Indifferentiable hashing to ordinary elliptic $\mathbb{F}_q$-curves of $j = 0$ with the cost of one exponentiation in $\mathbb{F}_q$

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Abstract. Let $\mathbb{F}_q$ be a finite field and $E_b\colon y^2 = x^3 + b$ be an ordinary (i.e., non-supersingular) elliptic curve (of j-invariant 0) such that $\sqrt{b} \in \mathbb{F}_q$ and $q \not\equiv 1 \pmod{27}$. For example, these conditions are fulfilled for the group $\mathbb{G}_1$ of the curve BLS12-381 ($b = 4$). It is a de facto standard in the real world pairing-based cryptography at the moment. This article provides a new constant-time hash function $H\colon \{0,1\}^* \to E_b(\mathbb{F}_q)$ indifferentiable from a random oracle. Its main advantage is the fact that H computes only one exponentiation in $\mathbb{F}_q$. In comparison, the previous fastest constant-time indifferentiable hash functions to $E_b(\mathbb{F}_q)$ compute two exponentiations in $\mathbb{F}_q$. In particular, applying H to the widely used BLS multi-signature with m different messages, the verifier should perform only m exponentiations rather than $2m$ ones during the hashing phase.

Key words: cubic residue symbol and cubic roots, hashing to ordinary elliptic curves of j-invariant 0, indifferentiability from a random oracle, pairing-based cryptography.

Introduction

Since its invention in the early 2000s, *pairing-based cryptography* [1] has become more and more popular every year, for example in secure multi-party computations. One of the latest reviews of standards, commercial products and libraries for this type of cryptography is given in [2, §4.1].

Let $\mathbb{F}_q$ be a finite field of $\mathrm{char}(\mathbb{F}_q) > 3$ and $E_b\colon y^2 = x^3 + b$ be an elliptic $\mathbb{F}_q$-curve whose the j-invariant is 0. The priority is given to the curves E_b, because the pairing computation on them is the most efficient (see [1, §4]). As is well known [1, Remark 2.22], only ordinary curves are safe to deal with the discrete logarithm problem. And according to [3, Example V.4.4] the ordinariness of E_b results in the restriction $q \equiv 1 \pmod 3$, i.e., $\omega := \sqrt[3]{1} \in \mathbb{F}_q$, where $\omega \neq 1$. Today, the most popular *pairing-friendly curve* in the industry is the Barreto–Lynn–Scott curve BLS12-381 [4, §2.1] for which $\lceil \log_2(q) \rceil = 381$.

Many pairing-based protocols (for example, the BLS multi-signature [5, §3], [6]) use a hash function of the form $H\colon \{0,1\}^* \to E_b(\mathbb{F}_q)$. There is the regularly updated draft [7] (see also [1, §8]) on the topic of hashing to elliptic curves. In order to be used in practice H must be *indifferentiable from a random oracle* [8, Definition 2] and *constant-time*, that is the computation time of its value is independent of an input argument.

Almost all such previously proposed hash functions are obtained as the composition $H :=$

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$e^{\otimes 2} \circ \mathfrak{h}$ of a hash function $\mathfrak{h}\colon \{0,1\}^* \to \mathbb{F}_q^2$ and the tensor square

$$e^{\otimes 2}\colon \mathbb{F}_q^2 \to E_b(\mathbb{F}_q) \qquad e^{\otimes 2}(t_1, t_2) := e(t_1) + e(t_2)$$

of some map $e\colon \mathbb{F}_q \to E_b(\mathbb{F}_q)$. Such a map is often called *encoding*. In this case the indifferentiability of H follows from [8, Theorem 1] if $\mathfrak{h}$ is indifferentiable and $e^{\otimes 2}$ is *admissible* in the sense of [8, Definition 4]. The fastest known encodings are Elligator 2 [9, §5] and the Wahby–Boneh "indirect" map [4]. Both (resp. H) can be implemented with the cost of one (resp. two) exponentiation(s) in $\mathbb{F}_q$.

This article essentially improves our ideas from [10]. More precisely, there provided that $\sqrt{b} \in \mathbb{F}_q$ we construct one more encoding e whose the tensor square $e^{\otimes 2}$ is admissible. Moreover, e equally requires only one exponentiation in $\mathbb{F}_q$. However in this work (also for $\sqrt{b} \in \mathbb{F}_q$) we directly provide an admissible map $h\colon \mathbb{F}_q^2 \to E_b(\mathbb{F}_q)$ approximately with the same cost as e and such that $h(t,t) = \pm e(t)$. In other words, the tensor square is superfluous in the given situation and hence we get rid of one exponentiation in $\mathbb{F}_q$. Let us also remark that h is given by quite simple formulas with small coefficients unlike the Wahby–Boneh encoding.

1 Geometric results

As mentioned above, we are only interested in $q \equiv 1 \pmod 3$, i.e., $\omega := \sqrt[3]{1} \in \mathbb{F}_q^*$, where $\omega \neq 1$. Further, for the sake of being definite, suppose that $\sqrt[3]{b} \notin \mathbb{F}_q$. The opposite case is much simpler, hence results of the article can be extended to it without problems. For $i \in \{0,1,2\}$ consider the elliptic curves $E_b^{(i)}\colon y_i^2 = b^i x_i^3 + b \simeq_{\mathbb{F}_q} E_{b^{2i+1}}$. Note that $E_b^{(1)}, E_b^{(2)}$ are two different cubic $\mathbb{F}_q$-twists of $E_b = E_b^{(0)}$.

