(q^{-1/4})$-non-malleable code for the split state model with $|\mathcal{L}| = (q^2+1)(q^5+1)$ and $|\mathcal{R}| = (q^3+1)(q^5+1)$. In particular, for every $0 < \varepsilon < 1$, there exists an explicit ε-non-malleable code with the encoding space cost $60\log(1/\varepsilon) + O(1)$.*

Proof. For every prime power q, there is an explicit construction of $GQ(q^2, q^3)$ (e.g. [37, Chapter 3]). By (2.1) and Lemma 18, we have $|E| = r|V_1| = \Theta(q^{10})$ and $\frac{(rs)^2\log(rs)}{\mu} = \Theta(q^{17/2}\log q)$. Thus according to Theorem 9, $(\mathrm{enc}_G, \mathrm{dec}_G)$ with $G = GQ(q^2, q^3)$ gives the desired code. □

5 Conclusion and problem

In this paper, we proposed a coding scheme based on bipartite graphs and showed that the non-malleability can be satisfied if the underlying bipartite graph is a biregular μ-spectral

expander with sufficiently small μ. Based on this, we provided explicit and efficient non-malleable codes via several types of biregular spectral expanders such as Ramanujan graphs, Wenger graphs and generalized quadrangles. In addition, it is shown that LDPC codes with appropriate Tanner graphs could be applied to construct explicit non-malleable codes for the split-state model as well.

In terms of practical applications, it is desirable to construct split-state non-malleable codes for k-bit messages with $k \geq 1$. As far as we know, there is no known graph-theoretic constructions of split-state non-malleable codes for $k > 2$. It would be of interest to generalize the graph-based codes in this paper and [36] for k-bit messages in the split state model.

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A Proof of Theorem 9

In this section, we prove Theorem 9. Although the proof is based on the discussion in [36], we give a full proof for reader's convenience. In the proof we adopt the following notations. Let X, Y be two sets and $f : X \to Y$ be a function. For each $y \in Y$, denote $f^{-1}(y) := \{x \in X : f(x) = y\}$. For a subset $S \subset Y$, denote $f^{-1}(S) := \cup_{s\in S} f^{-1}(s)$. Also let $G = (V_1, V_2, E)$ be an (r, s)-biregular graph with n vertices, $\mu(G) = \mu$ and $\overrightarrow{G} = (V_1, V_2, \overrightarrow{E})$ be the associated orientation of G. Then for any pair of subsets $S \subset V_1$ and $T \subset V_2$, let

$$\begin{aligned} E(S,T) &:= \left|\{(s,t) \in \overrightarrow{E} : s \in S, t \in T\}\right|, \\ D(S,T) &:= \frac{\sqrt{rs}}{\sqrt{|V_1||V_2|}} \cdot |S||T| - E(S,T). \end{aligned} \tag{A.1}$$

Now recall the statement of Theorem 9.

(Theorem 9) *Let $G = (V_1, V_2, E)$ be a sufficiently large (r, s)-biregular graph which is a μ-spectral expander. Suppose that $|E| = \Omega\Big(\frac{(rs)^2 \log(rs)}{\mu}\Big)$. Let $\mathcal{F}$ be the set of all functions $f = (g, h)$ with $g : V_1 \to V_1$ and $h : V_2 \to V_2$, where $f(v_1, v_2) := (g(v_1), h(v_2))$ for any $(v_1, v_2) \in V_1 \times V_2$. Then $(\mathrm{enc}_G, \mathrm{dec}_G)$ is an $O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big)$-non-malleable code with respect to $\mathcal{F}$.*

To derive the non-malleability in Theorem 9, we shall employ Theorem 3 and the following lemmas.