There is on $E_b^{(i)}$ the $\mathbb{F}_q$-automorphism $[\omega](x_i, y_i) := (\omega x_i, y_i)$ of order 3. Take the quotient $T := (E_b \times E_b^{(1)} \times E_b^{(2)})/[\omega]^{\times 3}$ with respect to the diagonal action of $[\omega]$. This is a *Calabi–Yau threefold* according to [11, §1.3]. It is readily seen that it has the affine $\mathbb{F}_q$-model

$$T\colon \begin{cases} y_1^2 - b = b(y_0^2 - b)t_1^3, \\ y_2^2 - b = b^2(y_0^2 - b)t_2^3 \end{cases} \qquad \subset \qquad \mathbb{A}^5_{(y_0, y_1, y_2, t_1, t_2)},$$

where $t_j := x_j/x_0$. By the way, the famous SWU (Shallue–van de Woestijne–Ulas) encoding [1, §8.3.4] deals with another Calabi–Yau $\mathbb{F}_q$-threefold.

We can look at T as a curve in $\mathbb{A}^3_{(y_0,y_1,y_2)}$ given as the intersection of two quadratic surfaces over $\mathbb{F}_q(t_1, t_2)$, where the latter denotes the rational function field in two variables t_1, t_2 over the constant field $\mathbb{F}_q$. Nevertheless, below it will be more convenient to work over the subfield $F := \mathbb{F}_q(s_1, s_2)$, where $s_j := t_j^3$.

Lemma 1 ([12]). *T/F is an elliptic curve having a short Weierstrass form $W\colon y^2 = x^3 + a_4 x + a_6$ with the coefficients*

$$a_4 := -3(b^2 s_1 s_2 + \omega^2 s_1 + \omega b s_2)(b^2 s_1 s_2 + \omega s_1 + \omega^2 b s_2),$$
$$a_6 := -(b^2 s_1 s_2 - 2s_1 + b s_2)(2b^2 s_1 s_2 - s_1 - b s_2)(b^2 s_1 s_2 + s_1 - 2b s_2).$$

In particular, the discriminant and j-invariant of W equal

$$\Delta = \bigl(2^2 3^3 b s_1 s_2 (bs_1 - 1)(b^2 s_2 - 1)(s_1 - bs_2)\bigr)^2,$$
$$j = \bigl(2^4 3^2 (b^2 s_1 s_2 + \omega s_1 + \omega^2 b s_2)(b^2 s_1 s_2 + \omega^2 s_1 + \omega b s_2)\bigr)^3/\Delta.$$

Theorem 1 ([12])**.** *There is a point $\psi \in W(F)$ with the coordinates*

$$x = b(2bs_1 - 1)s_2 - (3bs_1 - 2)s_1, \qquad y = 3\sqrt{b}(2\omega + 1)s_1(bs_1 - 1)(bs_2 - s_1).$$

It corresponds to a point $\varphi \in T(F)$ whose the coordinates are the irreducible fractions $y_i(t_1, t_2) := num_i/den$, where

$$num_0 := \sqrt{b}\cdot\bigl(b^2 s_1^2 - 2b^3 s_1 s_2 + 2bs_1 + b^4 s_2^2 + 2b^2 s_2 - 3\bigr),$$
$$num_1 := \sqrt{b}\cdot\bigl(-3b^2 s_1^2 + 2b^3 s_1 s_2 + 2bs_1 + b^4 s_2^2 - 2b^2 s_2 + 1\bigr),$$
$$num_2 := \sqrt{b}\cdot\bigl(b^2 s_1^2 + 2b^3 s_1 s_2 - 2bs_1 - 3b^4 s_2^2 + 2b^2 s_2 + 1\bigr),$$
$$den := b^2 s_1^2 - 2b^3 s_1 s_2 - 2bs_1 + b^4 s_2^2 - 2b^2 s_2 + 1.$$

Moreover, $\sum_{i=0}^2 y_i(t_1, t_2) + \sqrt{b} = 0$.

It is remarkable that the functions $y_i(t, t)$ are nothing but (up to the minus sign) those from [10, Theorem 1]. Besides, the important case $b = 4$ gives

$$num_0 = 2\cdot\bigl(2^4 s_1^2 - 2^7 s_1 s_2 + 2^3 s_1 + 2^8 s_2^2 + 2^5 s_2 - 3\bigr),$$
$$num_1 = 2\cdot\bigl(-2^4 3 s_1^2 + 2^7 s_1 s_2 + 2^3 s_1 + 2^8 s_2^2 - 2^5 s_2 + 1\bigr),$$
$$num_2 = 2\cdot\bigl(2^4 s_1^2 + 2^7 s_1 s_2 - 2^3 s_1 - 2^8 3 s_2^2 + 2^5 s_2 + 1\bigr),$$
$$den = 2^4 s_1^2 - 2^7 s_1 s_2 - 2^3 s_1 + 2^8 s_2^2 - 2^5 s_2 + 1.$$

In other words, $T/\mathbb{F}_q$ is an *elliptic threefold* whose the *elliptic fibration* is the projection to t_1, t_2. In these terms, $\varphi\colon \mathbb{A}^2_{(t_1,t_2)} \dashrightarrow T$ is an $\mathbb{F}_q$-section of the given fibration. In particular, $\mathrm{Im}(\varphi)$ is a rational $\mathbb{F}_q$-surface. In turn, W is a *global minimal* Weierstrass form for T. These and other notions of the theory of elliptic threefolds see, e.g., in [13]. For completeness, the much simpler theory of elliptic surfaces is well represented in [14].

If the point $\phi_0 := \bigl(\sqrt{b}, \sqrt{b}, \sqrt{b}\,\bigr)$ is chosen as the neutral element of the *Mordell–Weil group* $T(F)$, then as shown in [12] its 2-torsion subgroup $T(F)[2] = \{\phi_i\}_{i=0}^3$, where

$$\phi_1 := \bigl(\sqrt{b}, -\sqrt{b}, -\sqrt{b}\,\bigr), \qquad \phi_2 := \bigl(-\sqrt{b}, \sqrt{b}, -\sqrt{b}\,\bigr), \qquad \phi_3 := \bigl(-\sqrt{b}, -\sqrt{b}, \sqrt{b}\,\bigr).$$

The next theorem clarifies why ψ has the simplest coordinates among infinite order points from $W(F)$.

Theorem 2. *Consider F as the rational function field $k_1(s_2)$ (resp. $k_2(s_1)$) over the constant field $k_1 := \mathbb{F}_q(s_1)$ (resp. $k_2 := \mathbb{F}_q(s_2)$). Then, taking into account the lattice structure with respect to the height pairing,*

$$T(F) \simeq W(F) \simeq \mathrm{A}_1^* \oplus (\mathbb{Z}/2)^2, \qquad \textit{moreover}, \qquad W(F)/W(F)_{\mathrm{tor}} = \langle\psi\rangle.$$

Proof. Since T/k_j is obviously a rational surface, W/k_j is also so. With the help of [12] we get that the singular fibers of the Kodaira–Néron model of W/k_j have the types $\mathrm{I}_2, \mathrm{I}_2, \mathrm{I}_2, \mathrm{I}_0^*$ in Kodaira's notation. Consequently $W\big(\overline{k_1}(s_2)\big) \simeq W\big(\overline{k_2}(s_1)\big) \simeq \mathrm{A}_1^* \oplus (\mathbb{Z}/2)^2$ according to [14, Table 8.2]. Further, [12] allows to compute the canonical height of ψ, which turns out to equal $1/2$. This is also the minimal norm of the lattice A_1^*. Thus the theorem is proved. $\square$

We do not claim that $T(F)/T(F)_{\mathrm{tor}} = \langle \varphi \rangle$ with respect to ϕ_0 as the neutral element of $T(F)$, because this point does not correspond to that at infinity on W/F. We chose ϕ_0 just to describe $T(F)[2]$ in a more canonical way.

For the sake of compactness we put

$$\beta := -3\sqrt{b}, \qquad \infty := (1:0) \in \mathbb{P}^1, \qquad P_0 := (0, \sqrt{b}) \in E_b, \qquad \mathcal{O} := (0:1:0) \in E_b.$$

Denote by Num_i (resp. Den) the homogenization of num_i (resp. den) with respect to a new variable t_0. For $y \in \mathbb{F}_q$ consider on $\mathbb{P}^2_{(t_0:t_1:t_2)}$ the pencil of the $\mathbb{F}_q$-sextics

$$C_{i,y}\colon Num_i = Den\cdot y, \qquad C_{i,\infty} = C_\infty\colon Den = 0$$

and the $\mathbb{F}_q$-conics $D_{i,y} := \pi(C_{i,y})$, where

$$\pi\colon \mathbb{P}^2 \to \mathbb{P}^2 \qquad \pi(t_0 : t_1 : t_2) := (t_0^3 : t_1^3 : t_2^3).$$

Also, let $L_i\colon t_i = 0$,

$$R_0 := (1:0:0), \qquad R_1 := (0:1:0), \qquad R_2 := (0:0:1)$$

and $\mathbf{Q}_k := \pi^{-1}(Q_k)$, where

$$Q_0 := (0:b:1), \qquad Q_1 := (b^2:0:1), \qquad Q_2 := (b:1:0).$$

Below we formulate a few simple lemmas, which are readily checked. By the way, the indices $i \pm 1$ will always mean the operations $\pm$ modulo 3.