Lemma 20. *Let $G = (V_1, V_2, E)$ be an (r, s)-biregular graph and $\overrightarrow{G} = (V_1, V_2, \overrightarrow{E})$ the associated orientation of G. For given functions $g : V_1 \to V_1$ and $h : V_2 \to V_2$, define $f : V_1 \times V_2 \to V_1 \times V_2$ such that $f(v_1, v_2) := (g(v_1), h(v_2))$ for any $(v_1, v_2) \in V_1 \times V_2$. Let*

$$T := \frac{1}{2} \sum_{b \in \{0,1\}} \Pr\Big[\mathrm{dec}\big(f(\mathrm{enc}(b))\big) = 1 - b\Big].$$

Then we have

$$T = \frac{1}{2} + \delta \cdot \sum_{(v,w) \in \overrightarrow{E}} D\big(g^{-1}(v), h^{-1}(w)\big)$$

where

$$\delta := \frac{|V_2|}{2r(|V_2| - r)|V_1|} = \frac{|V_1|}{2s(|V_1| - s)|V_2|}.$$

Proof of Lemma 20. The proof here is analogous to the proof of [36, Proposition 6]. For $b \in \{0, 1\}$, let

$$Q_b := \Pr\Big[\mathrm{dec}_G\big(f(\mathrm{enc}_G(b))\big) = 1 - b\Big].$$

Notice that

$$\begin{aligned} Q_0 &= \Pr_{(v,w) \leftarrow (V_1 \times V_2) \setminus \overrightarrow{E}}\Big[(g(v), h(w)) \in \overrightarrow{E}\Big], \\ Q_1 &= \Pr_{(v,w) \leftarrow \overrightarrow{E}}\Big[(g(v), h(w)) \notin \overrightarrow{E}\Big], \end{aligned}$$

and thus $T = (Q_0 + Q_1)/2$. Now we turn to compute Q_b.

First let $b = 0$. For any $e = (v, w) \in \overrightarrow{E}$, the total number of non-edges of G mapped by f to e is

$$\begin{aligned} &|\{(x, y) \in (V_1 \times V_2) \setminus \overrightarrow{E} : f(x, y) = (g(x), h(y)) = (v, w)\}| \\ &= |g^{-1}(v)||h^{-1}(w)| - E\big(g^{-1}(v), h^{-1}(w)\big). \end{aligned}$$

Since (2.1) implies $|(V_1 \times V_2) \setminus \overrightarrow{E}| = (|V_2| - r)|V_1| = (|V_1| - s)|V_2|$, we have

$$Q_0 = \frac{\sum_{(v,w) \in \overrightarrow{E}} \Big\{|g^{-1}(v)||h^{-1}(w)| - E\big(g^{-1}(v), h^{-1}(w)\big)\Big\}}{(|V_2| - r)|V_1|}. \tag{A.2}$$

Next suppose $b = 1$. For any $e = (v, w) \in (V_1 \times V_2) \setminus \overrightarrow{E}$, the total number of edges of G mapped by f to e is $E\big(g^{-1}(v), h^{-1}(w)\big)$. Thus we have

$$\begin{aligned} Q_1 &= \frac{\sum_{(v,w)\notin \overrightarrow{E}} E\big(g^{-1}(v), h^{-1}(w)\big)}{r|V_1|} \\ &= \frac{|\overrightarrow{E}| - \sum_{(v,w)\in \overrightarrow{E}} E\big(g^{-1}(v), h^{-1}(w)\big)}{r|V_1|} \\ &= 1 - \frac{\sum_{(v,w)\in \overrightarrow{E}} E\big(g^{-1}(v), h^{-1}(w)\big)}{r|V_1|}, \end{aligned} \tag{A.3}$$

where the last equality follows from (2.1).

Summing up (A.2) and (A.3) completes the proof. □

Let $f = (g, h) : V_1 \times V_2 \to V_1 \times V_2$ be a given tampering function from $\mathcal{F}$. Recall that for each pair of $1 \le i \ne j \le 2$ and each vertex $v \in V_i$, $N(v) = \{u \in V_j : u, v \text{ are adjacent in } G\}$. Define the following partitions of V_1 and V_2.

$$G^1 := \left\{v \in V_1 : |g^{-1}(v)| > \frac{|V_1|}{rs}\right\}, \quad G^2 := \left\{v \in V_1 : |g^{-1}(v)| \le \frac{|V_1|}{rs}\right\},$$
$$H^1 := \left\{w \in V_2 : |h^{-1}(w)| > \frac{|V_2|}{rs}\right\}, \quad H^2 := \left\{w \in V_2 : |h^{-1}(w)| \le \frac{|V_2|}{rs}\right\}.$$

For $1 \le i, j \le 2$, let

$$R_{i,j} := \delta \cdot \sum_{(v,w)\in \overrightarrow{E}\cap(G^i\times H^j)} D\Big(g^{-1}(v), h^{-1}(w)\Big).$$

It follows from Lemma 20 that

$$T = \frac{1}{2} + \sum_{1\le i,j\le 2} R_{i,j}. \tag{A.4}$$

By (A.4), the proof of Theorem 9 is completed from the following Lemmas 21, 22 and 23.