Lemma 2. *The order* 3 *projective* $\mathbb{F}_q$*-transformations*

$$\tau\colon \mathbb{P}^2 \xrightarrow{\sim} \mathbb{P}^2 \qquad \tau(t_0 : t_1 : t_2) := (bt_2 : t_0 : t_1) \qquad \textit{and} \qquad \tau' := \pi \circ \tau \circ \pi^{-1}\colon \mathbb{P}^2 \xrightarrow{\sim} \mathbb{P}^2$$

give the isomorphisms

$$\tau\colon C_{i,y} \xrightarrow{\sim} C_{i+1,y}, \qquad \tau'\colon D_{i,y} \xrightarrow{\sim} D_{i+1,y}, \qquad \tau, \tau'\colon L_i \xrightarrow{\sim} L_{i+1}$$

as well as

$$\tau(R_i) = \tau'(R_i) = R_{i+1}, \qquad \tau'(Q_i) = Q_{i+1}.$$

It is worth noting that the curves $D_{i,\pm\sqrt{b}}$ (and hence $C_{i,\pm\sqrt{b}}$) are reducible over $\mathbb{F}_q$. Indeed,

$$D_{0,\sqrt{b}}\colon t_0(t_0 - bt_1 - b^2t_2) = 0, \qquad D_{0,-\sqrt{b}}\colon (t_0 - bt_1 + b^2t_2)(t_0 + bt_1 - b^2t_2) = 0. \qquad (1)$$

Lemma 3. *There are the following equalities. First,*

$$D_{i,y} \cap D_\infty = D_{i,0} \cap D_\infty = \{Q_k\}_{k=0}^2.$$

Second,

$$D_{0,y} \cap D_{1,y} = \{Q_k\}_{k=0}^2 \cup \big\{\big(b^2(y-\sqrt{b}) : b(y-\sqrt{b}) : 4y\big)\big\}$$

for $y \neq \pm\sqrt{b}$. Third,

$$\begin{aligned} D_{i,y} \cap L_i &= \{Q_i\}, & D_{0,y} \cap L_1 &= \big\{Q_1, \big(b^2(y-\sqrt{b}) : 0 : y-\beta\big)\big\}, \\ D_\infty \cap L_k &= \{Q_k\}, & D_{0,y} \cap L_2 &= \big\{Q_2, \big(b(y-\sqrt{b}) : y-\beta : 0\big)\big\} \end{aligned}$$

also for $y \neq \pm\sqrt{b}$.

Lemma 4. *The set of singular points*

$$\operatorname{Sing}(C_{i,y}) = \begin{cases} \mathbf{Q}_i & \text{if} \quad y \notin \{\pm\sqrt{b}, \beta, \infty\}, \\ \mathbf{Q}_i \cup \{R_i\} & \text{if} \quad y = \beta, \\ \cup_{k=0}^2 \mathbf{Q}_k & \text{if} \quad y = \infty. \end{cases}$$

Moreover, $R_i \in C_{i,\beta}$ is an ordinary point of multiplicity 3 and all other singularities are cusps regardless of y.

Theorem 3. *For $y \neq \pm\sqrt{b}$ the curves $C_{i,y}$ are absolutely irreducible.*

Proof. The cases $y \in \{\beta, \infty\}$ are immediately processed by Magma [12]. In compliance with Lemma 4 for another y the curve $C_{i,y}$ has only 3 cusps, hence it has no more than 3 different absolutely irreducible components F_0, F_1, F_2. Consider the transformations

$$\chi_k \colon C_{i,y} \xrightarrow{\sim} C_{i,y} \qquad \chi_0 := (\omega t_0 : t_1 : t_2), \qquad \chi_1 := (t_0 : \omega t_1 : t_2), \qquad \chi_2 := (t_0 : t_1 : \omega t_2).$$

Since they are of order 3, for any $k, \ell, m \in \{0,1,2\}$, $\ell \neq m$ the case $\chi_k \colon F_\ell \xrightarrow{\sim} F_m$, $F_m \xrightarrow{\sim} F_\ell$ is not possible, otherwise $F_\ell = F_m$. Also, given ℓ note that $\chi_k \colon F_\ell \xrightarrow{\sim} F_\ell$ for all k if and only if F_ℓ is a Fermat cubic or the line L_m for some m. Consequently either F_0, F_1 are Fermat cubics or F_0, F_1, F_2 are conics conjugate by χ_k for some (or, equivalently, any) k.

It is checked in [12] that the second case does not occur. In the first one, we obtain the decomposition $D_{i,y} = \pi(F_0) \cup \pi(F_1)$ into lines. However it is easily shown that the discriminant of the conic $D_{i,y}$ equals $\pm 4b^6(y-\sqrt{b})(y+\sqrt{b})^2$, hence it is non-degenerate for $y \neq \pm\sqrt{b}$. □

Hereafter we assume that $y \neq \pm\sqrt{b}$. Let $\sigma_{i,y} \colon C'_{i,y} \to C_{i,y}$ be the corresponding normalization morphisms. As is well known,

$$\#\sigma_{i,y}^{-1}(\mathbf{Q}_i) = \#\sigma_{i,\beta}^{-1}(R_i) = \#\sigma_\infty^{-1}(\mathbf{Q}_k) = 3, \qquad \sigma_{i,y} \colon C'_{i,y} \setminus \sigma_{i,y}^{-1}\big(\operatorname{Sing}(C_{i,y})\big) \xrightarrow{\sim} C_{i,y} \setminus \operatorname{Sing}(C_{i,y}).$$

Further, we have the coverings $\pi_{i,y} := \pi \circ \sigma_{i,y} \colon C'_{i,y} \to D_{i,y}$ whose the Galois group is clearly isomorphic to $(\mathbb{Z}/3)^2$.

Theorem 4. *For $y \notin \{\beta, \infty\}$ the geometric genus $g(C_{i,y}) = 7$. Also, $g(C_{i,\beta}) = 4$, $g(C_\infty) = 1$.*

Proof. Denote by r_y the number of ramified points $Q \in D_{i,y}$. Since $\pi_{i,y}$ is a Galois covering, the well defined ramification index $e_Q \in \{3, 9\}$ (see, e.g., [15, Corollary 3.7.2]). It is obvious that $Q \in L_k$ for some $k \in \{0, 1, 2\}$. Moreover, the case $e_Q = 9$ may occur only for $Q \in \{R_k\}_{k=0}^2$. From Lemmas 2, 3 it follows that

$$\#(D_{i,y} \cap L_i) = 1, \qquad \#(D_{i,y} \cap L_{i-1}) = \#(D_{i,y} \cap L_{i+1}) = \begin{cases} 1 & \text{if} \quad y = \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Moreover, $R_{i-1}, R_{i+1} \notin D_{i,y}$, but $R_i \in D_{i,y}$ if and only if $y = \beta$. Therefore $r_y = 5$ for $y \notin \{\beta, \infty\}$, $r_\beta = 4$, and $r_\infty = 3$. Besides, according to Lemma 4 for all points $Q \in D_{i,y} \cap (\cup_{k=0}^2 L_k)$ we have $e_Q = 3$. Applying the Riemann–Hurwitz formula [3, Theorem II.5.9] to $\pi_{i,y}$, we eventually obtain $g(C_{i,y}) = 3r_y - 8$. $\square$

2 New hash function

This paragraph clarifies how the $\mathbb{F}_q$-section $\varphi\colon \mathbb{A}^2_{(t_1,t_2)} \dashrightarrow T$ from Theorem 1 results in a constant-time map $h\colon \mathbb{F}_q^2 \to E_b(\mathbb{F}_q)$. First of all, for $a \in \mathbb{F}_q^*$ denote by $\left(\frac{a}{q}\right)_3 := a^{(q-1)/3}$ the *cubic residue symbol*, which is trivially a group homomorphism $\mathbb{F}_q^* \to \{\omega^i\}_{i=0}^2$.

Lemma 5 ([16, Remark 2.3])**.** *An element $a \in \mathbb{F}_q^*$ is a cubic residue if and only if $\left(\frac{a}{q}\right)_3 = 1$. Moreover, in this case*

$$\sqrt[3]{a} = \begin{cases} \text{[17, Proposition 1]} & \text{if} \quad q \equiv 1 \pmod 9 \text{ and } q \not\equiv 1 \pmod{27}, \\ a^{-(q-4)/9} = a^{(8q-5)/9} & \text{if} \quad q \equiv 4 \pmod 9, \\ a^{(q+2)/9} & \text{if} \quad q \equiv 7 \pmod 9. \end{cases}$$

To be definite, we put $\omega := \left(\frac{b}{q}\right)_3$ ($\neq 1$ by our assumption). Also, let us consider only $q \not\equiv 1 \pmod{27}$.

Letting $g_i := y_i^2 - b$ for $i \in \{0, 1, 2\}$, we get $T\colon \left\{g_j = b^j g_0 t_j^3 \text{ for } j \in \{1, 2\}\right.$. It is obvious that $\left\{\left(\frac{g_i}{q}\right)_3\right\}_{i=0}^2 = \{\omega^i\}_{i=0}^2$ whenever $g_i, t_j \in \mathbb{F}_q^*$. Besides, denote by $n \in \{0, 1, 2\}$ the position number of an element $t_1 \in \mathbb{F}_q^*$ in the set $\left\{\omega^i t_1\right\}_{i=0}^2$ ordered with respect to some order in $\mathbb{F}_q^*$. For example, if q is a prime, then this can be the usual numerical one.

One of crucial components of h is the auxiliary map

$$h'\colon T(\mathbb{F}_q) \to E_b(\mathbb{F}_q) \qquad h'(y_0, y_1, y_2, t_1, t_2) := \begin{cases} \left(\sqrt[3]{g_0},\ y_0\right) & \text{if} \quad g_0 = 0 \ \text{ or } \left(\frac{g_0}{q}\right)_3 = 1, \\ \left(\sqrt[3]{g_1},\ y_1\right) & \text{if} \quad \left(\frac{g_0}{q}\right)_3 = \omega^2, \\ \left(\sqrt[3]{g_2},\ y_2\right) & \text{if} \quad \left(\frac{g_0}{q}\right)_3 = \omega. \end{cases}$$

Unfortunately, in this form the value of h' is computed with the cost of two exponentiations in $\mathbb{F}_q$: the first for $\left(\frac{g_0}{q}\right)_3$ and the second for $\sqrt[3]{g_i}$. Instead, we give an equivalent definition of h' (up to the automorphisms $[\omega]^i$).