Lemma 21 ($i = 2$)**.**

$$R_{2,1} + R_{2,2} = O\Big(\frac{1}{r}\Big).$$

Proof. By the definition of $D(S, T)$ in (A.1),

$$\begin{aligned} R_{2,1} + R_{2,2} &\le \delta \cdot \sum_{(v,w)\in \overrightarrow{E}\cap(G^2\times V_2)} \frac{\sqrt{rs}}{\sqrt{|V_1||V_2|}} \cdot |g^{-1}(v)||h^{-1}(w)| \\ &\le \delta \cdot s \cdot \sum_{w\in V_2} \frac{\sqrt{rs}}{\sqrt{|V_1||V_2|}} \cdot \frac{|V_1|}{rs} \cdot |h^{-1}(w)| \end{aligned}$$

$$\leq \frac{|V_1|}{2s(|V_1|-s)|V_2|} \cdot s \cdot \frac{\sqrt{rs}}{\sqrt{|V_1||V_2|}} \cdot \frac{|V_1|}{rs} \cdot |V_2|$$
$$= O\left(\frac{1}{\sqrt{rs}} \cdot \sqrt{\frac{|V_1|}{|V_2|}}\right) = O\Big(\frac{1}{r}\Big)$$

where the second inequality follows from the definition of G^2. □

Lemma 22 ($i = 1$, $j = 2$)**.**

$$R_{1,2} = O\Big(\frac{1}{s}\Big).$$

Proof. Similar to Case 1, we have

$$\begin{aligned} R_{1,2} &\leq \delta \cdot \sum_{(v,w)\in\overrightarrow{E}\cap(G^1\times H^2)} \frac{\sqrt{rs}}{\sqrt{|V_1||V_2|}} \cdot |g^{-1}(v)||h^{-1}(w)| \\ &\leq \delta \cdot \sum_{(v,w)\in\overrightarrow{E}\cap(V_1\times H^2)} \frac{\sqrt{rs}}{\sqrt{|V_1||V_2|}} \cdot |g^{-1}(v)||h^{-1}(w)| \\ &\leq \delta \cdot r \cdot \sum_{v\in V_1} \frac{\sqrt{rs}}{\sqrt{|V_1||V_2|}} \cdot |g^{-1}(v)| \cdot \frac{|V_2|}{rs} \\ &\leq \frac{|V_2|}{2r(|V_2|-r)|V_1|} \cdot r \cdot \frac{\sqrt{rs}}{\sqrt{|V_1||V_2|}} \cdot |V_1| \cdot \frac{|V_2|}{rs} \\ &= O\left(\frac{1}{\sqrt{rs}} \cdot \sqrt{\frac{|V_2|}{|V_1|}}\right) = O\Big(\frac{1}{s}\Big). \end{aligned}$$

□

Lemma 23 ($i = j = 2$)**.**

$$R_{2,2} = O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big).$$

Since the proof of Lemma 23 is complicated, we first prove Theorem 9 assuming that Lemma 23 holds.

Proof of Theorem 9. By Theorem 3 and (A.4), Theorem 9 immediately follows from Lemmas 21, 22 and 23. □

Now we are going to prove Lemma 23. We will make use of the following lemma which plays a key role to prove the technical Lemma 23.