The case $q \equiv 4 \pmod 9$. Under this assumption

$$\left(\frac{\omega}{q}\right)_3 = \omega^{(q-1)/3} = \omega^{(q-4)/3} \cdot \omega = \omega^{3(q-4)/9} \cdot \omega = \omega.$$

Let $\theta := g_0^{(8q-5)/9}$ and $c_j := \sqrt[3]{(b/\omega)^j} \in \mathbb{F}_q^*$. We obtain

$$g_j = b^j g_0 t_j^3 = (c_j \theta t_j)^3 \qquad \text{if} \qquad \theta^3 = \omega^j g_0, \ \text{i.e.,} \ \left(\frac{g_0}{q}\right)_3 = \omega^{3-j}.$$

It is easily shown that

$$h' \colon T(\mathbb{F}_q) \to E_b(\mathbb{F}_q) \qquad h'(y_0, y_1, y_2, t_1, t_2) = \begin{cases} (\omega^n \theta, \ y_0) & \text{if} \quad \theta^3 = g_0, \\ (c_1 \theta t_1, \ y_1) & \text{if} \quad \theta^3 = \omega g_0, \\ (c_2 \theta t_2, \ y_2) & \text{if} \quad \theta^3 = \omega^2 g_0. \end{cases}$$

Since

$$\theta^3 = g_0^{-(q-4)/3} = g_0^{q-1-(q-4)/3} = g_0^{(2q+1)/3} = g_0^{2(q-1)/3} \cdot g_0,$$

this map is well defined everywhere on $T(\mathbb{F}_q)$. It is worth noting that θ can be computed with the cost of one exponentiation in $\mathbb{F}_q$ even if g_0 is given as a fraction u/v for $u \in \mathbb{F}_q$, $v \in \mathbb{F}_q^*$. Indeed,

$$\theta = (u/v)^{(8q-5)/9} = u^{(8q-5)/9} \cdot v^{(q-4)/9} = u^3 (u^8 v)^{(q-4)/9}. \tag{2}$$

The case $q \equiv 10 \pmod{27}$ (relevant for BLS12-381). Take any $\zeta := \sqrt[9]{1} \in \mathbb{F}_q^*$ such that $\zeta^3 = \omega$. In this case

$$\left(\frac{\zeta}{q}\right)_3 = \zeta^{(q-1)/3} = \omega^{(q-1)/9} = \omega^{(q-10)/9} \cdot \omega = \omega^{3(q-10)/27} \cdot \omega = \omega.$$

Let $\theta := g_0^{(2q+7)/27}$ and $c_j := \sqrt[3]{(b/\zeta)^j} \in \mathbb{F}_q^*$. Given $i \in \{0, 1, 2\}$ we obtain

$$g_j = b^j g_0 t_j^3 = (c_j \theta t_j)^3 / \omega^i \qquad \text{if} \qquad \theta^3 = \omega^i \zeta^j g_0, \ \text{i.e.,} \ \left(\frac{g_0}{q}\right)_3 = \omega^{3-j}.$$

It is easily shown that

$$h' \colon T(\mathbb{F}_q) \to E_b(\mathbb{F}_q) \qquad h'(y_0, y_1, y_2, t_1, t_2) = \begin{cases} (\omega^n \theta / \zeta^i, \ y_0) & \text{if} \quad \exists i \colon \theta^3 = \omega^i g_0, \\ (c_1 \theta t_1 / \zeta^i, \ y_1) & \text{if} \quad \exists i \colon \theta^3 = \omega^i \zeta g_0, \\ (c_2 \theta t_2 / \zeta^i, \ y_2) & \text{if} \quad \exists i \colon \theta^3 = \omega^i \zeta^2 g_0. \end{cases}$$

Since

$$\theta^3 = g_0^{(2q+7)/9} = g_0^{2(q-1)/9} \cdot g_0,$$

this map is well defined everywhere on $T(\mathbb{F}_q)$. It is worth noting that θ can be computed with the cost of one exponentiation in $\mathbb{F}_q$ even if g_0 is given as a fraction u/v for $u \in \mathbb{F}_q$, $v \in \mathbb{F}_q^*$. Indeed,

$$\begin{aligned} \theta = (u/v)^{(2q+7)/27} = u^{(2q+7)/27} \cdot v^{q-1-(2q+7)/27} = u^{(2q+7)/27} \cdot v^{(25q-34)/27} = \\ = u \cdot u^{2(q-10)/27} \cdot v^3 v^{5(5q-23)/27} = u v^8 (u^2 v^{25})^{(q-10)/27}. \end{aligned} \tag{3}$$

The cases $q \equiv 7 \pmod 9$ and $q \equiv 19 \pmod{27}$ are processed in a similar way. To be definite, throughout the rest of the article we will deal with the modified version of h'. Finally, we come to the map desired

$$h\colon \mathbb{F}_q^2 \to E_b(\mathbb{F}_q) \qquad\qquad h(t_1,t_2) := \begin{cases} P_0 & \text{if} \quad t_1t_2 = 0, \\ \mathcal{O} & \text{if} \quad den(t_1,t_2) = 0, \\ (h' \circ \varphi)(t_1,t_2) & \text{otherwise.} \end{cases}$$

We emphasize that in the definition of h' (a fortiori, in φ) the cubic residue symbol does not appear. Further, by returning the value of h in (weighted) projective coordinates, we entirely avoid inversions in the field. Besides, the constants ω, c_j (and ζ, $\zeta^{-1} = \zeta^8$ if $q \equiv 10 \pmod{27}$) are found once at the precomputation stage. By the way, in the formulas (2), (3) we take $u := num_0^2 - b{\cdot}den^2$ and $v := den^2$. Calculating the value θ every time no matter whether $t_0t_1uv = 0$ or not, we eventually obtain