Lemma 24 (Expander mixing lemma, [15], [23], [24])**.** *Let $G = (V_1, V_2, E)$ be an (r, s)-biregular graph with n vertices, $\mu(G) = \mu$. Then for any pair of subsets $S \subset V_1$ and $T \subset V_2$, we have*

$$|D(S,T)| \leq \mu\sqrt{|S||T|}. \tag{A.5}$$

Remark 25. The non-malleable codes from [36] used the following fact. Let $G = (V, E)$ be a d-regular (possibly non-bipartite) graph with $\lambda(G) = \lambda$. Then for any pair of subsets $S, T \subset V$,

$$\left|\frac{d}{n}|S||T| - e(S,T)\right| \leq \lambda\sqrt{|S||T|}. \tag{A.6}$$

Here $e(S,T)$ denotes the number of edges between S and T. However, if G is a bipartite graph, the estimation (A.6) cannot be used to prove the non-malleability for the coding schemes in [36] (see Appendix) and the coding scheme in this paper (see Definition 7), since in this case $\lambda(G) = d$ (see Lemma 4), which only implies $O(\sqrt{d})$-non-malleable codes. However we could see from Theorem 9 that using Lemma 24 can produce $o(1)$-non-malleable codes.

Proof of Lemma 23. Take partitions of G^1 and H^1 so that for each pair of $1 \leq k, l \leq \lceil \log_2(rs) \rceil$,

$$G^1(k) := \Big\{v \in G_1 : \frac{|V_1|}{2^{k-1}} \geq |g^{-1}(v)| \geq \frac{|V_1|}{2^k}\Big\},$$
$$H^1(l) := \Big\{w \in H_1 : \frac{|V_2|}{2^{l-1}} \geq |h^{-1}(w)| \geq \frac{|V_2|}{2^l}\Big\}.$$

For each pair of $1 \leq k, l \leq \lceil \log_2(rs) \rceil$, let

$$S_{k,l} := \delta \cdot \sum_{(v,w) \in \overrightarrow{E} \cap \left(G^1(k) \times H^1(l)\right)} D\Big(g^{-1}(v), h^{-1}(w)\Big).$$

Since $R_{1,1} = \sum_{1 \leq k,l \leq \lceil \log_2(rs) \rceil} S_{k,l}$, Lemma 23 follows from the following.

$$\sum_{1 \leq k,l \leq \lceil \log_2(rs) \rceil} S_{k,l} = O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big). \tag{A.7}$$

To that end, we divide the sum in (A.7) into two parts, namely,

$$\sum_{1 \leq k \leq l \leq \lceil \log_2(rs) \rceil} S_{k,l} \quad \text{and} \quad \sum_{1 \leq l < k \leq \lceil \log_2(rs) \rceil} S_{k,l}.$$

Case 1. This case is to prove the following estimation.

$$\sum_{1 \leq k \leq l \leq \lceil \log_2(rs) \rceil} S_{k,l} = O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big). \tag{A.8}$$

First we have

$$\begin{aligned}\delta^{-1} S_{k,l} &= \sum_{v \in G^1(k)} D\left(g^{-1}(v), \bigcup_{w \in N(v) \cap H^1(l)} h^{-1}(w)\right) \\ &\leq \sum_{v \in G^1(k)} \mu \sqrt{|g^{-1}(v)| \cdot \sum_{w \in N(v) \cap H^1(l)} |h^{-1}(w)|}\end{aligned}$$

$$
\begin{aligned}
&\le \mu\sqrt{\frac{|V_1|}{2^{k-1}}\cdot\frac{|V_2|}{2^{l-1}}}\sum_{v\in G^1(k)}\sqrt{|N(v)\cap H^1(l)|}\\
&\le 2\mu\cdot 2^{-\frac{l+k}{2}}\cdot\sqrt{|V_1||V_2|}\cdot\sqrt{|G^1(k)|}\cdot\sqrt{E\big(G^1(k),H^1(l)\big)}\\
&\le 2\mu\cdot 2^{-\frac{l+k}{2}}\cdot\sqrt{|V_1||V_2|}\cdot\sqrt{|G^1(k)|}\cdot\sqrt{\frac{\sqrt{rs}}{\sqrt{|V_1||V_2|}}\cdot|G^1(k)||H^1(l)|+\mu\sqrt{|G^1(k)||H^1(l)|}},
\end{aligned}
$$

where the second and last inequalities follow from Lemma 24.

By Jensen's inequality and (2.1), we obtain

$$
S_{k,l}\le O\Big(\frac{\mu}{\sqrt{|E|}}\Big)\cdot 2^{-\frac{l+k}{2}}\cdot|G^1(k)|\cdot\sqrt{|H^1(l)|}+O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big)\cdot 2^{-\frac{l+k}{2}}\cdot\Big(|G^1(k)|^3|H^1(l)|\Big)^{\frac{1}{4}}.
$$

To obtain (A.8), let

$$
\begin{aligned}
L&:=\sum_{1\le k\le l\le\lceil\log_2(rs)\rceil}2^{-\frac{l+k}{2}}\cdot|G^1(k)|\cdot\sqrt{|H^1(l)|},\\
K&:=\sum_{1\le k\le l\le\lceil\log_2(rs)\rceil}2^{-\frac{l+k}{2}}\cdot\Big(|G^1(k)|^3|H^1(l)|\Big)^{\frac{1}{4}}.
\end{aligned}
$$

By the definitions of L and K,

$$
\sum_{1\le k\le l\le\lceil\log_2(rs)\rceil}S_{k,l}=O\Big(\frac{\mu}{\sqrt{|E|}}\Big)\cdot L+O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big)\cdot K. \tag{A.9}
$$

First we estimate L. Notice that for each $k\le\lceil\log_2(rs)\rceil$,

$$
|G^1(k)|\cdot 2^{-\frac{k}{2}}\le 2^{\frac{k}{2}}\le 2\sqrt{rs}. \tag{A.10}
$$

Then by the Cauchy-Schwartz inequality,

$$
\begin{aligned}
L&\le\sum_{1\le k,l\le\lceil\log_2(rs)\rceil}2^{-\frac{l+k}{2}}\cdot|G^1(k)|\cdot\sqrt{|H^1(l)|},\\
&\le 2\sqrt{rs}\cdot\sum_{1\le l\le\lceil\log_2(rs)\rceil}\sqrt{2^{-l}|H^1(l)|}\\
&\le O\Big(\sqrt{rs\log(rs)}\Big)\cdot\sqrt{\sum_{1\le l\le\lceil\log_2(rs)\rceil}2^{-l}|H^1(l)|},
\end{aligned}
$$

where the second inequality follows from (A.10). On the other hand, the definition of $H^1(l)$ implies that

$$
|h^{-1}(H^1(l))|\ge\frac{|V_2||H^1(l)|}{2^l}. \tag{A.11}
$$

Since $H^1(1), \dots, H^1(\lceil \log_2(rs) \rceil)$ are disjoint subsets of V_2, we have

$$L = O\Big(\sqrt{rs\log(rs)}\Big) \cdot \sqrt{\sum_{1 \le l \le \lceil \log_2(rs) \rceil} \frac{|h^{-1}(H^1(l))|}{|V_2|}} = O\Big(\sqrt{rs\log(rs)}\Big), \tag{A.12}$$

where the last equation follows from (A.11).

Next we aim to bound K. Since we are assuming that $k \le l$, setting $t = l - k$, we obtain

$$\begin{aligned} K &\le \sum_{1 \le k \le l \le \lceil \log_2(rs) \rceil} \frac{2^{\frac{k-l}{4}}}{(|V_1|^3|V_2|)^{\frac{1}{4}}} \Big(|g^{-1}(G^1(k))|^3 \cdot |h^{-1}(H^1(l))|\Big)^{\frac{1}{4}} \\ &\le \sum_{t=0}^{\lceil \log_2(rs) \rceil} \frac{2^{-\frac{t}{4}}}{(|V_1|^3|V_2|)^{\frac{1}{4}}} \sum_{l=t}^{\lceil \log_2(rs) \rceil} \Big(|g^{-1}(G^1(l-t))|^3 \cdot |h^{-1}(H^1(l))|\Big)^{\frac{1}{4}}, \end{aligned}$$

where the first inequality follows from (A.11) and the following inequality.