Remark 1. *The map h is computed in constant time, namely in that of one exponentiation in $\mathbb{F}_q$.*

3 Indifferentiability from a random oracle

Theorem 5. *For any point $P \in E_b(\mathbb{F}_q) \setminus \{\pm P_0, \mathcal{O}\}$ we have*

$$|\#h^{-1}(P) - (q+1)| \leqslant 7\lfloor 2\sqrt{q} \rfloor + 6, \qquad |\#h^{-1}(P_0) - 3q| \leqslant \lfloor 2\sqrt{q} \rfloor,$$

$$|\#h^{-1}(-P_0) - 2(q+1)| \leqslant 2\lfloor 2\sqrt{q} \rfloor, \qquad |\#h^{-1}(\mathcal{O}) - (q+1)| \leqslant \lfloor 2\sqrt{q} \rfloor.$$

Proof. All the inequalities follow from the Hasse–Weil–Serre bound [15, Theorem 5.3.1] for the number of $\mathbb{F}_q$-points on a projective non-singular absolutely irreducible $\mathbb{F}_q$-curve.

First, suppose that $h(t_1,t_2) = \pm P_0$. Then $t_1t_2 = 0$ or $\theta = g_0 = 0$. In the first case, $h(0,t_2) = h(t_1,0) = P_0$. In the second one, $(1 : t_1 : t_2) \in C_{0,\pm\sqrt{b}}$. These curves decompose as $C_{0,\sqrt{b}} = L_0 \cup F_0$ and $C_{0,-\sqrt{b}} = F_1 \cup F_2$, where F_k are Fermat cubics (cf. the equations (1)). The latter are obviously elliptic curves (of j-invariant 0). In accordance with Lemma 3 we have $(C_{0,\pm\sqrt{b}} \cap C_\infty)(\mathbb{F}_q) = \emptyset$. Note also that $(F_1 \cap F_2)(\mathbb{F}_q) = (L_i \cap F_k)(\mathbb{F}_q) = \emptyset$ for all $i,k \in \{0,1,2\}$.

In turn, $(C_\infty \cap L_k)(\mathbb{F}_q) = \emptyset$ according to Lemma 3, hence $h^{-1}(\mathcal{O}) = C_\infty(\mathbb{F}_q)$. Besides, $\mathrm{Sing}(C_\infty)(\mathbb{F}_q) = \emptyset$ (see Lemma 4). As a result, we obtain the bijection $\sigma_\infty\colon C'_\infty(\mathbb{F}_q) \xrightarrow{\sim} C_\infty(\mathbb{F}_q)$. Finally, the geometric genus $g(C_\infty) = 1$ by virtue of Theorem 4.

Now take $P = (x,y) \in E_b(\mathbb{F}_q) \setminus \{\pm P_0, \mathcal{O}\}$. The case $y = \beta$ does not occur, because $\beta^2 - b = 8b$ is not a cubic residue in $\mathbb{F}_q$. In compliance with Lemmas 2, 3 we see that

$$(C_{i,y} \cap C_\infty)(\mathbb{F}_q) = (C_{i,y} \cap C_{i+1,y})(\mathbb{F}_q) = (C_{i,y} \cap L_i)(\mathbb{F}_q) = \emptyset, \qquad \#(C_{i,y} \cap L_k)(\mathbb{F}_q) \leqslant 3$$

for all $i,k \in \{0,1,2\}$. Besides, the x-coordinates of $h(t_1,t_2)$ and $h(\omega t_1,t_2)$ (resp. $h(t_1,\omega t_2)$) are always different if $i \in \{0,1\}$ (resp. $i = 2$), because $\theta(t_1,t_2) = \theta(\omega t_1,t_2) = \theta(t_1,\omega t_2)$. Therefore

$$h^{-1}\big(\{P, [\omega](P), [\omega]^2(P)\}\big) \;=\; \bigsqcup_{i=0}^{2} h^{-1}\big([\omega]^i(P)\big) \;=\; \bigsqcup_{i=0}^{2} C_{i,y}(\mathbb{F}_q) \setminus (L_{i-1} \cup L_{i+1}).$$

Since $\#h^{-1}\big([\omega]^i(P)\big) = \#h^{-1}\big([\omega]^{i+1}(P)\big)$, we obtain

$$3\cdot\#h^{-1}(P) = \sum_{i=0}^{2} \#C_{i,y}(\mathbb{F}_q) \setminus (L_{i-1} \cup L_{i+1}).$$

Consequently,

$$\sum_{i=0}^{2}(\#C_{i,y}(\mathbb{F}_q) - 6) \leqslant 3\cdot\#h^{-1}(P) \leqslant \sum_{i=0}^{2}\#C_{i,y}(\mathbb{F}_q).$$