$$|g^{-1}(G^1(k))| \ge \frac{|V_1||G^1(k)|}{2^k}. \tag{A.13}$$

By the definitions of $G^1(k)$ and $H^1(l)$, for each $0 \le t \le \lceil \log_2(rs) \rceil$, the sets $g^{-1}(G^1(l-t))$ and $h^{-1}(G^1(l))$, $t \le l \le \lceil \log_2(rs) \rceil$, are disjoint subsets of V_1 and V_2, respectively. Then it follows from Hörder's inequality that

$$\begin{aligned} K &\le \sum_{t=0}^{\lceil \log_2(rs) \rceil} \frac{2^{-\frac{t}{4}}}{(|V_1|^3|V_2|)^{\frac{1}{4}}} \left(\sum_{l=t}^{\lceil \log_2(rs) \rceil} \Big(|g^{-1}(G^1(l-t))|\Big)^{\frac{3}{4}} \cdot \left(\sum_{l=t}^{\lceil \log_2(rs) \rceil} |h^{-1}(H^1(l))| \right)^{\frac{1}{4}} \right. \\ &\le \sum_{t=0}^{\lceil \log_2(rs) \rceil} 2^{-\frac{t}{4}} = O(1). \end{aligned} \tag{A.14}$$

By (A.9), (A.12) and (A.14), we get

$$\begin{aligned} \sum_{1 \le k \le l \le \lceil \log_2(rs) \rceil} S_{k,l} &= O\Big(\frac{\mu}{\sqrt{|E|}} \cdot \sqrt{rs\log(rs)}\Big) + O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big) \\ &= O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big), \end{aligned}$$

where the last equality follows from the condition in Theorem 9 that $|E| = \Omega\Big(\frac{(rs)^2\log(rs)}{\mu}\Big)$. This concludes the discussion for Case 1.

Case 2. In this case, we aim to show the following.

$$\sum_{1 \le l < k \le \lceil \log_2(rs) \rceil} S_{k,l} = O\Big(\frac{\mu^{\frac{3}{2}}}{\sqrt{rs}}\Big). \tag{A.15}$$

To prove (A.15), we deal with the following equation.

$$\delta^{-1}S_{k,l} = \sum_{w\in H^1(l)} D\Bigg(\bigcup_{v\in N(w)\cap G^1(k)} g^{-1}(v),\ h^{-1}(w)\Bigg).$$

This follows from an analogous calculation as in Case 1.

Combining Cases 1 and 2 yields (A.7). This completes the proof of Lemma 23. □

B The graph code by Rasmussen and Sahai

In this section, we briefly review the graph code proposed in [36]. To define it more rigorously, we associate the underlying graph with a *digraph* as described below.

Definition 26 ([36])**.** Let $G = (V, E)$ be an undirected graph with no multiple edges (but each vertex may have at most one loop). Let $D_G = (V, E')$ be the associated symmetric digraph with vertex set V and edge set $E' \subset V \times V$ such that

$$E' = \{(u, w), (w, u) \in V \times V : \{u, w\} \in E\}.$$

Then the associated graph code $(\mathrm{enc}'_G, \mathrm{dec}'_G)$ consists of the functions

$$\mathrm{enc}'_G : \{0, 1\} \to V \times V, \qquad \mathrm{dec}'_G : V \times V \to \{0, 1\}$$

such that

$$\mathrm{enc}'_G(b) := \begin{cases} (u, w) \leftarrow (V \times V) \setminus E' & \text{if } b = 0; \\ (u, w) \leftarrow E' & \text{if } b = 1, \end{cases}$$

$$\mathrm{dec}'_G(v_1, v_2) := \begin{cases} 0 & \text{if } (v_1, v_2) \notin E'; \\ 1 & \text{if } (v_1, v_2) \in E'. \end{cases}$$

Theorem 27 ([36])**.** *Let $G = (V, E)$ be a d-regular graph with n vertices and $\lambda(G) = \lambda$. Assume that $n = \Omega(\frac{d^3 \log(d)}{\lambda})$. Let $\mathcal{F}$ be the set of all functions $f = (g, h)$ with $g : V \to V$ and $h : V \to V$. Then $(\mathrm{enc}'_G, \mathrm{dec}'_G)$ is an $O\Big(\frac{\lambda^{\frac{3}{2}}}{d}\Big)$-non-malleable code with respect to $\mathcal{F}$.*