Further, $\#C_{i,y}(\mathbb{F}_q) = \#C_{i+1,y}(\mathbb{F}_q)$ according to Lemma 2. Thus

$$3(\#C_{i,y}(\mathbb{F}_q) - 6) \leqslant 3\cdot\#h^{-1}(P) \leqslant 3\cdot\#C_{i,y}(\mathbb{F}_q)$$

and hence

$$|\#h^{-1}(P) - \#C_{i,y}(\mathbb{F}_q)| \leqslant 6.$$

At the same time, Theorem 4 says that $g(C_{i,y}) = 7$. Besides, $\mathrm{Sing}(C_{i,y})(\mathbb{F}_q) = \emptyset$ (see Lemma 4). As a result, $\sigma_{i,y}\colon C'_{i,y}(\mathbb{F}_q) \simeq C_{i,y}(\mathbb{F}_q)$. We eventually obtain

$$|\#h^{-1}(P) - (q+1)| \leqslant |\#h^{-1}(P) - \#C_{i,y}(\mathbb{F}_q)| + |\#C_{i,y}(\mathbb{F}_q) - (q+1)| \leqslant 6 + 7\lfloor 2\sqrt{q}\rfloor.$$

The theorem is proved. □

Corollary 1. *The map $h\colon \mathbb{F}_q^2 \to E_b(\mathbb{F}_q)$ is surjective at least for $q \geqslant 211$.*

Corollary 2. *The distribution on $E_b(\mathbb{F}_q)$ defined by h is ϵ-statistically indistinguishable from the uniform one* [8, Definition 3], *where $\epsilon := 16q^{-1/2} + O(q^{-1})$.*

Proof. For any point $P \in E_b(\mathbb{F}_q)$ put

$$\delta(P) := \left|\frac{\#h^{-1}(P)}{q^2} - \frac{1}{\#E_b(\mathbb{F}_q)}\right| \leqslant \left|\frac{\#h^{-1}(P)}{q^2} - \frac{1}{q}\right| + \left|\frac{1}{q} - \frac{1}{\#E_b(\mathbb{F}_q)}\right| =$$

$$= \frac{|\#h^{-1}(P) - q|}{q^2} + \frac{|\#E_b(\mathbb{F}_q) - q|}{q\cdot\#E_b(\mathbb{F}_q)} \leqslant \frac{|\#h^{-1}(P) - q|}{q^2} + \frac{\lfloor 2\sqrt{q}\rfloor + 1}{q(q+1-\lfloor 2\sqrt{q}\rfloor)} =$$

$$= \frac{|\#h^{-1}(P) - q|}{q^2} + \frac{2}{q^{3/2}} + O\Big(\frac{1}{q^2}\Big).$$

If $P \notin \{\pm P_0, \mathcal{O}\}$ from Theorem 5 we obtain

$$\delta(P) = \frac{16}{q^{3/2}} + O\Big(\frac{1}{q^2}\Big).$$

Similarly,

$$\delta(P_0) = \frac{2}{q} + O\Big(\frac{1}{q^{3/2}}\Big), \qquad \delta(-P_0) = \frac{1}{q} + O\Big(\frac{1}{q^{3/2}}\Big), \qquad \delta(\mathcal{O}) = \frac{4}{q^{3/2}} + O\Big(\frac{1}{q^2}\Big).$$

Thus

$$\sum_{P\in E_b(\mathbb{F}_q)}\delta(P) \leqslant (q + \lfloor 2\sqrt{q}\rfloor - 2)\Big(\frac{16}{q^{3/2}} + O\Big(\frac{1}{q^2}\Big)\Big) + \frac{3}{q} + O\Big(\frac{1}{q^{3/2}}\Big) = \frac{16}{q^{1/2}} + O\Big(\frac{1}{q}\Big).$$

The corollary is proved. □

For $t_2 \in \mathbb{F}_q$ consider the encoding $h_{t_2}\colon \mathbb{F}_q \to E_b(\mathbb{F}_q)$ of the form $h_{t_2}(t_1) := h(t_1, t_2)$. By definition, $h_0(t_1) = P_0$ for any $t_1 \in \mathbb{F}_q$. Nevertheless, by analogy with [10, Theorem 2] we can prove the next lemma. Its main difference is that $h_{t_2}(t_1) = h_{t_2}(\omega t_1)$ whenever $\sqrt[3]{g_2} \in \mathbb{F}_q$, hence 10 appears instead of 6.

Lemma 6. *For $t_2 \in \mathbb{F}_q^*$ and $P \in E_b(\mathbb{F}_q)$ we have $\#h_{t_2}^{-1}(P) \leqslant 10$ and hence $q/10 \leqslant \#\mathrm{Im}(h_{t_2})$.*

By this lemma [8, Algorithm 1] still works well in the case of h. Indeed, for $P \in E_b(\mathbb{F}_q)$ pick uniformly at random $t_2 \in \mathbb{F}_q$ and then find uniformly at random $t_1 \in h_{t_2}^{-1}(P)$. This gives

Remark 2. *The map h is samplable* [8, Definition 4].

Remarks 1, 2 and Corollary 2 imply that h is *admissible* in the sense of [8, Definition 4]. Finally, using [8, Theorem 1], we establish

Corollary 3. *Consider the composition $H := h \circ \mathfrak{h}\colon \{0,1\}^* \to E_b(\mathbb{F}_q)$ of a hash function $\mathfrak{h}\colon \{0,1\}^* \to \mathbb{F}_q^2$ and h. The hash function H is indifferentiable from a random oracle if $\mathfrak{h}$ is so.*

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